Towards a Global Study of Area Preserving Maps
Examples, Models and Applications

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Contents

- Statistical properties: Rates of escape and some difficulties
- Refined numerical tools
- Additional application: The outer cometary problem
- Some open questions

The problem

Area preserving maps appear, either **directly** or as **Poincaré maps** of Hamiltonian systems with 2 or 1+1/2 degrees of freedom.

Still some **global problems** are unclear, like

- **Measure and structure** of the set of confined points,
- **Presence of islands** inside chaotic domains,
- **Statistical properties**, like rates of escape.

An **ultimate goal** would be to have **global precise** results. Probably **beyond the possibilities of Mathematics in the near future**.

But a combination of **partial models** with **experiments** can lead to a **reasonable understanding**, at least of **simple problems** and show also many **open questions**.

CAP can also be interesting in some cases, when the **validity of a result** can be reduced to check a **finite number of inequalities**.

Phenomenology I: The Hénon map

Any quadratic area preserving map can be reduced to

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} = \begin{pmatrix} c(1-x^2)+2x+y \\ -x \end{pmatrix} \qquad x, y, c \in \mathbb{R}.$$

Fixed points at H$=(-1,1)$ and E$=(1,-1)$, **elliptic** if $c \in [0,2]$.

Normal Form analysis proves **non linear stability** if $c \in (0,2)\backslash\{1.5\}$.

Appears typically in **generic non-hyperbolic area preserving maps**, e.g., in **Poincaré sections of Hamiltonian systems with two degrees of freedom** in **many parts of the phase space**.

It serves as **paradigm** for many phenomena. See S-Vieiro, Global study of area preserving maps, and 2007 and S-Vieiro, in preparation.

Alternative formulations:

$$H_\alpha : \begin{pmatrix} x \\ y \end{pmatrix} \longmapsto R_{2\pi\alpha} \begin{pmatrix} x \\ y - x^2 \end{pmatrix}, \qquad \alpha \in (0, 1/2).$$

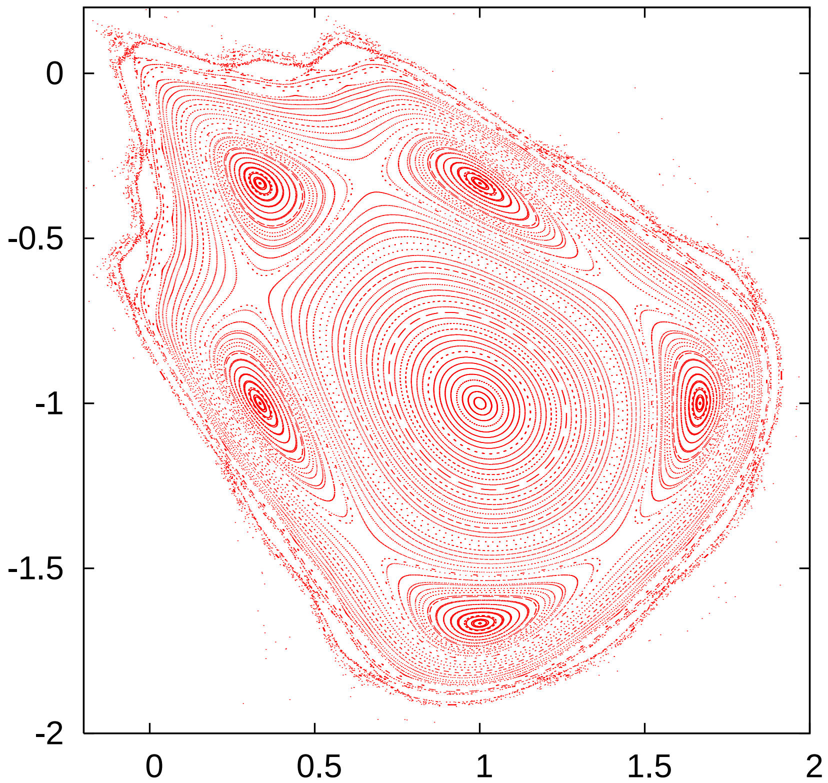
$$H_a : \begin{pmatrix} u \\ v \end{pmatrix} \to \begin{pmatrix} 1 - au^2 + v \\ -u \end{pmatrix},$$

