Study of an electro-mechanical device with figure-eight homoclinics in a domain with dissipation.

Dynamics, Bifurcations, and Strange Attractors
To the memory of Leonid Shilnikov.

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Contents of the talk

1. The electro-mechanical device.

2. General theory (previous work with Sergey Gonchenko).

   a) Description of the bifurcation diagram (qualitative description).

   b) A return map model (quantitative description).

3. Some numerical computations using the electro-mechanical system.

A simple electro-mechanical device (I)

We propose to study the dynamics of a parametrically driven dissipative pendulum with a magnetic kick forcing acting on it. That is,

1. Consider a gravitational pendulum

$$x'' + \sin(x) = 0, \qquad x \in (-\pi, \pi]$$

2. Add the effect of dissipation (damping) in an asymmetric way (asymmetric bulk shape)

$$x'' + a_{\pm} x' + \sin(x) = 0,$$

where $a_+ > 0$ is used when $x' > 0$ and $a_- > 0$ is used when $x' < 0$.

$\longrightarrow$ Note that under the damping effect:

- The separatrices are destroyed.
- The elliptic point becomes focus stable.

A simple electro-mechanical device (II)

3. By using a photoelectric cell system detect the passage of the pendulum through the minimum (at $x = 0$) and estimate the velocity. Then “kick” the pendulum (by exciting a strong magnetic or electric field) to increment its velocity by a quantity $\Delta x' = x'_{\rm new} - x'$ which depends on the instant velocity x' at the passage through the minimum:

$$\Delta x' = J_{\pm}(|x'|),\ J_{\pm}(0) = 0,$$

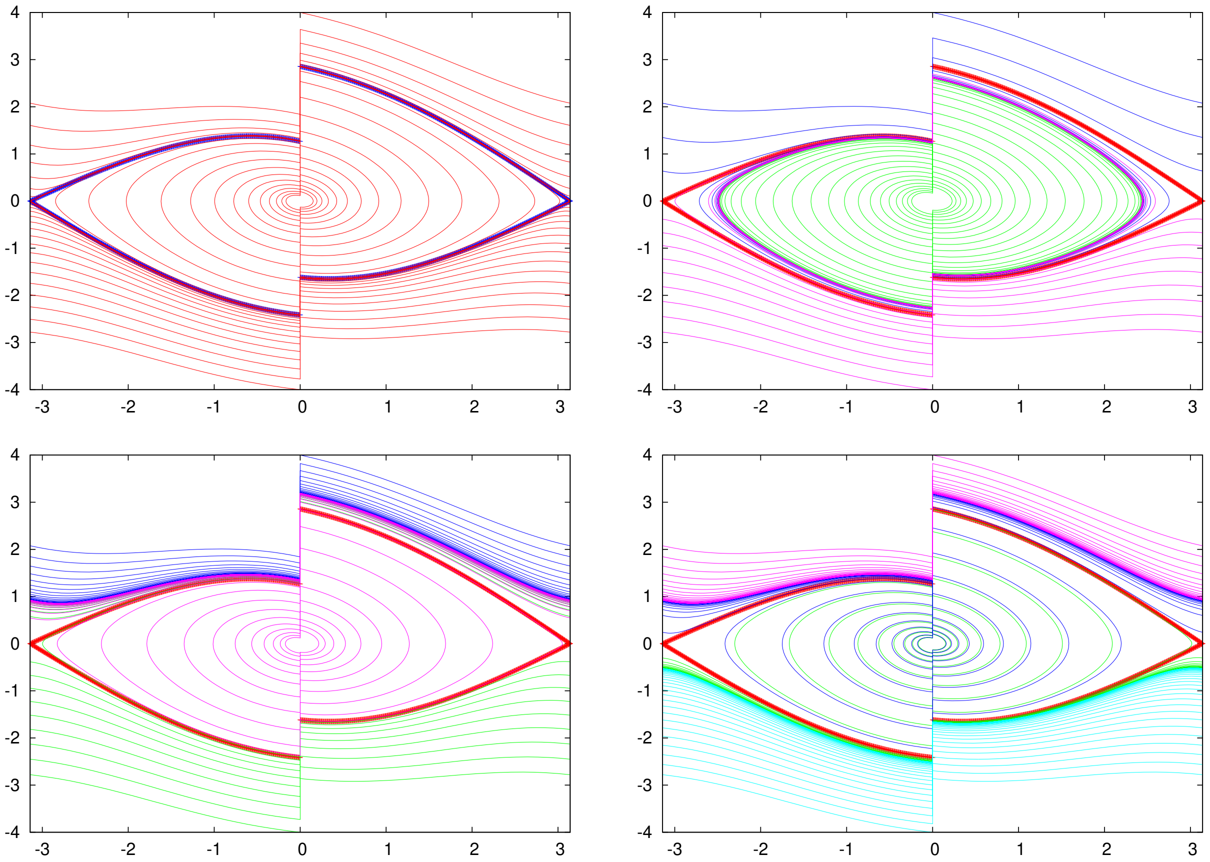
where $J_+(|x'|) \geq 0$ (resp. $J_-(|x'|) \leq 0$) are suitable “jump” functions used in the case of crossing $x = 0$ with positive velocity (resp. with negative velocity).

$\longrightarrow$ Note that with damping + kick:

- Using suitable “jump” functions we can recover the separatrices.
- The elliptic point becomes focus unstable.

Unfolding the homoclinic loop

For a fixed damping ($a_+ = 0.4$ and $a_- = 0.2$) there are $J_\pm$ such that the pendulum recovers the separatrices. Then we modify the jump function J_+ (resp. J_-) by a multiplicative factor $f_+ = 1 + \mu_1$ (resp. $f_- = 1 + \mu_2$). We consider the parameter space $\mu = (\mu_1, \mu_2)$.



$\mu = (0, 0), (-0.1, -0.05)$

$\mu = (0.1, -0.05), (0.1, 0.05)$

A simple electro-mechanical device (III)

4. Attach the pendulum to a periodic vertically oscillating support.

   This is a **non-autonomous perturbation** (the pendulum being parametrically excited).

   To fix ideas, without the “kick” the equation reads

   $$x'' + a_{\pm} x' + (1 + \epsilon \cos(6\,t)) \sin(x) = 0.$$

   But we also consider the kick effect as before.

Then:

1. pendulum + 2. damping + 3. kick + 4. oscillating support = complete model!

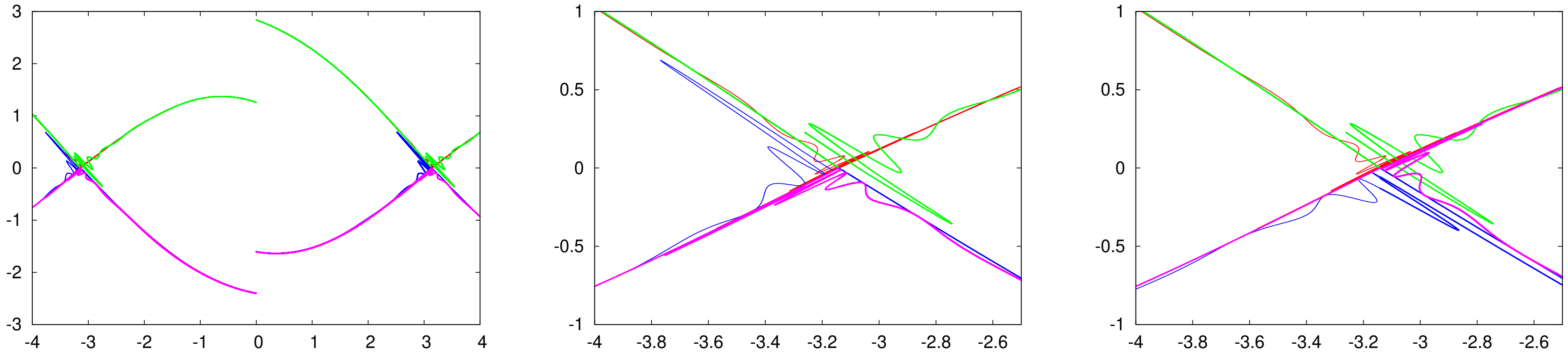
Finally, to study the model...

we consider $T_{\mu,\epsilon}$ the stroboscopic Poincaré map at a time $\tau = \pi/3$.

The complete device model

$a_+ = 0.4$, $a_- = 0.2$, $\epsilon = 0.2$, lower tangencies ($\mu_1 = 0$)

last tangency $\mu_2 = 0.00555$, first tangency $\mu_2 = -0.00579$



Statement of the bifurcation problem

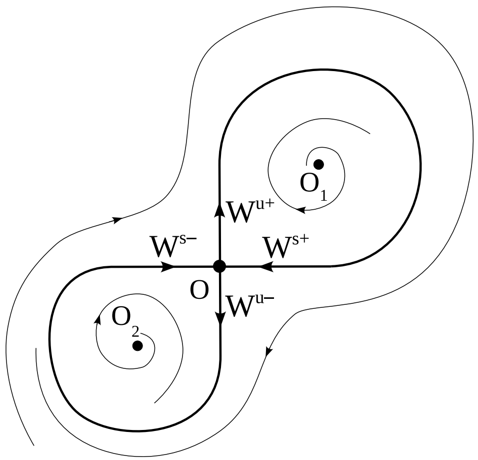
For a fixed damping (i.e. $a_\pm$ fixed), we look for the jump functions $J_\pm$ defining a suitable "kick" for which the separatrices are recovered. Then, for a given ϵ small enough, we add the non-autonomous effect. Finally we modify the jump functions as before and we study the parameter space (μ_1, μ_2) for the stroboscopic map $T_{\mu,\epsilon}$ (at a fixed ϵ!).

