A TUTORIAL ON KAM THEORY

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Abstract. This is a tutorial on some of the main ideas in KAM theory. The goal is to present the background and to explain and compare somewhat informally some of the main methods of proof.

It is an expanded version of the lectures given by the author in the Summer Research Institute on *Smooth Ergodic Theory* Seattle, 1999. The style is pedagogical and expository and it only aims to be an introduction to the primary literature. It does not aim to be a systematic survey nor to present full proofs.

Contents

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1. Introduction

The goal of these lectures is to present an introduction to some of the main ideas involved in KAM theory on the persistence of quasiperiodic motions under perturbations. The name comes from the initials of A. N. Kolmogorov, V. I. Arnol'd and J. Moser who initiated the theory. See [Kol54], [Arn63a], [Arn63b], [Mos62], [Mos66b], [Mos66a] for the original papers.

By now, it is a full fledged theory and it provides a systematic tool for the analysis of many dynamical systems and it also has relations with other areas of analysis.

The conclusions of the theory are, roughly, that in C^k – k rather high depending on the dimension – open sets of of dynamical systems satisfying some geometric properties – e.g. Hamiltonian, volume preserving, reversible, etc. – there are sets of positive measure covered by invariant tori. In particular, since sets with a positive measure of invariant tori is incompatible with ergodicity, we conclude that for the systems mentioned above, ergodicity cannot be a C^k generic property [MM74].

Of course, the existence of the quasiperiodic orbits, has many other consequences besides preventing ergodicity. The invariant tori are important landmarks that guide the motion.

Besides its applications to mechanics, dynamical systems and ergodic theory, KAM theory has grown enormously and has very interesting ramifications in dynamical systems and in Analysis. For example, averaging theory gives rise to estimates for very long times valid for all initial conditions, one can use partially hyperbolic tori to show existence of orbits that escape. On the analytical side, the theory leads to functional analysis methods that can be used to solve a variety of functional equations, many of which have interest in ergodic theory and in related disciplines such as differential geometry.

There already exist excellent surveys, systematic expositions and tutorials of KAM theory.

We quote in chronological order: [Arn63b], [Mos66b, Mos66a, Mos73], [AA68], [Rüs70], [Rüs], [Zeh75, Zeh76a], [Dou82a], [Bos86], [Sal86], [Pös82], [Pös92] [Yoc92], [AKN93], [dlL93]. [BHS96b], [Gal83a], [Gal86], [Way96], [Rüs98], and [Mar00].

Hence, one has to justify the effort in writing and reading yet another exposition.

I decided that each of the surveys above has picked up a particular point of view and tried to either present a large part of KAM theory from this point of view or to provide a particularly enlightening example.

Given the high quality of all (but one) of the above surveys and tutorials, there seems to be little point in trying to achieve the same goals. Therefore, rather than presenting a point of view with full proofs, this tutorial will have only the more modest goal of summarizing some of the main ideas entering into KAM theory and describing and comparing the main points of view. Therefore, it is not a substitute for the full papers we reference.

One of the disadvantages of covering such wide ground is that the presentation will have to be sketchy at some points. Hopefully, we have flagged a good fraction of these sketchy points and referred to the relevant literature. I would be happy if these lectures provide a road map (necessarily omitting important details) of a fraction of the literature that encourages somebody to enter into the field. Needless to say, this is not a survey and we have not made any attempt to be systematic nor to reach the forefront of research.

It should be kept in mind that KAM theory has experienced spectacular progress in recent years and that it is a very active area of research.

Let me mention some of these new developments (in no particular order and, with no claim of completeness of the list and omitting classical results – i.e. more that 15 years old –). They will not be covered in the lectures, which will be concerned only with the most classical results. Of course, the reader is not supposed to understand what they are about, only to notice that there are important developments happening in the field.

- The "lack of parameters" which was considered inaccessible has been solved very elegantly [JS92], [Eli88]. (See [BHS96b] for a recent survey, and also [BHS96a] [Sev99].) This has lead to remarkable progress in the existence of lower dimensional tori, specially elliptic tori – a theory of hyperbolic tori has been known for a long time – (see e.g. [JV97b], [JV97a].)
- As a corollary of this, one can get a reasonable KAM theory for volume preserving systems getting tori of codimension one. Hence blocking diffusion in many problems in hydrodynamics, etc. (See [CS90], [BHS96b], [DdlL90], [Xia92], [Yoc92].)
- The KAM theory for infinite dimensional systems has made remarkable progress.

  Note that in infinite dimensional systems, the most interesting tori are of lower dimension than the number of degrees of freedom.

  The subject of infinite dimensional KAM by itself would require a review of its own longer than these notes. We just refer to [CW93], [CW94], [Pös96], [Bou99a], [Bou99b], and [Kuk93] as representative references, where the interested reader can find further references.
- Many systems in applications – e.g in statistical mechanics – have the structure that they consists of arrays of systems connected by local couplings. [1] For these systems one can take advantage of this structure and develop a more efficient KAM theory than the simple application of the general results. [Way84], [Pös90] [FSW86]. Other KAM methods for these systems are developed in [AFS88] and [AF88], [AF91] which consider the existence of periodic solutions. See also [Way86] for an Nekhoroshev theorem for these systems. Conjectures and preliminary

[1] These systems, under the name of *coupled lattice maps* have also been the subject of very intense research when they have hyperbolicity properties, in some sense opposite to the situation considered in KAM theory

estimates (a challenge for rigorous proofs) on these systems can be found in [BGG85b], [BGG85a], [HdlL00],[CCSPC97].

- The non-degeneracy conditions needed for KAM theorems have been greatly weakened [Rüs90], [Rüs98]. See also [BHS96b], [BHS96a],[Sev95], [Sev96].
- Modern techniques of PDE's such as viscosity solutions have been used to study the Hamilton-Jacobi equation [Lio82], [CL83], [CEL84], leading to a weak version of KAM theory that has deep relations with Aubry-Mather theory [Fat97b], [Fat97a].
- There has been quite spectacular progress in the problem of *reducibility* of linear equations with quasiperiodic coefficients (That is, the study whether an equation with the form $\dot{x} = A(\phi + \omega t)x$ where $A \to \mathbf{T}^d \mathcal{M}_{n\times n}$ and $\omega \in \mathbf{R}^d$ is an irrational vector. can be transformed into constant coefficients. After the original work of [DS75], two important recent developments were [MP84] which introduced the deep idea of using transformations which are not close to the identity to eliminate small terms and [Ryc92] which introduced a renormalization mechanism. After that, many more new important refinements were introduced in several works (one needs to find ways to combine perturbative steps with non-perturbative ones). This is still a very active area and progress is being made constantly. We refer to the lectures of prof. Eliasson in this volume for up to date references. See also [Eli],[Kri99].
- The problem of reducibility is related to the problem of existence of pure point spectrum of one-dimensional Schrödinger operators with quasiperiodic coefficients. This area has experienced quite significant progress. Besides some of the papers mentioned in the previous paragraphs, let us mention [CS91], [FSW90], [Eli97].
- For Schrödinger operators in higher dimensions with random or quasiperiodic potential the theory of localization also has advanced greatly thanks to a multi-scale analysis which is quite reminiscent of KAM theory [FS83],[FS84]. Indeed, this analogy has been pursued quite fruitfully. [Alb93].
- Even if the symplectic forms that appear in mechanical systems admit a primitive (see later in Section 3.5), there are other symplectic forms without this feature. For such forms without a primitive, one has the possibility of finding persistent tori of more dimension than the degrees of freedom. This has important consequences and leads to very interesting examples in ergodic theory. See [Yoc92], [Her91] (See also [Par84], [Par89], [FY98].)
- KAM methods have been extended to elliptic PDE's – they are not evolution equations. The role of time in KAM has been taken by spatial variables. (See [Koz83], [Mos88], [Mos95].) This has also been related to a variational structure of the equations [Mos86], [Ban89].

- There are some proofs of KAM type theorems based on different principles, notably renormalization group, [BGK99], [Koc99], [Kos91]. This is perhaps related to some recent proofs that do not even use Fourier analysis [KS86], [KS87], [SK89], [KO93], [KO89b], [KO89a], [Sta88], [Hay90] [Sti94].
- More interestingly, renormalization group has been used to describe the breakdown of invariant circles, starting with [McK82] – which includes a beautiful picture in terms of fixed points and manifolds of operators and makes very detailed predictions about scalings at breakdown – or [ED81], which contains a simpler approach that gives less detailed predictions. Much of what is known at this level remains at the level of numerical well founded conjectures. Indeed, there are still quite important issues that are not even known at this level. Among the rigorous work in this area, we mention [Sti93], [Sti97].
- KAM theory has started to become a tool of applied mathematics with the advent of constructive methods to asses the reliability of numerical computations [CC95], [dlLR91], [Sch95], [Jor99].
- For some special cases of KAM theory, there has also been very important progress examining the limits of validity; the role of the arithmetic conditions has been clarified for complex mappings – specially quadratic – [Yoc95]. See also [PM00]. The study of the radius of convergence of the linearization in the same mappings [MMY97] has also been quite well understood.

  In some twist mappings, there has been a very significant advances in the study of non-existence of tori [Mat88], [MP85], [Jun91]. The domains of convergence of the perturbative expansions have been analyzed using tools similar to those used for analytic complex mappings starting in [Dav94] – a map which has features between those of a complex analytic map and those of a twist map – and then in [MS92], [BM95], [BG99].
- Two different techniques to study quasiperiodic orbits on twist maps are the variational methods of Mather [MF94] and the renormalization group [Koc99];

  In many cases, these theories have ranges of validity much greater than those covered by KAM theory and, therefore provide some glimpse into what happens at the breakdown of KAM theory.
- There has been great progress in using "direct methods", which are based on writing a perturbative expansion and showing it converges by studying more deeply the structure of small denominators.

  In the study of iterations of analytic functions, these methods led to the original proof of Siegel [Sie42], which was the first problem in which small denominators were understood. They were also used in the first proof of the optimal arithmetic conditions [Brj71].

In the study of Lindstedt series (see Section 2.1), the proof of convergence by exhibiting explicitly cancellations of the series was accomplished in [Eli96] (the preprint circulated much earlier). The proof of the convergence of the Lindstedt series in [Eli96] is much more subtle than that of [Sie42]. Contrary to the terms in the expansions considered in [Sie42], the terms in the Lindstedt series do grow very fast and one cannot establish convergence by just bounding sizes but one needs to exhibit cancellations in the terms.

Expositions and simplifications of this work relating it also to techniques of perturbative Quantum Field Theory can be found in [Gal94b], [GG95], [CF94] and extensions to some PDE's in [CF96].

Direct methods not only provide alternative proofs of known facts, but also have been used to prove several results, which at the moment do not seem to have proofs using rapidly convergent methods. To my knowledge, the following results established using direct methods do not have rapidly convergent proofs: The existence of some invariant manifolds contained in center manifolds in [Pös86] was proved using cancellations similar to those of Siegel. It seems that there are no rapidly convergent proofs of these results (however, see [Sto94a], [Sto94b] which solve a very related problem.)

The deeper cancellations of [Eli96] have been used to give a proof of the Gallavotti conjectures (which imply, among other consequences, the amusing result that an analytic Hamiltonian near an elliptic fixed point is the sum of two integrable systems – of course integrated in different coordinates.) [Eli89] and to prove the existence of quasi-flat intersections in [Gal94a]. A problem that remains open is the fact that the Lindstedt series for lower dimensional KAM tori involve less small divisors conditions than the KAM proof. (See [JdlLZ99] for a discussion of this problem.)

- Subjects closely related to KAM theory such as averaging and Nekhoroshev theory have also experienced a great deal of development.
- Even if this is somewhat out of the line of topics to be discussed here, we note that related fields such as averaging theory and Nekhoroshev estimates has also experienced very important developments. Let us just mention very quickly: An elegant proof of the theorem based on approximation by periodic orbits [Loc92], the proof of what are conjectured to be the optimal exponents [LN92], [Pös93], [DG96] – the later paper contains a unified point of view for KAM and Nekhoroshev theorems – and the proof of Nekhoroshev estimates in a neighborhood of an elliptic fixed point [GFB98], [FGB98], [Nie98], [Pös]. In a more innovative direction, Nekhoroshev type theorems for PDE's have been established [BN98], [Nek99], [Bam99b], [Bam99a].
- The list could (perhaps should) be continued, with other topics that are related to KAM theory and connecting it to other theories of mechanics, such as averaging theory, Aubry-Mather theory, quantum versions

of KAM theory, rigidity theory, exponential asymptotics or Arnol'd diffusion and many others which are not even mentioned mainly because of the ignorance of the lecturer, which he is the first to regret.

Needless to say in this tutorial, we cannot hope to do justice to all the topics above. (Indeed, I have little hope that the above list of topics and references is complete.) The only goal is to provide an entry point to the main ideas that will need to be read from the literature and, possibly, to convey some of the excitement and the beauty of this area of research.

Clearly, I cannot (and I do not) make any claim of originality or completeness. This is not a systematic survey of topics of current research. The modest goal I set set for these notes is to help some readers to get started in the beautiful and active subject of KAM theory by giving a crude road map. I just hope that the many deficiencies of this tutorial will incense somebody into writing a proper review or a better tutorial. In the mean time, I will be happy to receive comments, corrections and suggestions for improvement of this tutorial which I will make available electronically.

2. Some motivating examples

2.1. **Lindstedt series for twist maps.** One of the original motivations of KAM theory was the study of quasi-periodic solutions of Hamiltonian systems. In this Section we will cover some elementary and well-known examples.

One particularly motivating example is the so-called *standard map*. This is a map from $\mathbf{R} \times \mathbf{T}$ to itself. We denote the real coordinate by p and the angle one by q. Denoting by p_n, q_n the values of these coordinates at the discrete time n, the map can be written as:

$$
\begin{aligned}
p_{n+1} &= p_n - \varepsilon V'(q_n) \\
q_{n+1} &= (q_n + p_{n+1}) \mod 1,
\end{aligned}
\tag{2.1}
$$

where $V(x) = V(x+1)$ is a smooth (for our purposes in this Section, analytic) function. We will also use a more explicit expression for the map.

$$
T_\varepsilon(p,q) = \big(p - \varepsilon V'(q),\, q + p - \varepsilon V'(q)\big) \ . \tag{2.2}
$$

Substituting the expression for p_{n+1} given in the second equation of (2.1) into the first, we see that the system (2.1) is equivalent to the second order equation.

$$
q_{n+1} + q_{n-1} - 2q_n = -\varepsilon V'(q_n) \ , \tag{2.3}
$$

The first, "Hamiltonian", formulation (2.1) appears naturally in some mechanical systems (e.g., the kicked pendulum). The second, "Lagrangian", one (2.3) appears naturally from a variational principle, namely, it is equivalent to the equations

$$
\partial\mathcal{L}/\partial q_n = 0 \tag{2.4}
$$

with

$$\mathcal{L}(q) = -\sum_n \left[\frac{1}{2}(q_{n+1} - q_n - a)^2 + \varepsilon V(q_n) \right] . \tag{2.5}$$

The equations (2.4) – often called Euler-Lagrange equations – express that $\{q_n\}$ is a critical point for the action (2.5).

The model (2.5) has appeared in solid state physics under the name *Frenkel-Kontorova model*. (See e.g. [Aub83].) One physical interpretation (not the only possible one) that has lead to many heuristic insights is that q_n is the position of the n^{th} atom in a chain. These atoms interact with their nearest neighbors by the quadratic potential energy $\frac{1}{2}(q_{n+1} - q_n - a)^2$ (corresponding to springs connecting the nearest neighbors) and with a substratum by the potential energy $\varepsilon V(q_n)$. The parameter a is the equilibrium length of each spring. Note that a drops from the equilibrium equations (2.3) but affects which among all the equilibria corresponds to a minimum of the energy.

Another interpretation, of more interest for the theme of these lectures, is that q_n are the positions at consecutive times of a one-degree of freedom twist map. The general term in the sum $S(q_{n+1}, q_n)$ are the generating functions of the map. (See Section 3.7.) Then, the Euler-Lagrange equations for critical points of the functional are equivalent to the sequence $\{q_n\}$ being the projection of an orbit.

The first formulation (2.1) is area preserving whenever V' is a periodic function of the cylinder – not necessarily the derivative of a periodic function (i.e., the Jacobian of the transformation $(p_n, q_n) \mapsto (p_{n+1}, q_{n+1})$ is equal to 1). When, as we have indicated, V' is indeed the derivative of a periodic function, then the map is *exact*, a concept that we will discuss in greater detail in Section 3.5 and that has great importance for KAM theory.

If we look at the map (2.1) for $\varepsilon = 0$, we note that it becomes

$$\begin{aligned} p_{n+1} &= p_n \\ q_{n+1} &= q_n + p_n \ , \end{aligned} \tag{2.6}$$

so that the "horizontal" circles $\{p_n = \text{const}, n \in \mathbf{Z}\}$ in the cylinder are preserved and the motion of each q_n in each circle is a rigid rotation that is faster in the circles with larger p_n. Note that when p_0 is an irrational number, a classical elementary theorem in number theory shows that the orbit is dense on the circle. (A deeper theorem due to Weyl shows that it is actually equidistributed in the circle.)

We are interested in finding whether, when we turn on the perturbation ε, some of this behavior persists. More concretely, we are interested in knowing whether there are quasi-periodic orbits that persist and that fill a circle densely.

Problems that are qualitatively similar to (2.1) appear in celestial mechanics [SM95] and the role of these quasi-periodic orbits have been appreciated

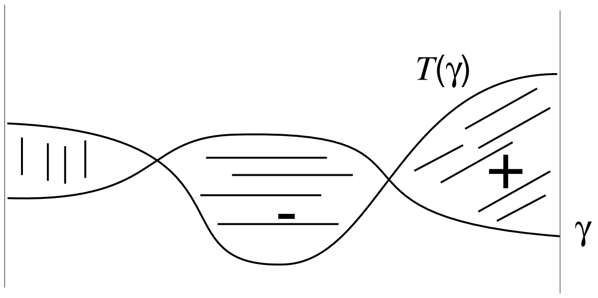

FIGURE 1. The flux is the oriented area between a circle and its image.

for many years. One can already find a rather systematic study in [Poi93] and the treatment there refers to many older works.

We note that the existence of quasi-periodic orbits is hopeless if one allows general perturbations of (2.6). For example, if we take a map of the form

$$\begin{aligned} p_{n+1} &= p_n - \varepsilon p_n \\ q_{n+1} &= q_n + p_{n+1} \ , \end{aligned} \tag{2.7}$$

we see that applying repeatedly (2.7), we have

$$p_n = (1-\varepsilon)^n p_0$$

so that, when $0 < \varepsilon < 2$, all orbits concentrate on the very small set $p = 0$ and that we get at most only one frequency. When $\varepsilon < 0$ or $\varepsilon > 2$, all the orbits except those in $p = 0$, blow up to infinity. Hence, we can have maps with radically different dynamical behavior by making arbitrarily small perturbations.

More subtly, the orbits of

$$\begin{aligned} p_{n+1} &= p_n + \varepsilon \\ q_{n+1} &= q_n + p_{n+1} \end{aligned} \tag{2.8}$$

escape towards infinity and never come back to themselves (in particular, can never be quasi-periodic).

The first example is not area preserving and the motion is concentrated in a smaller area (in particular, it does not come back to itself). The second example is area preserving but has non-zero "flux".

Definition 2.1. The "flux" of an area preserving map T of the cylinder is defined as follows: given a continuous circle γ on the cylinder, the flux of T is the oriented area between $T(\gamma)$, the image of the circle, and γ — see Figure 1.

The fact that the map is area preserving implies easily that this flux is independent of the circle (hence it is an invariant of the map). Clearly, if the map T had a continuous invariant circle, the flux should be zero, so we cannot find an invariant circle in (2.8) for $\varepsilon \neq 0$ since the flux is ε.

Remark 2.2. Note that if a map has a homotopically nontrivial invariant curve, then the flux is zero (compute it for the curve). Conversely, if the flux is zero, any homotopically non-trivial curve has to have an intersection with its image. (If it did not have any intersection, by Rolle's theorem, then the image would always be in above or below the curve.) The property that every curve intersects its image plays an important role in KAM theory and is sometimes called *intersection property*. Besides area preserving and zero flux, there are other geometric assumptions that imply the intersection property, notably, reversibility of the map (see [AS86])

As a simple calculation shows, that perturbation in (2.1) is of the form $V'(q_n)$, with V 1-periodic — therefore $\int_0^1 V'(q_n)\, dq_n = V(1) - V(0) = 0$ — the flux of (2.1) is zero.

We see that even the possibility that there exist these quasi-periodic orbits filling an invariant circle depends on geometric invariants.

Indeed, when we consider higher dimensional mechanical systems, the analogue of area preservation is the preservation of a symplectic form, the analogue of the flux is the Calabi invariant [Cal70] and the systems with zero Calabi invariant are called exact.

We point out, however, that the relation of the geometry to KAM theory is somewhat subtle. Even if the above considerations show that some amount of geometry is necessary, they by no means show what the geometric structure is, and much less hint on how it is to be incorporated in the proof.

The first widely used and generally applicable method to study numerically quasi-periodic orbits seems to have been the method of Lindstedt. (We follow in this exposition [FdlL92].)

The basic idea of Lindstedt's method is to consider a family of quasiperiodic functions depending on the parameter ε and to impose that it becomes a solution of our equations of motion. The resulting equation is solved – in the sense of power series in ε – by equating terms with same powers of ε on both sides of the equation. We will see how to apply this procedure to (2.1) or (2.3).

In the Hamiltonian formulation (2.1), (2.2) we seek $K_\varepsilon : \mathbf{T}^1 \to \mathbf{R} \times \mathbf{T}^1$ in such a way that

$$T_\varepsilon \circ K_\varepsilon(\theta) = K_\varepsilon(\theta + \omega) \ . \tag{2.9}$$

We set:

$$K_\varepsilon(\theta) = \sum_{n=0}^{\infty} \varepsilon^n K_n(\theta) \tag{2.10}$$

and try to solve by matching powers of ε on both sides of (2.9), (after expanding $T_\varepsilon \circ K_\varepsilon(\theta)$ as much as possible in ε using the Taylor's theorem).

[2] That is,

$$
\begin{aligned}
T_\varepsilon \circ K_\varepsilon(\theta) &= T_0 \circ K_0 + \varepsilon[T_1 \circ K_0 + (DT_0 \circ K_0)K_1] \\
&\quad + \varepsilon^2[T_2 \circ K_0 + (DT_0 \circ K_0)K_2 \\
&\qquad + (DT_1 \circ K_0)K_1 + \frac{1}{2}(D^2T_0 \circ K_0)K_1^{\otimes 2}] + \dots \ .
\end{aligned}
$$

In the Lagrangian formulation (2.3) we seek $g_\varepsilon : \mathbf{R} \to \mathbf{R}$ satisfying $g_\varepsilon(\theta + 1) = g_\varepsilon(\theta) + 1$ — or, equivalently, $g_\varepsilon(\theta) = \theta + \ell_\varepsilon(\theta)$ with $\ell_\varepsilon(\theta + 1) = \ell_\varepsilon(\theta)$, i.e., $\ell_\varepsilon : \mathbf{T}^1 \to \mathbf{T}^1$ — in such a way that

$$\ell_\varepsilon(\theta + \omega) + \ell_\varepsilon(\theta - \omega) - 2\ell_\varepsilon(\theta) = -\varepsilon V'(\theta + \ell_\varepsilon(\theta)) \ . \tag{2.11}$$

If we find solutions of (2.11), we can ensure that some orbits q_n solving (2.3) can be written as

$$q_n = n\omega + \ell_\varepsilon(n\omega) \ .$$

Note that the fact that, when we choose coordinates on the circle, we can put the origin at any place, implies that $K_\varepsilon(\cdot + \sigma)$ is a solution of (2.9) if K_ε is, and that $\ell_\varepsilon(\cdot + \sigma) + \sigma$ is a solution of (2.11) if ℓ_ε is. Hence, we can – and will – always assume that

$$\int_0^1 \ell_\varepsilon(\theta)\, d\theta = 0 \ . \tag{2.12}$$

This assumption, will not interfere with existence questions, since it can always be adjusted, but will ensure uniqueness.

If we now write [3]

$$\ell_\varepsilon(\theta) = \sum_{n=0}^{\infty} \ell_n(\theta)\varepsilon^n$$

and start matching powers, we see that matching the zero order terms yields

$$
\begin{aligned}
L_\omega \ell_0(\theta) \equiv \ell_0(\theta + \omega) + \ell_0(\theta - \omega) - 2\ell_0(\theta) = 0 \ , \\
\int_0^1 \ell_0(\theta)\, d\theta = 0 \ .
\end{aligned} \tag{2.13}
$$

The operator L_ω n (2.13), which will appear repeatedly in KAM theory, can be conveniently analyzed by using Fourier coefficients. Note that

$$L_\omega e^{2\pi i k\theta} = 2(\cos 2\pi k\omega - 1)\, e^{2\pi i k\theta} \ .$$

Hence, if $\eta(\theta) = \sum_k \hat{\eta}_k e^{2\pi i k\theta}$, then the equation

$$L_\omega \varphi(\theta) = \eta(\theta)$$

reduces formally to

$$2(\cos 2\pi k\omega - 1)\, \hat{\varphi}_k = \hat{\eta}_k \ .$$

---

[2] The notation is somewhat unfortunate since K_n could mean both the n term in the Taylor expansion and K_ε evaluated for $\varepsilon = n$. In the discussion that follows, K_1,K_2, etc. will always refer to the Taylor expansion. Note that K_0 is the same in both meanings.

[3] The same remark about the unfortunate notation we made in (2.10) also applies here.

We see that if $\omega \notin \mathbf{Q}$, the equation (2.13) can be solved formally in Fourier coefficients and $\ell_0 = 0$. (Later we will develop an analytic theory and describe precisely conditions under which these solutions can indeed be interpreted as functions.)

When $\omega \notin \mathbf{Q}$, we see that $\cos 2\pi k\omega \neq 1$ except when $k = 0$. Hence, even to write a solution we need $\hat{\eta}_0 = 0$, and then we can write the formal solutions as

$$\hat{\varphi}_k = \frac{\hat{\eta}_k}{2(\cos 2\pi k\omega - 1)} , \qquad k \neq 0 \tag{2.14}$$

Note, however, that the status of the solution (2.14) is somewhat complicated since $2\pi k\omega$ is dense on the circle and, hence, the denominator in (2.14) becomes arbitrarily small. Nevertheless, provided that η is a trigonometric polynomial, (See Exercise 2.6 , where this is established under certain circumstances) and ω is irrational, we can solve the equation (2.13). In case that the R.H.S. is analytic and that the number ω satisfies certain number theoretic properties, in Exercise 2.16, we can show that the solution is analytic.

The equation obtained by matching ε^1 is:

$$L_\omega \ell_1(\theta) = -V'(\theta) \ ; \qquad \int_0^1 \ell_1(\theta)\, d\theta = 0 \ . \tag{2.15}$$

Since $\int_0^1 V'(\theta)\, d\theta = 0$, we see that (2.15) admits a formal solution. (Again, we note that the fact that $\int_0^1 V'(\theta)\, d\theta = 0$ has a geometric interpretation as zero flux.)

Matching the ε^2 terms, we obtain

$$L_\omega \ell_2(\theta) = -V''(\theta)\ell_1(\theta) \ ; \qquad \int_0^1 \ell_2(\theta)\, d\theta = 0 \ , \tag{2.16}$$

and, more generally,

$$L_\omega \ell_n(\theta) = S_n(\theta) \ ; \qquad \int_0^1 \ell_n(\theta)\, d\theta = 0 \ , \tag{2.17}$$

where S_n is an expression which involves derivatives of V and terms previously computed. It is true (but by no means obvious) that

$$\int_0^1 S_n(\theta)\, d\theta = 0, \tag{2.18}$$

so that we can solve (2.17) and proceed to compute the series to all orders (when ω is irrational and S is a trigonometric polynomial or when ω is Diophantine (see later) and S is analytic). The fact that (2.18) holds was already pointed out in Vol II of [Poi93].

We will establish (2.18) directly by a seemingly miraculous calculation, whose meaning will become clear when we study the geometry of the problem. (We hope that going through the messy calculation first will give an appreciation for the geometric methods. Similar calculations will appear in Section 6.3.)

The desired result (2.18) follows if we realize that denoting $\ell_\varepsilon^{[\leq n]}(\theta) = \sum_{i\leq n} \varepsilon^i \ell_i(\theta)$, we have:

$$L_\omega \ell_\varepsilon^{[\leq n]} = \varepsilon^n S_n \tag{2.19}$$

Hence, multiplying (2.19) by $\left[1 + \ell_\varepsilon^{[\leq n]\prime}(\theta)\right]$ and integrating, we obtain

$$\begin{aligned}
0 = &\int_0^1 L_\omega \ell_\varepsilon^{[\leq n]}(\theta)\, d\theta + \int_0^1 L_\omega \ell_\varepsilon^{[\leq n]}(\theta)\, \ell_\varepsilon^{[\leq n]\prime}(\theta)\, d\theta \\
&+ \int_0^1 V'(\theta + \ell_\varepsilon^{[\leq n]}(\theta)) \left[1 + \ell_\varepsilon^{[\leq n]\prime}(\theta)\right] \, d\theta \\
&- \varepsilon^n \int_0^1 S_n(\theta)\, \ell_\varepsilon^{[\leq n]\prime}(\theta)\, d\theta \\
&- \varepsilon^n \int_0^1 S_n(\theta)\, d\theta \\
&+ O(\varepsilon^{n+1}) \ .
\end{aligned} \tag{2.20}$$

Now, we are going to use different arguments to show that all the terms in (2.20) except $\int S_n(\theta)\, d\theta$ vanish. This will establish the desired result.

By changing variables in the integral we have:

$$\int_0^1 V'(\theta + \ell_\varepsilon^{[\leq n]}(\theta)) \left[1 + \ell_\varepsilon^{[\leq n]\prime}(\theta)\right] \, d\theta = 0. \tag{2.21}$$

Furthermore, it is clear that $\int_0^1 L_\omega \ell_\varepsilon^{[\leq n]}(\theta)\, d\theta = 0$ because for any periodic function f $\int_0^1 f(\theta)\, d\theta = \int_0^1 f(\theta+\omega)\, d\theta = \int_0^1 f(\theta-\omega)\, d\theta$

Noting that

$$\int_0^1 \ell_\varepsilon^{[\leq n]}(\theta)\, \ell_\varepsilon^{[\leq n]\prime}(\theta) = \int_0^1 \frac{1}{2}\left(\left[\ell_\varepsilon^{[\leq n]}(\theta)\right]^2\right)' d\theta = 0$$

and that

$$\begin{aligned}
\int_0^1 \ell_\varepsilon^{[\leq n]}(\theta+\omega)\, \ell_\varepsilon^{[\leq n]\prime}(\theta)\, d\theta \quad &= \quad -\int_0^1 \ell_\varepsilon^{[\leq n]\prime}(\theta+\omega)\, \ell_\varepsilon^{[\leq n]}(\theta)\, d\theta \\
&= \quad -\int_0^1 \ell_\varepsilon^{[\leq n]\prime}(\theta)\, \ell_\varepsilon^{[\leq n]}(\theta-\omega)\, d\theta \ ,
\end{aligned}$$

we obtain that

$$\int_0^1 L_\omega \ell_\varepsilon^{[\leq n]}(\theta)\, \ell_\varepsilon^{[\leq n]\prime}(\theta)\, d\theta = 0.$$

It is also clear that, because ℓ_0 is a constant,

$$\varepsilon^n S_n(\theta) \ell_\varepsilon^{[\leq n]\prime}(\theta) = O(\varepsilon^{n+1}) \tag{2.22}$$

Hence, putting together (2.20) and the subsequent identities, we obtain the desired conclusion that $\int_0^1 S_n(\theta)\, d\theta$ vanishes.

□

Remark 2.3. There is a geometric interpretation for the vanishing of this integral. One can compute the flux over the curve in the Hamiltonian formalism predicted by $\ell_\varepsilon^{[\le n]}(\theta)$. The fact that the flux vanishes is equivalent to the fact that the integral vanishes.

Remark 2.4. Note that it is rather remarkable that for every irrational frequency we can find formal solutions (when the perturbation is a polynomial), or for Diophantine frequencies for analytic perturbations. Heuristically, this can be explained by the fact that, in area preserving systems, we do not have small parts of the system controlling the long term behavior (as it is the case in dissipative systems) and, hence, perturbations still have to leave open many possibilities for motion of the system.

When one applies the Lindstedt method to dissipative systems, [RA87], typically one sees that, except for a few frequencies, the perturbation equations do not have a solution.

Remark 2.5. The Lindstedt method can be used for dissipative systems [RA87]. (Code for easy to use, general purpose implementations is available from [RA92].) Then, one considers

$$T_\varepsilon \circ K_\varepsilon(\theta) = K_\varepsilon(\theta + \omega_\varepsilon) \ .$$

with $\omega_\varepsilon = \sum \varepsilon^n \omega_n$. One has to choose the terms $\omega_0, \dots, \omega_n$, so that the equations (2.17) have solutions. It is a practical and easily implementable method to compute limit cycles.

Exercise 2.6. Show that if V is a trigonometric polynomial, then l_n is also a trigonometric polynomial. Moreover, $\deg(l_n) \le An + B$ where A and B are constants that depend only on the degree of V. (For a trigonometric polynomial, $V(\theta) = \sum_{|k|\le M} \hat{V}_k \exp(2\pi i k\theta)$, the degree is M when $\hat{V}_M \ne 0$ or $\hat{V}_{-M} \ne 0$.)

As a consequence, if V is a trigonometric polynomial and ω is irrational, then the Lindstedt procedure can be carried out to all orders.

Remark 2.7. The above procedure can be carried out even in the case that the function $V(x)$ is $e^{2\pi i x}$.

In this case, we obtain the so-called semi-standard map. It can be easily shown that the trigonometric polynomials that appear in the series only contain terms with positive frequencies. This makes the terms in the Lindstedt series easier to analyze than those of the case $V(x) = e^{2\pi i x} + e^{-2\pi i x}$. Indeed, the analytical properties of the term of the series for $V(x) = e^{2\pi i x}$ very similar to those of the normalization problem for a polynomial.

We refer to [GP81] for numerical explorations, to [Dav94] for rigorous upper bounds of the radius of convergence and to [BM95], [BG99] for a method to transfer results from this complex case to the real one.

The convergence of the expansions obtained remains at this stage of the argument we have presented highly problematic. Note that, at every stage, (2.17) involves small divisors. Worse still, the S_n's are formed by multiplying terms obtained through solving small divisor equations. Hence, the S_n could be much bigger than the individual terms.

Poincaré undertook in [Poi93], Paragraph 148 a study of the convergence of these series. He obtained negative results for uniform convergence in a parameter that also forced the frequency to change. His conclusions read (I transcribe the French as an example of the extremely nuanced way in which Poincaré formulated the result.) Roughly, he says that one can conclude that the series does not converge, then points out that this has not been proved rigorously and that there are cases that could be left open, including quadratic irrationals. The conclusion is that, even if the divergence has not been proved, it is quite improbable.

> Il semble donc permis de conclure que les series (2) ne convergent pas.
>
> Toutefois le raisonement qui précède ne suffit pas pour établir ce point avec une rigueur compléte.
>
> En effect, ce que nous avons démontré au n° 42 c'est qu'il ne peut pas arriver que, pour toutes les valeurs de μ inferieurs a une certaine limite, il y ait une double infinité de solutions périodiques, et il nous suffirait ici que cette double infinité existâit pour une valeur de $\hat{\mu}$ determinée, different de 0 et généralment trés petite.
>
> [....]
>
> Ne peut-il pas arriver que les series (2) convergént quand on donne aux x_i^0 certaines valeurs convenablement choisies?
>
> Supposons, pour simplifier, qu'il y ait deux degrees de liberté; les series ne pourraient-elles pas, par example, converger quand x_1^0 et x_2^0 ont été choisis de telle sorte que le rapport $\frac{n_1}{n_2}$ soit incommensurable, et que son carré soit au contraire commensurable. (ou quand le rapport $\frac{n_1}{n_2}$ est assujetti á une autre condition analogue à celle que je viens d'ennoncer un peu au hassard)?
>
> Les raisonnements de ce Chapitre ne me permettent pas d'affirmer que ce fait ne se présentera pas. Tout ce qu'il m'est permis de dire, cest qu'il es fort inversemblable.

This was remarkably prescient since indeed the series do converge for Diophantine numbers. In particular, for algebraic irrationals (see Section 3.3, Theorem 3.6).

It is not difficult to show that, for Diophantine frequencies, these series satisfy estimates that fall short of showing analyticity

$$\|\ell_n\|_\sigma \leq (n!)^\nu \tag{2.23}$$

where ν is a positive number. These estimates are sometimes called Gevrey estimates and they appear very frequently in asymptotic analysis.

It is not difficult to construct examples (indeed we present one in Exercise 4.23) which have a similar structure and that the linearized equation that we have to solve at each step satisfy similar estimates. Nevertheless they saturate (2.23). Indeed, in many apparently similar problems with a very similar structure (e.g. Birkhoff normal forms near a fixed point, normal forms near a torus, jets of center manifolds) the bounds (2.23) are saturated. We will not have time to discuss these problems in these notes.

The proof of convergence of Lindstedt series was obtained in [Mos67] in a somewhat indirect way. Using the KAM theory, it is shown that the solutions produced by the KAM theory are analytic on the perturbation parameter. It follows that the coefficients of the expansion have to be the terms of the Lindstedt series and, therefore, that the Lindstedt series are convergent.

The example in Exercise 4.23 shows that the convergence that one finds in KAM theory has to depend on the existence of massive cancellations.

The direct study of the Lindstedt series was tackled successfully in [Eli96]. One needs to exhibit remarkable cancellations. The papers [Gal94b] and [CF94] contain another version of the cancellations above relating it to methods of quantum field theory.

We note that the transformations that reduce a map to its normal Birkhoff normal form either near a fixed point or near a torus were known to diverge for a long time. (See [Sie54], [Mos60].)

Examples of divergence of asymptotic series were constructed in [Poi93]. To justify their empirically observed usefulness, the same reference developed a theory of asymptotic series, which has a great importance even today.

It should be remarked that, at the moment of this writing, the convergence of Lindstedt series in slightly different situations (lower dimensional tori [JdlLZ99] or the jets for center manifolds of positive definite systems [Mie91], p. 39) are still open problems.

2.2. **Siegel disks.** The following example is interesting because the geometry is reduced to a minimum and only the analytical difficulties remain. Not surprisingly, it was the first small divisors problem to be solved [Sie42], albeit with a technique very different from KAM. This problem is quite paradigmatic both for KAM theory and for the theory of holomorphic dynamics. In these lectures, we will discuss only the KAM aspects and not the holomorphic dynamics. A very good introduction to the problems connected with Siegel theorem is including both the KAM aspects and the holomorphic dynamics aspect is [Her87]. More up to date references are [PM92], [Yoc95]. The lectures [Mar00] contain a great deal of material on the Siegel problem.

We consider analytic maps $f : \mathbf{C} \to \mathbf{C}$, $f(z) = az + N(z)$ with $N(0) = 0$, $N'(0) = 0$, and we are interested in studying their dynamics near the origin.

When $|a| \neq 1$, it is easy to show that the dynamics, up to an analytic change of variables is that of az. More precisely, there exists an $h : U \subset$

$\mathbf{C} \to \mathbf{C}$, $h(0) = 0$, $h'(0) = 1$ and

$$f \circ h = h(az) \tag{2.24}$$

in a neighborhood of the origin.

The proof for $|a| > 1$ can be easily obtained as follows (the case $0 < |a| < 1$ case follows by considering f^{-1} in place of f).

We seek a fixed point of $h \mapsto f \circ h \circ a^{-1}$ on a space of functions $h(z) = z + \Delta(z)$ with $\Delta(z) = O(z^2)$.

