De l'ordre al caos: el paper de les varietats invariants

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Temes a considerar

- **El Sistemes Dinàmics: perquè apareixen a tot arreu. Casos continus i discrets.**
- **L'esquelet del sistema: objectes invariants i la seva hiperbolicitat.**
- **Varietats invariants: el seu paper com a organitzadores de la dinàmica.**
- **Interseccions transversals que porten a impredictibilitat.**
- **Alguns exemples paradigmàtics elementals: conservatius i dissipatius.**
- **Alguns problemes oberts d'enunciat "elemental".**
- **Mecanismes que porten a la creació i destrucció de caos.**
- **Com mesurar la quantitat de caos.**
- **Algunes aplicacions a problemes no elementals.**
- **Les eines: importància de l'anàlisi i la geometria i d'algorismes numèrics i simbòlics eficients.**

Dynamical Systems

The object of study: **Everything which changes with time**.
It can be the motion of celestial bodies, space missions, fluid mechanics (like in meteorology, in oceanography or in a washing machine), the behavior of small and large sets of neurons, the physics of laser, chemical reactions, particle accelerators, confinement of plasma, biological populations, industrial production, propagation of epidemies, numerical iterative algorithms, etc.

We assume that there is a mathematical formulation of the evolution according to physical laws (despite in many problems they are not yet well understood).

Three main kinds of models:

- **Ordinary differential equations:** $\frac{dx}{dt} = f(t, x)$,
- **Evolution partial differential equations, like:**

$$\frac{\partial u}{\partial t} = F(t, x, u, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \ldots)$$

- **Discrete systems:** $x_{k+1} = f(k, x_k)$.

We can add other phenomena like **delay/memory and/or random effects** and we can include **integral terms** in the equations.

To determine the **existence, uniqueness or multiplicity** of solutions one should give **additional information, like initial/boundary data, historical data, etc**.

Furthermore the **equations describing the evolution of the system** can depend on **parameters**, like **physical, chemical, geological, astronomical, ... constants**.

Where do the variables live? On a **phase space or space of states** which can be a **subset of $\mathbb{R}^n$ or of a manifold, or a suitable function space (e.g., functions defined on a domain, with a given regularity class)**.

The **most convenient way to look at the dynamics** is to consider it in the **state and parameters space $\mathcal{E} \times \mathcal{P}$**. Here is where we can **study bifurcations, singularities, etc**.

The ultimate goal of DS: to be able to give a **good complete description of the dynamics and its dependence on parameters**.

Importance: To be **able to predict (perhaps only partially)** and, therefore, **if it is possible to change parameters**, **to "act" on or to "control" the evolution of the system**.

A remark: except in elementary cases it is **not possible to give explicit expressions of the solutions**. They require **numerical methods**.

In some formulations **PDE are reduced to a large system of ODE** and, in turn, **ODE are reduced to approximate solutions at some t_k leading to a discrete system**.

Reciprocally, **some discrete systems can lead to an ODE** and **large systems of ODE can lead to a PDE**.

Obviously, **in any scientific or technological problem** one should **cycle from phenomena to models, to analysis of the system, to predictions, again to the phenomena and so on**.

Nothing else than the **scientific method, as already claimed by Hooke, Newton, Huygens, Lagrange et al**.

The skeleton of the system

To begin with we should study **minimal objects left invariant by the system**, like **fixed points (steady states), periodic solutions, quasi-periodic solutions (moving in tori), ... and sets formed by union of these objects, like spheres, hyperboloids, etc**.

Next step is to study **the stability properties of these objects**.

In all cases we should study the **spectrum of suitable linear operators**. In the simplest case **fixed points of autonomous discrete systems,** $x_{k+1} = F(x_k)$**, and ODE,** $dx/dt = f(x)$, we should look for the **eigenvalues of** DF **or** Df at the fixed point.

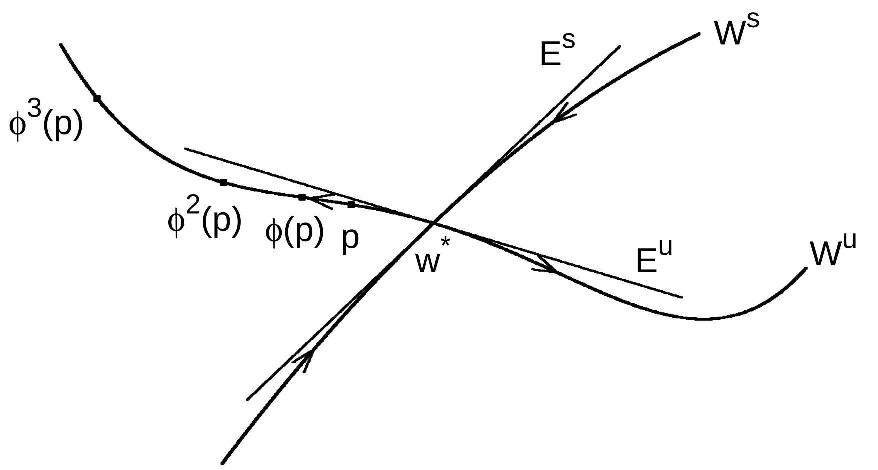
The eigenvalues $\lambda_j, j = 1, \ldots, n$ of DF can be classified as

- $|\lambda_j| > 1$ or **unstable part**,
- $|\lambda_j| < 1$ or **stable part**,
- $|\lambda_j| = 1$ or **central part**.

In the case of Df**, for ODE** they correspond to $\mathbf{Re}(\lambda) > 0$**,** $\mathbf{Re}(\lambda) < 0$**,** $\mathbf{Re}(\lambda) = 0$, respectively. The **associated linear subspaces are denoted as** E^u, E^s, E^c.

Invariant manifolds

Associated to these linear subspaces there are **invariant manifolds** W^u, W^s, W^c in general **nonlinear objects tangent to the linear subspaces**. The **unstable** W^u (resp. **stable** W^s) is formed by the points **tending to the fixed point by backward (resp. forward) iteration**. Specially important are the **codimension-1 manifolds**.



We can also talk about **centre-unstable** W^{cu} and **centre-stable** W^{cs} manifolds, study **existence, uniqueness, regularity, etc**.