II. General Theory

The general framework

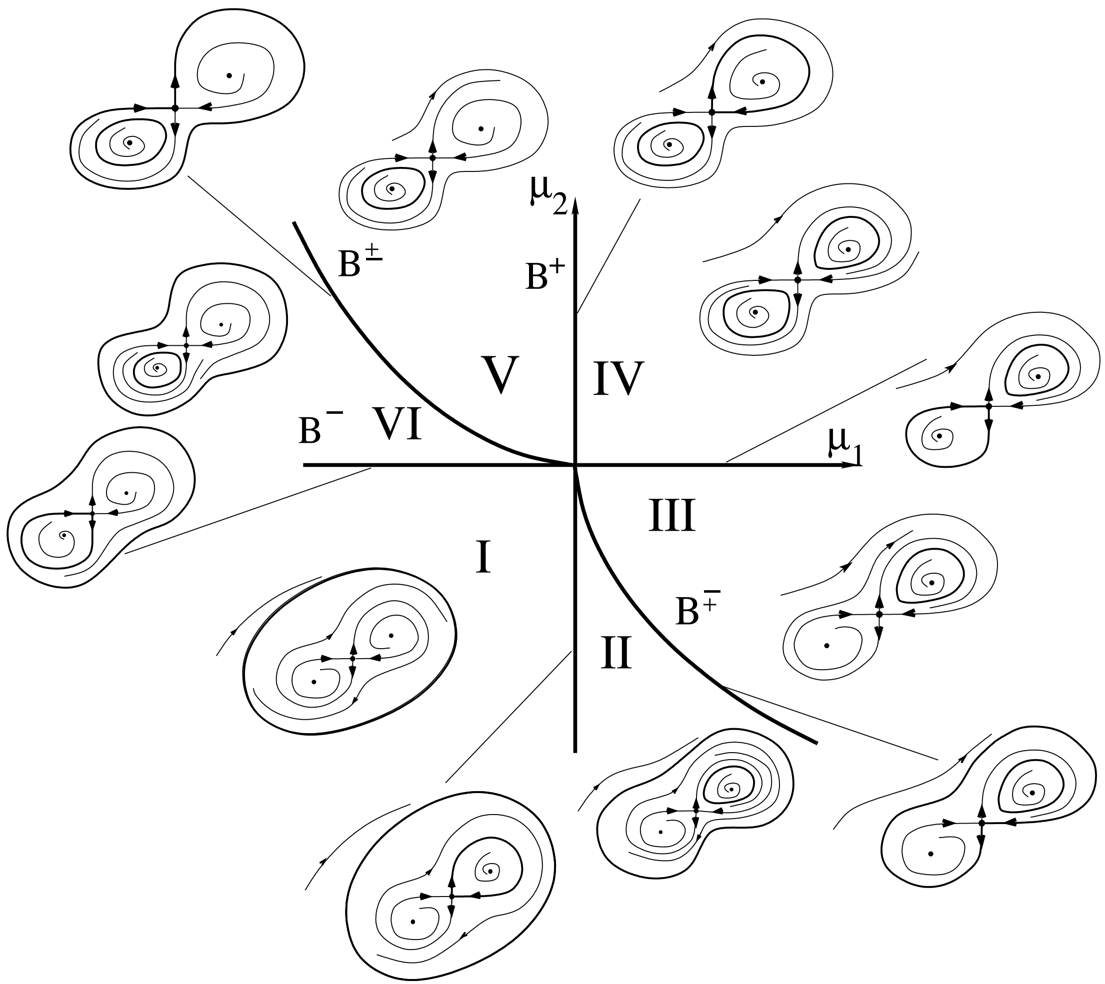
$T_{\mu,\epsilon}$ fits within the **Gonchenko-Simó-V.** [a] framework where we study the **bifurcation diagram** of the **Poincaré map** of a **non-autonomous** (2π time-periodic) $\mathcal{O}(\epsilon)$-perturbation of an **autonomous** family of vector fields f_μ.

- The non-autonomous perturbation is assumed to be fixed and sufficiently small (equivalently, ϵ is a small given value).
- The family of autonomous systems f_μ is a 2-parameter unfolding of the system f_0, which we assume to posses a **homoclinic figure-eight** to a **dissipative** saddle point.



[a] S.Gonchenko, C.Simó and A.Vieiro. Richness of dynamics and global bifurcations in systems with a homoclinic figure-eight. Nonlinearity 26(3):621–679, 2013.

The flow (autonomous $\epsilon = 0$) case



- **Six** regions.
- **Boundaries:**

  $W^{u+} = W^{s+}$ (I→II);

  $W^{u-} = W^{s+}$ (II→III);

  $W^{u-} = W^{s-}$ (III→IV);

  $W^{u+} = W^{s+}$ (IV→V);

  $W^{u+} = W^{s-}$ (V→VI);

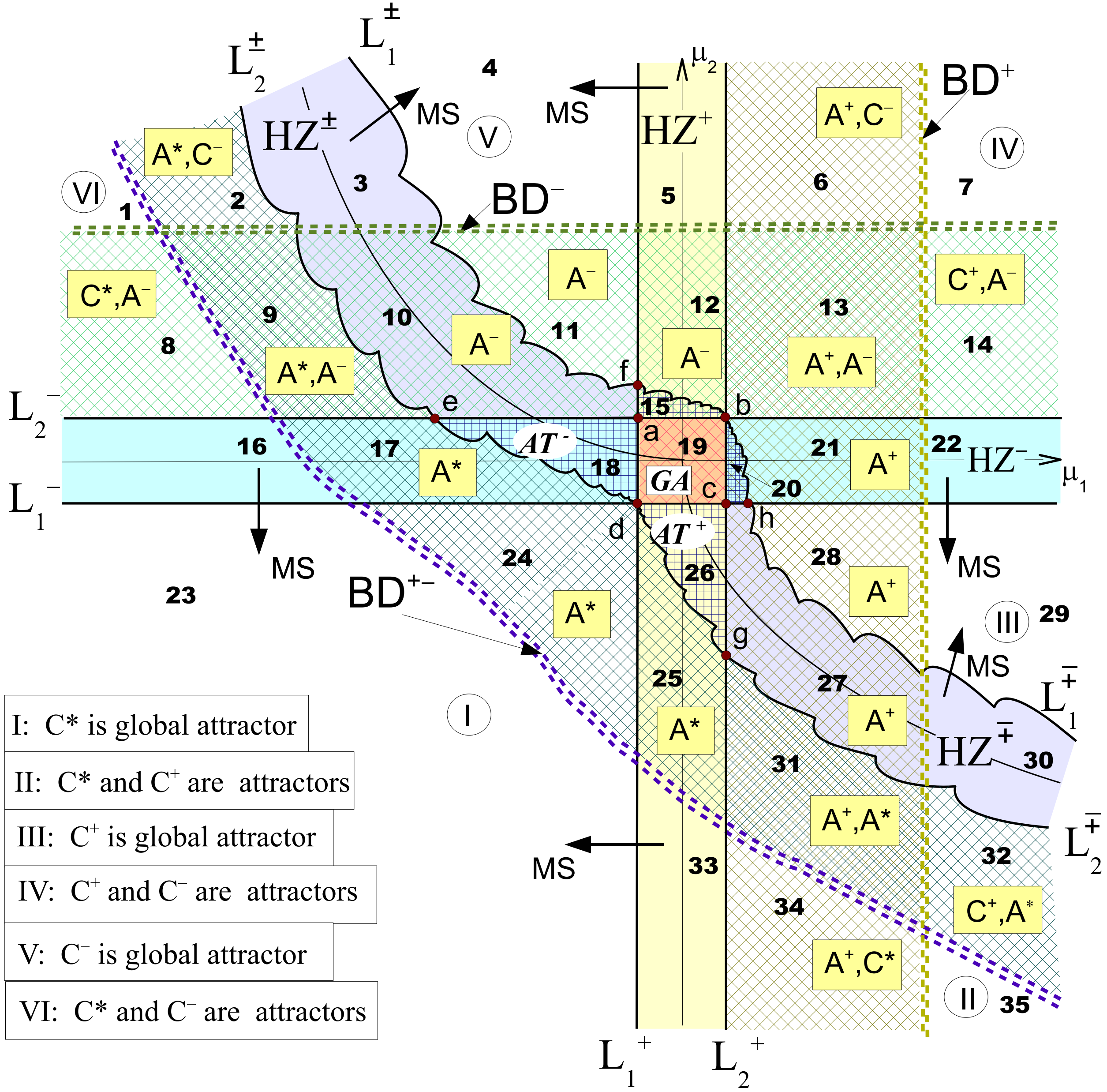
  $W^{u-} = W^{s-}$ (VI→I);

D. Turaev. On a case of bifurcation of a contour composed by two homoclinic curves of a saddle.

Methods of the qualitative theory of differential equations, Ed. Gorki, 1984, 162–175.

The diffeomorphism case: bifurcation diagram

We consider the effect of the non-autonomous perturbation and we look at the Poincaré map.



Properties of the bifurcation diagram of $T_{\mu,\epsilon}$

1. There appear **35 regions** with different dynamics!

2. These regions are separated by first/last tangency curves

$$\mathrm{L}_1^+,\ \mathrm{L}_2^+,\ \mathrm{L}_1^-,\ \mathrm{L}_2^-, \mathrm{L}_1^\pm,\ \mathrm{L}_2^\pm,\ \mathrm{L}_1^\mp,\ \mathrm{L}_2^\mp,$$

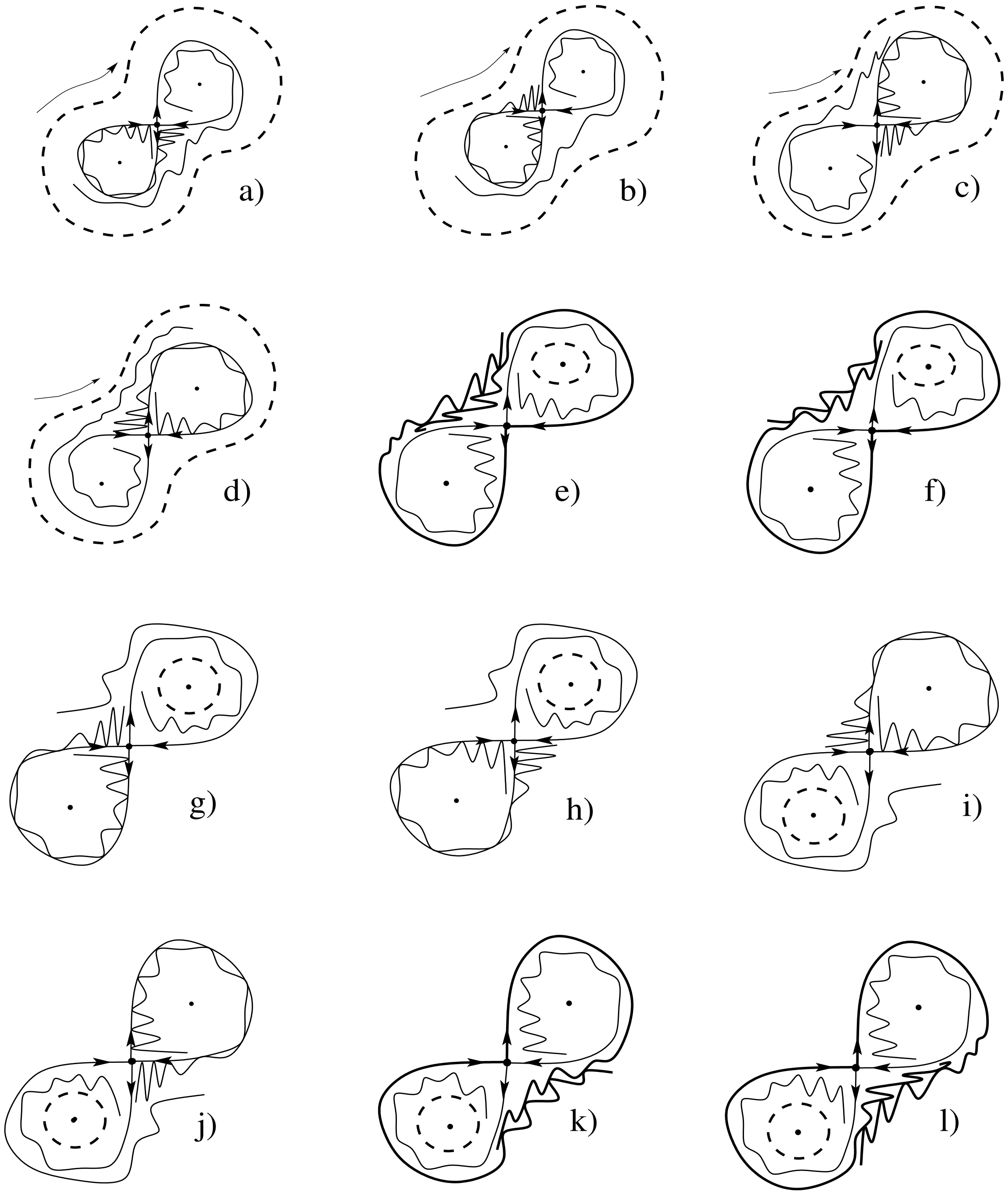
   and/or by "curves" that indicate transitions from "simple" dynamics to strange attractor (e.g. folding of an invariant curve can cause collision between tangent/normal bundles and create a SA)

$$\mathrm{BD}^+, \mathrm{BD}^-, \mathrm{BD}^{+-}.$$

3. Only the $\mathrm{L}_{1,2}^{+,-}$ are smooth. The curves $\mathrm{L}_{1,2}^{\pm,\mp}$ have a complicated structure (later) with infinitely many intervals of smoothness.

