Relative equilibria of the RTBP in curved spaces

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Dedicated to professor Amadeu Delshams

for his $2^2 \times 3 \times 5$ birthday

Per molts anys!

The problem

We want to study the **relative equilibria (RE) of restricted three-body problem (RTBP)** in $\mathbb{S}^2$ and $\mathbb{H}^2$ as an extension of the planar case $\mathbb{R}^2$.

To proceed in an **unified way** we consider the **curvature** κ as a parameter.

$$\mathbb{S}^2: \quad \kappa(x^2+y^2+z^2)+2\kappa^{1/2}z=0, \quad \kappa>0,$$

which reduces to $z=0$ for $\kappa=0$. $\mathbb{S}^2$ is the sphere of radius $\kappa^{-1/2}$ centered at $(0,0,-\kappa^{-1/2})$.

In a similar way

$$\mathbb{H}^2: \quad \kappa(x^2+y^2-z^2)+2|\kappa|^{1/2}z=0, \quad \kappa<0,$$

also **tangent to** $z=0$ **at** $(0,0,0)$.

We shall use (x,y) as **local coordinates**. In the $\mathbb{S}^2$ case we reduce to the upper hemisphere.

The general equations of motion

Florin Diacu has derived the equations of motion$^{(*)}$ depending on κ for the **n-body problem** with masses $m_i, i = 1, \dots, n$, which can be written as

$$\ddot{x}_i = \sum_{j=1, j\neq i}^{n} m_j [x_j - f_{ij} x_i] d_{ij}^{-3} - \kappa v_i x_i, \quad \ddot{y}_i = \sum_{j=1, j\neq i}^{n} m_j [y_j - f_{ij} y_i] d_{ij}^{-3} - \kappa v_i y_i,$$

for $i = 1, \dots, n$, being $f_{ij} = 1 - \kappa \rho_{ij}^2/2$, $v_i = \dot{x}_i^2 + \dot{y}_i^2 + \kappa B_i$, where

$$\rho_{ij}^2 = (x_i - x_j)^2 + (y_i - y_j)^2 + \frac{\kappa (A_i - A_j)^2}{(E_i + E_j)^2}, \quad d_{ij} = \rho_{ij}(1 - \kappa \rho_{ij}^2/4)^{1/2},$$

$$A_i = x_i^2 + y_i^2, \quad E_i = (1 - \kappa A_i)^{1/2}, \quad B_i = (x_i \dot{x}_i + y_i \dot{y}_i)^2 / E_i^2.$$

A **first integral** for these equations is $T_\kappa - U_\kappa = h$, where

$$T_\kappa = \frac{1}{2} \sum_{i=1}^{n} m_i v_i, \quad U_\kappa = \sum_{1 \le i < j \le n} m_i m_j f_{ij} d_{ij}^{-1}.$$

$^{(*)}$ F. Diacu: The classical N-body problem in the context of curved space, preprint 2014.

Planning

- To **restrict to the case** $n = 3$, $m_1 = 1 - \mu$, $m_2 = \mu$, $m_3 = 0$.

- To introduce **rotating coordinates** $\begin{pmatrix} x_i \\ y_i \end{pmatrix} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} \xi_i \\ \eta_i \end{pmatrix}$, where $\theta = \alpha t$, α being a given **angular velocity**.

- To derive the conditions for the **relative equilibria**, looking for **positions of the primaries, collinear and triangular equilibria** as well as some **limit cases**.

- To **scan a fine grid** in suitable variables and to do **numerical continuation** from some limit solutions, to cover a fine grid in (κ, μ).

- To provide analytic results on the **spectral stability** of the equilibria.

- To complete the previous results with a **numerical study of the stability** on the solutions obtained by continuation.

The equations for $n = 3$ in rotating frame

Let $\mathbf{U}_i = (\xi_i, \eta_i)^T$ and $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. In the case of three masses the equations in **the rotating frame** are written as

$$\ddot{\mathbf{U}}_i - 2\alpha J\dot{\mathbf{U}}_i - \alpha^2\mathbf{U}_i = \sum_{j\neq i} m_j[\mathbf{U}_j - f_{ij}\mathbf{U}_i]d_{ij}^{-3} - \kappa v_i\mathbf{U}_i, \quad i = 1, 2, 3,$$

where

$$v_i = \alpha^2 A_i + |\dot{\mathbf{U}}_i|^2 + 2\alpha\mathbf{U}_i^T J\dot{\mathbf{U}}_i + \kappa(\mathbf{U}_i^T\dot{\mathbf{U}}_i)^2/E_i.$$

For the **restricted** problem we set $m_1 = 1 - \mu$, $m_2 = \mu$, $m_3 = 0$.

The equations for **relative equilibria** are

$$\alpha^2 E_1^2\mathbf{U}_1 + m_2(\mathbf{U}_2 - f_{12}\mathbf{U}_1)d_{12}^{-3} = 0,$$
$$\alpha^2 E_2^2\mathbf{U}_2 + m_1(\mathbf{U}_1 - f_{12}\mathbf{U}_2)d_{12}^{-3} = 0,$$
$$-m_1\mathbf{U}_1 d_{13}^{-3} - m_2\mathbf{U}_2 d_{23}^{-3} + [m_1 f_{13}d_{13}^{-3} + m_2 f_{13}d_{13}^{-3} - \alpha^2 E_3^2]\mathbf{U}_3 = 0.$$

It is not restrictive to set $\mathbf{U}_1 = (r_1, 0)^T$, $\mathbf{U}_2 = (-r_2, 0)^T$, $\mathbf{U}_3 = (a, b)^T$, $r_1, r_2 \geq 0$.

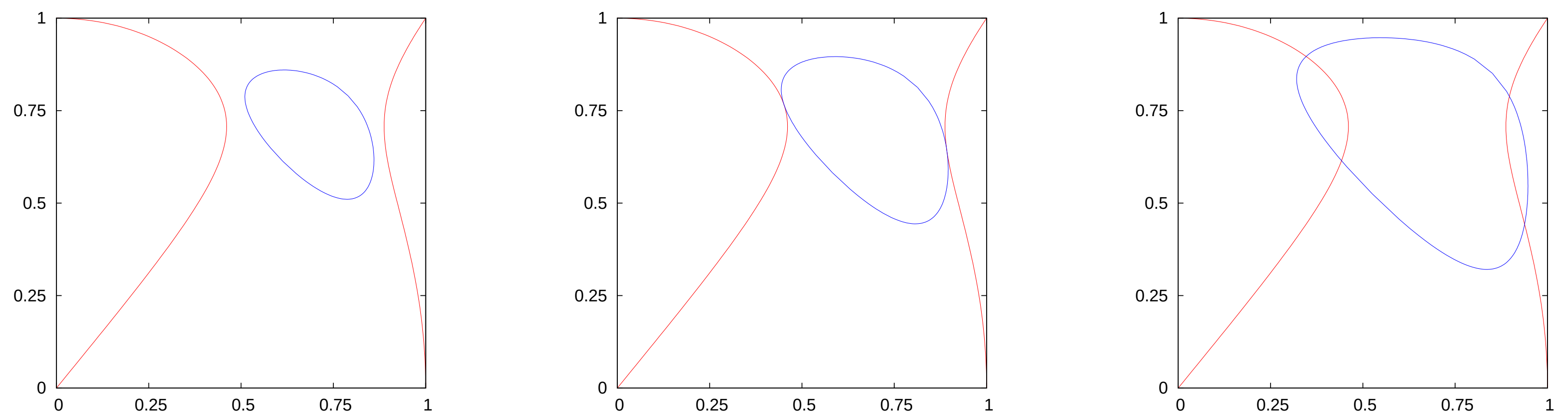
Remark. It is easy to check that one can **scale κ and α** and then to scale the coordinates accordingly. In some cases it can be useful to set $\kappa=1$ if $\kappa>0$ (or $\kappa=-1$ if $\kappa<0$). In other cases it can be useful to set $\alpha=1$.

