Experiments looking for theoretical predictions

Carles Simó

Dept. Matemàtica Aplicada i Anàlisi, UB, Barcelona, Catalonia

carles@maia.ub.es

Dynamics and Geometry

Lorenz International Center

Leiden, June 15 to June 19, 2015

Dedicated to professor Hendrik Wolter Broer

Een fijne verjaardag ...

vier maanden te laat

The problem

One of the **goals of Dynamical Systems** is to **understand the behavior of the orbits**. To this end one needs to **have information on the skeleton**: fixed points, periodic orbits, invariant curves/tori, invariant manifolds of hyperbolic objects (perhaps partially or weakly hyperbolic) and their relative position.

Changes, changing parameters, are produced at bifurcations:
Local: They require the analysis around a **given point/points**.
Semiglobal: Require the analysis around an **orbit or simple family of orbits** like **breakdown of a given curve/torus, splitting, homo/hetero tangencies**.
Global: We **do not know around which orbit they occur** like **breakdown of all invariant curves/tori** or of **all invariant tori with strong stickiness**.

Our goal: to present a few examples in this direction.

The challenge: to be able to predict for which parameter value they occur.

A simple motivation: the planar circular RTBP

It describes the motion of a **massless body** under the action of **bodies of masses** $1-\mu, \mu$ moving in **circular orbits** around the c.o.m. In a fixed frame the dynamics is described by

$$H(X_1, X_2, Y_1, Y_2, t) = \frac{1}{2}(Y_1^2 + Y_2^2) - \left(\frac{1-\mu}{r_1} + \frac{\mu}{r_2}\right),$$

where (X_1, X_2) is the position of the small body in an inertial frame and

$$r_1^2 = (X_1 - \mu\cos(t))^2 + (X_2 - \mu\sin(t))^2, \quad r_2^2 = (X_1 + (1-\mu)\cos(t))^2 + (X_2 + (1-\mu)\sin(t))^2.$$

It has 2 d.o.f., is 2π-time periodic and a **first integral (Jacobi)** given as $C = 2(M - H)$, $M = X_1 Y_2 - X_2 Y_1$ and H as before.

For μ small is a perturbation of the 2BP.

In the 2BP there are **parabolic orbits on $H = 0$**. Can be seen as a **fixed point of parabolic type** in suitable coordinates, usually called **fixed point at infinity**. The manifolds W^u **and** W^s **coincide**.

A suitable Poincaré section

We use the **pericenter passage**

$\Sigma = \{(X_1, X_2, Y_1, Y_2, t) \in \mathbb{R}^5 \mid r = (X_1^2 + X_2^2)^{1/2}$ local minimum wrt $t\}$.

For the **invariant manifolds at infinity** it is convenient to introduce **McGehee coordinates** (q, p, θ, ω) near infinity:

$$x + \mathrm{i}\, y = \frac{2}{q^2} e^{\mathrm{i}\,\theta}, \qquad \dot{x} + \mathrm{i}\, \dot{y} = \left(p + \mathrm{i}\, \left(q^2\omega/2 - 2/q^2\right)\right) e^{\mathrm{i}\,\theta}.$$

If α is the argument of the massless body in sidereal $(X_1 = r\cos(\alpha), X_2 = r\sin(\alpha))$ then $\theta = \alpha - t$.
One can derive the ODE for (q, p, θ, ω). In particular, for a fixed C there is a **periodic orbit at ∞, OP_∞** with $q = 0, p = 0, \omega = C/2, \theta \in \mathbb{S}^1$, $\dot{\theta} = -1$. As q goes to (or away) from ∞ as $\pm(8/9)^{1/4} t^{-1/3}$ (from Kepler) it is **parabolic in the DS sense**.

As said, for $\mu = 0$ the manifolds W^s_∞ and W^u_∞ **coincide**: $p^2 = q^2(1 - C^2 q^2/16)$ and intersect Σ on $p = 0, q = 4/C$. **For $\mu > 0$ do not coincide**: splitting, return maps, chaos, escape and capture, etc.

One can compute the **splitting for C large enough, μ small enough**

$$SF(\theta,\mu,C)=\mu\frac{\sqrt{\pi}}{8}C^{1/2}\exp(-C^3/24)\sin(\theta)(1+o(1)).$$

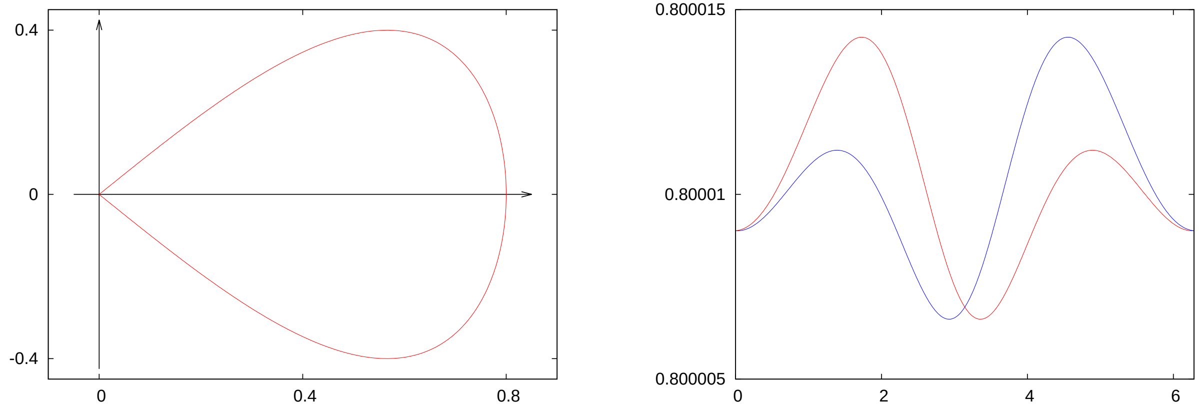
The usual methods: **derive a separatrix map**, approximate in suitable domains by a **standard map**, use the properties of that map

$$SM_\kappa:\begin{pmatrix} u \\ v \end{pmatrix}\to\begin{pmatrix} \bar{u}=u+\bar{v} \\ \bar{v}=v+\kappa\sin(u) \end{pmatrix},$$

