Restricted Four and Five Body Problems in the Solar System*

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Abstract

We focus on the dynamics of a small particle near the Lagrangian points of the Sun-Jupiter system. To try to account for the effect of other planets, such as Saturn or Uranus, we develop specific models based on the numerical computation of periodic and quasi-periodic (with two frequencies) solutions of the N-body problem and write them as perturbations of the Sun-Jupiter restricted Three Body Problem.

1 Introduction

The dynamics around the Lagrangian L_4 and L_5 points of the Sun-Jupiter system have been studied by several authors in the Restricted Three Body Problem using semi-analytical tools such as normal forms or approximate first integrals (see [5, 10, 3, 6, 11]).

On the other hand, it is known that Trojan asteroids move near the triangular points of the Sun-Jupiter system. The dynamics of these jovian Trojan asteroids has been studied by many authors (see, for example, [8, 9, 12]) using the Outer Solar System model, where the Trojans are supposed to move under the attraction of the Sun and the four main outer planets (Jupiter, Saturn, Uranus and Neptune). This is a strictly numerical model, so the semi-analytical tools mentioned above cannot be used in principle.

In this paper, we briefly present three intermediate models for the motion of a Trojan asteroid. These models try to simulate in a more realistic way the relative Sun-Jupiter motion and are written as perturbations of the RTBP, such that the semi-analytical tools can be applied.

The first model that we present is a natural improvement of the Sun-Jupiter RTBP that includes the effect of Saturn on the motion of Sun and Jupiter. In this model, Sun, Jupiter and Saturn move in a periodic solution of the (non-restricted) planar three body problem, with the same relative period as the real one. Then, it is possible to write the equations of motion of a fourth massless particle that moves under the attraction of these three. This is a restricted four body problem that we call Bicircular Coherent Problem (BCCP, for short). Its detailed construction and study can be found in [4].

In the second model, the periodic solution of the BCCP is used as the starting point of the computation of a 2-D invariant torus for which the osculating eccentricity of Jupiter's orbit is

*Proceedings of the International Conference on Libration Point Orbits and Applications held at the Parador d'Aiguablava, Girona, Spain, June 10-14, 2002

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the actual one. In this sense, the Sun-Jupiter relative motion is better simulated by this quasi-periodic solution of the planar three body problem. Afterwards, the equations of motion of a massless particle that moves under the attraction of these three main bodies (supposing that they move in the quasi-periodic solution) are easily derived. We call this restricted four body problem as the Bianular Problem (BAP, for short).

The third model is based on the computation of a quasi-periodic solution (with two basic frequencies) of the planar four body problem Sun, Jupiter, Saturn and Uranus. A restricted five body model can be constructed by writting the equations of a massless particle that moves under the influence of the four bodies. We call it Tricircular Coherent Problem (TCCP, for short).

2 The Bicircular Coherent Problem

It is possible to find, in a rotating reference frame, periodic solutions of the planar three body Sun-Jupiter-Saturn problem by means of a continuation method using the masses of the planets as parameters (see [4] for details). The relative Jupiter-Saturn period can be chosen as the actual one, and its related frequency is $\omega_{sat} = 0.597039074021947$.

Assuming that these three main bodies move on this periodic orbit, it is possible to write the Hamiltonian for the motion of a fourth massless particle as:

$$\begin{aligned} H &= \frac{1}{2}\alpha_1(\theta)(p_x^2+p_y^2+p_z^2) + \alpha_2(\theta)(xp_x+yp_y+zp_z) + \alpha_3(\theta)(yp_x - xp_y) \\ &\quad +\alpha_4(\theta)x + \alpha_5(\theta)y - \alpha_6(\theta)\left[\frac{1-\mu}{q_S} + \frac{\mu}{q_J} + \frac{m_{sat}}{q_{sat}}\right], \end{aligned} \tag{1}$$

where $q_S^2 = (x-\mu)^2+y^2+z^2$, $q_J^2 = (x-\mu+1)^2+y^2+z^2$ and $q_{sat}^2 = (x-\alpha_7(\theta))^2+(y-\alpha_8(\theta))^2+z^2$. The functions $\alpha_i(\theta)$ are periodic functions in $\theta = \omega_{sat}t$ and can be explicitly computed with a Fourier analysis of the numerical periodic solution of the three body problem.