The **conditions on r_1, r_2 for the primaries** are

$$\alpha^2 r_1 E_1^2 = m_2 d_{12}^{-3}(r_2 + r_1 f_{12}), \quad \alpha^2 r_2 E_2^2 = m_1 d_{12}^{-3}(r_1 + r_2 f_{12}).$$

Given r_1, r_2 one can **recover μ, α**, $\mu = F(r_1, r_2)$, $\alpha^2 = G(r_1, r_2)$ as

$$F(r_1, r_2) = \frac{r_1 E_1}{r_1 E_1 + r_2 E_2}, \quad G(r_1, r_2) = \frac{2}{(r_1 E_2 + r_2 E_1)^2 (r_1 E_1 + r_2 E_2)}.$$



Plots of **constant $\mu = 0.45$** (red curves) and values of $\alpha^2 = 1.1,\ 1.178016,\ 1.4$ (blue curves) from left to right, using $\kappa = 1$. They suggest the

Proposition 1. *Consider the system for* (r_1, r_2) *with* $\kappa = 1$ *and look for solutions in* $[0,1]^2$. *Then for any* $\mu \in (0,1)$, $\mu \neq 1/2$, *there exists* $\alpha_c(\mu)^2 > 1$, *such that*

1. *If* $0 < \alpha^2 < \alpha_c^2$ *the system has no solutions.*
2. *If* $\alpha^2 = \alpha_c^2$, *it has exactly two solutions.*
3. *If* $\alpha^2 > \alpha_c^2$, *it has exactly four solutions.*

If $\mu = 1/2$ *then the system has no solutions for* $0 < \alpha^2 < 1$, *four solutions for* $\alpha^2 > 1$, *and exactly one solution* $(r_1, r_2) = (1/\sqrt{2}, 1/\sqrt{2})$ *if* $\alpha = 1$.

Exchanging μ and $1-\mu$ the values of r_1 and r_2 are exchanged.

The places where **bifurcation occurs** in (r_1, r_2) can be obtained explicitly using $r_1 = \sin(\phi_1), r_2 = \sin(\phi_2)$ and are given as

$$\cos(2\phi_2) = -\frac{\cos(2\phi_1)}{1 + 3\cos^2(2\phi_1)}(2 \pm \sqrt{3}\sin(2\phi_1)),$$

using the $+$ for $\pi/6 \leq \phi_1 \leq \pi/3$ and $-$ for $0 \leq \phi_1 \leq \pi/2$.

Collinear relative equilibria in $\mathbb{S}^2$

Assume r_1, r_2, μ, α fixed such that **the primaries are in RE** with $\kappa = 1$. Let $\mathbf{U}_3 = (a, 0)$. The condition for a is

$$(1-\mu)\frac{(aE_1 - r_1E_3)}{|aE_1 - r_1E_3|^3} + \mu\frac{(aE_2 + r_2E_3)}{|aE_2 + r_2E_3|^3} = \alpha^2 aE_3.$$

As in the Lagrangian solutions **we should distinguish 3 cases**:
1) $a > r_1$; 2) $-r_2 < a < r_1$; 3) $a < -r_2$.
A special case appears for $r_1 = r_2$. Then $\mu = 1/2$. The cases $a > r_1$ and $a < -r_2$ are symmetrical. One has

Proposition 2. *Let $r_1 = r_2$ fixed. There is $r^* = 0.33784995\ldots$ such that*

1. *If $r_1 < r^*$ then there are* **5 collinear RE** *one of them between the primaries, and two additional ones at each side of the primaries.*
2. *If $r_1 = r^*$ then there are* **3 collinear RE**.
3. *If $r_1 > r^*$ then there is a* **unique collinear RE** *between the primaries.*

For $a \in (-r_1, r_1)$ one has $a = 0$. r^* is a zero of a **polynomial** *in $\mathbb{Q}(x)$.*

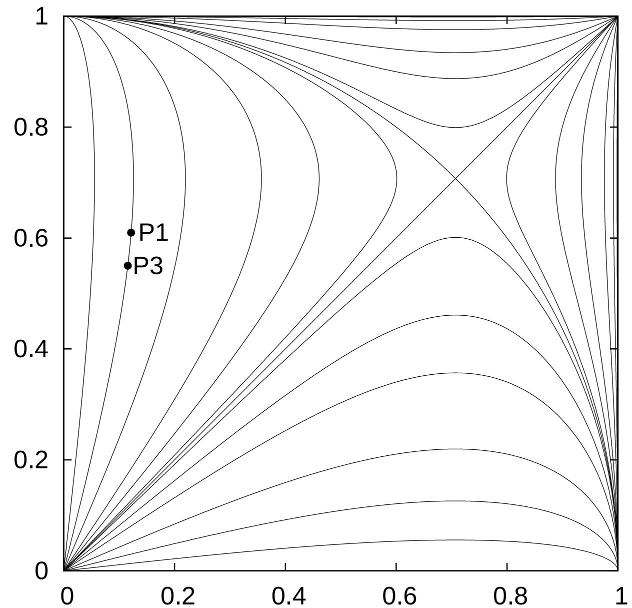
On the numerical studies

We use two different, independent, approaches

1. Take a **fine grid for** $(r_1, r_2) \in (0,1)^2$. Then using $\kappa = 1$ the values of (μ, α) are recovered. The solutions for a in the **collinear** case and (a, b) in the **triangular** one, are computed.
2. Starting at one of the **computed solutions** or at some limit case (e.g. **the planar problem**) we **fix a value of** μ and take $\alpha = 1$. Then a **continuation procedure** is started to obtain (κ, r_1, r_2, a, b) ($b = 0$ in the collinear case) as functions of an **arc-length parameter** s. To represent $(r_1, r_2) \in (0,1)^2$ one multiplies the obtained values by the **scaling factor** $\sqrt{\kappa}$. If (a, b) are represented, we use the same scaling.

Furthermore, using the **characteristic polynomials** to be shown later in the different cases, the eigenvalues are computed at the obtained solutions to decide on **stability properties**.
It is clear that it is enough to consider $r_1 \leq r_2$ but, for completeness, we shall display full diagrams.