When an invariant object has **unstable and/or stable manifolds** we talk about **hyperbolic invariant objects** (perhaps only in a **partial or weak sense**).

The set of the **minimal invariant objects** together with their **invariant manifolds** (if they exist) and its **relative positions** form the **skeleton of the dynamics**.

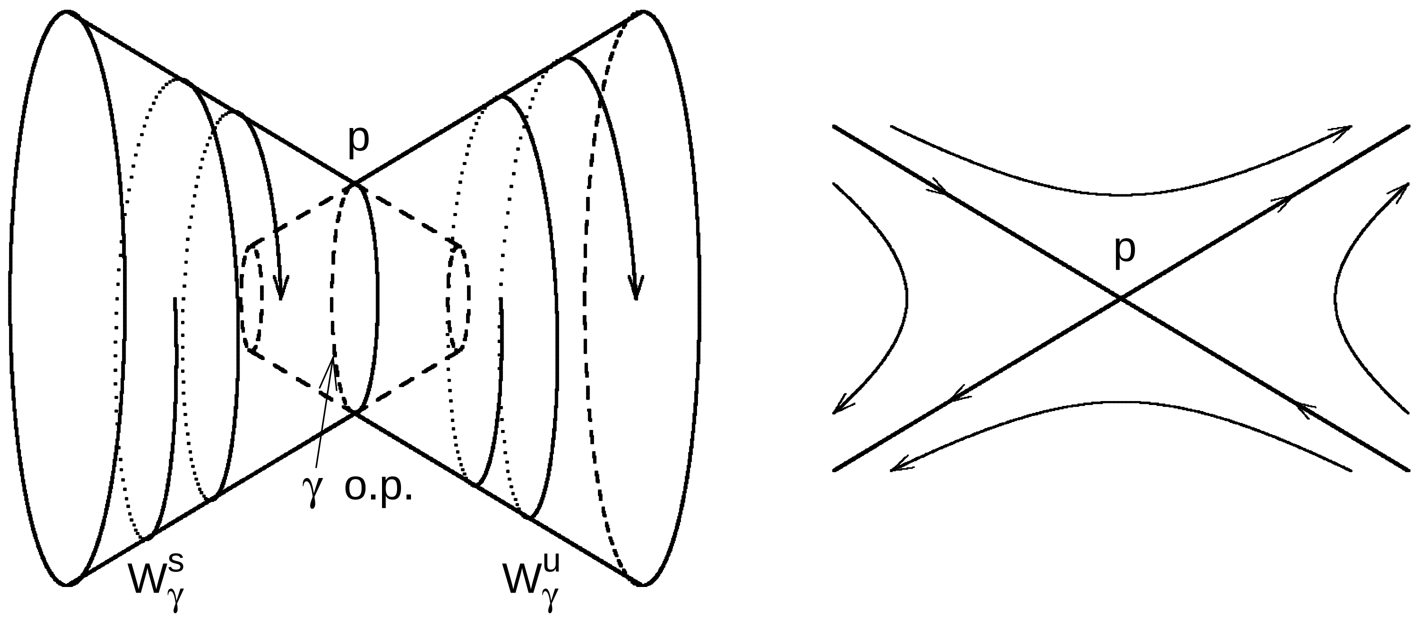
This **skeleton** is what **guides the dynamics of the system**. It can be seen as the **paths (or highways)** describing the **global behaviour of the system in the phase space**.

Hence, the **rules to understand the dynamics** are

1) Look for **the set of minimal invariant objects**,
2) Look for **its unstable/stable/central manifolds**,
3) Look for the **relative position and intersections of these manifolds**.

This study can be **not easy at all**. We can just mention the classical problem of the **relative equilibrium solutions of the n-body problem, for $n = 4, 5, 6, \ldots$** or the **Poincaré conjecture** about **density of periodic orbits among the bounded orbits in conservative systems**.

An example of **periodic solution** with **unstable and stable manifolds** and **how it is seen if we cut the dynamics by a suitable plane** (the so-called **Poincaré section**).

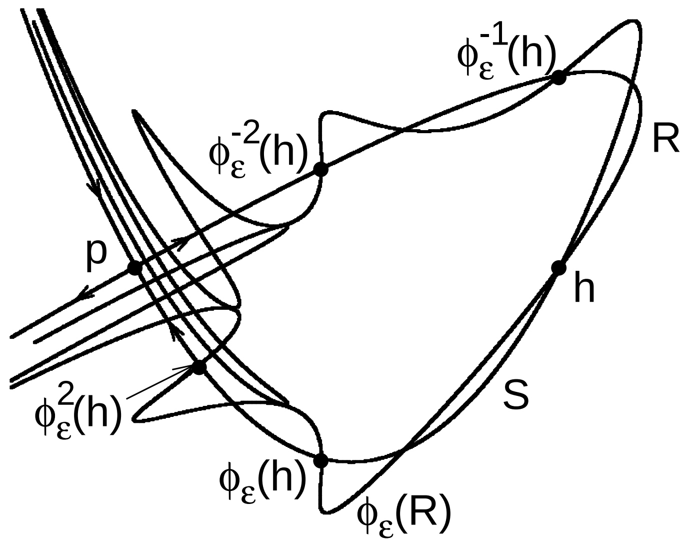


One can imagine the **complexity in case of more general objects, like (partially) hyperbolic invariant tori**.

But unstable manifolds which **go away from the invariant object** (like point p in the plot) **can fold and return near** p giving a **feedback**.