At that point, we want to mention that a Bicircular Coherent problem was already developed in [1] for the Earth-Moon-Sun case to study the dynamics near the Eulerian points.

3 The Bianular Problem

In this section, we compute a quasi-periodic solution, with two basic frequencies, of the planar Sun-Jupiter-Saturn three body problem. This quasi-periodic solution lies on a 2-D torus. As the problem is Hamiltonian, this torus belongs to a family of tori. We look for a torus, on this family, for which the osculating eccentricity of Jupiter's orbit is quite well adjusted to the actual one.

First, we rewrite the Hamiltonian of the planar three body problem in which the computations are done and we revisit a method for computing 2-D invariant tori. Afterwards, the desired quasi-periodic solution is found and the Hamiltonian of the Bianular Problem is explicitly obtained.

3.1 The reduced Hamiltonian of the Three Body Problem

We take the Hamiltonian of the planar three body problem written in the Jacobi coordinates in a uniformly rotating reference frame and we make a canonical change of variables (using the angular momentum first integral) in order to reduce this Hamiltonian from 4 to 3 degrees of freedom. We obtain:

$$
\begin{aligned}
H(Q_1,Q_2,Q_3,P_1,P_2,P_3) &= \frac{1}{2\alpha}\left(P_1^2+\frac{A^2}{Q_1^2}\right)+\frac{1}{2\beta}\left(P_2^2+P_3^2\right) \\
&\quad -K-\frac{\alpha}{r}-\frac{(1-\mu)m_{sat}}{r_{13}}-\frac{\mu m_{sat}}{r_{23}}
\end{aligned} \tag{2}
$$

where $\alpha=\mu(1-\mu)$, $\beta=m_{sat}/(1+m_{sat})$, $A=Q_2P_3-Q_3P_2+K$, $r=Q_1$, $r_{13}^2=(\mu Q_1-Q_2)^2+Q_3^2$ and $r_{23}^2=((1-\mu)Q_1+Q_2)^2+Q_3^2$, and K is the angular momentum of the system.

3.2 A method for computing 2-D invariant tori

We are interested in finding a quasi-periodic solution (with two frequencies) of a given vector field. We reduce this problem to the one of finding an invariant curve of a suitable Poincaré map. This invariant curve is seen as a truncated Fourier series and our aim is to compute its rotation number and a representation of it. We follow roughly the method developed by [2].

3.2.1 Numerical computation of invariant curves

Let be $\dot{x}=f(x)$ $(x,f\in\mathbb{R}^n)$ an autonomous vector field of dimension n (for example, the reduced field of the three body problem given at 3.1) and $\Phi(x,t)\equiv\Phi_t(x)$ its associated flow. Let us define the Poincaré map as the time T-flow $\Phi_T(\cdot)$, where T is a prefixed value ($T=T_{sat}$, the period of Saturn in the Sun-Jupiter system, in our case).

Let ω be the rotation number of the invariant curve. Let, also, $\mathcal{C}(\mathbb{T}^1,\mathbb{R}^n)$ be the space of continuous functions from $\mathbb{T}^1$ in $\mathbb{R}^n$, and let us define the linear map $T_\omega:\mathcal{C}(\mathbb{T}^1,\mathbb{R}^n)\to\mathcal{C}(\mathbb{T}^1,\mathbb{R}^n)$ as the translation by ω, $(T_\omega\varphi)(\theta)=\varphi(\theta+\omega)$.