First we show some **lines of constant μ**. In the plot for $\mu = 0.1, 0.2, 0.3, 0.4, 0.45, 0.49, 0.50$ and also for $1-\mu$. On $\mu = 0.2$ we mark **two relevant points** $P_1 \approx (0.121712, 0.609584)$, $P_3 \approx (0.115614, 0.550032)$.



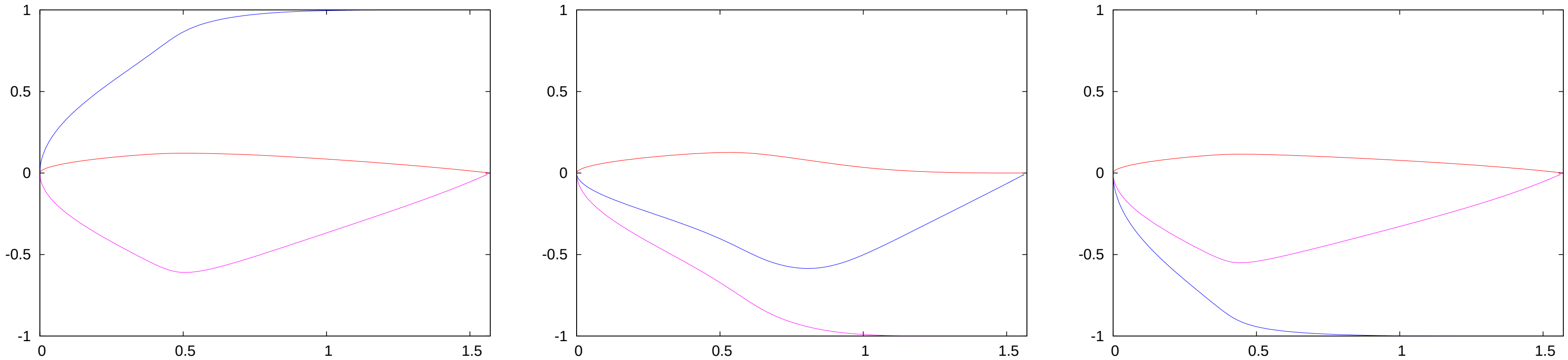
Continuation of solutions starting at $\kappa = 0$, e.g. $\mu = 0.2$ (see next plot):

1) For $a > r_1$ one has a **turning point at** P_1 and goes back to $(r_1, r_2) = (0, 0)$ **but** $a \to 1$.

2) For $-r_2 < a < r_1$, r_1 **returns to 0 but** $r_2 \to 1$.

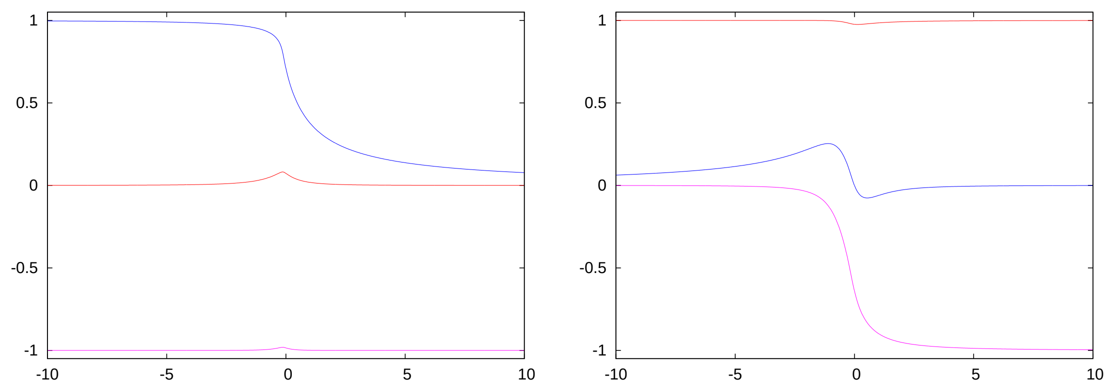
3) For $a < -r_2$ one has a **turning point at** P_3. Then (r_1, r_2) go back to $(0, 0)$ **but** $a \to -1$.

In the **axes** we represent r_1, r_2.



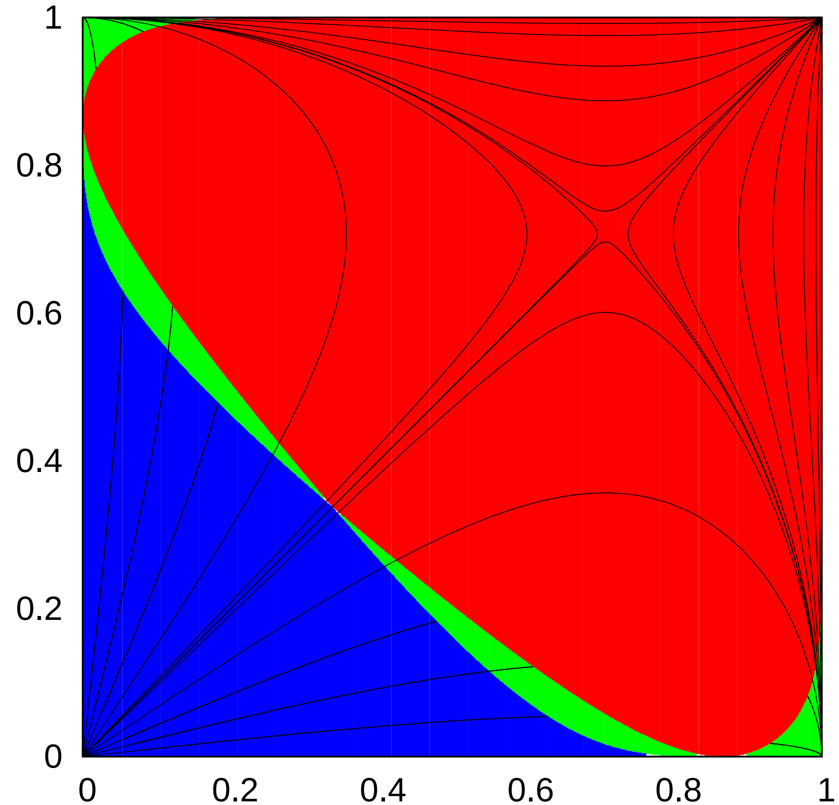
Evolution of r_1, r_2, a (red, magenta, blue) for cases 1), 2), 3) from left to right, **leaving from $\kappa = 0$** as a function of $\arctan(s)$.

Some solutions **cannot be continued to $\kappa = 0$**.



For $\mu = 0.3$. Left: **emerging from $(0, 1)$ and returning to it**. Right: **emerging from $(1, 0)$ and going to $(1, 1)$**. In red, magenta and blue the locations of $r_1, -r_2$ and a.

Note: If r_1, r_2, a (or $\sqrt{a^2 + b^2}$) go to 1, the mass **goes to the equator**.



