On the role of Dynamical Systems in Celestial Mechanics

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Symposium on

Dynamical Systems & Applications

September 29th - October 1st

Johann Bernoulli Institute for Mathematics

and Computer Science, Groningen

201010011000

Introduction

The goal of **Dynamical Systems**: To study the dynamics of a system in evolution. **Everything that moves** can be considered as a DS.

Mathematical models:

$$x' = \frac{dx}{dt} = f(t, x, \mu) \qquad \text{ODE}$$

$$\frac{\partial u}{\partial t} = F(t, x, u, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \ldots) \qquad \text{evolution PDE}$$

$$x \mapsto T(x) \qquad \text{discrete maps, diffeomorphisms}$$

and one can include the effect of **noise, history, etc**.

x moves in a **continuous or discrete** way in a **phase space or state space** $\mathscr{E}$ and **parameters** are in a space $\mathscr{P}$. The best way to study systems is to consider the product space $\mathscr{E} \times \mathscr{P}$.

One can include also many **numerical algorithms**.

Some **topics to study** in general:

- **topology** of the $\mathscr{E} \times \mathscr{P}$ space, **singularities**,
- identify **simple invariant objects** in $\mathscr{E}$, like fixed points (in a fixed or moving frame), periodic and quasi-periodic solutions ... ,
- **stability/instability** of these objects and **stable/unstable manifolds**,
- **connections** between unstable/stable manifolds, **homo/heteroclinic** orbits, complicated dynamics, **diffusion**, strange attractors, **chaos**,
- **dependence on parameters**, bifurcations, structural stability, persistence in measure
- **statistical properties**, etc

When applied to **Celestial Mechanics** one can be more specific.

Some topics in Celestial Mechanics

1) **Topology of $\mathscr{E}$**, of the **energy-momentum manifolds**, **central configurations** and **relative equilibria**,
2) **Regularity** of the solutions and the role of **collisions**,
3) **Stability of simple solutions** and **Normal Forms** around them,
4) **Periodic orbits**: how they appear/disappear, stability, some exceptional classes like **choreographies**, the problem of the **density** of p.o.,
5) **Invariant tori** of different dimensions, bifurcations, **creation/ /destruction**, local behaviour around them, **reducibility** properties,
6) **Invariant Cantorian sets**, like Aubry-Mather sets, the dynamics around them, their role in the rate of diffusion,
7) **Checking conditions** for the existence of bifurcations, tori, etc, **away from simple solutions**. The **jet transport** and applications,
8) **Splitting phenomena** between the invariant manifolds of quite different **invariant objects** (perhaps partially weakly hyperbolic), responsible of creation of **chaotic dynamics**,
9) **Escape/Capture boundaries** and the **mechanisms** creating them: **non-analytic** invariant manifolds of **invariant objects at infinity**,

10) **Normally Hyperbolic Invariant Manifolds**, like **centre manifolds** which include p.o., q-p.o. and chaotic zones, related **codimension 1** manifolds, "practical" stability,

11) **Return Maps** to the vicinity of some (perhaps partially, weakly) **hyperbolic invariant object**, like **separatrix maps** and extensions (multiseparatrix maps) or projections (scattering maps),

12) **The regular and chaotic solutions** taking into account the role of **resonances** and different **temporal scales**,

13) **Statistical Properties**, like rates of diffusion, rates of escape, mass transport, ergodicity in most of the phase space,

14) **Applications to Astronomy**, like motion of **comets and asteroids**, asteroids coming close to the Earth **(NEO)**, detection and analysis of orbits of **exoplanets**,

15) **Applications to Astrodynamics**, like orbits of **AES, space debris, etc** and **missions** far away from the Earth, close to **libration points**, close to **binary or larger systems**, formation flights, etc.

Relative equilibria solutions (RES) and topology

Equations of the N-body problem (Newton)

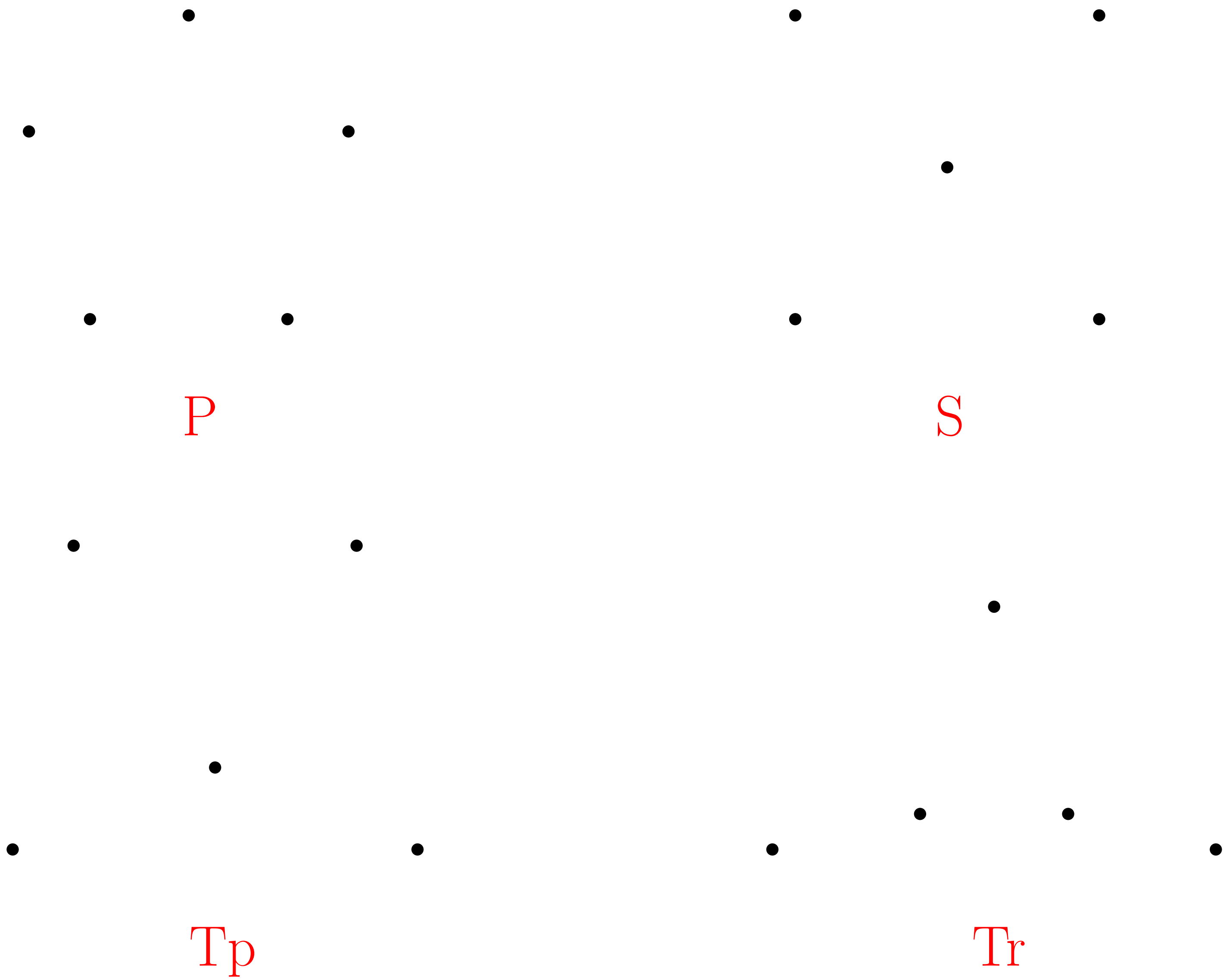
$$\frac{d^2\mathbf{q}_i}{dt^2} = \sum_{j=1, j\neq i}^{N} G m_j \frac{\mathbf{q}_j - \mathbf{q}_i}{r_{i,j}^3}, \quad r_{i,j} = ||\mathbf{q}_j - \mathbf{q}_i||_2, \quad \mathbf{p}_j = m_j d\mathbf{q}_j/dt\,.$$

Due to the **homogeneity** the total energy can be **normalised** to $0, \pm 1$.

The system has no fixed points at finite distance, but it has **relative equilibria**, e.g., when the bodies move in circles around the c.o.m. and the **centrifugal force cancels the gravitational attraction**.

For N**=3** Lagrange found that there are **5 non-equivalent RES**, independently of the values of the masses m_j.

For N**=4** there are **50 solutions** if the masses are equal (Albouy-Chenciner). But #(RES) **depends on the masses**, with a **minimum of 34 solutions**. The study of #(RES) as a function of m_j has only been done numerically. It is $\leq$ **50**. Rough theoretical bounds exist (≈ 8000, Moeckel).



RES for N **=5 equal masses**. P denotes a regular pentagon, S a square with a central mass, Tp a trapezoid with a central mass and Tr a triangle with 2 symmetric inner masses. The **60 collinear RES** are not displayed.

