The role of Michel Hénon in detecting regular and chaotic dynamics in conservative systems

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Quadratic APM

The model in different versions:

$$HM_\alpha : \begin{pmatrix} x \\ y \end{pmatrix} \to R_\alpha \begin{pmatrix} x \\ y - x^2 \end{pmatrix}, \ \alpha \in [0, \pi]; \ HMG_{a,b} : \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} 1 - ax^2 + y \\ bx \end{pmatrix}, \ b = -1;$$

$$F_c \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x + 2y + \frac{c}{2}(1 - (x+y)^2) \\ y + \frac{c}{2}(1 - (x+y)^2) \end{pmatrix}, \ c > 0,$$

with fixed points at $(-1, 0)$ **hyperbolic** H and $(1, 0)$ **elliptic** E if $0 < c < 2$, of **shear** type for $c = 2$ and **hyperbolic with reflection** for $c > 2$.

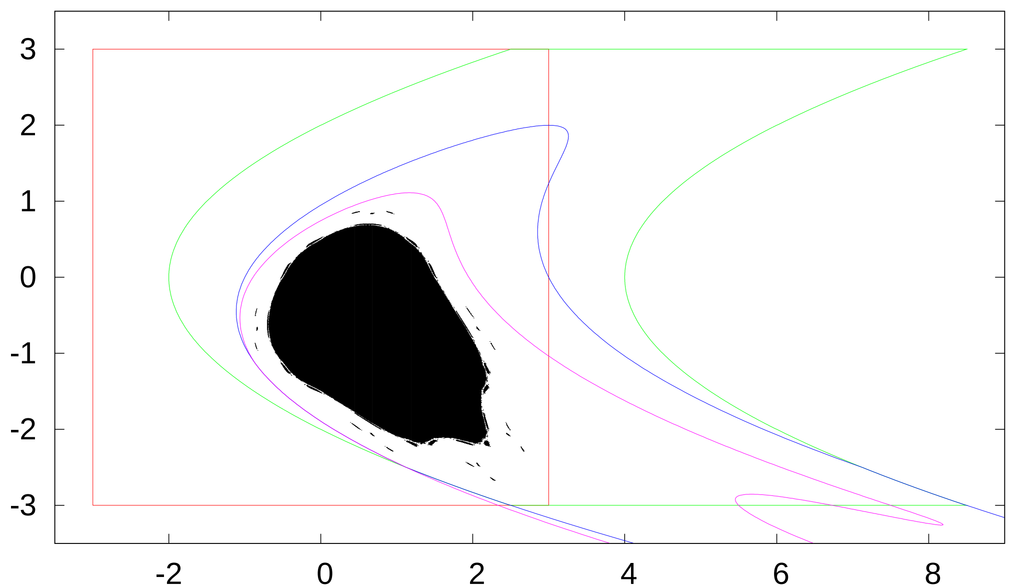
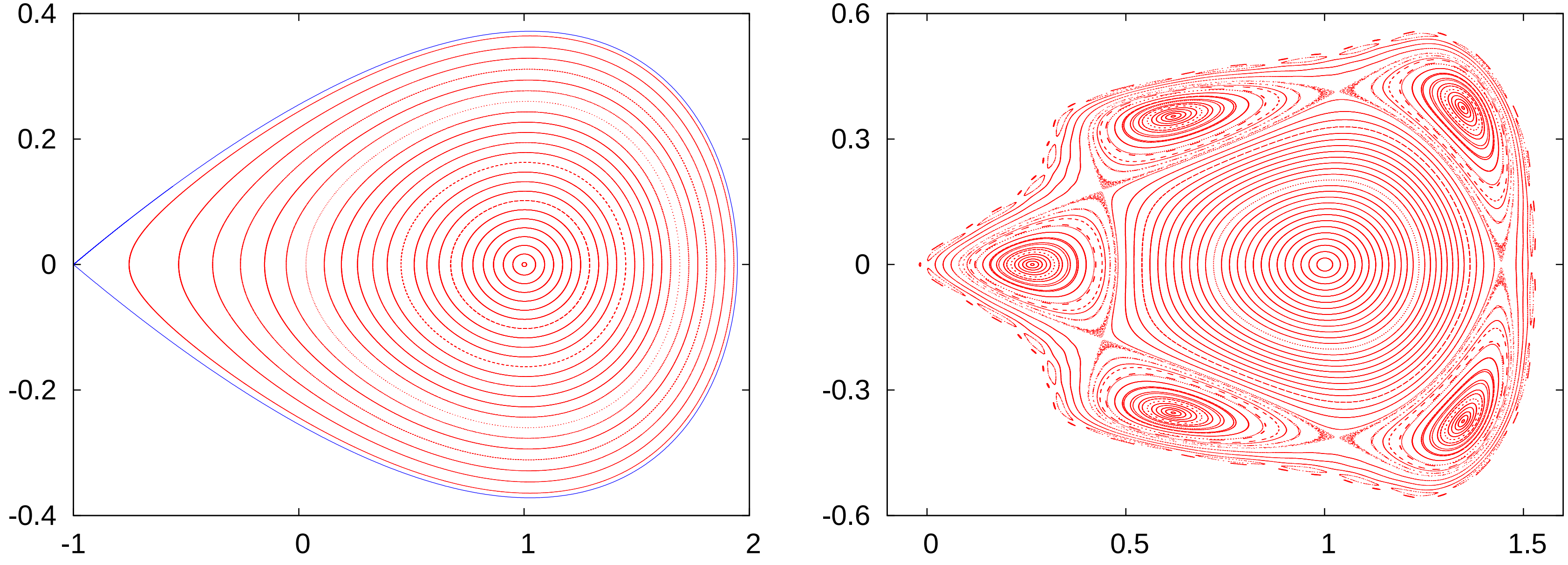


Image under HMG: quadratic shear plus rotation.

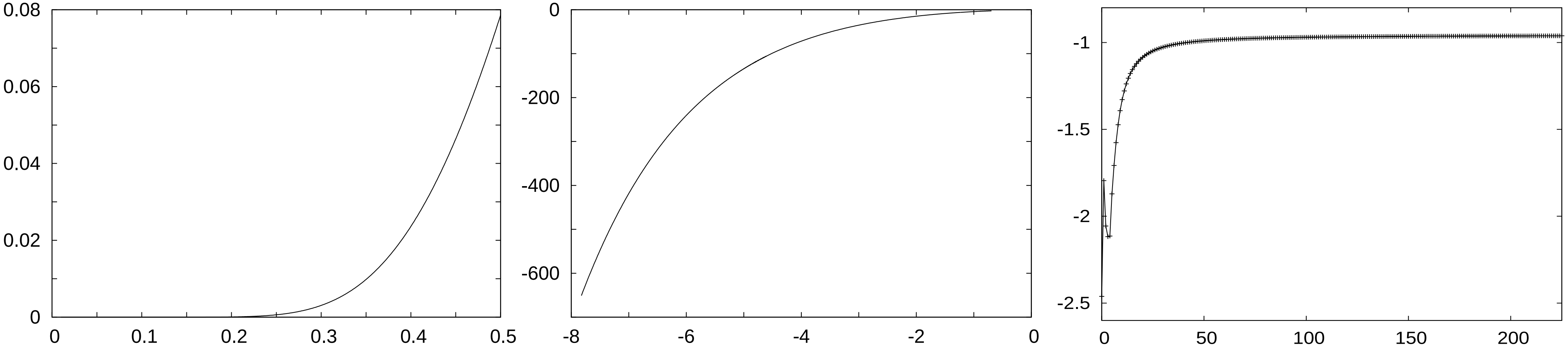


Bounded orbits for $c = 0.2$ and $c = 0.762$. The first case is close to the **time-1 map of a flow**, see later. The **invariant manifolds of H** seem to be coincident. In the other case **period-5 islands** and the associated **chaotic zones** are visible.