Let us define $F:\mathcal{C}(\mathbb{T}^1,\mathbb{R}^n)\to\mathcal{C}(\mathbb{T}^1,\mathbb{R}^n)$ as

$$
F(\varphi)(\theta)=\Phi_T(\varphi(\theta))-(T_\omega\varphi)(\theta)\qquad\forall\varphi\in\mathcal{C}(\mathbb{T}^1,\mathbb{R}^n). \tag{3}
$$

It is clear that the zeros of F in $\mathcal{C}(\mathbb{T}^1,\mathbb{R}^n)$ correspond to invariant curves of rotation number ω. The equation satisfied is

$$
\Phi_T(\varphi(\theta))=\varphi(\theta+\omega)\quad\forall\theta\in\mathbb{T}. \tag{4}
$$

The method we want to summarize in this section boils down to looking numerically for a zero of F. Hence, let us write $\varphi(\theta)$ as a real Fourier series,

$$
\varphi(\theta)=A_0+\sum_{k>0}(A_k\cos(k\theta)+B_k\sin(k\theta))\quad A_k,B_k\in\mathbb{R}^n\quad k\in\mathbb{N}.
$$

Then, we will fix in advance a truncation value N_f for this series (the selection of the truncation value will be discussed later on), and let us try to determine (an approximation to) the $2N_f+1$ unknown coefficients A_0, A_k and B_k, $1\le k\le N_f$. To this end, we will construct a discretized version of the map F, as follows: first, we select the mesh of $2N_f+1$ points on $\mathbb{T}^1$,

$$
\theta_j=\frac{2\pi j}{2N_f+1}\quad 0\le j\le 2N_f, \tag{5}
$$

and evaluate the function (3) on it. Let F_{N_f} be this discretization of F:

$$
\Phi_T(\varphi(\theta_j))-\varphi(\theta_j+\omega),\quad 0\le j\le 2N_f. \tag{6}
$$

So, given a (known) set of Fourier coefficients A_0, A_k and B_k $(1 \leq k \leq N_f)$, we can compute the points $\varphi(\theta_j)$, then $\Phi_T(\varphi(\theta_j))$ and next the points $\Phi_T(\varphi(\theta_j)) - \varphi(\theta_j + \omega)$, $0 \leq j \leq N_f$. From these data, we can immediately obtain the Fourier coefficients of $\Phi_T(\varphi(\theta)) - \varphi(\theta + \omega)$.

To apply a Newton method to solve the equation $F_{N_f} = 0$, we also need to compute explicitly the differential of F_{N_f}. This can be done easily by applying the chain rule to the process used to compute F_{N_f}. Note that the number of equations to be solved is $(2N_f + 1)n$ and that the unknowns are $(A_0, A_1, B_1, \ldots, A_{N_f}, B_{N_f})$, ω and the time T for which we fix the Poincaré map associated to the flow $(\Phi_T(\cdot))$. That is, we deal with $(2N_f + 1)n + 2$ unknowns. In each step of the Newton method, we solve a non-square linear system by means of a standard least-squares method. We want to mention that this system is degenerated unless we fix (keep constant during the computation) some of the unknowns.

Note that in the case of the reduced three body problem, an integral of motion is still left: the energy. We can easily solve the problem of the degeneracy, induced by it, fixing the time T of the Poincaré map.

3.2.2 Discretization error

Once we have solved equation (6) with a certain tolerance (error in the Newton method; we take tipically 10^{-11}), we still don't have any information on the error of the approximated invariant curve. The reason, as explained in [2], is that we have not estimated the discretization error; i.e., the error when passing from equation (4) to equation (6).

In order to do it, we compute

$$E(\varphi, \omega) = \max_{\theta \in \mathbb{T}} |\Phi_T(\varphi(\theta, z)) - \varphi(\theta + \omega, z)|$$

in a mesh of points, say, 100 times finer than the mesh (5) and consider it as an estimation of the error of the invariant curve.

If $||E||_\infty > 10^{-9}$, the solution obtained is not considered good enough and another one with the same initial approximation for the Newton method but with a greater discretization order N_f is computed. The process is repeated until the sub-infinity norm of the discretization error is smaller than 10^{-9}.

3.3 Finding the desired torus

The initial approximation to the unknowns in the Newton method is given by the linearization of the Poincaré map around a fixed point (a periodic orbit, for the flow) X_0. We use the periodic orbit computed in Section 2 for the BCCP model:

$$X_0 = \Phi_{T_{sat}}(X_0),$$

where, $\Phi_{T_{sat}}(\cdot)$ is the time T_{sat}-flow corresponding to Hamiltonian (2).