That is, we seek fixed points of the operator

$$\mathcal{T}(\Delta) = a\,\Delta \circ a^{-1} + N \circ (\mathrm{Id} + \Delta) \circ a^{-1} \ .$$

We note that, on a space of functions with $\|\Delta\|_r = \sup_{|z| \le r} |\Delta(z)/z^2|$, the operator $\mathcal{T}$ is a contraction if r is sufficiently small. Note that then $\mathcal{T}^r(0)$ converges uniformly on a ball and the limit is analytic.

Remark 2.8. Note that the previous argument works without any significant change when $f : \mathbf{C}^d \to \mathbf{C}^d$ and a is a matrix all eigenvalues of which have modulus less than 1. Indeed, a very similar result for flows already appears in Poincaré's thesis [Poi78], where it was established using the majorant method. (Remember that the concept of Banach spaces had not been yet formalized, so that fixed point proofs were unthinkable). The method in [Poi78] can be adapted without too much difficulty to cover the theorem started above. Hence, the situation when all the eigenvalues are smaller than one is sometimes called *the Poincaré domain*.

The situation that remains to be settled is that when $|a| = 1$.

Remark 2.9. Building up on case for $|a| < 1$, there is a lovely proof by Yoccoz [Her87] using complex function theory that one can extend the conjugacies for $|a| < 1$ to a positive measure set with $|a| = 1$. Several elements of this proof can be used to obtain a very fast algorithm to compute the so called *Siegel radius*. (See the definition in Proposition 2.13.)

Another cute proof of a particular case of Siegel's theorem is in [dlL83] adapting a method of [Her86]. This method can be applied to a variety of one-dimensional problems.

The method of [Sie42] has been quite refined and extended in [Brj71], [Brj72].

We will not discuss the above proofs here, because, in contrast with KAM ideas that have a wide range of applications, they seem to be rather restricted.

It is typical of complex dynamics that there are very few possibilities for the dynamics. Either it is very unstable or it is a rigid rotation (up to a change of variables).

We will prove something more general.

Lemma 2.10. *Let $f : \mathbf{C}^d \to \mathbf{C}^d$ be analytic in a neighborhood of the origin and*

$$f(0) = 0 \ , \qquad Df(0) = A \ ,$$

where A is a diagonal matrix with all the diagonal elements of unit modulus (hence $\|A^{-n}\| = 1\ \forall n \in \mathbf{Z}$).

Assume that there is a domain U, $0 \in U$ and a constant $K > 0$ such that for all $n \in \mathbf{N}$

$$\sup_{z\in U} |f^n(z)| \le K \ . \tag{2.25}$$

Then there exists an analytic function $h : U \to \mathbf{C}^d$ such that $h(0) = 0$,

$$h'(0) = \mathrm{Id} \qquad , h \circ f = A \circ h. \tag{2.26}$$

Of course, by the implicit function theorem a solution of (2.26) implies that there is a solution of (4.4) (h in (4.4) is the inverse of h in (2.26)).

Note also that the assumption (2.25) implies, by Cauchy estimates that $|Df^n(0)| \le K'$, hence, that all the eigenvalues are inside the closed unit circle and that the eigenvalues on the unit circle have trivial Jordan blocks. If rather than assuming (2.25) for $n \in \mathbf{N}$, we assumed it for $n \in \mathbf{Z}$, this would imply the assumption that A is diagonal and has the eigenvalues on the unit circle.

Proof. Consider

$$h^{(n)}(z) = \frac{1}{n} \sum_{i=0}^{n} A^{-n} f^n(z) \ .$$

Note that, using the definition of A and (2.25) and we have:

$$h^{(n)}(0) = 0 \ , \quad h^{(n)\prime}(0) = 1 \ , \tag{2.27}$$

$$\sup_{z\in U} |h^{(n)}(z)| \le K \ , \tag{2.28}$$

$$h^{(n)} \circ f(z) = Ah^{(n)}(z) + (1/n)[A^{-(n+1)} f^{n+1}(z) - z] \ . \tag{2.29}$$

By (2.28), $h^{(n)}$ restricted to U is a normal family and we can find a subsequence converging uniformly on compact sets to a function h. Using (2.27), we obtain that $\tilde{h}(0) = 0$, $\tilde{h}'(0) = 1$.

Note also that, since $|f^n(z)|$ is bounded independently of n, by (2.25) and so is z for $z \in U$, we have that

$$\frac{1}{n}[A^{-(n+1)} f^{n+1}(z) - z]$$

converges to zero uniformly on any compact set contained in U as $n \to \infty$. Therefore, taking the limit $n \to \infty$ of (2.29), we obtain $h \circ f = A \circ h$. $\square$

Exercise 2.11. Show that one can always assume that U is to be simply connected. (Somewhat imprecisely, but pictorially, if we are given are given a U with holes, we can always consider $\tilde{U}$ obtained by filling the holes of U. The maximum modulus principle shows that f^n is uniformly bounded in $\tilde{U}$.)

In one dimension, show that the Riemann mapping that sends U into the unit disk and 0 to itself should satisfy (2.26) except the normalization of the derivative.

Proposition 2.12. *If the product of eigenvalues of A is not another eigenvalue, then the function $\tilde{h}$ satisfying* (2.26) *is unique even in the sense of formal power series.*

Note that, when $d = 1$ the condition of Proposition 2.12 reduces to the fact that A is not a root of unity. In particular, it is satisfied when the modulus of A is not equal to one. When the modulus equals to 1, the hypothesis of Proposition 2.12 reduces to a not being a root of unity, which is the same as $a = \exp(2\pi i\theta)$ with $\theta \in \mathbf{R} - \mathbf{Q}$.

Proof. If we expand using the standard Taylor formula for multi-variable functions,

$$f(z) = \sum_{n=0}^{\infty} f_n z^{\otimes n}$$

(where f_n is a symmetric n-linear form taking values in $\mathbf{C}^d$) and seek a similar expansion for $\tilde{h}$, we notice that

$$A\tilde{h}_n - \tilde{h}_n A^{\otimes n} = S_n \ ,$$

where S_n is a polynomial expression involving only the coefficients of f and $\tilde{h}_1 = \mathrm{Id}, \dots, \tilde{h}_{n-1}$.

As it turns out, the spectrum of the operator $\mathcal{L}_A$ acting on n-multilinear forms by

$$\tilde{h}_n \mapsto A\tilde{h}_n - \tilde{h}_n A^{\otimes n} \tag{2.30}$$

is:

$$\mathrm{Spec}(\mathcal{L}_A) = a_i - a_{\sigma_1} \dots a_{\sigma_n}, \quad i \in \{1, \dots, d\}, \quad \sigma_1, \dots, \sigma_n \in \{1, \dots, d\}, \tag{2.31}$$

where a_i denotes the eigenvalues of A.

See, e.g., [Nel69] for a detailed computation which also leads to interesting algorithms. We just indicate that the result can be obtained very easily when the matrix is diagonalizable since one can construct a complete set of eigenvalues of (2.30) by taking products of eigenvalues of A. The set of diagonalizable matrices is dense on the space of matrices. Hence the desired identity between the spectrum of (2.30) and the set described in (2.31) holds in a dense set of matrices. We also note that the spectrum is continuous with respect to the linear operator.

□

When $d = 1$ and $|a| = 1$, as we mentioned before, the condition for Proposition 2.12 (usually referred to as *non-resonance condition*) reduces to:

$$a = e^{2\pi i\omega} \ , \qquad \omega \in \mathbf{R} - \mathbf{Q} \ .$$

We note that, even if $a(a^{n-1} - 1) \neq 0$, it can be arbitrarily close to zero, because $e^{2\pi i\omega(n-1)}$ is dense in the unit circle. Hence, we also have small divisors in the computation of the $\tilde{h}_n$'s.

We note that when $d > 1$, we can have small divisors if there is some $|a_i| > 1$, $|a_j| < 1$ even if they are real. When all $|a_j| = 1$, $a_j = e^{2\pi i \omega_j}$, the non-resonance condition amounts to

$$\sum_j k_j \omega_j \neq \omega_i \ , \ \forall k_j \in \mathbf{N}, \sum_j k_j \geq 2 \tag{2.32}$$

We now investigate a few of the analyticity properties of h. Of course, the power series expansion converges in a disk (perhaps of zero radius) but we could worry about whether it is possible to perform analytic continuation and obtain h defined on a larger domain.

Proposition 2.13. *If f is entire, the maximal domain of definition of h is invariant under A.*

In particular, when $d = 1$, $|a| = 1$, $a^n \neq 1$, the domain of convergence is a disk. (The radius of the disk of convergence of the function h such that $h'(0) = 1$ is called the Siegel radius.)

Moreover, when $d = 1$, $|a| \leq 1$, $a^n \neq 1$, the function h is univalent in the domain of convergence.

Proof. To prove the first point, we just observe that if f is entire and h is analytic in the neighborhood of a point z_0, we can use the functional equation (2.24) to define the function h in a neighborhood of $A\, z_0$.

Hence, if h was defined in domain D and $z_0 \in D$ was connected to the origin by a path $\gamma \subset D$, we see that Az_0 is connected to the origin by $a\gamma \subset aD$. We conclude that it is defined in $AD \cup D$ and that the analytical continuation is unique. If we consider the maximal domain of definition $AD \cup D \subset D$. Hence $AD = D$.

The second statement follows by observing that the only domains invariant under an irrational rotation are disks.

To prove univalence, we assume that if $h(z_1) = h(z_2)$ and one of them – say z_2 – different from 0. We want to conclude that $z_1 = z_2$.

Using (2.24), we obtain $h(az_1) = h(az_2)$. Repeating the process, $h(a^n z_1) = h(a^n z_2)$.

Hence, when $z \in \{a^n z_2\}$, we have

$$h(z) = h(z\alpha) \tag{2.33}$$

with $\alpha = z_1/z_2$. Since the set where (2.33) holds has an accumulation point: when $|a| < 1$, it accumulates to 0, when $|a| = 1$ since it is an irrational rotation, the orbits are dense on circles), we conclude that it holds all over the unit disk. Taking derivatives at $z = 0$, using $h'(0) = 1$, we obtain $\alpha = 1$. ◻

Exercise 2.14. Show that the conclusions of Proposition 2.13 remain true if we consider $d > 1$ and A a diagonalizable matrix with all eigenvalues in the unit disc and satisfying (2.32). Namely

i') The domain of definition is a polydisk.

ii') The function is univalent in its domain of definition.

Exercise 2.15. Once we know that the domain of the function h in (2.24) is a disk, the question is to obtain estimates of the radius.

Lower bounds are obtained from KAM theory.

Obtain upper bounds also using the fact that by the Bieberbach-De Branges theorem, the Taylor coefficients of a univalent function satisfy upper bounds that depend on the radius of the disk. On the other hand, we know the coefficients explicitly.

Also obtain upper bounds when $f(z) = az + z^2$ using the area formula for univalent functions Area $h(B_r(0)) = \pi \sum_{i=1}^{\infty} |h_i|^2 r^{2i-2}$ knowing that the range of h – orbits that are bounded – cannot include any point outside of the disk of radius 2 and that we know the coefficients h_k.

This exercise is carried out in great detail in [Ran87], which established upper and lower bounds of the radius for rotation by the golden mean.

It turns out to be very easy to produce examples where the series diverges. We will discuss what we think is oldest one [Cre28] (reproduced in [Bla84]). Other examples of [Cre38] can be found in [SM95] Chapter 25 in a more modern form. A different line of argument appears in [Ily79], [Ily] using more complex analysis.

Consider $f(z) = az + z^2$ with $a = e^{2\pi i\omega}$, then its n^{th} iteration is

$$f^n(z) = a^n z + \cdots + z^{2^n} \ .$$

If we seek fixed points of f^n, different from zero, they satisfy $(a^n - 1) + \cdots + z^{2^n - 1} = 0$. The product of the $2^n - 1$ roots of this equation is $a^n - 1$. Hence, there is at least one root with modulus smaller or equal to $|a^n - 1|^{1/(2^n-1)}$. It is possible to find numbers $\omega \in \mathbf{R} - \mathbf{Q}$ such that

$$\liminf_{n\to\infty} [\text{dist}(n\omega, \mathbf{N})]^{1/(2^n-1)} = 0 \ .$$

Hence, the f above has periodic orbits different from zero in any neighborhood of the origin. This is a contradiction with f being conjugate to an irrational rotation in any neighborhood of the origin. This shows that the perturbation expansions may diverge if the rotations are very well approximated by rational numbers.

For complex polynomials in one variable it has been shown [Yoc95], (see also [PM92]) that if ω does not satisfy the Brjuno conditions (2.34) below, the series for the quadratic polynomial diverges. The Theorem 4.1 which we will prove later will establish that if the condition is met, then the series for all the non-linearities converges.

We say that ω satisfies a Brjuno condition when there exists an Ω increasing and log convex (the later properties are just for convenience and can always be adjusted) such that

$$\Omega(n) \geq \sup_{k\leq n} |a^k - 1|^{-1}$$

$$\sum_n \frac{\log \Omega(2^n)}{2^n} < \infty \iff \sum_n \frac{\log \Omega(n)}{n^2} < \infty \tag{2.34}$$

The equivalence of the two forms of the condition is very easy from Cauchy test for the convergence of series. An example of functions $\Omega(n)$ satisfying (2.34) is:

$$\Omega(n) = \exp(An/(\log(n)\log\log(n)\cdots[\log^k(n)]^{1+\epsilon}))$$

for large enough n, where by $\log^k$ we denote the function log applied k times.

Indeed, [Yoc95] shows that if ω fails to satisfy the condition (2.34) then $f(z) = e^{2\pi i\omega}z + z^2$ is not linearizable in any neighborhood of the origin.

Remark 2.16. In [Yoc95] one can find the result that if, a function $f(z)$ with $f(0) = 0$, $f'(0) = a$, with $|a| = 1$ is not linearizable, near 0, then, the quadratic function $az + z^2$ is not linearizable.

Similar results for other families of functions do not seem to be in the literature. Of course, one can easily construct families for which a non-Brujno number is linearizable. It does not seem easy to characterize families of functions for which the above is true.

In the case of one dimensional variables, one can use the powerful theory of continued fractions to express the Brjuno condition in an equivalent form.

If $\omega \in \mathbf{R} - \mathbf{Q}$ can be written $\omega = [a_0, a_1, a_2, \cdots, a_n, \cdots]$ with $a_i \in \mathbf{N}^+$, we call $[a_0, a_1, \cdots, a_n] = p_n/q_n$ the convergents.

Brjuno condition is equivalent to

$$\mathcal{B}(\omega) \equiv \sum_n (\log q_{n+1})/q_n \leq \infty \tag{2.35}$$

A very similar condition

$$\sum_n (\log\log q_{n+1})/q_n \leq \infty \tag{2.36}$$

has been found in [PM91] [PM93] to be necessary and sufficient for the existence of the Cremer's phenomenon of accumulation of periodic orbits near the origin in the sense that if condition (2.36) is satisfied, then, all non-linearizable functions have a sequence of periodic orbits accumulating at the origin. If condition (2.36) is not satisfied, there exists a non-linearizable germ with no periodic orbits other than zero in a neighborhood of zero.

Remark 2.17. We note that the formula (2.35) has very interesting covariance properties under modular transformations. They have been used quite successfully in [MMY97].

Without entering in many details, we point out that another function very closely related to the one we have defined satisfies (setting $\tilde{\mathcal{B}}(x) = +\infty$ when $x \in \mathbf{Q}$)

$$\begin{aligned}\tilde{\mathcal{B}}(\omega) &= -\log(x) + x\tilde{\mathcal{B}}(1/x) \quad x \in (0, 1/2)\\ \tilde{\mathcal{B}}(\omega)(-x) &= \tilde{\mathcal{B}}(x) \quad x \in (-1/2, 0)\\ \tilde{\mathcal{B}}(\omega)(x+1) &= \tilde{\mathcal{B}}(x)\end{aligned}$$

Similar invariance properties are true for the sum appearing in (2.36). Nevertheless, it does not seem to have been investigated as extensively.

Unfortunately, this one dimensional theory does not have analogues in higher dimensions. Some preliminary numerical explorations for the higher dimensional case were done in [Tom96].

Remark 2.18. There is a very similar theory of changes of variables that reduce the problem to linear – or some canonical – form for differential equations.

Of course, these normalizations resemble the normalizations of singularity theory and are basic for many applied questions such as *bifurcation theory*.

Similarly, there is a theory of these questions in the C^∞ or C^r categories under assumptions, which typically include that there are no eigenvalues of unit length. This theory usually goes under the name of Sternberg theory.

The reduction of maps and differential equations to *normal form* by means of changes of variables can also be done when the map is required to preserve a symplectic – or another geometric – structure and one requires that the change of variables preserve the same structure.

We will not discuss much of these interesting theories. For more information on many of these topics we refer to [Bru89], [Bib79].

3. Preliminaries

In this Section, we will collect some background in analysis, number theory and (symplectic and volume preserving) geometry. Experts will presumably be familiar with most of the material and will only need to read this as it is referenced in the following text. Of course, this chapter is not a substitute for manuals in geometry or on analysis. I have found [Thi97], [AM78], [GP74] useful for geometry and [Ste70] [Kat76] useful for analysis. Many of the techniques are discussed in other papers in KAM theory which we will mention as we proceed. Specially the papers [Mos66b], [Mos66a] contain an excellent background in many of the analytical techniques.

In the previous discussion of Lindstedt series we saw that we had to consider repeatedly equations of the form

$$L_\omega \varphi = \eta.$$

(The formal solution was given in (2.14).)

In this Section, we will also study equations

$$D_\omega \varphi = \eta, \quad \text{where} \quad D_\omega = \sum_i \omega_i \frac{\partial}{\partial \theta_i} \ ,$$

which also appears in KAM theory.

A first step towards obtaining proofs of the KAM theorem is to devise a theory of these equations. That is, find conditions in ω and η so that the function defined by (2.14) has a precise meaning.

The guiding heuristic principles are very simple:

1) The smoother the function η, the faster its Fourier coefficients $\hat{\eta}_k$ decay.

2) Some numbers ω are such that the denominators appearing in the solution (2.14) do not grow very fast with k.
3) Hence, for the numbers alluded to in 2), we will be able to make sense of the formal solutions (2.14) when the function considered is smooth.

We devote Sections 3.1, 3.3, 3.4 to making precise the points above. We will need to discuss number theoretic properties (usually called Diophantine properties) that quantify how small the denominators can be as a function of k. We will also need to study characterizations of regularity in terms of Fourier coefficients.

Since the result in KAM theory depends on the geometric properties of the map – as illustrated in (2.7) and (2.8) – it is clear that we will need to understand which geometric properties enter in the conclusions. Moreover, many of the traditional proofs indeed use a geometric formalism. Hence, we have devoted a Section 3.5 to collect the facts we will need from differential geometry.

3.1. **Preliminaries in analysis.** In modern analysis, it is customary to measure the regularity of a function by saying that it belongs to some space in a certain scale of spaces. Some scales that are widely used on compact manifolds are:

$$
\begin{aligned}
C^r &\equiv \Big\{\eta \,\Big|\, D^r\eta \text{ is continuous, } \|\eta\|_{C^r} \equiv \max\big(\sup_x |\eta(x)|, \dots, \\
&\qquad\qquad \sup_x |D^r\eta(x)|\big)\Big\}\ , \\
C^{r+\alpha} &\equiv \Big\{\eta \,\Big|\, \|\eta\|_{C^r} \equiv \max\Big(\sup_x |\eta(x)|, \dots, \sup_x |D^r\eta(x)|, \\
&\qquad\qquad \sup_{x\neq y} \frac{|D^r\eta(x) - D^r\eta(y)|}{|x-y|^\alpha}\Big)\Big\}\ , \\
A_\delta &\equiv \Big\{\eta \,\Big|\, \eta \text{ analytic on } |\operatorname{Im}\theta| < \delta,\ \text{continuous on } |\operatorname{Im}\theta| \le \delta, \\
&\qquad\qquad \|\eta\|_\delta \equiv \sup_{|\operatorname{Im}\theta|\le\delta} |\eta(\theta)|\Big\}\ , \\
H^s &\equiv \Big\{\eta \,\Big|\, \eta \in L^2,\ (-\Delta+1)^{s/2}\eta \in L^2,\ \|\eta\|_{H^s} \equiv \|(-\Delta+1)^{s/2}\eta\|_{L^2}\Big\}\ ,
\end{aligned}
$$

where $r = 0, 1, 2, \dots,\ \alpha \in [0,1],\ \delta \in \mathbf{R}^+,\ s \in \mathbf{R}$ and we have used Δ to denote the Laplacian

Note that this notation (even if in wide usage) has certain ugly points. C^{r+0} and C^{r+1} are ambiguous and can be considered according to the first or the second definition. Indeed, C^{r+0} consider according to the two definitions agrees as a space (that is, the functions in one are functions in the other and the topologies are the same), but the norms differ (they are equivalent). On the other hand, C^{r+1} differs when the we interpret it in the first or in the second sense. To avoid that, we will use $C^{r+\mathrm{Lip}}$ to identify the second definition.

All the above scales of spaces have advantages and disadvantages. Against $C^{r+\alpha}$ we note that, even if for $r = 0$, these are the Hölder spaces which can be defined in great generality (e.g. metric spaces), when $r \geq 1$, the definition needs to be done in a differentiable system of coordinates. This is because, for $r \geq 1$, $D^r\eta(x)$ and $D^r\eta(y)$ are multilinear operators in T_xM and T_yM, so that the differences in the definition are comparing operators in different spaces. Even though the different choices of coordinates lead to equivalent norms, some of the geometric considerations are somehow cumbersome. Also the composition operator — ubiquitous in KAM theory — has properties which are cumbersome to trace in $C^{r+\alpha}$. For example, the mapping $x \to f(x + \cdot)$ can be discontinuous in $C^{r+\alpha}$ when f is $C^{r+\alpha}$.

It is somewhat unfortunate that the notations C^r ($r \in \mathbf{N}$) and $C^{r+\alpha}$, ($r \in \mathbf{N}, \alpha \in [0,1) \cup \{\mathrm{Lip}\}$ suggest that one can consider perhaps C^s ($s \in \mathbf{R}$) which includes both. If one proceeds in this way, one obtains very bad properties for the scale of spaces. In colorful words, "the limit of the space $C^{k+\alpha}$ as $\alpha \to 0$ is not C^k". More precisely, several important inequalities such as interpolation inequalities which relate the different norms in a scale fail to hold. Many characterizations – e.g. in terms of approximations by analytic functions – break down for the case that r is an integer.

A possible way of breaking up the unfortunate C^r vs. $C^{r+\alpha}$ notation is to introduce the spaces called Λ_r in [Ste70], or $\hat{C}^r$ in [Zeh75], [Mos66b], [Mos66a]. In general we define:

$$
\begin{aligned}
&\Lambda_0 = C^0 \\
&\Lambda_1 = \Big\{ f \,\Big|\, \sup_{\substack{1>|h|>0\\ x\in\mathbf{R}}} \frac{|f(x+h) + f(x-h) - 2f(x)|}{h} \equiv \|f\|_{\Lambda_1} < \infty \Big\} \\
&\Lambda_r = \{ f \mid Df \in \Lambda_{r-1},\ \|f\|_{\Lambda_r} = \max(\|f\|_{C^0}, \|Df\|_{\Lambda_{r-1}}) \Big\} \quad r \in \mathbf{N} \\
&\Lambda_{r+\alpha} = C^{r+\alpha} \qquad r+\alpha \notin \mathbf{N}
\end{aligned}
\tag{3.1}
$$

Here $[r]$ means the integer part of r and $\{r\}$ means the fractional part of r.

There are many reasons why the Λ_α spaces are the natural scale of spaces to consider when one is considering a space that includes the usual $C^{r+\alpha}$. For example, one can obtain very nice approximation theory, interpolation inequalities, and generalize naturally to several variables. Note that

$$C^1 \subsetneq C^{0+\mathrm{Lip}} \subsetneq \Lambda_1 \ .$$

Again, we point out that it is not easy to define these spaces on manifolds except through patches. Choosing different patches leads to different norms. Fortunately, all of them are equivalent and, hence define the same topology in the spaces.

Note that C^r norms can be defined naturally on any smooth Riemannian manifold. (The norm of derivatives can be defined since it is the norm of multilinear operators in the tangent bundle.)

The main inconvenience of C^r (r is, by assumption, an integer) is that the characterization by Fourier series is rather cumbersome. It is easy to show integrating by parts that

$$
\begin{aligned}
\hat{\eta}_k &\equiv \int_0^1 \eta(\theta) e^{-2\pi i k\theta}\, d\theta \\
&= (-2\pi i)^{-r} k^{-r} \int_0^1 D^r \eta(\theta) e^{-2\pi i k\theta}\, d\theta \ .
\end{aligned}
$$

Hence, if $\eta \in C^r$, we have

$$
\sup_k \left(|\hat{\eta}_k|\, |k|^r \right) \le C_r \|\eta\|_{C^r} \ . \tag{3.2}
$$

where C_r is a constant that depends only on r.

In the other direction, we have for any $\delta > 0$

$$
\begin{aligned}
\|\eta\|_{C^r} &\le \tilde{C}_r \sum_k |\hat{\eta}_k|\, |k|^r = \tilde{C}_r \sum_k \frac{1}{|k|^{1+\delta}} |\hat{\eta}_k|\, |k|^{r+1+\delta} \\
&\le \tilde{C}_r \left(\sum_k \frac{1}{|k|^{1+\delta}} \right) \sup_k \left(|\hat{\eta}_k|\, |k|^{r+1+\delta} \right) \\
&\le \tilde{\tilde{C}}_{r,\delta} \sup_k \left(|\hat{\eta}_k|\, |k|^{r+1+\delta} \right) \ .
\end{aligned} \tag{3.3}
$$

Both inequalities (3.2), (3.3) are essentially optimal in the following sense. Inequality (3.2) is saturated by trigonometric polynomials, while the usual square wave — or iterated integrals of it — shows that it is impossible to reduce the exponent on the right hand side of (3.3) to $r+1$. This discrepancy is worse when we consider functions on $\mathbf{T}^d$, $d > 1$. In that case, to obtain convergence of the series, in (3.3) one needs to take $\delta > d$. This shows that studying regularity in terms of just the size of the coefficients will lead to less than optimal results.

Exercise 3.1. Show that given any sequence a_n of positive numbers converging to zero, the set of continuous functions f with $\limsup_k |\hat{f}_k|/a_k = \infty$ is residual in C^0.

The spaces of analytic functions A_δ are better behaved in respect of characterizations of norms of the function in terms of its Fourier coefficients. Integrating over an appropriate contour, we have Cauchy inequality

$$
|\hat{\eta}_k| \le e^{-2\pi\delta|k|} \|\eta\|_{A_\delta} \ . \tag{3.4}
$$

On the other hand,

$$
\begin{aligned}
\|\eta\|_{A_{\delta-\sigma}} &\le \sum_{k\in\mathbf{Z}} e^{2\pi(\delta-\sigma)|k|}|\hat{\eta}_k| \\
&\le \left(\sum_{k\in\mathbf{Z}} e^{-2\pi\sigma|k|}\right)\sup_k e^{2\pi\delta|k|}|\hat{\eta}_k| \qquad (3.5)\\
&\le C\sigma^{-1}\sup_k e^{2\pi\delta|k|}|\hat{\eta}_k|
\end{aligned}
$$

Of course, for Sobolev spaces, the characterization in terms of Fourier coefficients is extremely clean:

$$\|\eta\|_{H^s} = \left(\sum_{k\in\mathbf{Z}}(|k|^2+1)^s|\hat{\eta}_k|^2\right)^{1/2} .$$

Sobolev spaces have other advantages. For example, they are very well suited for numerical work and they also work nicely with partial differential operators. Many of the tools that we used in Λ_α spaces also carry through to Sobolev spaces.

For example, we have the interpolation inequalities:

$$\|u\|_{H^j} \le K\|u\|_{H^m}^{j/m}\|u\|_{H^0}^{1-j/m} . \tag{3.6}$$

This inequality is a particular case of the following Nirenberg inequality

$$\|D^i u\|_{L^r(\mathbf{R}^n)} \le C\|u\|_{L^p(\mathbf{R}^n)}^{1-i/m}\cdot\|D^m u\|_{L^q(\mathbf{R}^n)}^{i/m} , \tag{3.7}$$

where $1/r = (1-i/m)(1/p)+(i/m)(1/q)$. We refer to [Ada75], p. 79.

These interpolation inequalities both for Λ_α and for Sobolev spaces are part of the more general "complex interpolation method" and the scales of spaces are "interpolation spaces". Even if this is quite important for certain problems of analysis in these spaces, we will not go into these matters here.

As we will see later, some of the abstract versions of KAM as an implicit function theorem work perfectly well for Sobolev spaces. I think it is mainly a historical anomaly that these spaces are not used more frequently in the KAM theory of dynamical systems. (Notable exceptions are [Her86], [KO89b].) Of course, for the applications of Nash-Moser theory to PDE's or geometric problems, Sobolev spaces are used quite often.

One of the most useful tools in the study of $C^{r+\alpha}$ spaces is that they can be characterized by their approximation properties by analytic functions.

The following characterization of Λ_r spaces (remember that they agree with the Hölder spaces $C^{[r]+\{r\}}$ when $\{r\}\neq 0$) comes from [Mos66b, Mos66a] (see also [Zeh75], Lemma 2.2).

Lemma 3.2. *Let $h\in C^0(\mathbf{T}^1)$. Then $h\in\Lambda_r$ if and only if for some $\sigma>0$ we can find a sequence $h_i\in A_{\sigma 2^{-i}}$ such that*

i) $\|h_i - h_0\|_{C^0}\to 0$

ii) $\sup_{i\ge 1}(2^{ir}\|h_i-h_{i-1}\|_{A_{\sigma 2^{i-1}}})<\infty$

Moreover, it is possible to arrange that the sup in ii) is equivalent to $\|h\|_{\Lambda_r}$ if one chooses the h_i appropriately.

If we denote the sup in ii) by M, we have that for $h \in \Lambda_r$ it is possible to find a sequence h_i in such a way that $M \le C_{\sigma,r}\|h\|_{\Lambda_r}$. Conversely, for any sequence h_i as above, we have $\|h\|_{\Lambda_r} \le \tilde{C}_{\sigma,r}M$. Given a function $h \in \Lambda_r$ there are canonical ways of producing the desired h_j. For example, in [Ste70] and [Kra83] is shown that one can use convolution with the Poisson kernel to produce the h_j. In that case, the sup in ii) can be taken to define a norm equivalent to $\| \ \|_{\Lambda_r}$.

Another important feature of the Λ_α spaces is that they admit a very efficient approximation theory.

The first naive idea that occurs to one when trying to approximate a function by a smoother one is just to expand in Fourier series and to keep only a finite number of terms corresponding to the harmonics of small degree. Indeed, for some methods of proof of the KAM theorem that emphasize geometry this is the method of choice. (See Section 6.2.) Unfortunately, keeping only a finite number of the low order Fourier terms is a much less efficient method of approximation (from the point of view of the number of derivatives required) than convolving with a smooth kernel. Recall that summing a Fourier series is just convolution with the Dirichlet kernel,

$$\begin{aligned}\sum_{k=-N}^{N} \hat{\eta}_k e^{2\pi i k\theta} &= \int_0^1 \eta(\varphi)\mathcal{D}_N(\theta-\varphi)\,d\varphi = (\eta * D_N)(\theta)\\ \mathcal{D}_N(\theta) &= \frac{\sin(2N+1)\pi\theta}{\sin\pi\theta}\ ,\end{aligned}$$

which is large and oscillatory and hence generates more oscillations upon convolution than smooth kernels.

Hence the method of choice of approximating functions by smoother ones is to choose an positive analytic function $K : \mathbf{R}^d \to \mathbf{R}$ decaying at infinity rather fast and with integral 1 and define $K_t(x) \equiv \frac{1}{t^d}K(x/t)$.

We define smoothing operators S_t by convoluting with the kernels K_t. That is:

$$S_t\phi = K_t * \phi.$$

The properties of these smoothing operators that are useful in KAM theory are (we express them in terms of the Λ_r spaces introduced in (3.1)):

$$\text{(3.8)}\quad \begin{aligned} &\text{i)} && \lim_{t\to\infty}\|S_t u - u\|_{\Lambda_0} = 0\ , \quad u \in \Lambda_0\\ &\text{ii)} && \|S_t u\|_{\Lambda_\mu} \le t^{\mu-\lambda}C_{\lambda\mu}\|u\|_{\Lambda_\lambda}\ , \quad u \in \Lambda_\lambda,\ 0 \le \lambda \le \mu\\ &\text{iii)} && \|(S_t - 1)u\|_{\Lambda_\lambda} \le t^{-(\mu-\lambda)}C_{\lambda\mu}\|u\|_{\Lambda_\mu}\ , \quad x \in \Lambda_\mu\ , 0 \le \lambda \le \mu\end{aligned}$$

We note that a slightly weaker version of these properties is:

$$\text{(3.9)}\quad \begin{aligned} &\text{ii}') && \|S_t u\|_{\Lambda_{t-1}} \le k(\ell)\|u\|_{\Lambda_\ell} \qquad t \ge 0\\ &\text{iii}') && \|(S_\tau - S_t)u\|_{\Lambda_{\tau-1}} \le t^{-\ell}k(\ell)\|u\|_{\Lambda_\ell} \qquad u \in \Lambda_\ell \quad \tau \ge t \ge 1\end{aligned}$$

Note that it is easy to show that ii) $\Rightarrow$ ii$'$), iii) $\Rightarrow$ iii$'$). In [Zeh75] operators S_t satisfying (3.8) are said to constitute a C^∞ smoothing and those satisfying i), ii$'$), iii$'$) a C^ω smoothing.

There are other smoothing operators and other scales of spaces that satisfies the same inequalities. Indeed, in the most abstract version of KAM theory, which we discuss in Section 5, one can even abstract these properties and obtain a general proof which also applies to many other situations.

One important consequence of the existence of smoothing operators is the existence of interpolation inequalities (see [Zeh75]). Even if this inequality were proved directly long time ago, and can be obtained by different methods, it is interesting to note that they are a consequence of the existence of smoothing operators. As we mentioned, this happens in other situations and for other spaces than Λ_r. In the following, we denote $\|u\|_r \equiv \|u\|_{\Lambda_r}$.

Lemma 3.3. *For any* $0 \le \lambda \le \mu$, $0 \le \alpha \le 1$, *denoting*

$$\nu = (1-\alpha)\lambda + \alpha\mu$$

we have for any $u \in \Lambda_\mu$*:*

$$\|u\|_\nu \le C_{\alpha,\lambda,\mu} \|u\|_\lambda^{1-\alpha} \, \|u\|_\mu^\alpha \tag{3.10}$$

Proof. We clearly have:

$$\|u\|_\nu \le \|S_t u\|_\nu + \|(\mathrm{Id} - S_t)u\|_\nu \ .$$

Applying *ii*) of (3.8) to the first term and *iii*) to the second, we obtain:

$$\|u\|_\nu \le t^{\nu-\lambda} C_{\alpha\lambda,\mu} \|u\|_\lambda + t^{-(\mu-\nu)} C_{\alpha\mu,\nu} \|u\|_\mu$$

and we obtain (3.10) by optimizing the right hand side in t.

□

These inequalities are descendents of inequalities for derivatives of functions which were proved, in different versions, by Hadamard and Kolmogorov and others. For Λ_r, $r \notin \mathbf{N}$ and for C^r, $r \in \mathbf{N}$, the proofs can be done by elementary methods and extend even to functions defined in Banach spaces [dlLO99]. For analytic spaces, these interpolation inequalities are classical in complex analysis and are a consequence of the fact that the $\log|f(z)|$ is sub-harmonic when $f(z)$ is analytic [Rud87].

In KAM theory the interpolation inequalities (3.10) are useful because if we have a smooth norm ($\| \ \|_\mu$) blowing up and a not so smooth one ($\| \ \|_\lambda$) going to zero, we can still get that other norms smoother than λ still converge.

All the above results about Λ_α spaces of functions on the real line can be generalized to spaces of functions on $\mathbf{R}^n$. Indeed, one of the nicest things of these spaces is that the theory for them can be reduced to the study of one dimensional restrictions of the function. We refer to [Ste70, Kra83] for more details.

For analytic spaces, the theory can be also extended with minor modifications. In KAM theory we often have to consider functions defined in

$\mathbf{T}^m \times \mathbf{R}^n$ (often $n = m = d$). In such a case, it is very convenient to use expansions which are Taylor expansions in the real variables and Fourier expansions in the angle:

$$f(\theta, I) = \sum_{j \in \mathbf{N}^n, k \in \mathbf{Z}^m} f_{j,k} I^j \exp(2\pi i k \cdot \theta) \tag{3.11}$$

For these functions, it is convenient to define norms

$$\|f\|_\sigma = \sup_{|I| \le e^{2\pi\sigma}, |\operatorname{Im}(\theta)| \le \sigma} |f(\theta, I)| \tag{3.12}$$

With this definition, we have the Cauchy bounds

$$\begin{aligned} |f_{j,k}| &\le \exp(-2\pi\delta(|j| + |k|))\|f\|_\sigma \\ \left\| \frac{\partial^{|r|+|s|}}{\partial I^r \partial \theta^s} f \right\|_{\sigma-\delta} &\le C_{r,s,n,m} \delta^{-|r|-|s|} \|f\|_\sigma \end{aligned} \tag{3.13}$$

The proof of these inequalities is quite standard in complex analysis and will not be given in detail here. It suffices to express the derivatives as integrals over an $n+m$ dimensional torus which is close to the boundary of the domain in which $f(\theta, I)$ is controlled by $\|f\|_\sigma$. The only subtlety is that for some $l \in \{1, \dots, m\}$, $k_l > 0$ one needs to choose the torus $\operatorname{Im}(\theta_l) = -\sigma$. (Similarly for the case when $k_l < 0$ one needs to choose the torus $\operatorname{Im}(\theta_l) = \sigma$.)

It is also obvious that, under these supremum norm the spaces constitute a Banach algebra, that is:

$$\|fg\|_\sigma \le \|f\|_\sigma \|g\|_\sigma \ . \tag{3.14}$$

Therefore, if $\|f\|_\sigma < 1$, then $\|(1+f)^{-1}\|_\sigma \le (1 - \|f\|_\sigma)^{-1}$.

3.2. **Regularity of functions defined in closed sets. The Whitney extension theorem.** In KAM theory, we often have to study functions defined in Cantor sets. In particular, sets with empty interior. In this situation, the concept of Whitney differentiability plays an important role.

A reasonable notion of smooth functions in closed sets is that they are the restriction of smooth functions in open sets that contain them. This definition is somewhat unsatisfactory since the extension is not unique.

In the paper [Whi34a], one can find an intrinsic characterization of smooth functions in a closed set.

Definition 3.4. We say that a function f is C^k in the sense of Whitney in a compact set $F \subset \mathbf{R}^d$ when for every point $x \in F$ we can find polynomials P_x of degree less that k such that

$$\begin{aligned} &f(x) = P_x(x) \quad x \in F \\ &|D^i P_x(y) - D^i P_x(x)| \le |x-y|^{r-i} \sigma(|x-y|) \quad x, y \in F \end{aligned} \tag{3.15}$$

where σ is a function that tends to zero.

It is clear that if a function is the restriction of a C^k function the Taylor polynomials will do.

The deep theorem of [Whi34a] is that the converse is true. That is,

Theorem 3.5. *Let $F \subset \mathbf{R}^d$ be a compact set.*

If for a function f we can find polynomials satisfying (3.15) *and such that $f(x) = P_x(x)$ then the function f can be extended to an a C^r function in $\mathbf{R}^d$.*

Note that if a function is C^r in $\mathbf{R}^d$, then one can find polynomials satisfying (3.15) by taking just the Taylor expansions of f.

Contrary with what happened with the ordinary derivatives, the polynomials satisfying (3.15) may not be unique. (For example, if we take F to be the x-axis in $\mathbf{R}^2$, we can take polynomials with a a very different behavior in the y direction.)