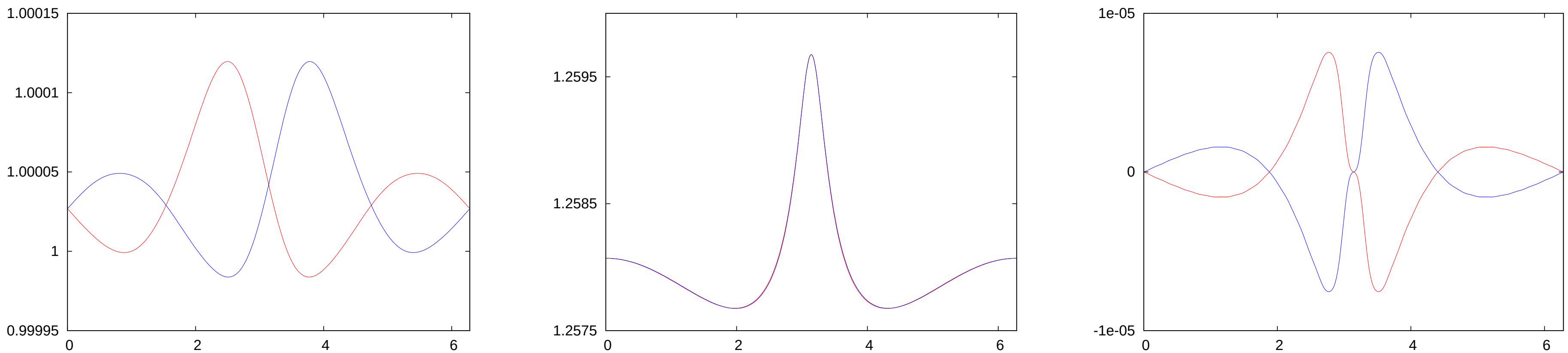
to predict the location of **invariant rotational curves IRC** preventing from escape, **until $\kappa=\kappa_G\approx 0.971635$, the Greene's constant**, at a distance from the pericenter of the case $\mu=0$ $(4/C)$ given by

$$\mu\frac{32}{C^5}+\mu\cos(\theta)\frac{3\cdot 2^{18}}{C^{13}}+\left(\frac{3}{\kappa_G}\right)^{2/5}\frac{\pi^{3/5}}{8}C^{4/5}\mu^{2/5}\exp(-C^3/60)+\dots\ .$$

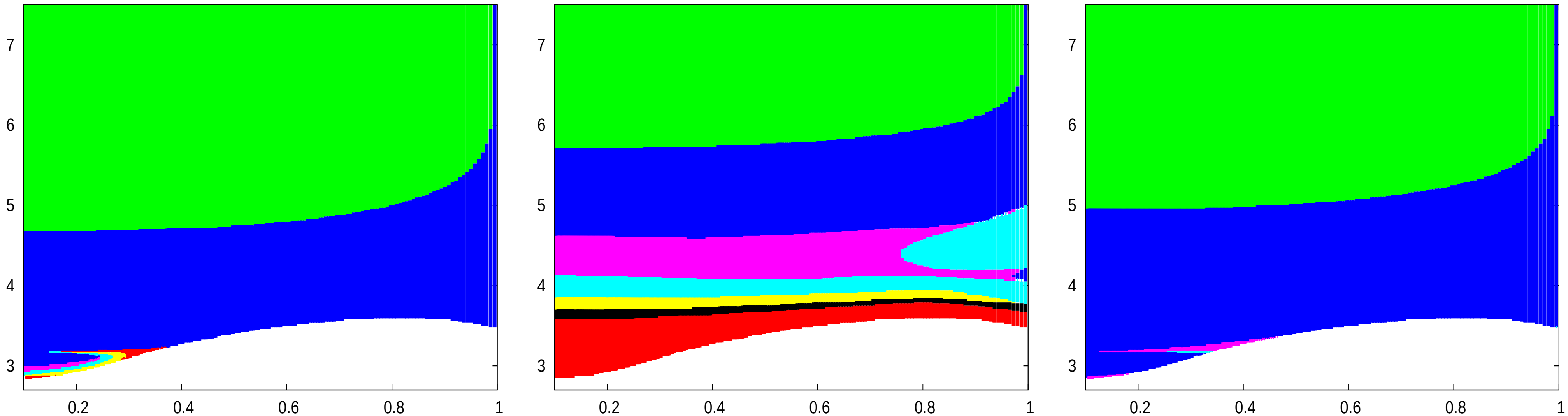
But **how large has to be C** and **how small has to be μ** ?



Left: Invariant manifolds in (q, p) for $\mu = 0$. Rigth: $W_{\infty}^{u,s} \cap \Sigma$ in (θ, q) for **Sun-Jupiter,** $C = 5$.