4. Multiple attractors can coexist.

Homoclinic tangencies – phase space



a) $\mu \in L_2^-$, $\mu_1 < 0$;

b) $\mu \in L_1^-$, $\mu_1 < 0$;

c) $\mu \in L_1^+$, $\mu_2 < 0$;

d) $\mu \in L_2^+$, $\mu_2 < 0$;

e) $\mu \in L_2^{\mp}$;

f) $\mu \in L_1^{\mp}$;

g) $\mu \in L_1^-$, $\mu_1 > 0$;

h) $\mu \in L_2^-$, $\mu_1 > 0$;

i) $\mu \in L_2^+$, $\mu_2 > 0$;

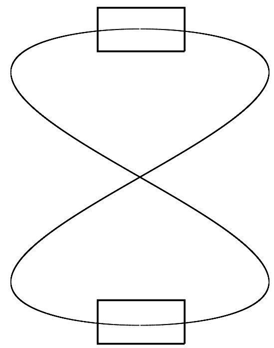
j) $\mu \in L_1^+$, $\mu_2 > 0$;

k) $\mu \in L_1^{\pm}$;

l) $\mu \in L_2^{\pm}$.

A quantitative model: dissipative separatrix map

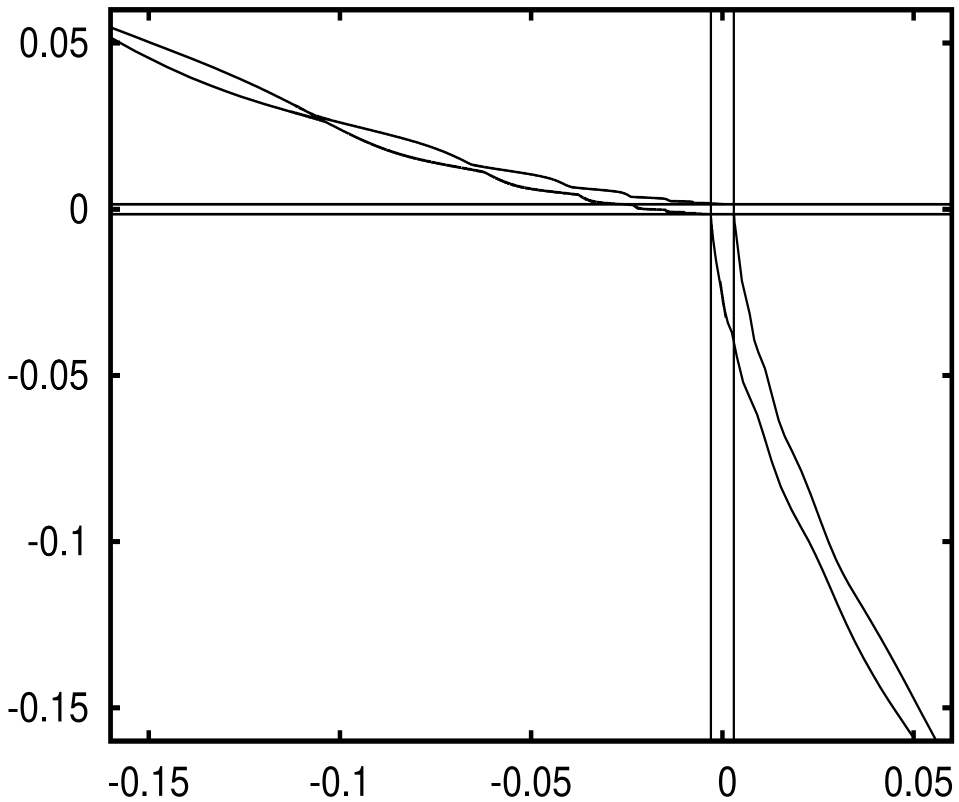
$$M_{a,b,\psi,A,\omega} : \begin{pmatrix} z \\ \eta \\ s \end{pmatrix} \mapsto \begin{pmatrix} z + \omega_j + A\log(|y|) \\ \mathsf{sign}(y)|y|^{\psi} \\ \mathsf{sign}(y)s \end{pmatrix}$$

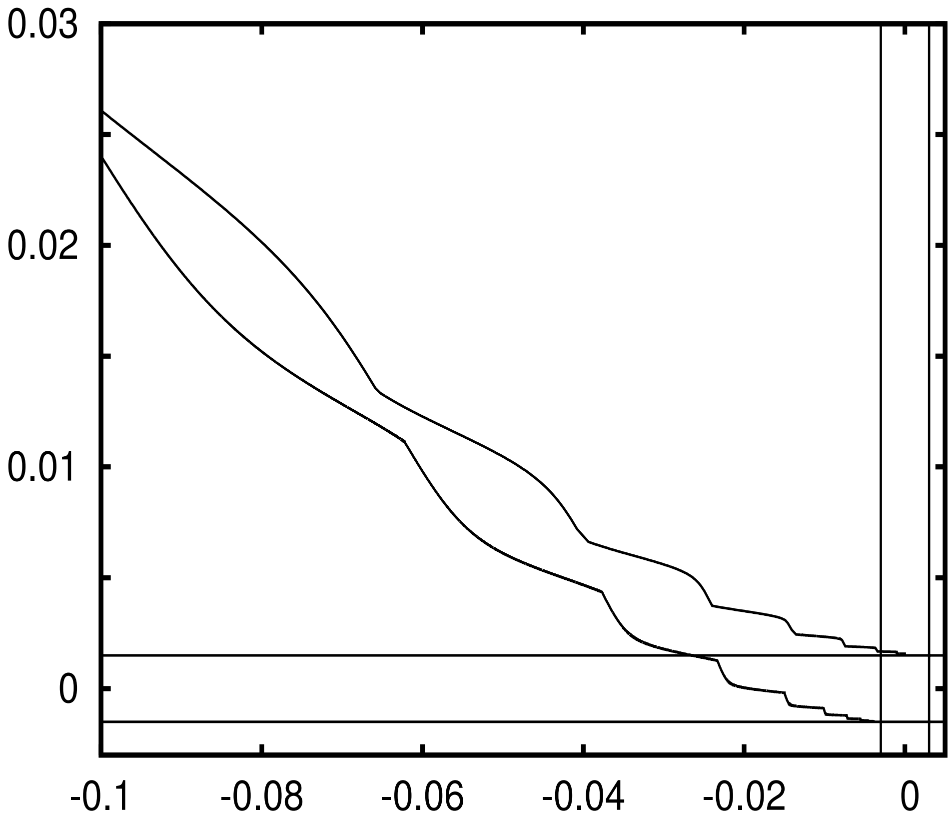


- → FD = **two annuli**: the index j equals 1 if $s = 1$ and $j = 2$ if $s = -1$.
- → $\psi = \lambda/\gamma$ accounts for the dissipation in the passage near the saddle.
- → **Returning time** = constant ω_j +"flying" time $A\log(y)$ near the saddle.
- → $y = a_j + \eta + b_j \sin(2\pi z)$, and for both η (distance w.r.t. W^u) and y (distance w.r.t. W^s) the positive orientation points towards the saddle.
- → If $a_j = b_j = 0$ both branches $W^{u/s}$ coincide.
  For $b_j = 0$ it mimics the vector field provided $|a_j| < (\psi - 1)/\psi^{\psi/(\psi-1)}$.
  Then b_j play the role of ϵ (they undulate the inv. manifolds).
- → In the **simulations**: $\omega_j = 0$, $A = 2$, $\psi = 1.6$, $b_1 = 0.003$, $b_2 = 0.0015$.
  Then a_1, a_2 are taken as **leading parameters** ranging in $[-0.15, 0.15]$.

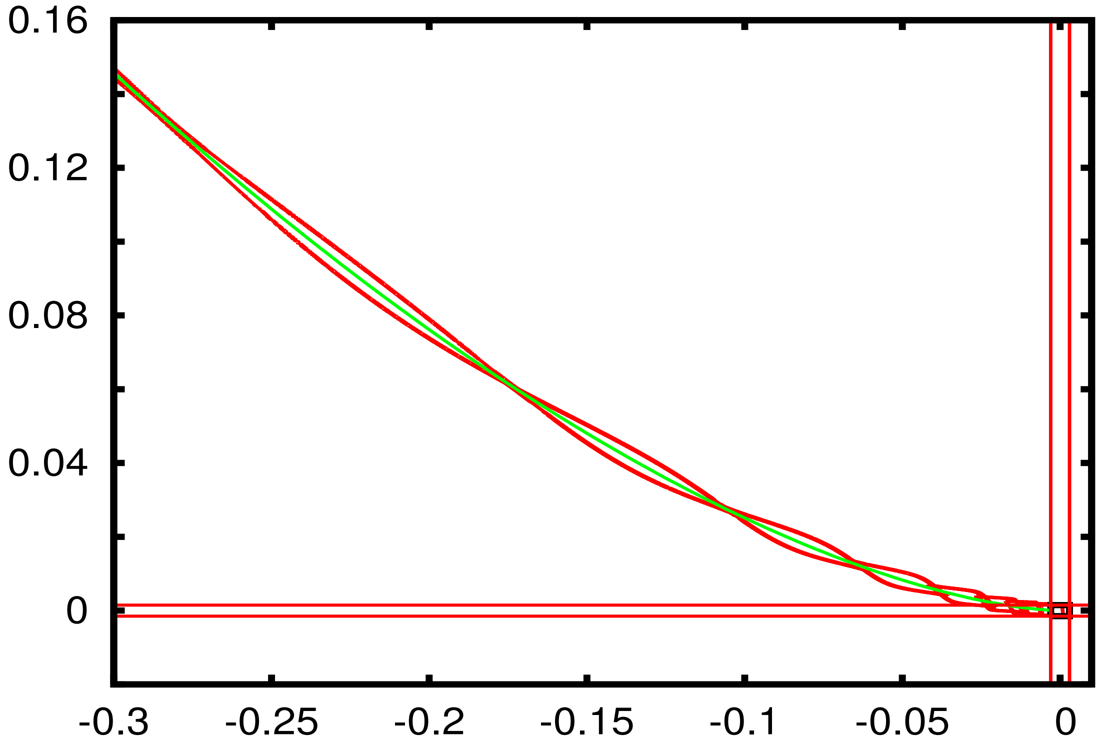
A preliminary numerical exploration of the model

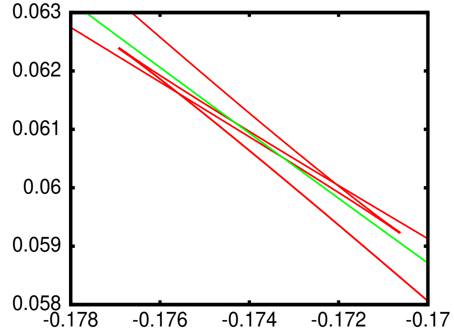
In the (a_1, a_2)-parameter space we compute first/last primary homoclinic quadratic tangency curves between $W^{u\pm} = \{\eta = 0, s = \pm 1\}$ and $W^{s\pm} = \{y = 0, s = \pm 1\}$. The curves $L^{\pm}_{1,2}$ and $L^{\mp}_{1,2}$ are the envelope of different bifurcating curves (related to different primary quadratic tangencies) that bound a "diagonal" strip with "stair-type" structure. Essentially **8 curves**.