There are other variants of the definitions in which rather than using $D^i P_x$ one introduces another polynomial P^i_x which is then, required to satisfy compatibility conditions with the other polynomials.

The assumption that F is compact can be removed. It suffices to require (3.15) in each compact subset of F, allowing σ to depend on the compact subset.

In [Ste70] one can find a version of this theorem in which the extensions can be implemented via a linear extension operator. (There is a different extension operator $\mathcal{E}_k$ for each k.) In [Ste70], one can also find versions for $C^{k+\alpha}$. The C^∞ version can be found in [Whi34b].

Even if adapting Whitney's theorem from real valued function to functions taking values in a Banach space is well known, (e.g [Fed69] p. 225 ff.) I do not know how to prove a similar result when F lies on an infinite dimensional space.

3.3. **Diophantine properties.** In this Section, we want to study the existence of vectors $\omega \in \mathbf{R}^n$ so that we can obtain upper bounds of $[\mathrm{dist}(\omega \cdot k, \mathbf{N})]^{-1}$ and of $|\omega \cdot k|^{-1}$ when $k \in \mathbf{Z}^n - \{0\}$. These are the small divisors that appear respectively in the solution of the equations (3.27), (3.26), which appear often in KAM theory.

When we are studying problems such as those in Section 2.2, we need only to consider $k \in \mathbf{N}^n$.

When $n = 1$ for (3.27) (and for $n = 2$ for (3.26)) one can get quite good results using classical tools of number theory, notably continued fractions, which we will not review here, in spite of their importance in 1-dimensional dynamics.

For example, the classical result of Liouville states

Theorem 3.6. *Let $\omega \in \mathbf{R} - \mathbf{Q}$ satisfy $P(\omega) = 0$ with P a polynomial of degree ℓ with integer coefficients. Assume that $P'(\omega) = 0, \dots, P^{(j}(\omega) = 0$,*

$P^{(j+1)}(\omega) \neq 0$. *Then for some $C > 0$*

$$\left|\omega - \frac{m}{n}\right| \geq Cn^{-\ell/(j+1)} \qquad \forall\ m, n \in \mathbf{Z}\ . \tag{3.16}$$

Proof. The zeroes of polynomials are isolated, hence $P(\frac{m}{n}) \neq 0$ when $\frac{m}{n}$ is close enough to ω. This together with the fact that $n^\ell P(\frac{m}{n}) \in \mathbf{Z}$ implies that $|n^\ell P(\frac{m}{n})| \geq 1$ and, therefore, $|P(\frac{m}{n}) - P(\omega)| \geq n^{-\ell}$. On the other hand, by the Taylor's theorem,

$$\left|P\Big(\frac{m}{n}\Big) - P(\omega)\right| \leq C\left|\omega - \frac{m}{n}\right|^{j+1}$$

for some $C > 0$. (The R.H.S. is the remainder of Taylor's theorem.) This yields the desired result for $\frac{m}{n}$ close to ω. For $\frac{m}{n}$ far from ω, the result is obvious. □

Theorem 3.6 was significantly improved by Roth, who showed that, if ω is an algebraic irrational, $|\omega - \frac{m}{n}| \geq C_\varepsilon n^{-2-\varepsilon}$ for every $\varepsilon > 0$.

The numbers that satisfy the equation (3.16) in the conclusions of Theorem 3.6 are quite important in number theory and in KAM theory and are called Diophantine. As we will see in Lemma 3.9, Diophantine numbers occupy positive measure, hence, there are some of them which do not satisfy the hypothesis of Theorem 3.6.

Definition 3.7. A number ω is called *Diophantine* of type (K, ν) for $K > 0$ and $\nu \geq 1$, if

$$\left|\omega - \frac{p}{q}\right| > K\,|q|^{-1-\nu} \tag{3.17}$$

for all $\frac{p}{q} \in \mathbf{Q}$. We will denote by $\mathcal{D}_{K,\nu}$ the set of numbers that satisfy (3.17). We denote by $\mathcal{D}_\nu = \cup_{K>0}\mathcal{D}_{K,\nu}$.

A number which is not Diophantine is called a *Liouville number*.

The numbers ω for which $|\omega - \frac{m}{n}| \geq Cn^{-2}$ are called "constant type" and the previous result shows that quadratic irrationals are constant type. It is an open problem to decide whether $\sqrt[3]{2}$ is constant type or not. Indeed, it would be quite interesting to produce any non-quadratic algebraic number which is of constant type.

In higher dimensions, there are two types of Diophantine conditions that appear in KAM theory, namely:

$$|\omega \cdot k|^{-1} \leq C|k|^\nu \quad \forall k \in \mathbf{Z}^n - \{0\} \tag{3.18}$$

$$|\omega \cdot k - \ell|^{-1} \leq C|k|^\nu \quad \forall (k, \ell) \in \mathbf{Z}^n \times \mathbf{Z} - \{(0, 0)\} \tag{3.19}$$

The first condition (3.18) appears when we consider the KAM theory for flows, the second one (3.19) when we consider KAM theory for maps. As we will see the arguments are very similar in both cases.

Remark 3.8. One important difference between these Diophantine conditions is that the first condition (3.18) is maintained – with only different constants – if the vector ω is multiplied by a constant. Nevertheless, the second one is not. Indeed, if we take advantage of this to set one of the coordinates to 1, then, we see that (3.18) becomes (3.19) for the vector in one dimension less obtained by keeping the coordinates not set to 1.

The arguments that study geometry of these Diophantine conditions are identical. Nevertheless, we point out that the scale invariance of (3.18) will have some consequences later, namely that KAM tori for flows often appear in smooth one-dimensional families, whereas those for maps are isolated.

For us, the most important result is

Lemma 3.9. *Let $\Omega : \mathbf{R} \to \mathbf{R}$ be an increasing function satisfying*

$$\sum_{r=1}^{\infty} \Omega(r)^{-1} r^{n-1} < a(n) \tag{3.20}$$

where $a(n)$ is an explicit function of the dimension n. Then the set $\mathcal{D}_\Omega$ of $\omega \in \mathbf{R}^n$ such that

$$\left(\inf_{\ell \in \mathbf{N}} |\omega \cdot k - \ell| \right)^{-1} \le \Omega(|k|) \qquad \forall\ k \in \mathbf{Z}^n \setminus \{0\} \tag{3.21}$$

has the property that, given any unit cube $\mathcal{C}$

$$|\mathcal{C} \cap \mathcal{D}_\Omega| \ge 1 - a(n)^{-1} \sum_{r=1}^{\infty} \Omega(r)^{-1} r^{n-1} \tag{3.22}$$

where $|\ |$ denotes the Lebesgue measure.

Note that when we take $\Omega(|k|) = K^{-1}|k|^\nu$, (3.21) reduces to (3.17). The condition (3.20) is satisfied for $\nu > n$ and for K sufficiently big. This shows that the set of Diophantine numbers $\mathcal{D}_\nu$ has full measure for $\nu > n$. Indeed

$$|\mathcal{C} \cap \mathcal{D}_{K,\nu}| \ge 1 - K b(\nu, n) \tag{3.23}$$

Proof. We will denote by σ_n constants that depend only on the dimension n. The same symbol can be used for different constants.

For $k \in \mathbf{Z}^n \setminus \{0\}$, $\ell \in \mathbf{Z}$ we consider the set

$$\mathcal{B}_{k,\ell} = \{\omega \in \mathbf{R}^n \mid |\omega \cdot k - \ell| \le \Omega(|k|)^{-1}\}$$

consisting of the ω's for which the desired inequality (3.21) fails precisely for k, ℓ. The desired set will be the intersection of the complements of these sets.

Geometrically $\mathcal{B}_{k,\ell}$ is a strip bounded by parallel planes which are at a distance $2\Omega(|k|)^{-1}|k|^{-1}$ apart (see Figure 3.3). Thus, given a unit cube $\mathcal{C} \in \mathbf{R}^n$, the measure of $\mathcal{C} \cap \mathcal{B}_{k,\ell}$ cannot exceed $\sigma_n \Omega(|k|)^{-1}|k|^{-1}$.

We also observe that given $k \in \mathbf{Z}^n - \{0\}$, there is only a finite number of ℓ such that $\mathcal{C} \cap \mathcal{B}_{k,\ell} \ne \emptyset$. Indeed, this number can be bounded by $\sigma_n |k|$.

Therefore, for any $k \in \mathbf{Z}^n \setminus \{0\}$

$$\sum_{\ell \in \mathbf{Z}} |\mathcal{B}_{k,\ell} \cap \mathcal{C}| \le \sigma_n \Omega(|k|)^{-1} \ ,$$

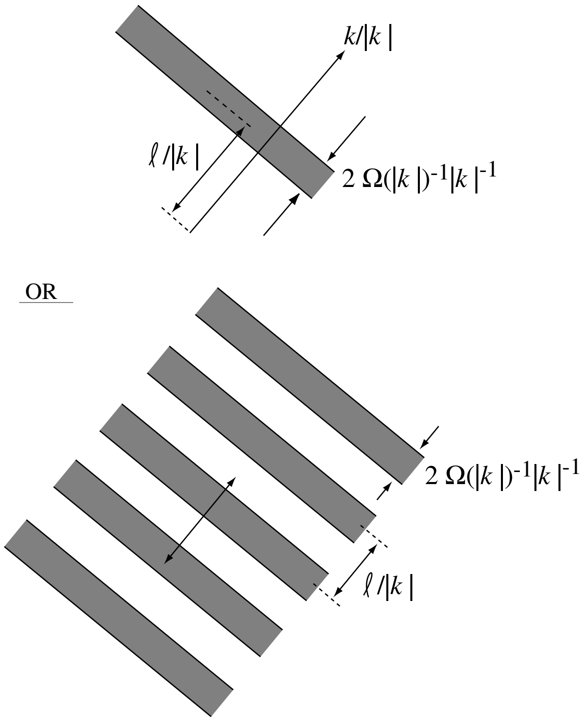

FIGURE 2

hence,

$$
\begin{aligned}
1 - |\mathcal{C} \cap \mathcal{D}_\Omega| &= \sum_{k \in \mathbf{Z}^n} \sum_{\ell \in \mathbf{Z} \setminus \{0\}} |\mathcal{B}_{k,\ell} \cap \mathcal{C}| \\
&\leq \sigma_n \sum_{k \in \mathbf{Z}^n \setminus \{0\}} \Omega(|k|)^{-1} \\
&\leq \sigma_n \sum_{r=1}^{\infty} \Omega(r)^{-1} r^{n-1} \ .
\end{aligned}
\tag{3.24}
$$

Under the hypothesis that the R.H.S. of the above equation is smaller than 1, the conclusions hold. □

An important generalization of the above argument [Pja69] leads to the conclusion that a submanifold of Euclidean space that has curvature (or torsion or any other higher order condition) in such a way that planes cannot have a high order tangency to it (see below or see the references) then the submanifold has to contain Diophantine numbers. Even if the proof is relatively simple, the abundance of Diophantine numbers in lower dimensional curves has very deep consequences since it allows one to reduce the number of free parameters needed in KAM proofs.

Lemma 3.10. *Let Σ be a compact C^{l+1} submanifold of $\mathbf{R}^n$.*

Assume that at every point $x \in \Sigma$ of the manifold

$$T_x\Sigma + T_x^2\Sigma + \cdots + T_x^l\Sigma = T_x\mathbf{R}^n \tag{3.25}$$

where by $T_x^j\Sigma$ we denote the j tangent plane to Σ.

Then, we can find a constant C_Σ that depends only on the manifold such that:

$$|\Sigma - \mathcal{D}_\Omega| \le C_\Sigma \sum_{r=1}^{\infty} \Omega(r)^{1/l} r^{n-1}$$

where by $|\cdot|$ we denote the Riemannian volume of the manifold.

The geometric meaning of the hypothesis (3.25) is that the manifold is not too flat and that it has curvature and torsion (or torsion of high order) so that every neighborhood of a point has to explore all the directions in space. In particular, we will have a lower bound on the area of the portion of the manifold that can be trapped in a resonant region, which in the space of ω is a flat plane.

The remaining details of the proof is left as an exercise for the interested reader. See also the lectures on number theory in this volume. The proof follows by noting that because of (3.25) we can bound the measure of the regions $\sum_{l\in\mathbf{Z}} \Sigma \cup \mathcal{B}_{k,l} \le C_\Sigma \Omega(k)^{-1/l}$. The worst case happens when the manifold is tangent to a very high order to one of the resonant regions. Since the order of tangency – as well as the constants involved – are uniformly bounded, we obtain the desired result.

Remark 3.11. Notice that the formulation of the Diophantine properties (3.18) and (3.19) also makes sense if we allow ω to take complex values. This sometimes appears when we study complex maps and it is a useful tool. Notice that the argument we have presented works very similarly for the case of ω taking complex values. Indeed, the norm of the inverse can be bounded by the norm of inverse of the real part (or the norm of the inverse of the imaginary part) so, when the real or imaginary parts of an ω vector are Diophantine, the vector is Diophantine.

Sometimes, when studying problems with polynomials we will also need the inequalities only for $k \in \mathbf{N}^n$. Needless to say, these are much easier to satisfy since the signs have less possibilities to compensate and lead to small numbers.

Exercise 3.12. Construct a complex vector which is Diophantine, but whose imaginary and real parts are not Diophantine.

Remark 3.13. The same simple minded argument used in the proof of Lemma 3.9 can be used to obtain estimates not only on the Lebesgue measure of the set of Diophantine numbers but also other geometric properties (for example Hausdorff measure), of sets satisfying Diophantine properties, and that are forced to belong to a manifold, have a resonance, etc.

3.4. **Estimates for the linearized equation.** In this subsection, we will consider estimates for the following equations (3.26), (3.27) that occur very frequently in KAM theory. We have encounter them already in the study of Lindstedt series and we will encounter them again as linearized equations.

We will consider equations of the form:

$$D_\omega \varphi = \eta \qquad \left(D\omega \equiv \omega_1 \frac{\partial}{\partial \theta_1} + \cdots + \omega_n \frac{\partial}{\partial \theta_n} \right) \tag{3.26}$$

$$\begin{aligned} L_\omega \varphi = \eta \qquad & \Big(L_\omega \varphi(\theta_1, \dots, \theta_n) \\ & \equiv \varphi(\theta_1 + \omega_1, \dots, \theta_n + \omega_n) - \varphi(\theta_1, \dots, \theta_n) \Big) \ , \end{aligned} \tag{3.27}$$

where $\eta : \mathbf{T}^n \to \mathbf{R}$ is given and the unknown function to be found is φ.

For the sake of simplicity we will only discuss in detail (3.26). The same considerations apply for (3.27) and we will indicate the minor differences – in fact simplifications – that enter in the discussion of (3.27).

Recall that these equations have a formal solution in terms of Fourier series. Namely, if

$$\eta(\theta) = \sum_{k \in \mathbf{Z}^n} \hat{\eta}_k e^{2\pi i k \cdot \theta} \ , \qquad \hat{\eta}_0 = 0 \ ,$$

then any reasonable solution of (3.26) for which one can define unique Fourier coefficients (e.g. any distribution) has to satisfy:

$$\hat{\varphi}_k 2\pi i k \cdot \omega = \hat{\eta}_k \ .$$

Hence, if $k \cdot \omega \neq 0$, then

$$\hat{\varphi}_k = \frac{\hat{\eta}_k}{2\pi i k \cdot \omega} \ . \tag{3.28}$$

We restrict our attention to cases when $k \cdot \omega \neq 0$ for any $k \in \mathbf{Z}^n - \{0\}$. In that case φ is determined by (3.28) up to an additive constant since we can take any $\hat{\varphi}_0$. To avoid unnecessary complications, we will set $\hat{\varphi}_0 = 0$.

It is not difficult to see that, unless we impose some quantitative restriction on how fast $|k \cdot \omega|^{-1}$ can grow, the solutions given by (3.28) may fail to be even distributions. E.g., take $\hat{\eta}_k = e^{-|k|}$ and arrange that there are infinitely many k for which $|k \cdot \omega|^{-1} \geq e^{e^{|k|}}$.

Exercise 3.14. Given any sequence a_n of positive terms tending to infinity construct an $\omega \in \mathbf{R}^n - \mathbf{Q}^n$ such that, for infinitely many $k \in \mathbf{Z}^n$

$$|\omega \cdot k|^{-1} \geq a_{|k|}. \tag{3.29}$$

Show that the ω constructed above are dense (even if, as we have shown, they will be of measure zero for sequences a_n which grow fast enough).

We will consider ω which satisfy

$$|k \cdot \omega|^{-1} \leq \gamma |k|^\nu \ . \tag{3.30}$$

These numbers were studied in Section 3.3.

It is not difficult to obtain some crude bounds for analytic or finite differentiable functions (we will do better later). Recall that for $\eta \in A_\delta$

$$|\hat{\eta}_k| \le e^{-2\pi\delta|k|}\|\eta\|_{A_\delta} \ ,$$

while for $\eta \in C^r$

$$|\hat{\eta}_k| \le (2\pi)^{-r}|k|^{-r}\|\eta\|_{C^r} \ .$$

Hence, if ω satisfies (3.30), we have for $\eta \in A_\delta$

$$|\hat{\varphi}_k| \le (2\pi)^{-1}\gamma|k|^{\nu}e^{-2\pi\delta|k|}\|\eta\|_{A_\delta} \ ,$$

and for $\eta \in C^r$

$$|\hat{\varphi}_k| \le (2\pi)^{-r-1}\gamma|k|^{\nu-r}\|\eta\|_{C^r} \ .$$

These estimates do not allow us to conclude that φ belongs to the same space as η, but allow us to conclude that it belongs to a slightly weaker space.

As mentioned before, the characterization of the analytic spaces in terms of their Fourier series is very clean, so that we can obtain estimates of the solutions in these spaces. Then, we will use Lemma 3.2 to obtain the results for Λ_r spaces.

Since for $0 < \sigma < \delta$ we have:

$$\|e^{2\pi i k\cdot\theta}\|_{\delta-\sigma} \le e^{2\pi(\delta-\sigma)|k|} \ ,$$

we have

$$\begin{aligned}
\|\varphi\|_{A_{\delta-\sigma}} &\le \sum_{k\in\mathbf{Z}^n\setminus\{0\}} |\hat{\varphi}_k| e^{2\pi|k|(\delta-\sigma)} \\
&\le \sum_{k\in\mathbf{Z}^n\setminus\{0\}} \frac{1}{2\pi|k\cdot\omega|}\|\eta\|_{A_\delta}e^{-2\pi\sigma|k|} \\
&\le \frac{1}{2\pi}\gamma\|\eta\|_{A_\delta}\sum_{k\in\mathbf{Z}^n\setminus\{0\}}|k|^{\nu}e^{-2\pi\sigma|k|} \\
&\le C\gamma\|\eta\|_{A_\delta}\sum_{\ell\in\mathbf{N}}\ell^{\nu+n-1}e^{-2\pi\sigma\ell} \\
&\le C\gamma\sigma^{-(\nu+n)}\|\eta\|_{A_\delta} \ ,
\end{aligned} \tag{3.31}$$

where in the fourth inequality we have just used that we do first the sum in the k with $|k| = \ell$ (the number of terms in this sum can be bounded by $C\ell^{n-1}$). We denote by C constants that depend only on ν and the dimension n and are independent of γ, k, etc.

Similarly, using that

$$\|e^{2\pi i k\cdot\theta}\|_{C^s} \le C|k|^s \ ,$$

we have[4]

$$\begin{aligned}
\|\varphi\|_{C^s} &\leq C\gamma\|\eta\|_{C^r}\sum_{k\in\mathbf{Z}^n}|k|^{\nu-r+s}\\
&\leq C\gamma\|\eta\|_{C^r}\sum_{\ell\in\mathbf{N}}\ell^{\nu-r+s+n-1}\ .
\end{aligned}$$

The sum in the R.H.S. converges provided that

$$r > \nu + s + n\ .$$

Actually, one can do significantly better that these crude bounds if one notices that the small divisors have to appear rather infrequently (see [Rüs75, Rüs76b]).

Note that $\omega\cdot(k+\ell) = \omega\cdot k + \omega\cdot\ell$. Hence, if $\omega\cdot k$ happens to be very small, $\omega\cdot(k+\ell)\approx\omega\cdot\ell$, so that if $|\ell| << |k|$, $\omega\cdot(k+\ell)\approx\omega\cdot\ell$.

In other words, the really bad small divisors appear surrounded by a ball on which the divisors are not that small. Hence, if instead of estimating the size as in (3.31) using the estimates (3.18) in the third step we use a Cauchy-Schwartz inequality, which takes into account the sum of terms, not just the the sup and that can profit from the fact that (3.18) cannot be saturated very often, we obtain the result of [Rüs75, Rüs76b], which reads:

Lemma 3.15. *Assume that ω satisfies* (3.18)*, with $\nu \geq n-1$ and that $\tilde\omega$ satisfies* (3.19)*. Let $\eta,\tilde\eta$ be analytic functions with zero average.*

Then, we can find $\varphi,\tilde\varphi$ solving (3.26)*,* (3.27)*. Namely*

$$\begin{aligned}
&D_\omega\varphi = \eta\\
&L_{\tilde\omega}\tilde\varphi = \tilde\eta.
\end{aligned}\tag{3.32}$$

and φ, $\tilde\varphi$ have zero average.

Moreover, we have for all $\delta>0$:

$$\begin{aligned}
&\|\varphi\|_{\sigma-\delta}\leq C\delta^{-\nu}K_{\nu,n}\|\eta\|_\sigma\\
&\|\tilde\varphi\|_{\sigma-\delta}\leq C\delta^{-\nu}K_{\nu,n}\|\tilde\eta\|_\sigma
\end{aligned}\tag{3.33}$$

Where the C are the same constants that appear in (3.18)*,* (3.19) *and K are constants that depend (in a very explicit formula) only on the exponent in* (3.18)*,* (3.19) *and the dimension of the space.*

If we assume that η, $\tilde\eta$ are in Λ_r, $r>\nu$, we obtain:

$$\begin{aligned}
&\|\varphi\|_{\Lambda_{r-\nu}}\leq CK_{\nu,n}\|\eta\|_{\Lambda_r}\\
&\|\tilde\varphi\|_{\Lambda_{r-\nu}}\leq CK_{\nu,n}\|\tilde\eta\|_{\Lambda_r}
\end{aligned}\tag{3.34}$$

We just note that the part (3.34) is a consequence of (3.33) using the the characterization of differentiable functions by properties of the approximation by analytic functions in Lemma 3.2.

When studying analytic problems, one can be sloppy with the exponents obtained and still arrive at the same result. However, as (3.34) shows, taking

[4]Here, C depends on s even if it is independent of k. We, however do not include the s dependence in the notation to avoid clutter.

care of the exponents is crucial if we are studying finitely differentiable problems and want to obtain regularity which is close to optimal.

Exercise 3.16. Read the argument in [Rüs76b]. Do you obtain some improvement using the Hölder inequality in place of Cauchy-Schwartz?

Exercise 3.17. In the study of Lindstedt series (e.g. (2.17)) we encountered second order equations for φ given η of the form:

$$\varphi(x+\omega)+\varphi(x-\omega)-2\varphi(x)=\eta(x) \tag{3.35}$$

where φ and η are periodic and ω is a Diophantine number.

Develop a theory of the equation (3.35) along the theory of the theory developed in Lemma 3.15.

Do it either by treating it directly in Fourier series or by factoring it as two equations:

$$\begin{aligned} w(x)-w(x-\omega)&=\eta(x)\\ \varphi(x+\omega)-\varphi(x)&=w(x) \end{aligned} \tag{3.36}$$

Are there any differences between the estimates or the solvability conditions you get by the two methods?

What happens if instead of using the naive estimates presented in the text you use the estimates of [Rüs76a]?

3.5. **Geometric structures.** There are several structures that play an important role in KAM theory. In this Section, we will discuss symplectic and, more briefly, volume preserving and reversible systems (there are other geometric structures that have come to play a role in KAM theory, but we will not discuss them here).

In this Section, the emphasis will be on the geometric structures and not on the differentiability properties, so we will assume that vector fields generate flows, for which variational equations are valid, etc. (i.e., that they have some mild differentiability properties).

Here we will use Cartan calculus of differential forms rather than the old-fashioned notation. Since Cartan calculus uses only geometrically natural operations, it is conceptually simpler. This is a great advantage in mechanics, where one frequently uses changes of variables, restriction to submanifolds given by regular values of the integrals of motion, etc..

The traditional notation — in which one writes functions as functions of the coordinates, e.g., $H(p,q)$ — is perfectly adequate when the coordinates are fixed. On the other hand, when one changes coordinates, one has to decide whether $H(p',q')$ denotes the same function of new arguments or whether $H(p',q')$ is a different function of p' and q' which produces the same numerical value as the old function H produced with the old variables p and q. The ambiguity increases enormously when one needs to compute partial derivatives — a great deal of the complications in traditional books and papers on mechanics and thermodynamics arises from this.

For KAM theory these considerations are not so crucial because many of the operations one has to perform require using Fourier coefficients and the like, which forces the fixing of a certain system of coordinates. Nevertheless, we think the conceptual simplification provided by the geometric notation is worth the effort required in introducing it.

Now let us start with some important definitions.

Definition 3.18. A *symplectic structure* in a manifold is given by a 2-form ω_2 satisfying the conditions

i) ω_2 is non degenerate
ii) ω_2 is closed, i.e., $\mathrm{d}\omega_2 = 0$.

A *volume form* in a manifold of dimension n is an n-form ω_n that satisfies

i′) ω_n is non degenerate.

Naturally, an n-form ω_n in an n-dimensional manifold automatically satisfies

ii′) $\mathrm{d}\omega_n = 0$.

Much of the geometric theory goes through just under the conditions i) and ii) — or i′) and ii′). When we do not need to distinguish between the symplectic and the volume preserving cases, we will use ω to denote either ω_2 or ω_n.

Properties i) and i′) allow us to identify a vector field v with a 1- and $(n-1)$-form, respectively, by

$$\mathrm{i}_v\omega_2 := \omega_2(v,\cdot) = \gamma_1\ , \qquad \mathrm{i}_v\omega_n = \gamma_{n-1}\ . \tag{3.37}$$

We will denote the identifications (3.37) by $\mathcal{I}_{\omega_2}$ and $\mathcal{I}_{\omega_n}$, respectively.

Fundamental examples of a symplectic form ω_2 on $\mathbf{R}^k \times \mathbf{R}^k$ and a volume form ω_n on $\mathbf{R}^n$ are

$$\begin{aligned} \omega_2 &= \sum_{i=1}^{k} \mathrm{d}p_i \wedge \mathrm{d}q_i\ , \\ \omega_n &= \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_n\ . \end{aligned} \tag{3.38}$$

Remark 3.19. The name symplectic seems to have been originated as a pun on the name complex. Indeed, there is a sense in which symplectic geometry is a complexification of Riemannian geometry. This is actually quite deep and there is a wonderful new area of research using methods of complex analysis in symplectic topology.

Since these notes are focused on KAM theory, it suffices to note that in mechanics one often finds the matrix $J \equiv= \begin{pmatrix} 0 & \mathrm{Id}_d \\ -\mathrm{Id}_d & 0 \end{pmatrix}$ which satisfies $J^2 = -1$ and which, therefore, is quite analogous to multiplication by i in complex analysis.

The identification of vector fields with forms plays a very important role because it allows us to describe the vector fields whose flow preserves the structure. Denote by Φ_t a family of diffeomorphisms of the manifold generated by the time-dependent vector field v_t, i.e.,

$$\frac{d}{dt}\,\Phi_t = v_t \circ \Phi_t \ , \qquad \Phi_0 = \mathrm{Id} \ .$$

In particular, if v_t is independent of t, Φ_t is a flow: $\Phi_{t+s} = \Phi_t \circ \Phi_s$. (Again we recall that in this Section we are assuming the objects to be differentiable enough, in this case v_t to be C^1.)

Using the definition of Lie derivative, Cartan's so called "magic formula" to express the Lie derivative

$$L_X\gamma = d(\mathrm{i}_X\gamma) + \mathrm{i}_X(d\gamma), \tag{3.39}$$

the closedness of ω and the definition of $\mathcal{I}_\omega$ we obtain:

$$\frac{d}{ds}|_{s=0}\Phi^*_{t+s}\omega = \Phi^*_t L_{v_t}\omega = \Phi^*_t(\mathrm{d}\,\mathrm{i}_{v_t}\omega + \mathrm{i}_{v_t}\,\mathrm{d}\omega) = \Phi^*_t\,\mathrm{d}\,\mathcal{I}_\omega v_t \ .$$

Thus, if ω is invariant under the flow Φ_t (i.e., $\Phi^*_t\omega = \omega$), we conclude that $\mathcal{I}_\omega v_t$ is closed.

The above result is quite interesting because the Φ_t-invariance of ω seems at first sight to be a non-linear and non-local constraint for the flow Φ_t. The vector field v_t is perfectly linear and local.

Of particular importance for KAM theory are the vector fields (called *exact symplectic*, resp. *exact volume preserving*) for which $\mathcal{I}_\omega v_t$ is exact, i.e.,

$$\mathcal{I}_\omega v_t = \mathrm{d}\gamma_t$$

with γ_t a function (symplectic case) or an $(n-2)$-form (volume preserving case). Sometimes these are called *globally Hamiltonian* vector fields to indicate that one can find a Hamiltonian that generates them globally and not only locally. All the flows that preserve the symplectic or volume structure can be expressed locally as a Hamiltonian flow, but perhaps not globally. We will come back to this in more detail when we consider some extra structure of the space.

Of course, when we are considering local problems, by Poincaré's lemma, we do not need to distinguish between symplectic and exact symplectic vector fields.

In the symplectic case, for (3.38), we have that $\mathcal{I}_{\omega_2}v \equiv \mathrm{i}_v\omega_2 = -\mathrm{d}H$ reduces to the standard Hamilton's equations

$$v_{p_i} = -\frac{\partial H}{\partial q_i} \ , \qquad v_{q_i} = \frac{\partial H}{\partial p_i} \ .$$

The function H is called the *Hamiltonian* of the vector field v. Vector fields satisfying locally $\mathrm{i}_v\omega_2 = -\mathrm{d}H$ for some function H are called *locally Hamiltonian vector fields*. If the function H can be defined globally, the vector field v is called *globally Hamiltonian*.

An important consequence of the preservation of symplectic or volume form is that if a diffeomorphism f preserves the form ω and $\mathrm{i}_X\omega = -\mathrm{d}H$, we have

$$\text{(3.40)}\quad \mathrm{i}_{f_*X}\omega = \mathrm{i}_{f_*X} f_*\omega = f_*(\mathrm{i}_X\omega) = -f_*\mathrm{d}H = -\mathrm{d}f_*H = -\mathrm{d}(H \circ f^{-1})\ ,$$

so that f_*X is also a Hamiltonian flow for $H \circ f^{-1}$.

In old fashioned language, this was described as saying that "canonical transformations preserve the form of Hamilton's equations" or some similar sentence. (In old fashioned books the name canonical transformations referred to diffeomorphisms preserving the symplectic form, or sometimes to what we have referred to as exact symplectic.)

The importance of the formula (3.40) is that to make canonical changes of variables to a Hamiltonian vector field, it suffices to make changes of variables in the Hamiltonian functions. This is conceptually much simpler and computationally more efficient. As we will see, canonical perturbation theory owes its success to this remark. Note that this calculation goes through both for symplectic and volume forms. Using Cartan calculus, it is possible to develop perturbation theories for symplectic and volume preserving flows which are completely analogous.

Notice that in 2 dimensions the volume form and the symplectic structure are the same and that, when $n = 2k$,

$$\omega_n^{\wedge k} := \omega_2 \wedge \omega_2 \wedge \cdots \wedge \omega_2 \qquad (k \text{ times})$$

is a volume form.

Clearly, a flow that preserves ω_2, also preserves $\omega_n^{\wedge k}$. This fact is usually referred to in mechanics as *Liouville's theorem* and is of fundamental importance since it makes a connection of mechanics with ergodic theory. Indeed, ergodic theory was introduced in the study of the relations of this observation with statistical mechanics.

In the study of Hamiltonian flows, it is also of interest to study the form μ_E defined in the regular energy surfaces $\Sigma_E = \{H = E\}$ – assumed that dH is not degenerate so that it is an smooth manifold – by $\omega^{\wedge k} = \mu_E \wedge dH$. Since H is invariant under the flow, so is dH and μ_E is invariant.

The intermediate forms, $\omega_2 \wedge \cdots \wedge \omega_2$ (ℓ times, $\ell < k$) are also invariant. It seems that not much use has been made of them ([Poi93], Chapters XXII–XXVII is devoted to this question).

One of the first consequences of the identification between vector fields and forms (3.37) is a simple proof of the Darboux theorem. (See [MS95]. The original proof along this lines was done for the volume case in [Mos65].)

Theorem 3.20. *Given a symplectic or volume preserving form ω and a point x_0, there exists a local diffeomorphism f on a neighborhood of x_0 to $\mathbf{R}^n$ such that $f^*\omega$ is of the form in* (3.38)*.*

Note that the Darboux theorem implies that there are no local symplectic or volume invariants (so that the recent but already very rich theory of symplectic invariants and obstructions is eminently global). Moreover, an argument of [Mos65] shows that for volume preserving geometry in a compact manifold the only invariant is the total volume. This is in great contrast with Riemannian geometry where the "theorema egregium" of Gauss shows that there are local invariants for isometry [5].

Of particular importance for KAM theory will be the study of exact transformations. They can only be defined on manifolds where ω is exact, i.e., manifolds for which

$$\omega = \mathrm{d}\theta \ .$$

One important example is

$$\omega_2 = \sum_{i=1}^{k} \mathrm{d}p_i \wedge \mathrm{d}q_i \ , \qquad \theta = \sum_{i=1}^{k} p_i \, \mathrm{d}q_i$$

with $q \in \mathbf{T}^k$, $p \in \mathbf{R}^k$, so that $M = \mathbf{T}^k \times \mathbf{R}^k$.

More generally, if $M = T^*N$ is the cotangent bundle of the k-dimensional manifold N, and $\pi : T^*N \to N$ is the projection, one can define θ intrinsically as the only 1-form in T^*N with the property

$$\gamma^*\theta = \gamma$$

for all 1-forms γ on N [AM78], Pro. 3.2.11. (Here, γ is considered as a map from N to T^*N, satisfying $\pi \circ \gamma = \mathrm{Id}$, so that γ^* maps the 1-forms in T^*N into 1-forms in N.) One can easily check that this is equivalent to the standard prescription of taking a local trivialization of T^*N with coordinates (p, q) and then setting $\theta = \sum_{i=1}^{k} p_i \, \mathrm{d}q_i$. One needs to check that the definition is independent of the system of coordinates chosen.

For volume preserving maps, our main example will be

$$M = \mathbf{T}^{n-1} \times \mathbf{R} \ , \qquad \theta = p \, \mathrm{d}q_1 \wedge \cdots \wedge \mathrm{d}q_{n-1} \ ,$$

where $(q_1, \ldots, q_{n-1}) \in \mathbf{T}^{n-1}$, $p \in \mathbf{R}$. We note that given ω, θ is determined up to a closed form.

When ω is exact (i.e., $\omega = \mathrm{d}\theta$) we use that $f^*\mathrm{d} = \mathrm{d}f^*$ to obtain that for any diffeomorphism f preserving ω

$$\mathrm{d}(f^*\theta - \theta) = 0 \ .$$

We say that the ω-preserving diffeomorphism f is *exact* when

$$f^*\theta - \theta = \mathrm{d}S \ . \tag{3.41}$$

Once we fix θ, S is defined up to a form of zero exterior derivative; in the symplectic case, this means up to a constant.

Conversely, it turns out that the function S determines to a large extent the diffeomorphism. If we know S in the whole manifold and and the diffeomorphism restricted to a *Lagrangian* submanifold, it is possible to reconstruct the diffeomorphism in the whole manifold. (See [Har99].)

[5] Note that the condition $d\omega = 0$ is some sort of curvature condition, so that perhaps it is fairer to compare symplectic geometry to a sort of Riemannian geometry of flat manifolds.

In exact symplectic (or volume) manifolds ($\omega = d\theta$), there is a very close relationship between exact families and hamiltonian flows. Families of exact diffeomorphisms are generated by a Hamiltonian flow and vice versa.

To show the first statement, note that if f_t is an smooth family, we have that ${f_t}^*\theta - \theta = \mathrm{d}S_t$ and we can choose S_t smooth in t. If we take derivatives of this relation with respect to t and introduce the vector field $\mathcal{F}_t$ generating f_t by $\frac{d}{dt}f_t = \mathcal{F}_t \circ f_t$, we have

$$ {f_t}^* L_{\mathcal{F}_t}\theta = \mathrm{d}\dot{S}_t \ , \tag{3.42} $$

where L denotes the Lie derivative and $\dot{S}_t = \frac{d}{dt}S_t$. Using Cartan's formula for the Lie derivative, we have

$$ {f_t}^*[\mathrm{d}\mathrm{i}_{\mathcal{F}_t}\theta + \mathrm{i}_{\mathcal{F}_t}d\theta] = \mathrm{d}\dot{S}_t \ . \tag{3.43} $$

Therefore,

$$ \mathrm{i}_{\mathcal{F}_t}\omega = \mathrm{d}\,[({f_t}^*)^{-1}\dot{S}_t - \mathrm{i}_{\mathcal{F}_t}\theta] \ . \tag{3.44} $$

Hence, we conclude that a family of exact maps is generated by a Hamiltonian flow of Hamiltonian given by the formula:

$$ H_t = \mathrm{i}_{\mathcal{F}_t}\theta - ({f_t}^*)^{-1}\dot{S}_t \ . \tag{3.45} $$

In the exact symplectic case, the last formula reads

$$ H_t = \mathrm{i}_{\mathcal{F}_t}\theta - \dot{S}_t \circ f_t^{-1} \ . $$

The converse is proved by a very similar calculation. Note that if we are given H_t and an exact ω-preserving diffeomorphism f_0 with an initial primitive S_0, and f_t is generated by the Hamiltonian flow of H_t, then the deformation f_t is also exact, and the primitive S_t satisfies the differential equation

$$ \dot{S}_t = f_t^*(\mathrm{i}_{\mathcal{F}_t}\theta - H_t) \ . \tag{3.46} $$

Notice that the obstruction for a symplectic or volume preserving diffeomorphism f to be exact is just the cohomology class with real coefficients of $f_*\theta - \theta$. For example, in the map we considered before,

$$ f(q,p) = (q, p + a) $$

with a a constant n-vector, we have

$$ \begin{aligned} f_*\theta_1 - \theta_1 &= \sum_{i=1}^{k} a_i\,\mathrm{d}q_i \ , \\ f_*\theta_{n-1} - \theta_{n-1} &= a\,\mathrm{d}q_1 \wedge \cdots \wedge \mathrm{d}q_{n-1} \ . \end{aligned} $$

In this case, one can see that the cohomology obstruction vanishes if and only if the flux that we considered in Definition 2.1 vanishes.

If f is a diffeomorphism close to the identity in the C^r ($r = 1, 2, \ldots, \infty$) topology, it is not hard to show for M as in the example that there is an exact family of vector fields interpolating with the identity.

This can also be proved for analytic functions. however, it is far from trivial. See [KP94].

The reason why exactness plays an important role in KAM theory can be understood from the simple example (already mentioned) in $\mathbf{R} \times \mathbf{T}$,

$$f(p,q) = (p+\varepsilon, q+p) \ , \qquad f^n(p,q) = (p+n\varepsilon, q+np) \ ,$$

which does not admit any quasiperiodic orbits for $\varepsilon \neq 0$ (all the orbits escape to infinity).

A consequence of great importance for us later is that if we choose a function, resp. an $(n-1)$-form, γ and form a vector field by

$$v = \mathcal{I}_\omega \, \mathrm{d}\gamma \ ,$$

then the time one map of the vector field, Φ_1, is exact. This gives a convenient way to generate transformations close to the identity.