Left: Invariant manifolds for Sun-Jupiter and $C = 4$. Middle: idem for $C = 3.18$: the manifolds are, relatively, very close. Right: The previous plot **skipping the symmetrical component**. The **splitting function** SF is the difference between manifolds.



Some characteristics of the **splitting function** SF as a function of $((2\mu)^{1/3}, C)$. Left: the **dominant term** in the SF. Middle: **number of relevant harmonics** needed to have a good approximation to SF. Right: the number of zeros of SF, i.e., **primary homoclinic points**.

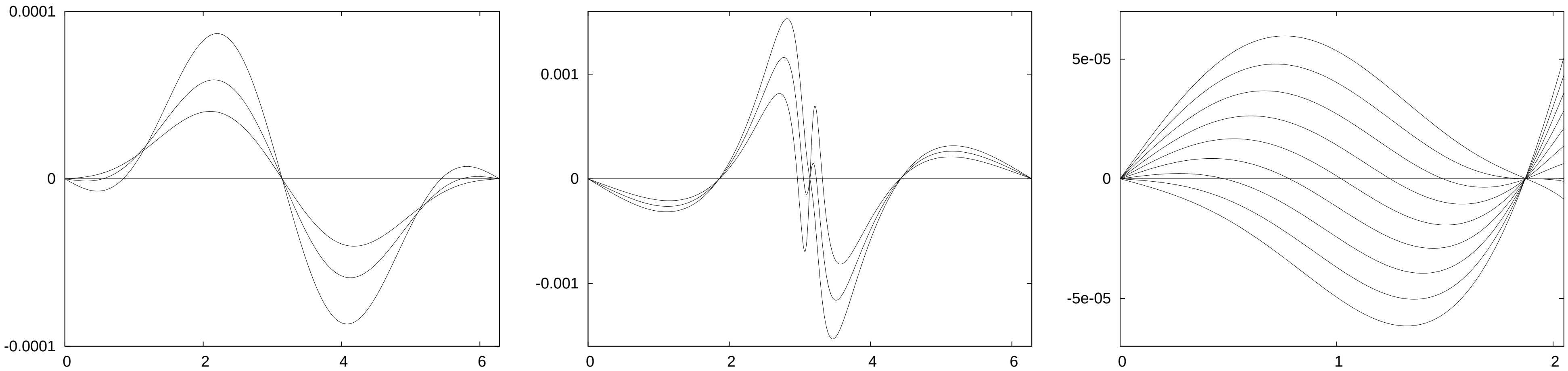
Color codes:

Left: green, blue, magenta, light blue, yellow, red for $k = 1, 2, 3, 4, 5, > 5$.

Middle: green, ..., yellow, black, red for $k = 1, \ldots, 6, > 6$.

Right: green, blue, magenta, light blue for $k = 2, 4, 6, 8$.

In the **white** domains $W^{u,s}_{\infty} \cap \Sigma$ are no longer graphs.



Evolution of $SF(\theta, \mu, C)$ for $\mu = 0.5 \times 10^{-1.7} \approx 0.0099763$ (close to 10 times the Sun-Jupiter mass ratio) to show the **changes in the number of zeros as a function of C** in a fundamental domain.

Left: For $C = 5.0, 4.9, 4.8$ going from 2 to 4 zeros.

Middle: For $C = 3.23, 3.22, 3.21$ going from 4 to 6 zeros.

Right: For C between 3.184 and 3.168, step 0.002. For $C \in (3.180, 3.182)$ a **new zero emerges** for $\theta \approx 0$ (and near 2π), going from 6 to 8 zeros. Decreasing C it moves to the right until **3 zeros coalesce** and only 1 remains. Symmetrical for $\theta > \pi$, then going from 8 to 4 zeros.

These changes can imply quite strong changes in the location of the IRC which prevent from escape.

First example: A standard-like map with two harmonics

As seen, e.g., in the Sun-Jupiter case for $C = 4.8$ there are interesting cases with SF having **two relevant harmonics**. For simplicity, we can consider (as in the RTBP) that $SF(\theta, \mu, C)$ is **antisymmetric with respect to θ**.

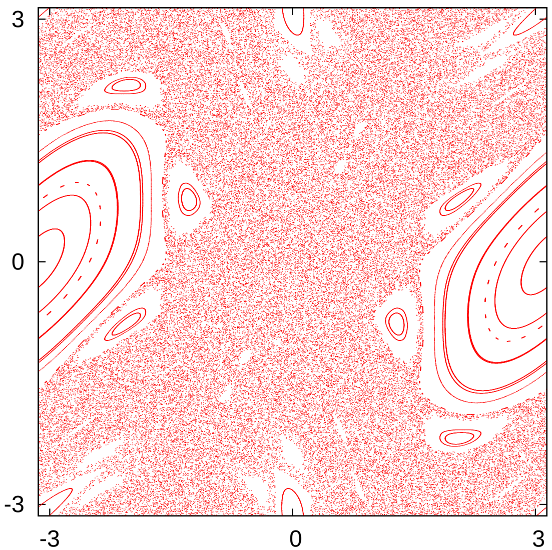
We can apply the usual methods **to obtain, from splitting and return time, a separatrix map and approximate by a standard-like map** in a domain not too close to the broken separatrix.