This leads us to consider the **possible intersections between manifolds**. Note that **unstable manifolds can only intersect stable ones and vice versa**. Specially important are the

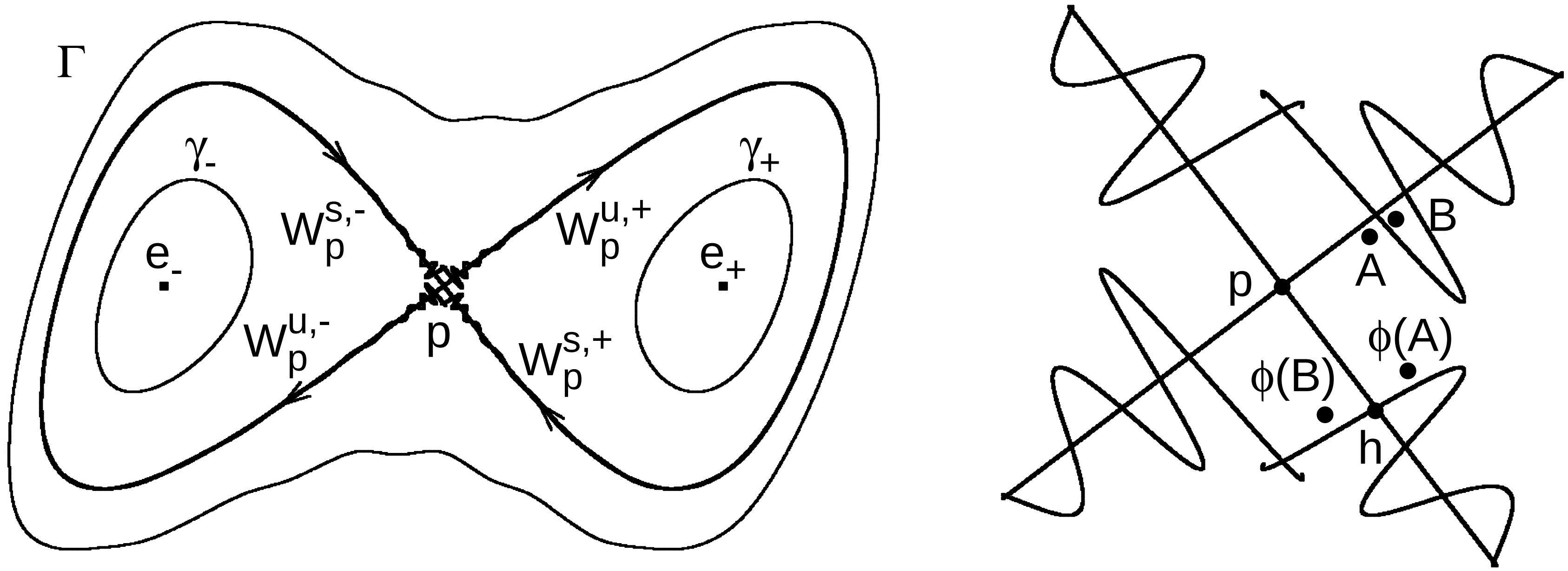
Transversal intersections



An example of **fixed point** p of a **conservative map in the plane** with **invariant manifolds** having **transversal intersections**.

When, for instance, the **unstable manifold of a point** p**,** W_p^u **intersects the stable manifold of the same point,** W_p^s**,** we talk about **homoclinic intersections**. If the invariant manifolds **correspond to different points,** W_p^u **and** W_q^s**,** we talk about **heteroclinic intersections**.

In the above plot, if the **right hand sides of** W_p^u **and** W_p^s **coincide** (i.e., **non-transversal** intersection), the points **inside the "loop"** are **confined**. Compare with **points in** R above.



Again for a **conservative map**, an example of **double homoclinic loop** with **transversal intersections** and of **inner and outer invariant curves** γ_+, γ_-, Γ.

The **magnification** on the right describes the fate of **two nearby points A, B** after **iteration along the right loop**. Both are **near the homoclinic point** h. This can be described by a **return map**.

After **passing again near** p they go **along** $W_p^{u,+}$ **and** $W_p^{u,-}$**, respectively**.

The behaviour of initial points **as close as we like** becomes **unpredictable** because of the **transversal intersections**.

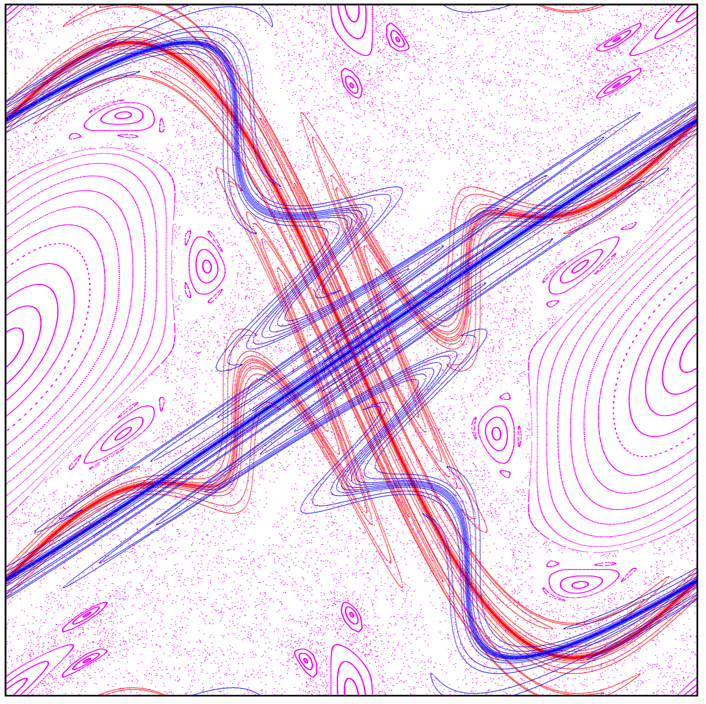
Paradigmatic examples

As **very simple examples** we consider a couple of 2D models

- The **standard map**: $(x, y) \to (\bar{x}, \bar{y}) = (x + \bar{y}, y + \varepsilon \sin(2\pi x))$ **in** $\mathbb{T}^2$,

- The **dissipative Hénon map**: $(x, y) \to (1 - ax^2 + y, bx)$ **in** $\mathbb{R}^2$.

Despite its **simplicity** they help to **understand more complicated models** and appear as **models of the models** even in higher dimension.