Since commutators of vector fields are an ingredient of the variational equations, it is quite interesting to study how commutators interact with the geometric structures (volume forms and symplectic).

Recall that the commutator of two vector fields can be considered as the commutator of the vector fields considered as differential operators. That is, the commutator of C^1 vector fields is defined as

$$[X,Y] = XY - YX, \tag{3.47}$$

when we consider the vector fields as first order differential operators in a manifold i without boundary. (It is somewhat surprising, but of course true, that the commutator is first order operator, the R.H.S. of (3.47) looks like a second order operator!) The commutator can also be defined as

$$[X,Y] = \lim_{t\to 0} t^{-2}(Y_{-t} \circ X_{-t} \circ Y_t \circ X_t - \mathrm{Id})$$

where X_t denotes the flow generated by X and similarly for Y. We have also taken the usual liberty of employing additive notation rather than a more geometric one to denote comparisons.

The following well known result relates the commutators to geometry. We have followed the presentation of [BdlLW96].

Lemma 3.21. *Let ω be a non-degenerate closed form as before.*

(i) *If X, Y are locally Hamiltonian vector fields, then $[X,Y]$ is a globally Hamiltonian vector field with Hamiltonian $\mathrm{i}_Y(\mathrm{i}_X\omega) = \omega(X,Y)$.*
(ii) *If X has H as a Hamiltonian and Y is locally Hamiltonian, then $-L_Y H$ is a Hamiltonian for $[X,Y]$.*
(iii) *If Y has F as a Hamiltonian and X is locally Hamiltonian, then $L_X F$ is a Hamiltonian of $[X,Y]$.*

Proof. First recall the identities $L_X \mathrm{d}\alpha = \mathrm{d}L_X\alpha$ and

$$\mathrm{i}_{[X,Y]}\alpha = L_X \mathrm{i}_Y \alpha - \mathrm{i}_Y L_X \alpha$$

which are valid for each m-form α and vector fields X and Y. Also, observe that a locally Hamiltonian vector field X satisfies

$$L_X\omega = 0$$

which follows easily from Cartan's "magic formula" (3.39).

To prove (i), compute:

$$\begin{aligned} \mathrm{i}_{[X,Y]}\omega &= L_X\mathrm{i}_Y\omega - \mathrm{i}_Y L_X\omega = L_X\mathrm{i}_Y\omega \\ &= \mathrm{i}_X\mathrm{d}\,\mathrm{i}_Y\omega + \mathrm{d}\,\mathrm{i}_X\mathrm{i}_Y\omega = \mathrm{d}\,\mathrm{i}_X\mathrm{i}_Y\omega \\ &= \mathrm{d}(\omega(Y,X)) = -\mathrm{d}(\omega(X,Y))\ . \end{aligned}$$

For (ii), we know that $\mathrm{i}_X\omega = -\mathrm{d}H$ and use (i):

$$\begin{aligned} -\mathrm{d}(-L_Y H) &= L_Y(\mathrm{d}H) = -L_Y\mathrm{i}_X\omega = -\mathrm{i}_Y\mathrm{d}\,\mathrm{i}_X\omega - \mathrm{d}\,\mathrm{i}_Y\mathrm{i}_X\omega \\ &= \mathrm{i}_Y\mathrm{d}(\mathrm{d}H) - \mathrm{d}(\omega(X,Y)) = \mathrm{i}_{[X,Y]}\omega\ . \end{aligned}$$

The proof of (iii) is analogous to that of (ii): from $\mathrm{i}_Y\omega = -\mathrm{d}F$ we obtain

$$-\mathrm{d}(L_X F) = -L_X(\mathrm{d}F) = L_X\mathrm{i}_Y\omega = \mathrm{i}_{[X,Y]}\omega\ .$$

□

Let X and Y be Hamiltonian vector fields with Hamiltonians H and F, respectively (i.e., $\mathrm{i}_X\omega = -\mathrm{d}H$, $\mathrm{i}_Y\omega = -\mathrm{d}F$). The *Poisson bracket* of H and F is defined as

$$\{H,F\} := -L_X F\ , \tag{3.48}$$

or, equivalently, as

$$\{H,F\} = -\mathrm{d}H(X) = (\mathrm{i}_Y\omega)(X) = \omega(Y,X)\ .$$

The antisymmetry of ω yields

$$\{H,F\} = -\{F,H\} \tag{3.49}$$

as well as the formula $\{H,F\} = L_Y H$.

In coordinates,

$$\{H,F\} = \sum_{i=1}^{k}\left(\frac{\partial H}{\partial q_i}\frac{\partial F}{\partial p_i} - \frac{\partial H}{\partial p_i}\frac{\partial F}{\partial q_i}\right)\ . \tag{3.50}$$

Using (3.50), one can easily check that the Poisson bracket satisfies the Jacobi identity,

$$\{H,\{F,G\}\} + \{F,\{G,H\}\} + \{G,\{H,F\}\} = 0\ ,$$

which, together with the linearity and the antisymmetry of the Poisson bracket, implies that the functions on the phase space of a dynamical systems with the Poisson bracket are a Lie algebra. Moreover, the Poisson bracket is a derivation of this Lie algebra.

This means:

$$\{H,\{F,G\}\} = \{H,F\}G + F\{H,G\}. \tag{3.51}$$

The property (ii) (or (iii)) of Lemma 3.21 implies that the Hamiltonian vector field corresponding to $\{H, F\}$ is equal to $-[X, Y]$:

$$\mathrm{i}_{[X,Y]} = \mathrm{d}\{H, F\}\ .$$

This means that the map from the functions on the phase space to their Hamiltonian vector fields (i.e., $H \mapsto X$ such that $\mathrm{i}_X\omega = -\mathrm{d}H$) is a morphism of Lie algebras (the Lie-algebraic operations being respectively the Poisson bracket and the commutator of vector fields).

Note that $L_X F$ means the Lie derivative of F along the flow of X and, similarly, $L_Y H$ is the Lie derivative of H along the flow of Y.

By the identities above,

$$L_X F = -L_Y H,$$

which indicates that the derivative of a Hamiltonian form along the Hamiltonian flow of another Hamiltonian form is related by a sign change to the situations when the roles are reversed. This is a somewhat surprising property of Hamiltonian systems.

One way to look at the above calculations is to realize that the exact transformations are a group and that the vector fields $\mathcal{I}_\omega\, \mathrm{d}\gamma$ are a Lie algebra. (In the old fashioned language, the vector fields of the form $\mathcal{I}_\omega\, \mathrm{d}\gamma$ were called "infinitesimal transformations" or, given that "infinitesimal" is a somewhat dirty word in some circles, "transformations close to the identity.")

Unfortunately, even if this point of view is heuristically correct, it is not without problems. First of all, composition of transformations is not a differentiable operation in almost any precise sense. Indeed, note that $f \circ (g + \Delta) - f \circ g \approx f' \circ g\Delta$, so that the derivative of composition should be a multiplication by $f' \circ g$. Hence, if we consider composition in a space as C^r, Λ_δ, etc., then $f' \circ g$ may not belong to this space; if we consider $C^{r+\alpha}$ spaces, then the composition is not even continuous!

More importantly, the exponential of the Lie algebra does not cover an open neighborhood of the identity. That is, in any arbitrarily small neighborhood of the identity in Λ_δ, there exist exact maps that cannot be written as time one maps of a differentiable vector field.

Another important geometric structure that plays a role in KAM theory is the so called *reversible* systems. They appear in any applied problem in which time can be "run backwards" (i.e., if $\gamma(t)$ is a trajectory, then $\gamma(T - t)$ also is). This happens in mechanical problems without friction or in electric circuits without resistors and in other problems. Examples of reversible systems also appear in finite dimensional truncations of fluid mechanics problems when there is no viscosity. In general, physical problems in which there is no dissipation are often reversible. When the systems are not mechanical, there is no reason why we should have also a symplectic structure. In particular, in the example of circuits, it is possible to find interesting examples with odd dimensions.

A map is said to be *reversible* when there exists an involution R (that is, $R^2 = \mathrm{Id}$) for which $A^{-1} = R^{-1}AR$, i.e., A is conjugate by R to its own inverse.

Since $R^{-1} = R$, the above condition can be expressed as $A^{-1} = RAR = RAR^{-1}$. Note also that reversibility implies that $S = AR$ is an involution. Hence, A is a product of two involutions, $A = SR$. One can also check that the product of two involutions is reversible with respect to either of them, so that one can just as well define a reversible map as the product of two involutions, even if this obscures the physical interpretation and the origin of the name.

Sometimes one does not require that R is an involution. These systems are sometimes called *weakly reversible*. The KAM theory only needs weak reversibility. (Actually, in many occasions that KAM theory applies, we can use KAM theory to show that the systems are actually reversible.)

For flows, the definition is similar: the flow f_t is *reversible* if there exists an involution R such that $R^{-1}f_tR = f_t^{-1} = f_{-t}$. Taking derivatives, we obtain the reversibility condition in terms of the vector field $\mathcal{F}_t$ generating the flow: $R_*\mathcal{F}_t = -\mathcal{F}_t$.

One very important example of a reversible system is a mechanical system without friction whose forces depend only on the position of the particles. If we reverse the velocities and keep the positions the same, the system runs backwards. Hence we can take $R(x, v) = (x, -v)$. Clearly, R is an involution. Reversible mappings have recently received a great deal of interest in the context of statistical mechanics since many slightly dissipative models are reversible. This reversibility leads to very amusing consequences such as pairing rules for Lyapunov exponents. See [BCP98] for some applications to Statistical Mechanics and references.

Good surveys of reversible systems in general are [Sev86] and [AS86] and recent developments in the KAM theory for reversible systems are covered in [Sev98].

3.6. **Canonical perturbation theory.** The goal of perturbation theory is to understand the dynamics of a "perturbed" system which is close to another well understood system. Usually these well understood systems are chosen among "integrable" systems but this is not necessarily the case. As we will see in later proofs of KAM theorem, sometimes we want to take as unperturbed systems systems of a particular kind that have an interesting feature. In the case of integrable systems, the feature of interest for the study is quasi-periodic orbits.

The most naive approach to perturbation theory is to develop the solutions in powers of the perturbation parameter. That is, if we have a vector field

$$X_\varepsilon = X_0 + \varepsilon X_1 + \varepsilon^2 X_2 + \cdots \ ,$$

we try to find solutions of

$$\dot{x}_\varepsilon = X_\varepsilon(x_\varepsilon) \ , \qquad x_\varepsilon(0) = a(\varepsilon)$$

by setting

$$x_\varepsilon(t) = x_0(t) + \varepsilon x_1(t) + \cdots \ ,$$
$$a(\varepsilon) = a_0 + \varepsilon a_1 + \varepsilon^2 a_2 + \cdots \ , \tag{3.52}$$

substituting in the equation and solving.

That is,

$$
\begin{aligned}
\dot{x}_0 &= X_0(x_0) &, \quad & x_0(0) = a_0 \\
\dot{x}_1 &= X_1(x_0) + DX_0(x_0)x_1 &, \quad & x_1(0) = a_1 \\
\dot{x}_2 &= X_2(x_0) + DX_1(x_0)x_1 &, \quad & x_2(0) = a_2 \\
&\quad + DX_0(x_0)x_2 + \tfrac{1}{2}D^2X_0(x_0)x_1^{\otimes 2} & & \\
&\quad \cdots \quad . & &
\end{aligned} \tag{3.53}
$$

Provided that X_ε is analytic in ε and its argument, this series was shown to converge by Cauchy (but before that, it was used regularly by Newton).

Note that all the equations in the hierarchy have the form

$$\dot{x}_n - DX_0(x_0)x_n = R_n \ , \qquad x_n(0) = a_n \ ,$$

where R_n is a polynomial expression involving only terms $x_0 \dots x_{n-1}$ and (known!) derivatives of X_i.

In spite of its ancient pedigree and the theorems of convergence this method has shortcomings.

It is an easy exercise that taking the second order problem

$$\ddot{x}_\varepsilon = -(1 + 2\varepsilon + \varepsilon^2)x_\varepsilon \ , \qquad x_\varepsilon(0) = 1 \ , \ \dot{x}_\varepsilon(0) = 0 \ ,$$

the solution is

$$
\begin{aligned}
x_0(t) &= \cos t \\
x_1(t) &= -t \sin t \\
x_2(t) &= -t^2 \cos t \\
&\cdots \quad .
\end{aligned}
$$

This series indeed converges to the right solution $x_\varepsilon(t) = \cos((1+\varepsilon)t)$ as well as one can expect (it is entire in ε and in t) but, if one truncates, one can see that the approximate solution thus obtained blows up. Indeed, the more terms one takes, the more severe the blow up is. On the other hand, the true solution remains bounded for all times.

Hence, these series are unable to predict long term behavior, even in those extremely favorable examples where the function is linear and the solutions are entire. Of course, this phenomenon only becomes worse if one considers other more complicated non-linear problems.

This phenomenon caused consternation when the phenomena above appeared in the study of the solar system and the instability of the solar system was confirmed to all orders in perturbation theory. The terms with powers in t became dominant for t of order of centuries, which gave then the name “secular” terms (in Latin “saeculum” means – among other things – century).

A more careful examination of the convergence proof and the quantitative estimates that lead to it, shows that one cannot trust this perturbation theory to order n except when $\varepsilon^n t \ll 1$. For non-entire perturbations, one should not use this naive perturbation method except when $\varepsilon t \ll 1$. (Fortunately, we will be using some more effective methods that can give information on perturbations over longer time scales.)

The Lindstedt series we have seen in Section 2.1 originated with the goal of obtaining a perturbation series that produced series which were always periodic or quasi-periodic (that is, free of secular terms).

A much more effective method to ascertain the long term behavior of systems is the following:

We try to find transformations g_ε in such a way that

$$g_{\varepsilon *} X_\varepsilon = X_0 \ . \tag{3.54}$$

This method is not restricted to Hamiltonian systems. Indeed, the very influential book [BM61] develops many applications to non-Hamiltonian systems (one can also find there Lindstedt series for dissipative systems).

This method is, however, very well suited for Hamiltonian systems because it is very easy to keep track of families of transformations of Hamiltonian systems and vector fields.

In the case where g_ε are canonical transformations and X_ε are Hamiltonian vector fields (i.e., $\mathrm{i}_{X_\varepsilon}\omega = \mathrm{d}H_\varepsilon$), as we saw in (3.40), the equation (3.54) reduces to

$$H_\varepsilon \circ g_\varepsilon = H_0 \ . \tag{3.55}$$

One should emphasize that in contrast with the more elementary "secular method", the validity of this method is not limited by the length of the orbit but rather by whether the orbit leaves the region where the transformation g_ε is defined.

In some cases, especially when there is some contraction (of course, this never happens for Hamiltonian systems), one can use the perturbation theory itself to show that this region is never left by the trajectories.

Note that if (3.55) is solved, then we have

$$g_\varepsilon^{-1} \circ \Phi_t^\varepsilon \circ g_\varepsilon = \Phi_t^0 \ ,$$

where Φ_t^ε and Φ_t^0 denote the flows of H^ε and H^0, respectively.

To solve (3.55), it is of paramount importance to parameterize the families g_ε in such a way that (3.55) can be solved order by order.

One possibility followed in old fashioned books (but not very practical in many applications) is to parameterize g_ε by their generating functions (see Section 3.7). One shortcoming of generating functions is that one needs to assume existence of a system of global coordinates which are mixed variables (or work in patches). Another shortcoming of this method is that the rule of composition is awkward and it involves solving implicit equations (see Section 3.7).

Another alternative, which is more geometrical is that of the Lie series. The basic idea is that we try to consider transformations as time one maps of Hamiltonian vector fields. Some more detailed tutorials on Lie series are [MH92],[Mey91],[DF76]. Some reviews of canonical perturbation theory from the point of view of Physicists including a variety of applications are [Car81], [Omo86].

It is customary to write the time one maps of a vector field $\mathcal{L}$ as $\exp(\mathcal{L})$. This notation is motivated by the remark that the space vector fields can be considered as the Lie algebra of the space of diffeomorphisms. Also, if we identify g with the operator

$$U_g : L^2(M) \to L^2(M) \;\; : g \mapsto U_g\varphi = \varphi \circ g \ ,$$

then $\exp(\mathcal{L}) = U_g$ in the usual sense of operator theory when $\mathcal{L}$ is a complete flow preserving volume.

This notation is very suggestive and one would also like to use tools of Lie group theory such as Baker-Cambell-Hausdorff formula

$$\begin{aligned}\exp(\varepsilon\mathcal{L})\exp(\varepsilon\mathcal{L}^1) = \exp\Big(&\varepsilon(\mathcal{L}+\mathcal{L}^1) + \varepsilon^2\frac{1}{2}[\mathcal{L},\mathcal{L}^1] + \varepsilon^3\mathcal{T}_3 \\ &+ \cdots + \varepsilon^n\mathcal{T}_n + \cdots\Big) \ ,\end{aligned} \tag{3.56}$$

where [,] denotes the commutator and $\mathcal{T}_n$ is a sum of iterated commutators of $\mathcal{L}, \mathcal{L}^1$.

Even if the sums in (3.56) cannot be considered as convergent, the formula can be justified in an appropriate weak sense ([dlLMM86]), and it is true that when applying the formula up to order n, we have, for sufficiently differentiable vector fields,

$$\begin{aligned}\left\| \left[\exp(\varepsilon\mathcal{L})\exp(\varepsilon\mathcal{L}^1) - \exp\left(\sum_{n=1}^{N}\varepsilon^N\mathcal{T}_n\right)\right]\varphi\right\|_{C^r} \le \\ \le \varepsilon^{N+1}C_{\mathcal{L},\mathcal{L}^1,N}\|\varphi\|_{C^{N+r+2}} \ .\end{aligned} \tag{3.57}$$

In spite of (3.57), it is not true that $\exp\left(\sum_{n=1}^{N}\varepsilon^n\mathcal{T}_n\right)\varphi$ converges as $N\to\infty$ even for an analytic φ (a sketch of a proof will be given later). It is, however, not difficult to obtain bounds $C_{\mathcal{L},\mathcal{L}^1,N} \le (N!)^k$ for some $k>0$, so that (3.57) can be used quite quantitatively.

In connection with (3.56) it is interesting to note that the commutator of two locally Hamiltonian vector fields is globally Hamiltonian (see Proposition 3.21). Hence, even if $\mathcal{L}$, $\mathcal{L}^1$ are only locally Hamiltonian, all the $\mathcal{T}_n$'s are globally Hamiltonian and can, therefore, be described by the Hamiltonian function.

There are several variants of the method of Lie transforms that have been considered in the literature depending on how we write our candidate map in terms of exponentials (time-one maps) of Hamiltonian vector fields. In order of historical appearance some of the methods proposed in the literature are:

$$g_\varepsilon = \exp(\varepsilon\mathcal{L}_1 + \varepsilon^2\mathcal{L}_2 + \cdots + \varepsilon^n\mathcal{L}_n + \cdots) \ , \tag{3.58}$$

$$g_\varepsilon = \cdots \exp(\varepsilon^n\mathcal{L}_n)\cdots\exp(\varepsilon^2\mathcal{L}_2)\exp(\varepsilon\mathcal{L}_1) \ , \tag{3.59}$$

$$g_\varepsilon = \cdots \exp\left(\sum_{i=2^n}^{2^{n+1}-1} \varepsilon^i\mathcal{L}_i\right)\cdots\exp(\varepsilon^3\mathcal{L}_3 + \varepsilon^2\mathcal{L}_2)\exp(\mathcal{L}_1) \ . \tag{3.60}$$

(See [Dep70],[DF76],[dlLMM86] respectively.)

The recursive equation for the perturbation expansions can be computed rather straightforwardly if we use with abandon — we can if we interpret the formulas in the asymptotic sense – the formulas $\exp\mathcal{L} = \sum_{i=0}^\infty \frac{1}{n!}\mathcal{L}^n$, think of the $\mathcal{L}^n$ as differential operators and rearrange the expressions according to the rules of of non-commutative algebra.

For example, in (3.58) we obtain:

$$\begin{aligned}
&\exp(\varepsilon\mathcal{L}_1\cdots + \varepsilon^n\mathcal{L}_n)H_\varepsilon = H_0\\
&\mathcal{L}_1 H_0 + H_1 = 0\\
&\left(\tfrac12\mathcal{L}_1^2 + \mathcal{L}_2\right)H_0 + \mathcal{L}_1H_1 + H_2 = 0\\
&\left[\tfrac16\mathcal{L}_1^3 + \tfrac12(\mathcal{L}_1\mathcal{L}_2 + \mathcal{L}_2\mathcal{L}_1) + \mathcal{L}_3\right]H_0\\
&\qquad + \left(\tfrac12\mathcal{L}_1^2 + \mathcal{L}_2\right)H_1 + \mathcal{L}_1H_2 + H_3 = 0 \ .
\end{aligned}$$

A point that we would like to emphasize is that the equation that we obtain in the three schemes (3.58), (3.59), (3.60) for Lie series is always

$$\mathcal{L}_nH_0 + H_n = R_n \ , \tag{3.61}$$

where R_n is an expression that depends only on previously computed terms.

Using (3.49), we can transform (3.61) into

$$-\mathcal{H}_0L_n + H_n = R_n \ . \tag{3.62}$$

Note that, if we have a theory for the solutions of equations of the form (3.62), we can proceed along the perturbation schemes above.

Note that if we take

$$H_0(p,q) = \omega\cdot p \ ,$$

then (3.62) reduces to the equation (3.27) that we have studied (under Diophantine assumptions on ω) in Section 3.4. Both the data and the unknown in (3.62) have an extra variable, but since it enters as a parameter, we can discuss the regularity of the equation in terms of the theory that we have developed.

Perhaps more importantly, we note that if we have a good theory of approximate solutions of (3.62) we can solve the hierarchies of equations approximately. This is important in practice as well as in some proofs on KAM theorem.

We also note that an integrable system $H_0(p)$ can be written using the Taylor expansion

$$H_0(p) = H_0(0) + \omega \cdot p + O(p^2)$$

Hence, we can solve very approximately (3.62) in a sufficiently small neighborhood of $\{p = 0\}$. This is what is actually used in KAM theory.

These algorithms are also practical tools that can and have been implemented numerically. The next two remarks are concerned with some issues about numerical implementations.

In [dlLMM86] one can find an appendix where it is shown that the theories based in the three schemes above (and in others) are equivalent in the sense that they give results which are equivalent in the sense of asymptotic series.

Remark 3.22. We emphasize that although all the schemes (3.58), (3.59), (3.60) are formally equivalent in the sense that they require solving the same equations, they are not at all equivalent from the point of view of efficiency and stability of the numerical implementation or from the point of view of detailed estimates or even convergence.

As we pointed out, the exponential of vector fields does not cover any neighborhood of the origin in the group of diffeomorphisms so that (3.58) does not provide with a good parameterization of a neighborhood of the identity and, perhaps relatedly, it is known to be outperformed in stability etc. by (3.59) [DF76].

The method (3.60) [dlLMM86] is actually convergent in many cases. Indeed, the KAM theorem asserts it does converge in certain cases as we will see. For example, it is convergent for the perturbation series that are based in Kolmogorov's method' that will be discussed in Section 6.1.

The only numerical implementations of (3.60) that I know of are some tentative ones carried out by A. Delshams and the author, but it seems that the scheme (3.60) has a very good chance to be very efficient and stable. Indeed, it seems to be the only method for which it is possible to establish convergence.

Remark 3.23. Sometimes in the numerical solution of the equations (3.58), (3.59), (3.60) it is sometimes advantageous – both from the point of view of speed and of reliability – not to proceed order by order but rather to take groups of orders $2^n - -2^{n+1} - 1$.

This is tantamount to solving the equations by a Newton method in the space of families. It has the disadvantage over the order by order algorithm that at every stage one has to solve a different equation. This inconvenience is sometimes offset by the advantage that one linear equation allows one to study many orders and because the equations that need to be solved may be more stable than those of other methods.

These quadratic algorithms can be used for all the three methods described above. Nevertheless, they are somewhat easier to implement in (3.60) which has some quadratic convergence already in place.

We emphasize that all the methods can be studied either order by order or quadratically.

I think that it would be quite important to have a better theory of these algorithms.

One lemma that we will be using later is that it is possible to approximate the action of the Lie transform on functions by just the first term in the series of the exponential.

Lemma 3.24. *Let H, G be functions on $\mathbf{T}^n \times \mathbf{R}^n$ endowed with the canonical symplectic structure. We use the notation of* (3.12) *for the analytic norms of functions.*

Assume that:

i) $\|H\|_\sigma$ *is finite.*

ii) *For a constant C which depends only on the dimension, we have for $\delta > 0$*

$$\delta^2 > C\|G\|_\sigma \ . \tag{3.63}$$

Then, for another constant $\tilde{C}$ depending only on the dimension, we have:

$$\|H \circ \exp \mathcal{L}_G - H - \{H, G\}\|_{\sigma-\delta} \leq \tilde{C}\delta^{-4}\|G\|_\sigma^2 \|H\|_\sigma \tag{3.64}$$

Proof. By Cauchy estimates, (3.13), we have:

$$\|\nabla G\|_{\sigma-\delta/2} \leq \hat{C}\delta^{-1}\|G\|_\sigma. \tag{3.65}$$

with $\hat{C}$ a constant that depends only on the dimension.

The constant in (3.63) is chosen so that the R.H.S. of (3.65) is smaller than $\delta/2$.

Therefore, all the trajectories of the Hamiltonian flow generated by G which start in the region

$$\mathcal{D}_{\sigma-\delta} \equiv \{|I| \leq e^{2\pi(\sigma-\delta)}, |\operatorname{Im}(\phi)| \leq \sigma - \delta\}$$

do not leave the region $\mathcal{D}_{\sigma-\delta/2}$ for a time smaller than one (note that they are moving at an speed that does not allow them to transverse the region separating the domains in a unit of time). Hence,

$$\exp(\mathcal{L}_G)(\mathcal{D}_{\sigma-\delta}) \subset \mathcal{D}_{\sigma-\delta/2} \ .$$

In particular, we can define the composition $H \circ \exp(\mathcal{L}_G)$ in $\mathcal{D}_{\sigma-\delta}$.

For any point (I, ϕ), we can estimate the difference along a trajectory by using the Taylor theorem with remainder along a trajectory. It suffices to estimate the second derivative of H and the square of the displacement. The second derivative of H can be estimated by Cauchy estimates (3.13) $\|\nabla^2 H\|_{\sigma-\delta/2} \leq \tilde{C}\delta^{-2}\|H\|_\sigma$.

The displacement can be estimated by $\|\nabla G\|_{\sigma-\delta/2}$, which by Cauchy estimates (3.13) can be estimated by $\tilde{C}\delta^{-1}\|G\|_\sigma$.

Putting these two estimates together, obtains the desired result.

□

Remark 3.25. Analogues of Lemma 3.24 are true in any analytic symplectic manifold. One just needs to define appropriately norms of analytic functions, Cauchy inequalities, etc. In the versions of KAM theory that we will cover in this tutorial, the version we have stated is enough, but the reader is encouraged to formulate and prove the more general versions.

It is also possible to develop a canonical perturbation theory for maps. Again, the main idea is to change variables so that the system becomes close to the system which is "well understood".

The perturbative equation in this case becomes

$$g_\varepsilon^{-1} \circ f_\varepsilon \circ g_\varepsilon = f_0 \ . \tag{3.66}$$

We should think of those equations as equations for g_ε given f_ε.

These equations have been dealt with traditionally by parameterizing f_ε using the generating functions method, and similarly for the g_ε.

A more geometric method to use in perturbation theory is the method of deformations which was introduced in singularity theory. (In the book [MH92], one can also find this method introduced in the Lie transform method.) It seems particularly well suited to discuss conjugacy equations of a geometric nature. (See [dlLMM86, BdlLW96] for some global geometric applications.) We write

$$\frac{d}{d\varepsilon} f_\varepsilon = \mathcal{F}_\varepsilon \circ f_\varepsilon \ , \qquad \mathcal{F}_\varepsilon = \mathcal{I}_\omega(\mathrm{d}F_\varepsilon) \ .$$

We refer to f_ε as a family, $\mathcal{F}_\varepsilon$ as the generator and to F_ε as the Hamiltonian and adopt the typographical convention of using the same letter to denote the objects associated with the same family but using lowercase to denote the family, calligraphic font to denote the generator and capital to denote the Hamiltonian.

We note that, under the assumption that $\mathcal{F}_\varepsilon$ is C^1, given the generator and the initial point f_0 of the family, we can reconstruct f_ε in a unique way. Hence, given $F_\varepsilon \subset C^2$, and f_0 we can reconstruct f_ε.

If we express equation (3.66) in terms of the generators, it becomes

$$-\mathcal{G}_\varepsilon + \mathcal{F}_\varepsilon + f_{\varepsilon *}\mathcal{G}_\varepsilon = 0 \ . \tag{3.67}$$

Expressed in terms of Hamiltonians, it reads

$$-G_\varepsilon + F_\varepsilon + f_{\varepsilon *}G_\varepsilon = 0 \ . \tag{3.68}$$

(In the Hamiltonian case, we recall $f_{\varepsilon *}G_\varepsilon = G_\varepsilon \circ f_\varepsilon$.)

There are several advantages in expressing equation (3.66) in terms of the generators and the Hamiltonians:

- The equations in terms of the generators are linear. This is natural if we think that the vector fields are infinitesimal quantities which can, therefore, enter only linearly.

- The geometric structure — not only symplectic, but also volume preserving and contact (which we have not and will not discuss in these lectures) are taken care without any extra constraint.
- These equations are geometrically natural and can be formulated globally.

The proof that (3.67) and (3.68) are equivalent to (3.66) follows easily from the observation that

$$
\begin{aligned}
k_\varepsilon &= f_\varepsilon \circ g_\varepsilon \\
\Longleftrightarrow \mathcal{K}_\varepsilon &= \mathcal{F}_\varepsilon + f_{\varepsilon *}\mathcal{G}_\varepsilon \quad ; \quad k_0 = f_0 \circ g_0 \\
\Longleftrightarrow K_\varepsilon &= F_\varepsilon + f_{\varepsilon *}G_\varepsilon \quad ; \quad k_0 = f_0 \circ g_0 \ .
\end{aligned}
\tag{3.69}
$$

Even if the equations (3.68) is linear in the Hamiltonian F_ε, we should keep in mind that f_ε depends on F_ε through the very non-linear process of solving the corresponding ODE.

Nevertheless, one can approximate (3.68) by

$$F_\varepsilon - G_\varepsilon + f_{0*}G_\varepsilon = 0 \ . \tag{3.70}$$

When $f_0(I,\phi) = (I, \phi+\omega)$, this equation – for a fixed I – has the form of (3.27) the difference equations which were studied in Section 3.4. Since I can be considered as just a parameter in the data for the equation, we can use the regularity theory derived for (3.27).

If G_ε is a solution of (3.70), we note that

$$F_\varepsilon - G_\varepsilon + f_{\varepsilon *}G_\varepsilon = (f_{\varepsilon *} - f_{0*})G_\varepsilon \ . \tag{3.71}$$

The intuition is that if F_ε is small, we can think that G_ε (obtained by solving a linear equation with F_ε as R.H.S.) is small and that $f_{\varepsilon *} - f_0$ (obtained by solving a differential equation which involves derivatives of F_ε) is also small. Hence, the term in the R.H.S. of (3.71) is "quadratically" small.

Using the estimates in Lemma 3.15 and mean value theorem etc., we can prove the estimate in the analytic spaces

$$\|(f_{\varepsilon *} - f_{0*})G_\varepsilon\|_{\sigma-\delta} \le C\delta^{-2\nu-4}\|F_\varepsilon\|_\sigma \ .$$

Similarly, for the finitely differentiable case,

$$\|(f_{\varepsilon *} - f_{0*})G_\varepsilon\|_{\Lambda^r} \le C\|F_\varepsilon\|^2_{\Lambda^{r+\nu+4}} \ .$$

Note also that if we write

$$F_\varepsilon = \varepsilon F_1 + \varepsilon^2 F_2 + \cdots$$

and try to find

$$G_\varepsilon = \varepsilon G_1 + \varepsilon^2 G_2 + \cdots \ ,$$

then (3.68) can be turned into a hierarchy of equations for the G_n's. All the equations are of the form

$$G_n - f_{0*}G_n + F_n = R_n \ ,$$

where R_n is an expression involving previously computed terms.

Remark 3.26. For later developments, it is important to note that both (3.66) and (3.68) (and (3.67), (3.68)) have a "group structure".

This means that if we can find an approximate solution g_ε (e.g., by solving the first order equations), we can perform the (3.70), (3.62) change of variables and set

$$\begin{aligned} \tilde{f}_\varepsilon &= g_\varepsilon^{-1} \circ f_\varepsilon \circ g_\varepsilon \\ \tilde{H}_\varepsilon &= H_\varepsilon \circ g_\varepsilon \ . \end{aligned} \tag{3.72}$$

If we solve the problem for $\tilde{f}_\varepsilon$, $\tilde{H}_\varepsilon$, i.e.,

$$\begin{aligned} \tilde{g}_\varepsilon^{-1} \circ \tilde{f}_\varepsilon \circ \tilde{g}_\varepsilon &= f_0 \\ \tilde{H}_\varepsilon \circ \tilde{g}_\varepsilon &= H_0 \ , \end{aligned} \tag{3.73}$$

then, we have solved the original problem since joining (3.72) and (3.73), we obtain

$$\begin{aligned} (g_\varepsilon \circ \tilde{g}_\varepsilon)^{-1} \circ f_\varepsilon \circ g_\varepsilon \circ \tilde{g}_\varepsilon &= f_0 \\ H_\varepsilon \circ g_\varepsilon \circ \tilde{g}_\varepsilon &= H_0 \ . \end{aligned}$$

The importance of the above observation, which will be appreciated later, is that, by making successive changes of variables, we can eliminate all the linear terms of the error by solving an equation which is just the linearized equation *at the integrable system*.

This is an important difference with the standard Newton method since the standard Newton method requires that we solve the linearized equation in a neighborhood.

The fact that we can obtain a method that, for all purposes is like a Newton method but which nevertheless only requires that we know how to solve one linearized equation depends crucially on the fact that the equations that we are studying have a particular structure which is called *group structure* and that will be discussed much more in Section 5, in particular, Remark 5.6 and Exercise4.23.

3.7. **Generating functions.** One of the reasons why Hamiltonian mechanics is so practical is because of the ease with which one can generate enough canonical transformations.

In old fashioned books ([Whi88], [Gol80], [LL76]) one can find that canonical transformations are described in terms of generating functions. We will describe those briefly and only for purposes of comparing with older books. It should be remarked however, that generating functions, even if not so useful from the point of view of transformation theory (there are better tools such as Lie transforms) are still quite useful tools in the variational formulation of Hamiltonian mechanics, providing thus a valuable link to Lagrangian mechanics. Moreover, some of the constructions that appear in generating functions are quite natural in optics. See [BW65].

The equation

$$f^*\theta - \theta = \mathrm{d}S$$

is written in old fashioned notations as

$$p' \,\mathrm{d}q' - p \,\mathrm{d}q = \mathrm{d}S \ , \tag{3.74}$$

where $p \,\mathrm{d}q := \sum_{i=1}^k p_i \,\mathrm{d}q_i$, etc. This should be interpreted as saying that we consider the coordinate functions p_i, q_i and the transformed functions $p'_i = p_i \circ f$, $q'_i = q_i \circ f$. Then, $\theta = p \,\mathrm{d}q$ and $f^*\theta = p' \,\mathrm{d}q'$; S is a function on the manifold.

When q, q' are a good coordinate system (i.e. p can be expressed as a function of q and q', $p = p(q, q')$), we can define a function $\mathcal{S} : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$ by setting $\mathcal{S}(q, q') := S(q, p(q, q'))$. Usually, in old fashioned notations, this is described as "expressing S in terms of q and q'" or simply by writing "$S = S(q, q')$" or something to that effect. Very often the same letter is used for S, $\mathcal{S}$.

Remark 3.27. In old fashioned notation in mechanics, the same letter is used for the functions that give the same result irrespective of the arguments. Of course, even if this is almost manageable and one understand what is meant by $S(q, p)$, $S(q, q')$, by paying attention to the arguments this notation wrecks havock when one tries to evaluate at concrete points. For example, what is meant by $S(2, \pi)$ when one is considering at the same time $S(q, p)$, $S(q, q')$?

Note however, that the assumption that q, q' is a system of coordinates is far from trivial. To begin with, it is not obvious that the manifold on which we are working admits a system of coordinates. Even if it does, or if we work just on a neighborhood so that we have local coordinates, there are other conditions to be imposed. For example, it is false for the identity and for transformations close to identity, it may be a system of coordinates with undesirable properties. It is, however, true for $(p, q) \mapsto (p, q + p)$ and small perturbations.

In that case, when we compute the differential in (3.74), we have

$$\mathrm{d}S = \partial_1 \mathcal{S}(q, q') \,\mathrm{d}q + \partial_2 \mathcal{S}(q, q') \,\mathrm{d}q' \ ,$$

hence

$$p = -\partial_1 \mathcal{S}(q, q') \ , \qquad p' = \partial_2 \mathcal{S}(q, q') \ . \tag{3.75}$$

We think of (3.75) as of an equation for p', q' in terms of p, q. If the implicit function theorem applies (for which it suffices that q, q' provide a good system of coordinates on the manifold) and indeed the equations (3.75) can be solved differentiably, $\mathcal{S}$ determines the transformation. Note that the implicit function theorem will apply in a C^2 open set of functions $\mathcal{S}$, so that we can think of this procedure as giving a chart of some subset of the space of symplectic mappings. Also note that we parameterize the transformation by one scalar function. Moreover, the changes of variables given by (3.75) are automatically symplectic. Keeping track of transformations – in an open set – which satisfy some non-linear and non-local constraints (preserving the symplectic structure) by just keeping track of a function is a great simplification.

However, one important shortcoming of these generating functions is that for the identity transformation, q, q' is not a good system of coordinates on the manifold and we cannot use (3.75) to represent the identity or near identity transformations. As we have seen, near identity transformations play an important role in canonical perturbation theory, so, it is necessary to devise variants of the method to incorporate them.

In the case that the coordinate functions p, q are global (or that we just work on a neighborhood), we can write

$$p\,\mathrm{d}q = -q\,\mathrm{d}p + \mathrm{d}(pq) \ .$$

Hence (3.74) reads

$$p'\,\mathrm{d}q' + q\,\mathrm{d}p = \mathrm{d}(S + pq) \ . \tag{3.76}$$

In the case that p, q' is a good system of coordinates (as happens in a neighborhood of the identity), we can write

$$S + pq = \tilde{\mathcal{S}}(p, q')$$

and from (3.76) we see that

$$q = \partial_1 \tilde{\mathcal{S}}(p, q') \ , \qquad p' = \partial_2 \tilde{\mathcal{S}}(p, q') \ .$$

Again, we can consider this as a system of implicit equations defining p', q' in terms of p, q.

Note that if q is an angle, then $\mathcal{S}(q + k, q' + \ell) = \mathcal{S}(q, q')$ for all $k, \ell \in \mathbf{Z}^n$. On the other hand, $\tilde{\mathcal{S}}(p, q' + \ell) = \mathcal{S}(p, q') + p\,\ell$. Even if this generating function works in neighborhoods of the identity, it does not work at all for the map $(p, q) \mapsto (-q, p)$.

One can use similar procedures to obtain many other generating functions.

For example, one can use for a partition of $\{1, \dots, d\}$ into two sets $\mathcal{A}$ and $\mathcal{B}$ the formula:

$$-\sum_{i\in\mathcal{B}} p_i'\,\mathrm{d}q_i' + \sum_{i\in\mathcal{A}} p_i\,\mathrm{d}q_i = \sum_{i\in\mathcal{B}} q_i'\,\mathrm{d}p_i' - \sum_{i\in\mathcal{A}} q_i\,\mathrm{d}p_i + \mathrm{d}\left(-\sum_{i\in\mathcal{B}} p_i' q_i' + \sum_{i\in\mathcal{A}} p_i q_i\right)$$

to change some of the p_i's for q_i's in the push-forward.