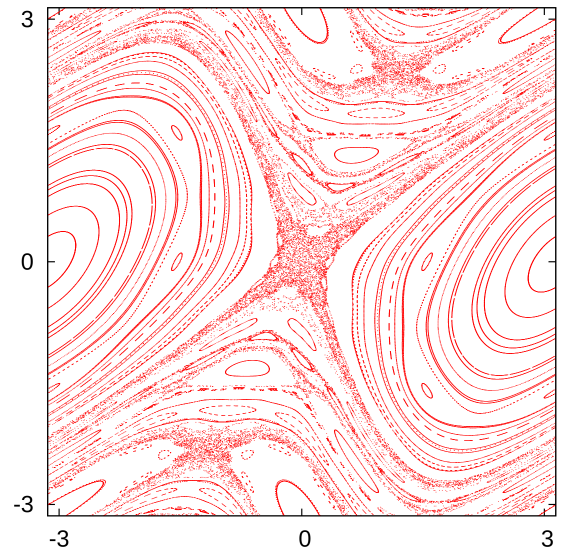
This suggest to look at the **following model**

$$SM_{\kappa,\alpha}\begin{pmatrix} u \\ v \end{pmatrix} \to \begin{pmatrix} \bar{u} = u + \bar{v} \\ \bar{v} = v + \kappa(\cos(2\pi\alpha)\sin(u) + \sin(2\pi\alpha)\sin(2u)) \end{pmatrix},$$

with $\alpha \in (-1/4, 1/4]$ as **additional parameter**. $\alpha \in (1/4, 3/4)$ reduces to the previous range (via $u \to u + \pi$). $\alpha = 0$ is simply the classical standard map and $\alpha = 1/4$ reduces to the classical with κ replaced by 2κ (via $(u, v) \to (2u, 2v)$).

Differences in the dynamics due to α are seen in the plots for $\kappa = 1.6$ and $\alpha = 0, 0.02$ (left,right). The changes in the **chaotic zone emerging from $(0,0)$ are large**. On the right one can see a **secondary chaotic zone associated to period 2**. Between them **there are still IRC**.

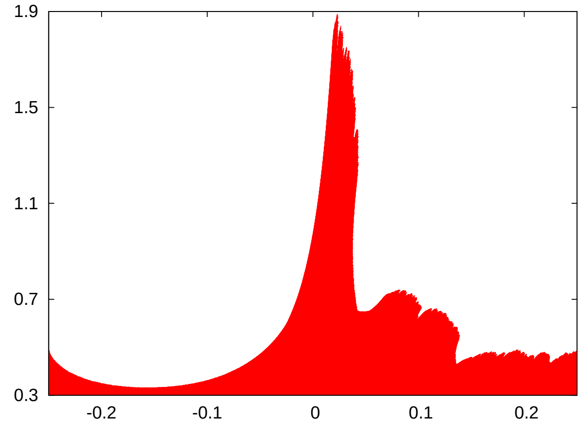




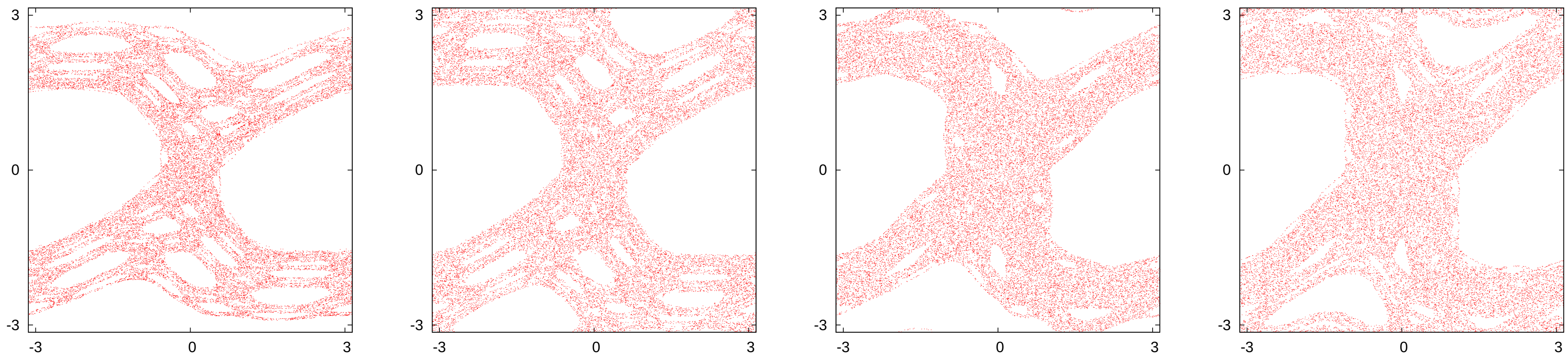
One of the reasons is the **asymptotic expression of the splitting**

$$SF(\theta, \kappa, \alpha) \approx \sum_{k \geq 1} A_k \kappa^{B_k} \exp(-kc/\gamma(\kappa)) \cos(kd/\gamma(\kappa) + \psi_k(\kappa)) \sin(k\theta),$$

where $\gamma(\kappa)$ is the log of **dominant eigenvalue** and $\mathrm{i}\, c + d$ is the **singularity with minimal $|c|$ of the separatrix of a limit Hamiltonian**, located at $\pm \mathrm{i}\, \pi/2 \pm \log(\sqrt{1+\Delta} + \sqrt{\Delta})$ (except by a scaling factor), where $\Delta = 2\tan(2\pi\alpha)$ for $\alpha \in (0, 1/4)$.



Domain in (α, κ) of **existence of IRC**. One can not define a **Greene's-like constant** $\kappa_G(\alpha)$. For a fixed α **increasing** κ **IRC can disappear and reappear, typically with different rotation number**.



For $\alpha = 0.04$ from left to right: $\kappa = 0.7$ **with IRC**, $\kappa = 0.8$ **without IRC**, $\kappa = 1.2$ **with IRC**, $\kappa = 1.4$ **without IRC**.