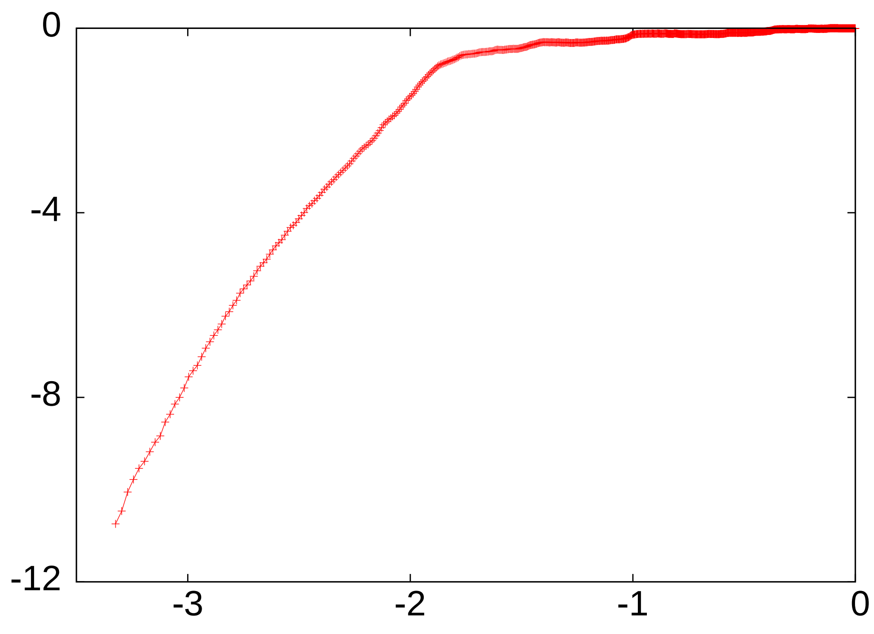
The **SM for** $\varepsilon = 0.25$: **invariant manifolds, invariant curves around the elliptic point, several islands and chaotic zones** can be seen.

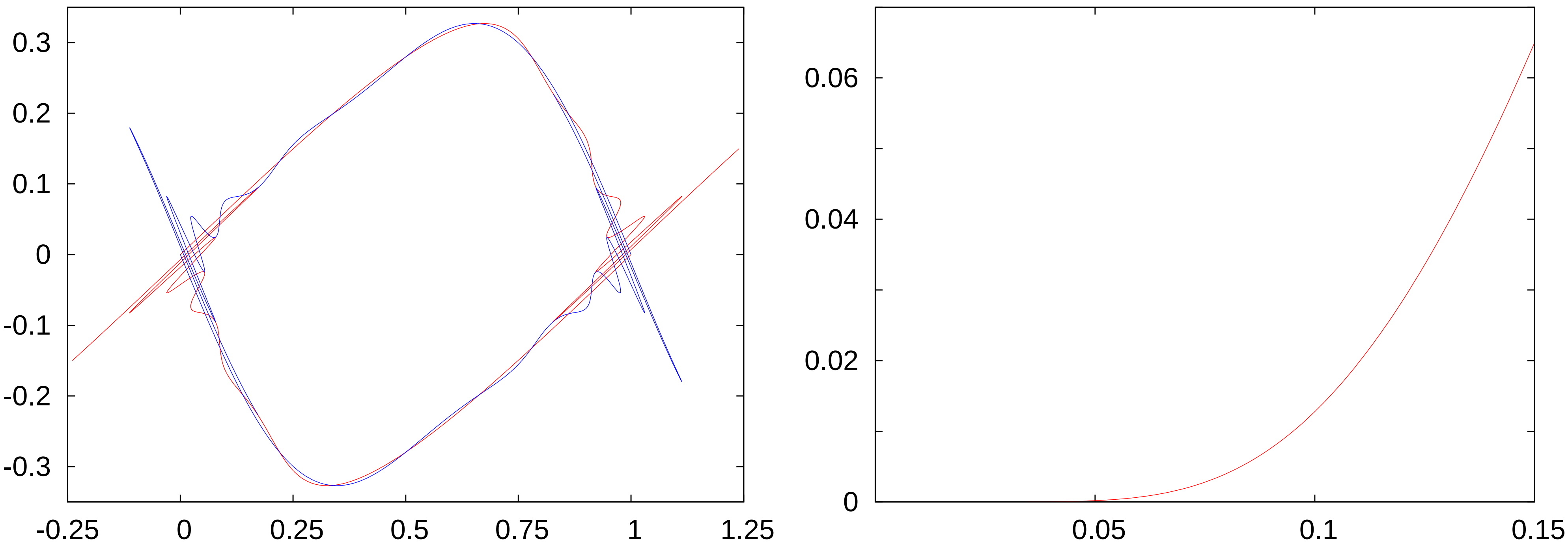
First we look at **iterates of the SM for different** ε

The **critical value up to which there exist invariant rotational curves is** $\varepsilon_{\mathbf{crit}} \approx 0.971635/(2\pi) \approx 0.154641$.

For $\varepsilon = 0.15$ we show **iterates of a point near** $(0,0)$ for increasing number of iterations, similar for a **point near a period-2 hyperbolic point** and for **both of them**. The **gap** is clear.