It is easy to see, by looking at the eigenvalues of $D\Phi_{T_{sat}}(X_0)$, that there are two different non-neutral normal directions to the periodic orbit X_0. Thus, two families of tori arise from it. We call them Family1 and Family2.

We compute a first torus for each family (they are called Torus1 and Torus2) with the method described in 3.2. Once we have a first torus, we want to continue the family it belongs to. This family can be parameterized by the angular momentum K. It is straightforward from the computations that there is a strong relationship between the angular momentum, K, and the osculating orbital elements of Jupiter's and Saturn's orbits. As we want to simulate in a more realistic way the Sun-Jupiter relative motion, we are more interested in adjusting Jupiter's orbital elements than Saturn's ones. As we have one degree of freedom (we are allowed to set K), we select the osculating eccentricity of Jupiter's orbit as targeting value. Thus, by means of a continuation method, we try to find another torus inside Family1 and Family2 for which the osculating eccentricity of Jupiter (at a given moment) is exactly 0.0484.

In order to continue the families, we add to the invariant curve equations the following one:

$$\mathrm{eccen}(Q_1, Q_2, Q_3, P_1, P_2, P_3, K) = e,$$

where $\mathrm{eccen}(\cdot)$ is a function that gives us Jupiter's osculating eccentricity at a given moment (we evaluate it when Sun, Jupiter and Saturn are in a particular collinear configuration), and e is a fixed constant that is used as a control parameter. We try to continue each family increasing the parameter e to its actual value.

In Family1, we start from Torus1 increasing little by little the parameter e in order to have a good enough initial point for the Newton method in each step of the continuation process. What is observed is that when e increases, the number of harmonics (N_f) has also to be increased if we want the discretization error of the invariant curve to be smaller than a certain tolerance (tipically we take 10^{-9}). We stop the continuation when the number of harmonics is about 180. At this moment, if we look at the orbital elements of Jupiter's and Saturn's orbits, we see that they do not evolve in the desired direction, but they are getting farther from the real ones. Thus, increasing Jupiter's eccentricity inside Family1 forces us to move away from the desired solution and we fail in finding an adequate torus in Family1.

For Family2, we proceed in the same way as before but starting from Torus2. In this case, we are able to increase e up to its actual value ($e = 0.0484$), the number of harmonics doesn't grow up very much (actually, if we ask the invariant curve to have an error smaller than 10^{-9}, N_f increases from 6 to 9) and the solution obtained is of the planetary type. In Figure 1, we plot the variation of the angular momentum K of the planar SJS Three Body Problem when the parameter e is increased during the continuation process.

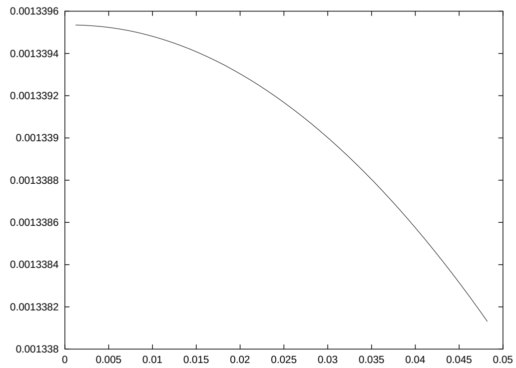

Figure 1: Plot of the evolution of the angular momentum K when the parameter e is increased from 0.00121 (corresponding to Torus2) to 0.0484 (the desired final value) in the continuation of Family2.