Numerical tools: invariant manifolds, homo/hetero, splitting, rotation numbers, Lyapunov exponents, detection of islands, massive simulations.

Second example: A driven standard map

A different modification can be made to SM_κ by **keeping one harmonic, but assuming κ driven by an external angle θ**

$$SM_{\kappa_0,\kappa_1,\gamma} : \begin{pmatrix} u \\ v \\ \theta \end{pmatrix} \to \begin{pmatrix} \bar{u} = u + \bar{v} \\ \bar{v} = v + \kappa(\bar{\theta}) \sin(u) \\ \bar{\theta} = \theta + 2\pi\gamma \end{pmatrix},$$

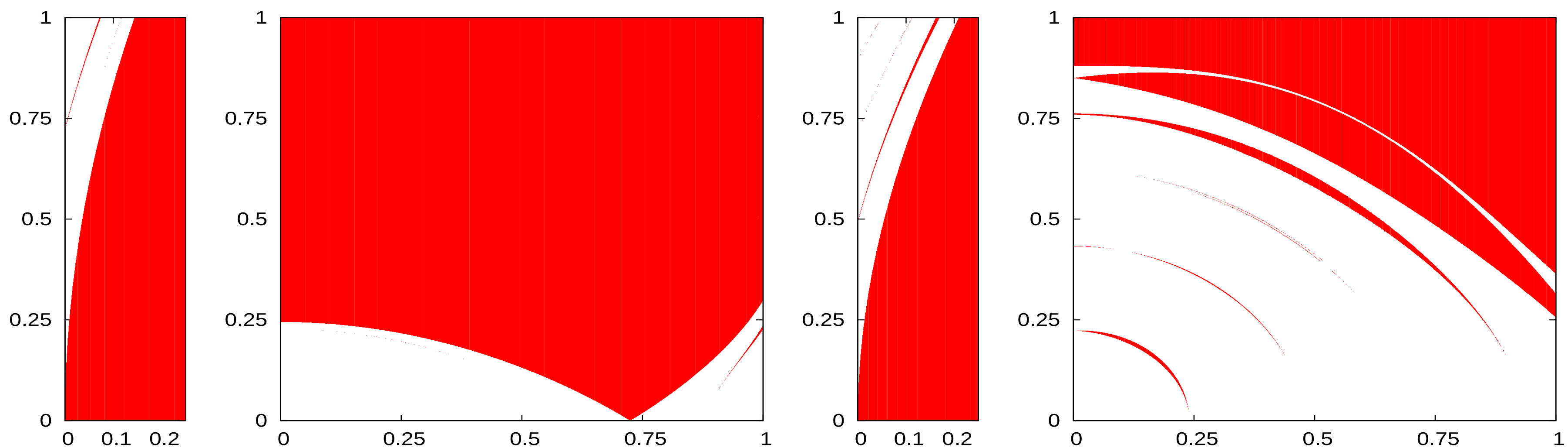
where $\kappa(\bar{\theta}) = \kappa_0 + \kappa_1 \sin(\bar{\theta})$.

This is a **skew product in** $\mathbb{T}^3$. For $\kappa_0 < \kappa_G$ and κ_1 small one can **expect 2D invariant tori** coming for an IRC with rotation number ρ provided (ρ, γ) **satisfy a Diophantine condition**. In a similar way **hyperbolic/elliptic** periodic points for $\kappa_1 = 0$ should give **hyperbolic/elliptic invariant curves** for $SM_{\kappa_0,\kappa_1,\gamma}$.

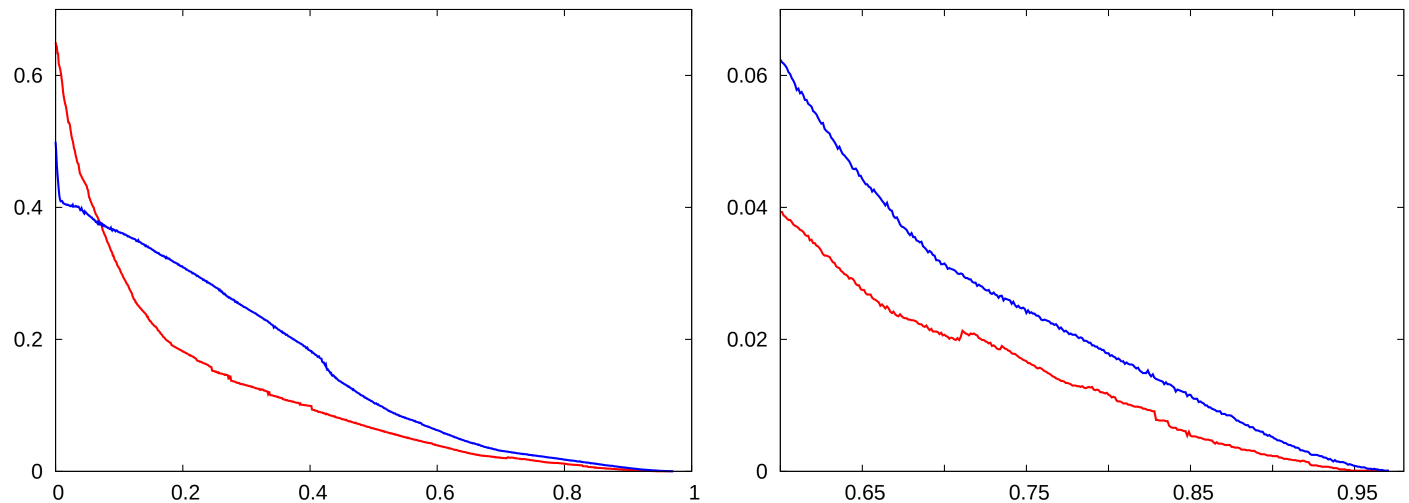
Some questions are:

- **For a given κ_0 up to which value of κ_1 some $\mathbb{T}^2$ survives** ?
- **How this depends on γ** ?