Number of collinear RE as a function of (r_1, r_2). Blue, green and red correspond to 5, 3 and 1 RE. Green and blue meet $r_1 = r_2$ at r^*.
Some **lines of μ=ctant**, leaving from $(0,0)$ and for cases 1) to 3) are shown. On top of $r_1 = r_2$ we plot the line for $\mu = 0.2$ in case 1), for $\mu = 0.4, 0.49$ and 0.4999 in case 2) and for $\mu = 0.1, 0.3$ in case 3).
Other μ=ctant lines **emerge from** $(0,1)$ **and return to it** (green left upper domain) or **from** $(1,0)$ **and go to** $(1,1)$.

Triangular RE in $\mathbb{S}^2$

Given r_1, r_2 and computing μ, α with $\kappa = 1$ **one should solve for** (a, b). For the general case one can be guided by

Proposition 3. *Let $r_1 = r_2$. There exist* $0 < \beta_1 < \beta_2 < 1 - 2^{-3/2}$ *s.t.*

1. *If $r_1^2 \in (0, \beta_1) \cup (\beta_2, 1 - 2^{-3/2})$ there are* **exactly 4** *triangular RE.*
2. *If $r_1^2 \in (1 - 2^{-3/2}, 1)$ there are* **exactly 2** *triangular RE.*
3. *If $r_1^2 \in (\beta_1, \beta_2)$* **there are no** *triangular RE.*
4. *If $r_1^2 = \beta_1$ or $r_1^2 = \beta_2$ there are* **exactly 2** *triangular RE* $(a, b) = (0, \pm\sqrt{(3 - 4r_1^2)/(4E_1^2)})$.
5. *If $r_1^2 = 1 - 2^{-3/2}$ there is a unique RE* **of the form** $(a, b) = (0, 0)$, *that is, it becomes* **collinear**.

The values β_1, β_2 are the real zeros of $P(\beta) = 2^{14}\beta^3(1 - \beta)^5 - 27$.

Bifurcation points: $\sqrt{\beta_1} \approx 0.39625585$, $\sqrt{\beta_2} \approx 0.79732539$, $\sqrt{1 - 2^{-3/2}} \approx 0.80401904$.

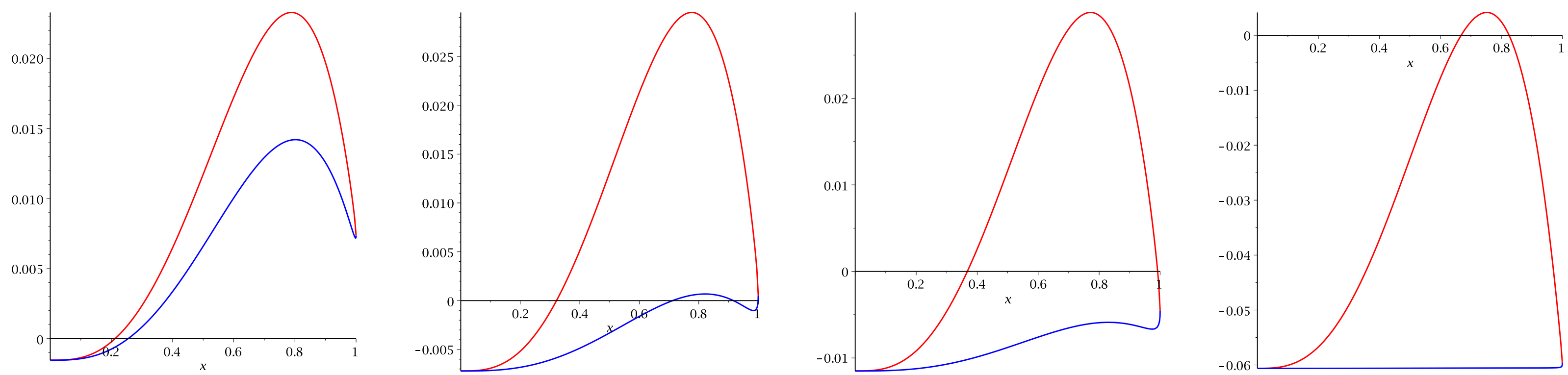
For the general case, **after several transformations**, it is enough to study the solutions of **a single equation**

$$P_{\pm}(V) := V^3(r_2\sqrt{1-\sigma_1 V} \pm r_1\sqrt{1-V})^2 - \sigma_2 = 0,$$

where $\sigma_1 = (E_2/E_1)^{2/3}$ and $\sigma_2 = (r_1E_2 + r_2E_1)^8 E_1^2$.
After finding the **solutions of both** $P_{\pm}(V) = 0$

$$E_3 = (r_2 f_{13} + r_1 f_{23})/(r_1E_2 + r_2E_1), \quad a = (E_2 f_{13} - E_1 f_{23})/(r_1E_2 + r_2E_1)$$

and then, if $a^2 + E_3^2 \le 1$, **we recover** $b = \pm\sqrt{1 - a^2 - E_3^2}$.



For $r_2 = 0.4$ and $r_1 = 0.05, 0.16, 0.2, 0.38$ (from left to right) plots of $P_+(V)$ **and** $P_-(V)$ as red and blue curves.

In **the general case one has the following**

Proposition 4. *For fixed values of κ, r_1 and r_2 the* **number of triangular relative equilibria is at most 8**. *Furthermore there are no solutions with $\kappa(r_1 + r_2) = 1$.*

In the numerical simulations we shall see that there are values of (κ, r_1, r_2) for which **there are exactly 8 solutions**.

The previous formulation makes also possible to study **limit cases** like $r_1 = 0$ or $r_2 = 1$.

The Triangular/Collinear RE (T/CRE)

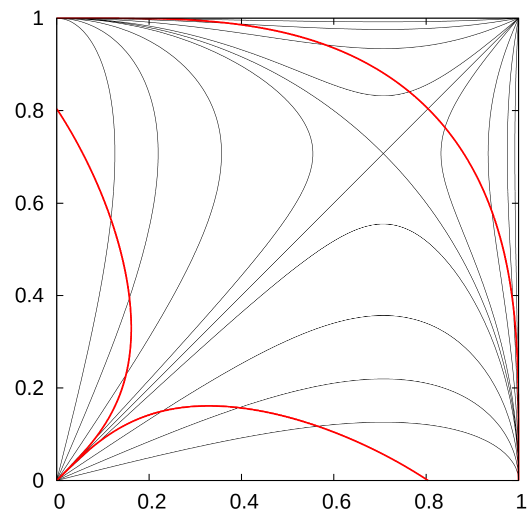
Looking for **triangular RE** it can happen to find some for which $b = 0$. This is evident in **continuation**: leaving from L_5 for $\kappa = 0$ **one reaches L_4 passing through collinear, $\kappa > 0$** except if $\mu = 1/2$. Concretely

Proposition 5. *A T/CRE, $(a, 0)$ exists if $t = \sigma_1^{1/2}$ and r_1, r_2 satisfy*