For N=**5** results of a numerical exploration show that the total number of non-equivalent RES for equal masses is **354**. For other sets of masses the number of RES has been found to range between **294 and 450**.

N	3	4	5	6	7	8	9	10
RES(N)	5	50	354	4104	53640	676080	13905360	250185600
$\frac{\mathrm{RES}(N)}{\Gamma(1.2N+1)}$	0.37	0.58	0.49	0.54	0.56	0.47	0.57	0.52

Estimated RES(N) for equal masses. The number seems to largely exceed the factorial.

Up to $N = 7$ all RES seem to have **some symmetry**. For $N = 8, 9, 10$ one has found, respectively, 2,3 and 12 **geometric configurations without any symmetry**.

It is not know how the **number of RES changes as a function of** N and even if it is **finite for all** N.

A related problem is the structure of the **energy-momentum manifolds for the N-body problem**, that is

$$I_{hc} = \{(\mathbf{q}_j, \mathbf{p}_j), j = 1, \ldots, N, \mid \quad \text{energy} = h, \quad \text{angular momentum} = c\}.$$

In fact I_{hc} are **stratified objects** instead of differentiable or topological manifolds.

Due to **homogeneity** the relevant parameter to study I_{hc} is hc^2. Changes in the **topology** can be related to the values of hc^2 **at the RES**.

For N**=3** the number of **connected components** of I_{hc} (**Hill's regions**) depends on hc^2. It can be **1,2,3**. Unfortunately:

Theorem If $N > 3$ then I_{hc} has a unique connected component for all values of hc^2.

Collisions and regularisation

Solutions of the N-body equations are not defined on $\Delta = \cup_{1\le i<j\le N}\Delta_{ij}$, $\Delta_{ij} = \{\mathbf{q}_i = \mathbf{q}_j\} = \{r_{i,j}\}\mathbf{=0}$, the **set of collisions**.

Some special cases: a) **Binary collision** (BC), not a problem for the Newtonian potential; b) **Total collision** (GC). And between these extreme cases we can find intermediate ones: c) **Triple collision** (TC); d) **Simultaneous binary collision** (SBC).

Relevant questions: **Is it possible to regularise collisions?**

First one should understand **what means to regularise**. There are different approaches:
1) **Analytic or Siegel's** regularisation, for solutions analytic in $t^{1/m}$, m odd,
2) **Surgery or Easton's** regularisation, using **isolating blocks** and a suitable homeomorphism,
3) **Geometric regularisation**, by recovering (at least) continuous dependence w.r.t. initial conditions.

Interesting problems appear for other **homogeneous potentials**, even for the two-body problem.

Let $w(t) = (\mathbf{q}(t), \mathbf{p}(t))$ be a solution of the N-body problem (or any other having singularities), defined in $(t_0, 0)$, $t_0 < t_1 < 0$ and ending in collision when $t \rightarrow 0_-$. Let $w_c^{(i)} = w(t_1)$.

Assume that there are **initial conditions** (i.c.), $w_i^{(i)}$, in **any neighbourhood** of $w_c^{(i)}$, **not leading** to any singularity and such that $w(t; t_1, w_i^{(i)})$ is **defined until** some fixed t_3 **>0**. Let $w_i^{(f)} = w(t_2; t_1, w_i^{(i)})$ for some fixed $0 < t_2 < t_3$.

Definition: For **any sequence** $\{w_i^{(i)}\}$ of i.c. not leading to collision (except, perhaps, simple BC) with $\lim_{i\to\infty} w_i^{(i)} = w_c^{(i)}$ assume that $\lim_{i\to\infty} w_i^{(f)}$ **exists**. Then we **define** $w_c^{(f)}$ **as that limit** and look at it as $w(t_2; t_1, w_c^{(i)})$. We denote this extension as the **natural or geometric regularisation**.

For regularisable problems one can ask about the **regularity** of the map $w_i^{(i)} \mapsto w_i^{(f)}$ extended to $w_c^{(i)}$.

Some results:

a) The general **triple collision** is **not geometrically regularisable** except, at most, for a zero measure set of masses.

b) All **SBC problems** are at least $\mathcal{C}^0$ geometrically regularisable.

c) Consider four bodies and let $m_1 - m_2$ and $m_3 - m_4$ masses **colliding simultaneously** at $t = 0$ and $\mathbf{q}_j, j = 1, \ldots, 4$ their positions. Let

$$\mathbf{Q}_1 = \mathbf{q}_2 - \mathbf{q}_1, \quad \mathbf{Q}_2 = \mathbf{q}_4 - \mathbf{q}_3, \quad \mathbf{Q} = \mathbf{q}_{34} - \mathbf{q}_{12},$$

$\mathbf{q}_{12}$ and $\mathbf{q}_{34}$ the c.o.m. of $\mathbf{q}_1, \mathbf{q}_2$ and $\mathbf{q}_3, \mathbf{q}_4$.

Definition The problem is said to be **1D–reducible** if $\mathbf{Q}_1, \mathbf{Q}_2, \mathbf{Q}$ have constant direction along the motion. This happens in some subproblems.

Theorem: In the 1D–reducible 4-body problems the SBC is $C^{8/3-\epsilon}$ **regularisable** for any $\epsilon > 0$, but it is not $C^{8/3}$ regularisable.

Conjecture: Assume the limit directions of $\mathbf{Q}_1, \mathbf{Q}_2, \mathbf{Q}$, say $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}$ exist. Then, for all $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}$ and all masses $m_j > 0$, except by sets of zero measure, the **SBC is $C^{8/3-\epsilon}$ regularisable** in the 4–body problem. The exceptional cases have a higher regularity.

The TC in the planar 3-body problem

To learn about **passages near TC** it is useful to study the flow on the 4D **non-rotating TC manifold** $\mathcal{N}$. Suitable changes: a) **Blow up of variables** and then appear **10 critical points** $L_{\pm}^{i,s}, E_j^{i,s}, j = 1, 2, 3$ (5 for collision, 5 for ejection); b) **compactification** by adding "hard" binaries $B_j^{i,s}, j = 1, 2, 3$ to get $\bar{\mathcal{N}}$. In total **16 critical points** (8 for collision, X^i, 8 for ejection, X^s) all of them hyperbolic and the flow is **gradient-like**.

$$\dim W^{u,s}(L_{+,-}^{i,s}) = 2, \quad \dim W^u(B_j^i) = \dim W^s(B_j^s) = 4, j = 1, 2, 3,$$

$$\dim W^u(E_j^i) = \dim W^s(E_j^s) = 3, \dim W^s(E_j^i) = \dim W^u(E_j^s) = 1.$$

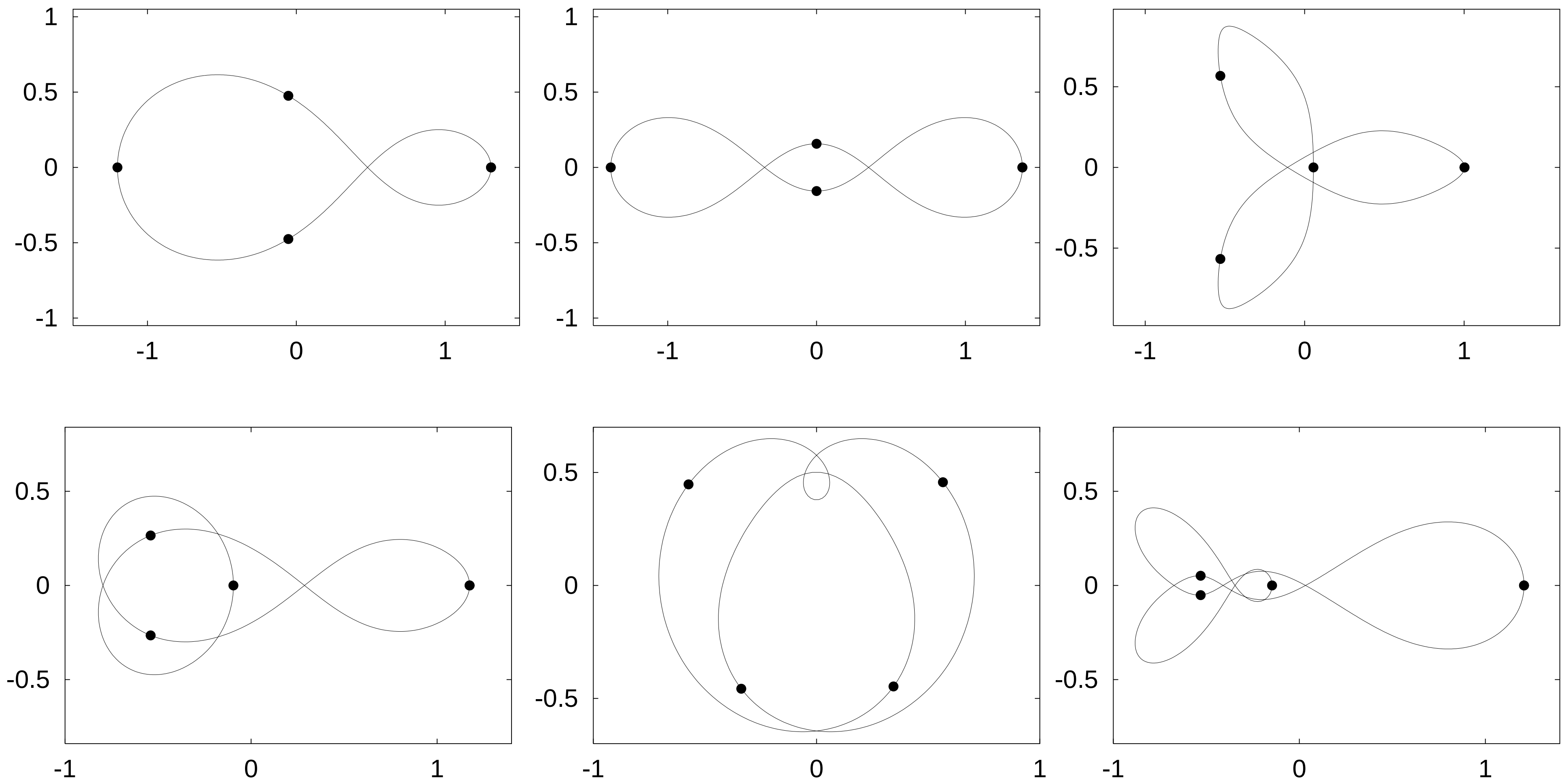
Theorem If two of the values of the masses are close enough and there are not connections $L_{+,-}^i \to L_{+,-}^s$ then points $B_j^i, E_j^i, j = 1, 2, 3$ connect to all points X^s, and $L_{+,-}^i$ connect to $B_j^s, E_j^s, j = 1, 2, 3$.
Conjecture This holds for all positive masses.