If **c is small** (like in the left plot) after a suitable **scaling of coordinates and time** the dynamics of F_c is close to the one of the **time-1 flow of the ordinary differential equation (ODE)**

$$\frac{d\xi}{dt} = \eta, \quad \frac{d\eta}{dt} = 1 - \xi^2,$$

with Hamiltonian $H(\xi, \eta) = \frac{1}{2}\eta^2 - \xi + \frac{1}{3}\xi^3$.



The **splitting of the separatrices of H, $\sigma(c)$**. Left: in true scale; middle: $\log(\sigma(c))$ **as a function of** $\log(c)$. Put $h(c) = \log(\lambda(c))$ and

$$\sigma(c) = \frac{9}{2} \times 10^6 \pi^2 \exp\left(-\frac{2\pi^2}{h(c)}\right) \times \Omega(h), \quad \Omega(h) = \sum_{m \geq 0} \omega_{2m} h^{2m}.$$

Right part:

$$\log_{10}\left(\omega_{2m}\frac{(2\pi^2)^{2m}}{(2m+6)!}\right) \quad \text{as a function of} \quad m,$$

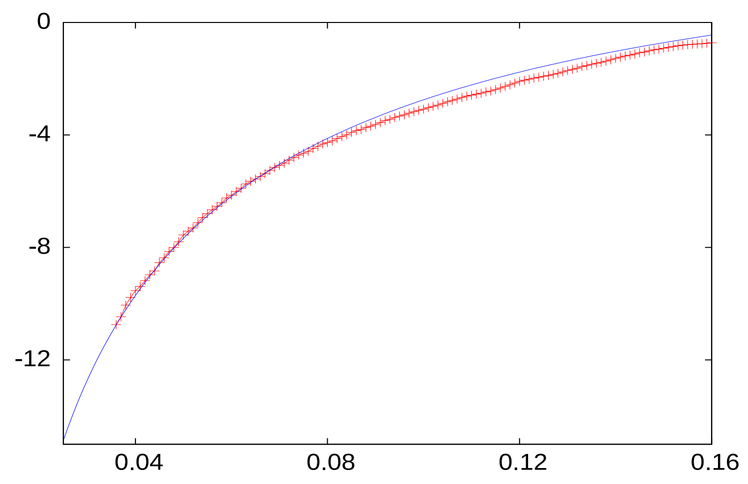
giving evidence of the **Gevrey character** of the correction term (**open problem**) with Borel transform having radius of convergence $2\pi^2$.

Relation of the Hénon map with the Standard map

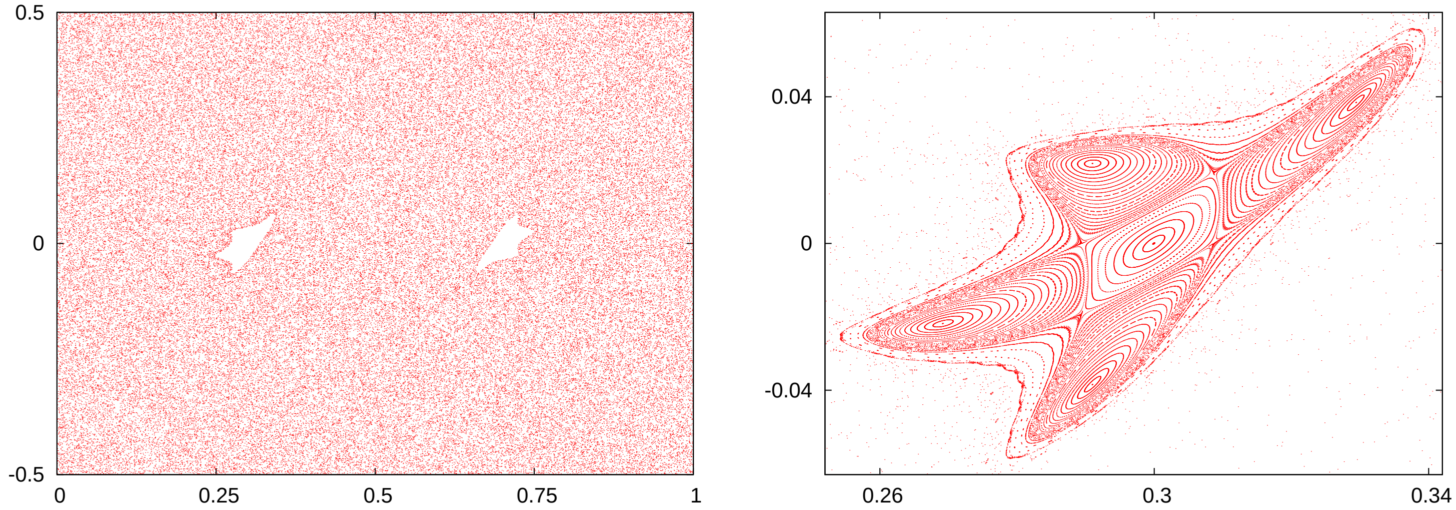
We recall the **Chirikov's standard map**

$$M_k : \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} = \begin{pmatrix} x + \bar{y} \\ y + k\sin(2\pi x) \end{pmatrix},$$

defined in $\mathbb{T}^2$ or in $\mathbb{S} \times \mathbb{R}$. Invariant rotational curves exist until **Greene's threshold**, $k \approx k_G = 0.971635\ldots/(2\pi) \approx 0.15464$.