A first question is **to select γ**. As extreme cases we take $\gamma_1 = (\sqrt{5}-1)/2$, **the golden number**, which is 1-1 resonant with the ρ of the last IRC to be destroyed for SM_κ and $\gamma_2 = \sqrt{769} - 27$ which satisfies, for $|k_1|, |k_2| \leq 2^{20}$ and $k_0 \in \mathbb{Z}$ the **Diophantine condition** $|k_1\gamma_1 + k_2\gamma_2 + k_0| > 0.233126\ldots(|k_1|+|k_2|+|k_0|)^{-2}$. (The best case for $\sqrt{p}$, p a prime < 1000). If $\kappa_1 = 0$ $(0,0)$ is **fixed hyperbolic** if $\kappa_0 > 0$. $(0,\pi)$ is **2-periodic elliptic** if $|\kappa_0| < 2$. This **changes for** $SM_{\kappa_0,\kappa_1,\gamma}$, when κ_1, γ play a role.



Two left: For $\gamma = \gamma_1$, for fixed and period 2. Two right: Idem for $\gamma = \gamma_2$. The horizontal (vertical) variable is κ_0 (κ_1). **The red domains correspond to hyperbolic invariant curves** of the skew product.



The horizontal (vertical) variable is κ_0 (κ_1). Left: **An upper bound for the value of κ_1, as a function of κ_0, such that $\mathbb{T}^2$ still exist**. Right: A **magnification for $\kappa_0 > 0.6$**. Red (blue) for $\gamma = \gamma_1$ ($\gamma = \gamma_2$).

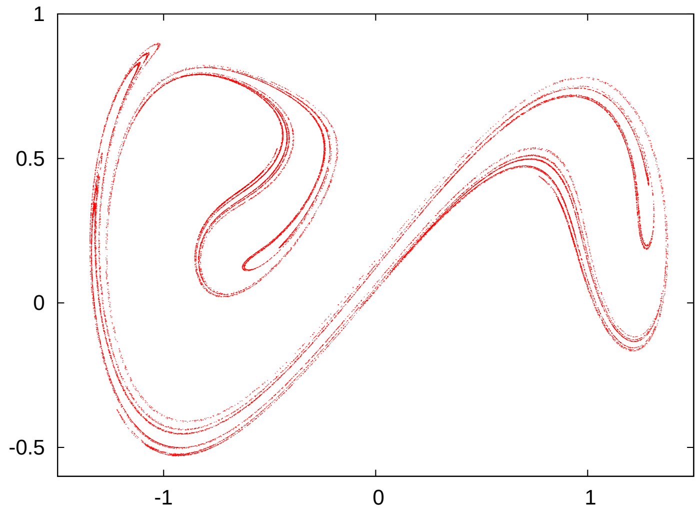
We note that, near $\kappa_0 = \kappa_G$ in the γ_1 case, values $\kappa_1 < 10^{-3}$ are **enough to destroy all the $\mathbb{T}^2$ for $\kappa_0 > 0.922$**, and $\kappa_1 < 10^{-4}$ to **destroy all the $\mathbb{T}^2$ for $\kappa_0 > 0.947$**. In the γ_2 case one needs these values of κ_1 for **$\kappa_0 > 0.945$ and $\kappa_0 > 0.967$**, respectively.

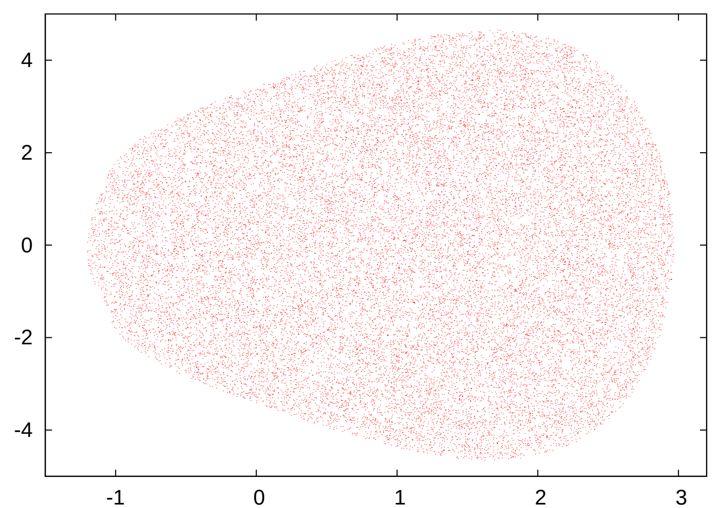
In the γ_1 case for a **moderate value $\kappa_0 = 0.2$** just **$\kappa_1 = 0.182$ is enough to destroy all the $\mathbb{T}^2$**.