Theorem If $m_3 = \varepsilon, m_1 = m_2 = (1 - \varepsilon)/2$ the set of ε for which $L_{+,-}^i \to L_{+,-}^s$ occurs is countable, and $\varepsilon(n) = \pi^2/n^2 + \mathcal{O}(n^{-3})$ for n large.
Conjecture In the mass triangle $L^i \to L^s$ occur only in a countable set of lines.

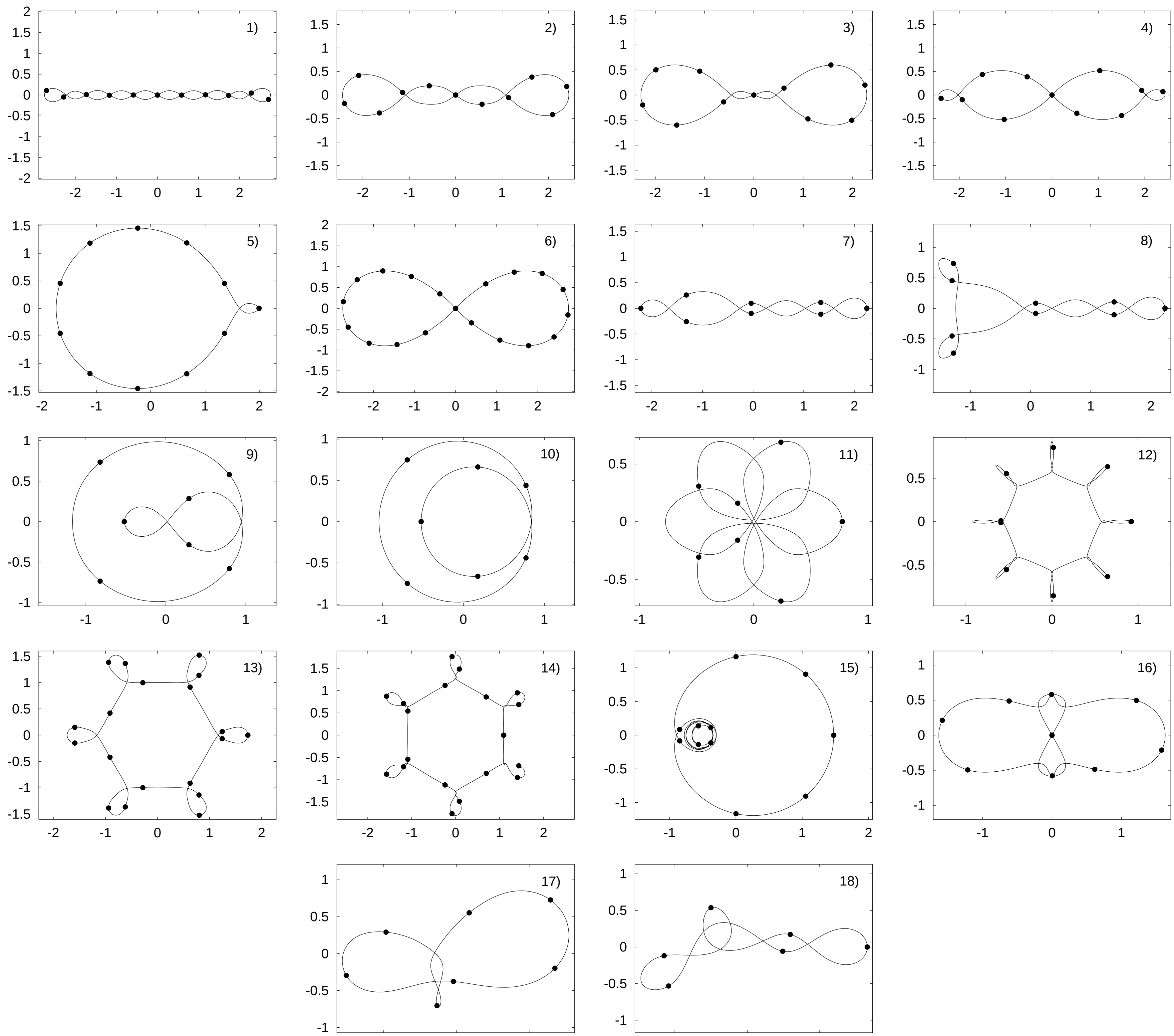
Some choreographies

Planar simple choreographies: Periodic solutions of planar N-body problem with equal masses with all the bodies moving in the same path.

A sample of choreographies for $N = 4$ is presented. Newtonian case.



And some additional examples come.


1)
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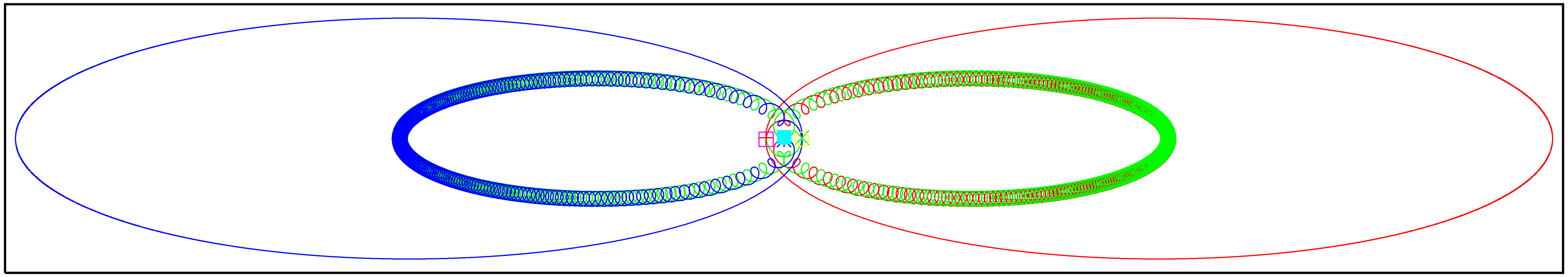
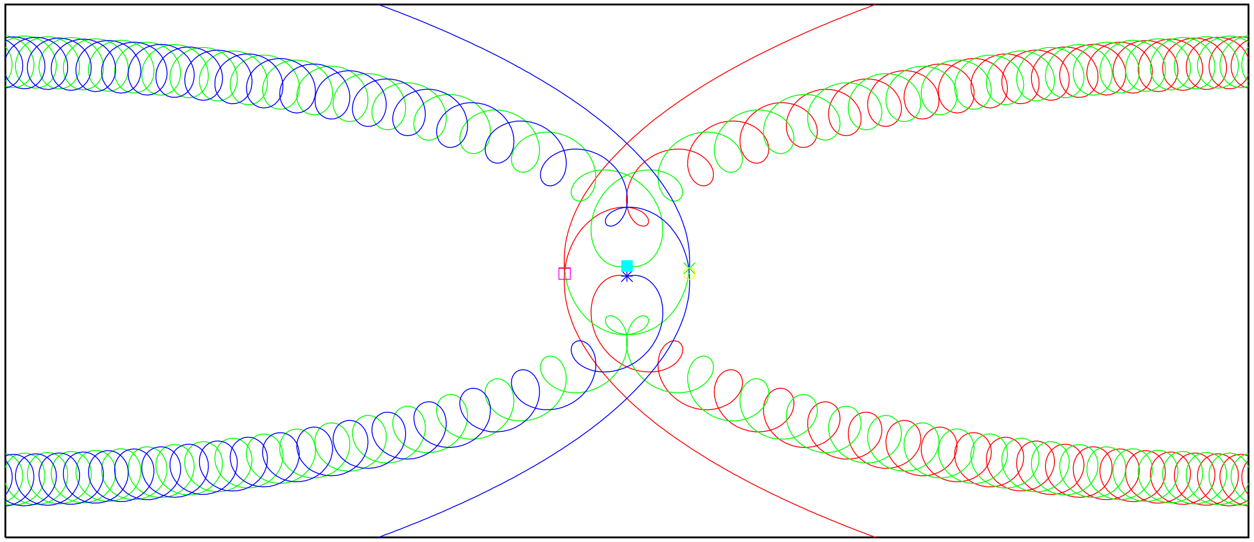
How many?