1) $\quad 1 - t^4 = [1 + t^2 + 2t(r_1 r_2 - E_1 E_2)]^2$, *with* $r_1 < a$,

2) $\quad 1 + t^4 = [1 + t^2 - 2t(r_1 r_2 - E_1 E_2)]^2$, *with* $-r_2 < a < r_1$,

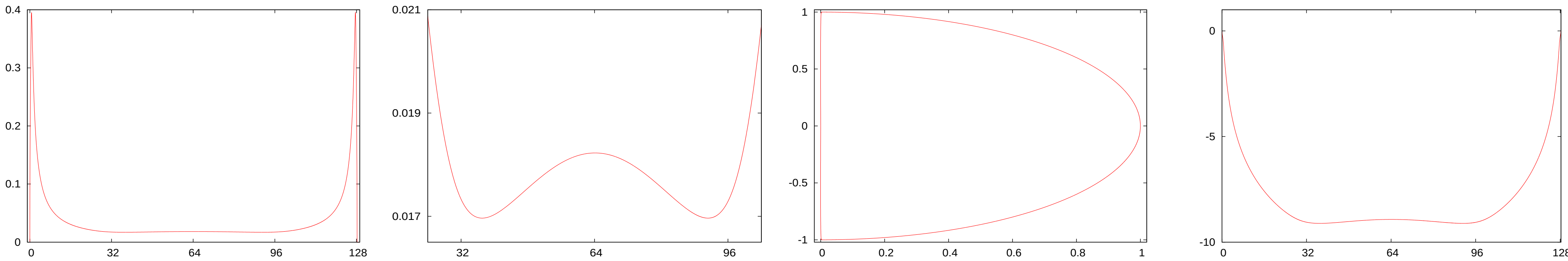
3) $\quad t^4 - 1 = [1 + t^2 + 2t(r_1 r_2 - E_1 E_2)]^2$, *with* $a < -r_2$.



In red the curves for case 1) (from $(0, 0)$ to $(0, r^{**})$), for case 2) (from $(0, 1)$ to $(1, 0)$) and for case 3) (from $(0, 0)$ to $(r^{**}, 0)$), where $r^{**} = \sqrt{1 - 2^{-3/2}}$.

Triangular RE: Numerical studies

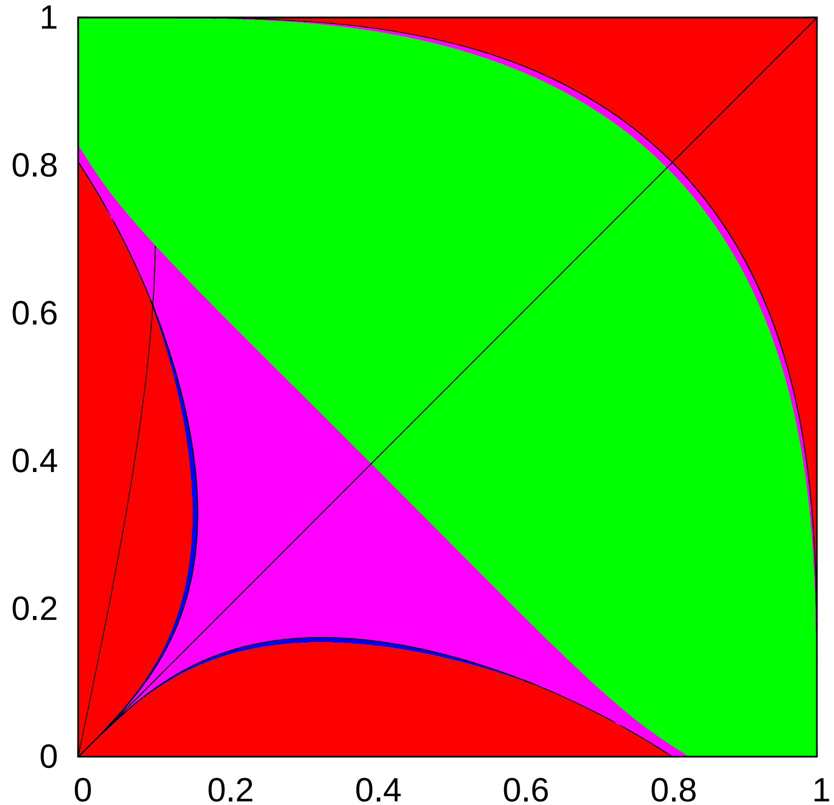
First we show some **evolution by continuation** from planar L_5.



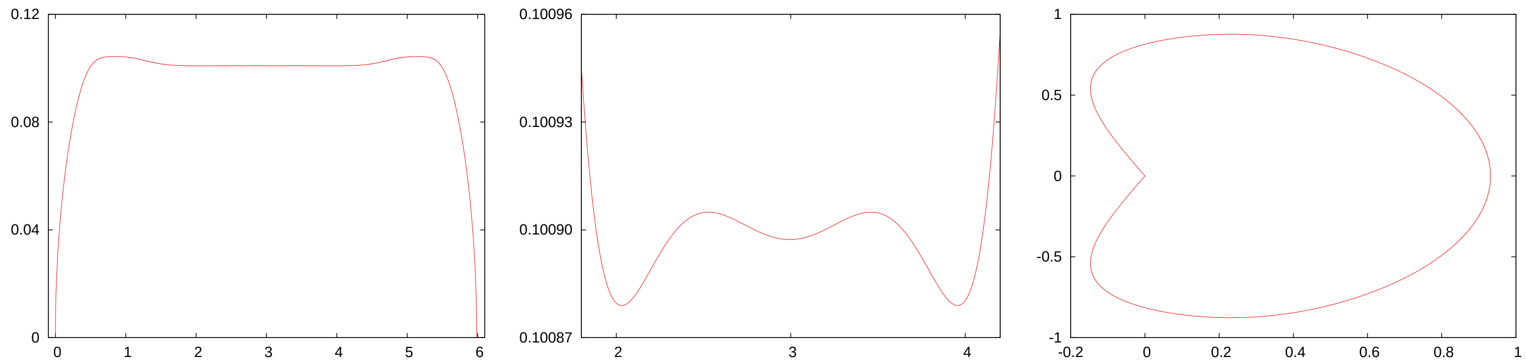
For $\mu = 0.499$ from left to right: 1. $r_1 = r_1(s)$; 2. magnification of 1. in the flat direction; 3. (a, b) along the continuation; 4. the distance to the equator as $\log_{10}(1 - \sqrt{a^2 + b^2})$.

The point on the middle corresponds to $b = 0$. From 1. and 2. it is clear that there are values of r_1 (and the corresponding r_2 on $\mu = 0.499$) for which there exist **6 triangular RE** (3 with $b > 0$ and 3 with $b < 0$). Note that from $s = 0$ ($\kappa = 0$) to $b = 0$ there are **2 extrema**.