We can see the projection of the final torus into the configuration space in Figure 2. This solution of the planar Sun-Jupiter-Saturn TBP is what we call the Bianular solution of the TBP. This torus is parameterized with the angles $(\theta_1, \theta_2) = (\omega_1 t + \theta_1^0, \omega_2 t + \theta_2^0)$, where the frequencies

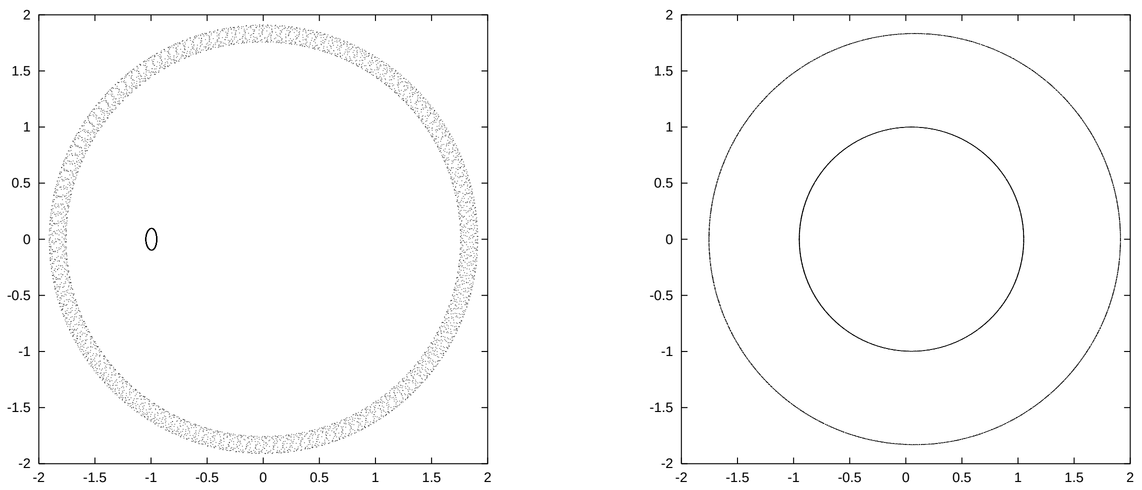

Figure 2: Projection into the configuration space of the Bianular solution of the planar three body problem Sun-Jupiter-Saturn in the rotating reference frame (left plot) and in an inertial system (right plot).

are $\omega_1 = \omega_{sat} = 0.5970390074021947$ and $\omega_2 = \frac{\omega_1 \bar{\omega}}{2\pi} = 0.194113943490717$ ($\bar{\omega}$ is the rotation number of the invariant curve), and $\theta^0_{1,2}$ are the initial phases.

3.4 The Hamiltonian of the BAP Model

Finally, it is possible to obtain the equations of a massless particle that moves under the attraction of the three primaries. The corresponding Hamiltonian is:

$$\begin{aligned} H_{BAP} &= \frac{1}{2}\alpha_1(\theta_1,\theta_2)(p_x^2+p_y^2+p_z^2) \\ &+\alpha_2(\theta_1,\theta_2)(xp_x+yp_y+zp_z) \\ &+\alpha_3(\theta_1,\theta_2)(yp_x-xp_y)+\alpha_4(\theta_1,\theta_2)x \\ &+\alpha_5(\theta_1,\theta_2)y-\alpha_6(\theta_1,\theta_2)\left[\frac{1-\mu}{q_S}+\frac{\mu}{q_J}+\frac{m_{sat}}{q_{sat}}\right], \end{aligned}$$

where $q_S^2 = (x-\mu)^2+y^2+z^2$, $q_J^2 = (x-\mu+1)^2+y^2+z^2$, $q_{sat}^2 = (x-\alpha_7(\theta_1,\theta_2))^2+(y-\alpha_8(\theta_1,\theta_2))^2+z^2$, $\theta_1 = \omega_1 t + \theta_1^0$ and $\theta_2 = \omega_2 t + \theta_2^0$.

The auxiliar functions $\alpha_i(\theta_1,\theta_2)_{\{i=1\div 8\}}$ are quasi-periodic functions that can be computed by a Fourier analysis of the solution found in 3.3.

4 The Tricircular Coherent Problem

In this section, we describe the computation of a quasi-periodic solution with two basic frequencies of the planar four body problem Sun, Jupiter, Saturn and Uranus (SJSU). We adapt the method described in Section 3.2 for computing invariant curves of maps to this case.