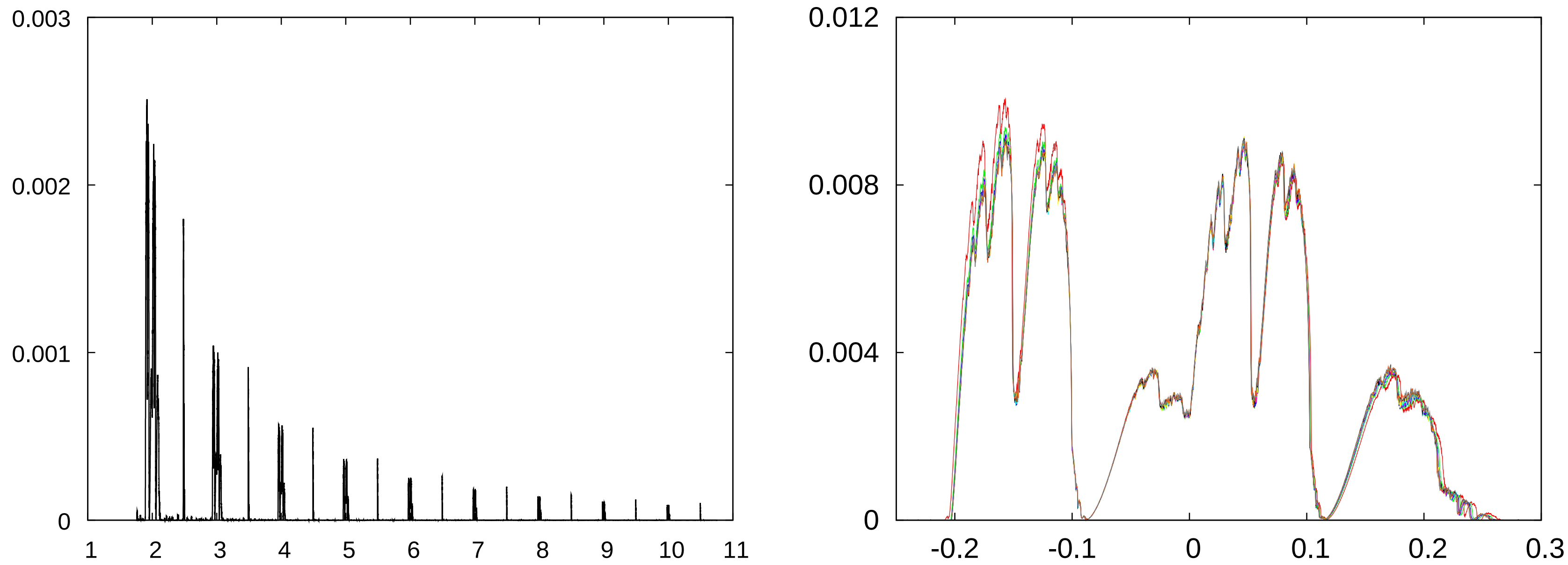


The figure shows a fit of the **measure of the set of chaotic points** and an exponentially small fit $Ak^B \exp(-C/\log(\lambda(k)))$.
This **kind of fits** will appear many times in next slides.



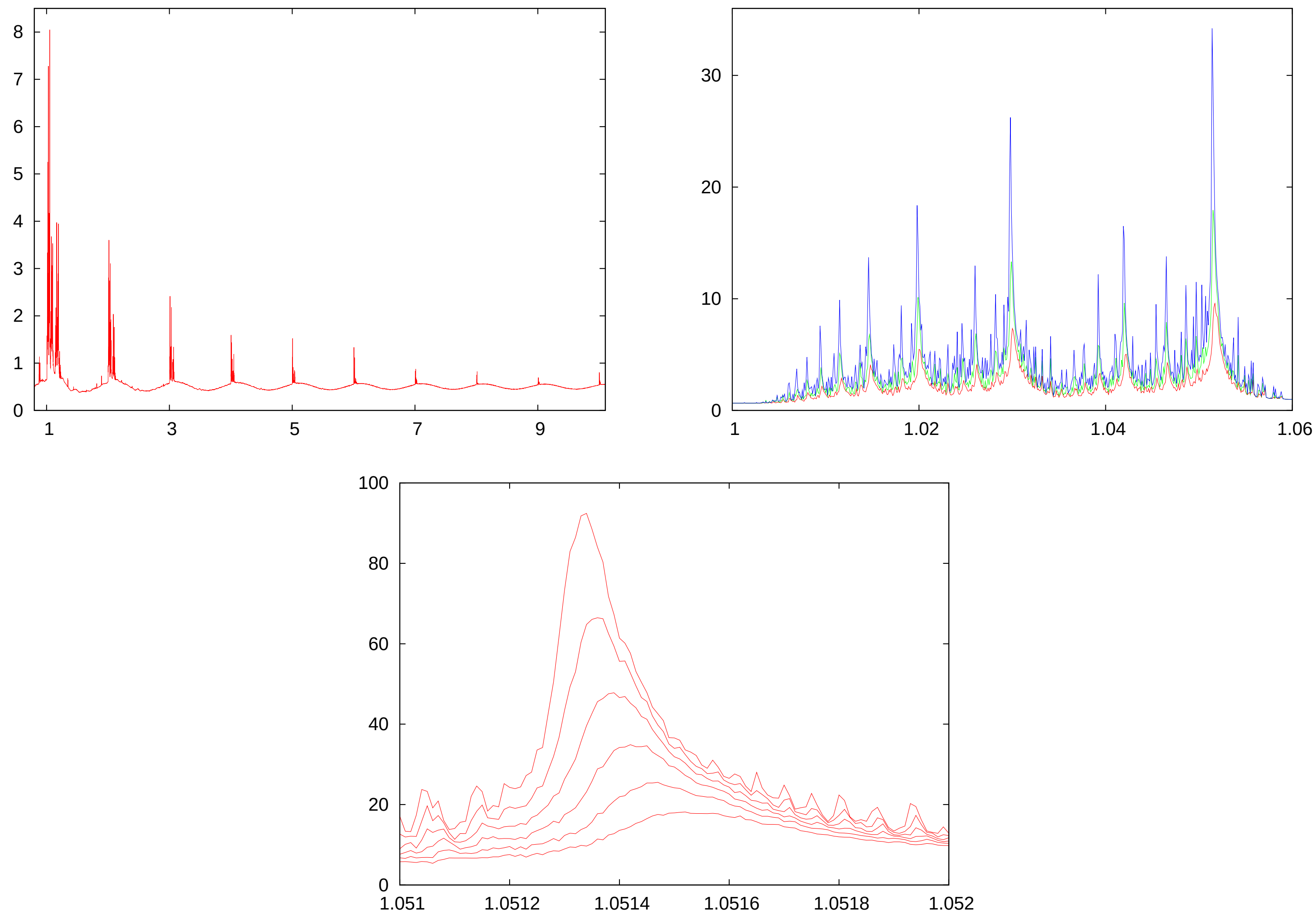
Left: Iterates of a point in the **unstable manifold of the origin**. Right: The **islands which cannot be penetrated** in the previous iterations. The parameter is $k = 1.05123$. Quite similar to the **invariant curves around islands of period 4 in the Hénon map**.

In fact the following result holds:
Proposition: There exist a **limit behaviour of the dynamics** around the two fixed points, and around the period 2 and period 4 orbits of M_k **under suitable scalings** in x, y and k for k **near** n**,** $n \in \mathbb{Z}$ **and** $n - 0.5 \in \mathbb{Z}$. These limit maps **are conjugated to area preserving orientation-preserving or orientation-reversing Hénon maps**, depending on a suitably scaled parameter.



On the left we show the **relative measure, $A_r(k)$**, in the phase space of M_k of the **set of points with zero Lyapunov exponent** as a function of the parameter $k = 1.75(0.00005)10.75$. This gives an indication of the **size of the relevant islands**. Observe the **scalings in size and domain**, different near integers and near integers plus $1/2$.

On the right the different **bumps for k close to integer** are shown after a **suitable nonlinear scaling** (which depends on k). They look like **two concatenated copies of the Hénon map**. The deepest minima are related to the **islands of period 4** and to the location of the **elliptic fixed point with $\rho = 1/3$**.



The plots show a **normalized diffusion coefficient** as a function of k and the number of iterates N. We look at M_k on the cylinder $\mathbb{S} \times \mathbb{R}$. Assuming **standard diffusion and uncorrelated iterations** one has $k^2 N/2$ **as variance of a sample**.

For concreteness we compute the **standard deviation of a sample** divided by $k\sqrt{2N}$. Under the above assumptions one should obtain **a value of 1/2**. An improvement is done to **correct for correlation**. Using Fourier techniques this produces a minor modification **involving Bessel functions**, which agree very well with the **undulations seen in the top left plot**.

The **diffusion anomalies** seen as **rather narrow bumps whose location and height depend on k and N** are associated to the existence and destruction of **invariant curves around periodic islands on the corresponding Hénon model**.