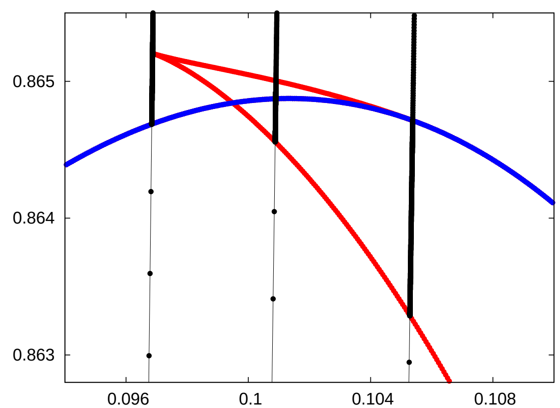
Putting together the **evolutions for a large set of values of μ** we obtain a global view.



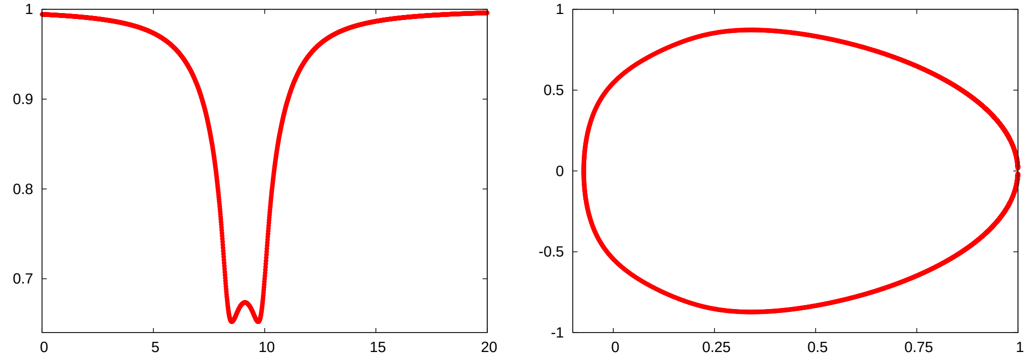
Number of triangular RE with $b > 0$, as a function of (r_1, r_2). In the **green, red, magenta and blue** domains there are, 0, 1, 2 **and 3 solutions**, resp.. Similar for $b < 0$. The boundaries between green-magenta and red-blue domains correspond to **turning points**. The black curves between magenta-red ($r_1 + r_2 > 1$) and magenta-blue ($r_1 + r_2 < 1$) are **T/CRE**. Also in black the case $r_1 = r_2$ and the line for $\mu = 0.172$. Close to that line **other phenomena occur in a tiny domain**.



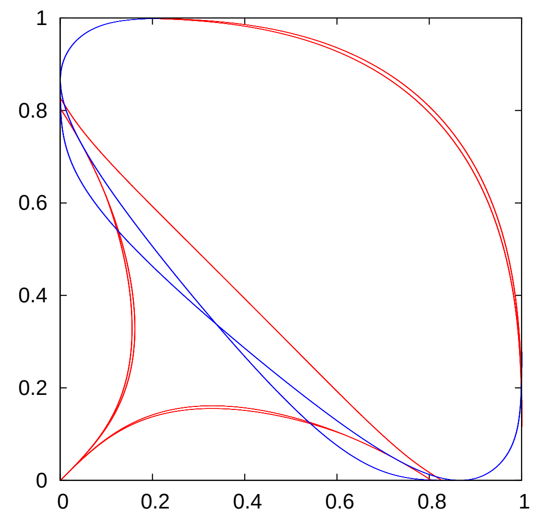
Triangular RE for $\mu = 0.172$. Left: r_1 as a function of s. Middle: detail of the **central part** on the left plot. The central minimum is a **T/CRE**. Before it **3 extrema** (compare $\mu = 0.499$). Right: evolution of (a, b).



A tiny domain: red = **turning points**; blue = **T/CRE**, black = $\mu = 0.1654, 0.172, 0.1794$. **Evolution for the central black**: **up** from $(0, 0)$ to magenta-green, **down** to lower red, **up** to upper red, **down** to blue. Then a **symmetric** path. Summarizing: **8 solutions** for a range of μ.



Emerging and returning to $(1, 1)$, $\mu = 0.5603$. Left: $r_1 = r_1(s)$. Right: (a, b). **Away** from $r_1 = 1$ reaches a **turning point**, passes through **T/CRE** and b changes sign and follow a **symmetric** path.



Location of bifurcations for collinear (blue) and triangular (red) RE. **Up to 11 different RE** can coexist for some (r_1, r_2).

Spectral stability

First we consider the stability of the **primaries**. We have

Proposition 6. *Let μ, α^2 be given and $r_1 > 0, r_2 > 0$ satisfying the equilibrium equations. The RE of the two body problem is spectrally* **stable if** $\kappa^2 r_1^2 r_2^2 < 1/4$ *and, it is* **unstable if** $\kappa^2 r_1^2 r_2^2 > 1/4$*.*

For the RTBP the **characteristic polynomial** is of the form $p(\lambda) = \lambda^4 + \beta_2\lambda^2 + \beta_0$. Letting aside degenerate cases one can have **EE, EH, HH, CS**, where E denotes **elliptic**, H **hyperbolic** and CS **complex saddle**.

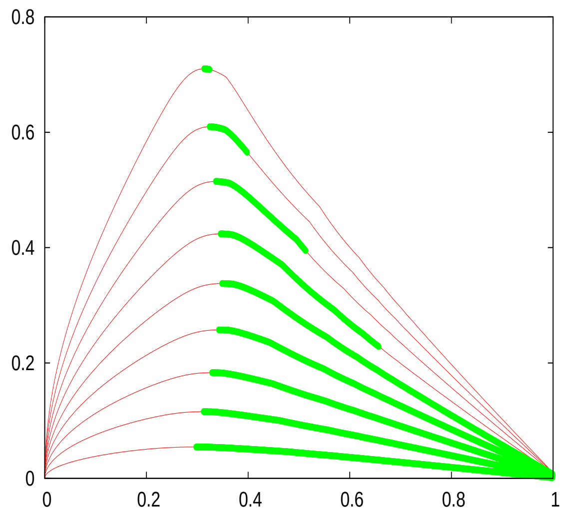
In the **collinear case**, with third body at $(a, 0)$, one has

$$\beta_2 = \alpha^2(2 - \kappa a^2) + G_y, \quad \beta_0 = (\alpha^2 E_3^2 + G_y)(\alpha^2(1 - 2\kappa a^2) - 2G_y),$$

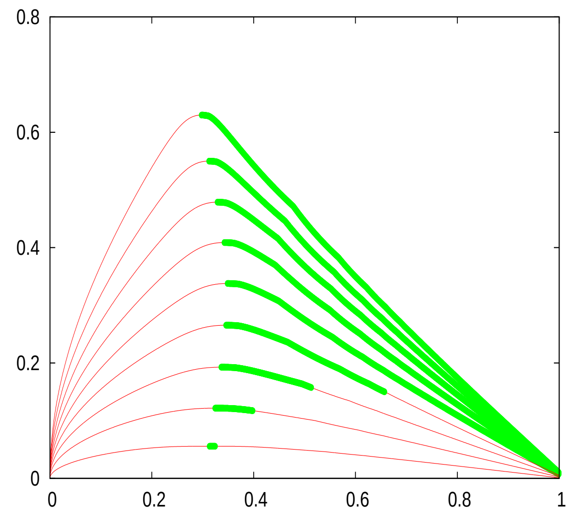
where $G_y = -(m_1 f_{13} d_{31}^{-3} + m_2 f_{23} d_{32}^{-3})$.