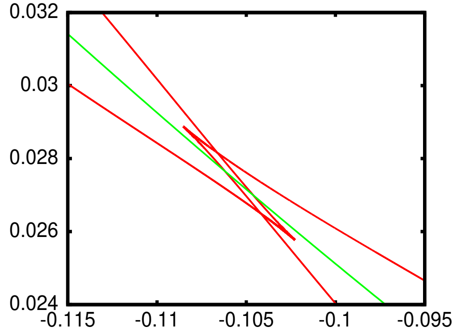


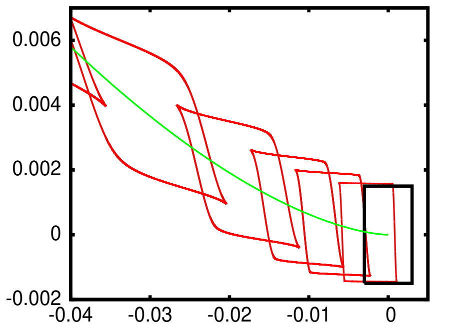


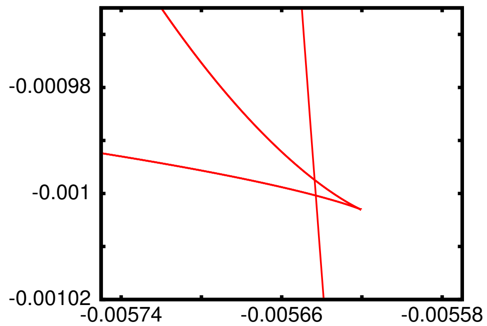
Bifurcating curves within $HZ^{\pm}$

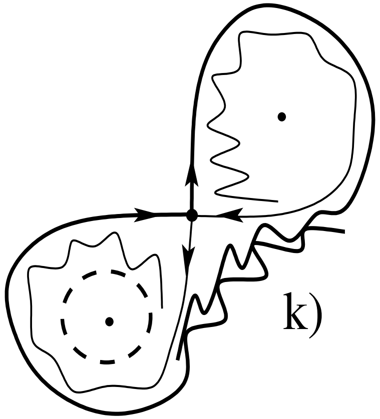


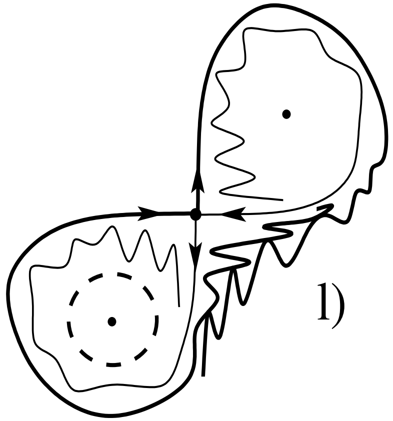


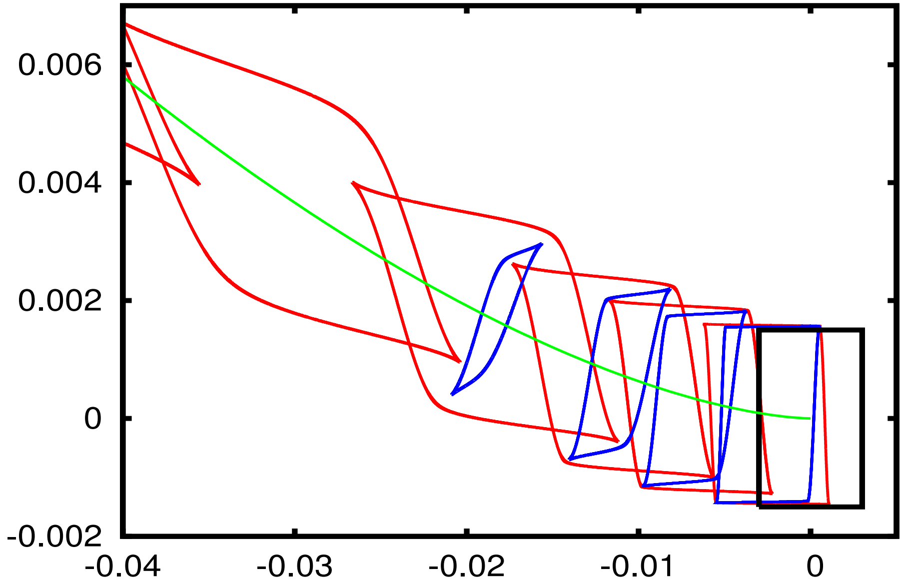




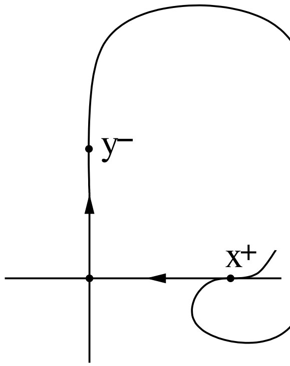








Cubic single-round homoclinic tangencies

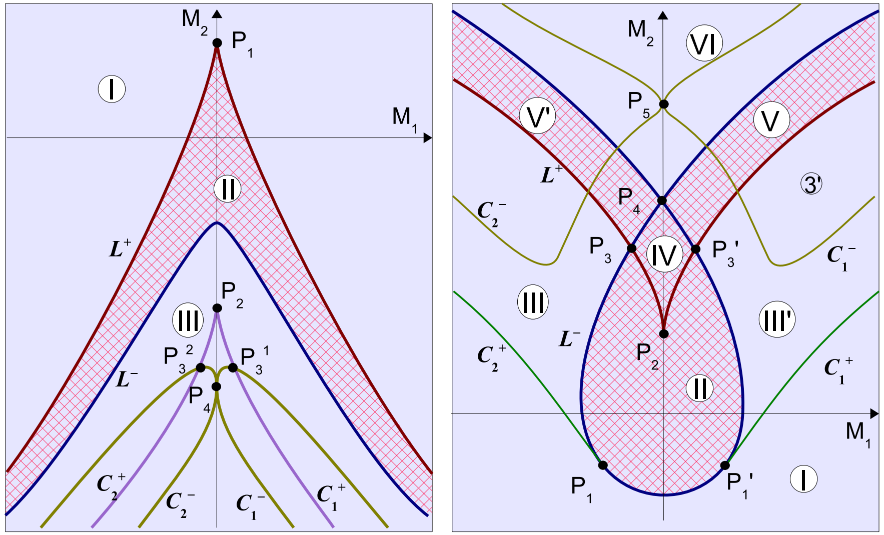


Outer map:

$$\begin{aligned} \bar{x} - x^{+} &= ax + b(y - y^{-}), \\ \bar{y} &= cx + d(y - y^{-})^{3}. \end{aligned}$$

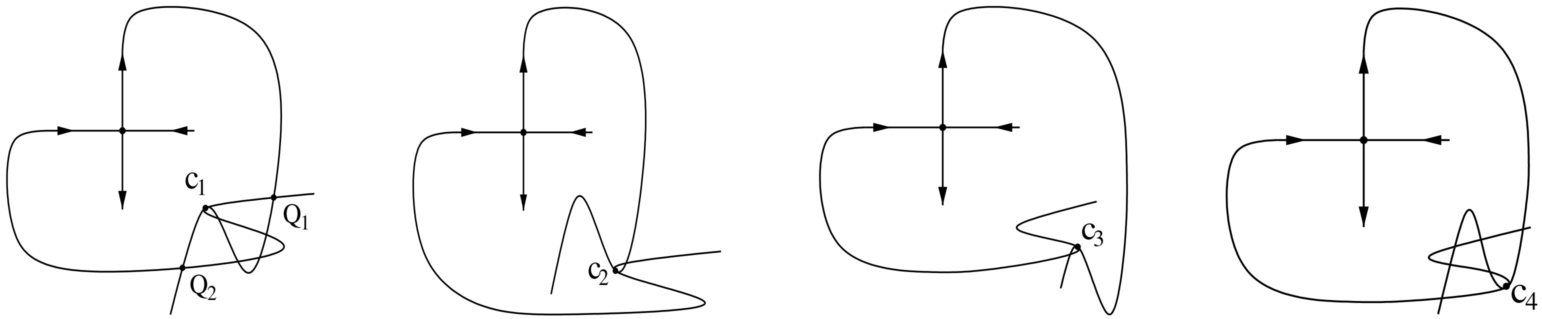
Single round k-p.o, k large, **limit** return map:

$$\begin{aligned} \bar{X} &= Y, \\ \bar{Y} &= M_1 + M_2 Y + \text{sign}(d) Y^3. \end{aligned}$$