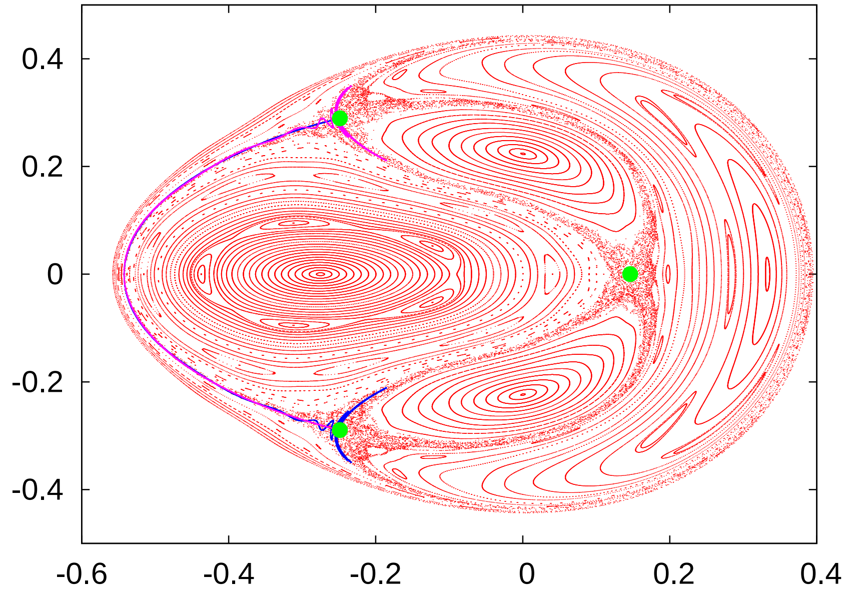
Islands around period 1 points of M_k in $\mathbb{T}^2$ become **accelerated modes** in $\mathbb{S} \times \mathbb{R}$. Note that for **islands around period 2 and period 4** of M_k in $\mathbb{T}^2$ there is a **cancellation** with tiny effects on the diffusive properties.

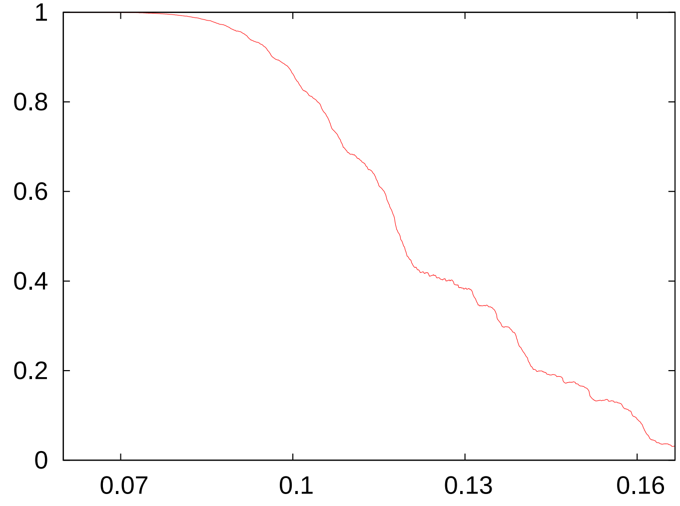
The **Cantor sets produced at the breakdown** of invariant curves around the main islands, and the **chaotic domains of the associated hyperbolic points** play a key role.

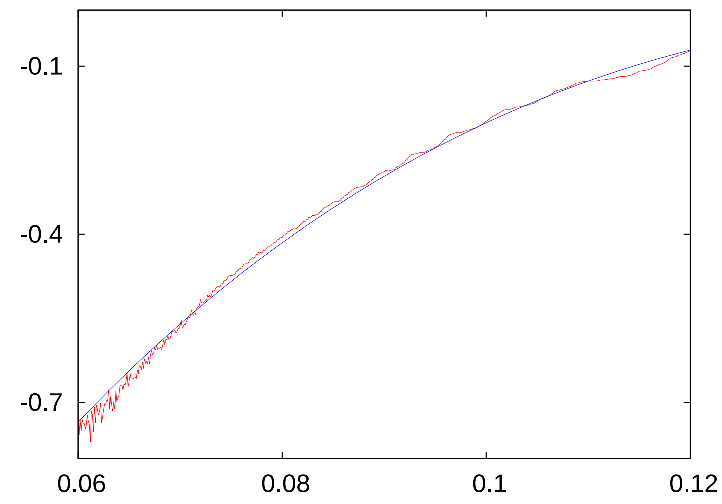
Some active work is now in progress (with N. Miguel and A. Vieiro), but there are still **many open questions**.

The Hénon-Heiles Hamiltonian, closely related systems and a generalization

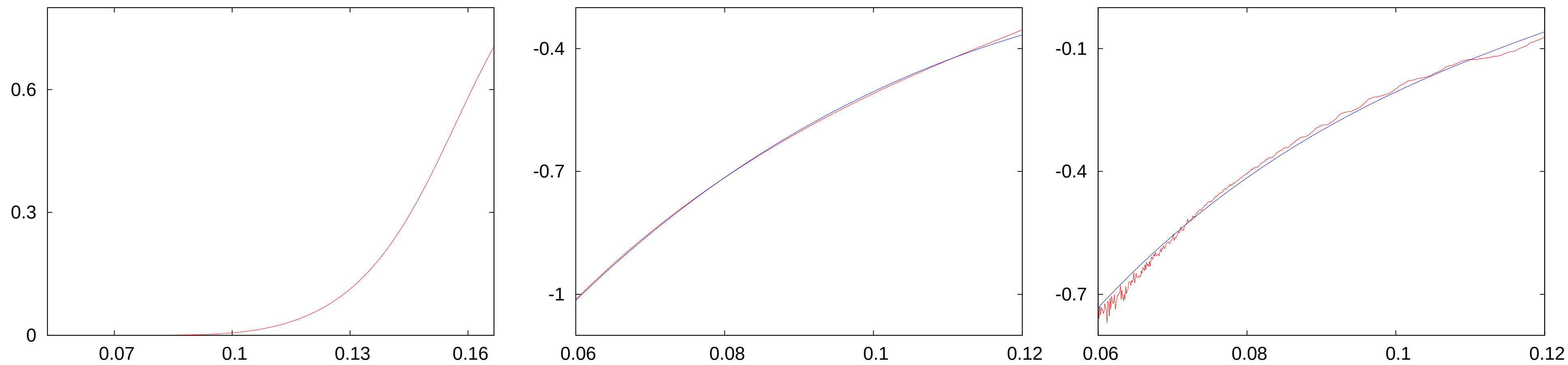
Recall $HH = (x^2 + y^2 + p_x^2 + p_y^2)/2 + x^3/3 - xy^2$. Plot on $y = 0$ for $h = 0.1$.







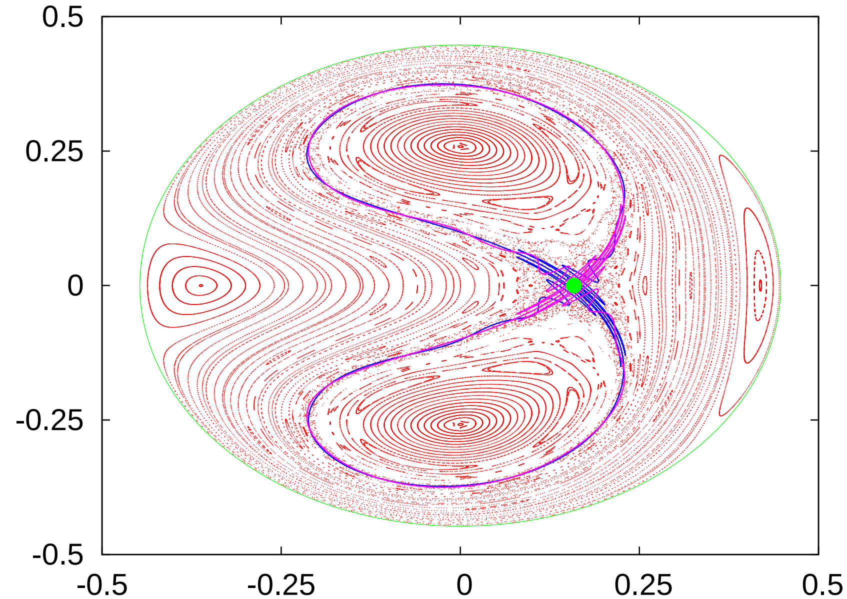
Left: **regular fraction** $\psi(h)$ **vs** h; right: $h \log(1 - \psi(h))$ **vs** h and a fit.