Proposition 7. *For any μ, α a collinear RE in $\mathbb{S}^2$ cannot be of type HH.*

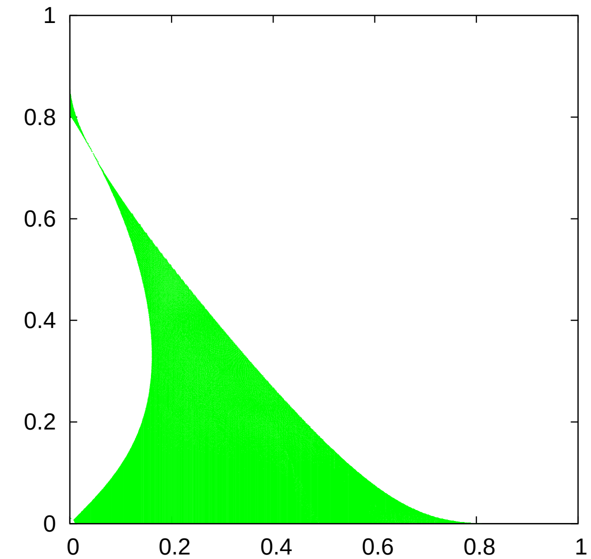
Numerical results for collinear RE



$r_1 < a$



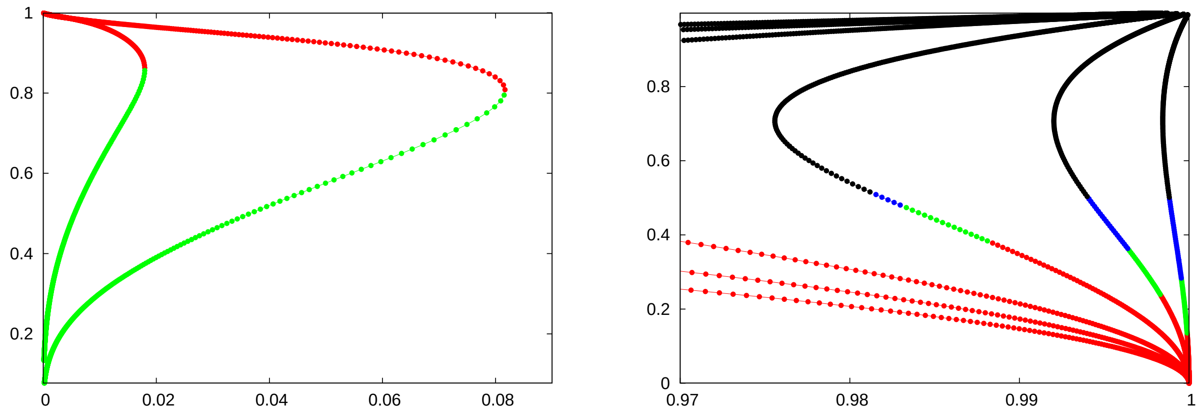
$a < -r_2$



Left and middle: Stability of **collinear RE** leaving from planar as a function of a normalized s. The vertical variable is r_2. Red: **EH type**. Green: **EE type**. The first transition from EH to EE occurs at the **turning point**. Curves for $\mu = 0.1, \ldots, 0.9$ from top to bottom. For $-r_2 < a < r_1$ one has always **EH type**.

Right plot: Domain of **EE collinear RE** as a summary of the left plot. The upper boundary contains the **boundary green-red** for $\mu \le 1/2$, between (r^*, r^*) and $(0, \sqrt{3}/2)$ and the **boundary blue-green** for $\mu \geq 1/2$ between the symmetric points.

Consider now the collinear RE which are **not connected to the planar case under continuation**.



Left: changes in stability for the branch **leaving from and returning to** $(0,1)$. Right: changes in stability for the branch **connecting** $(1,1)$ **and** $(1,0)$. In all cases for $\mu < 1/2$. Color code: red, green and blue for **EH, EE and CS**. The **black parts** mean that already the **two-body problem is unstable**. On the left (resp. right) part the values of μ are 0.05 and 0.3 (resp. 0.1, 0.2, 0.3, 0.4, 0.45 and 0.49).

Stability in the triangular case

In the triangular case the **coefficients** of the characteristic polynomial are
$\beta_2 = \alpha^2$,

$$\beta_0 = b^2 \left(\frac{9m_1 m_2 f_{13} f_{23} (r_2 E_1 + r_1 E_2)^2}{d_{31}^5 d_{32}^5} - 3\kappa\alpha^2 \left[\frac{m_1 f_{13} r_1^2}{d_{31}^5} + \frac{m_2 f_{23} r_2^2}{d_{32}^5} \right] \right).$$

In particular, for the **T/CRE** one has $\beta_0 = 0$, as should correspond to a **bifurcation** case.

For general triangular RE the spectral stability depends only on β_0:

- If $\beta_0 < 0$ we are in an **EH** case.

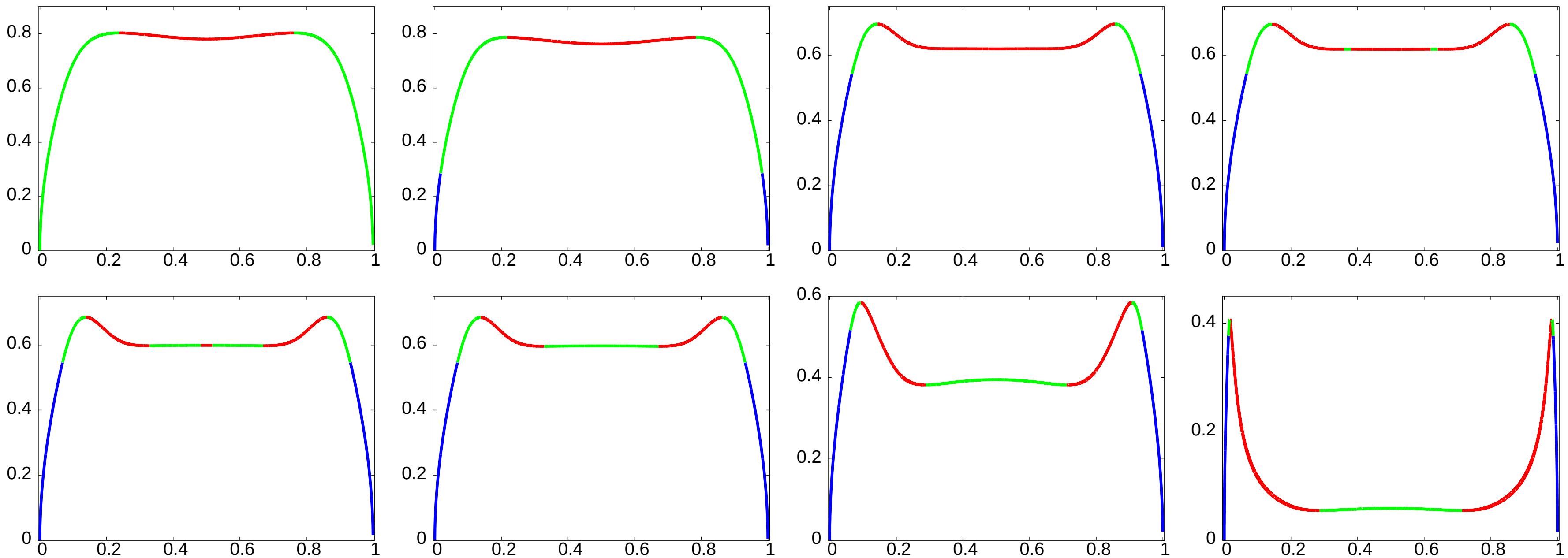
- If $0 < \beta_0 < \alpha^4/4$ we are in an **EE** case.