A relevant question is whether the number even for $N = 3$ is **finite or not**. The answer is **NOT** (a routine Computer Assisted Proof can prove it).





Top: A choreography of the **3-body problem**. Bottom: A magnification of the central part. In each one of the binary portions, the bodies in the binary make **200 revolutions around their centre of masses**.

Jet transport and application to NEO

Consider an **IVP for an ODE** like $\dot{x} = f(t,x)$, $x(t_0) = x_0$, Assuming f **analytic** in a neighbourhood of $(t_0, x_0) \in \Omega \subset \mathbb{R} \times \mathbb{R}^n$ or $\Omega \subset \mathbb{C} \times \mathbb{C}^n$

Goal: to easily obtain the **Taylor expansion** $x(t_0 + h)$ for suitable values of h and use it as a one-step method.

For a very large class of functions the evaluation of f can be **split in simple expressions**

$$\begin{aligned}
e_1 &= g_1(t,x), \\
e_2 &= g_2(t,x,e_1), \\
&\vdots \\
e_j &= g_j(t,x,e_1,\ldots,e_{j-1}), \\
&\vdots \\
e_m &= g_m(t,x,e_1,\ldots,e_{m-1}), \\
f_1(t,x) &= e_{k_1}, \\
&\vdots \\
f_n(t,x) &= e_{k_n}.
\end{aligned}$$

Each expressions e_j contains a sum of arguments, a product or quotient of two arguments or an **elementary function** (like $\sin, \cos, \log, \exp, \sqrt{}, \ldots$) of a **single argument**.

Basic idea: to compute in a **recurrent way** the **power series expansion** of all the e_j. The g_j have to be seen as **operations with (truncated) power series**.

Input: t and the coefficients of order 0 of the components of x_0.

Step s: from arguments of g_j at order s we obtain order s of e_j. In particular for $f_j(t,x)$, which gives **order** $s+1$ for x_j (dividing by $s+1$).

The **representation** of $x(t_0+h)=(x_i)$ is $x_i=\sum_{s=0}^{N} a_i^{(s)} h^s$ for suitable N, h, such that the **truncation error** $\sum_{s>N} a_i^{(s)} h^s$ can be considered as **negligible** in front of the (unavoidable) **round off error**.

Example: $a(t)=\sum_{k\geq 0} a_k t^k$, $a_0\neq 0, \alpha\in\mathbb{R}$ and $b(t)=a(t)^\alpha=\sum_{k\geq 0} b_k t^k$:

$$b_0=a_0^\alpha, \quad b_n=-\frac{1}{na_0}\sum_{k=0}^{n-1} b_k a_{n-k}[k-\alpha(n-k)],\ n>0,$$

the determination being fixed by the one used for b_0. To compute to order N has a **cost** $\mathcal{O}(N^2)$. This is true for the **most expensive elementary operations and functions**.

Similar recurrences can be obtained for **any elementary function**.

For (near) integrable Hamiltonian systems and $\Lambda_{\max} \approx 0$ (zero maximal Lyapunov exponent) the **errors in actions** are $\mathcal{O}(t^{1/2})$ and **in angles** are $\mathcal{O}(t^{3/2})$ due to **random walk-like** behaviour of round off.

Jet transport: Assume the i.c. are $x_0 + \xi$, where ξ are some variations of i.c. It is enough **to replace all operations** with numbers by operations with **polynomials in ξ** up to the desired order. It is elementary to include as components of ξ all **relevant parameters**.

Can be **implemented in efficient way**, to produce **rigorous estimates of the tails** at every step and to obtain **intervals** which contain the **correct values of all the coefficients**.

If the initial data are in some **uncertainty set**

a) The coordinates can be **adapted** according to the **shape** of the initial uncertainty set.

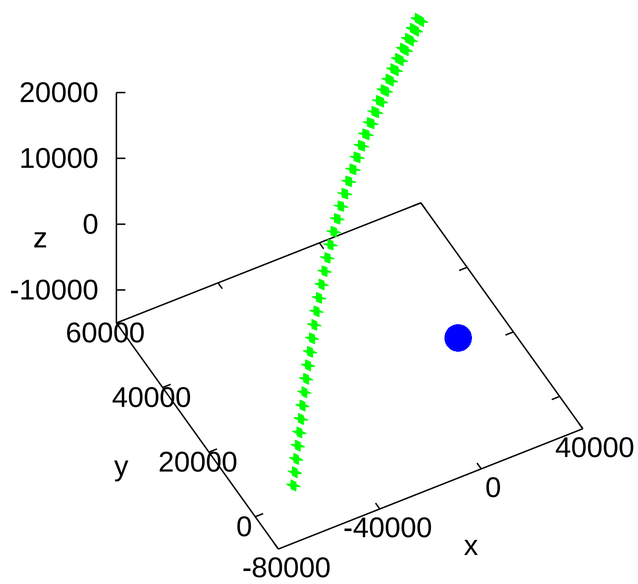
b) The effect of the **uncertainty in parameters** can be easily included.

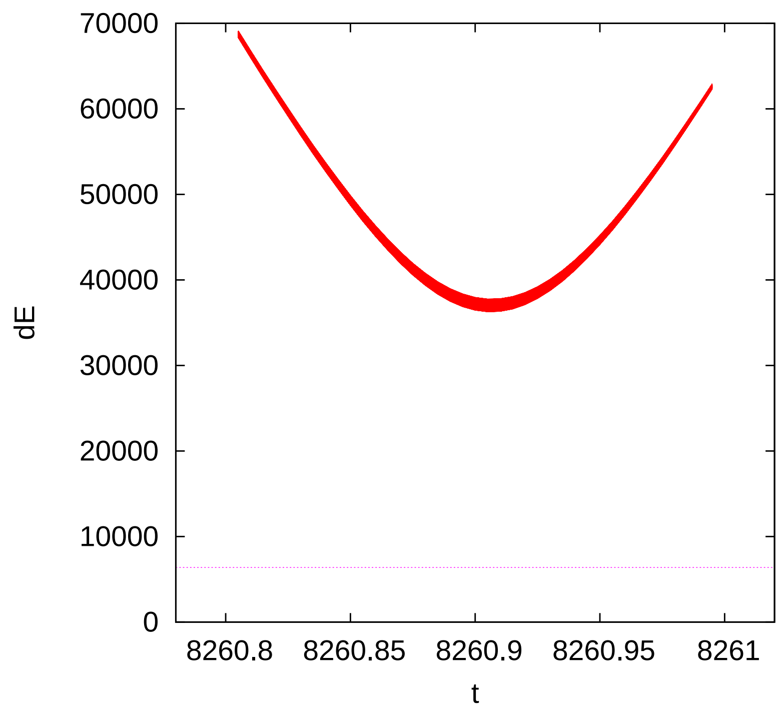
c) It can be be modified to take into account **different distributions** for the uncertainty.

The case of the **asteroid (99942) Apophis**, which will experience **close approaches with the Earth in 2029** and, perhaps, between 2036 and 2037. We want to understand the **effect of initial uncertainties**. Standard deviations are ≈ 1.5 km and ≈ 3 km **across and along the orbit**.

In 2029 Apophis will get at about **36000 km w.r.t. the centre of the Earth** (on Friday, April 13, near 9 pm UT). An accurate description of that passage can be obtained by means of a **3rd order integration** in ξ.

Most significant changes of Apophis orbit at the close encounter: **inclination, semi-major axis and thus period**, slowing down ≈ 3 km/s.





Escape/Capture boundaries

One of the outstanding problems in Celestial Mechanics is the **detection and computation** of **capture and escape boundaries**.

They are related to the **existence of some invariant objects at infinity** which have **invariant manifolds**.

But these invariant objects **are not hyperbolic**. They are only **parabolic** in the sense of Dynamical Systems.