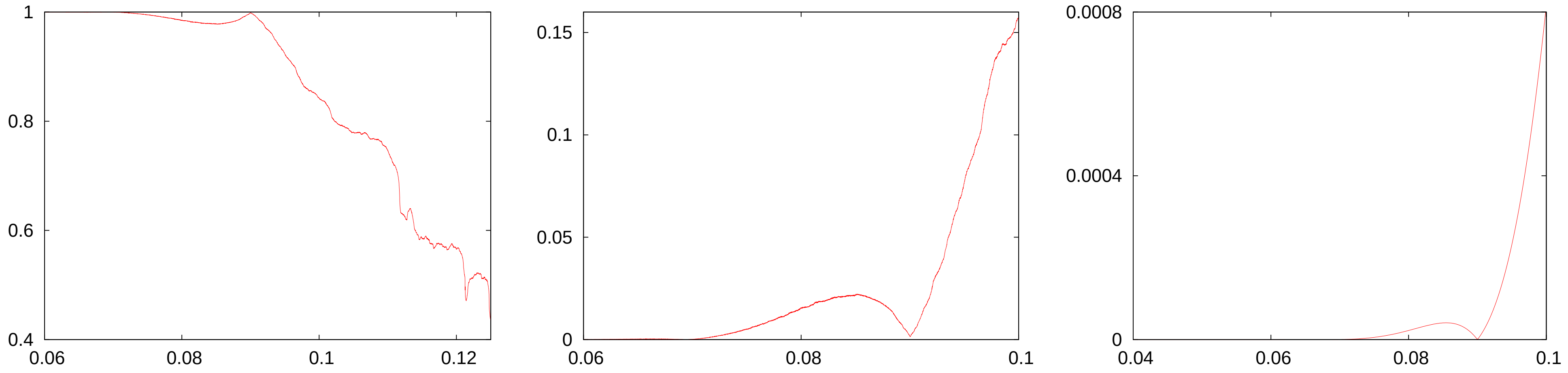


Left: **splitting angle $\sigma(h)$ of $W^{s,u}$ from fixed points of the Poincaré map**, measured on $y = 0, x < 0$, vs h.
Middle: $h \log(\sigma(h))$ in an intermediate range, excluding values of h for which $\sigma(h)$ is **too small** to be confident on the dp values, and large values of h, for which the **destruction of invariant curves around period 5 orbits plays an additional role**. We also plot (in blue) a **fit** $Ah^B \exp(-C/\log(\lambda(h)))$.
Right: Comparison of $h \log(1 - \psi(h))$ and a multiple of **the $\sigma(h)$ fit**, with changes in A and B (following from an analysis of the separatrix map).

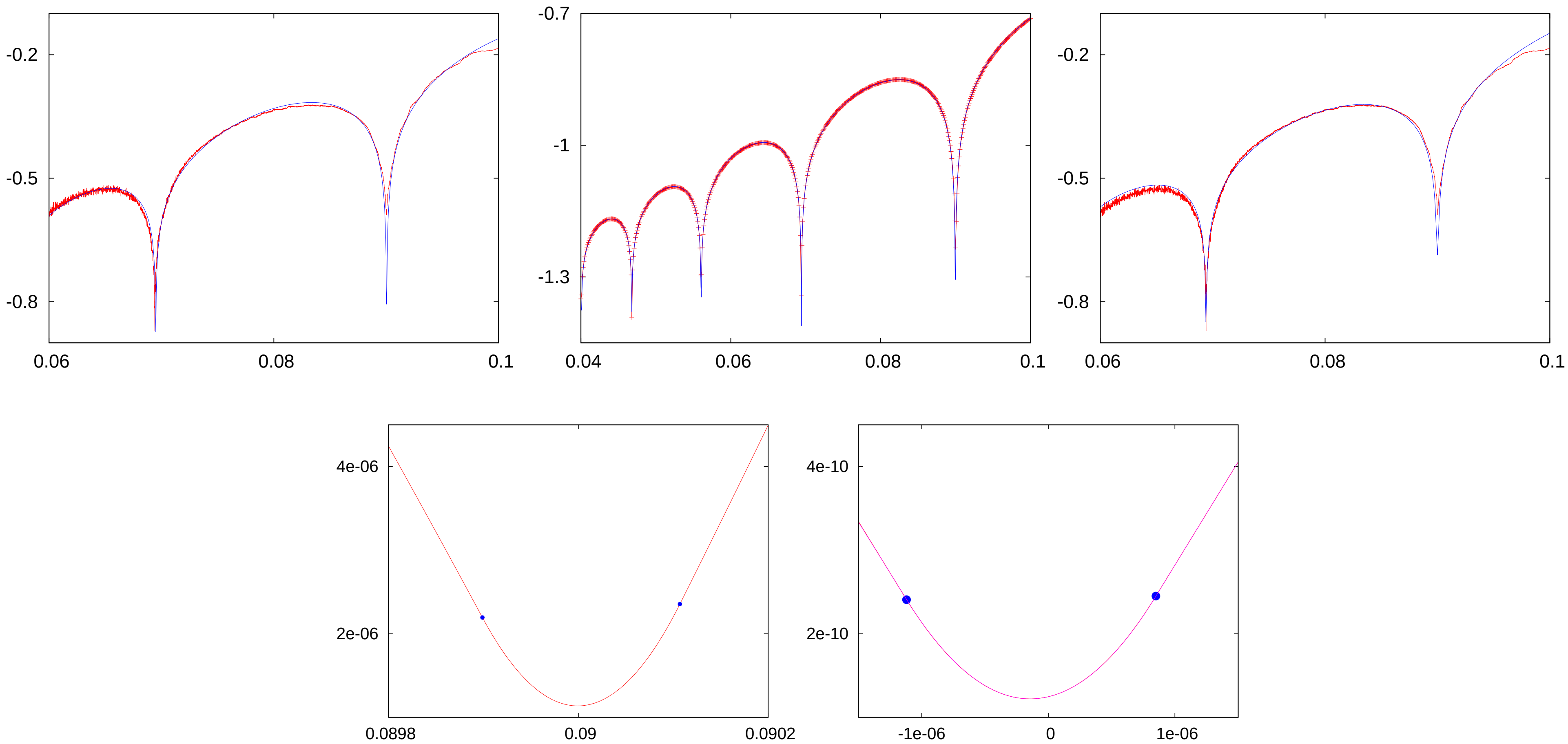
There is a nice relation between the measure of the chaotic set and the splitting of the manifolds of the periodic hyperbolic orbits. As can be expected now (not in 1962!), this is exponentially small.

What about $HC0 = (x^2 + y^2 + p_x^2 + p_y^2)/2 - xy^2$? Plot on $y = 0$ for $h = 0.1$.