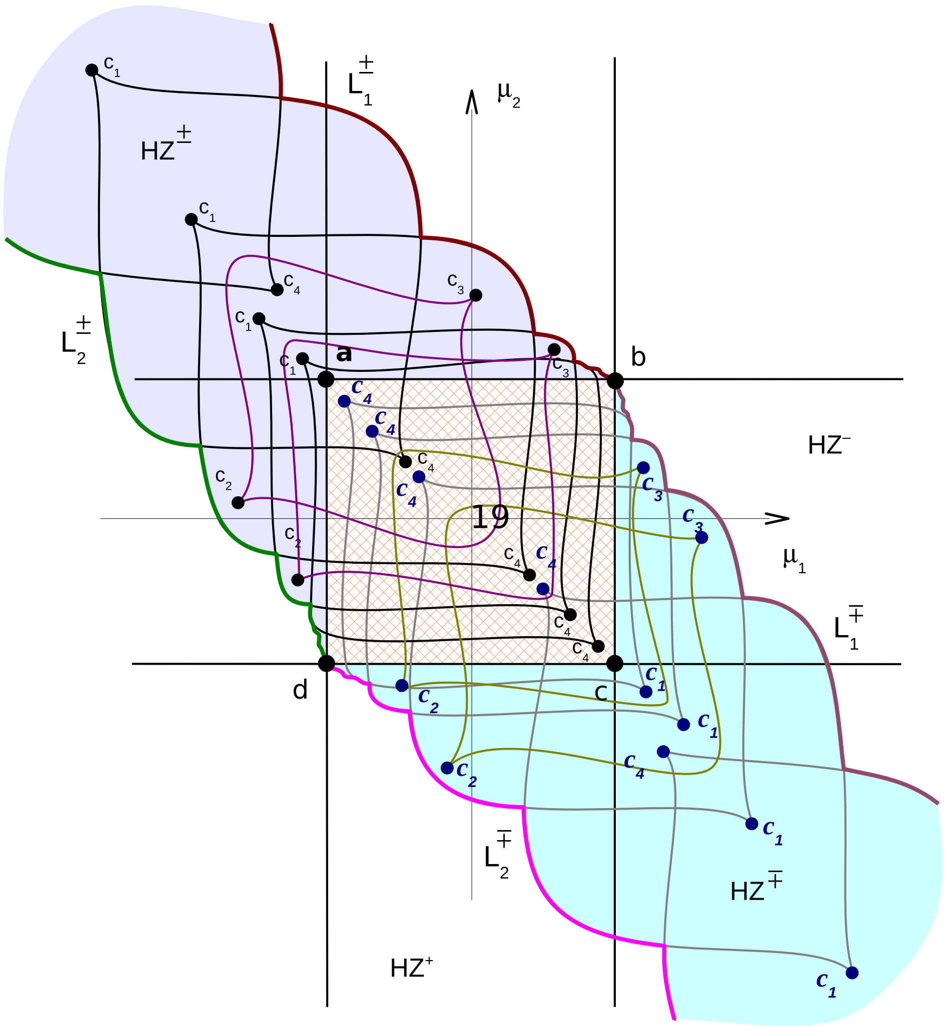
(a) (b)

In our system, $c_1, \ldots, c_4$ cubic tangencies inside $HZ^{\pm}$ and $HZ^{\mp}$.



Lemma. All the cubic tangencies $c_1, \ldots, c_4$ are of **spring-area type** ($d < 0$).

Accumulation of links inside $HZ^{\pm}$



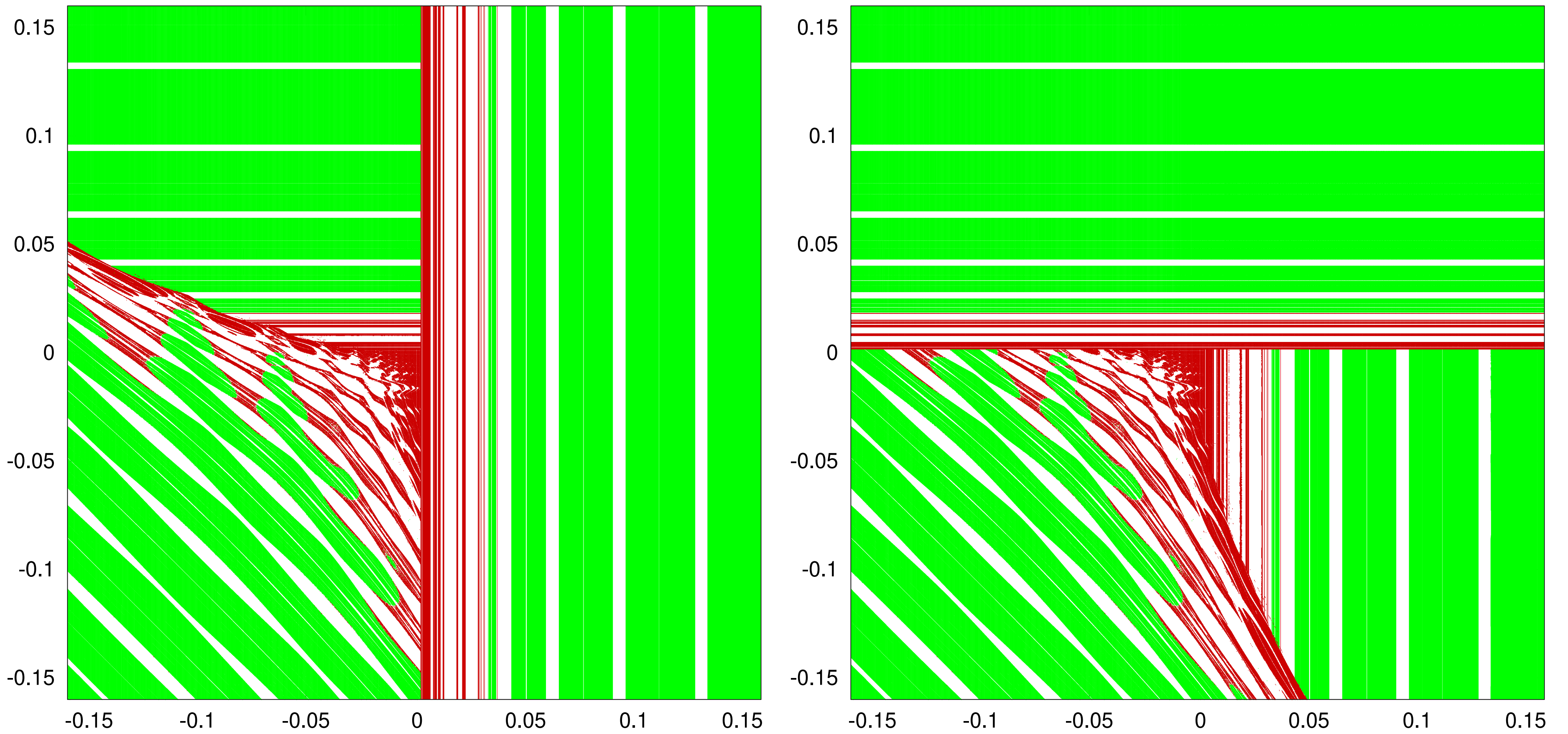
Lemma.

1. The primary cubic tangencies c_1 can exist only if $W^{u+} \cap W^{s+} = \emptyset$ and $W^{u-} \cap W^{s-} = \emptyset$ (i.e. in the regions **3** and **10** of the bif. diagram).
2. The primary cubic tangencies c_2 can exist if $W^{s+} \cap W^{u+} = \emptyset$ (i.e. in the regions **3**, **10** and **18**).
3. The primary cubic tangencies c_3 can exist if $W^{s-} \cap W^{u-} = \emptyset$ (i.e. in the regions **3**, **10** and **15**).
4. In the region **19** of the bif. diagram only primary cubic tangencies c_4 can exist.

Corollary. The cusp points c_1, c_2, c_3 and c_4 accumulate to the points a, d, b and c resp.

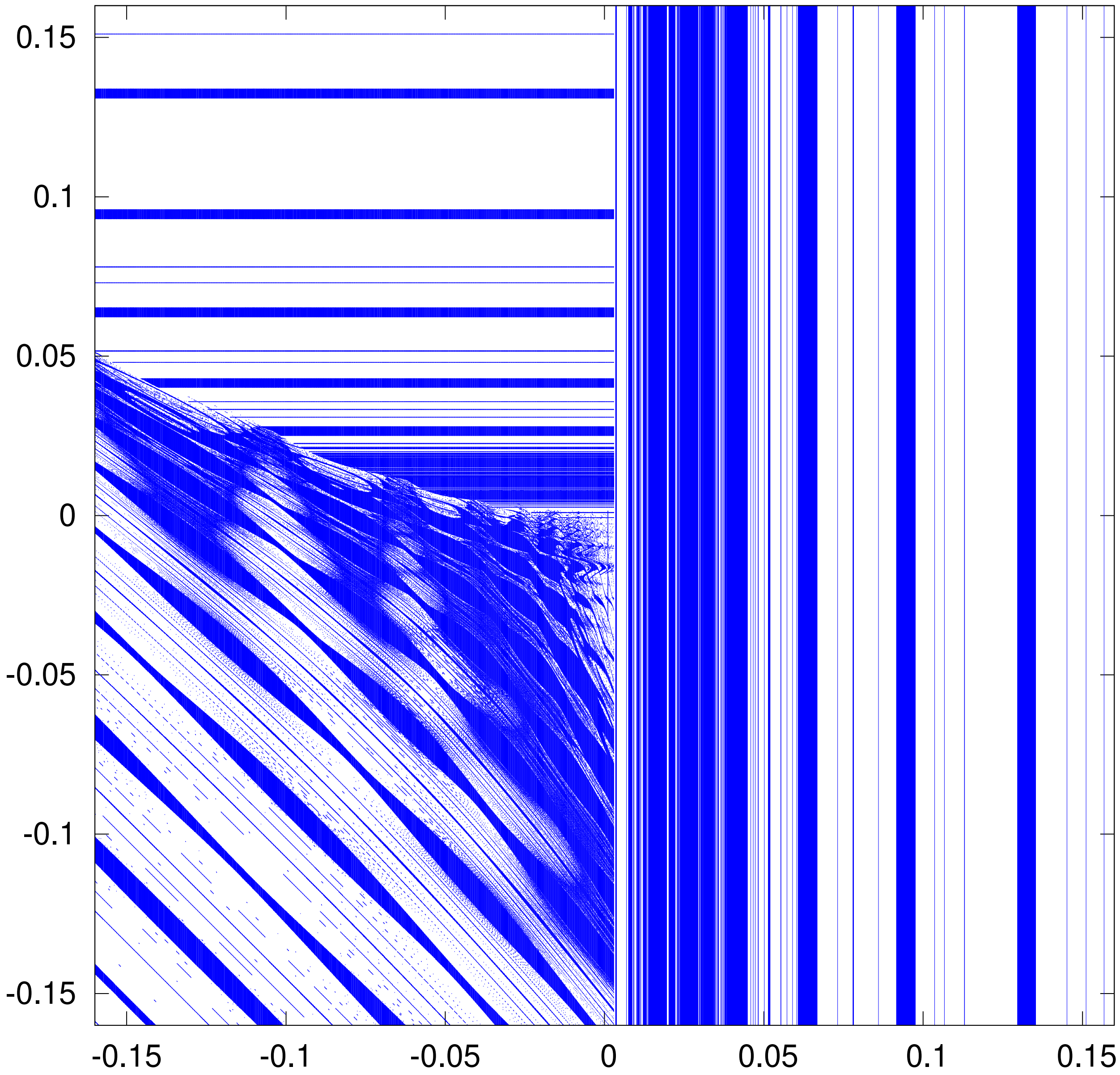
Further analysis of the model: MLE

For each (a_1, a_2)-parameters we take $z_0 = 0.5$, $\eta_0 = 0$ and $s_0 = 1$ (left) or $s_0 = 1$ (right) as i.c. (i.e. on W^u) and we compute the Max. Lyap. exp. Λ.