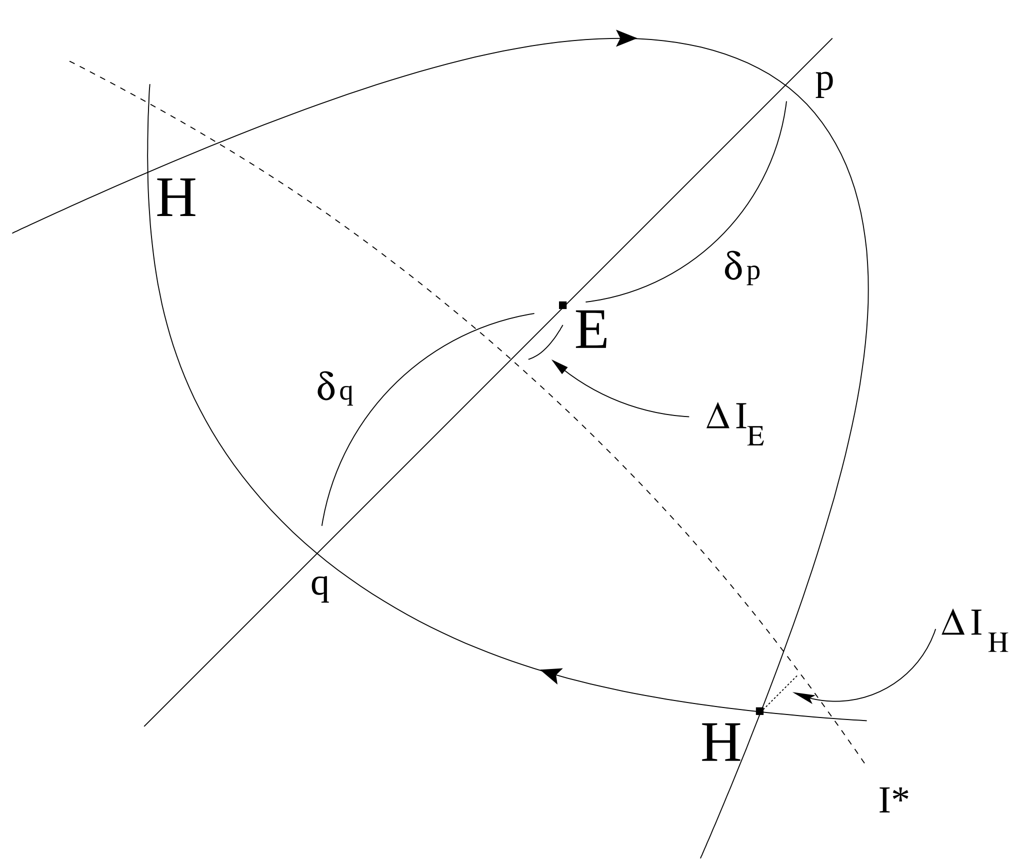
where $a = \cos^2(2\pi\alpha) - 2\cos(2\pi\alpha)$, $a \neq 0$, $c = \sqrt{1+a}$.

With suitable scalings and for small c the map can be approximated by the **time-d map of a Hamiltonian flow**

$$K(q,p) = p^2 - q + \frac{1}{6}q^3, \quad \dot{q} = 2p, \qquad \dot{p} = 1 - \frac{1}{2}q^2, \quad d = \sqrt{c/\sqrt{2}},$$

The level $K = \frac{2}{3}\sqrt{2}$ containing the H point $q = -\sqrt{2}, p = 0$ corresponds to a **separatrix, enclosing the elliptic point** $q = \sqrt{2}, p = 0$. The **splitting** for the manifolds of the map is **exponentially small in d**. Also **invariant curves** of the map exist at an exponentially small distance of the manifolds.