- If $\beta_0 > \alpha^4/4$ we are in a **CS** case.

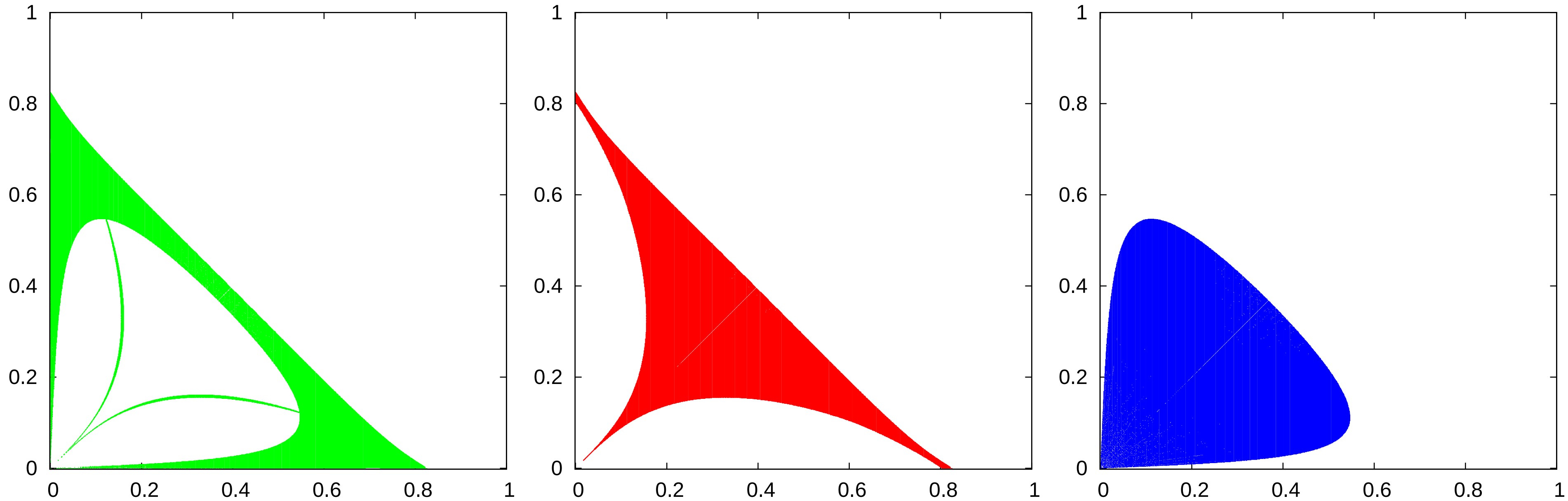
Proposition 8. *For any μ, α a triangular RE in $\mathbb{S}^2$ cannot be of type HH.*

Numerical results for triangular RE

First we show the **evolution of the stability**, displaying r_2 as depending on s, for some values of μ.



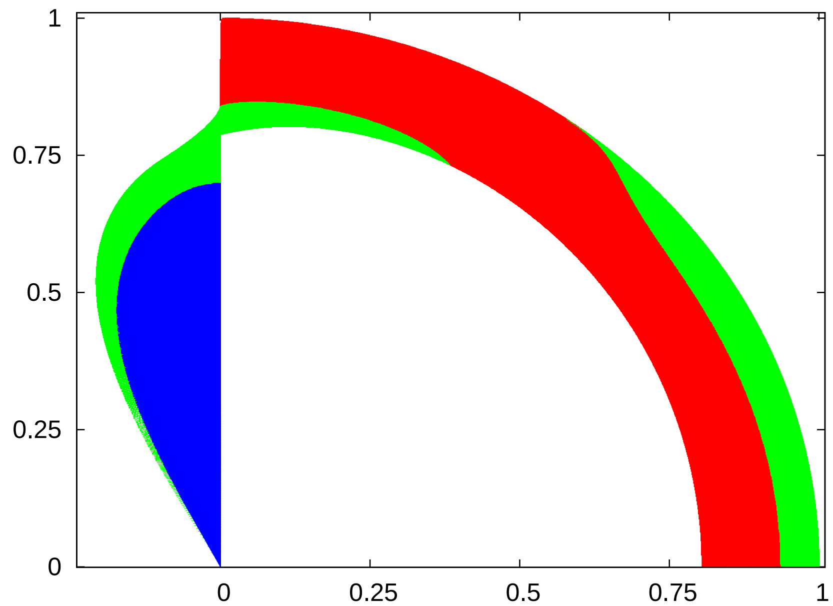
From left to right $\mu =$ **0.030, 0.050, 0.165 and 0.166** on top and **0.179, 0.180, 0.300 and 0.490** at bottom. **Color code**: red, green and blue for **EH, EE and CS**, resp. We identify **transitions** due to **Routh**, and to the passage through **turning points** shown in the description of the triangular RE. Some domains are rather **narrow**. **No change at T/CRE** because of the symmetry.



A **summary** of the stability properties for triangular RE leaving from planar.

From left to right: values of (r_1, r_2) for which **at least one** of the solutions obtained by continuation from $\kappa = 0$ to $\kappa > 0$ is of **EE, EH or CS type**, respectively.

For **identical values of** (r_1, r_2) (after normalization of the continuation results, i.e., after taking $(\sqrt{\kappa}r_1, \sqrt{\kappa}r_2)$ from the computed values) the **different triangular RE** which appear can have **different stability properties**.



The type EE, EH, CS (green,red,blue) for the triangular RE **seen in the (a, b) variables**. For the EE case it can happen that **the same point corresponds to different solutions**, i.e., different values of (r_1, r_2). Only the upper part $b \geq 0$ is shown.

For the solutions **leaving from $(1, 1)$ and returning to it** after reaching the magenta-green boundary, they **start as EH, become EE at that boundary** and keep being **EE until the T/CRE**. After this apply symmetry.