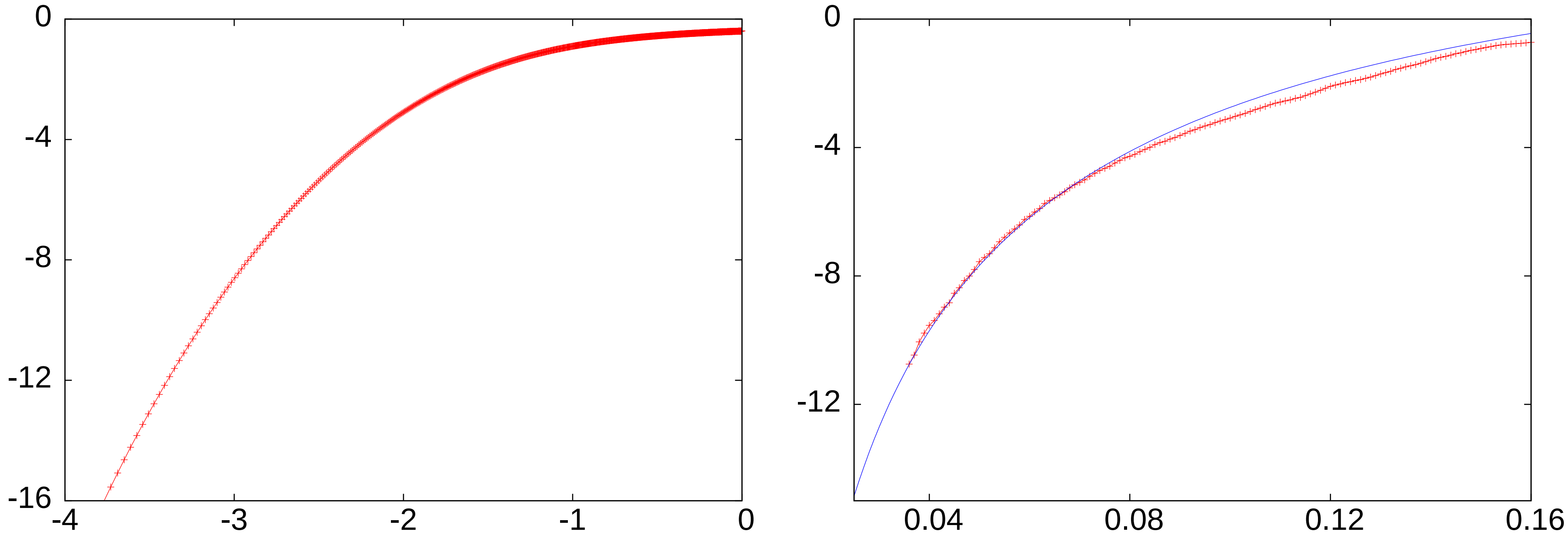
From $\varepsilon = 0.1$ **on** it is clear the **existence of a chaotic domain**. The figure shows a **measure of the relative area of that domain** as a function of ε in $\log - \log$ scale. **Saturation** is clear for $\varepsilon > 0.15$.





On the left we show **part of the invariant manifolds** of the fixed hyperbolic point H. The W^s **is shown in blue**, the W^u **is shown in red**. For better representation we **do not take mod 1**. Note the **infinitely many intersections near** H, the plot that **defeated Poincaré**. Even for $\varepsilon = 0.15$ **the (splitting) angle between** W^s **and** W^u **at** $x = 0.5$ **is not easy to see**.

On the right: **the splitting angle** $\gamma(\varepsilon)$ **at** $x = 0.5$ **as a function of** ε. Note that $\gamma(\varepsilon)$ **tends to zero, when** $\varepsilon \to 0$**, faster than any power of** ε. In the plot it **seems to be zero for** $\varepsilon < 0.5$**, but it is positive**.



The left plot shows the **splitting angle** as a function of ε in $\log - \log$ scale. This is **the angle between the unstable and stable manifolds of the hyperbolic fixed point when they intersect on $x = 1/2$**. This is a quite useful **measure of the degree of chaos and unpredictability**.

The right plot compares, for **moderate values $\varepsilon < 0.16$** the log of the **measure of the chaotic domain** (in red) and a **multiple of the measure of one oscillation of W^s with respect to W^u** (in blue).

Note that the **behaviour of the splitting angle** is **beyond all orders of ε** and **algebraically undecidable**. In present case of the form $A\varepsilon^B \exp(-C/\sqrt{\varepsilon})$ **for small ε** and computable A, B, C.

Some "elementary" open problems

1. Positive measure of the closure of $W^{u,s}$ for APM

We have seen some manifolds, $W^{u,s}$, for the **standard map**, a typical example of **Area Preserving Map (APM)**.

Question: is it true that the closure has positive measure?

Computational **evidence** is in favour of a **positive answer**, but no proof is available yet.

2. Density of periodic points

In the same example one can see **many periodic orbits**.

Question: is it true that the periodic orbits are dense?

This conjecture was already raised by **Poincaré** in a larger context: **the set of bounded orbits in conservative systems**, as mentioned at the beginning of the talk.

3. What the manifolds do in the complex domain?

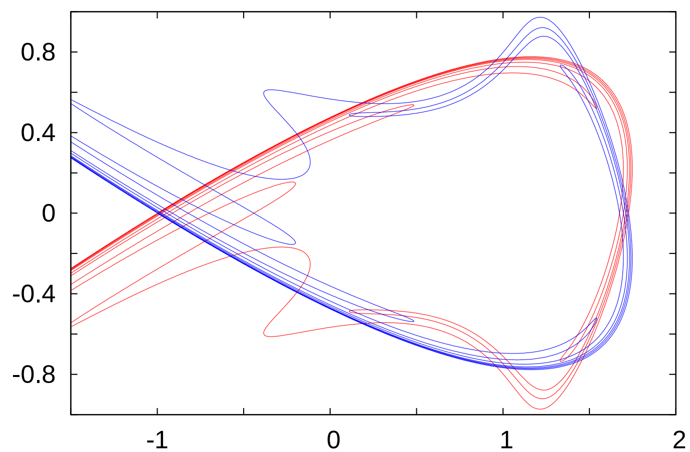
Again for the standard map, it is clear that **invariant manifolds** $W_H^{u,s}$ of the hyperbolic fixed point H are **away** from the **elliptic fixed point** E. They are separated by **invariant KAM curves**.

But we could consider the map, with $\varepsilon \in \mathbb{R}$, **allowing x, y to be complex**.
Question: is it then true that if $x, y \in \mathbb{C}$ then E belongs to the closure of $W_H^{u,s}$?
Same comments as we did for question 1.