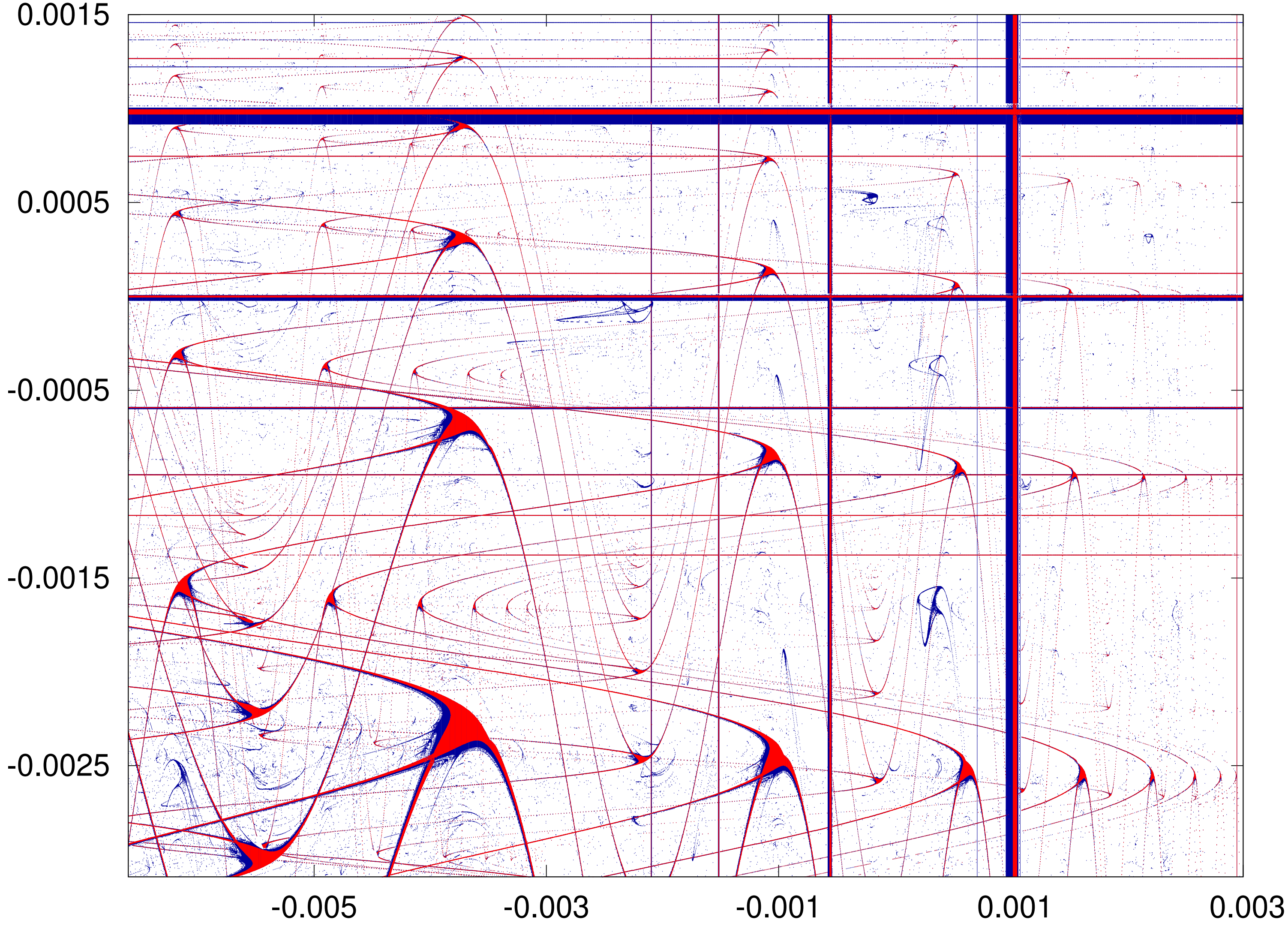


Red points correspond to $\Lambda > 0$ (**chaotic attractor**), **green** points to $\Lambda = 0$ (**invariant curve**) and **white** points to $\Lambda < 0$ (**periodic sink**).

Stability regions ($\Lambda < 0$) related to periodic sinks



Stability region: magnification

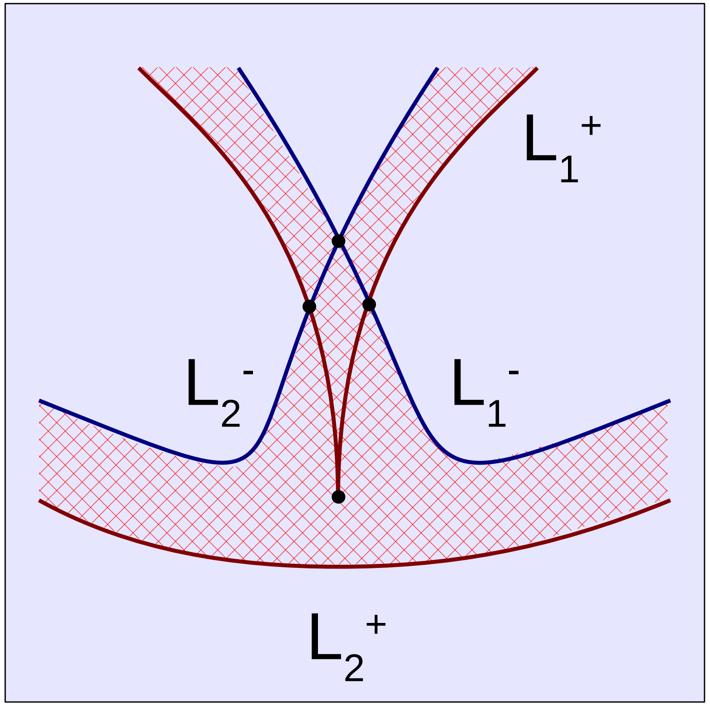


Blue: set of (a_1, a_2)-parameters with $\Lambda < 0$ for the i.c. $(0.5, 0, 1)$. The attractor is a **periodic sink**.

Red: parameters for which there is a **2-periodic sink** as attractor.

The cross-road scenario

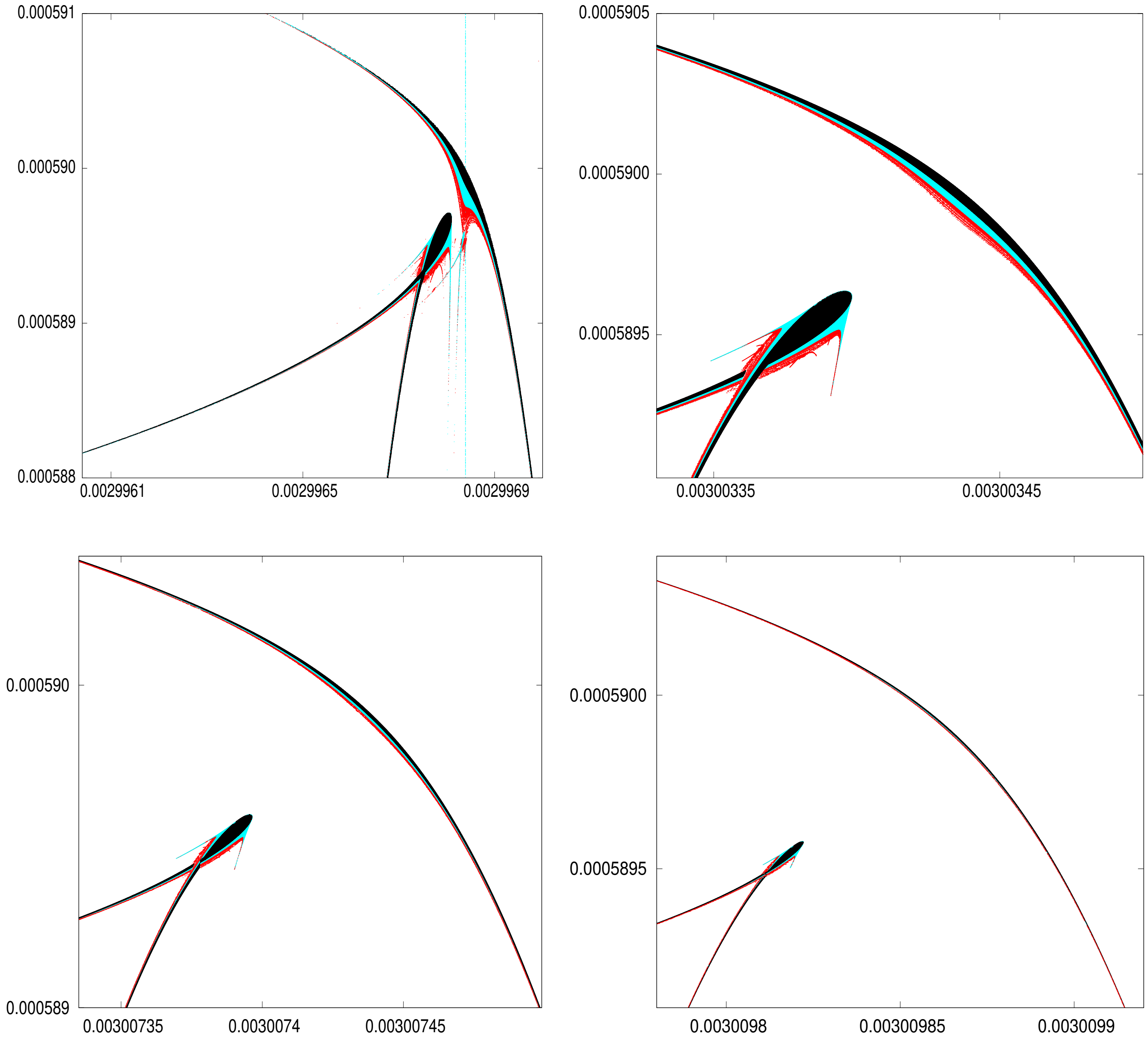
If k is not large enough (depending on the parameters) other configurations might appear (non-local effects and role of high order terms in the return map). One of this, which is commonly observed in numerical explorations and related to the spring-area configuration, is the cross-road scenario.

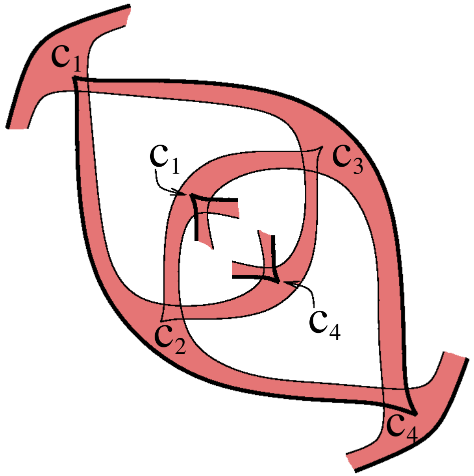


H. Broer, C. Simó and J.C. Tatjer. Towards global models near homoclinic tangencies of dissipative diffeomorphisms. Nonlinearity, 1998, 11, 667–770.

J.P. Carcassès, C. Mira, M. Bosch, C. Simó and J.C. Tatjer. “Crossroad area-spring area” transition (I)-(II). Parameter plane representation. Int. J. Bifur. and Chaos, 1991, 1.