It is well known that this fact was **partially analysed by Moser and McGehee**. The manifolds **exists** and they are **analytic except, perhaps, at infinity**. Related results are due to **C.Robinson**.

Standing question: Which is the **regularity class** of these manifolds? How can we **compute them** with rigorous error control, so that they can be used to obtain **capture and escape boundaries**?

The simplest problem to analyse is the **Sitnikov problem**:

$$q' = \Psi q^3 p, \quad p' = \Psi q^4 \left(1 + \Psi^2 q^4\right)^{-3/2}, \quad \Psi = (1 - e\cos(E))/4, \quad {}' = d/dE.$$

We look for a **parametric representation** of the manifolds of the p.o. as

$$p(E,e,q) = \sum_{k\geq 1} b_k(e,E) q^k = \sum_{k\geq 1}\sum_{j\geq 0}\sum_{i\geq 0} c_{i,j,k} e^i \,\mathrm{sc}(jE)\, q^k,$$

where $b_k(e,E)$ are trigonometric polynomials in E with polynomial coefficients in e, $c_{i,j,k}$ are rational coefficients, sc denotes sin or cos functions.

Theorem: The manifolds $W_{\pm}^{u,s}$ are **exactly Gevrey-1/3** in q uniformly for $E \in \mathbb{S}^1$, $e \in (0,1]$. Concretely, let a_n denote the norm of b_n. Then there exist constants c_1, c_2, $0 < c_1 < c_2$ such that, for $n \geq 5$ except for $n = 6, 7, 10$ one has

$$\mathbf{c_1}\rho^{\mathbf{n}} < \mathbf{a_n}/\boldsymbol{\Gamma}((\mathbf{n}+\mathbf{1})/\mathbf{3}) < \mathbf{c_2}\rho^{\mathbf{n}}, \quad \rho = (3/4)^{1/3}.$$

Recall: a **formal power series** $\sum_{n\geq 0} a_n \xi^n$ **is of Gevrey class** s **if** $\sum_{n\geq 0} a_n (n!)^{-s} \xi^n$ **is analytic around the origin**.

Theorem: The formal expansion gives an **asymptotic representation** of the invariant manifolds of p.o._∞. Concretely, the **truncation of the series at order** n has an error which is bounded by the sum of the norms of next three terms

$$C(a_{n+1}q^{n+1} + a_{n+2}q^{n+2} + a_{n+3}q^{n+3}), \quad C \approx 1.$$

Given q the **optimal order is** $n_{\mathbf{opt}} \approx 4/q^3$. Using optimal order **the error bound is** $< N \exp(-4/(3q^3)), N < 1$.

The method opens the way to other **more relevant problems, like 2DCR3BP, 3DCR3BP, 3DER3BP, general 3BP, etc**.

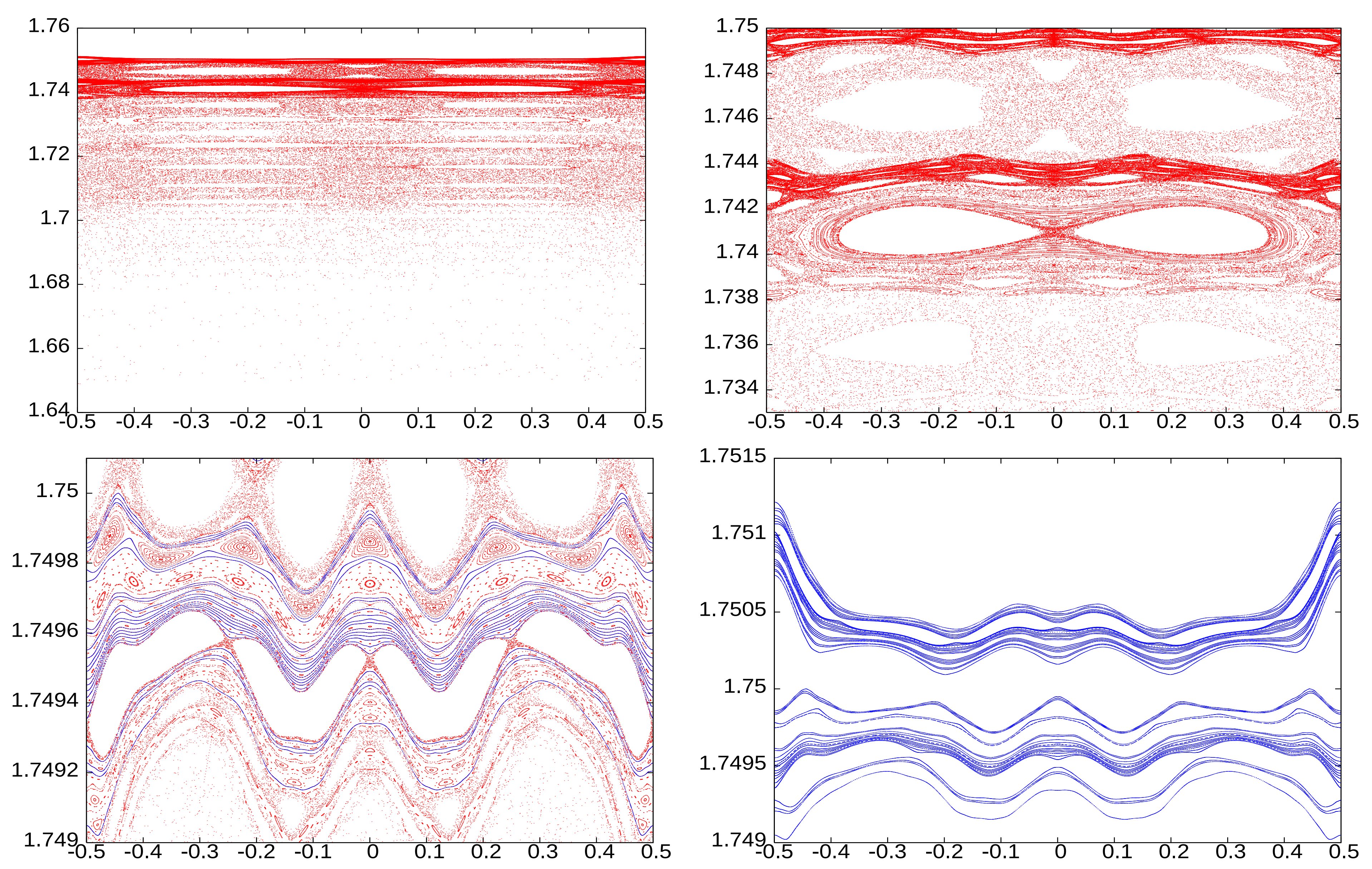
In these problems there are **parameters** playing **quite different roles**: small eccentricity and masses play an $\approx$ **linear role** while some **energy** plays and **exponentially small role**.

One recovers and enlarges previous results about **splitting of separatrices** obtained so far for **2DCR3BP, 2DER3BP, planar general 3BP**.

Combined with **return maps: separatrix maps and extensions** one obtains **barriers or "practical" barriers for escape**.

In next page these ideas are applied to a **comet** outside Jupiter orbit, in the planar RTBP, $\mu = 10^{-3}, C = 3.6$, and **escape is not possible if** $e <$**0.75**.

The **blue curves** identify approximate locations of **KAM tori**.



Poincaré map obtained by **pericentre passage** in (revolutions,radius) and details. For **small values of the pericentre distance** the effect of Jupiter is strong and some orbits **end in escape**.

Additional notes

I would like to present some elementary asymptotic considerations. Assume some phenomenon is measured by a function φ depending on the **small parameter** ε and can be represented by an **asymptotic expansion**

$$\varphi(\varepsilon) \sim \sum_{m\geq 0} a_m \varepsilon^m, \qquad \text{with} \qquad \left| \sum_{m\leq n} a_m \varepsilon^m - \varphi(\varepsilon) \right| < |a_{n+1}| \varepsilon^{n+1}.$$

If we assume $|a_n|$ monotonically increasing, the **best bound** for the error is obtained for $|a_{n+1}|\varepsilon^{n+1}$ minimum. Let $b(\varepsilon)$ be the bound.