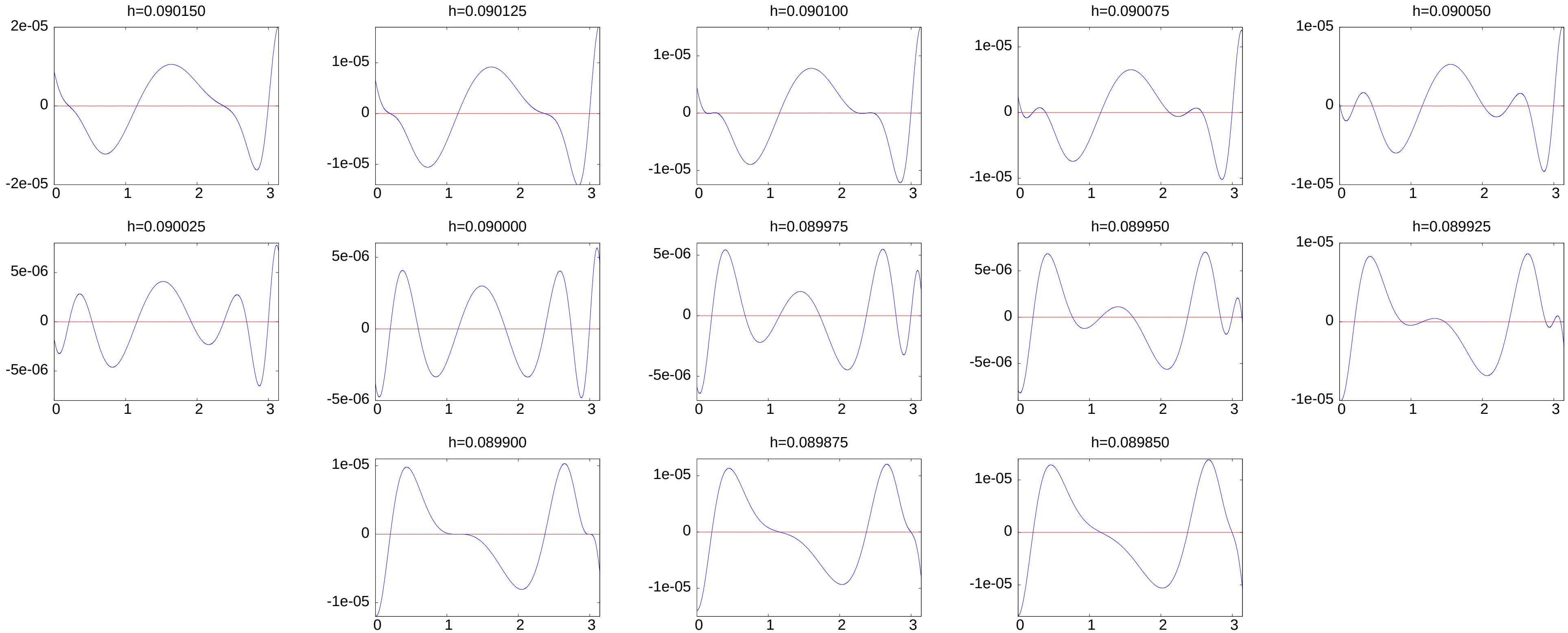


Left: $\psi(h)$ vs h. Middle $1 - \psi(h)$ vs h. Right: the **splitting $\sigma(h)$ of the manifolds** as seen in the top plot. **Why $1 - \psi(h)$ and $\sigma(h)$ are not uniformly tending to zero when $h \to 0$?**



Remark: In the present case we do not measure $\sigma(h)$ as an angle but **as the sum of the (absolute values of the) areas of lobes**.

Top row: Left: $h\log(1-\psi(h))$ and a fit (not as usual!). Middle: $h\log(\sigma(h))$ and related fit. Right: $h\log(1-\psi(h))$ and a multiple of the fit for $\sigma(h)$, as in the HH case. Lower row: Left: A magnification of the **behaviour of $\sigma(h)$ near the rightmost minimum** $h \approx 0.089999$. Right: Similar for **next minimum** $h \approx 0.06941635$ and a fit to a theoretically derived formula.



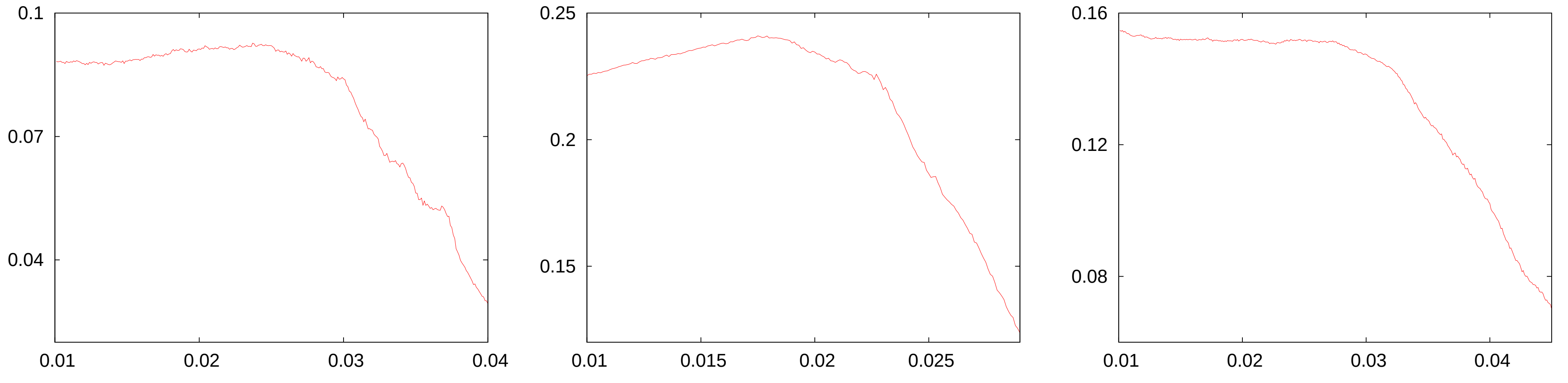
A **movie of the position of W^u wrt W^s as a function of h** around the rightmost minimum. It shows **cubic tangencies and new homoc.**

Why?: The oscillation of W^u wrt W^s is periodic (in suitable variables) with **first harmonic** amplitude $\mathcal{O}(\exp(-C/\log(\lambda(h))))$ where $C = 2\pi\Gamma_1$ **if some limit 1dof Hamiltonian has a separatrix with closest singularities to $\mathbb{R}$ of the form** $\pm\Gamma_1 \mathrm{i}$. If they are $\pm\Gamma_1\mathrm{i} \pm \Gamma_2$ the exp term has to be multiplied by $\cos(D/\log(\lambda(h)) - \varphi(h))$ **with** $D = 2\pi\Gamma_2$. This has to be included in the fit. When $\cos \approx 0$ the **second harmonic plays a role**.

What about $H = (x^2 + y^2 + z^2 + p_x^2 + p_y^2 + p_z^2)/2 + Ax^2y + By^2z + Cz^2x$?