First, we write the reduced Hamitonian of the four body problem in which the computations are done. Second, we give an heuristic approximation of the initial point used in the Newton

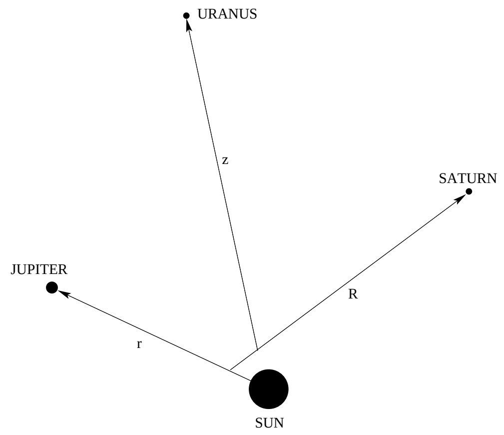

Figure 3: Jacobi coordinates for the Sun, Jupiter, Saturn and Uranus four body problem.

method for computing an invariant curve when the mass of Uranus is equal to zero. Then, we compute a first torus for this case. Finally, by means of a continuation method (taking the mass of Uranus as the parameter), we compute a quasi-periodic solution of the SJSU planar problem.

4.1 The Hamiltonian of the SJSU problem

We take the Hamiltonian of the planar four body problem written in the generalized Jacobi coordinates (Figure 3) in a uniformly rotating reference frame and we make a canonical change of variables (using the angular momentum first integral) in order to reduce this Hamiltonian from 6 to 5 degrees of freedom. We obtain the following Hamiltonian:

$$\begin{aligned} H &= \frac{1}{2\alpha}\left(P_1^2+\frac{A^2}{Q_1^2}\right)+\frac{1}{2\beta}\left(P_2^2+P_3^2\right)+\frac{1}{2\gamma}\left(P_4^2+P_5^2\right)-K-\frac{\alpha}{r} \\ &\quad -\frac{(1-\mu)m_{sat}}{r_{13}}-\frac{\mu m_{sat}}{r_{23}}-\frac{(1-\mu)m_{ura}}{r_{14}}-\frac{\mu m_{ura}}{r_{24}}-\frac{m_{sat}m_{ura}}{r_{34}}, \end{aligned} \tag{7}$$

where $\alpha = \mu(1-\mu)$, $\beta = m_{sat}/(1+m_{sat})$, $\gamma = (1+m_{sat})m_{ura}/(1+m_{sat}+m_{ura})$, $A = Q_2P_3-Q_3P_2+Q_4P_5-Q_5P_4+K$, $r = Q_1$, $r_{13}^2 = (Q_2-\mu Q_1)^2+Q_3^2$, $r_{23}^2 = (Q_2+(1-\mu)Q_1)^2+Q_3^2$, $r_{14}^2 = (Q_5+\beta Q_3)^2+(Q_4+\beta Q_2-\mu Q_1)^2$, $r_{24}^2 = (Q_5+\beta Q_3)^2+(Q_4+\beta Q_2+(1-\mu)Q_1)^2$ and $r_{34}^2 = (Q_5-(1-\beta)Q_3)^2+(Q_4-(1-\beta)Q_2)^2$, and K is the angular momentum of the system.

4.2 Computation of a first torus

As a first approximation, we suppose that Uranus has mass equal to zero and that it is moving in a Keplerian orbit around the Sun. Using the method for computing invariant curves, described in 3.2, we obtain a 2-D invariant torus for this case.

If we suppose that, at $t=0$, Sun, Jupiter, Saturn and Uranus are in a particular collinear configuration, it is easy to obtain an approximate initial condition for the Newton method by taking the Keplerian one for Uranus, and the periodic orbit X_0 (used in the construction of the BCCP in Section 2) for Sun, Jupiter and Saturn. As we have already seen, the period of this (periodic) orbit is $T = T_{sat} = \frac{2\pi}{\omega_{sat}}$, where $\omega_{sat} = 0.597039074021947$ is the relative frequency of Saturn in the Sun-Jupiter rotating system. This will be the first frequency of the 2-D invariant torus.