4. The Gevrey character of many series involved in dynamics.
First we return for a while to the **conservative Hénon map**, given before but with $b = -1$. An alternative representation is

$$H_c \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x + 2y + \frac{c}{2}(1 - (x+y)^2) \\ y + \frac{c}{2}(1 - (x+y)^2) \end{pmatrix},$$

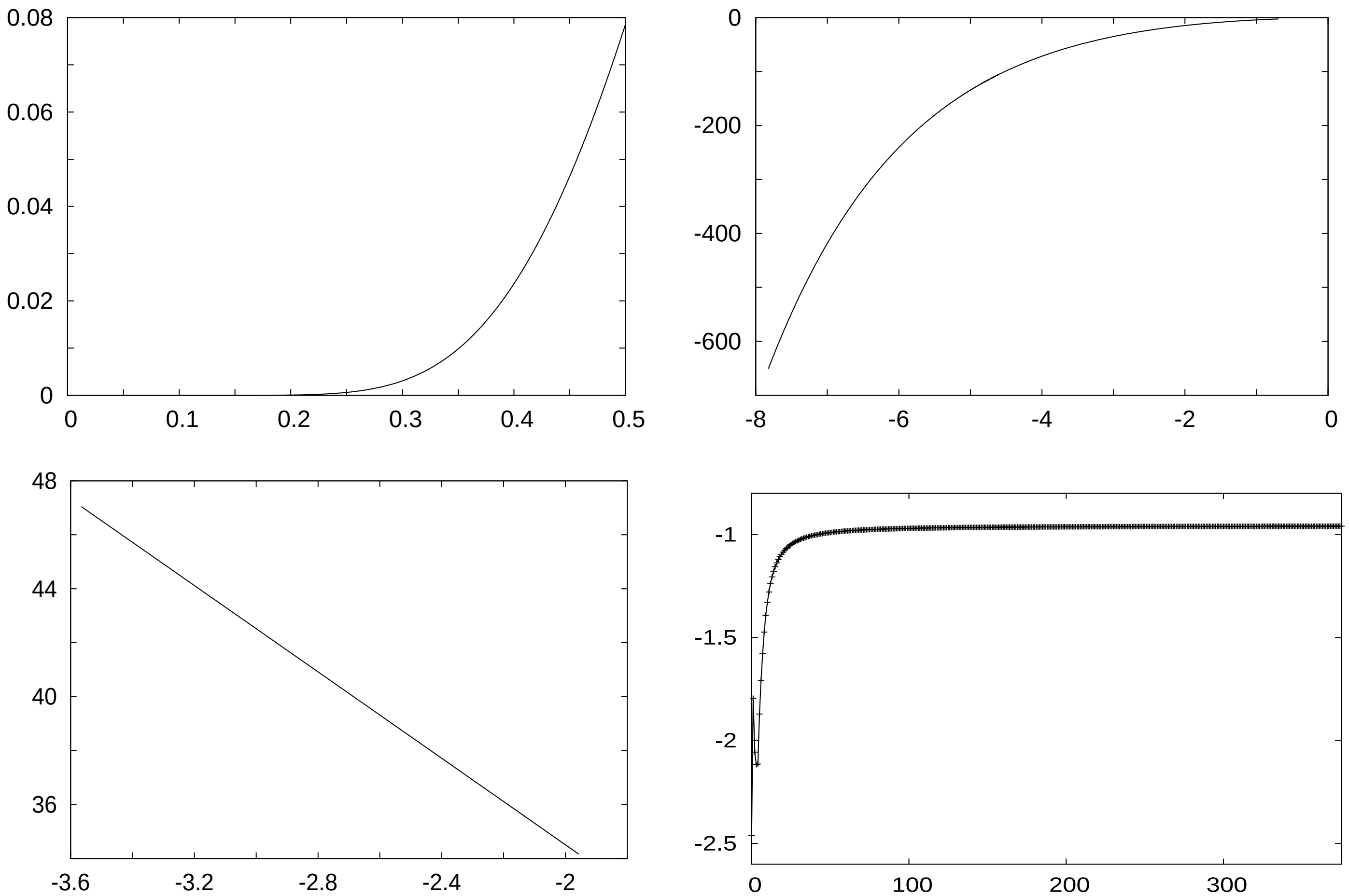
where it is enough to consider $c > 0$. The figure shows a part of $W_H^{u,s}$ for $c = 0.77$. We address our attention to the **splitting angle** $\gamma(c)$ at the first intersection on $y = 0$.



Let $\lambda(c)$ be the **dominant eigenvalue at H** and $h = h(c) = \log(\lambda(c))$. For the Hénon map (and much more general maps) one has proved for $\gamma(c)$ an **exponentially small upper bound** and numerical evidence allows to conjecture

$$\sigma(c) = \frac{9}{2} \times 10^6 \pi^2 h^{-8} \exp(-2\pi^2/h(c)) \times \Omega(h)$$

for a suitable **correction factor $\Omega(h)$** formally $\sum_{m\geq 0} \omega_k h^k$.