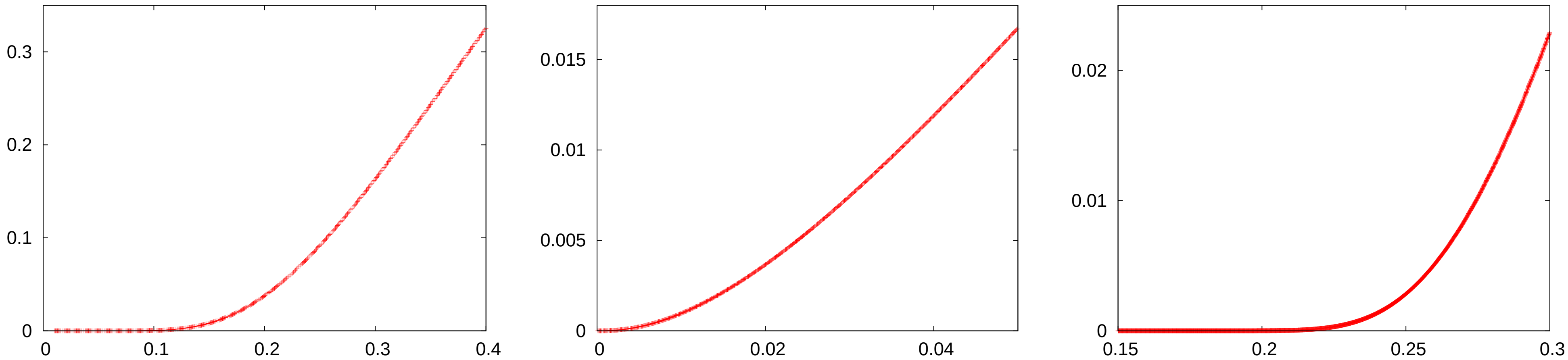
Some **examples**:

1) For $a_n = (n!)^\gamma$ (Gevrey classes) we obtain $n \simeq \varepsilon^{-1/\gamma}$ and $b(\varepsilon) \simeq (2\pi)^{\gamma/2} \varepsilon^{-1/2} \exp(-\gamma\varepsilon^{-1/\gamma})$, a typical **exponentially small** behaviour.

2) If $a_n = (n^\beta!)^\gamma$, $\beta > 1$, then n satisfies the equation $\gamma\beta^2 n^{\beta-1} \log(n) \simeq |\log(\varepsilon)|$ for $\varepsilon \to 0$ and $b(\varepsilon) \simeq \exp(K|\log(\varepsilon)|^{\beta/(\beta-1)} / \log(|\log(\varepsilon)|)^{1/(\beta-1)})$, where $K = -(1-\beta^{-1})((\beta-1)/\gamma\beta^2)^{1/(\beta-1)}$, for $\varepsilon \to 0$.

3) For $a_n = (\log n)^n$ we obtain $n(\varepsilon) \simeq \exp\left(\frac{1}{\varepsilon} - 1 - \frac{\varepsilon}{2} - \frac{2\varepsilon^2}{3} - \frac{9\varepsilon^3}{8} + \mathcal{O}(\varepsilon^4)\right)$ and $b(\varepsilon) \simeq \exp\left(-\frac{\varepsilon}{e}\exp(\frac{1}{\varepsilon})\left(1 + \frac{\varepsilon}{2} + \frac{11\varepsilon^2}{24} + \frac{9\varepsilon^3}{16} + \mathcal{O}(\varepsilon^4)\right)\right)$. This case shows really a **quite sharp behaviour** with respect to ε:

ε	0.4	0.2	0.15	0.12	0.1
$b(\varepsilon)$	$8.5 \cdot 10^{-2}$	$4.6 \cdot 10^{-6}$	$3.3 \cdot 10^{-21}$	$6.8 \cdot 10^{-86}$	$4.4 \cdot 10^{-372}$



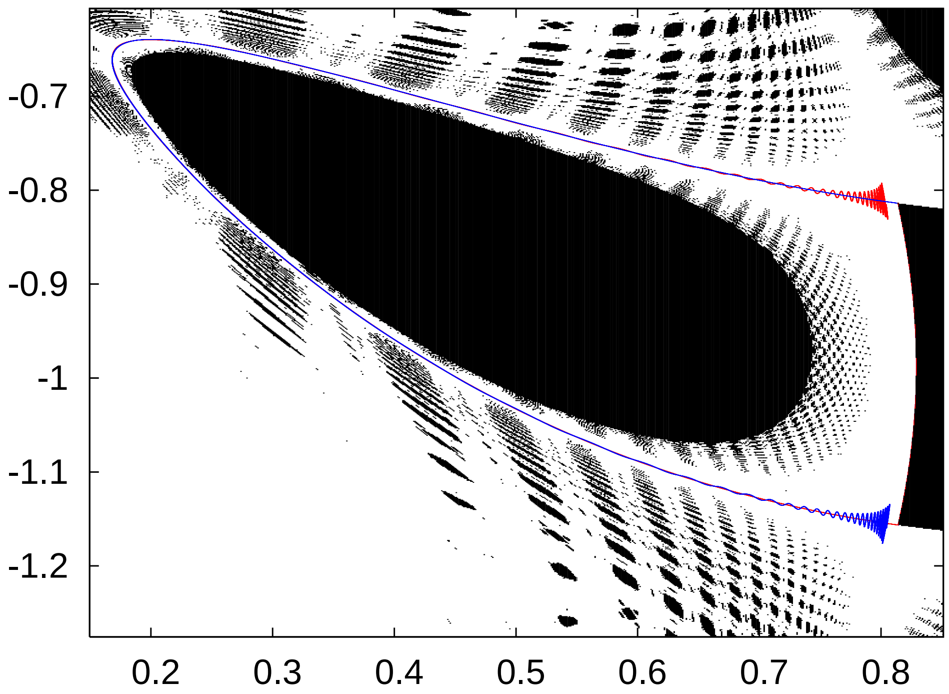
Case 1 Case 2 Case 3

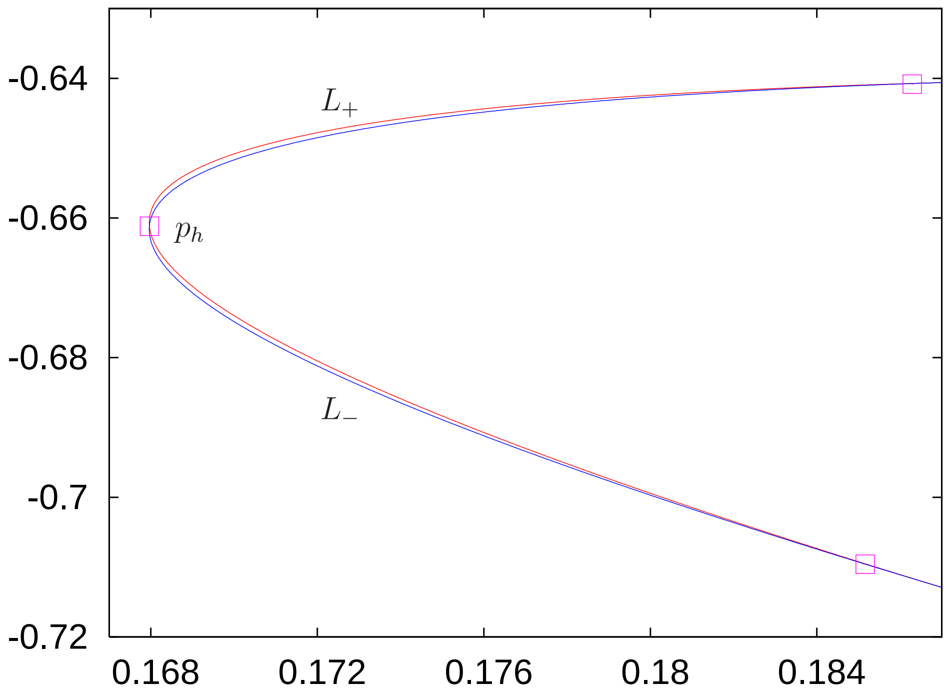
All of them $\mathcal{C}^\infty$ **flat functions** but the behaviour is quite different.

The Hénon map near the 4:1 resonance

$$H_c : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} c(1-x^2)+2x+y \\ -x \end{pmatrix}$$

A **4:1 resonance** appears for $c=1$. We look at $c=1.015$.





The dynamics of H_c^4 can be interpolated by the flow of a Hamiltonian

$$\mathcal{H}(x,y,\delta), \qquad \delta = (c-1)^{1/4},$$

which shows a **Gevrey character**.

Practical stability and codimension 1 manifolds

The **triangular libration points** of the RTBP (and many other examples) are **practically stable** in the sense that the rate of escape is **exponentially small** w.r.t. the distance ρ to the fixed point. Time to move to a finite distance behaves like $\exp(c/\rho^d), c > 0, d > 0$ **(Nekhorosev-like estimates)**.

This follows from **Normal Forms estimates** and/or **averaging theory**.