Third example: Strange attractors with large dimension

Consider **dissipative (rather weakly dissipative) systems**. We take the simplest version of the **forced and damped Duffing model**

$$\ddot{x} = x - x^3 - \delta\dot{x} + \gamma\cos(\omega t).$$

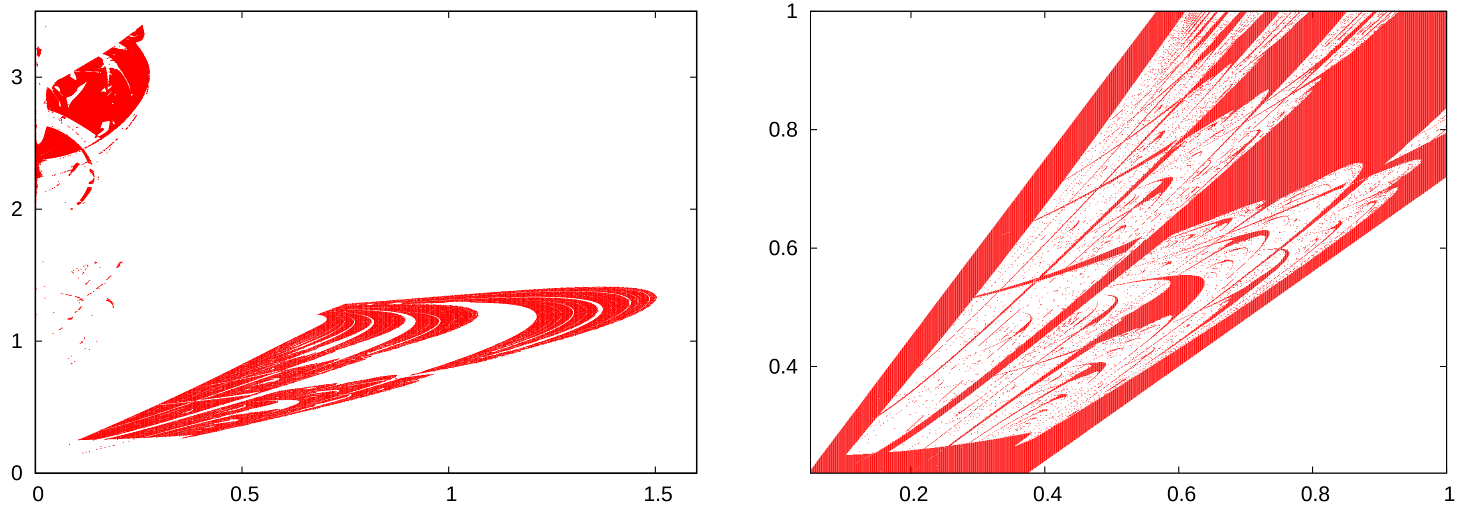




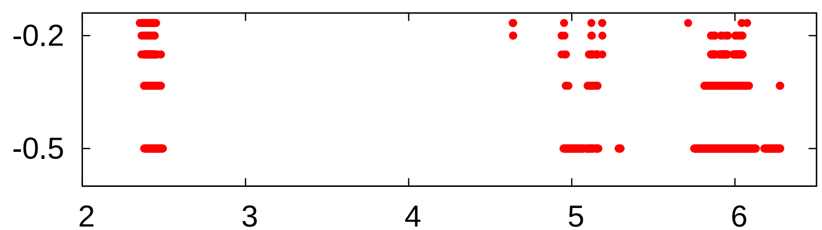
Left: The **classical values** (widely used) $\delta = 0.2, \gamma = 0.3$. Right: A **near conservative case** $\delta = 10^{-6}, \gamma = 2.4$. Attractors for $\omega = 1$ and 25,000 time-2π iterates plotted after a **transient of 10^6 (left), 10^9 (right)**.

Both have **zero measure** but quite different **fractal dimension**.

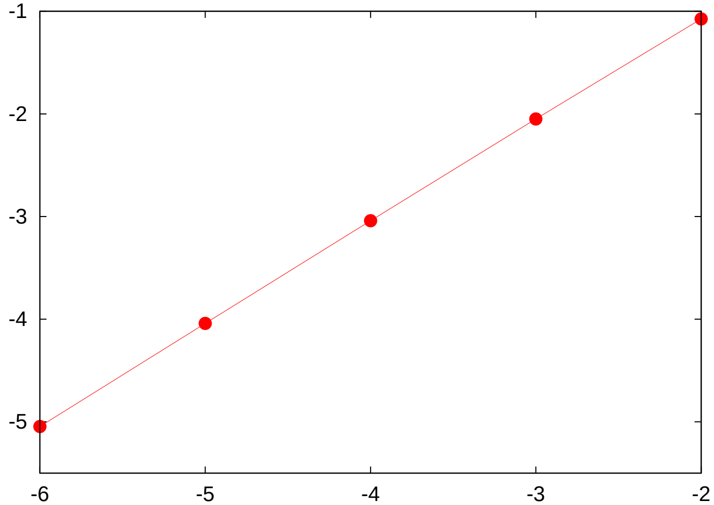
Next we check a **wide domain** in δ, γ, guided by **Lyapunov** exponent after a 10^5 **transient**, starting near $(0, 0)$.