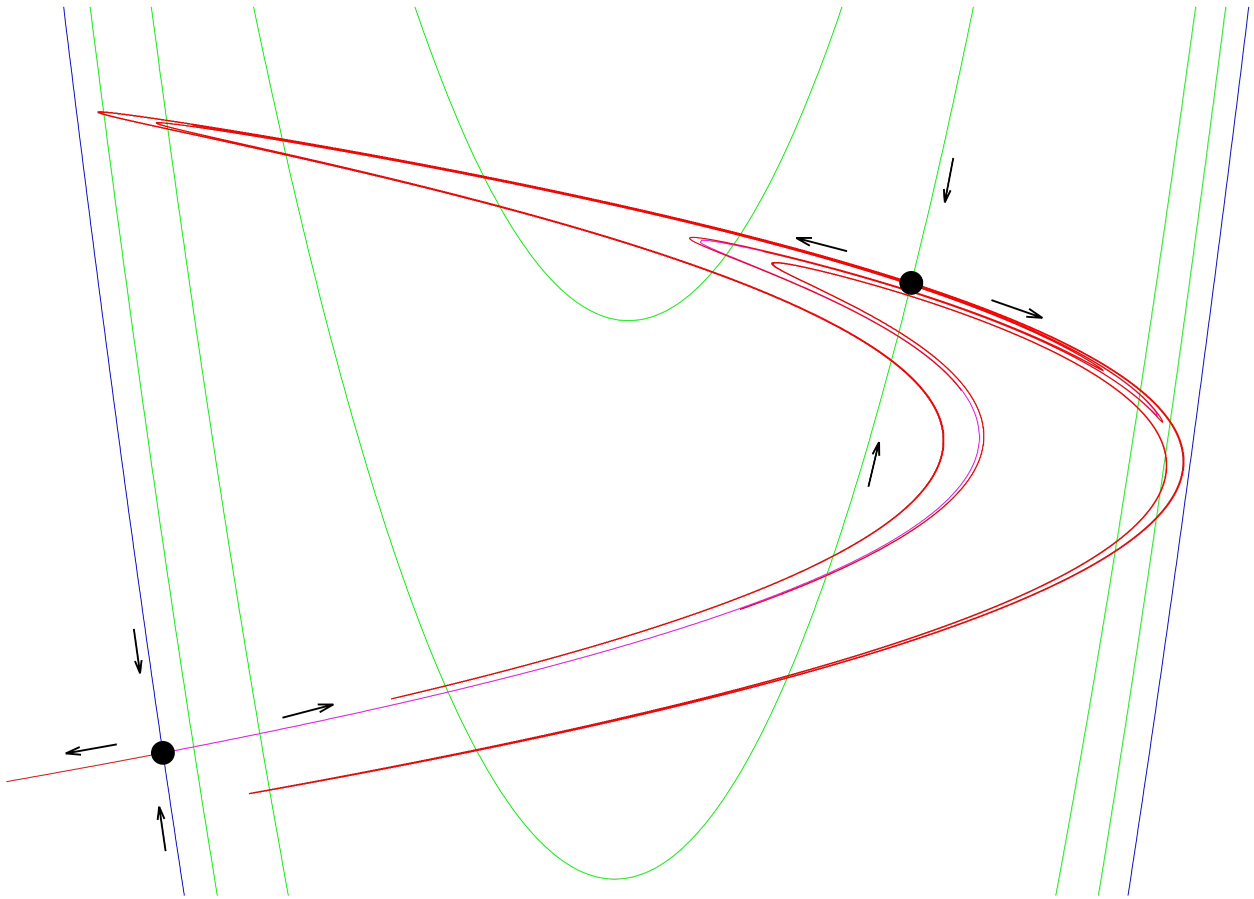


In the plots we show **different representations** of $\gamma(c)$. Top left: γ as a function of c, showing that γ **seems negligible for** $c < 0.2$. Right: $\log(\gamma)$ as a function of $\log(c)$, which allows to see **how small is** $\gamma(c)$ **for** c **approaching zero**.
Bottom left: $\log(\gamma)+2\pi^2/h(c)$ as a function of $\log(h(c))$, showing an **almost linear dependence with slope -8**, in agreement with the proposed formula. Bottom right: the values of $\log_{10}(\omega_{2m}(2\pi^2)^{2m}/(2m+6)!)$ as a function of m, to give evidence of the **Gevrey-1 character of** $\Omega(h)$.

We recall that a **formal power series** $\sum_{k\geq 0} a_k t^k$ **is said to be of class Gevrey-**α if $\sum_{k\geq 0} a_k t^k/(k!)^\alpha$ **is a convergent power series**.

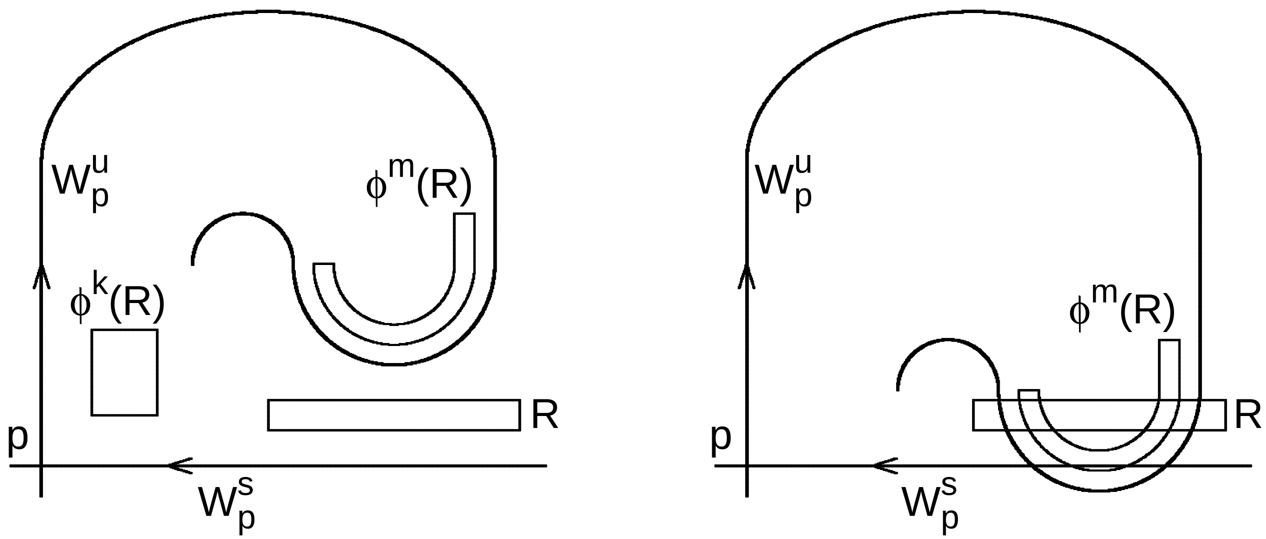
Question: Are the **splitting correction factors** of some Gevrey class?
Strongly related: are the manifolds of **invariant parabolic objects** of some Gevrey class?
Again related: analytic maps **close to the identity** can be approximated by a $\mathcal{C}^\infty$ flow (**Takens theorem**). Is this approximation of **some Gevrey class**?
In all cases: **is it possible to produce sharp estimates of** γ?

Creation/destruction of chaos



An example of **strange attractor** (for the **dissipative** Hénon map $a = 1.4$, $b = 0.3$). The closure of the W^u **of the upper hyperbolic point (in red), contains the attractor**. Homoclinic intersections with W^s **(in green)**. The W^s **of the lower hyperbolic point** is the **boundary of the basin of attraction**. Note the **fractal structure of the SA**.

One of the **basic ingredients** is the **creation of transversal homoclinic points** to an hyperbolic point p by **changes in the position of W^u and W^s across a tangency**.