Transition to spring-area: larger (return) periods

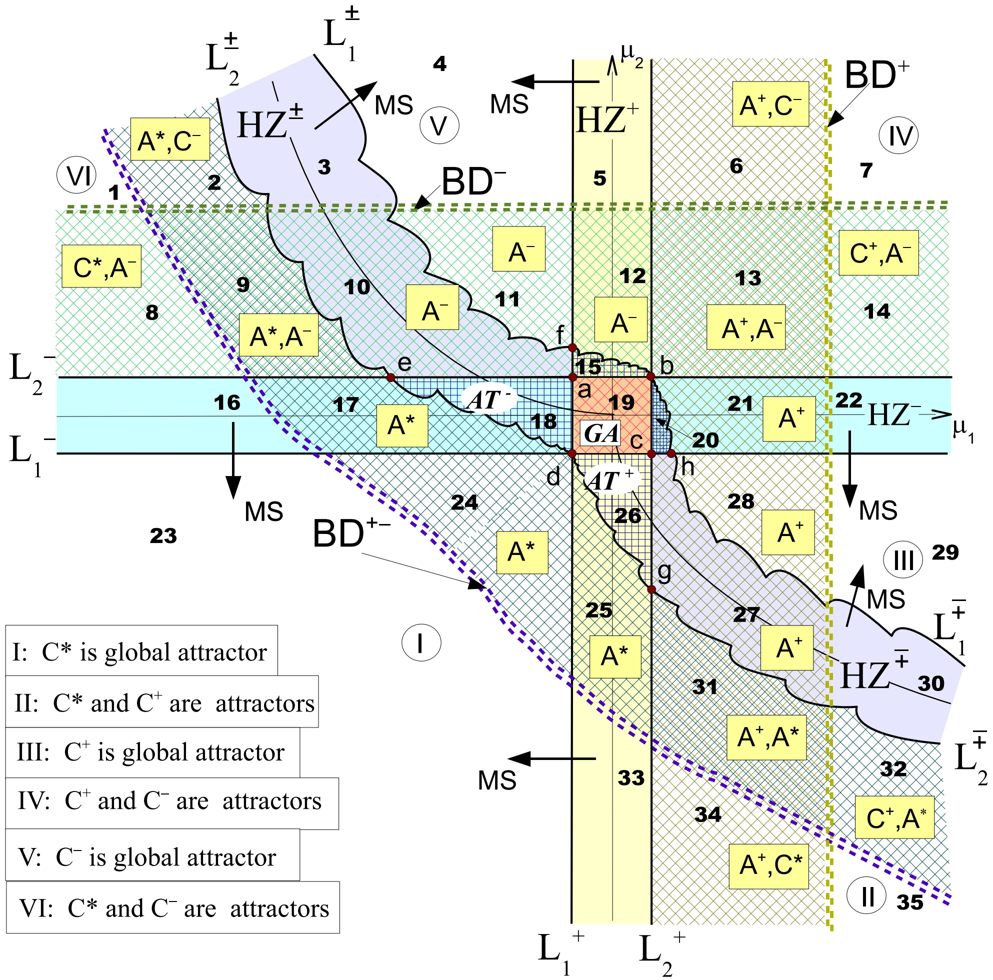


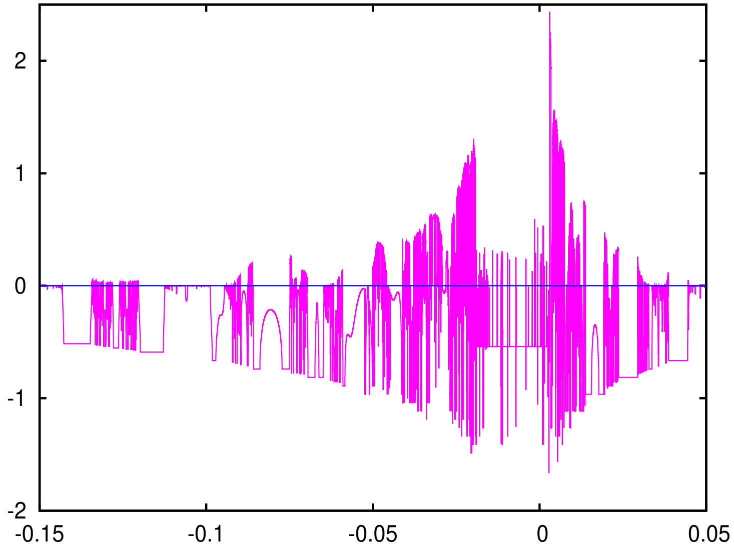


Note the progressive destruction of the previous cross-road domain.

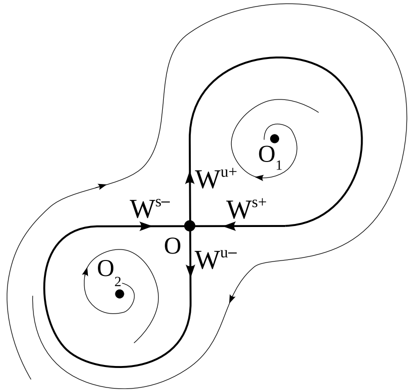
Attractors

Next slides show the different attractors in the return map model.





Lyapunov exp. on $a_2 = 0$.



(Unperturbed) figure-eight.

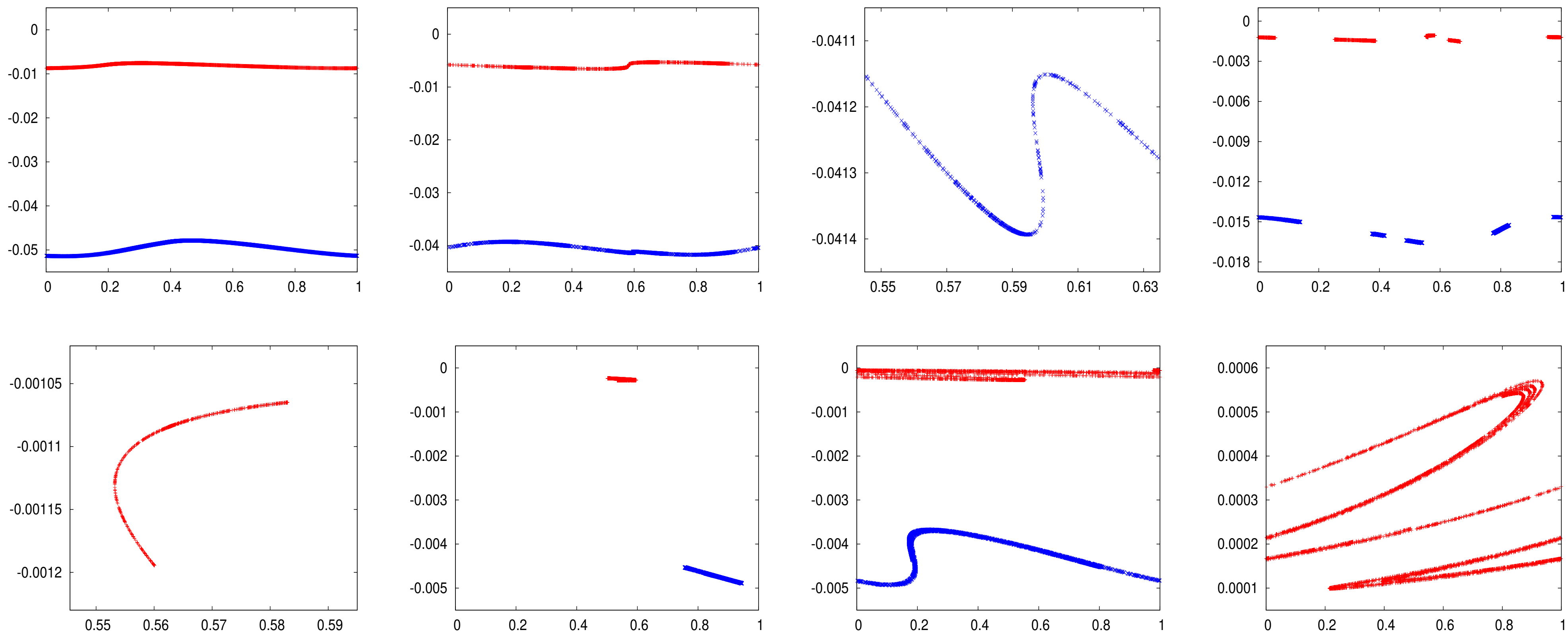
Comments on the attractors

1. Only the regions I,II,...,VI are related to non-chaotic dynamics (like the flow). The global attractors are invariant curves C^+, C^- and/or C^*.

2. In the chaotic regions, the closure of the invariant manifolds can contain a **quasi-attractor**: a nontrivial attracting invariant set which contains stable p.o. (sinks) and/or SA (maybe made by several pieces). Arbitrarily small perturbations of the parameters when a SA is found can give rise to sinks.

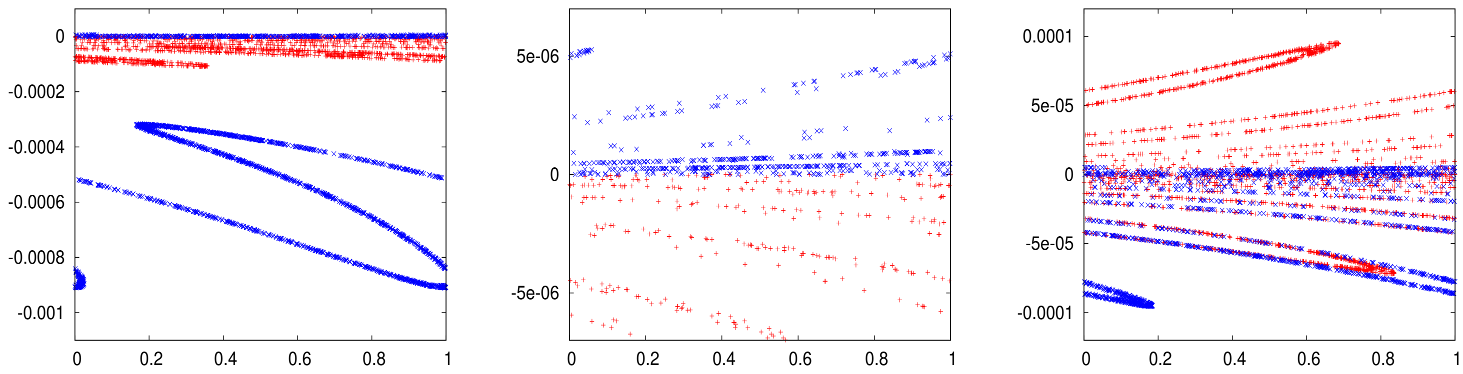
3. There appear strange attractors of different nature:
   - → A^+, A^- and A^* are born under the break-down of the closed invariant curves C^+, C^- and C^*: Due to the folding of the curve it becomes tangent to stable foliation of the saddle fixed point.
   - → The global attractors AT^+, AT^- and GA are “homoclinic attractors” related to the intersection of (some or all) the invariant manifolds.
   - → SA can also appear at the end of a period doubling cascade of sinks. These attractors have local character.