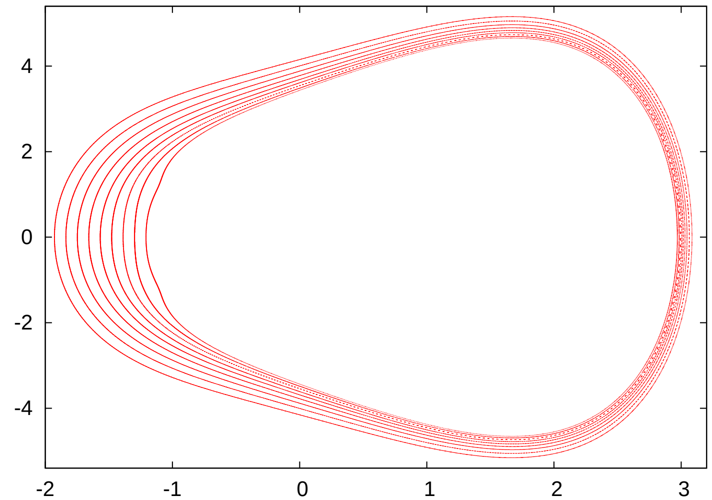


Left: Values (δ, γ) **giving rise to SA**. Other values exist with $\delta \approx 0$ near $\gamma = 5$ and $\gamma = 6$. Right: For the domain closer to $(0, 0)$ we show the **negative of a magnification** of the previous plot. Red parameters give rise to **sinks**. Structure reminiscent of the **dissipative Hénon map**.



Values $(\gamma, 1/\log_{10}(\delta))$ for which **SA is detected after** 2^{30} **to** 2^{33} **iterates** for 10 different initial data. We show next some results for $\gamma = 2.4$ for $\delta \to 0$ **and also** $\delta = 0$.





Left: Variation of the **Lyapunov dimension** D_L as a function of δ for $\gamma = 2.4$. Variables plotted: $\log_{10}(\delta)$ **and** $\log_{10}(2 - D_L)$.

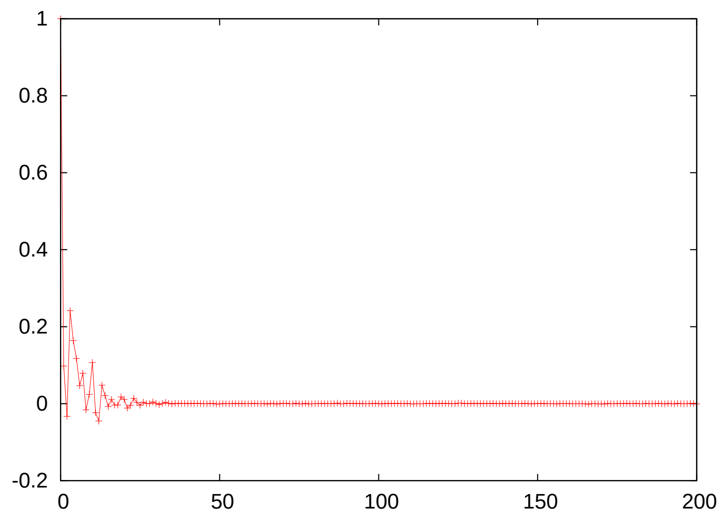
Right: For the **conservative case** $\gamma = 2.4$ some **invariant curves** of the time-2π map, around a **large chaotic domain** which looks like the orbit plotted for $\delta = 10^{-6}, \gamma = 2.4$.

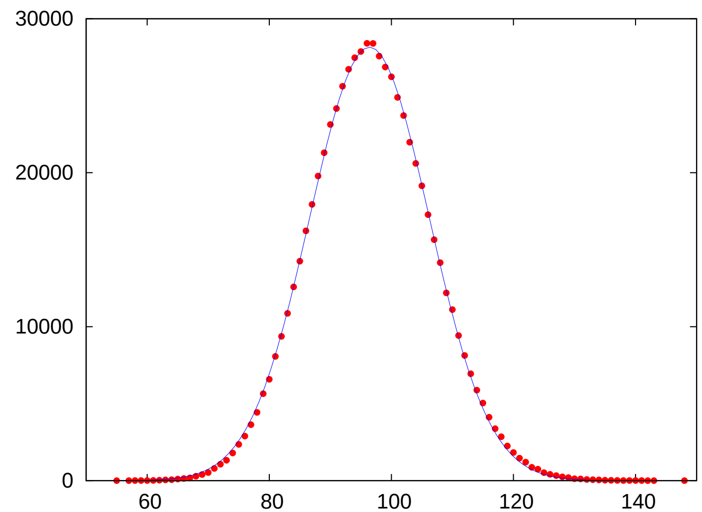
If an n-dimensional system has **Lyapunov exponents** $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ and k is the **maximum index such that** $\sum_{i \leq k} \lambda_i > 0$, then the **Lyapunov dimension** is defined as $D_L = k + (\sum_{i \leq k} \lambda_i > 0)/|\lambda_{k+1}|$.

In present case $\lambda_1 > 0, \lambda_2 < 0, \lambda_1 + \lambda_2 = -2\pi\delta$.

In the **conservative case** we can look at statistics of the iteratess. First at the **autocorrelation** of some variable (e.g. x) as a function of the delay. The left plot shows a **quick decrease of the correlation**.

Next we look at **how the iterates are distributed in the chaotic domain**. We use 1024×1024 pixels covering the domain. Computing 2^{26} iterates of an initial point, **201,539 fall in the 3,803 boundary pixels**; the remaining N_r ones **fall in the $n_i = 692,161$ inner pixels**.





Right: **Number of inner pixels** in which k **iterates fall**, as a function of k. **Assuming** iid **in each of the** n_i **pixels** with binomial probability $1/n_i$, **the normal character of the data** fits the assumption.

Summarizing:

Some elementary models have been presented. Changes in the dynamics which cannot be associated to local or semiglobal bifurcations have been detected.

The mechanisms leading to these changes are understood in a local or semiglobal basis, both geometrically and analytically.

I consider that to predict when these global phenomena occur is one of the present challenges in Dynamical Systems.

Thanks for your attention