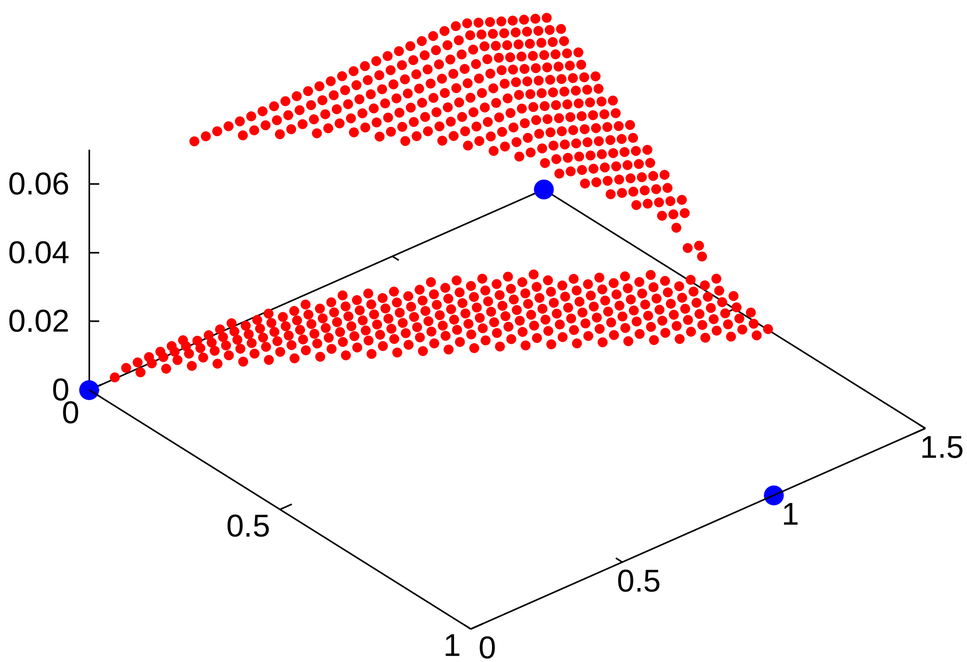
It can be seen as a **generalisation of HH to 3 dof**; allows to **study the 1:1:1 resonance**; allows to **break all symmetries** if $A \le B \le C$ with $<$ in all cases. Clearly, we can normalise $A + B + C = 3$.

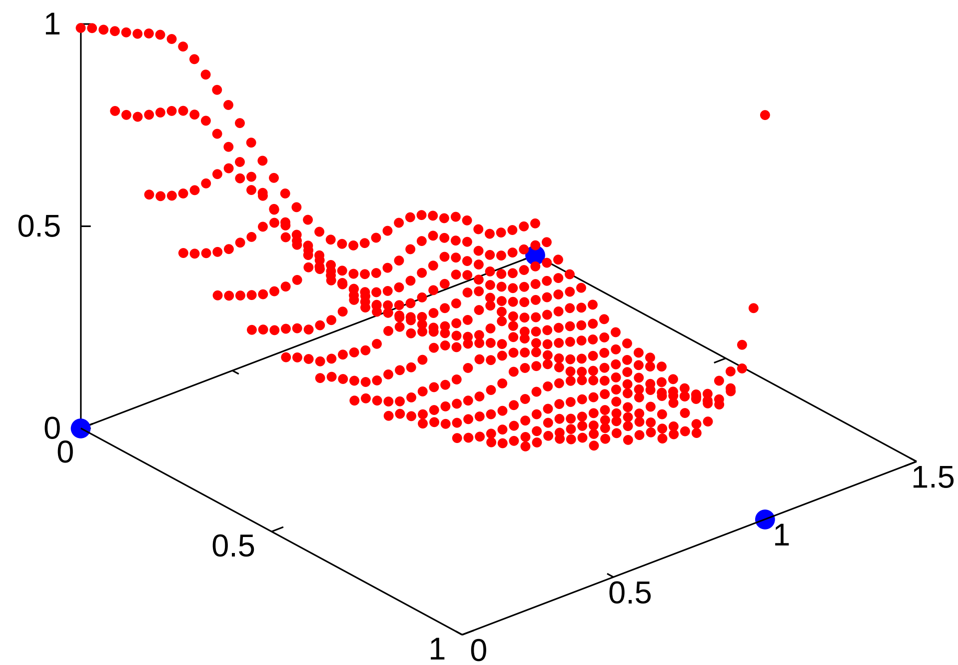
A problem: we cannot expect $\psi(h) \to 1$ as $h \to 0$. To distinguish regular from chaotic via Lyapunov exponents **is computationally expensive**. **Sketch of solution**: 1) scale by $\varepsilon = \sqrt{h}$; 2) compute NF in (I, φ); 3) slow angles $\alpha_j = \varphi_j - t$; 4) average wrt t; 5) scale time by h; 6) let $h \to 0$. We obtain an **homogeneous quartic Hamiltonian** H_4. The **regular fraction** $\psi(h)$ as a function of the initial h for values (A, B, C) equal to $(0, 3/2, 3/2)$, $(0, 1, 2)$, $(1/2, 1, 3/2)$ is shown from left to right.



What about **integrability**? Standard tools (Ziglin, Morales-Ramis, etc) can be used. Given (A, B, C) there exist **motion is lines in the configuration space**. The solutions are given by **elliptic functions, doubly periodic, with a pole of order 2**. Looking at the **monodromy (in general, jet transport) along a suitable circuit γ (in present case, just local)** allows to detect **non-integrability** for all (A, B, C).

An **algebraic measure** $1/(1+\log(||M(\gamma)-I||+1))$ of regularity, using the maximum of $||M(\gamma)-I||$ on the real lines (left plot) can be compared to the **dynamical measure ψ** (right plot). Planar variables: A, B.



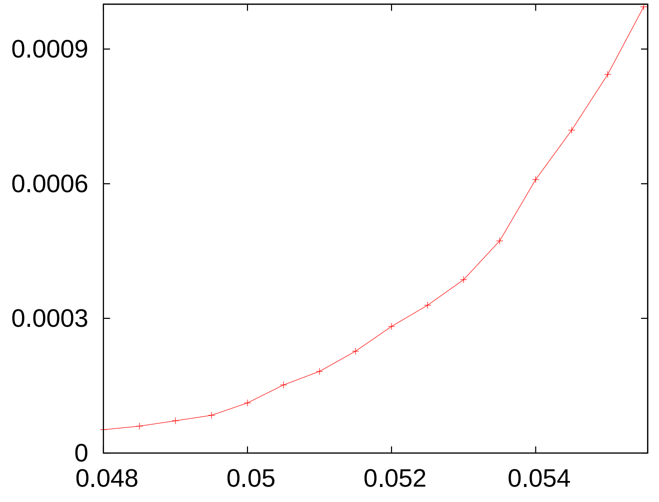


From the dynamical measure ψ it seems that there are two cases $(A, B, C) = (0, 0, 3)$, $(A, B, C) = (1, 1, 1)$ **much more regular**. No **lack of regularity is detected using** H_4.