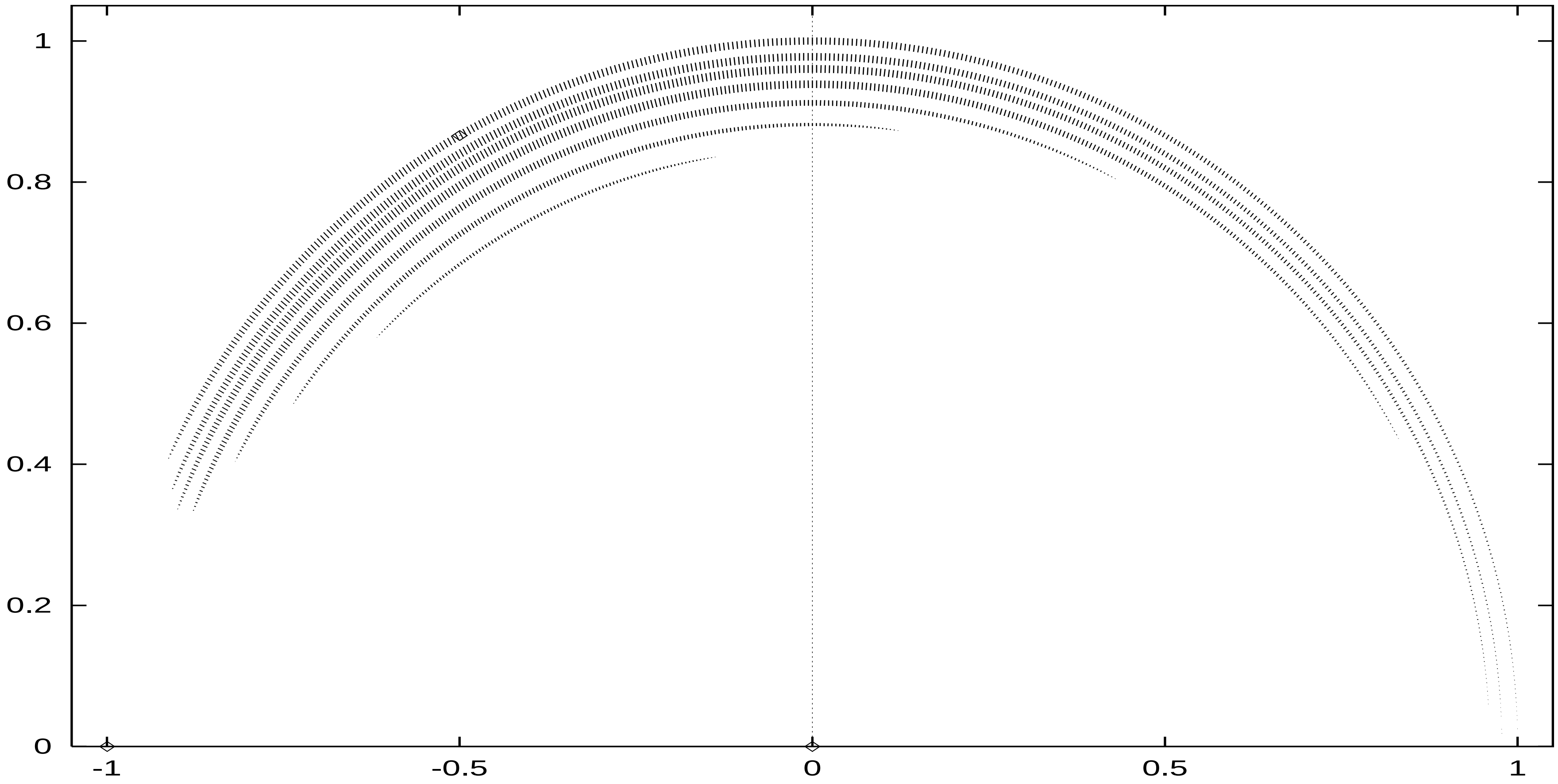
But plenty of numerical simulations (in many families of problems) show a **practically "stable domain"** much larger than the one following from these estimates.

We consider the case $\mu = 0.0002$ because:

a) it is **small enough** so that **perturbation theory** can be useful,

b) it is **large enough** so that some escapes do not require **too much computational time**,

c) it is also close to the **Saturn–Titan** mass-ratio.



Take points with **zero synodical velocity** and (changing C)

$$X = \mu + (1+\rho)\cos(2\pi\alpha), \quad Y = (1+\rho)\sin(2\pi\alpha), \quad Z$$

Escape criterion: $Y(t) < Y^*$ for some $Y^* < 0$ (e.g. $Y^* = -0.5$).

Time span: 10^4 revolutions of primaries (circa 400 years). No essential differences with $10^5, 10^6$.

Displayed results: $Z =$0.0, 0.3, 0.4, 0.5, 0.6, 0.7 and 0.8

Comments on results

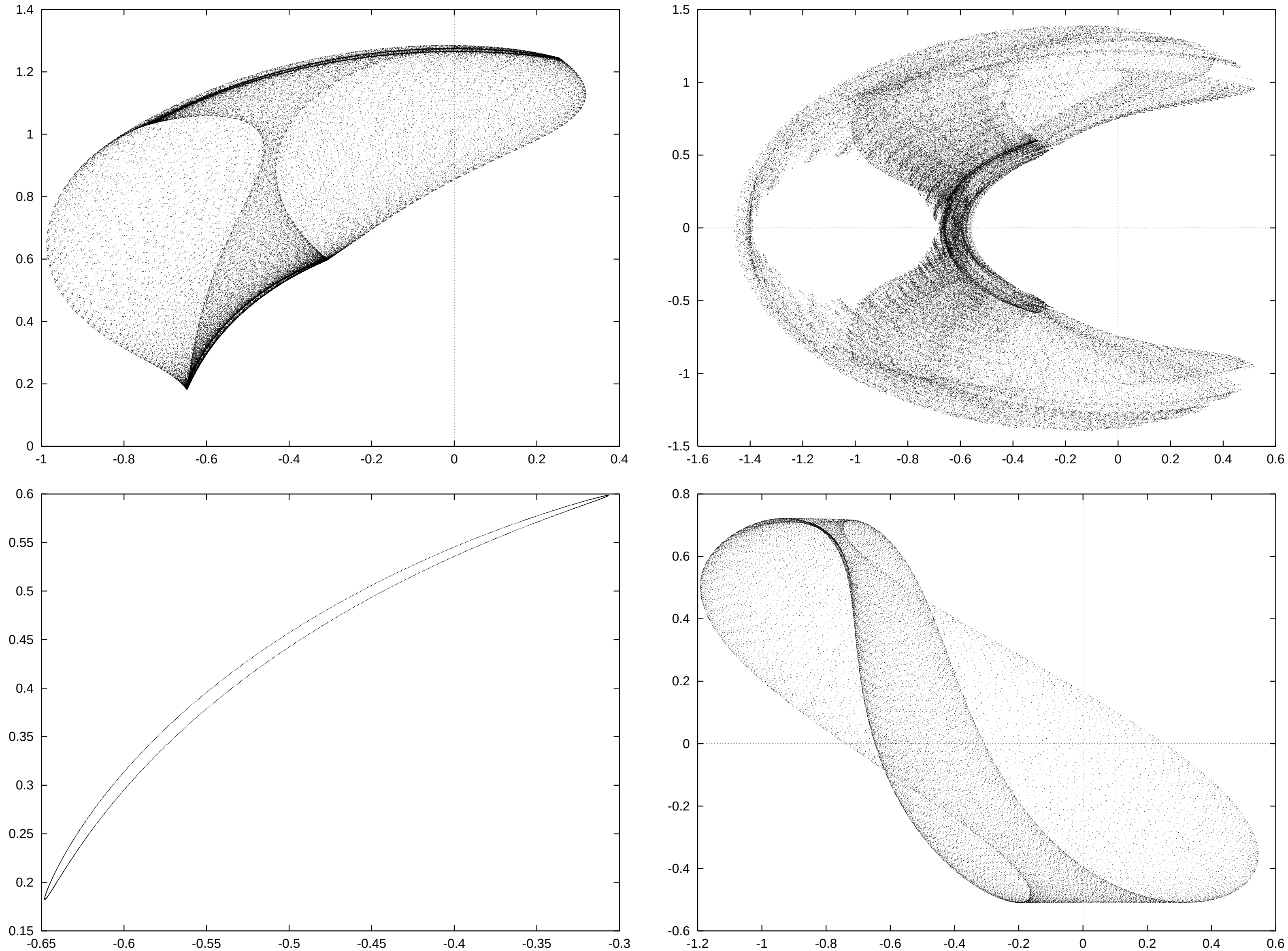
1) For the **planar case** the domain of non-escaping points is bounded by the intersection of $W^{u,s}(W^c_{L_3})$ (a **codimension 1 manifold** of the planar problem) and $v_{\text{syn}}\mathbf{=0}$. If the ZVC on the L_3 level is given by $\rho = K_\pm(\alpha)\mu^{1/2} + \mathcal{O}(\mu)$ then the domain boundary is $\rho \approx \frac{1}{2}K_\pm(\alpha)\mu^{1/2} + \mathcal{O}(\mu)$.

2) The full set of points in the "practical stability" zone with $v_{\text{syn}} = 0$, in the **spatial RTBP**, is on a **thin shell**. Its shape is **near circular** in (X, Y) and **parabolic** in the vertical direction. Cutting the "stable" zone by $\alpha = 1/3$, the central point, as a function of Z, is close to $\rho = -0.245Z^2$.