Even if these procedures are quite customary in old fashioned mechanics treatises, they will not be very useful for us. Again, we emphasize that even if the q, q' generating function can be defined in any exact manifold, the others seem to require some extra structure, which can be arranged in small neighborhoods.

We note however, that the function S has a well defined intrinsic meaning as evidenced in [BW65] — this is sometimes described as Hamilton-Jacobi equation or "the action as a function of coordinates" depending on what interpretation one gives. We refer to [Har99] for much more information on this primitive function.

In Hamiltonian optics [BW65], S represents the phase of the wave. Indeed, Hamiltonian mechanics was developed as a byproduct of Hamiltonian optics. This explains why so much of Hamiltonian mechanics, especially in earlier treatises is based on studying S and its relatives.

More modern treatments ([Arn89], [AM78]) prefer to start from the symplectic geometry and postulate it without any other motivation that it eventually works. This is certainly expeditious.

4. Two KAM proofs in a model problem

In this section we will discuss one of the technically simplest applications of the KAM methodology: the Siegel center theorem.

The main goal of this application is to show in action perhaps the most basic heuristic principle of the KAM method:

Quadratic convergence can overcome small divisors.

Roughly speaking this means that if we have a method of improvement that reduces the error to something that is quadratic in the original error, even if the solution requires solving an equation which involves small denominators, we can still obtain convergence.

The fact that the convergence does indeed take place is rather subtle. In our opinion, the only way to appreciate the subtlety of the convergence achieved by KAM theory is to give a serious try to several other seemingly reasonable schemes and see them fail. At the end of the proof, we have suggested several of these schemes as exercises.

Besides those exercises, we have also included some exercises which admit easy solutions and provide extensions to the material in the text.

We also emphasize that the fact that one can get a quadratically convergent method solving only one small denominator equation is far from trivial and it requires that the equations we consider have some special structure. This will be elaborated in more detail in Section 5 and in particular in Remark 5.6.

In this section, we will present two versions of the Siegel theorem – one using just Diophantine conditions in one dimension, and another using approximation functions and decomposition in scales in higher dimensions.

The second proof will be formulated as a set of exercises. The main ideas of this section follow [Mos66a], [Zeh77] and [Arn88]. Indeed, we follow these references rather closely.

These two proofs will illustrate the main features of KAM proofs and contain the essential analytic and number theoretic difficulties even if they do not involve any geometry.

We will start with a one dimensional problem. See [Mos66a] for more details on the proof we present and [Zeh77] for a higher dimensional version.

Theorem 4.1. *Let $f : U \subset \mathbf{C} \to \mathbf{C}$ be and analytic function of the form*

$$f(z) = az + \hat{f}(z) \tag{4.1}$$

with $\hat{f}(z) = O(z^2)$.

Assume that

$$|(a^n - 1)^{-1}| \le n^\nu K \tag{4.2}$$

and that

$$\|\hat{f}\|_1 \le \rho(\nu) K^2 \ , \tag{4.3}$$

where $\rho(\nu)$ is an explicit function.

Then there exists a unique function

$$h(z) = z + \hat{h}(z)$$

with $\hat{h}(z)$ analytic in a disc of radius

$$\sigma = 1 - 2\rho(\nu)$$

such that

$$f \circ h(z) = h(az) \ . \tag{4.4}$$

Moreover, we have

$$\|\hat{h}\|_\sigma \le \|\hat{f}\|_1 C \ . \tag{4.5}$$

Remark 4.2. The uniqueness for h claimed in Theorem 4.1 means that if there are two functions satisfying this they have to agree in an open set of the origin. As we have seen already, the condition (4.2) and (4.4) determine the jet of h uniquely.

Remark 4.3. Condition (4.2) is automatic when $|a| \neq 1$. In that case, we have presented a simple proof already. So we will restrict ourselves to the case when $|a| = 1$.

Remark 4.4. It is a standard observation that, assuming that f is defined in a ball of radius 1 and small is the same as considering a small neighborhood.

Heuristically, in a small neighborhood, the linear part is the dominant term and it is natural to try to describe the behavior of the whole system in terms of the behavior of the linear one.

More precisely, given f, consider for λ small

$$f_\lambda = \lambda^{-1} f(\lambda z) \ .$$

Notice that f_λ has the same linear part and is defined in $\lambda^{-1} B_r$ if f is defined on B_r. Since $|\hat{f}(z)| = O(|z|^2)$, we have $\|\hat{f}_\lambda\|_{B_1} \le C\lambda$.

If we apply Theorem 4.1 to f_λ, we obtain a h_λ. Then h will satisfy (4.4).

Remark 4.5. Condition (4.2) is not optimal. Later we will discuss how to obtain the same result when the arithmetic condition (4.2) is replaced by the Brjuno condition, which is indeed optimal as shown in [Yoc95],[PM92].

The fact that if Brjuno condition fails one can construct counterexamples is considerably deeper and out of the scope of these notes. See the references above.

Before embarking in the proof, we note that all the methods are based in estimates for the equation

$$\varphi(az) - a\varphi(z) = \eta; \quad \varphi(0) = 0 \tag{4.6}$$

in which we consider η and a as given and we are to determine φ.

The analysis of this equation is very similar to the analysis of (3.27) in Section 3.4. Since it is not completely identical, we need to start by revising slightly the definitions of norms and the setup.

We define the norm of an analytic function by [6]

$$\|f\|_r = \sup_{|z|\le r} |f(z)|.$$

Lemma 4.6. *Assume that a satisfies* (4.2). *Then, if $\eta(0) \equiv \eta_0 = 0$ we can find a solution of* (4.6). *Moreover*

$$\|\varphi\|_{re^{-\delta}} \le CK|\delta|^{-\tau}\|\eta\|_r \tag{4.7}$$

Proof. This follows from the results in Section 3.4.

It suffices to write $z = \exp(2\pi i\theta)$. Then, the result stated is a particular case of Lemma 3.15 applied to a Fourier series which only has positive terms. □

Exercise 4.7. Give a direct proof of the Lemma 4.6

One can follow the sketch in the beginning of Section 3.4.

Start by observing that the solution of (4.6) is $\varphi_k = \eta_k(a^k - a)^{-1}$. Estimate $|\varphi_k|$ using the above formula, (4.2), and the estimates for $|\eta_k|$ in terms of $\|\eta\|_r$ obtained using Cauchy estimates.

Estimate $\|\varphi\|_{re^{-\delta}}$ by the sup of the coefficients. Then, one ends up with the desired result with $\tau = \nu + 1$.

Since we are dealing with analytic estimates, this is enough to get through the proof. The ambitious reader is invited to carry out an analysis similar to that in [Rüs76b] and obtain the optimal exponent.

Now, we proceed to the proof of Theorem 4.1. The proof we present here follows [Zeh77] – it is a particular case of the results of that paper.

Proof. Proceeding heuristically for the moment, we can think of (4.4) as an implicit equation in a space of functions

$$0 = \mathcal{T}(f, h) \equiv f \circ h - h \circ a$$

(by a we denote either the constant or the function $a(z) = az$). Note that $\mathcal{T}(a, \mathrm{Id}) = 0$.

[6] These norms are slightly inconsistent with those in Section 3.1 in which we took $\|f\|_\sigma = \sup_{|z|\le e^\sigma} |f(z)|$. The convention of Section 3.1 is more natural when one is using at the same time Fourier series and Taylor series. For the present section, the convention we now take is more natural.

We consider f fixed (but close to a) and we are given an approximate solution h

$$\mathcal{T}(f,h) \equiv f \circ h - h \circ a = R \ , \tag{4.8}$$

where R is the remainder which we would like to think of as small (the precise sense in which it is small will not be made explicit in this heuristic discussion).

We would like to obtain a Δ that eliminates most of R so that $\mathcal{T}(f, h + \Delta) \ll R$. This amounts to a Newton's method. Since

$$\mathcal{T}(f,h+\Delta) \approx \mathcal{T}(f,h) + D_2\mathcal{T}(f,h)\Delta \ ,$$

we are lead to consider the equation for Δ

$$R + D_2\mathcal{T}(f,h)\Delta = 0 \ . \tag{4.9}$$

In our case – remember that we are, for the moment, just proceeding heuristically, but this step is not difficult to justify – we have that the derivatives will be:

$$D_2\mathcal{T}(f,h)\Delta = (f' \circ h)\Delta - \Delta \circ a \ .$$

Hence, in our case (4.9) becomes:

$$(f' \circ h)\Delta - \Delta \circ a = -R \ . \tag{4.10}$$

If the factor $f' \circ h = a + \hat{f}' \circ h$ were just a, the equation (4.10) would reduce to those considered in Lemma 4.6.

One way that succeeds in reducing the annoying $\hat{f}' \circ h$ to a constant is the following: (in the exercises we examine several seemingly natural methods which do not work).

Take derivatives with respect to z of (4.10) and obtain the identity

$$f' \circ h \, h' - a h' \circ a = R' \ . \tag{4.11}$$

If rather than looking for Δ, we look for w defined by $\Delta = h' \, w$ (remember h is close to the identity, so that indeed $1/h'$ is an analytic function so that looking for Δ and for w is equivalent), equation (4.10) becomes

$$f' \circ h \, h' \, w - h' \circ a \, w \circ a = -R \ . \tag{4.12}$$

Substituting (4.11) in (4.12), we are lead to

$$a \, h' \circ a \, w - h' \circ a \, w \circ a = -R - R' \, w \tag{4.13}$$

or

$$a \, w - w \circ a = -(h' \circ a)^{-1} R - (h' \circ a)^{-1} R' \, w \ . \tag{4.14}$$

If we ignore the term $(h' \circ a)^{-1} R' \, w$ (the intuition, which we will later turn into rigorous estimates, says that $h' \circ a$ is of order one, R and R' are small, hence w is small and $R'w$ is much smaller), we simplify the problem to studying

$$aw - w \circ a = -(h' \circ a)^{-1} R \ , \tag{4.15}$$

which indeed is an equation of the type we considered in Lemma 4.6.

Hence, the prescription that we have derived heuristically to obtain a more approximate solution is:

1. Take w solving (4.15);
2. Form $\Delta = h'w$;
3. Then, $h + \Delta$ should be a better solution to the problem.

Now, we turn to making all the previous ideas rigorous. We will need to show that the procedure improves (that is, show estimates for the remainder after one step given estimates on the remainder before starting). We will also need to show that the procedure can be repeated infinitely often and that it leads to a convergent procedure.

If we are given a system with an remainder and run the procedure outlined above, the following lemma will establish bounds for the new remainder in terms of the original one.

We will follow standard practice in KAM theory and denote by C throughout the proof constants that depend only on the dimension and other parameter which are fixed in our proof. In our case, since we are paying special attention to the dependence of the domain loss parameter on the size of the Diophantine constants and the smallness assumptions, C will not depend on them. Other KAM proofs which emphasize other features may allow C to stand for constants that could also depend on the Diophantine constants.

Lemma 4.8. *Let f be as in Theorem 4.1, $h(z) = z + \hat{h}(z)$, ($\hat{h}(z) = O(|z|^2)$) defined in a ball of radius $\frac{1}{2} < \sigma < 1$ satisfy*

$$\|\hat{h}'\|_\sigma \leq M \leq 1/2 \ . \tag{4.16}$$

with

$$\sigma + M < 1 \ , \tag{4.17}$$

$$\|f \circ h - h \circ a\|_\sigma \leq \varepsilon \ . \tag{4.18}$$

Assume furthermore that $\delta > 0$ is such that

$$KC\delta^{-\nu-1}\varepsilon + \sigma e^{-\delta} < \sigma \ . \tag{4.19}$$

Then, the prescription above can be carried out and we have:

$$\|f \circ (h+\Delta) - (h+\Delta) \circ a\|_{\sigma e^{-\delta}} \leq KC\delta^{-\nu-1}\varepsilon^2 + 2\|f\|_1 (KC\delta^{-\nu-1}\varepsilon M)^2 \ . \tag{4.20}$$

Remark 4.9. Notice that since for $\delta \geq 0$, $\sigma(1 - e^{-\delta}) \leq \sigma\delta$, condition (4.19) is implied by

$$\sigma\delta \geq CK\delta^{-\nu-1}\varepsilon \ . \tag{4.21}$$

which, once we have σ, just tells us that δ cannot be smaller than a power of ε.

Remark 4.10. Note that if we assume without loss of generality that $\|\hat{f}\|_1 \leq 2$, $K \leq K^2$, $\delta < 1$, the R.H.S. of (4.20) is less or equal to

$$CK^2\varepsilon^2\delta^{-2(\nu+1)} \ . \tag{4.22}$$

Proof. To check that the prescription can indeed be carried out, we just need to check that the function $f \circ (h + \Delta)$ can be defined. Hence, our first goal will be to obtain estimates on Δ and show that the image of the ball of radius $re^{-\delta}$ under $h + \Delta$ is contained in the domain of f. Indeed, the estimates for the range will allow us also to obtain estimates for the derivative of f via the Cauchy theorem which will later prove to be useful.

Then, we will obtain the estimates in (4.20) and (4.22) provided that we have suitable estimates on $\|\Delta\|_{\sigma e^{-\delta}}$.

To obtain the estimates on $\|\Delta\|_{\sigma e^{-\delta}}$, we note that using the Banach algebra property of the norms and the inductive assumption (4.16), we can bound the R.H.S. of (4.15) by

$$\|(h' \circ a)^{-1} R\|_\sigma \leq (1 - 1/2)^{-1} \|R\|_\sigma \ .$$

By Lemma 4.6 we have that

$$\|w\|_{\sigma e^{-\delta}} \leq KC\delta^{-\nu}\|R\|_\sigma \ .$$

By Cauchy estimates, (see Lemma 3.13, but take into account that now we are in an slightly different situation), we have:

$$\|h' \circ a\|_{\sigma e^{-\delta}} \leq K\delta^{-1}\|h\|_\sigma \ , \tag{4.23}$$

$$\|R'\|_{\sigma e^{-\delta}} \leq K\delta^{-1}\varepsilon \ . \tag{4.24}$$

Hence, taking into account (3.14), and that we had called $\|R\|_\sigma = \varepsilon$, we obtain from the previous results:

$$\|\Delta\|_{\sigma e^{-\delta}} \leq KC\delta^{-\nu-1}\varepsilon M \ , \tag{4.25}$$

$$\|R'w\|_{\sigma e^{-\delta}} \leq KC\delta^{-\nu-1}\varepsilon^2 \ . \tag{4.26}$$

Note that the assumption (4.16),

$$\|h + \Delta\|_{\sigma e^{-\delta}} < 1 \ ,$$

so that, as claimed, the composition in (4.20) indeed makes sense.

To obtain the estimates in (4.20), we consider the term to be estimated in (4.20) and the obvious identity obtained just by adding and subtracting terms to it and grouping the result conveniently.

$$\begin{aligned} f \circ (h + \Delta) &- (h + \Delta) \circ a \\ &= f \circ h - h \circ a + f' \circ h\,\Delta - \Delta \circ a \\ &+ [f \circ (h + \Delta) - f \circ h - f' \circ h\,\Delta] \ . \end{aligned} \tag{4.27}$$

The first four terms in the R.H.S. of (4.27), using (4.11) and (4.12) amount to:

$$R + h' \circ aw + R'w - ah' \circ aw = R'\,w \ .$$

The term in braces in (4.27) can be estimated because, by a calculus identity (Taylor theorem with the Lagrange form of the remainder)

$$\begin{aligned} & f(h(z)+\Delta(z)) - f(h(z)) - f'(h(z))\Delta(z) = \\ & \qquad = -\int_0^1 (s-1)\, f''(h(z)+s\Delta(z))\, \Delta^2(z)\, ds \ . \end{aligned} \tag{4.28}$$

Since, again by Cauchy bounds and (4.16) we have

$$\|f''(h(z)+s\Delta(z))\|_{\sigma e^{-\delta}} \le C\delta^{-2}\|f\|_1 \ ,$$

we can bound the $\| \ \|_{\sigma e^{-\delta}}$ of (4.28) by

$$1/2\|f\|_1(KC\delta^{-\nu-1}\varepsilon M)^2 \tag{4.29}$$

If we estimate (4.27) putting together (4.26) and (4.29), and remembering the standing assumptions on M, $\|f\|_1$, we obtain (4.20).

□

To finish the proof of Theorem 4.1, we just need to show that if $\|\hat{f}\|_1$ is sufficiently small, we can repeat the iterative procedure arbitrarily often and that we converge to a limit which satisfies (4.5).

We will denote by subindices n the objects after n steps of the iterative process (assuming that it can be carried out this far). For example, σ_n will be the domain of definition of h_n and we have $\sigma_{n+1} = \sigma_n e^{-\delta_n}$. To simplify the discussion, we will use the condition (4.21) which implies (4.16) and the bounds (4.22).

The main thing that we have to do is to choose the δ_n's. Notice that if we choose δ_n going to zero slowly, we lose more domain than needed and end up with a weaker theorem –of course, if we lose too fast, we end up with an empty domain. On the other hand, the smaller that we choose δ_n, the worse (4.22) becomes.

A reasonable compromise that is neither too fast so that we end up with no domain nor too slow so that we can still converge is to choose an exponential rate of decay. In the exercises, we will explore other choices.

We will choose

$$\delta_n = \delta_0 2^{-n} \ , \tag{4.30}$$

and then, will show how to choose δ_0.

With this choice of δ_n, (4.22) implies easily

$$\varepsilon_{n+1} \le CK^2\varepsilon_n^2\delta_0^{-2\mu}A^{2n} \tag{4.31}$$

where $\mu = \nu + 1$, $A = 2^\mu$.

We assume by induction that the iterative step can be carried out n times (i.e., that hypothesis (4.21) is verified for the first n steps). We will show that, under certain assumptions on the size of δ_0, ε_0, which will be independent of n, hypothesis (4.21) will be verified for $n+1$. Moreover, we will show that ε_{n+1} decreases very fast. Then, by repeated application of (4.31) we have:

$$\begin{aligned}\varepsilon_{n+1} &\le CK^2\delta_0^{-2\mu}\varepsilon_n^2 A^n\\ &\le (CK^2\delta_0^{-2\mu})^{1+2}\,A^{n+2(n-1)}\,\varepsilon_{n-1}^{2\cdot 2}\\ &\le \cdots\\ &\le (CK^2\delta_0^{-2\mu})^{1+2+2^2+\cdots+2^{n-1}\cdot 1}\,A^{n+2(n-1)+\cdots+2^{n-1}}\,\varepsilon_0^{2^{n+1}}\ .\end{aligned} \tag{4.32}$$

Note that $1+2+2^2+\cdots+2^n \le 2^{n+1}$ and without loss of generality, we can assume that $CK^2\delta_0^{-2\mu} > 1$. Similarly,

$$\begin{aligned} n+2(n-1)+\cdots+2^{n-1}\cdot 1 &\\ &= 2^n[n2^{-n}+(n-1)2^{-(n-1)}+\cdots+2^{-1}\cdot 1]\\ &\le 2^n\sum_{k=1}^{\infty} k2^{-k} = 2^n\cdot 2 = 2^{n+1}\ ,\end{aligned}$$

hence

$$\varepsilon_{n+1} \le \left(CK^2\delta_0^{-2\mu}\varepsilon_0 A\right)^{2^{n+1}}\ . \tag{4.33}$$

Notice that if $\rho \equiv CK^2\delta_0^{-2\mu}A\varepsilon_0 < 1$, then (4.33) converges to zero extremely fast (faster than any exponential).

The equation that we need to satisfy to be able to perform the next step is

$$\delta_{n+1} \equiv \delta_0 2^{-(n+1)} \ge CK\delta_0^{-\mu}2^{-n\mu}\varepsilon_{n+1} = CK\delta_0^{-\mu}2^{-n\mu}\rho^{2^{n+1}}$$

or

$$CK\delta_0^{-\mu-1} \le 2^{n\mu-(n+1)}\rho^{-2^{n+1}}\ . \tag{4.34}$$

By now, it should be clear that if we take $\delta_0 = \frac{1}{2}$ (so that $\sigma_n \ge e^{-1}$), if we assume that ε_0 is sufficiently small, we can satisfy (4.34).

Moreover, since by (4.25),

$$\|\Delta_n\|_{e^{-1}} \le KC\delta_0 2^{\mu n}\rho^{2^n}\ ,$$

we see that $\sum \Delta_n < \infty$ Hence

$$\Delta \equiv \sum \Delta_n$$

converges uniformly in the space of functions in the disk of radius e^{-1} and we can easily bound $\|\Delta\|_{e^{-1}}$.

□

At the end of this subsection, we have collected some exercises that explore alternatives for the present proof and for another that will be presented.

Let us highlight some of the remarkable points of the proof.

Remark 4.11. We call attention to the remarkable fact that the derivatives of (4.10) could be used to transform the equation (4.11) into a much simpler equation (with an error which is small if the remainder is small and of *quadratic order*).

This is what allowed us to solve the step with quadratic error. In turn, this quadratic error was crucial in being able to deal with the small divisors (see the following remark). See exercise 4.23 for an example of a problem with very similar analytical properties but without group structure for which the result is false.

The possibility of performing this remarkable simplification comes from the group structure of the equations, as was emphasized in [Zeh76a].

This remarkable cancellation has other justifications, for example, in the context of Lagrangian principles. Indeed, one can see that it is related to the symmetry that we used in (2.18). With a bit of hindsight we can see that the factor $(1+\ell^{[<n]'})$ used there is really an *infinitesimal translation* on the right for the data of the problem and that the cancellation is just a reflection of the fact that the original problem is invariant under translations (see the classical Noether theorem about variational principles with continuous symmetries). In Quantum Field Theory the identities that come from changes of variables are called Ward identities. The relation between Ward identities and the identities used in Lindstedt series has been emphasized in [Gal94a], [BGK99], which are papers designed to bridge the gap between the language of Quantum Field Theory and KAM theory. Of course, in QFT one often does not consider the objects as defined per se, but rather as formal power series.

Remark 4.12. Once we have the iterative step and the estimates that give quadratic convergence, the rest is (even if miraculous and quite remarkable) by now well understood.

Indeed, there are several abstract formulations, some of which we will discuss later. See Section 5.

In what follows, we will emphasize the steps required to reach the quadratic convergence and leave to the reader the checking that the convergence indeed takes place.

Experience shows that, once one has worked out a few quadratically converging arguments it becomes faster and more reliable to work out a proof by oneself than to read the proofs by others. It is certainly more instructive for the reader and more comfortable for the writer. In this case, the reader should be assured by the existence of properly written papers that we reference where he/she is encouraged to look for extra details.

Obtaining the quadratically convergent algorithm in classical KAM theory is not obvious since it depends on cancellations given by the geometry or the structure of the problem which eliminate some terms which would result in a linearly convergent method.

Note that in the classical KAM theory, we are constrained by the fact that we know only how to solve one linearized equation (in contrast with the usual Newton method, where we can solve the linearized equation in a whole neighborhood.

In the remainder of this subsection, we will present a proof of the multidimensional case of Theorem 4.1 following [Arn88] chapter 28. The one-dimensional version of this proof is covered in [SM95] chapter 25.

Theorem 4.13. *Let $f : U \subset \mathbf{C}^d \to \mathbf{C}^d$, be analytic in a polydisk. $f(0) = 0$. Denote $Df(0) = A$ and assume that A is diagonal and $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_d)$, the spectrum of A satisfies:*

$$|\sigma^k - \sigma_i|^{-1} \le C|k|^\nu \quad k \in \mathbf{N}^d \quad |k| \ge 2, \quad i \in \{1, \cdots n\} \tag{4.35}$$

(where we use the customary multi-index notation $\sigma^k = \sigma_1^{k_1} \cdot \sigma_2^{k_2} \cdot \dots \cdot \sigma_d^{k_d}$, $|k| = k_1 + \cdots k_d$).

Then, we can find an $h : V \subset \mathbf{C}^d \to \mathbf{C}^d$, $h(0) = 0$, $Dh(0) = \mathrm{Id}$ such that in a neighborhood we have:

$$h^{-1} \circ f \circ h = A \tag{4.36}$$

The conclusion of (4.36) is again that f is just the linear map in other coordinates.

Remark 4.14. Notice that we are not assuming that the σ_i have modulus 1. The multidimensional case can have several interesting examples in which some of the σ_i are smaller than 1 and others are greater than 1. In the case that there are no eigenvalues equal to 1 and that no product of eigenvalues is another eigenvalue, Sternberg theorem will guarantee us that there exists a C^∞ change of variables that reduces the system to a linear one. To obtain that the change of variables is analytic, we need not only that the products are not eigenvalues but also some quantitative estimates on how far they are such as (4.35). Note also that the C^∞ changes of variables produced by Sternberg theorem are not unique, whereas, as pointed out before, the analytic ones are unique.

Remark 4.15. Note that, implicitly, condition (4.35) requires that there is no eigenvalue 0, hence A is invertible.

It is very easy to show that if one eigenvalue is 0 one should not expect the conclusion to be true.

Remark 4.16. If we write $\sigma_j = \exp(2\pi i \omega_j)$ – with ω_j possibly complex numbers – we see that the (4.35) is equivalent to the fact that ω satisfies (3.19) but we only need it for $k \in \mathbf{N}^d$ rather than $k \in \mathbf{Z}^d$.

We will discuss the different stages of the proof but leave many details to the reader since this will provide some training and, moreover, it can be found in the references indicated.

Proceeding heuristically, we will note that if $h(z) = z + \hat{h}(z)$, we have $h^{-1}(z) = z - \hat{h}(z) + O([\hat{h}]^2)$. (Here in the O notation we allow to include derivatives. For example, $\hat{h}'\hat{h}''$ will be a term allowed in $O([\hat{h}]^2)$.)

If we assume that $f(z) = Az + \hat{f}(z)$ and that $\hat{f}$ is small, if we want to make the changes of variables that reduce f to linear with an smaller error, we have

$$h^{-1} \circ f \circ h(z) = A(z) + \hat{f}(z) - \hat{h} \circ A(z) - A\hat{h}(z) + O([\hat{h}]^2, [\hat{f}][\hat{h}]) \tag{4.37}$$

This suggests the following iterative step

1) Solve the following equation for $\hat{h}$

$$\hat{f}(z) = \hat{h} \circ A(z) - A\hat{h}(z) \tag{4.38}$$

2) Consider now the the map

$$\tilde{f} = h^{-1} \circ f \circ h(z)$$

If all works according to plan, $\tilde{f}$ will be much closer to the linear map A.

The approximations we have taken can be readily estimated by adding and subtracting as follows. (Ignore for the moment questions of domains of definition. Suffice it to say that the simple minded identities we obtain are supposed to hold in a domain near the origin. Later we will need to worry about how big we can choose the domain.)

$$f \circ h(z) = Az + A\hat{h}(z) + \hat{f}(z) + R_1(z) \tag{4.39}$$

where $R_1(z) = \hat{f} \circ h(z) - \hat{f}(z)$.

$$(\mathrm{Id} - \hat{h}) \circ f \circ h(z) = Az + \hat{h}(z) + \hat{f}(z) + R_1(z) - \hat{h}(Az) + R_2(z) \tag{4.40}$$

where $R_2(z) = \hat{h}(Az) - \hat{h} \circ f \circ h(z)$.

$$h^{-1} \circ f \circ h(z) = (\mathrm{Id} - \hat{h}) \circ f \circ h(z) + R_3(z) \tag{4.41}$$

where $R_3(z) = \Big(h^{-1} - (\mathrm{Id} - \hat{h})\Big) \circ f \circ h(z)$.

Hence, from (4.39), (4.40), (4.41), we obtain:

$$h^{-1} \circ f \circ h(z) = Az - \hat{h} \circ (Az) + \hat{f}(z) + \hat{h}(z) + R_1(z) + R_2(z) + R_3(z)\ . \tag{4.42}$$

Now we turn to the task of obtaining estimates that quantify how this step indeed improves the situation and how we can use it repeatedly to converge to a solution. We highlight the main arguments.

i) Estimates on $\hat{h}$ obtained solving (4.38).

By carrying out exactly the same procedure indicated before estimating the sizes of the Taylor coefficients by the size of the function, solving the small divisor equation for the coefficients and then, estimating the size in an slightly smaller domain we obtain:

$$\|\hat{h}\|_{re^{-\delta}} \leq C\delta^{-\tau}\|\hat{f}\|_r \tag{4.43}$$

Note that using (4.38) we immediately obtain estimates

$$\|\hat{h}\circ A\|_{re^{-\delta}} \leq C\delta^{-\tau}\|\hat{f}\|_r \tag{4.44}$$

Having control of $\hat{h}$ both in the polydisk and in its image under A will be quite important to be able to check that compositions, etc. make sense.

ii) Estimates obtained using (4.43) and the implicit function theorem.

Note that this requires that we assume some smallness condition in $\hat{h}$ that ensures that we can indeed define the compositions.

$$\|h^{-1} - (\mathrm{Id} - \hat{h})\|_{re^{-2\delta}} \leq C\delta^{-2\tau-1}\|\hat{h}\|^2_{re^{-\delta}} \tag{4.45}$$

and, similarly using (4.44) and the implicit function theorem (again, we need some conditions that ensure that we can define the compositions needed to apply the implicit function theorem).

$$\|h^{-1} - (\mathrm{Id} - \hat{h})\|_{re^{-2\delta}} \leq C\delta^{-2\tau-1}\|\hat{h}\circ A\|^2_{re^{-\delta}} \tag{4.46}$$

For the conditions that allow us to use the implicit function theorem, it is enough to assume that we have:

$$\begin{aligned} \|\hat{h}\|_{re^{-\delta}} &\leq K\delta \\ \|\hat{h}\circ A\|_{re^{-\delta}} &\leq K\delta \end{aligned} \tag{4.47}$$

which, in view of (4.43) (4.44) are implied by:

$$\begin{aligned} \|\hat{f}\|_{re^{-\delta}} &\leq K\delta^{\tau+1} \\ \|\hat{f}\circ A\|_{re^{-\delta}} &\leq K\delta^{\tau+1} \end{aligned} \tag{4.48}$$

iii) Easy estimates using the mean value theorem.

Note that we can estimate

$$\|f\circ h_1(z) - f\circ h_2(z)\|_r \leq \sup_{z\in\Sigma}|f'(z)|\|h_1 - h_2\|_r\ , \tag{4.49}$$

where Σ is a convex domain that includes the image of the polydisk of radius r under h_1 and h_2. In particular, we can take Σ to be the polydisk of radius $r + \max(\|h_1\|_r, \|h_2\|_r)$.

If we use Cauchy estimates, can obtain

$$\|f\circ h_1(z) - f\circ h_2(z)\|_r \leq \delta^{-1}\|f\|_{r'}\|h_1 - h_2\|_r \tag{4.50}$$

where $r' = [r + \max(\|h_1\|_r, \|h_2\|_r)]\, e^{\delta}$.

As it turns out, to be able to make sure that the domains match we need a condition of the same form as (4.48).

Hence, one can prove a lemma that ensures that, provided (4.48) holds, we can perform the step and obtain estimates

$$\|\tilde{f} - A\|_{re^{-3\delta}} \leq CK^2\delta^{-\tau'}\|f - A\|_r^2 \ , \tag{4.51}$$

where, as usual, we have denoted by C a constant that depends only on the dimension, K is the constant in the Diophantine inequality (4.2) and τ' is an exponent related to the Diophantine exponent (roughly twice, since we are squaring the result of Lemma 4.6 and we are applying Cauchy bounds twice).

This statement is usually called the *iterative lemma*.

Once we have an iterative lemma, we need to show

iv) Choose $\delta_n = \delta_0 2^{-n}$. If you assume that $r_0 \geq 3\delta_0$ and that $\|f - A\|_{r_0}$is sufficiently small, then the iterative lemma can be applied repeatedly to obtain a sequence $\{f_n\}$ defined on $r_n = r_{n-1}e^{-\delta_{n-1}}$. This sequence satisfies

$$\|f_n - A\|_{r_n} \leq C\alpha^{2^n} \tag{4.52}$$

for some $0 < \alpha < 1$, which can be made arbitrarily small by assuming $\|f_0 - A\|_{r_0}$ is sufficiently small.

v) We need to show that the compositions $h^{(n)} \equiv h_n \circ h_{n-1} \circ \cdots h_0$ converge on a non-trivial domain.

This follows because $h_n - h_{n-1} = \hat{h}_n \circ h_{n-1} \circ \cdots h_0$ and we can estimate

$$\|\hat{h}_n\|_{r_n e^{-2\delta_n}} \leq \|\hat{f}\|_{r_n} 2^{n\tau'}$$

and using (4.48) show that

$$\|\hat{h}_n \circ h_{n-1} \circ \cdots h_0\|_{r_n e^{-3\delta_n}} \leq \|\hat{h}_n\|_{r_n e^{-2\delta_n}}$$

Besides the fact that the quadratic convergence is allowing us to dominate the small divisors, we want to highlight some features of the algorithm.

Note again that we can only solve the linearized equation at precisely the identity. Nevertheless, the progress that we are making, allows us to reduce the problem closer to the identity so that we are starting at a problem which is even more favorable. Again, this is the group structure of the problem. The successive changes of variables has been applied very often in the proofs involving Hamiltonian systems with preference to the proofs that involve just solving functional equations. This is due, in part to the fact that Hamiltonian systems have a very nice transformation theory. It is also true that reducing to normal forms, even if only approximately has very interesting byproducts. For example, the Nekhoroshev theorem.

Note that the analytic part of convergence was extremely similar. We obtained the estimates which are *quadratic* but which contain the bad term which has to grow unbounded. All these estimates were proved under some inductive assumptions that allow one to perform the algorithm. The quadratic nature of the estimates can be used to show that if we start with small enough error, the growing terms due to the solution of the linearized equation do not spoil the convergence and that indeed we recover the inductive assumption that allows us to keep on improving our linearization. Once we obtain that the remainder goes to zero extremely fast, it is possible to show that the composition of the transformations converges.

Exercise 4.17. When A has a non-trivial Jordan block and the spectrum satisfies (4.35), show that the cohomology equation $A\hat{h} - \hat{h} \circ A = \hat{f}$ is solvable as a formal power series.

What type of estimates do you obtain?

Are the estimates you obtain enough to prove Theorem 4.13 without the assumption that A is diagonalizable?

If not, can you construct a counterexample?

Exercise 4.18. Obtain optimal estimates in the Rüssman style for the linear equation for analytic functions in the several variables case. The case when A is a diagonalizable matrix with all eigenvalues equal to 1 is very similar to the one we have discussed so far. Much more interesting are the cases when the matrix has eigenvalues of modulus 1 and non-trivial Jordan blocks.

In preparation of arguments to come, note that, when A has eigenvalues of modulus different from 1, if the domain of φ is a polydisk, the domain of $\varphi \circ A$ is a different set.

Exercise 4.19. Formulate the improved estimates of Exercise 4.18 in the language of approximation functions Ω. Do they lead to some improvement in Brjuno conditions?

Exercise 4.20. Some of the estimates in the proof of Theorem 4.1 we have presented are rather wasteful.

Notice in particular that we estimated in (4.24)

$$\|R'\|_{\sigma e^{-\delta}} \le K\delta^{-1}\varepsilon \ .$$

We can observe that, as we iterate, the remainder vanishes at higher and higher orders. This will allow us to use sharper Cauchy estimates, which we detail below.

Note that if a function $f(z) = z^N g(z)$, we have $f'(z) = Nz^{N-1}g(z) + z^N g'(z)$. Also, we have $\|f\|_1 = \|g\|_1$. Hence

$$\begin{aligned} \|f'\|_r &\le Nr^{N-1}\|g\|_r + Cr^N r^{-1}\|g\|_1 \\ &\le CNr^{N-1}\|f\|_1 \end{aligned} \tag{4.53}$$

Carry on the proof using these improved estimates and see if one obtains something better.

Exercise 4.21. There is a certain arbitrariness in the speed at which domain is lost in the proofs.

What happens is you take $\delta_n = \delta_0 n^{-\alpha^n}$ with $\alpha > 0$?

What happens with $\delta_n = \delta_0 n^{-\alpha}$, $\alpha > 1$, or $\delta_n = \delta_0 n^{-1}(\log n)^{-\alpha}$, $\alpha > 1$?

Exercise 4.22. Fix $a = \exp\left\{2\pi i \frac{\sqrt{5}-1}{2}\right\}$ and consider

$$f_N(z) = az + z^N \ .$$

What are the asymptotic of the Siegel radius as $N \to \infty$?

Exercise 4.23. In the classical Newton method, we use the fact that if the derivative $D_2\mathcal{T}(a, \mathrm{Id})$ is invertible, then $D_2\mathcal{T}(f, h)$ is invertible when (f, h) are in a neighborhood of $(a \,\mathrm{Id})$ and, moreover, the norm of the inverse is bounded.

We can try to apply the same ideas involved in the proof that in the classical case the invertibility of the derivative is an open condition $(A + B)^{-1} = A^{-1}\sum_{i=0}(-BA^{-1})^i$. (sometimes called the Neumann series) to solve the equation

$$f' \circ h\Delta - \Delta \circ a = -R$$

by iterating the solution of

$$a\Delta - \Delta \circ a = -R - \hat{f}' \circ h\Delta$$

Try to carry out the procedure and decide whether it can be applied as an ingredient in a KAM proof. (e.g. one can try to take more stages in the proof as one progresses etc.

To the best of the knowledge of the author it cannot be made to work (unless one uses cancellations similar to those used in the quadratically convergent methods or those of the direct methods) but attempting this will give an appreciation of the cleverness of the use of rapidly convergence methods.

Of course, if there is a proof that succeeds in accomplishing this, the result will be quite interesting.

5. Hard implicit function theorems.

Before proceeding to more geometric considerations, it will be convenient to abstract some of the properties that made the previous argument work and isolate them in an abstract implicit function theorem. This will streamline a good deal of the arguments and illustrate quite strikingly the principle that the quadratic convergence can dominate the small divisors.

Even if implicit function theorems take care very nicely of the analysis of the convergence, they ignore the geometric considerations and particularities of the problem at hand. This particularities are crucial to obtain the general framework of the implicit function theorem. Nevertheless, it is useful to introduce the difficulties one at a time.

Later we will have to spend time making sure that we can fit a problem or an algorithm to solve a problem into the functional framework of a theorem.

We emphasize however that the usefulness of these implicit function theorems is not restricted to KAM theory and they have been used in a variety of problems in geometry, PDE, etc. and that in any case, they are a very useful strategic guide on how to organize the proofs of the problem at hand.

There are different versions of implicit function theorems adapted to work in KAM theory. We just mention [Zeh75], [Ham82], [Hör90].(See also [Hör85]). The main variation we have included is that we have used the approximation functions (introduced seemingly in [Rüs80]) in the implicit function theorem. Some parts of the exposition are based on [dlLV00]. A very good recent exposition – regretfully, not easy to obtain – of Nash-Moser theorems including detailed comparisons and examples of applications, specially to PDE's is [HM94]. Also very important for the relation with PDE's are [AG91], [Hör90]. (Of course, one should also consider the work of [CW93], [Bou99b], even if it has not been formulated as an abstract implicit function theorem and I am not sure it fits easily into the existing ones.)

The theorem that most closely models the problem we have discussed so far (and those that we will discuss later) is that of [Zeh75], which he calls *analytic smoothing* which we now, reproduce, with an small improvement to deal with the Brjuno conditions rather than just the Diophantine conditions.

Note that to abstract the spaces of analytic functions defined on balls of different radius, we will consider not just a single Banach space, but rather a family of Banach spaces. In the following, it will be good to keep in mind the proof of Theorem 4.1 as motivation for the definitions and the assumptions.

Theorem 5.1. *We will consider scales of Banach spaces $\{X_\sigma\}_{\sigma\in[0,1]}$ such that for $0 \le \sigma' \le \sigma \le 1$ we have:*

$$X_0 \supseteq X_{\sigma'} \supseteq X_\sigma \supseteq X_1 \tag{5.1}$$

$$\|x\|_{X_{\sigma'}} \le \|x\|_{X_\sigma} \ , \tag{5.2}$$

and analogously for $\{Y_\sigma\}_{\sigma\in[0,1]}$, $\{Z_\sigma\}_{\sigma\in[0,1]}$.