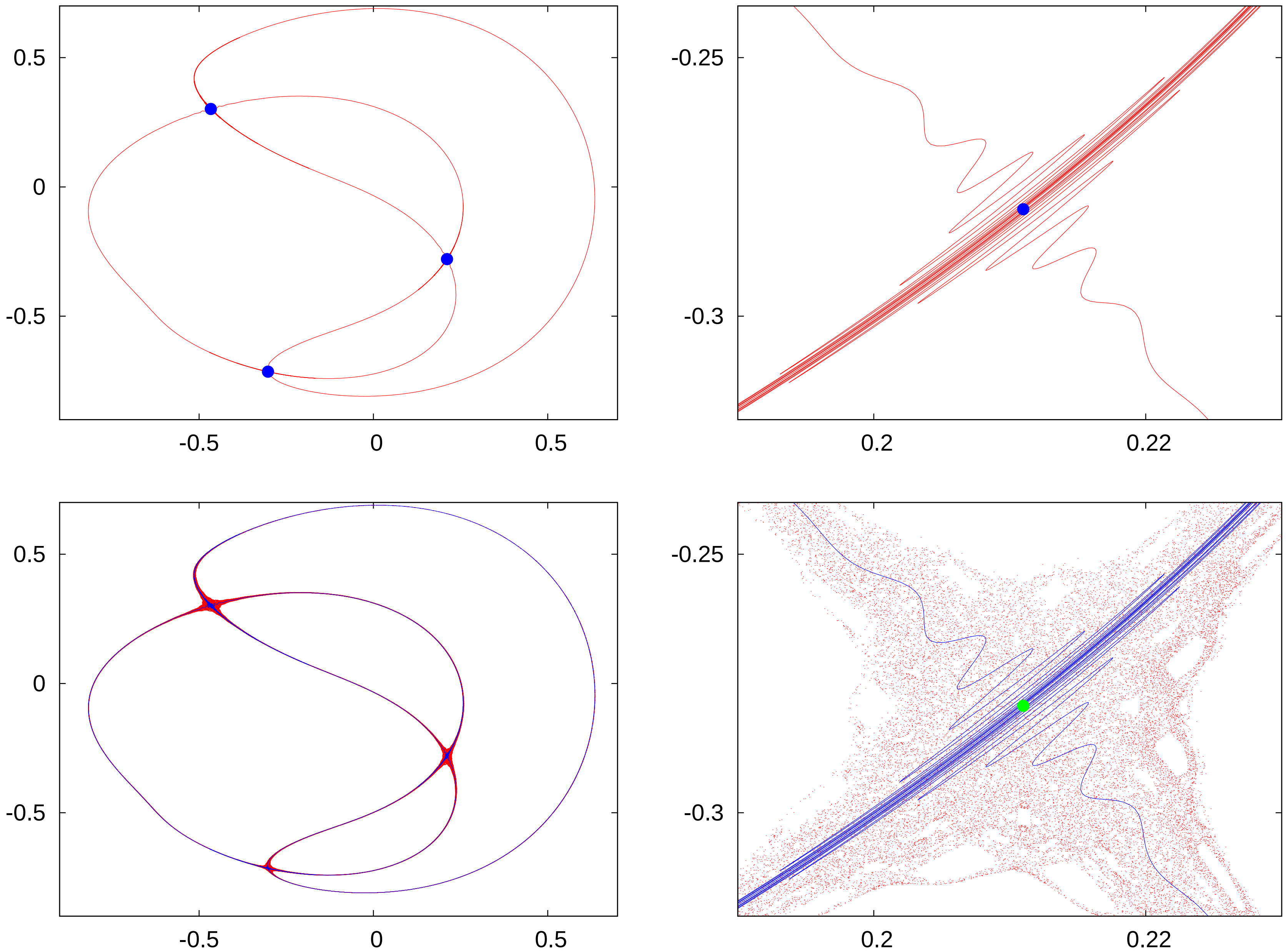
The case $(0, 0, 3), h < 1/12$ could be expected. It coincides with $HC0$ **times an harmonic oscillator**. What happens in the $(1, 1, 1), h < 1/18$ case ?

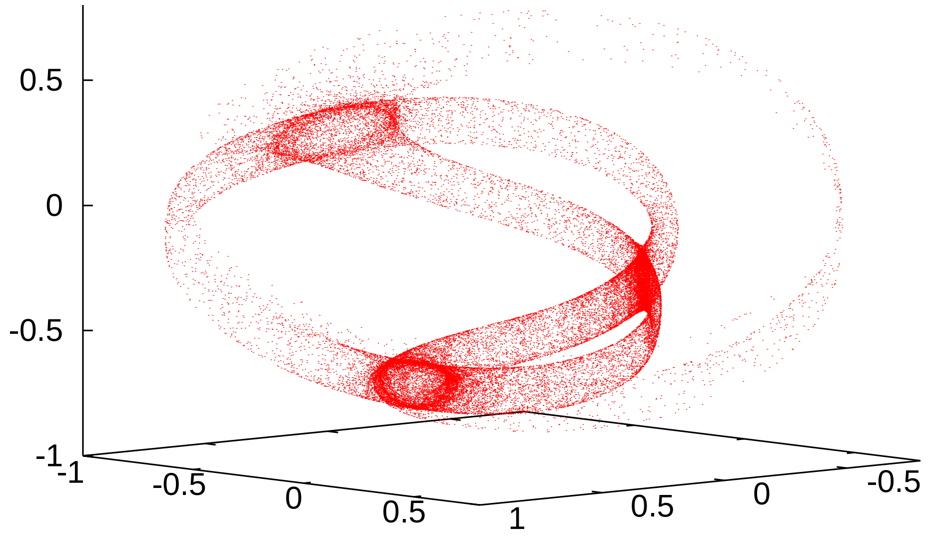
From the **symmetry** one has that $u = x + y + z$ comes from the **1dof** $(u^2 + p_u^2)/2 + u^3/3$. Hence, it has **an extra integral**. The behaviour is similar to the 1:1 case. H_4 **that would detect only** $O(h)$ **chaos is unable to detect the exponentially small one**.

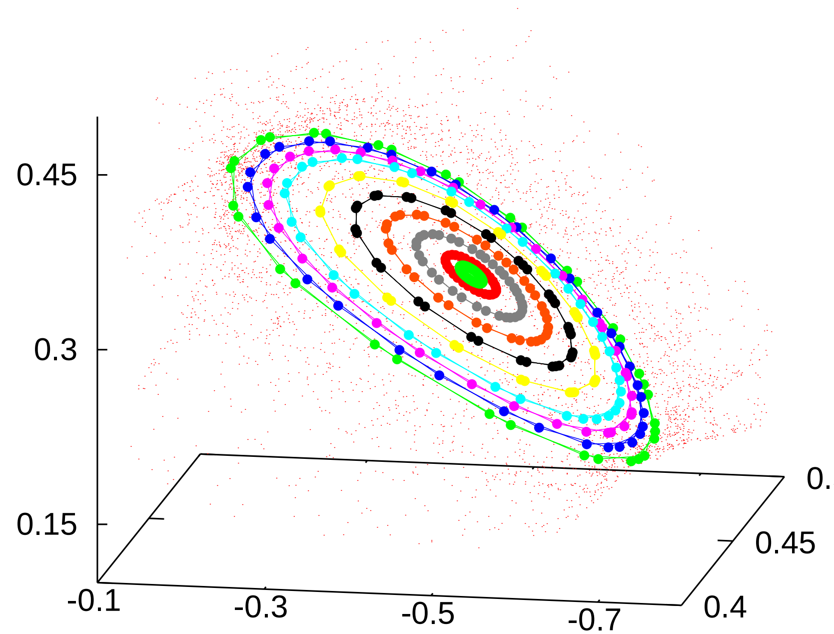
Surprisingly $1 - \psi$ for $(1, 1, 1)$ is **very small** (see plot) even for h close to $1/18$. It requires **large samples** to be confident on ψ.



Some plots of the dynamics in the $(1,1,1)$ case in the section $x=0$ which contains **3 fixed points p_j of the Poincaré map**. We remark that the **1D $W^{u,s}$ of these points also lie on $x=0$**. Energy: $h=0.333^2/2$.







Left: a 3D view of **iterates in $x = 0$, showing the heteroclinic chains between the $W^{u,s}$ of the centre manifolds of $p_j, j = 1, 2, 3$.**

Right: Some **invariant curves in $W^c(p_i)$** together with some iterates close to the $W^{u,s}$ of some of these curves.

Summarizing: **still an infinite amount of things to understand!**

But we are **guided by Michel Hénon insight in displaying key paradigmatic models**.

Understanding these models gives the basic ideas to understand many other phenomena.

Thanks Michel!!!