3) There are families of **unstable 2D tori** which play also a role in the **boundary of the "stability" region**, see next plots. If the initial Z is small these tori reach a **vicinity of** L_3 and their sections through $Z = 0$ have **positive** X **values**. For large initial Z, the sections through $Z = 0$ of these 2D tori have **negative** X **values**.

4) **Continuation** of these 2D shows that they belong to the W^c **of a family of p.o. which is partially hyperbolic**: $W^c_{\text{p.o.family}}$. The $W^{u,s}$ of that object is a **codimension 1 manifold**. Several of them play a role in different parts.

$\alpha = 1/3, Z = 0.8, \rho = -0.1506066340$ vs $\rho = -0.1506066339$ and $\mathbb{T}^2$.



Libration point orbits

Our interest in now to understand the structure of the set of **non-escaping orbits around $L_{1,2}$** in the RTBP. One has to **skip the unstable terms** of the Hamiltonian around an E×E×H point. This is done by **reduction to the 4D centre manifold W^c**.

After linear symplectic change the Hamiltonian is

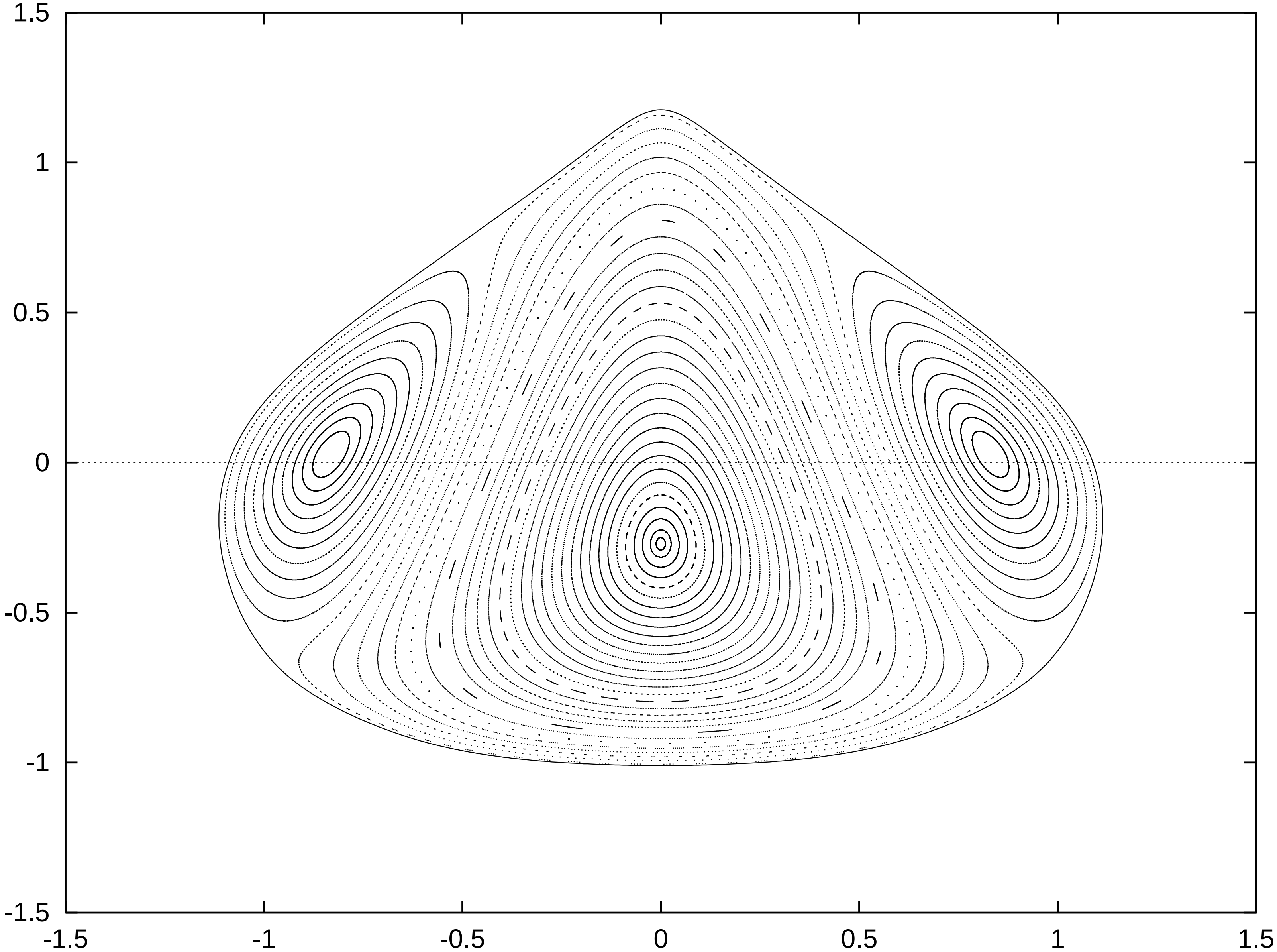
$$M = \lambda x_1 y_1 + \frac{1}{2}\omega_1(x_2^2 + y_2^2) + \frac{1}{2}\omega_2(x_3^2 + y_3^2) + \sum_{j\geq 3} M_j(x_1, x_2, x_3, y_1, y_2, y_3),$$

where M_j denotes a homogeneous polynomial of degree j.

One must **cancel all terms of total degree 1 in x_1, y_1**. No small divisors show up, there is **no convergence in general** (but the divergence is mild). We obtain

$$\widetilde{M} = M^0(x_2, x_3, y_2, y_3) + \sum_{j_1+j_2>1} x_1^{j_1} y_1^{j_2} M^{j_1,j_2}(x_2, x_3, y_2, y_3)$$

and $x_1 = y_1$**=0 gives** W^c.



Poincaré section of the flow of the Hamiltonian reduced to W^c of the L_2 point in the Earth-Moon case, for a given level of the energy, h. The Moon is located outside the figure, in the negative vertical axis of the plot.

One can **read off** the structure from this plot on a section Σ which is $\approx \{z = 0\}$ in the initial variables.

The **boundary p.o.** corresponds to the **planar Lyapunov orbit**, unstable inside W^c for that value of h.

Near the **centre of the plot** there is a fixed point, corresponding to the **vertical Lyapunov orbit**.

The **two additional fixed points** correspond to **halo orbits**.

Invariant curves around fixed points correspond to **tori** (also named Lissajous orbits for that problem).

The zones **between the domains of curves** contain the **intersections of $W^{u,s}$** of the planar Lyapunov orbits with Σ. **Tiny chaos** appears there, hard to see.

The plot helps the preliminary **design of space missions around the libration point**. Improvement using perturbations of other bodies is required. In particular the **solar effects** are quite relevant.

Summary and Outlook

The systematic use of **invariant objects** and, when applicable its **centre, stable, unstable manifolds**, that is the **skeleton of the system**, provides useful tools to understand the **global dynamics**.

I consider the **most challenging problem** to study **statistical properties** like rates of diffusion, rates of escape, mass transport, “practical” ergodicity.

One has to face **diffusion problems** which are **highly heterogeneous and highly anisotropic**.

Addenda

Going back near 120 years, when people realized that **most of the problems in dynamics cannot be solved explicitly** but some particular solutions, like the **periodic solutions**, can be extremely valuable to give a hint on the dynamics, we can recall **a well-known sentence of Poincaré**:

"ce qui nous rends ces solutions périodiques si précieuses, c'est qu'elles sont, pour ainsi dire, la seule brèche par où nous puissions essayer de pénétrer dans une place jusqu'ici réputée inabordable"

(from the end of paragraph 36 of the first volume of his Méthodes nouvelles de la Mécanique Céleste, 1892), that is:

"what makes these periodic solutions so precious is that they are, so to say, the only breach through which we can try to penetrate into a region reputed, up to now, to be unapproachable".

Now, already in the **XXI century**, the words of **Poincaré still apply** to give valuable information. But the sentence has to be **updated**. We can say, concerning conservative systems:

The **stability properties** of these orbits play a key role. In the **linearly stable** case we can expect to have nearby **KAM tori**. In the **unstable case** what we have are the **invariant manifolds**, stable/unstable, of the centre manifold of the periodic orbit. They also play a **key role**. The **splitting** properties of these manifolds, the existence of **homo/heteroclinic** connections between different invariant objects, gives the way to **organize globally** the orbits in the phase space. **Poincaré itself was aware** of the role of splitting and homoclinic phenomena.

Furthermore, the same ideas can be applied to **conservative and dissipative systems, continuous or discrete, finite or infinite dimensional**. And it is fully relevant to consider all the elements of the **"skeleton"** of the dynamics (fixed points, periodic and quasi-periodic solutions, invariant manifolds and their relative position) as depending on **parameters**. It is not just the **phase space** $\mathcal{E}$ what is relevant, but the product of **phase and parameter spaces** $\mathcal{E} \times \mathcal{P}$. This gives the framework to analyze bifurcations and unfold degenerate cases.