A sample of attractors I ($a_2 = 0$)



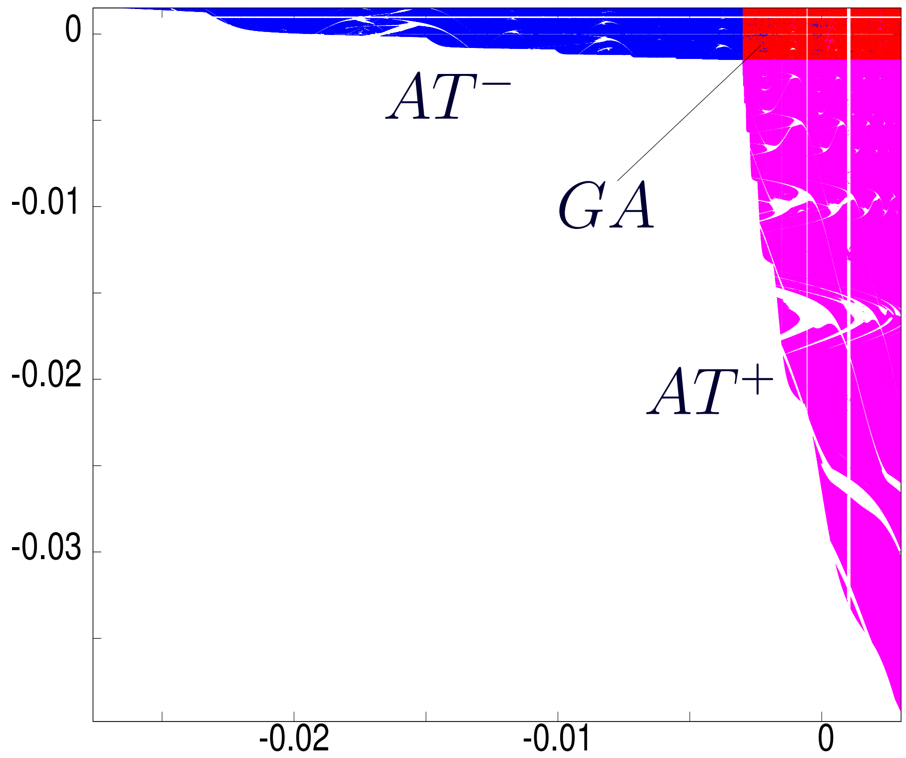
1st row: invariant curve ($a_1 = -0.145$), SA of type A^* with a global nature ($a_1 = -0.129$), detail of the fold in the previous SA ($a_1 = -0.129$) and a SA of type A^* with a local periodic nature ($a_1 = -0.073$). **2nd row:** Detail of the Hénon-like structure of the previous SA ($a_1 = -0.073$), SA of type A^* with a local nature ($a_1 = -0.034$), globalization of the previous SA ($a_1 = -0.033$) and a SA of type A^- ($a_1 = 0.006$).

A sample of attractors II ($a_2 = -0.001$)



Left: Tail attractor of type AT^- ($a_1 = -0.0095$). Center: Magnification of the previous figure. Right: Global SA of type GA ($a_1 = 0$).

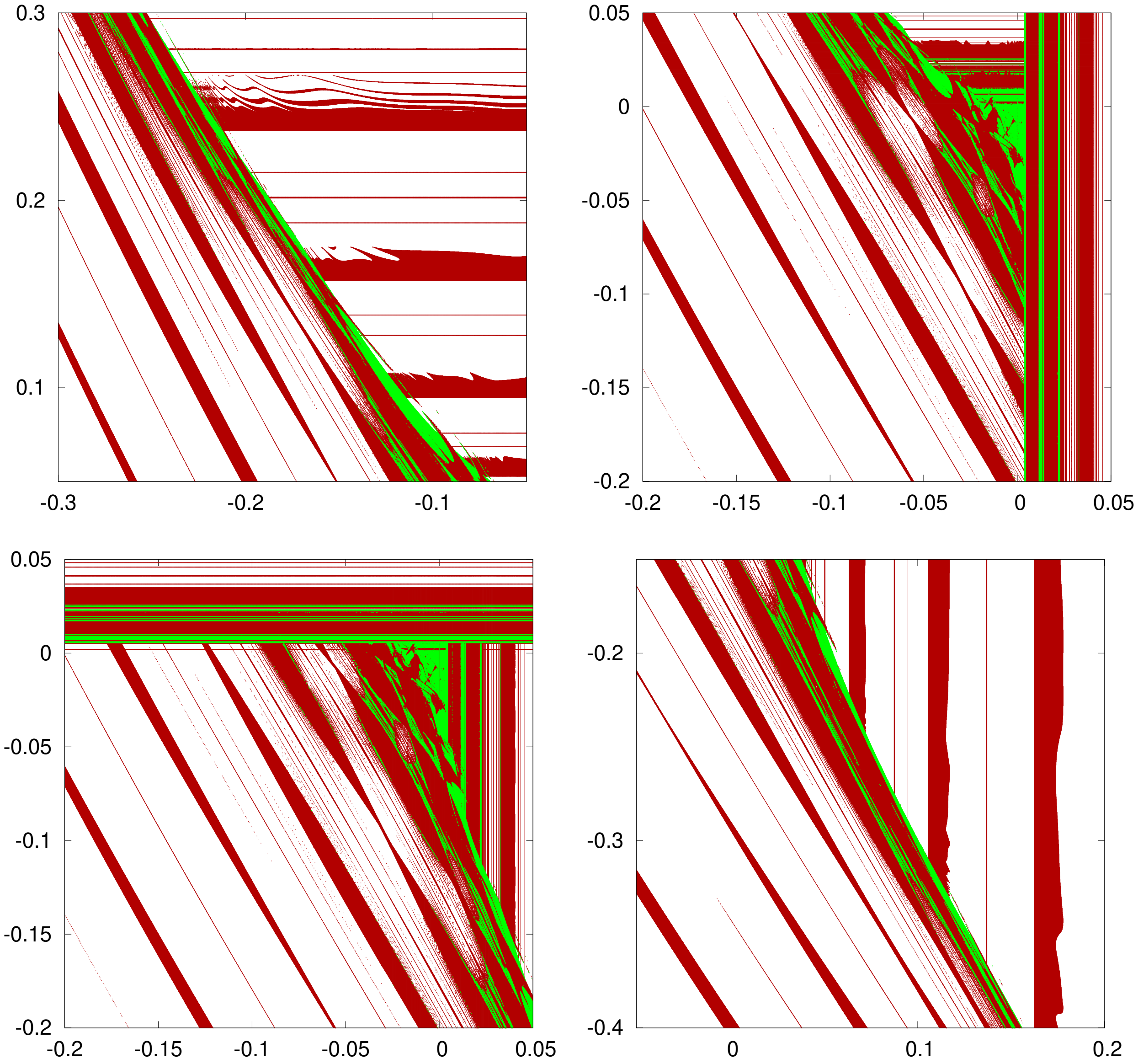
We can identify the points e and g of the bif. diagram. The white domains contained in these colored regions correspond to sinks.



III. Direct computations using the mechanical device system:

pendulum + damping + kick + oscillating support

Maximal Lyapunov exponent indicator



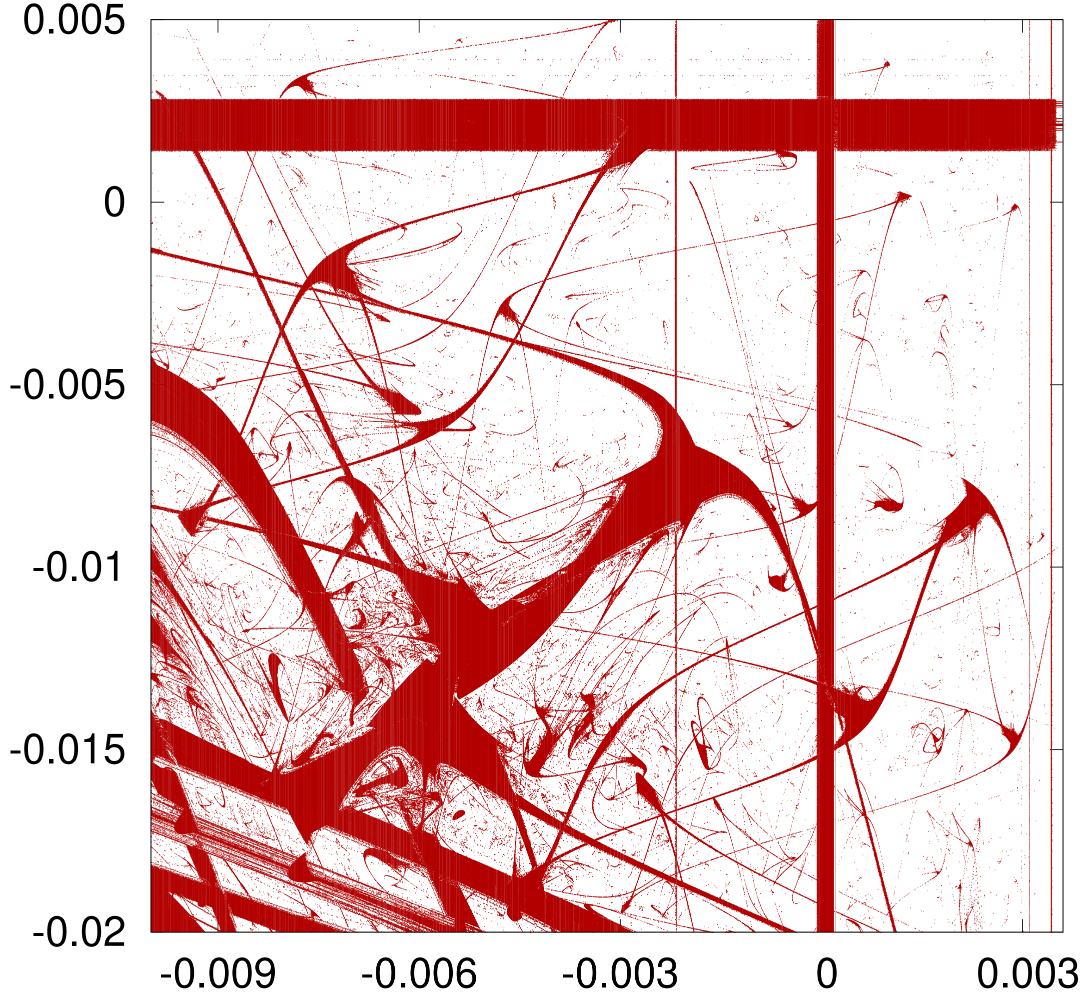
$a_+ = 0.4, a_- = 0.2$
$\mu \in [-0.2, 0.05]^2$

Red–periodic sink

White–invariant curve.

Green–chaotic attractor

Magnification of the stability region



Set of parameters for which the global attractor corresponds to periodic sinks (stability domains related to the cascades of cusp points).

Thanks for your attention!!