Assume that we have $F : X_0 \times Y_0 \to Z_0$

1) *$F(f_0, u_0) = 0$ for some $f_0 \in X_1, u_0 \in Y_1$.*
2) *The domain of F contains the sets*

$$\mathcal{B}_\sigma = \{(f,u) \in X_\sigma \times Y_\sigma \bigm| \|f - f_0\|_{X_\sigma} \le A,\ \|u - u_0\|_{X_\sigma} \le B\} \ .$$

3) *$F(\mathcal{B}_\sigma) \subset Z_\sigma$ and it is continuous when the range and the domain are given the natural topologies.*

In what follows, $M \ge 1, \gamma > 0, \alpha \ge 0$ will denote fixed constants.

Assume furthermore:

H1) *F satisfies a so called "Taylor estimate". More precisely:*

H1.1) *The mapping*

$$F(f,\cdot) : Y_\sigma \cap B_\sigma \to Z_{\sigma'}$$

is Frechet differentiable for every $\sigma' < \sigma$.

Denote by $D_2F(f,u)$ the Frechet derivative and

$$Q(f; u, v) \equiv F(f,u) - F(f,v) - D_2F(f,v)(u - v) \tag{5.3}$$

H1.2) *We have the bounds:*

$$\|Q(f;u,v)\|_{\sigma'} \le \Upsilon(\sigma-\sigma')\|u-v\|_\sigma^2 \tag{5.4}$$

where Υ is a decreasing function. (We will assume without loss of generality and to avoid complications in algebraic expressions, that $\Upsilon > 1$.) The function Υ is called an approximation function. *It will also enter in subsequent hypothesis and in (H4) it will be required to satisfy certain conditions.*

H3) *Approximate right inverse We can find an approximate right inverse for the derivative.*

That is we can find a linear operator η that maps Z_σ into $X_{\sigma'}$ for all $\sigma' < \sigma$ and that satisfies:

$$\begin{aligned} &\|\eta(f,u)z\|_{\sigma'} \le \Upsilon(\sigma-\sigma')\|z\|_\sigma \\ &\|D_2F(f,u)\eta(f,u)z - z\|_{\sigma'} \le \Upsilon(\sigma-\sigma')\|F(f,u)\|_\sigma\|z\|_\sigma \end{aligned} \tag{5.5}$$

H4) *The approximation function satisfies the* Brjuno-Rüssmann conditions:

The function Υ in (5.5) *satisfies that there is a sequence $\delta_n > 0$ such that $\sum_n \delta_n = 1/2$, $\sum 2^{-n}|\log(\delta_n/2)| < \infty$ and such*

$$\sum_n 2^{-n}\log(\Upsilon(\delta_n)) < \infty \tag{5.6}$$

Then, there exists a constant C, depending only on M, α and Υ such that if u_0 is an approximate solution. That is:

$$\|F(f,u_0)\|_1 \equiv \varepsilon \tag{5.7}$$

is sufficiently small, then, we can find $u^* \in X_{1/2}$ solving exactly the equation

$$F(f,u^*) = 0$$

Moreover,

$$\|u-u^*\|_{1/2} \le C\|F(f,u_0)\|_1 \tag{5.8}$$

Remark 5.2. The theorem in [Zeh75] included also a hypothesis H2 that allowed one to obtain information on the dependence of the solutions u in terms of f.

We have eliminated the dependence of u on f from the conclusions of the main theorem and relegated it to remarks (see Remark 5.12). Hence, we suppressed H2 from the main theorem, but kept the numbering to allow easy comparisons. On the other hand, the hypothesis H4 here is different from that of [Zeh75], but it plays the same role.

Remark 5.3. There are several equivalent formulations of hypothesis H4). For all practical purposes, it suffices to take δ a fixed exponential sequence. See the exercises.

The proof of this theorem is very simple since we have abstracted away many of the complications of the previous theorem. We will present it and then, we will highlight some of the subtle points and indicate some of the applications.

Remark 5.4. One of the important features of the proof in [Zeh75], which we have eliminated for this pedagogical presentation, is that the final result is expressed in a form which is independent of the space considered. This requires one to assume that $\Upsilon(t) = Ct^{-\alpha}$ for some positive C, α. This case has very important consequences such as the finitely differentiable case. We will develop these improvements in the exercises.

Proof. We use a quasi-Newton method defined by the iterative procedure

$$u_{n+1} = u_n - \eta(f, u_n)F(f, u_n) \tag{5.9}$$

in which η takes the place of the inverse of the derivative in the regular Newton method.

We set $\sigma_{n+1} = \sigma_n - \delta_n$ and $\sigma_0 = 1$. We will obtain recursively estimates of $\|F(f, u_n)\|_{\sigma_n}$ $\|u_n\|_{\sigma_n}$, and of $\|u_n - u_{n+1}\|_{\sigma_{n+1}}$.

Since $\sigma_n \geq 1/2$, the later estimates will imply that u_n converges in $X_{1/2}$.

Adding and subtracting, we have:

$$\begin{aligned} F(f, u_{n+1}) = &F(f, u_{n+1}) - F(f, u_n) - D_2F(f, u_n)\eta(f, u_n)F(f, u_n) \\ &+ F(f, u_n) + D_2F(f, u_n)\eta(f, u_n)F(f, u_n) \ . \end{aligned} \tag{5.10}$$

We can estimate the terms in the second line in (5.10) using the second part of (5.5)

$$\|F(f, u_n) - D_2F(f, u_n)\eta(f, u_n)F(f, u_n)\|_{\sigma_{n+1}} \leq \Upsilon(\delta_n)\|F(f, u)\|^2_{\sigma_n} \ .$$

Using the first part of (5.5), we obtain: for $\tau_n = (\sigma_n + \sigma_{n+1})/2$

$$\|\eta(f, u_n)F(f, u_n)\|_{\tau_n} \leq \Upsilon(\delta_n/2)\|F(f, u_n)\|_{\sigma_n} \ . \tag{5.11}$$

This estimate allows us to apply (5.4) to the terms in the first line of (5.10).

Hence, we obtain (bounding $\Upsilon(\delta_n/2) > \Upsilon(\delta_n)$)

$$\|F(f, u_{n+1})\|_{\sigma_{n+1}} \leq 2\Upsilon(\delta_n/2)^2\|F(f, u_n)\|^2_{\sigma_n} \ . \tag{5.12}$$

If we iterate (5.12), we obtain

$$\begin{aligned} \|F(f, u_{n+1})\|_{\sigma_{n+1}} \leq &2\Upsilon(\delta_n/2)^2 \times (2\Upsilon(\delta_{n-1}/2)^2)^2 \times \cdots \times \\ &\times (2\Upsilon(\delta_0/2)^2)^{2^n}\|F(f, u_0)\|^{2^{n+1}}_{\sigma_0} \\ = &2^{1+2+\cdots+2^n}\Upsilon(\delta_n/2)^2 \times \Upsilon((1/2)\delta_{n-1})^{2^2} \times \cdots \times \\ &\times \Upsilon((1/2)\delta_0)^{2^{n+1}}\|F(f, u_0)\|^{2^{n+1}}_{\sigma_0} \ . \end{aligned} \tag{5.13}$$

We can estimate the logarithm of the factor of $\|F(f, u_0)\|^{2^{n+1}}_{\sigma_0}$ in the R.H.S. of (5.13) by:

$$\begin{aligned} 2^{n+1}\big[\log(2) + \log\Upsilon((1/2)\delta_n)2^{-n} + \cdots + \log\Upsilon((1/2)\delta_{n-1})2^{-(n-1)} \\ + \cdots + \log\Upsilon((1/2)\delta_0)2^0\big] \ . \end{aligned}$$

We see that under our assumption H4) (see (5.6)), the term in braces can be bounded by a constant (the sum of the series). Hence (5.13) yields

$$\|F(f, u_{n+1})\|_{\sigma_{n+1}} \leq (A\|F(f, u_0)\|_{\sigma_0})^{2^{n+1}} \tag{5.14}$$

where A is a constant depending on the properties of the approximation function and the other constants involved in the set up of the problem.

We see that, if we $\|F(f, u_0)\|_{\sigma_0}$ is sufficiently small, the right hand side of (5.14) converges to zero extremely fast.

Using (5.11), we have:

$$\|u_n - u_{n+1}\|_{\sigma_{n+1}} \leq \Upsilon(\delta_n/2)(A\|F(f, u_0)\|_{\sigma_0})^{2^n} \tag{5.15}$$

where A is also a constant depending only on the properties of the approximation function and the other constants involved in the set up of the problem. (It will be different from the A in (5.14), but we follow the standard practice of denoting all such constants by the same letter.)

The R.H.S. of (5.15) is a convergent series because by our assumption (5.6) the general term of the series is bounded. Therefore, $\log \Upsilon(\delta_n/2)2^{-n} \leq B$ (where, again, B is another constant depending only on the constants of the problem and the approximation function).

When $A\|F(f, u_0)\|_{\sigma_0} < 1$, the second factor converges to zero faster than any exponential.

Note also that, if $\|F(f, u_0)\|_{\sigma_0}$ small enough, the series obtained summing (5.15) has a sum as small as desired. In particular, we can verify that the limit is close to u_0 in $X_{1/2}$.

Hence:

$$\Upsilon(1/2\delta_n)(A\|F(f, u_0)\|_{\sigma_0})^{2^n} \leq (Ae^B\|F(f, u_0)\|_{\sigma_0})^{2^n}$$

This establishes the claim. □

Exercise 5.5. Many classical proofs of the classical implicit function theorem are based not in the Newton method, which is quadratically convergent, but rather in a contraction mapping principle (which is called linearly convergence since the remainder after one step is only a fixed factor smaller than the remainder before the step.)

Can one base a method that beats small denominators on a linearly convergent procedure?

Similarly, one can get algorithms whose convergence is faster than quadratic. (For example, solving the equation given by the second order Taylor expansion or interpolating several of the previous steps of the algorithm.) Can one base a hard implicit function theorem on these algorithms?

It is interesting to check how the previous result compares with the proof we have presented of Theorem 4.1. The scales of spaces are just spaces of analytic functions on balls of different radii. The approximate inverse corresponds to the solving of the linearized equation by comparing it with the equation obtained by taking derivatives of the remainder. Checking that the scales map into each other is roughly the same as our inductive hypothesis.

In the presentation of Theorem 4.1, we have, of course taken

$$\Upsilon(\delta) = M\delta^{-\tau} \tag{5.16}$$

This is a very important particular case of the whole theorem since it not only appears in interesting situations but also leads to further consequences which we will discuss in the following remarks. (Of course, the reader should also consult [Zeh75] and the other references.)

The choice of a general Υ satisfying (5.6) corresponds to the small divisors satisfying (2.34), whereas (5.16) corresponds to Diophantine conditions. (For more details see [Rüs90], [DeL97], [dlLV00].)

Remark 5.6. The existence of approximate inverses is a general feature of conjugacy problems or of problems having a group structure.

As pointed out in [Zeh75] p. 133 ff. existence of approximate inverses assuming only the existence of an inverse in the trivial case is a general feature of conjugacy problems, at least at the heuristic level. This indeed gives a guiding principle for the cancellations that we found e.g. in the proof of Theorem 4.1 in which we used that comparing the prescription suggested by the heuristic Newton method with the derivative of the remainder the linearized equation suggested by the heuristic Newton method can be reduced to constant coefficients up to quadratically small errors.

Notice that the functionals we are solving are conjugacy equations. Hence they satisfy the identity

$$F(f, u \circ v) = F(F(f, u), v) \tag{5.17}$$

If we take $v = \mathrm{Id} + \hat{v}$ and we think of $\hat{v}$ as infinitesimal, we obtain

$$D_2F(f, u)u'\hat{v} = D_2F(F(f, u), \mathrm{Id})\hat{v} \tag{5.18}$$

If we assume that $\eta D_2(f_0, \mathrm{Id}) = \mathrm{Id}$, we obtain that:

$$\eta D_2F(f, u)u'\hat{v} = \mathrm{Id} + \eta[D_2F(F(f, u), \mathrm{Id}) - D_2F(f_0, \mathrm{Id})] \tag{5.19}$$

Notice that we can expect that, if D_2F satisfies some Lipschitz conditions on the first argument, the term in braces in the R.H.S. of (5.19) satisfies the bounds we wanted for an approximate inverse provided that η satisfies the desired bounds.

The importance of this remark is that by knowing the existence of η, which is just an inverse of $D_2F(f_0, \mathrm{Id})$ we can deduce, for functionals with a group structure, the existence of approximate inverse in a whole neighborhood, which the hypothesis needed by Theorem 5.1.

Of course, F satisfies assumption (5.17) when $F(f, u) = u^{-1} \circ f \circ u$ but it could also be the action by u on vector fields or more complicated objects and indeed it happens quite frequently when one is considering geometrical problems.

Remark 5.7. strategy for KAM (Discussed in more detail in Section 6.1) can be formulated as reducing the Hamiltonian to a Hamiltonian of a particular kind.

Hence, we are not interested in just solving the equation $F(f, u) = 0$ but rather $F(f, u) = N$, where N is a submanifold of infinite codimension. Indeed, this is the problem that is considered in [Zeh75] and especially in [Zeh76a].

Remark 5.8. Even if most of the classical KAM problems (certainly all that will be discussed in this notes) are conjugacy problems and, therefore, have the group structure, this is not completely necessary to have a quadratic algorithm not completely necessary to have a quadratic scheme.

A review of problems in geometry which are not conjugacy problems can be found in [Ham82].

A very interesting recent development is the observation that variational problems with symmetry also present another general structure that allows to obtain quadratic convergence. See, for example [Koz83], [Mos88] for PDE's or, in the context of KAM [SZ89]. (We will present an account of that work in Section 6.3.)

Much more interesting is the fact that in [CW93], [CW94], another mechanism to obtain quadratic convergence was introduced. At the moment, I do not know of a functional analytic framework that encompasses these remarkable results.

Remark 5.9. In the applications of the implicit function theorems to problems of persistence of tori – and to some geometric problems – we are not interested in the equation $F(f, u) = 0$ but rather in the equation $F(f, u) \in N$ where N is an appropriate submanifold.

See exercises 5.22, 5.23

Remark 5.10. Note that the structure of Theorem 5.1 is that the input is just an approximate solution (with some extra mild requirements) and that the output is an exact solution not too far from the original approximate solution.

In the most commonly quoted applications, the input is the exact solution for an integrable system, which is an approximate solution for a quasi-integrable system. Nevertheless, other applications are possible. Among them, we mention:

1) Numerical algorithms:

If carefully implemented and successfully, numerical algorithms produce approximate solutions (i.e. ssomething that, when plugged into the equations satisfies them approximately).

Hence, using a theorem with the structure of Theorem 5.1, one can justify that the approximate solutions produced by a computer algorithm indeed correspond to a true solution nearby. In numerical analysis, this is sometimes called a-posteriori bounds. (See [BZ82], [dlLR91], [CC95].) We discuss some numerical issues involved in Section 7.

2) Justification of asymptotic expansions (e.g. LLindstedt series)

These expansions produce objects that satisfy the equations approximately. Hence, a theorem similar to Theorem 5.1, can be used to justify asymptotic expansions. That is, show that one can indeed find tori which are not far away from the truncations of the Lindstedt series. For the KAM tori, one can find this type of arguments in [Mos67]. In these case, it is also shown that the Lindstedt series converge (since the torus should be analytic as a function of the parameter). We emphasize that to justify the asymptotic nature of the series one just needs that the series produce objects that satisfy the equation with smaller errors and that are not too complicated. The Lindstedt series of lower dimensional tori are studied by this method in [JdlLZ99]. In that case, we do not know whether the series converges or not, but following the argument sketched here, it is possible to show that they are asymptotic in a certain complex domain.

3) Establishing continuity or Whitney regularity of the solutions with respect to parameters – assuming that F is more regular in both its arguments –.

This application is worked out explicitly in [dlLV00]. The latter arguments require some certain amount of uniqueness, which is not provided by the theorem in the way we have stated and proved it, but which we obtain in Exercise 5.21.

4) Obtaining a result for finitely differentiable problems out of the analytic ones.

An application that can already be found in [Mos66b], [Mos66a] is that, as we saw in Lemma 3.2, we can characterize finitely differentiable functions by their approximation properties by analytic functions. We just sketch the argument.

Given a smooth f, we study the problem $F(f, u)$ by considering a sequence of problems $F(f_n, u_n) = 0$ where f_n are constructed approximating the smooth function f by analytic functions.

Using that $||f_n - f_{n+1}||_{2^{-(n+1)}} \le C2^{-(n+1)r}$ it is often possible (using the structure of F) to show that

$$||F(f_n, u_n) - F(f_{n+1}, u_n)||_{2^{-(n+1)}} = ||F(f_{n+1}, u_n)||_{2^{-(n+1)}} \le C2^{-(n+1)r'}$$

We consider u_n as an approximate solution for the problem with f_{n+1}. In the case that Υ is a power, it follows that

$$||u_n - u_{n+1}||_{2^{-(n+2)}} \le C2^{-(n+1)r''}$$

from which, appealing again to Lemma 3.2 we obtain that there $u = \lim u_n$ which solves the desired equation and which is analytic.

This method has the advantage that one always works with analytic functions for which estimates are often easier and, as we have seen sharper if one needs to use Fourier coefficients.

We refer to [Mos66b], [Mos66a], [Zeh75], [Zeh76a] for more details (such as how to get the induction started), somewhat different versions of the argument, and applications to concrete problems.

The quantitative estimates needed to carry out this strategy are explained in Exercise5.18.

5) Bootstrapping the regularity.

A solution which is moderately smooth, if approximated by an analytic one is an analytic approximate solution.

Applying a theorem of this sort, one can conclude that given an analytic problem, if there is a sufficiently smooth solution (so that the smoothings are indeed very good approximations), then there is an analytic one. Of course, if one does have uniqueness of the problem, one obtains that any solution that has a certain regularity, is analytic.

Of course, if we start with a problem that is very regular, we can also show that given a solution which is beyond a certain critical regularity, there will be another one which is as as the problem allows, and if there is uniqueness, we conclude that all the solutions beyond a certain regularity are as smooth as the problem allows.

Arguments of this type are worked out explicitly in [SZ89]. Again, we refer to Exercise5.18 for some of the quantitative estimates needed.

Remark 5.11. Notice that the Theorem 5.1 only assumes the existence of an approximate right inverse.

One should not expect that the solution one produces in the theorem to be unique. Indeed, in some problems such as the Nash embedding theorem which motivated a good deal of the original research one only has an approximate right inverse and, indeed the solution is not unique. In many geometric problems, the results we seek are in any case invariant under diffeomorphisms, so that it is to be expected that the solution is not unique.

Under moderate assumptions – e.g. under the existence of an approximate left inverse – one gets uniqueness. See the remarks in [Zeh75] and see Exercise 5.21. These assumptions are often satisfied in KAM theory or uniqueness of the objects we are interested in can be obtained by other means. (Often one seeks geometrical objects in coordinate systems, so that the geometric objects may be unique even if their coordinate representation is not.)

One situation when these considerations play a role is the proof of the KAM theorem following Kolmogorov's strategy. (See Section 6.1.)

In this method, we seek a change of variables in which the resulting system manifestly has an invariant torus. That is, we try to reduce the system to the the Kolmogorov normal form (6.6). Such change of variables is manifestly not unique since the normal form does not specify what are the higher order terms and one can make changes of variables that only depend to higher order in the actions. Therefore, one cannot expect uniqueness in the change of variables nor in the term of the normal form and a formulation of the theorem based on this formalism cannot aspire to obtain uniqueness.

Nevertheless, it is true that, under moderate non-degeneracy assumptions, the torus that has a prescribed frequency is unique.

Remark 5.12. The theorem of [Zeh75] has an extra hypothesis H2 that requires that F is Lipschitz in the first argument Then, one obtains Lipschitz dependence on the solution on the function f (in some appropriate spaces).

We note that, in the case that there is no uniqueness, the only claim made is that the algorithm (5.9) leads to a solution that depends in a Lipschitz manner on f. Clearly, when there is no uniqueness one could make different choices of solutions for different f and end with a u that depends discontinuously on f.

A detailed treatment of these ideas can be found in [Zeh76b].

We point out that there are other methods to obtain smooth dependence with respect to parameters that do not involve following the proof of Theorem 5.1 and checking the differentiability with respect to parameters of all the steps.

1) One can also obtain quickly higher regularity with respect to parameters by applying Theorem 5.1 in spaces that consists of smooth families of functions. Of course, one needs that the approximate inverse also maps smooth families into smooth families. This is somewhat tricky since approximate inverses are not uniquely defined, so one could make different choices for different values of the parameter and spoil even continuity.

Nevertheless, for problems with group structure, the prescription given by (5.18) gives a way of accomplishing the solution in spaces of smooth families of functions. Arguments of this sort are carried out in detail in [dlLO00] to solve a problem in differential geometry.

2) When there is uniqueness, one can follow other sort of arguments such as finding formal derivatives for the solution and then, showing that these formal derivatives satisfy the hypothesis of Whitney theorem [dlLV00].

Remark 5.13. When one has some regularity – at least Lipschitz – with respect to the parameters, one can start discussing issues – important in the applications – such as the measures in the space of parameter covered.

Exercise 5.14. Write precisely the reduction of Theorem 4.1 to Theorem 5.1 by making explicit choices of spaces, etc.

Exercise 5.15. A challenging variant of the previous exercise is to show that, if the number ω satisfies the conditions (2.34), the approximate inverse we constructed in the proof of Theorem 4.1 satisfies (5.6).

If independent study fails, see [Rüs90], [DeL97] for estimates that go from arithmetic conditions to approximation functions.

Remark 5.16. In practical applications, e.g. when one is computing numerically solutions to a problem defined implicitly one of course, does not compute the inverse of the matrix, but rather solves numerically the system.

In numerical practice, this usually entails a factorization of the matrix. Traditionally, one uses the LU factorization (Gaussian elimination), even if in KAM theorems that tend to be ill conditioned one should, perhaps, prefer the SVD decomposition.

In any case, it is convenient not to have to recompute these factorizations – which may much more costly than the application to a function –. Of course, we would not like to lose the quadratic convergence which, e.g. in continuation methods that require great precision is much more practical that a method that converges more slowly.

The following two schemes, which avoid having to recompute factorizations but which get convergence faster than linear are studied in [Mos73] p. 151. The second one comes from [Hal75]. A geometric interpretation of these methods as a Newton method in the space of jets is discussed in [McG90].

$$\begin{aligned} u_{n+1} &= u_n - \eta_n F(f, u_n) \\ \eta_{n+1} &= \eta_n - \eta_n(\mathrm{Id} - D_2 F(f, u_n))\eta_n \end{aligned} \tag{5.20}$$

$$\begin{aligned} u_{n+1} &= u_n - \eta_n F(f, u_n) \\ \eta_{n+1} &= \eta_n - \eta_n(\mathrm{Id} - D_2 F(f, u_{n+1}))\eta_n \end{aligned} \tag{5.21}$$

(In numerical applications, one does not compute the product of matrices in (5.20), (5.21). Note that it suffices to apply the matrices to vectors.)

Exercise 5.17. Show in finite dimensions that, under smoothness assumptions and smallness assumptions: (5.20) leads to

$$\|F(f, u_{n+1})\| \le C|F(f, u_n)|^{(\sqrt{5}+1)/2}$$

and (5.21) leads to

$$\|F(f, u_{n+1})\| \le C|F(f, u_n)|^2.$$

Applications of these schemes to hard implicit function theorems and other modifications of the basic algorithm will be developed in the following exercises.

The following exercises are designed to show that the quadratic convergence is rather forgiving and that there are many variants that also work. We have also included some variants in which the results fail so that the reader can start to develop a feeling for the range of applicability of the techniques.

Exercise 5.18. Consider the following improvements to Theorem 5.1 (either separately or several at the same time, for the most ambitious reader).

- Modify the hypothesis and the conclusions so that the approximate solution is assumed to satisfy

$$\|F(f, u_0)\|_{\delta_0} \equiv \varepsilon$$

instead of (5.7) and the conclusion about u^* reads

$$\|u - u^*\|_{\delta_0/2} \le C(\delta_0)\|F(f, u_0)\|_{\delta_0} \tag{5.22}$$

instead of (5.8).

Hint: This result can be deduced from the statement of the theorem just by a relabeling of the spaces.

- Show that in case that we take $\Upsilon(t) = Ct^{-\alpha}$ for some $C, \alpha > 0$, we have $C(\delta_0) = C' t^{-\alpha'}$ for some $C', \alpha' > 0$.

  The previous two items are quite important since they allow to obtain finite differentiability out of the analytic result. The strategy to obtain that is explained in Remark5.10.

  They are worked out in [Zeh75]. It can also be worked out from the statement that we have given by a rescaling argument.
- Consider that in (5.4), (5.5) we have three different Υ functions. For example, three different powers.

  (This appears in practice. Some of the powers come from the Diophantine approximations whereas others come from the differentiation of composition and the like.)
- Modify the second equation of (5.5) to read

$$\|dF(f,u)\eta(f,u)z - z\|_{\sigma'} \le \Upsilon(\sigma - \sigma')\|F(f,u)\|_\sigma^{\kappa'}\|z\|_\sigma$$

  for some $\kappa' > 0$.
- Modify (5.4) to read

$$\|Q(f;u,v)\|_{\sigma'} \le \Upsilon(\sigma - \sigma')\|u - v\|_\sigma^{1+\kappa} \tag{5.23}$$

  for some $\kappa > 0$.
- One can also have a different approximate inverse during the iteration.

$$\begin{aligned}\|dF(f,u_n)\eta_n(f,u_n)z - z\|_{\sigma'} \le &\Upsilon(\sigma - \sigma')\|F(f,u_n)\|_\sigma\|z\|_\sigma \\ &+ \exp(-a(1+\kappa'')^n)\end{aligned}$$

  for some $\kappa'' > 0$.

  A variant is to choose

$$\begin{aligned}\|dF(f,u_n)\eta_n(f,u_n)z - z\|_{\sigma'} \le &\Upsilon(\sigma - \sigma')\|F(f,u_n)\|_\sigma\|z\|_\sigma \\ &+ \exp(-4^n(\sigma - \sigma'))\end{aligned} \tag{5.24}$$

  This appears in some proofs (e.g. in Arnol'd type proofs) when one tries to do some truncation of the problem. This improvement is not too tricky to do by itself, but it is not so easy to understand how it does work with the others. It is quite enlightening to understand how it works with the method of obtaining finite differentiability.

  with some of the others.

Under these modifications, one has to modify slightly the conditions (5.6).

Exercise 5.19. Formulate precisely the assumptions of domain loss etc. to obtain a proof of the implicit function theorem using an iteration as in (5.21).

Exercise 5.20. Taking into account the improvement suggested in (5.23) give a proof of the theorem using the scheme of (5.20).

Exercise 5.21. Show that if one supplements the assumption H3 of Theorem 5.1 with the existence of a left approximate inverse satisfying the same estimates, one obtains that the solution is unique in an appropriate sense.

Formulate a precise theorem in which the domains in which uniqueness holds are explicitly specified.

(Some version of this is done in [Zeh75].)

Exercise 5.22. State and prove an implicit function theorem in which we do not attempt to solve $F(f, u) = 0$ but rather $F(f, u) \in N$ as explained in Remark 5.9.

In that generality, one should not expect uniqueness, hence, continuity and differentiability with respect to parameters is presumably not very clean.

Exercise 5.23. When considering the normal form problem one should also modify the assumption H3 of Theorem to be: 5.1

$$\|dF(f, u)\eta(f, u)z - z\|_{\sigma'} \leq \Upsilon(\sigma - \sigma')d_\sigma(N, F(f, u))\|z\|_\sigma$$

where d_σ denotes the distance between sets measured with the norm $\|\cdot\|_\sigma$.

This observation appears in [Mor82].

Exercise 5.24. A classical theorem in KAM theory is the theorem of [Arn61] which states that given a diffeomorphism of the circle with a rotation number ρ, which is Diophantine and sufficiently close to the rotation by ρ in an analytic topology, then, there is an analytic change of variables that transforms it in the rotation by ρ.

Formulate it in terms of an abstract implicit function theorem.

The main difficulty is that, when we start proving this theorem, we do not know that the set diffeomorphisms with rotation number ρ is a manifold. (We know it after we prove the theorem!.)

Note also that the conjugacy is not unique since all rotations conjugate a rotation to itself.

I know several ways to do it, but all of them require some dirty tricks. (A good source for those – and for almost anything having to do with circle maps – is [Her79] and [Her83]).

Remark 5.25. Note that, the estimates we have made to prove Theorem 5.1 do not use that $\|\cdot\|_\sigma$ is a norm. They would have worked just as well if $\|\cdot\|_\sigma$ had been a semi-norm.

Of course, in order that the result is meaningful, we would need that the family of seminorms $\{\|\cdot\|_\sigma\}_{\sigma\in(0,1/2)}$ defines a useful space, i.e., they define a Fréchet space. See [Ham82] for more details about such improvement and also for applications.

The original proof of KAM theorems for finite differentiability were based on different schemes than the proof we have presented.

Note that, for example, the proof of Theorem 4.13 follows a different scheme. At every step, the linear operator we have to solve does have an inverse (not just an approximate inverse). The problem is that the operator is unbounded and, hence, simple-minded iterations such as those of the classical Newton method do not work.

This situation happens also in PDE's. A notable example was the celebrated Nash embedding theorem [Nas56].

The method used in [Mos66b] and [Mos66a] was to combine steps of the linearized operator with smoothings. The method allows a norm – in a space of somewhat smooth functions – to blow up, whereas a norm – in a space of rougher functions – decreases. By using interpolation inequalities, one can recover good behavior of some intermediate norms. (The decrease may not be exactly quadratic, but it is still is faster than exponential.) This technique has been highly formalized in [Ham82], which also includes a wealth of applications, mainly to geometry. See [Ham82], Section 3, for a comparison with the methods of Zehnder [Zeh75].

In the following, we present a proof along these lines, which follows rather closely [Sch69]. This book also contains a very nice discussion of the Nash embedding theorem and on other problems of nonlinear functional analysis.

In the sequel, we shall refer to a certain range $m-\alpha \le r \le m+10\alpha$ of spaces Λ_r (defined in Section 3.1), and to a certain constant $M > 1$. We suppose that M is sufficiently large so that the smoothing operators S_t satisfy

$$
\begin{aligned}
\|S_t u\|_\rho &\le M t^{\rho - r}\|u\|_r \quad u \in \Lambda_r \\
\|(\mathrm{Id} - S_t)u\|_r &\le M t^{r-\rho}\|u\|_\rho \quad u \in \Lambda_\rho
\end{aligned}
\tag{5.25}
$$

for $m-\alpha \le r \le \rho \le m+10\alpha$ ($\|\cdot\|_r$ stands for the norm in Λ_r).

Theorem 5.26. *Let B_m be the unit ball in Λ_m and $f : B_m \to \Lambda_{m-\alpha}$ be a map that satisfies:*

(i) *$f(B_m \cap \Lambda_r) \subset \Lambda_{r-\alpha}$, for $m \le r \le m+10\alpha$;*
(ii) *$f_{|B_m\cap\Lambda_r} : B_m \cap \Lambda_r \to \Lambda_{r-\alpha}$ has two continuous Fréchet derivatives, both bounded by M, for $m \le r \le m+10\alpha$;*
(ii) *There exists a map $L : B_m \to \mathcal{B}(\Lambda_m, \Lambda_{m-\alpha})$, where $\mathcal{B}(\Lambda_m, \Lambda_{m-\alpha})$ is the space of bounded linear operators on Λ_m to $\Lambda_{m-\alpha}$ such that:*
  (ii.a) *$\|L(u)h\|_{m-\alpha} \le M\|h\|_m, \quad u \in B_m, h \in \Lambda_m$;*
  (ii.b) *$df(u)L(u)h = h, \quad u \in B_m, h \in \Lambda_{m+\alpha}$;*
  (ii.c) *$\|L(u)f(u)\|_{m+9\alpha} \le M(1 + \|u\|_{m+10\alpha}), \quad u \in B_m \cap \Lambda_{m+10\alpha}$.*

Then, if $E := \|f(0)\|_{m+9\alpha}$ is sufficiently small, there exists $u \in \Lambda_m$ such that $f(u) = 0$.

Proof. Let $\kappa > 1$, $\beta, \mu, \nu > 0$ be real numbers to be specified later. We will need that they satisfy a finite set of inequalities relating them and the constants appearing in the assumptions of the problem.

We construct a sequence $\{u_n\}_{n\ge1} \subset \Lambda_m$ by taking $u_0 = 0$ and

$$u_{n+1} = u_n - S_n L(u_n) f(u_n)$$

where $S_n = S_{t_n}$ and $t_n = e^{\beta\kappa^n}$. Later on, we will prove that this sequence satisfies, for $n \ge 1$:

(p1;n) $u_{n-1} \in B_m$,
(p2;n) $\|u_n - u_{n-1}\|_m \le e^{-\mu\alpha\beta\kappa^n}$,
(p3;n) $u_n \in \Lambda_{m+10\alpha}$ and $1 + \|u_n\|_{m+10\alpha} \le e^{\nu\alpha\beta\kappa^n}$.

Notice that, then, $\{u_n\}_{n\ge1} \subset \Lambda_m$ converges to some $u \in \Lambda_m$ and, moreover:

$$\begin{aligned} \|f(u_n)\|_{m-\alpha} =& \|df(u_n)(u_{n+1} - u_n) - df(u_n)(\mathrm{Id} - S_n)L(u_n)f(u_n)\|_{m-\alpha} \\ \le& M\|(u_{n+1} - u_n)\|_m + M^2 t_n^{-9\alpha}\|L(u_n)f(u_n)\|_{m+9\alpha} \\ \le& Me^{-\mu\alpha\beta\kappa^{n+1}} + M^2 e^{(\nu-9)\alpha\beta\kappa^n}. \end{aligned} \tag{5.26}$$

Hence, the R.H.S. of the previous inequality (5.26) converges to zero when n goes to infinity, provided that

$$\nu < 9. \tag{5.27}$$

We are going two prove by induction the three properties satisfied by the sequence $\{u_n\}_{n\ge1}$. For $n = 1$, condition(p1;1) is trivial.

Condition (p2;1) reads

$$\begin{aligned} \|u_1 - u_0\|_m =& \|S_0 L(0)f(0)\|_m \le Mt_0^\alpha \|L(0)f(0)\|_{m-\alpha} \\ \le& M^2 e^{\alpha\beta} E \\ \le& e^{-\mu\alpha\beta\kappa}, \end{aligned} \tag{5.28}$$

where the last inequality holds if

$$E \le M^{-2} e^{-(1+\mu\kappa)\alpha\beta}. \tag{5.29}$$

Condition (p3;1) reads

$$\begin{aligned} 1 + \|u_1\|_{m+10\alpha} =& 1 + \|S_0 L(0)f(0)\|_{m+10\alpha} \le 1 + Mt_0^\alpha \|L(0)f(0)\|_{m+9\alpha} \\ \le& 1 + M^2 e^{\alpha\beta} \le 2M^2 e^{\alpha\beta} \\ \le& e^{\nu\alpha\beta\kappa}, \end{aligned}$$

where the last inequality holds if

$$1 \le \frac{1}{2} e^{\alpha\beta(\nu\kappa-1} M^{-2}, \tag{5.30}$$

that is, if $\nu\kappa > 1$ and β is sufficiently large.

Suppose now that conditions (p1;j),(p2;j) and (p3;j) are true for $j \le n$. Then,

$$\|u_n\|_m \le \sum_{j=1}^{\infty} e^{-\mu\alpha\beta\kappa^j} \le \sum_{j=1}^{\infty} e^{-\mu\alpha\beta(\kappa-1)j} = \frac{e^{-\mu\alpha\beta(\kappa-1)}}{1 - e^{\mu\alpha\beta(\kappa-1)j}} < 1, \tag{5.31}$$

If we require that

$$\frac{e^{-\mu\alpha\beta(\kappa-1)}}{1 - e^{\mu\alpha\beta(\kappa-1)j}} < 1 \tag{5.32}$$

which holds when

$$\mu\beta >> 1$$

we obtain that the R.H.S. of (5.31) is bounded from above by 1 and, therefore, we recover (p1:n+1).

To prove (p3;n+1) note that

$$\begin{aligned} 1+\|u_{n+1}\|_{m+10\alpha} &\le 1+\sum_{j=0}^{n}\|S_j L(u_j) f(u_j)\|_{m+10\alpha} \\ &\le 1+M^2\sum_{j=0}^{n} e^{(1+\nu)\alpha\beta\kappa^j} \ . \end{aligned}$$

Hence,

$$\begin{aligned} (1+\|u_{n+1}\|_{m+10\alpha})e^{-nu\alpha\beta\kappa^{n+1}} &\le e^{-nu\alpha\beta\kappa^{n+1}} + M^2\sum_{j=0}^{n} e^{(1+\nu-\nu\kappa)\alpha\beta\kappa^j} \\ &\le 1 \ , \end{aligned}$$

where the last inequality holds (and so (p3;n+1)) if $\nu > \frac{1}{\kappa-1}$ and β is sufficiently large.

Finally, we come to the proof of (p2;n+1). We have:

$$\begin{aligned} \|(u_{n+1}-u_n)\|_m =& \|S_n L(u_n) f(u_n)\|_m \le M^2 e^{\alpha\beta\kappa^n}\|f(u_n)\|_m \\ \le& M^2 e^{\alpha\beta\kappa^n}(\|f(u_{n-1}) - df(u_{n-1})S_{n-1}L(u_{n-1})f(u_{n-1})\|_m \\ &\quad + M\|(u_n-u_{n-1})\|_m^2) \\ \le& M^5(e^{(\nu-9+\kappa)\alpha\beta\kappa^{n-1}} + e^{(1-2\mu)\alpha\beta\kappa^n}) \end{aligned}$$

Therefore, if

$$M^5(e^{(\nu-9+\kappa)\alpha\beta\kappa^{n-1}} + e^{(1-2\mu)\alpha\beta\kappa^n})e^{-\mu\alpha\beta\kappa^{n+1}} \tag{5.33}$$

we recover (p2;n+1)).

The condition (5.33) is true when $\kappa < 2$, $\mu > \frac{1}{2-\kappa}$, $\nu > 9-\kappa-\mu\kappa^2$ and β is sufficiently large.

Therefore, we have established that, when the parameters μ, ν, κ satisfy (5.27), (5.29), (5.30), (5.32), (5.33) then we can carry out the induction and establish the theorem.

This is satisfied if we take $1 < \kappa < 2$, $\mu > \frac{1}{2-\kappa}$ and $\frac{1}{\kappa} < \frac{1}{\kappa-1} < \nu < 9-\kappa-\mu\kappa^2 < 9$. For instance, $\kappa = \frac{3}{2}$, $\mu = \frac{20}{9}$ and $\nu = \frac{9}{4}$) then, choose β sufficiently large). $\square$

Remark 5.27. The above methods of proof can also produce results for C^∞ functions. This is significantly more complicated than the ideas used so far and we will not discuss them.

Remark 5.28. In many applications the embeddings of scales of spaces considered are not just continuous but also compact. This allows one to improve several of the steps. See [Hör90] which also includes very nice ideas on how to use paradifferential calculus and several interesting new ideas to obtain very sharp results on the differentiability.

6. Persistence of invariant tori for quasi-integrable systems.

In this section, we will present several proofs of the theorem that made KAM theory famous. This theorem is very useful in mechanics and in ergodic theory.

Basically the theorem says that an integrable system which is not degenerate (See below for a precise definitions) and sufficiently differentiable has the property that many of the quasi-periodic orbits persist under small perturbations.

The theorem has versions for Hamiltonian flows and for exact symplectic maps.