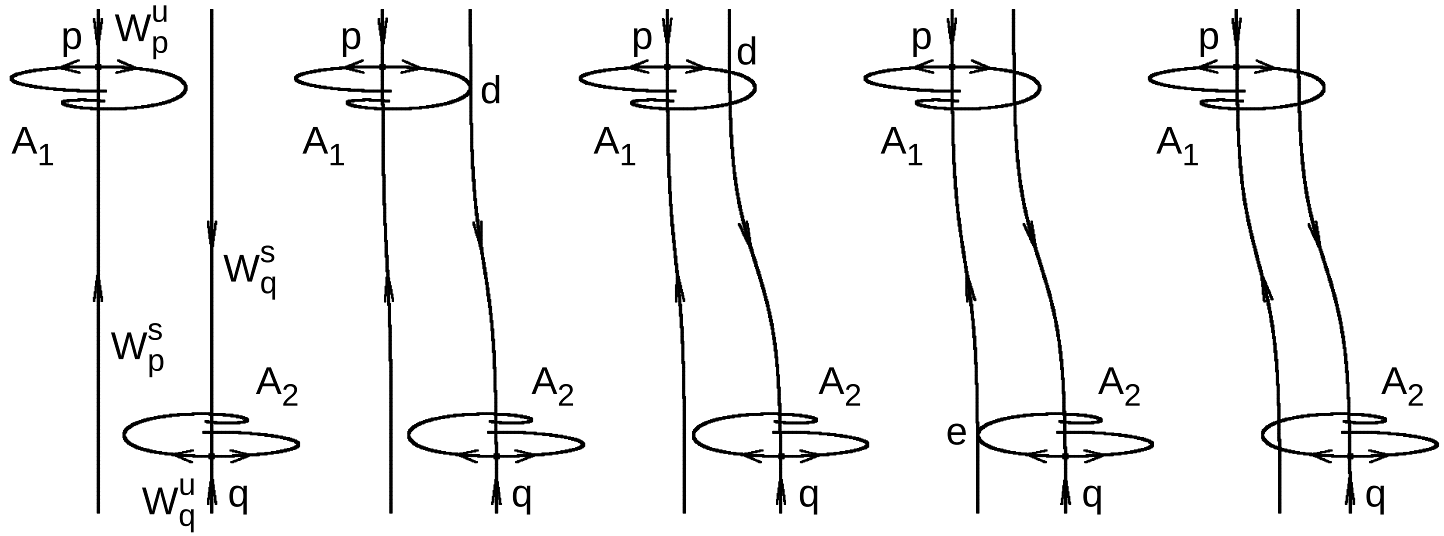


On the **left** points in the domain R **never return** to R. On the **right** some image $\phi^m(R)$ **intersects** R. The intersection has a **left and a right part** to be **coded as 0 and 1**.

If **some easy conditions are satisfied** one can prove then:

a) To **any sequence of 0's and 1's** corresponds a point whose **dynamics passes through the domains coded 0 and 1 as prescribed**. In particular we have **unpredictability**.

b) There are **infinitely many periodic orbits** and are **dense**.



A sketch of the **evolution of the attractors**, associated to the **change of position of the stable manifolds** W_p^s, W_q^s for a **dissipative map**:

Plot 1: Two points with **homoclinics** producing **two independent strange attractors (SA)**.

Plot 2: A **tangency of** W_q^s **and** W_p^u. This is a first **bifurcation**.

Plot 3: The **transverse intersection of** W_q^s **and** W_p^u **destroys the upper SA**. Only the **lower SA subsists**.

Plot 4: A **tangency of** W_p^s **and** W_q^u. This is a second **bifurcation**.

Plot 5: The **transverse intersection of** W_p^s **and** W_q^u produces a **coupling of SA giving rise to a global SA**.

How to measure the amount of chaos

The basic problem is **how to generalise the role of eigenvalues to a general solution**.

The idea is **to measure the exponential rate of separation of nearby orbits**.

Concretely, one uses the **maximal Lyapunov exponent** which in the case of maps and for a point x is defined (**in any dimension**) as

$$\Lambda(x) = \max_{||\xi||=1} \lim_{n\to\infty} \frac{1}{n} \log(||DF^n(x)\xi||).$$

The limit exists for all x, ξ except, perhaps, for a set of zero measure.

Large $\Lambda(x)$ imply more chaotic behaviour. For **attracting fixed points** one has $\Lambda(x) < 0$ and then $\Lambda(x)$ coincides with **the logarithm of the eigenvalue of DF of largest modulus**.

There are methods to **compute several Lyapunov exponents, to speed up the convergence to the limit and minor changes in the definition allow to study the case of ODE and of PDE**.

Non elementary applications of invariant manifolds

- Station keeping of spacecrafts on unstable orbits,
- Minimization of the fuel consumption in transfers,
- Detection of the boundaries of existence of asteroids, of rings of planets, of confined comets, ...,
- Effect of a punctual excitation on a system (a neuron or a network of neurons, a laser, an electrical or mechanical device, etc) and how it returns to a previous state,
- To determine "practical stability domains" in particle accelerators,
- To understand the origin of turbulence in fluids,
- To look for the evolution/diffusion of biological populations,
- Analysis of chemical reactions in Hamiltonian limits of quantum models,
- To discover the different basins of attraction in problems with multiplicity of attractors (hysteresis...),
- To analyze rates of diffusion in heterogeneous anisotropic models,
- etc.

On the tools

Dynamical systems use a **large variety of tools**.

For **theoretical aspects**: **existence, uniqueness, regularity results using analysis; dependence with parameters and bifurcations using algebraic geometry; the role of the geometry/topology of the phase space; subtle questions of number theory to study resonances; tools of functional analysis to study spectra; probabilistic methods for invariant measures, entropy and statistical properties; Lie group methods to obtain normal forms, etc**.

For **concrete, effective, applications**: **symbolic methods to obtain normal forms, local expansions, analysis of bifurcations; numerical solution of equations** $F(x, \lambda) = 0$ **and continuation; integration of ODE and higher order variational equations (jet transport); computation and continuation of periodic and quasiperiodic solutions; globalization of invariant manifolds; analysis of the complexity of algorithms and it numerical stability; data analysis and computer graphics; solution of very large systems by matrix-free methods (Krylov); etc**.

Summarizing ...

It is

- **Interesting,**
- **Fascinating,**
- **Challenging,**
- **With applications to all sciences,**
- **With many open problems,**
- **With theoretical questions,**
- **With numerical questions,**

the study of Dynamical Systems.

Thanks for your attention