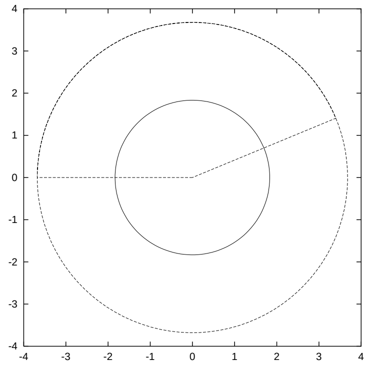

Figure 4: Heuristic first approximation of the Uranus motion (exterior orbit) around the Sun. The inside orbit corresponds to a periodic orbit of Saturn. The lines show the starting and final points of the Uranus orbit for an interval of time of T_{sat}.

If we integrate the flow corresponding to Hamiltonian (7) in the time interval $t \in [0, T_{sat}]$ taking as initial condition a point of the periodic orbit X_0 for the Sun-Jupiter-Saturn system and the Keplerian approximation for Uranus, we find that the orbit corresponding to Sun, Jupiter and Saturn obviously closes (it is a periodic orbit of period T_{sat}) and the one corresponding to Uranus turns one lap-odd (see Figure 4).

We are interested in measuring the angle that Uranus covers in its trajectory (actually, the angle between the two straight lines in Figure 4). We relate this angle with the rotation number of the invariant curve that we are looking for. It is possible, from the initial an final points of the integration, to compute the value of the angle: $\omega_0 = 2.749448441$.

Let us note that this angle is very close to the following number:

$$\tilde{\omega} = \frac{2\pi\omega_{ura}}{\omega_{sat}} (\text{mod } 2\pi) = 2.750807556, \tag{8}$$

where $\omega_{ura} = 0.858425538978989$ is the relative frequency of Uranus in the Sun-Jupiter rotating system. Thus, if we impose $\tilde{\omega}$ to be the rotation number of the invariant curve, the second frequency of the 2-D invariant torus that we are computing will be ω_{ura}, Uranus's frequency.

4.3 The Tricircular solution of the SJSU problem

Once we have computed an invariant torus for the case $m_{ura} = 0$, we proceed by a continuation method to increase the parameter m_{ura} up to its actual value. During the continuation, the two frequencies, $\omega_1 = \omega_{sat}$ and $\omega_2 = \omega_{ura}$ are kept constant. Thus, we obtain a quasi-periodic solution (that moves on a 2-D torus parameterized by the two angles $\theta_1 = \omega_1 t + \theta_1^0$ and $\theta_2 = \omega_2 t + \theta_2^0$) of the reduced four body field (7).

In Figure 5, we can see the projection into the configuration space of this torus in a rotating frame (left plot) and in an inertial frame (right plot). We call this solution "Tricircular Coherent Solution" of the SJSU four body problem.

4.4 The Hamiltonian of the TCCP Model

Finally, it is possible to write the equations of motion of a massless particle that moves under the attraction of the four primaries, supposing that they move on the tricircular solution. The