The simple minded versions that we will discuss can be stated as follows:

Theorem 6.1. *Consider the symplectic manifold $M = \mathbf{R}^n \times \mathbf{T}^n$ endowed with the canonical symplectic form.*

Let $H : M \to \mathbf{R}$ be an analytic function such that:

$$H(I, \phi) = h(I) + R(I, \phi) \tag{6.1}$$

Let $\omega \in \mathbf{R}^n$ satisfy (3.18)*, and $\omega = \nabla h(I_0)$ for some I_0.*

Assume that for I in a neighborhood of I_0 we have:

$$\left| \det \frac{\partial^2}{\partial I_i \partial I_j} h(I) \right| \geq \kappa > 0. \tag{6.2}$$

Then, if $\|R\|_\sigma$ is sufficiently small, the Hamilton equations for (6.1) *admit a quasiperiodic solution of frequency ω.*

This solution lies on an analytic torus $\mathcal{T}$, which it fills densely. Moreover, if $\|R\|_\sigma$ is sufficiently small $\mathcal{T}$ is arbitrarily close to the torus $\{I_0\} \times \mathbf{T}^n$

The version for exact symplectic maps reads as follows:

Theorem 6.2. *Consider the symplectic manifold $M = \mathbf{R}^n \times \mathbf{T}^n$ endowed with the canonical symplectic form.*

Consider the map $F_0 : M \to M$ given by:

$$F_0(I, \phi) = (I, \phi + \Delta(I)) \tag{6.3}$$

where $\Delta : \mathbf{R}^n \to \mathbf{R}^n$ is an analytic function,

$$\Delta_j(I) = \frac{\partial}{\partial I_j} \Phi(I) \tag{6.4}$$

Assume that $\omega \in \mathbf{R}^n$ satisfies (3.19) *and*

$$\omega = \Delta(I_0)$$

for some I_0 and that

$$\left| \det \frac{\partial}{\partial I_j} \Delta_i(I) \right| \geq \kappa > 0 \tag{6.5}$$

in a neighborhood of I_0.

Let $F : M \to M$ be an analytic, exact symplectic map.

If $\|F - F_0\|_\sigma$ is sufficiently small, then the map F admits a quasiperiodic orbit of frequency ω. This orbit is dense in an analytic torus which (if $\|F - F_0\|_\sigma$ is sufficiently small) is arbitrarily close in the analytic topology to the torus $\{I_0\} \times \mathbf{T}^n$ which is filled densely by the orbit of frequency ω of F_0.

Remark 6.3. The condition (6.4) is imposed so that the unperturbed map is exact symplectic.

An obvious consequence of (6.4) is that the matrix in (6.5) is symmetric.

In what follows we will indicate several proofs of the above theorems.

The ideas and techniques of the proofs in both cases are roughly the same. Moreover, one can pass from one to the other by an ingenious construction [Dou82b], so that they are indeed equivalent in a precise sense.

Since proofs of these theorems have been in the literature for several decades, and many of the estimates have been covered in the previous sections, we will leave many of the details to the reader, indicating the most interesting ones as exercises.

Of course, the theorems, as stated above are quite far from the state of the art, but we hope that they still contain enough difficulty to illustrate the techniques of the theory and to fulfill the pedagogical goal of these notes. We will also present references to places in the literature where more elaborate arguments, which we will try to sketch, lead to sharper results.

6.1. **Kolmogorov's method.** The original paper has been translated in [Kol54]. A translation of a much less detailed account can be found in an appendix of [AM78]. Very good modern implementations of the method can be found in [BGGS84], [Bar70]. A generalized discussion of the ideas, putting them in a much wider context can be found in [Mos67], [Zeh76a]. (See Remark 6.6.)

We observe that a Hamiltonian system of the form [7]e use the notation $O(I^2)$ to denote functions $A(I, \phi)$ such that $A(0, \phi) = 0$, $\frac{\partial}{\partial \phi} A(0, \phi) = 0$ and similarly for other orders

$$H(I, \phi) = \omega I + O(I^2) \tag{6.6}$$

has Hamiltonian equations of motion

$$\begin{aligned} \dot{\phi} &= \omega + O(I) \\ \dot{I} &= O(I^2) \end{aligned}$$

Hence $\phi = \phi_0 + \omega t$, $I = 0$ is a solution.

A quasi-integrable system has the form

$$H(I, \phi) = h(I) + R(I, \phi)$$

with $R(I, \phi)$ "small" in some sense that will be made precise later.

Clearly we can consider Hamiltonians defined up to constants.

[7]W

We will write

$$h(I) = \omega I + h_2 I^2 + \cdots + h_n I^n + h_{[>n]}(I)$$

where $h_i I^i$ stands for the homogeneous polynomial of degree i in the Taylor expansion of h (think of I^i as standing for all the monomials of degree i), and $h_{[>n]}(I) = o(I^n)$.

Similarly, we will write, performing a Taylor expansion in the variable I,

$$R(I, \phi) = R_0(\phi) + R_1(\phi) I + \cdots + R_n(\phi) I^n + R_{[>n]}(I, \phi) \ .$$

Then, we can write the quasi-integrable Hamiltonian as

$$H(I, \phi) = R_0(\phi) + \omega I + R_1(\phi) I + h_2 I^2 + R_2(\phi) I^2 + H_{[>2]}(I, \phi) \tag{6.7}$$

We observe that, if $R_0(\phi)$ and $R_1(\phi)$ were zero, we would be in the situation described in (6.6). To add a bit of color to the description of the proof, we will refer to these terms as the "*bad*" terms since their presence spoils the easy argument for existence of quasi-periodic orbits.

The idea of the proof of Theorem 6.1 by this method is to find a canonical transformation C – which will be close to the identity – in such a way that $H \circ C^{-1}$ will not have the bad terms.

The canonical transformation will be constructed as the limit of a sequence of canonical transformations $C^{(n}$ defined recursively by:

$$C^{(n+1} = \exp\left(-\mathcal{L}_{G^{(n}}\right) \circ (T^{(n)})^{-1} \circ C^{(n} \ ,$$

where $T^{(n}$ is a canonical transformation of the form

$$T^{(n}(I, \phi) = (I + k_n, \phi) \ , \qquad (k_n \text{ constant}) \ . \tag{6.8}$$

and $\exp\left(-\mathcal{L}_{G^{(n}}\right)$ is the time one map of the Hamiltonian flow corresponding to the Hamiltonian $-G^{(n}$. The theory of these canonical transformations was developed in Section 3.6. [8]

We will denote by $H^{(n}$ the Hamiltonian expressed in the coordinates given by $C^{(n}$. That is, $H^{(n} \circ C^{(n} = H$. Hence, $H^{(n+1} = H^{(n} \circ T^{(n} \circ \exp(\mathcal{L}_{G^{(n}})$.

We will choose the $G^{(n}$ and the $T^{(n}$ in such a way that they reduce as much as possible the bad terms of the Hamiltonian $H^{(n}$.

We will try to find the Hamiltonians of these transformations among linear functions in I

$$G^{(n}(I, \phi) = G_0^{(n}(\phi) + G_1^{(n}(\phi) I \ . \tag{6.9}$$

This is a reasonable choice to try first since this is the form of the terms that we want to eliminate. As we will see rather quickly, it works. (If not, we would have gone back and chosen a more complicated $G^{(n}$.)

[8] Notice that the canonical transformation in (6.8) cannot be generated by a time one map of a vector field generated by a Hamiltonian function since it is not an exact symplectic transformation. One can develop the proof considering the exponential of a locally Hamiltonian vector field that combines $\exp\left(-\mathcal{L}_{G^{(n}}\right) \circ (T^{(n)})^{-1}$. (See [BGGS84].) We prefer to keep the translations separate with a view to proving translated curve theorems later.

Even if for the proof that we have discussed here it is enough to verify that the above form works, the reader that plans to study new problems may be interested in the fact that there is a theory to predict what terms will work and we have sketched it in Remark 6.6. See also [Mos73] p. 138.

We first describe semi-formally the step to construct the transformation $G^{(n}$. Since

$$H^{(n} \exp G^{(n} = H^{(n} + \{H^{(n}, G^{(n}\} + \text{“second order in } G^{(n}\text{”} \,, \tag{6.10}$$

we try to eliminate the bad terms in the main part of (6.10). Expanding (6.10) more explicitly, taking into account (6.9) and (6.7), we have:

$$\begin{aligned} H^{(n} + \{H^{(n}, G^{(n}\} = \; & \omega I \\ & + \{\omega I, G_0^{(n}\} + R_0^{(n}(\phi) \\ & + \{\omega I, G_1^{(n}(\phi) I\} + \{h_2^{(n} I^2, G_0^{(n}(\phi)\} + R_1^{(n}(\phi) I \\ & + \{H_{[>1]}^{(n}(I,\phi), G_1^{(n}(\phi) I\} \\ & + \{H_{[>2]}^{(n}(I,\phi), G_0^{(n}(\phi)\} + H_{[>1]}^{(n}(I,\phi) \\ & + \text{“second order in } G^{(n}\text{”} \,. \end{aligned} \tag{6.11}$$

Notice that the “bad” terms of (6.11) (i.e. those that do not include I or include it only to the first power) are precisely those on the second and third lines (up to “second order in $G^{(n}$” terms).

The goal will be to choose $G^{(n}$ in such a way that the bad terms in the resulting Hamiltonian are much smaller than those in the original system. If we manage to eliminate the bad terms in the main part of (6.11), the Hamiltonian in (6.10) will only have bad terms which are “second order in $G^{(n}$”.

We claim that it is always possible to find a $G_0^{(n}$ in such a way that we eliminate the bad term with no powers of I. Equating the second line of (6.11) to zero, we obtain the following equation for $G_0^{(n}$:

$$\{\omega I, G_0^{(n}\} + R_0^{(n}(\phi) = 0 \tag{6.12}$$

Equation (6.12) is, of the form (3.26) for which we have developed a theory in Lemma 3.15. Note that $\{\omega I, G_0^{(n}\} = L_\omega G_0^{(n}(\phi)$ where L_ω is defined in (3.26).)

The main conclusion of the theory of Lemma 3.15 is that the equation (6.12) can always be solved (with an slightly regular function), if the R.H.S. has average zero. Notice that, since Hamiltonians are defined only up to the addition of a constant, we can always ensure that $R_0^{(n}$ has average zero and, hence, that equation (6.12) can be solved for $G_0^{(n}$.

Eliminating the second bad term in (6.11) is more subtle. The equation to eliminate this term is

$$\{\omega I, G_1^{(n}(\phi) I\} + \{h_2^{(n} I^2, G_0^{(n}(\phi)\} + R_1^{(n}(\phi) I = 0 \tag{6.13}$$

The $G_0^{(n}$ appearing in (6.13) is known since we found it by solving (6.12) so that the equation (6.13) is only an equation for $G_1^{(n}$ and all the other terms in it are known.

Noting that all the terms have the structure of the dot product of a vector (depending on ϕ) with I and eliminating this vector, we can write the equation (6.13) as

$$L_\omega G_1^{(n}(\phi) = -2h_2^{(n}\nabla G_0^{(n}(\phi) - R_1^{(n}(\phi)\ . \tag{6.14}$$

The equation for each of the components of (6.14) is just one equation of the form (3.26).

We see that (6.14) will have a solution when and only when the average of the right-hand side is equal to zero. The average of the term $h_2^{(n}\nabla G_0^{(n}(\phi)$ is automatically zero. Hence, we conclude that if

$$\int_0^1 R_1^{(n}(\phi)\, d\phi = 0\ , \tag{6.15}$$

we can indeed solve (6.14) and, hence, eliminate the second part of bad terms.

Of course, (6.15) is very restrictive. It is very easy to construct perturbations that do not satisfy the condition. Here is when the translations $T^{(n}$ come into play. Given any Hamiltonian of the form (6.7), provided that the $h_2^{(n}$ satisfies the non-degeneracy assumptions, it is possible to choose a translation $T^{(n}$ of the form (6.8) in such a way that the average of $R_1^{(n}$ vanishes. This is an application of the implicit function theorem provided that $h_2^{(n}$ is an invertible matrix and that, of course, all the R terms are small. Notice that the "vertical" translation by k_n is roughly given by $k_n \approx -\frac{1}{2}\left(h_2^{(n)}\right)^{-1}\bar{R}_1^{(n}$. (We call attention to the fact that the conditions that need to be adjusted in (6.15) is exactly the number of parameters at our disposal when we apply a translation.)

The magnitude of the translation required to adjust the average of $R_1^{(n}$ can be bounded by a constant times the size of $R_1^{(n}$ (provided that $h_2^{(n}$ is invertible and that the other terms are small, so that we can apply the implicit function theorem).

Hence, the algorithm for the iterative proof is:

1. To determine the translation so that $H^{(n}\circ T^{(n}$ satisfies the normalization
$$\int_0^1 \frac{\partial}{\partial I} H^{(n}\circ T^{(n}\Big|_{I=0}\, d\phi = \omega\ .$$
2. For the "new" $H^{(n}$ (i.e. for $H^{(n}\circ T^{(n)}$) find $G_0^{(n}$ and $G_1^{(n}$ in such a way that we eliminate the two "bad terms" in (3.52) up to quadratic error.

We have already seen that step 2 involves small divisors and unbounded operators. Nevertheless, we have also seen several times that the quadratic convergence can overcome the effect of small denominators (for Diophantine numbers). Compared with the previous cases we have dealt with, the only new complication of the present algorithm is that we have to deal with the extra complication of having to adjust the translation so that (6.14) becomes solvable.

The main complication of the translation is that terms that were high order generate lower order terms. For example, a "good term" $H(I,\phi) = f(\phi)I^2$, with $f(\phi)$ a ϕ-dependent quadratic form becomes upon translation

$$H \circ T = f(\phi)I^2 + 2f(\phi)Ik + f(\phi)k^2 \ . \tag{6.16}$$

The last two terms of (6.16) are "bad".

The fact that find a translation to eliminate the average in (6.14) depends on the fact that the quadratic term $h_2^{(n}$ is invertible. We need to keep track of the fact that this remain so under the successive changes of variables. This is not so difficult since the condition is an open condition.

From the analytic point of view, we note that the procedure involves solving (twice) equations of the form (3.26) and applying the implicit function theorem. As we did in Theorem 4.1, the *second order* terms can be estimated in analyticity domains using Cauchy estimates.

In summary, we have sketched a procedure that given a perturbation that satisfies certain non-degeneracy conditions, makes a change of variable that reduces the bad terms and whose resulting error is smaller. More precisely, given estimates of the bad terms in a domain, we can obtain estimates of the resulting bad terms in an slightly smaller domain. The estimates will be of the form $||New||_{\sigma e^{-\delta}} \le C\delta^{-\tau}||Original||_\sigma$ Note also that, in order to match domains etc. we need that δ and the size of the remainder are suitably related.

The proof consists in showing that if the original error is sufficiently small, then we can carry out indefinitely the iterative procedure sketched above and it converges in a non-trivial domain.

Here we sketch the main considerations that need to be taken into account converting the above remarks into a proof. The reader is urged to either work them out alone or to use this as a reading guide for excellent expositions in the literature (some of them are discussed below).

A) We start by deciding that we consider domains loses of the form $\delta_0 2^{-n}$, and that we will do estimates on domains parametrized by a r_n defined by $r_{n+1} = r_n - \delta_0 2^{-n}$.
B) We will need to assume inductively that
   B.1) We have bounds:

$$\|(h_2^{(n})^{-1}\| \le C_1 \ ,$$

   and that the derivatives of R are sufficiently small so that they do not affect the application of the implicit function theorem (to ensure the existence of the translation $T^{(n)}$).

We take C_1 to be twice the initial constant: $C_1 := 2\,\|(h_2^{(0)})^{-1}\|$. We will need to check that, if the initial error is small enough, the iterative procedure keeps the assumption being valid.

B.2) Assume inductively that:

$$\|R^{(n}\|_{r_n} \le C_2 \ ,$$

with C_2 being twice the initial value: $C_2 := 2\,\|R^{(0}\|_{r_0}$.

B.3) We will also assume that we have bounds similar to those in the study of the Siegel problem

$$\|\nabla G^{(n}\|_{r_n} \le \delta_0 2^{-n\tau} \tag{6.17}$$

The goal of the latter bounds (6.17) is to ensure that when we perform the composition of $H^{(n} \circ T^{(n} \circ \exp(\mathcal{L}_{G^{(n}})$, the composition is still defined in the smaller domain.

C) Using assumption B.1, we are able to control the size of the translation by $\|R^{n)}\|_{r_n}$ times an universal constant.

Given B.2, we see that the size of the remainder of $H^{(n} \circ T^{(n}$ is still of the same order of magnitude as $\|R^{(n}\|_{r_n}$. (The new lower order terms generated are bounded by the size of the translation.)

D) Solving the small divisors equation, we obtain $G_1^{n)}$, $G_0^{n)}$. We can bound

$$\|G_1^{(n}\|_{r_{n+1}} + \|G_0^{(n}\|_{r_{n+1}} \le CK^2\,\frac{2^{n\tau'}}{\delta_0^{\tau'}}\left(\|R_0^{(n}\|_{r_n} + \|R_1^{(n}\|_{r_n}\right)^2 \ ,$$

The factor $2^{n\tau'}\delta_0^{\tau'}$ is the usual small divisor factor when we take domain losses as in A).

E) The heuristics can be justified by adding and subtracting and applying the mean value theorem pretty much in the same way that we did in the proof of Siegel theorem but using the estimates we developed in Section 3.6.

We obtain:

$$\|R_0^{(n+1}\|_{r_{n+1}} + \|R_1^{(n+1}\|_{r_{n+1}} \le CK^2\,\frac{2^{n\tau'}}{\delta_0^{\tau'}}\left(\|R_0^{(n}\|_{r_n} + \|R_1^{(n}\|_{r_n}\right)^2 \ . \tag{6.18}$$

F) The rest is essentially mopping up:

F.1) We need to show that the quadratic convergence implied by (6.18) implies that the inductive assumptions in B) remain valid (if we start with a small enough error). This is accomplished in a similar manner as that in the Siegel theorem (the only delicate one is (6.17) and this is exactly the same as in the Siegel domain).

F.2) We need to show that the accumulated transformation converge. Again, this is not very delicate since the quadratic convergence implies that $C^{(n}$ are converging to the identity extremely rapidly.

We urge the reader to compare the above sketch with the papers [BGGS84] and with [Bar70] which contain very readable full proofs.

The main difference in the strategies of those papers with the presentation here is that [Bar70] uses generating functions to deal with canonical transformations. Both of [BGGS84] [Bar70] do not make a distinction between the translations and the exact exact transformations and they use just one locally hamiltonian transformation that accomplishes the effect of the two steps that we discussed. This is, of course, perfectly fine for the problem at hand. we have, however, preferred to keep the two types of transformations separate with a view in translated curve theorems.

A very pedagogical proof of a particular case of the result (that nevertheless contains the most essential difficulties) is [Thi97]. The paper [Zeh76a] contains a detailed reduction of the proof based in the Kolmogorov method to an abstract implicit function theorem very similar to Theorem 5.1.

Remark 6.4. The Kolmogorov method of proof has the advantage that it is quite direct and very well suited to functional analysis. We always deal with the same linearized equation with the same frequency. In particular, it leads to very good regularity results.

The main disadvantages arise from the fact that every different frequencies require different transformations. Moreover, the form (6.6) is not unique.

Natural question, which are important for applications, but that do not follow directly from the results are what is the measure covered by the tori and determining whether tori of similar frequencies are close together. (Indeed, so far, we have not shown that there is only one torus with the a given frequency. Note that there are many hamiltonians with the same form (6.6).)

The question about the measure occupied by tori can be answered by showing that the mapping which associates to a frequency ω satisfying (3.18) the torus with frequency ω produced in the Theorem 6.1 is Lipschitz.

Moreover, the tori can be expressed as the graph of a function of ϕ,

$$I = W_\omega(\phi). \tag{6.19}$$

Clearly, given one torus, there is only one function W, whereas, given one torus, there will be several hamiltonians of the form (6.6) and several transformations reducing the original flow to them.

It is true that $W_\omega(\phi)$ turns out to be Lipschitz with Lipschitz constant close to $\|h_2^{-1}\|$.

The proof of these Lipschitz properties can be obtained rather easily if we note that the system of the form (6.6) is also an approximate solution to the equation for $\tilde{\omega}$ in the plane of ω. Hence, if ω and $\tilde{\omega}$ are close enough, we can consider the torus for frequency ω as an approximate solution for the equation that would produce a torus of frequency $\tilde{\omega}$. The error of the approximation is controlled by $\omega - \tilde{\omega}$. Hence, applying the procedure, we see that we produce a solution which differs not more than something that can be controlled by $\omega - \tilde{\omega}$. This type of argument also leads to uniqueness results of the torus with a given frequency. More details on this type of argument can be found in [Dou88], [Sev95].

As we will see later, it is true that the map $(\omega, \phi) \mapsto W_\omega(\phi)$ introduced above is differentiable in the sense of Whitney.

Remark 6.5. Another aspect in which the method of proof we have discussed is not optimal is that it requires very strong non-degeneracy conditions.

Notice that we want to ensure that the size of the translation required to adjust the error to zero average is commensurate with the error. In a degenerate situation, the size of the translation would be a root of the size of the error and then, the method as we have presented it, would collapse.

As a matter of fact, one can get a better non-degeneracy condition if one does not fix the frequency, but fixes it up to a multiple. Hence, the only thing that we require is that $\mathrm{Span}(\omega) + \mathrm{Range}(h_2) = \mathbf{R}^n$.

One can also use clever tricks to reduce degenerate situations to non-degenerate ones. For example, in [BH91].

As we will see later, one can do significantly better than that by using other methods. For example, [Sev95].

Remark 6.6. There is an interesting interpretation of the method of proof we have presented above in terms of geometry in infinite dimensional spaces.

This interpretation can certainly serve as a heuristic guide and many KAM theorems can be fit into this form. It was proposed in [Mos67] and developed quite forcefully in [Zeh76a], which developed in this language the main KAM theorems. In [Ham82], a similar philosophy is applied to many geometric problems.

The idea is to think of (6.6) as defining a manifold $\mathcal{N}$ in the space of Hamiltonians $\mathcal{H}$. All the elements of this manifold have a feature that we are interesting in studying. In this case, having an invariant torus of frequency ω.

We also have an action Ψ of a group. In this case, the action by canonical transformations. The proof we have sketched shows that given a neighborhood $\mathcal{U}$ of $\mathcal{N}$ in $\mathcal{H}$ all the elements of $\mathcal{U}$ have an orbit under Ψ that intersects $\mathcal{N}$.

Even if this is not completely trivial to make precise, (one has to define the topologies of the spaces of hamiltonians and mappings, check that they are manifolds, check the properties of the action of the group of transformations on it, etc.) it can serve as a heuristic principle to decide which theorems are possible. (Note that, if we were considering a finite dimensional problem, we could just decide what was true by deciding whether the tangent spaces of $\mathcal{N}$ and of the orbits of the action span the tangent space of $\mathcal{H}$.)

We note that this line of reasoning and these heuristic principles apply to other problems outside mechanics. Indeed, a good part of singularity theory can be formulated in this way. Similarly, many problems in geometry and PDE can be reduced to implicit function theorems by applying this heuristic picture. (See [Ham82].) [9]

[9] Incidentally, in singularity theory one has a very powerful implicit function theorem [Mat69], which allows to deal in some cases with operators that loose half of the derivatives.

The idea of deciding which theorems in KAM theory could be true by just looking at when the tangent spaces span leads very quickly to the problem of counting parameters. (See the discussion in [Mos67].) Roughly one needs that the normal form $\mathcal{N}$ and the group acting contain enough free parameters to overcome all the obstructions imposed by the geometry.

One of the important developments of later years is that in this counting of parameters, one should include the frequency [Eli88] or the perturbation parameter [JS92]. One reason why this is not obvious is that these extra parameters have a Cantor structure, hence at first sight, notions based on the geometry of tangent spaces etc. do not seem workable. Nevertheless, it turns out to be true that one can use these Cantor parameters very much in the same way as continuous families supplementing the standard geometric arguments based on implicit function theorems with measure theoretic estimates. Indeed, the next method of proof which we discuss can be used to cope with this type of problems. We refer to [Sev99] for an account of recent developments in the lack of parameters problem and, relatedly on the problem of study of degenerate systems.

Exercise 6.7. Try to carry out the proof choosing the translation $T^{(n}$ given by $k_n = -\frac{1}{2}\left(h_2^{(n}\right)^{-1}\bar{R}_1^{(n}$. Notice that in such a choice we kill the average of $R_1^{(n}$ up to second order terms in $R^{(n}$.

6.2. **Arnol'd method.** In [Arn63a], V. I. Arnol'd introduced a method prove the persistence of quasi-periodic solution quite different from the method of proof by Kolmogorov that we have discussed in the previous Section 6.1.

Rather than trying to perform a change of variables that produces one torus, the method of [Arn63a] produces changes of variables that reduce the system to approximately integrable in a region of space. Hence, the method of [Arn63a] produces all the tori at the same time.

The main complication that arises with respect to the method of Kolmogorov is that the intermediate steps require to study transformations that are defined in rather complicate regions. The fist transformation is defined in a domain that excludes the low order resonances. (The places where $\partial_I H(I) \cdot k \ll 1$ for $|k|$ not big.) In successive steps of the iterative procedure, one performs another transformation that reduces the system much more closely to integrable, but in a more complicated region since we need to take into account more resonances. At the end of the process one ends up with a transformation defined on a Cantor family of invariant tori. (A set which is locally diffeomorphic to the Cartesian product of a Cantor set and a torus. Each of the torus in a connected component of the set is invariant.)

The method of [Mat69] paper has, to my knowledge not been used in KAM theory, even if [dlLMM86] was very inspired by it.

An alternative way to describe the whole process is to say that we have a smooth canonical transformation defined in the whole space which reduces the perturbed differential equation to integrable in a smaller set. At intermediate steps of the iteration we just keep estimates of how the system differs from integrable in a smaller and smaller set with increasingly complicated geometry. In the limit, we obtain control on just a Cantor family of tori, on which the system can be considered as integrable.

The basic strategy, which we will detail later, has several advantages with respect to the Kolmogorov one:

One of them is that one obtains more information on the way that the tori are organized. For example, it follow rather naturally that the tori constitute a family that is differentiable in the sense of Whitney. (This was observed in [CG82] [Gal83a]. Similar results can were obtained by other methods in [Pös82].)

Another advantage is that if we stop the process after a finite number of steps, we may still have quite good information about the system. For example, under the assumption that $\frac{\partial^2}{\partial I_i \partial I_j} H(I)$ is a positive definite matrix, in [Neh77] (as a matter of fact, the assumption in [Neh77] is sharper but more complicated to state than the positive definiteness, which is enough in many applications and which is the assumption used in many more modern proofs.) one can find the result that, denoting by $\epsilon = \|R\|_\sigma$ in (6.1) we have, for times $t \le \exp(A\epsilon^{-a})$, **all** the orbits of the perturbed system (6.1) remain at a distance not more than ϵ^b of those of the perturbed system.

The method of exclusion of parameters near the resonances and continuing the transformation in the rest of the space, has had many applications in other KAM problems. For example, in the problem of changing a system with quasi-periodic coefficients into constant coefficients, usually called the reducibility problem, most of the papers (see specially the early ones [DS75]) are quite influenced by the method. We refer to the lectures of Prof. Eliasson in this meeting for an up to date review of this problem. The strategy of [Arn63a] was also employed in the first proofs that started to study the problem of lack of parameters and the related problem of studying systems which are rather degenerate.

From the point of view of the regularity assumptions needed the main shortcoming of the method is that the analytic part of the proof is based on truncating the Fourier series of the perturbation which produces bad results in finitely differentiable systems. Even it it is not too difficult, I know of no place in the literature where the Arnol'd strategy is implemented for finitely differentiable systems. (I wrote some very preliminary notes on that for a graduate course.)

Another shortcoming arises from the fact that one of the elements of the iterative step is the domain of the definition on which the changes of variables are defined. Keeping track of this domain is much more complicated than keeping track of the sizes of the functions. Hence, the proofs are more complicated and often one obtains worse estimates on the sizes of perturbations allowed and other quantitative results. (Nevertheless the method was used in the first proofs of several sharp estimates such as [Neˇi81] [Way84].)

I do not think that the method of [Arn63a] has been formalized in such a way that it leads to an abstract implicit function theorem in the style of Theorem 5.1 which takes care of the detailed estimates in applications or, at least provided with a detailed strategy to carry them out.

Besides [Arn63a], a very pleasant and instructive modern exposition of this method of proof is [Gal83a] (see also [CG82].) The Nekhoroshev theorem proved by this method is nicely explained in [BGG85c] and a unified exposition of Nekhoroshev and KAM theorems is in [DG96]. Other proofs of Nekhoroshev theorems are covered in [Pös93], [Loc92].

In somewhat more, but still insufficient, detail: At the nth step of Arnold's method, we keep track of:

1) An excluded set, on which we do not expect to define the transformation.
2) In the complement of the excluded set we have defined a transformation $C^{(n}$ in such a way that
$$H \circ C^{(n} = \bar{H}^{(n}(I) + R^{(n} \ ,$$
3) We keep track of $\|\nabla H^{(0} - \nabla \bar{H}^{(n}\|_{\sigma_n}$ and $\|R^{(n}\|_{\sigma_n}$. We assume by induction that $\|\nabla H^{(0} - \nabla \bar{H}^{(n}\|_{\sigma_n}$ remains bounded and that $\|R^{(n}\|_{\sigma_n}$ is bounded by a superexponentially decreasing function. (The $\|\cdot\|_{\sigma_n}$ norms will refer to complex extensions of the excluded set, not a fixed set.)

Filling in more details about 1): The excluded set consists of bands given by

$$\left|\frac{\partial \bar{H}^{(i}}{\partial I}(I) \cdot k\right| \geq C_{n,i}|k|^{-\nu} \ , \quad 2^{i-1} < |k| \leq 2^i \ . \tag{6.20}$$

In particular, it is a set with piecewise smooth boundary and the angles of the corners are bounded from below by $C4^{-n}$ (where, C again is a constant that depends on the inductive assumptions).

This lower bound on the angles comes from the fact that a bound of this sort is what one would get for planes whose normals are integer vectors of total length 2^n and the fact that $\bar{H}^{(i}$ are uniform diffeomorphisms and, therefore only change the angles by a factor which remains uniformly bounded through all the iteration.

We denote the excluded set by $\mathcal{E}_n$ and

$$\begin{aligned} \mathcal{D}_{n,\sigma} &:= \{\, z \in \mathbf{C}^d \times \mathbf{C}^d/\mathbf{Z}^d \,|\, d(z, \mathcal{E}_n) \leq \sigma \,\} \\ \|f\|_{n,\sigma} &:= \sup_{z \in \mathcal{D}_{n,\sigma}} |f(z)| \ . \end{aligned}$$

Once we fix a sequence $\{\sigma_n\}$ (we will take $\sigma_n = \sigma_0(1 - \frac{1}{2}\sum_{n=0}^{\infty}(\frac{1}{3})^n)$), we denote the norm $\|\cdot\|_{n,\sigma_n}$ by $\|\cdot\|_n$.

The main difference between these norms and the regular ones is that, due to to the small angles, the Cauchy estimates are worse. Nevertheless, given the lower bound on the angles, they do not get too much worse:

$$\|\nabla f\|_{\sigma_n e^{-\delta_n}} \leq C \left(\delta_n e^{-4^n}\right)^{-1} \|f\|_{\sigma_n} \ .$$

To go from one step to the next, we exclude an slightly larger region and define a new transformation $C^{(n+1} = C^{(n} \circ \exp(\mathcal{L}_{G^{(n}})$ so that the new remainder is much smaller (here, we will need to make a small modification to our usual notion of smaller, meaning quadratic times powers of the domain loss).

We see that

$$\begin{aligned} H^{(n} \circ \exp(\mathcal{L}_{G^{(n}}) = \bar{H}^{(n} + R^{(n} + \{\bar{H}^{(n}(I), G^{(n}\} \\ + \{R^{(n}(I), G^{(n}\} + O((G^{(n})^2) \end{aligned}$$

(a precise estimate for $O((G^{(n})^2)$ appears in Lemma 3.24).

A new idea of the method is to modify the prescription of Newton method by restricting only to a finite number of frequencies and include a truncation of the Fourier series so that, at every stage, we only have to deal with a finite (but growing) number of denominators. The error incurred by the truncation can be estimated if we increase the order of truncation at the right speed.

We write:

$$\begin{aligned} R^{(n\,[\leq 2^n]}(I, \phi) &= \sum_{|k| \leq 2^n} \hat{R}_k^{(n} e^{2\pi i k \phi} \\ R^{(n\,[> 2^n]}(I, \phi) &= \sum_{|k| > 2^n} \hat{R}_k^{(n} e^{2\pi i k \phi} \end{aligned}$$

Hence, we solve:

$$\{\bar{H}^{(n}(I), G^{(n}\} + R^{(n\,[\leq 2^n]}(I, \phi) = \Delta^{(n}(I) \ . \tag{6.21}$$

The equation (6.21) can be solved by setting

$$\hat{G}_k^{(n}(I) = \hat{R}_k^{(n}(I) / (\frac{\partial \bar{H}^{(n}}{\partial I} \cdot k) \ , \quad |k| \leq 2^n \ . \tag{6.22}$$

By the definition of the excluded set, we can bound the denominators (6.22) over the complement of the excluded set.

Notice also that we can bound

$$\|R^{(n\,[> 2^n]}\|_{\sigma_n e^{-\delta_n}} \leq \|R^{(n}\|_{\sigma_n} e^{-\delta_n 2^n} \ .$$

This allows us to define the generator of the transformation that eliminates $R^{(n\,[\leq 2^n]}$ (up to quadratic orders).

We have estimates

$$\|G^{(n}\|_{\sigma_n e^{-\delta}} \leq C \delta^{-\tau} \|R^{(n}\|_{\sigma_n} \ ,$$

where, as usual, τ is roughly ν plus something depending on the dimension. We use the letter τ to denote similarly constants that depend only on the Diophantine exponent and the dimension.

To study the domain of $\exp(\mathcal{L}_{G^{(n}})$, we note that if we set

$$C_{n+1,\,i} = \delta^{-\tau} 2^n \|R^{(n}\|_{\sigma_n} + C_{n,\,i} \ ,$$

we can define the transformation from the set

$$\Big| \frac{\partial \bar{H}^{(i}}{\partial I} \cdot k \Big| \geq C_{n+1,\,i} |k|^{-\nu} \ , \quad 2^{i-1} < |k| \leq 2^i \quad i = 1, 2, \ldots, n \tag{6.23}$$

to the set

$$\Big| \frac{\partial \bar{H}^{(i}}{\partial I} \cdot k \Big| \geq C_{n,\,i} |k|^{-\nu} \ , \quad 2^{i-1} < |k| \leq 2^i \quad i = 1, 2, \ldots, n \ .$$

In that case, we have

$$\bar{H}^{(n+1}(I) = \bar{H}^{(n}(I) + \bar{\Delta}^{(n+1}(I) \ ,$$

from which it is clear that

$$\|\bar{H}^{(n+1}\|_{\sigma_{n+1},\,\mathcal{D}_n} \leq \|\bar{H}^{(n}\|_{\sigma_n} + \|R^{(n}\|_{\sigma_n}$$

and

$$\|\nabla \bar{H}^{(n+1} - \nabla \bar{H}^{(n}\|_{\sigma_{n+1},\,\mathcal{D}_n} \leq C 2^{\tau} \|R^{(n}\|_{\sigma_n} \ .$$

Most importantly, we have:

$$\|R^{(n+1}\|_{\sigma_{n+1},\,\mathcal{D}_{n+1}} \leq C 2^{n\tau} \|R^{(n}\|_{\sigma_n, \mathcal{D}_n} + 2^{-\delta_n 2^n} \ . \tag{6.24}$$

To define the next excluded set, the only thing we have to do is to add to the excluded regions corresponding to

$$\Big| \frac{\partial \bar{H}^{(n+1}}{\partial I} \cdot k \Big| \geq C |k|^{-\nu} \ , \quad 2^n < |k| \leq 2^{n+1} \ .$$

Of course, excluding more regions makes the suprema in the left-hand side of (6.24) and all the other estimates even smaller.

The recursion (6.24) leads still to superexponential convergence choosing $\delta_n = \delta_0 (2/3)^n$. Establishing this was proposed in Exercise 5.18, see (5.24).

Once we have the superexponential convergence of the reminders, we obtain that the $C_{n,\,i}$'s remain bounded and so does $\|\nabla \bar{H}^{(n}\|_{\sigma_n}$ Indeed, $\|\nabla \bar{H}^{(0} - \nabla \bar{H}^{(n}\|_{\sigma_n}$ is small (arbitrarily small if we assume that $\|R^{(0}\|_{\sigma_0}$ is sufficiently small. Similarly, it is easy to check that $\|(\nabla^2 \bar{H}^{(n)})^{-1}\|_{\sigma_n}$ remains bounded and that the bound is close to the one for $\|(\nabla^2 \bar{H}^{(0)})^{-1}\|_{\sigma_n}$ if $\|R^{(0}\|_{\sigma_0}$ is sufficiently small. Hence, under the assumption that $\|R^{(0}\|_{\sigma_0}$ is sufficiently small, we can verify the inductive assumption on $\|(\nabla^2 \bar{H}^{(n)})^{-1}\|_{\sigma_n}$.

The passage to the limit in this procedure is somewhat subtle.

In the original coordinates, we have to study the sets $(C^{(n)})^{-1}\mathcal{E}^n$. These sets will be dense. By increasing slightly the excluded sets at each stage so that we exclude also the mismatches of the domain, we can arrange that $(C^{(n)})^{-1}\mathcal{E}^n$ are increasing. (note that this extra exclusion will be decreasing superexponentially since the transformations that we need to carry out in each step are decreasing superexponentially) Hence, $(C^{(n)})^{-1}(\mathbf{T}^d \times \mathbf{R}^d - \mathcal{E}^n)$ is a decreasing sequence of compact sets. On the other hand, their measure remains bounded away from zero as follows from the fact that

$\|(\nabla^2 \bar{H}^{(n)})^{-1}\|_{\sigma_n}$ remains uniformly bounded so that we can use the same arguments as in Section 3.9).

It is slightly more subtle, but we can also estimate the derivatives of the transformations $C^{(n}$ to show that the derivatives remain bounded (it follows by an argument very similar to that used in the proof of Theorem 4.13 part v)) This shows that the sets $(C^{(n)})^{-1}(\mathbf{T}^d \times \mathbf{R}^d - \mathcal{E}^n)$ get closer and closer to being invariant. The limiting set will be invariant.

If one keeps track of all the derivatives in the closed sets, one can show that the limiting transformation $C^{(\infty}$ is differentiable in the sense of Whitney (see [CG82] or [Gal83b]). An interesting remark [Val98] is that one can use the fact that the gaps between the sets are much larger than the corrections to show directly the Whitney extension theorem. This remark could be important when studying infinite dimensional systems (e.g. PDE's). In infinite dimensions, the Whitney extension theorem is not a available, but the method of [Val98] could still work to produce tori that lie in a smooth family.

For more details of this method of proof we refer to the original paper [Arn63a], and the more expository paper [Arn63b], which also contains applications to celestial mechanics.

An early development of the method with several improvements is [Sva80]. More modern expositions (including the Whitney differentiability) of Arnol'd method are [CG82] and [Gal83b]. An exposition of the Arnol'd method that, at the same time proves Nekhoroshev's theorem and clarifies the geometry of the domains, is [DG96].

The method also lies at the heart of several other papers. One paper that incorporates the exclusion of parameters but is free of many geometric complications is [DS75]. This paper also shows that the method can allow some frequencies that are not Diophantine (they allow $|\omega \cdot k|^{-1} \geq \exp \frac{A|k|}{\log |k|^{1+\varepsilon}}$).

The method of transformations and exclusion of parameters is the basis of many modern developments in KAM theory related to lower dimensional tori, e.g. [Eli89], [JS92], [JV97a].