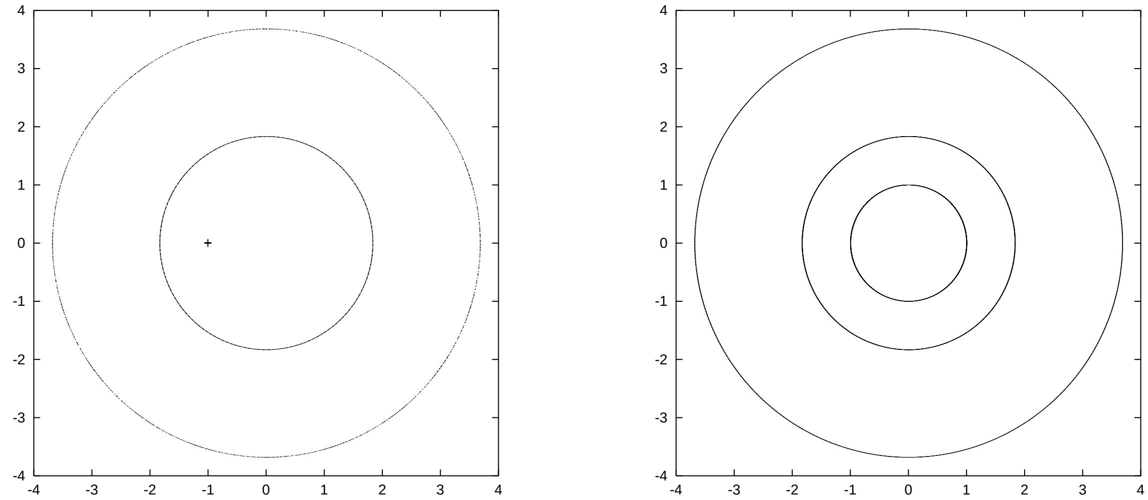

Figure 5: Quasi-periodic solution for the four body Sun-Jupiter-Saturn-Uranus Problem. The exterior orbit concerns to Uranus, the one in the middle to Saturn and the interior one (seen as a small point that librates around the point $(-1, 0)$, in the left plot) is the relative Sun-Jupiter's orbit. The left plot is represented in the rotating coordinates and the right one in an inertial reference frame.

corresponding Hamiltonian is:

$$\begin{aligned} H_{TCCP} &= \frac{1}{2}\alpha_1(\theta_1,\theta_2)(p_x^2+p_y^2+p_z^2)+\alpha_2(\theta_1,\theta_2)(xp_x+yp_y+zp_z)+\alpha_3(\theta_1,\theta_2)(yp_x-xp_y) \\ &\quad +\alpha_4(\theta_1,\theta_2)x+\alpha_5(\theta_1,\theta_2)y-\alpha_6(\theta_1,\theta_2)\left[\frac{1-\mu}{q_S}+\frac{\mu}{q_J}+\frac{m_{sat}}{q_{sat}}+\frac{m_{ura}}{q_{ura}}\right], \end{aligned}$$

where $q_S^2=(x-\mu)^2+y^2+z^2$, $q_J^2=(x-\mu+1)^2+y^2+z^2$, $q_{sat}^2=(x-\alpha_7(\theta_1,\theta_2))^2+(y-\alpha_8(\theta_1,\theta_2))^2+z^2$, $q_{ura}^2=(x-\alpha_9(\theta_1,\theta_2))^2+(y-\alpha_{10}(\theta_1,\theta_2))^2+z^2$, $\theta_1=\omega_{sat}t+\theta_1^0$ and $\theta_2=\omega_{ura}t+\theta_2^0$.

The auxiliar functions $\alpha_i(\theta_1,\theta_2)_{\{i=1\div 10\}}$ are quasi-periodic functions that can be computed by a Fourier analysis of the tricircular solution found in 4.3.

5 Conclusions

We have seen a particular case of a methodology for constructing semi-analytic models of the Solar System and write them as "perturbations" of the Sun-Jupiter RTBP. For instance, if a quasi-periodic solution of the N-Body Problem with m frequencies is known, it is then possible to write the Hamiltonian of the Restricted Problem of $(N+1)$ bodies as:

$$\begin{aligned} H &= \frac{1}{2}\alpha_1(\theta)(p_x^2+p_y^2+p_z^2)+\alpha_2(\theta)(xp_x+yp_y+zp_z) \\ &\quad +\alpha_3(\theta)(yp_x-xp_y)+\alpha_4(\theta)x+\alpha_5(\theta)y-\alpha_6(\theta)\sum_{i=0}^{N} G\frac{m_i}{\rho_i}, \end{aligned}$$

where the functions $\alpha_i(\theta)$ are also quasi-periodic with the same m frequencies ($\theta\in\mathbb{T}^m$) and ρ_i is the distance between the particle and the i-th body written in a "rotating-pulsating" reference system.

All these models (as BCCP, BAP and TCCP) are specially written in order that semi-analytical tools (such as Normal Forms or numerical First Integrals techniques) can be applied.

Acknowledgements

This work has been partially supported by the Spanish CICYT grant BFM2000–0623, the Catalan CIRIT grant 2001SGR–00070 and DURSI.

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