6.3. **Lagrangian proof.** In this section, we study a proof of of the Theorem 6.2 which has a different flavor from the proofs already presented. We will present the proof only in the case $d = 1$ and only for the particular case of the map given in (2.2). Similar proofs in any dimension and for more general maps are in the literature and we refer to that.

The proof differs substantially from the previous proofs of Theorem 6.1 in that it does not use compositions. Of course, the proof we presented of Theorem 4.1 does not require compositions either, even if the proofs we have presented so far for Theorems 6.1 do rely on transformation theory/ More interesting is that it is based on Lagrangian formalism. (That is, on second order equations rather than in systems of first order equations. The structure that is used is the fact that the equations solve a Lagrange variational principle, not that they come from a Hamiltonian formalism.)

The proof we present is based on unpublished notes of J. Moser for a course he gave in Zürich. A generalization of these results is included in the paper [SZ89]. We follow very closely the presentation in one of the chapters of [Ran87] (which in turn followed the presentation of the Moser's course.)

In [Ran87], one can also find the implementation of computer assisted proofs based on this method. In particular the result that the map given in (2.2) for $V(x) = \varepsilon\frac{1}{2\pi}\sin(2\pi x)$ has an invariant circle with golden mean rotation for $\varepsilon = .93$ (this was later improved to $\varepsilon = 0.935$). This is very close to the values for which [Jun91] showed that there can be no invariant circle. We discuss some of these issues in Section 7.
10

The Lagrangian formalism for KAM theory has several other applications. For example, many elliptic PDE's have a very natural Lagrangian formalism but not a simple Hamiltonian one. (Note that in this case, the independent variable is multidimensional, while in Mechanics, the independent variable is the time, which is one-dimensional) There is no easy canonical transformation theory for elliptic PDE's.

We will try to find solutions to (2.11) which read

$$\ell_\varepsilon(\theta+\omega) + \ell_\varepsilon(\theta-\omega) - 2\ell_\varepsilon(\theta) = -\varepsilon V'(\theta + \ell_\varepsilon(\theta)) \ . \tag{6.25}$$

We refer to Section 2.1 for the interpretation of this equation as a parameterization of a set in which the motion is quasiperiodic of frequency ω.

Somewhat informally, what we will do is to show that there is a procedure that, given an an approximate solution of (6.25) (which is not too badly behaved) we can produce another function that solves the equation even more approximately. Then, we will have to show that the whole process can be iterated indefinitely and that it converges to a solution.

Of course, making precise the notion of close, will involve introducing analytic norms. The statement that the result of the algorithm is closer to being a solution will mean to prove that in an slightly smaller domain, we will have the usual bounds which are quadratic in the previous error and have powers of the loss of analyticity. The fact that the iterative step can be performed and that it will lead to the desired improvement will require that certain expressions are not too large. (This is what we alluded to when mentioning that the solution is well behaved.) Of course, we will need to check that, if the initial error is small enough, the quadratic convergence that ensues, allows us to recover the inductive hypothesis indefinitely.

The theorem whose proof we will sketch is:

[10] The conjecture in [Gre79], given a theoretical – but not yet rigorous – basis in [McK82] is that there are smooth invariant circles with rotation golden mean when $\varepsilon < \varepsilon^*$ and not when $\varepsilon > \varepsilon*$. For $\varepsilon = \varepsilon^*$ there is an invariant circle which is not very differentiable. It is believed that $\varepsilon^* \approx .971635$. Of course, in other families it could – probably does – happen that the set of parameters for which one can find an smooth invariant circle is a more complicated set, perhaps with infinitely many components.

Theorem 6.8. *Let $\ell_0 : \mathbf{T}^1 \to \mathbf{T}^1 \times \mathbf{R}^1$ be such that $\|\ell_0\|_\sigma < \infty$. Assume that*

$$\begin{aligned} \|\ell_0' + 1\|_\sigma &\le M_+, \\ \|(\ell_0' + 1)^{-1}\|_\sigma &\le M_- \end{aligned} \tag{6.26}$$

$$\frac{1}{2}\|V''\|_{\sigma e^{(M_+ + \frac{3}{M_-})}} \le D \tag{6.27}$$

and

$$\|\ell_0(x+\omega) - 2\ell_0(x) + \ell_0(x-\omega) - V'(\ell_0(x)+x)\|_\sigma \le \epsilon \tag{6.28}$$

where M_+, M_-, D, and ϵ are finite positive constants.

Let $\Gamma(M_+, M_-, D, K, \nu)$ be a function which will be made rather explicit during the proof, where K and ν are the constant and the exponent appearing in the Diophantine properties of ω.

If

$$\epsilon \le \Gamma(M_+, M_-, D, K, \nu)$$

then, there is a periodic function ℓ, $\ell(x+1) = \ell(x)$, solving (6.25).

Moreover

$$\|\ell - \ell_0\|_{\sigma_0/2} \le C\epsilon$$

where C is a constant that depends on M_+, M_-, D, K, ν.

The proof will be done using a quasi-Newton method. The method will be rather similar to the proof of Theorem 4.1. We try to solve the infinitesimal equation suggested by the Newton method. This will lead to an equation which is not immediately solvable with the method of Section 3.4. Nevertheless, by manipulating the equation with the remainder, we will arrive at a factorization of the equation that will be solvable by applying repeatedly the theory for the equation (3.27).

Denote

$$\mathcal{T}(\ell)(x) \equiv \ell(x+\omega) - 2\ell(x) + \ell(x-\omega) - V'(\ell(x)+x) \tag{6.29}$$

We assume that we are given an approximate solution ℓ such that

$$\mathcal{T}(\ell) = R \tag{6.30}$$

where R is small.

The prescription of the Newton method would be to find a Δ periodic solving

$$\Delta(x+\omega) + \Delta(x-\omega) - (2 + V''(\ell(x)+x))\Delta(x) = -R(x) \tag{6.31}$$

This equation (6.31) is not readily solvable in terms of Fourier coefficients (as indicated in Exercise 3.17) since the term $(2+V''(\ell(x)+x))\Delta(x)$ is not diagonal in Fourier coefficients. Our next task is to manipulate the equation so that it becomes solvable using the Fourier methods. The manipulations that will follow, even if rather straightforward and indeed convenient for numerical work will perhaps look mysterious, but in later sections we will argue that the success is due to natural geometric reasons.

If we take derivatives with respect to x of (6.30), we obtain, denoting by $T_\omega(x) = x+\omega$ and by $g'(x) = \ell'(x)+1$

$$g' \circ T_\omega + g' \circ T_{-\omega} - (2+V'' \circ g)g' = R' \tag{6.32}$$

Substituting the expression for $2+V''\circ g$ from (6.32) into (6.31), we obtain

$$g'\Delta \circ T_\omega + g'\Delta \circ T_{-\omega} - [-R' + g' \circ T_\omega + g' \circ T_{-\omega}]\Delta = -g'R \tag{6.33}$$

Ignoring the term $R'\Delta$, which is quadratic in the error, yields the system of equations:

$$\left(\frac{\Delta}{\ell'+1}\right) \circ T_{-\omega} - \left(\frac{\Delta}{\ell'+1}\right) = \frac{W}{(\ell'+1)(\ell_0'+1)\circ T_{-\omega}} \tag{6.34}$$

with:

$$W \circ T_\omega - W = (\ell'+1)R \tag{6.35}$$

The above system of equations consists of equations of the form (3.27) which can be studied using Lemma 3.15. We first solve (6.35) for W and we take the solution and substitute it in the R.H.S. of (6.34). We then solve (6.34) for $\frac{\Delta}{\ell'+1}$, out of which Δ is obtained just multiplying by $\ell'+1$.

Of course, in order to carry out the above plan, we need to check that the equations we plan to solve are indeed solvable (i.e. that their R.H.S. has average zero). Later, we will have to worry about obtaining estimates of the solution thus obtained.

The fact that (6.35) can be solved is a calculation which we have done in Section 2.1 when we wanted to show the solvability of equation (2.17). Once we have that the equation (6.35) is solvable up to an additive constant, we can determine the additive constant in W in such a way that the R.H.S. of (6.34) has average zero. (Note that adding a constant $\bar{W}$ to W changes the average of the R.H.S. of (6.34) by $\bar{W} \cdot \int((\ell'+1)(\ell'+1)\circ T_{-\omega})^{-1}$. The integral is not zero, since we assumed by induction that the denominator in the integrand is bounded away from zero and hence, it is positive. Indeed, we can have a bound for it under the assumptions given in our inductive hypothesis.

The fact that this procedure works can be shown using the familiar method.

Adding and subtracting, we show that given some inductive assumptions on bounds on $||\ell'+1||_\sigma$, $||(\ell'+1)^{-1}||_\sigma$, the above procedure leads to a new quadratic remainder. That is, as usual, we have

$$||R^{n+1}||_{\sigma_n-\delta_n} \le ||R^n||^2_{\sigma_n} CK^2\delta_n^{-\tau}$$

The bounds we assumed on the derivatives deteriorate slightly, but again the quadratic convergence ensures that they remain bounded during the iteration.

Remark 6.9. The remarkable cancellations (see (6.32), (6.31)) between the derivative of the remainder and the linearized equation which allowed us to obtain a quadratically convergent method solving only the linear equation (and performing easy multiplication and divisions by known functions) are not a coincidence. In [SZ89] one can find how they work for twist maps if we uses the equations given by the generating functions, linking them to a Lagrangian formalism.

Indeed there are deeper reasons. For example, the cancellations apply to partial differential equations. See [Koz83], [Mos88].

Remark 6.10. If one is interested in obtaining the existence of invariant circles for numerical values that are as close to the optimal value as possible, one should be prepared to cope with the difficulty of having the quality of the solution get worse and worse. Indeed, the domains of analyticity shrink and function becomes more and more close to having zeros. Indeed, there are precise predictions – not rigorous but supported by numerical evidence that at the breakdown of the invariant circle for the map given in (2.2), all the difficulties happen at the same time and indeed all the quantities that need to be estimated blow up as powers of the distance of the parameter to the critical one.

See [BCCF92] for numerical results and [dlL92] for a non-rigorous explanation and precise conjectures.

6.4. **Proof without changes of variables.** In this section, we will present another proof of theorem 6.2 This proof is based on [GJdlLV00]. A version of the method for lower dimensional tori was presented in [JdlLZ99].

The proof actually proves something more general since the main result does not require that the map is exact. Of course, without assuming exactness, we cannot expect to have invariant tori as was shown in the examples. The conclusion of the main theorem is that for symplectic maps that satisfy all the other assumptions of Theorem 6.2 there is a torus that gets translated rigidly in the direction of the actions. The points of the torus are, roughly, rotated.

This is a generalization to higher dimensions of the translated curve theorem of [Rüs76b].

If we assume that the map is exact, it will be very easy to show that the translation has to vanish and that the torus is indeed invariant and that the motion on it is conjugated to a rigid rotation.

We will consider the symplectic manifold $\mathbf{T}^d \times \mathbf{R}^d$ endowed with the standard symplectic structure.

We will consider a map $F : \mathbf{T}^d \times \mathbf{R}^d \to \mathbf{T}^d \times \mathbf{R}^d$ which is symplectic (not necessarily exact) analytic (and other conditions somewhat weaker than those of Theorem 6.2 which we will formulate when the heuristic discussion motivates them). We fix ω Diophantine and seek a mapping $K : \mathbf{T}^d \to \mathbf{T}^d \times \mathbf{R}^d$ and a vector $a \in \mathbf{R}^d$ in such a way that

$$F \circ K(\theta) = K(\theta + \omega) + (0, a) \tag{6.36}$$

Note that, of course, the equation (6.36) expresses that the image of the torus is translated in the direction of the action by a rigid displacement a.

In the case that F is an integrable map, all the tori given by a parameterization

$$K_0(\theta) = (\theta, I_0) \tag{6.37}$$

are "vertically" translated. So, we expect that the functions we will have to consider will be close to that.

Later, we will show that if F is exact (and there are other conditions), then, a should vanish. This is very similar to the line of argument in [Rüs76b]). When $d = 1$, it can be seen that the zero flux condition indeed implies that $a = 0$. We note that, even if the proof does not use the exactness of the symplectic structure, it uses the symplectic structure. Under appropriate redefinition of translation, one can have similar theorems in other symplectic manifolds.

We will sketch the proof of the following theorem:

Theorem 6.11. *Assume that F is an analytic symplectic map of $\mathbf{T}^d \times \mathbf{R}^d$ endowed with the canonical symplectic structure and that ω is a Diophantine number.*

Assume that F is close to an integrable map and that it satisfies the hypothesis of non-degeneracy of Theorem 6.2. Assume that we can find an approximate (K, a) solution of (6.36)*.*

If the residual of (6.36) *is small enough (depending on properties of F and of $\|K - K_0\|$) where K_0 is the solution in* (6.37)*), then we can find an exact solution of* (6.36)*.*

In particular, if we take as approximate solution K_0 as in (6.37) *the hypothesis are satisfied when F is sufficiently close to an integrable map.*

The fact that F is close to integrable is not really necessary as it will transpire from the proof. At this stage it is only introduced to avoid using a more complicated notion of non-degeneracy than that used in Theorem 6.2. As the proof in Section 6.3 it can apply to all the maps of the form (2.2). Even that can be generalized by formulating appropriately the degeneracy. See [GJdlLV00].

A very simple calculation (a more general version appears in [JdlLZ99]) shows that if we have an exact system, then the translation a for an true solution has to be zero.

Proposition 6.12. *If the K solving* (6.36) *is close to K_0 in an analytic norm and F is exact, then $a = 0$.*

Before starting the discussion of the Theorem 6.11, we discuss the proof of Proposition 6.12.

Proof. Let α be the symplectic potential form $\alpha = \sum_i I_i \, d\phi_i$. Assume also that $F^*\alpha = \alpha + dS$.

We consider the loops in the ith angle coordinate given by

$$L_{\theta_1,\cdots,\theta_{i-1},\theta_{i+1},\cdots,\theta_d}(\theta) := K(\theta_1, \cdots, \theta_{i-1}, \theta, \theta_{i+1}, \cdots, \theta_d),$$

where $\theta_1, \cdots, \theta_{i-1}, \theta_{i+1}, \cdots, \theta_d \in \mathbf{T}$.

Because of (6.36) and the exactness of the map, we have

$$\int_{L_{\theta_1,\cdots,\theta_{i-1},\theta_{i+1},\cdots,\theta_d}} \alpha = a_i + \int_{L_{\theta_1+\omega_1,\cdots,\theta_{i-1}+\omega_{i-1},\theta_{i+1}+\omega_{i+1},\cdots,\theta_d+\omega_d}} \alpha$$

And, integrating over the variables $\theta_1, \cdots, \theta_{i-1}, \theta_{i+1}, \cdots \theta_d$, we obtain

$$a_i = 0$$

□

Remark 6.13. This Theorem (and the Proposition) are much weaker than what can be proved by the method.

For example, the hypothesis that the system is close to integrable can be replaced by several quantitative statements about the approximate solution. This is quite important for several applications. We note that approximate embeddings can be obtained with the computer. One can try to solve a discretized Fourier series or, a big advantage for the present method, just compute orbits and compute the Fourier transform. This improvement is discussed in more detail in Remark 6.19.

Once this improvement is in place, it should be apparent that the way that the torus is embedded does not play any role. We do not need that the system is close to integrable or that the tori are close to integrable. (This is the main difference with the exposition of [Bos86].)

In particular, we can justify the existence of tori which have different topology than the tori of the unperturbed system. (Recently there has been some interest in these secondary tori since there are numerical experiments that suggest that secondary tori are very important for the statistical properties of coupled systems [HdlL00].)

Also, an important advantage of the method is that it allows one to deal will more degenerate situations than the twist mapping. Indeed, one can use it to deal with non-twist maps and with even more degenerate situations.

We refer the reader to [GJdlLV00] for these precisions as well as for more details about the proof. We also refer to [JdlLZ99] for another application of similar techniques to discuss lower dimensional tori.

Now, we start describing the main ideas of the proof. Again, we refer to [GJdlLV00] for more details.

The method of proof will be an iterative procedure in which we start from (6.36) being satisfied with a certain error and return a solution that satisfies the equation with an smaller error. Of course, as usual in this theory, what we mean by smaller error is that the size of the new error will be bounded (in a smaller domain than the original one) by the square of the size of the original error times a factor that is the domain loss parameter to a negative power. Of course, by now the convergence of the procedure should be well understood. Actually, since we do not need to make changes of variables and we do not need to keep track of much the geometric structures, the inductive hypothesis will be very mild.

We will begin with a heuristic discussion.

If we start with an approximate solution of (6.36) that is:

$$F \circ K(\theta) - K(\theta + \omega) - (0, a) = R(\theta) \tag{6.38}$$

where R is small in some appropriate norm that we will make precise later.

The Newton method prescription, would be to change K into $K + \Delta$, a into $a + \alpha$ in such a way that

$$DF \circ K(\theta)\ \Delta(\theta) - \Delta(\theta + \omega) - (0, \alpha) = -R(\theta)\ . \tag{6.39}$$

Unfortunately, this equation is not readily solvable by easy methods such as comparing Fourier coefficients since it involves the non-constant coefficient factor $DF \circ K(\theta)$.

Hence, we try to compare it with the equation obtained taking derivatives of (6.38):

$$DF \circ K(\theta)\partial_\theta K(\theta) - \partial_\theta K(\theta + \omega) = \partial_\theta R(\theta). \tag{6.40}$$

At this point, we are going to introduce some notation (which is not completely necessary but which will make the geometry more concrete). We define $D(\theta) := DF \circ K(\theta)$ and let $K_1(\theta)$ an orthogonal basis for $\partial_\theta K(\theta)$. The previous equation (6.40) reads $D(\theta)K_1(\theta) - K_1(\theta + \omega) = R_1(\theta)$. As usual, we define the matrix

$$J = \begin{pmatrix} 0 & \mathrm{Id}_d \\ -\mathrm{Id}_d & 0 \end{pmatrix} ,$$

which is the representation in coordinates of the symplectic form.

We define then the symplectic matrices.

$$M(\theta) = [K_1(\theta), JK_1(\theta)] \tag{6.41}$$

Notice that from the fact that K_1 is almost invariant under $DF(\theta)$ we obtain that:

$$DF(\theta)M(\theta) = M(\theta + \omega) \begin{pmatrix} \mathrm{Id} & A(\theta) \\ 0 & B(\theta) \end{pmatrix} + O(R) \tag{6.42}$$

We will introduce the assumption that $M(\theta)$ is invertible for all θ.

This is reasonable assumption in view of the fact that, for integrable systems [11]ndeed, this is the only reason why we assumed that F was close to integrable. If we formulate the theorem assuming that M is invertible, we could have eliminated the assumption of close to integrability. Later, we will need to formulate the non-degeneracy assumption using this matrix M.

This is explained in more detail in Remark 6.19 and in the references quoted there. using (6.37) we have:

$$M(\theta) = \begin{pmatrix} \mathrm{Id}_d & 0 \\ 0 & \mathrm{Id}_d \end{pmatrix} .$$

[11]I

Recall also that the assumption that the map F preserves the symplectic form is equivalent to

$$JDF(x) = \left[DF(x)^t\right]^{-1} J \; . \tag{6.43}$$

This gives that $B(\theta) = \mathrm{Id}_d$.

We note that in the integrable case, the matrix $A(\theta)$ will be a constant $d \times d$ matrix A and, the twist condition implies that A is invertible. Hence, in the proof of the theorem, we will assume that $A(\theta)$ is not very far from a constant, invertible matrix in the sense that $\bar{A}(\theta)$ is an invertible matrix. Indeed, this is the only non-degeneracy condition that we will assume.

We call attention to the fact that the non-degeneracy assumption only amounts to the invertibility of M and the fact that $\bar{A}(\theta)$ is invertible. These assumptions could be checked a posteriori on a numerically computed solution or on an approximate solution produced by any other means. Other than that, we do not need any property of the map F. See Remark 6.19 and the references quoted there for an explanation of this alternative approach.

Remark 6.14. It is an easy exercise to show that, under Diophantine conditions we can reduce the block $A(\theta)$ to a constant, so that the matrix is is indeed reducible, Nevertheless, for the applications that we have in mind, this does not help. Indeed, by doing it, we incur in extra small denominator estimates, which can worsen the result.

Remark 6.15. A more geometric interpretation of the previous calculations is to say that $DF \circ K(\theta)$ is a reducible matrix whenever K is a parameterization of an invariant torus by a rigid rotation.

We want to give a geometric argument that shows that the linearization of the equations around an invariant torus is reducible. The argument will show that for an approximate solution, the equation will be approximately reducible and, hence that one can start an iterative procedure in which in the iterative step we improve the solution of the main equation and its reducibility.

That is, our goal find a system of coordinates on the tangent of the torus so that the matrix representing $DF \circ K(\theta)$ has constant coefficients.

Since the vectors along the direction of θ are moved just by a rotation in the torus, this is an invariant field that can be lifted to the space by the embedding. By the preservation of the symplectic structure, we also have that the plane spanned by the the vector and its symplectic conjugate is also preserved. We can see that in the plane spanned by a vector and its symplectic the matrix has to be upper diagonal (one vector is preserved.) The dilation along the symplectic conjugate has to be the inverse of the dilation along the preserved direction due to the requirement that the two-area in the plane is preserved. This gives us the diagonal blocks of the matrix. The upper diagonal does not bother.

This system of coordinates provides with a system in which the derivative is upper triangular.

Once that we have that the diagonal blocks are constant, then it is easy to see that the linearized equation can be solved by using equations of the form (3.27).

The above geometric interpretation makes it clear that we do not need the symplectic form to be constant. Moreover, it is clear that it does not require that the symplectic form has action-angle variables and that it can accommodate certain singularities.

The algorithm is now very easy. If we write $\Delta(\theta) = M(\theta)w(\theta)$ and substitute in (6.39) we obtain

$$D(\theta)M(\theta)w(\theta) - M(\theta+\omega)w(\theta+\omega) - (0,\alpha) = -R(\theta) \tag{6.44}$$

which using (6.41) becomes:

$$M(\theta+\omega)\left[\begin{pmatrix} \mathrm{Id}_d & A(\theta) \\ 0 & \mathrm{Id}_d \end{pmatrix} w(\theta) - w(\theta+\omega)\right] - (0,\alpha) = -R(\theta) - N(\theta)w(\theta)\ . \tag{6.45}$$

Therefore, ignoring the last term of the R.H.S. of (6.45), which is quadratic, we are lead to the study of the equation:

$$\begin{pmatrix} \mathrm{Id}_d & A(\theta) \\ 0 & \mathrm{Id}_d \end{pmatrix} [w(\theta) - w(\theta+\omega)] = -M(\theta+\omega)^{-1}\left[R(\theta) - (0,\alpha)\right] \tag{6.46}$$

We claim that this equation for w, α can be studied using the methods that we have developed in Section 3.4. This will constitute our iterative step. Of course, after this heuristic derivation, we will need to go back and justify the estimates of the step and show that it can be iterated. This, even if being the essential part of the proof, we hope will bring no surprises anymore for the reader.

If we write (6.46) in components, denoting the components of $w(\theta) = (w_\phi(\theta), w_I(\theta))$ and by Π_ϕ,Π_I the projections over the components, we have:

$$\begin{aligned} w_\phi(\theta) + A(\theta)w_I(\theta) - w_\phi(\theta+\omega) &= -\Pi_\phi M(\theta+\omega)^{-1}\left[R(\theta) - (0,\alpha)\right]) \\ w_I(\theta) - w_I(\theta+\omega) &= -\Pi_I M\left(\theta+\omega\right)^{-1}\left[R(\theta) - (0,\alpha)\right]\big) \end{aligned} \tag{6.47}$$

If we look at the second equation in (6.47) (recall that it is an equation for w_ϕ and α) we see that it is an equation of the form (3.27) which we have already studied. We chose α in such a way that the R.H.S. has average 0. (This can be done if M is close to the identity, but otherwise it can be made into an assumption to be checked a posteriori on the approximate solution.)

Note that we have bounds

$$|\alpha| \le C\|R\|_\sigma\|M^{-1}\|_\sigma\|(M^{-1} - \mathrm{Id})^{-1}\|_\sigma \tag{6.48}$$

If we assume for convenience (somewhat sharper assumptions could also work, see Remark 6.19) that the factors in the R.H.S. of (6.48) satisfy:

$$\|M^{-1}\|_\sigma \|(M^{-1} - \mathrm{Id})^{-1}\|_\sigma \le 333\ . \tag{6.49}$$

Then we can apply Lemma 3.15 to obtain w_I up to a constant, which we will determine in the next equation.

We have, denoting by $\tilde{w}_I$ the solution with zero average:

$$\|\tilde{w}_I\|_{\sigma-\delta} \le C\delta^{-\nu}\|R\|_\sigma\ . \tag{6.50}$$

If we look at first equation of (6.47) we see that, at this stage of the argument is an equation only for w_ϕ and the average of of w_I. Hence, we write it as

(6.51)

$$w_\phi(\theta) - w_\phi(\theta+\omega) = -\Pi_\phi M(\theta+\omega)^{-1}\left[R(\theta) - (0,\alpha)\right]) - A(\theta) w_I(\theta)\ .$$

If we assume that

$$\|(\bar{A})^{-1}\| \le 333\ , \tag{6.52}$$

we can determine $\bar{w}_I$ so that the terms in the R.H.S. of (6.51) have average 0. We have:

$$|\bar{w}_I| \le C\|R\|_\sigma$$

We will furthermore assume that

$$\|A\|_\sigma \le C \tag{6.53}$$

Hence, we can apply Lemma 3.15 and obtain a w_ϕ with zero average which satisfies:

$$\|w_\phi\|_{\sigma-2\delta} \le C\delta^{-2\nu}\|R\|_\sigma\ . \tag{6.54}$$

Note that the power of δ in this case is twice as high as that in the previous one since the R.H.S. of (6.51) involves the solution of the previous one.

From (6.50)(6.54), using the inductive assumptions on the size of M, we obtain

$$\|\Delta\|_{\sigma-2\delta} \le C\delta^{-\tau}\|R\|_\sigma.$$

From this, the rest of the proof of the translated tori theorem is very similar to the previous proofs, in particular to the proof of Theorem 4.1.

Under the assumption that

$$\|K\|_\sigma + \|\Delta\|_{\sigma-4\delta} \le \Sigma - \delta \tag{6.55}$$

where Σ denotes the size of the domain of analyticity of F, we can define the composition $F \circ (K + \Delta)$ and indeed the range of $K + \Delta$ is at least a distance δ from the boundary of the domain of definition of F.

Note that adding and subtracting and using Taylor's theorem to control the terms neglected to derive (6.39), (and Cauchy bounds to control the size or the derivatives involved) we get:

$$\|\tilde{R}\|_{\sigma-4\delta} \le C\delta^{-\tau'}\|R\|_\sigma^2\ . \tag{6.56}$$

From this, we can conclude as in the previous cases that if the original remainder is small, then the iteration can be carried out an arbitrarily large number of times, moreover, the final remainder in its domain of definition keeps decreasing.

Note also that this proof – in contrast with those based on composition – does not require any subtle inductive hypothesis to ensure that the domains of the composition match. These assumptions, that we had to consider in the proofs based on composition are subtle because they require that the errors decrease faster than the analyticity losses.

In this case, the only assumptions that we have to check are (6.55), (6.49), (6.52), (6.53).

We can see that if we start with an small enough residual, the iterative procedure does not change A or M much, so that using in the step bounds which are twice the ones at the start, the estimates of the step remain valid if the original error is small enough.

Remark 6.16. We emphasize that the only thing that we need to get the proof started is an approximate solution of the functional equation.

This can be obtained in a variety of ways. For example if the system was close to integrable, one could take as an initial guess the parameterization of the integrable system.

Other choices are possible. One could use a few steps of the Lindstedt series. In such a case, the proof will establish that the Lindstedt series is asymptotic.

More audaciously, one could use the results of a non-rigorous, numerical algorithm. Provided that one can verify rigorously that one has an approximate solution, one then obtains a rigorous proof of the existence of these circles. These issues will be explored in more detail in Section 7.

We also note that the present proof does not require much from the function except that it gives a parameterization of an invariant torus. In particular, it can apply to tori of topological types not present in the original system.

Remark 6.17. Another proof without changes of variables can be found in [Bos86] which is based in unpublished work of M. Herman. This proof contains also a translated curve theorem.

The main difference with the proof presented here is that that method parameterizes the curves by the graph of a function. When studying tori that are not graphs, it requires that one performs a preliminary change of variables.

Remark 6.18. The twist hypothesis in this method of proof can be bargained away considerably.

Remark 6.19. A variant of the method that is useful in the study of lower dimensional tori or for some degenerate situations, is to take as a starting point of the procedure not just the K but the K and the M, which, respectively, almost solve the equation and almost reduce the equation to constant coefficients.

The iterative step, uses the M to solve the equation and then updates the M so that it reduces the new linearized equation to an even higher approximation.

By intertwining the improvement in K and in M it is possible to achieve quadratic convergence.

One advantage of this improvement is that, if one studies this for lower dimensional tori, both the K and the M can be computed perturbatively. The approximate K is a polynomial in the perturbation parameter, nevertheless, the M is a polynomial in the square root. Hence, the iterative method based on both approximations can capture the singularity structure much better than the approximation we have discussed here.

We refer to [JdlLZ99] for more details about the method for lower dimensional tori.

Remark 6.20. One feels that these methods of reducing the equation to constant coefficients is a bit of overkill. When one tries to invert an operator, diagonalizing it is rather more than what is needed.

Indeed, the great advances in KAM for PDE's started when the emphasis went from diagonalizing the linear operator as was done usually in KAM theory to just using estimates from the inverse. (See [CW93],[Bou99a]) Even if we will not discuss it in these notes, when one considers elliptic lower dimensional tori some of the resonances that appear in some proofs are obstructions to the diagonalization of the operator, not to the invertibility. Therefore, they can be eliminated from the proofs of the existence of the torus if one relies on inverting the operator rather than just diagonalizing it. (See [Bou97].) Related to this issue we call attention to the lectures of prof. Eliasson in this meeting. He shows that even if it could happen that the tori are not reducible, using his non-perturbative results, they are arbitrarily close to reducible. This is enough to continue the iterative procedure.

Remark 6.21. One issue that still is quite puzzling to me is that if one performs the Lindstedt series for lower dimensional tori, one encounters only small denominators related to the frequencies of the motion on the torus. This is significantly less small denominators than those appearing in the proofs mentioned above in which one needs to take into account denominators which happen when harmonics of the intrinsic frequencies of the torus are close to a normal frequency. The proofs in which one also proves reducibility of the lower dimensional torus, require even more small denominators conditions. In them, one has to take into account the cases when differences of two normal frequencies become a combination of the frequencies of motion in the torus.

In [JdlLZ99], one can find a proof of the fact that the Lindstedt series is asymptotic and defines an analytic function in a large sector. (One has to exclude an exponentially thin wedge.) Nevertheless, the convergence or not of these series has not been settled.

Note that at the same time that one develops the series for the torus, one develops also a series for the reducing matrix which also does not present other small divisors than those of the intrinsic frequencies. The convergence of this series has not been settled either.

7. Some remarks on computer assisted proofs

The existence or non-existence of invariant tori in a system appearing in a concrete application could have enormous practical importance.

For example, there are many systems such as accelerators or plasma devices that are modeled rather well by Hamiltonian systems. The existence of tori in these systems has very drastic effects in their long term behavior. For example, if the system is a two dimensional map, the existence of invariant circles, will imply that one region of phase space will remain trapped for ever. This is of great interest for plasma devices whose goal is to confine a plasma or for accelerators that try to keep a beam of particles in place. Indeed, many of these devices are designed in such a way that they maximize the *abundance* and robustness of invariant tori. One hopes that, even if the Hamiltonian approximation is not completely accurate, the KAM tori will survive somehow.

In celestial mechanics, one is interested also in finding regions with invariant tori since they are suitable for parking orbits.

The judicious numerical experimentation with dynamical system has been a great source of insight and inspiration, even if, of course, much of the work is non-rigorous and, hence, does not fit well with this tutorial. We refer the reader to [Hén83], [Sim98] for some study of the issues involved in numerical computations and to [Mei92] for a point of view closer to the physical applications

Of course, in these applications, one also wants to get, besides the existence, information about the shape of the torus and more details about its properties. What we want to discuss in this section is how some of these non-rigorous calculations can be turned into theorems.

The basic observation is that some of the KAM proofs we have presented here have the structure that they formulate a functional equation and show that, given an approximate solution which is not too bad from the analytic point of view, then there is a true solution, which, moreover is not too far from the approximate solutions. These constructive methods do not require that the system is close to integrable.

Note that these proofs do not care about how we have produced the approximate solutions. The only thing that we need to verify rigorously is that these approximate solutions indeed solve the functional equation to up to an small error and that their analyticity properties are adequate. Hence the problem of justifying that these computed solution correspond to a true one reduces to showing rigorously that these numerically specified functions indeed solve the desired functional equation with a good accuracy and verifying rigorously their analyticity properties.

Of course, given one polynomial with a few coefficients one could imagine studying its properties with respect to an easy equation such as (6.30) with a pencil and a notepad. (See [Her86] for an example of these verifications with pencil and paper.) Nevertheless, if the number of coefficients approaches those needed for what is considered good accuracy in numerical calculations (this is often a few hundred or a few thousand coefficients), using a notepad becomes impossible.

One would like to use a computer. The problem with using a computer is that, as they are used most commonly computers do not deal with real numbers and they do not perform on them the mathematical arithmetic operations. In their normal mode of working, computers deal only with a finite set of numbers, the *representable* numbers [12]n modern computers, there are almost universally around 2^{64} representable numbers, those which can be written in 8 bytes – there are a few delicate and complicated issues such as denormalized numbers. Most computers also use for certain calculations numbers with 80 bits, which are, 2^80 numbers.

There is a rather detailed standard by IEEE [IEE85] on how to perform arithmetic in numbers. It does not only specify the precision to be used, but also rounding and how to report troubles such as attempted division by zero or overflow. This standard is now almost universally implemented in the chips and the languages (rater inexplicably Java did not include it) and there are good tests of compliance so that one can asses one's arithmetic. See [Kah96]. On these representable numbers, we perform arithmetic operations which are approximations of the arithmetic operations among real numbers.

These operations produce an approximation to the true answer if at all possible [13]he process of taking the true result and producing a representable number is called *rounding*. Returning an representable number that is larger than the true result is called rounding up, similarly rounding down, rounding to nearest, rounding towards zero etc. The IEEE standard mentioned above specifies that the user can control the properties of the rounding and of the exceptions by setting a control word. or if it is impossible to give a reasonable answer in terms of representable numbers (e.g. if you ask to multiply by 10 the largest representative number or to divide by zero) they do not return an answer, but they *raise an exception* which typically does something drastic such as causing the program to terminate abruptly, perhaps copying the state of the memory to a file (*dumping a core*) that can be examined to trace the problem. (A good discussion of the subtleties involved in the implementation of floating point arithmetic is [Knu97].)

One problem with this approximate way of proceeding is that approximate of approximate may not be approximate enough. Much less if one repeats

[12]I
[13]T

the process of approximation a large number of times. Of course, given that a computer nowadays produces over one hundred million operations in a second, we have to worry about the effect that performing millions of approximations may lead us away from a good approximation.

As every good numerical analyst knows, producing numbers is not too difficult. Unless the computer catches fire, you will get numbers. The real difficult issue is to produce numbers that can be trusted. More difficult even is to device methods that ensure the numbers produced can be trusted. One should keep in mind that most of the technology and research happens at the borderline regions when the algorithms are about to break. (If they were safely solvable, we would fix the situation going to a more challenging problem.)

The problem of reliability of arithmetic calculations is significantly more pressing for the problems involving small divisors. We have seen already that the Lindstedt series manage to converge only through massive amounts of cancellations. Cancellations are one of the worst enemies of accuracy in floating point calculations. Since computers keep a fixed number of digits, adding numbers that cancel almost exactly, will lead to a catastrophic lack of precision (e.g. if we have 1.00001 and 1.00000 exact up to six digits, their difference will only have one exact digit.) Many of the problems with small divisors are such that the numerics deteriorates in a complicated way until the algorithms blow up or start behaving erratically.

Exercise 7.1. One of the standard programs to asses the characteristics of a computer is

```
epsilon = 1.0;
oneplus = 1.0 + epsilon;
count = 0;
while (oneplus > 1.0){
epsilon /= 2.0;
oneplus = one + epsilon;
count++;
}
printf("%d", count);
```

Run it in your computer.

Run also

```
epsilon = 1.0;
count = 0;
while ( 1.0 + epsilon > 1.0){
epsilon /= 2.0;
count++;
}
printf("%d", count);
```

Chances are that the results will be quite different. Explain why.

Exercise 7.2. The computer program Mathematica uses a numerical scheme in which high precision numbers drop precision if the last figures cannot be kept.

This leads to some unexpected effects.

Run

```
a = N[Pi,40]
Do[ a = 2*a -a , {100}]
```

and discuss the results.

One way of obtaining reliable results from a computer without sacrificing too much performance is to use *interval arithmetic* (See [Moo79], [KM84].) The idea is that a real variable is represented by two representable numbers which are supposed to mean an upper and a lower bound for the value of the variable we are interested in.

Once one has bounds for the values of a variable, one can operate on these bounds in such a way that one always keeps obtaining bounds. The only subtlety is that when adding upper bounds, one has to round up, adding lower bounds, one has to round down, etc. This can be done by reprogramming pieces of the arithmetic, or, in systems that conform to the IEEE standard by setting appropriately the control word. This quickly leads to an arithmetic among intervals that can produce bounds of arithmetic expressions given bounds on the variables.

One can pass from bounds on arithmetic expressions to bounds on sets in functional spaces. For example, one can specify a set in function space. For example, if we specify a set of analytic functions by

$$U_{v_1,\dots,v_n;\varepsilon} = \{f(z) \quad | \quad f(z) = \sum_{i=0}^{N} f_i z^i + f_e(z), f_i \in v_i, ||f_e||_1 \le \varepsilon\} \tag{7.1}$$

where v_i are intervals (i.e. pairs of representable numbers) and ε is a representable number. (There are, of course, many variants. One can for example, take into account that some errors are high order, use other norm for the error or even several norms at the same time.)

It is reasonably easy to imagine how can one define operations on sets of the type in (7.1) such that the numerical operations bound the real operations on sets. With a bit more of imagination, one can do compositions, integrals, and other operations. In particular, one can implement the operations involved in the evaluation of the terms in (6.25).

If starting with the numerically produced non-rigorous guess one can use the rigorous interval arithmetic to verify the hypothesis of Theorem 6.8 – or some other theorem enjoying a similar structure – then, one can guarantee that there is a true solution near the computed one. This strategy has been implemented in [Ran87], [dlLR91]. Similar ideas have been implemented in [CC95].

Indeed, by now, starting with the inspiring proof of [Lan82] (it relied on the usual contraction mapping theorem rather than in the hard implicit function theorems) there has been a number of significant theorems proved with similar techniques. A survey of these developments is [KSW96].

One of the main difficulties of the method is that it requires to spend a great deal of time in coding carefully the problems. One can hope that some of the tasks could be automated but there are difficulties. Even if automatic translation of arithmetic expressions produces a valid answer, arithmetic expressions that are equivalent under the ordinary rules of arithmetic are not equivalent under interval arithmetic. For example in intervals

$$(a + b) \times c \subset a \times c + b \times c \tag{7.2}$$

and the inclusion can be strict. A classic problem in interval arithmetic is to find fast algorithms to compute accurately the image of the unit disk under a polynomial.

Exercise 7.3. Give a proof of (7.2) and find examples when it is strict.

I personally think that computer assisted proofs and is a very interesting area in which it is possible to find a meaningful collaboration between Mathematicians (proving theorems of the right kind), Computer Scientists (developing good software tools that relieve the tedium of programming the variants required) and applied scientists that have challenging real life problems.

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