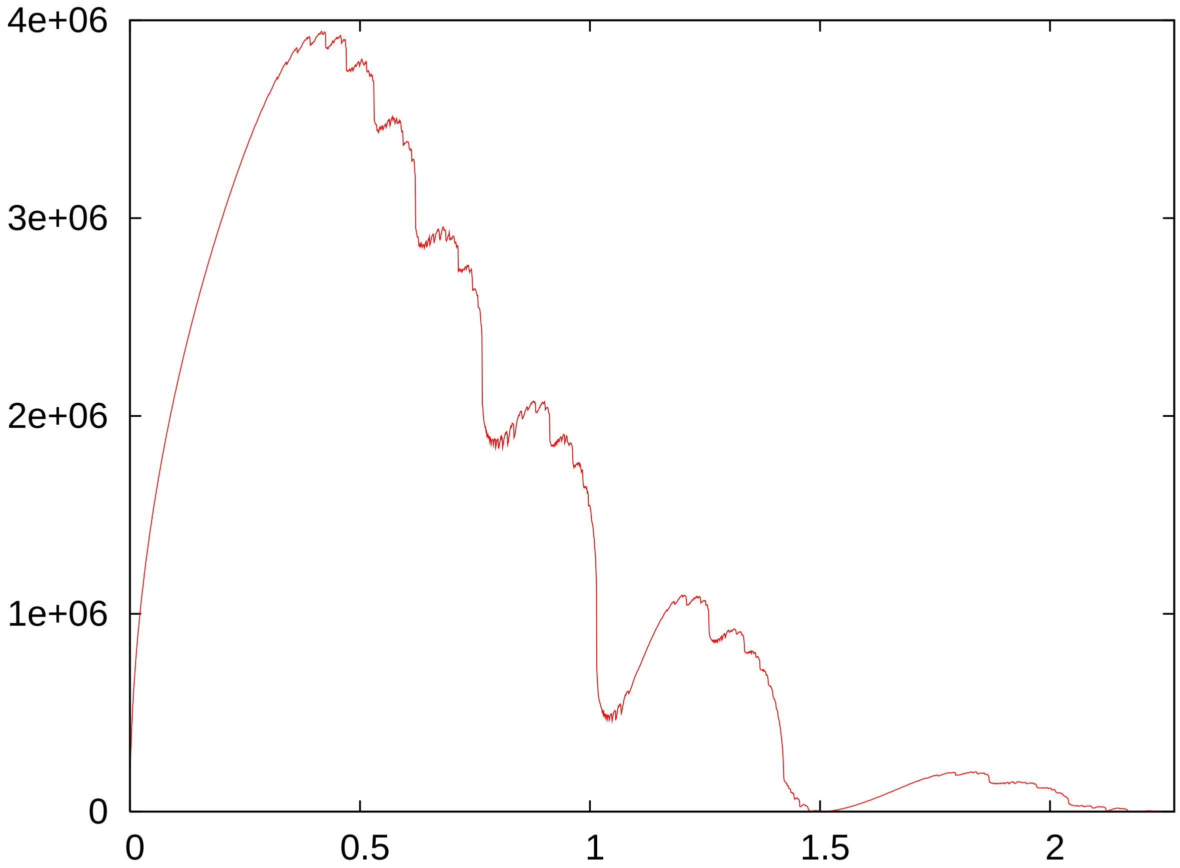




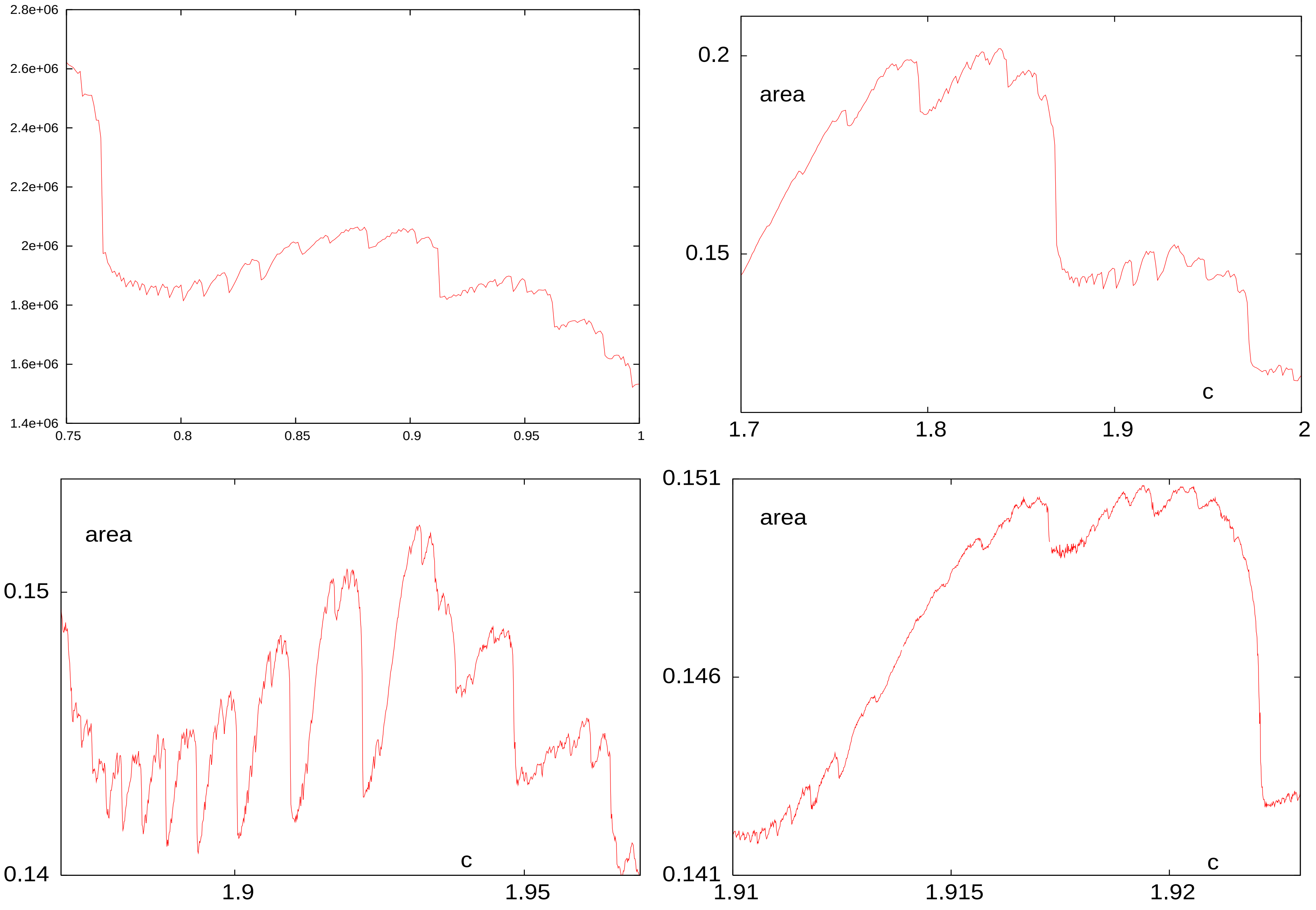
Left: **Phase portrait** for $c = 0.75$. Outside that domain, points **escape** to infinity, close to W_H^u. Right: A sketch of a **resonant zone**. The dotted curve corresponds to the **mean radius of the resonance**. It displays also the location of elliptic and hyperbolic periodic points and the **inner and outer separatrices**. Notice the **lack of symmetry**.

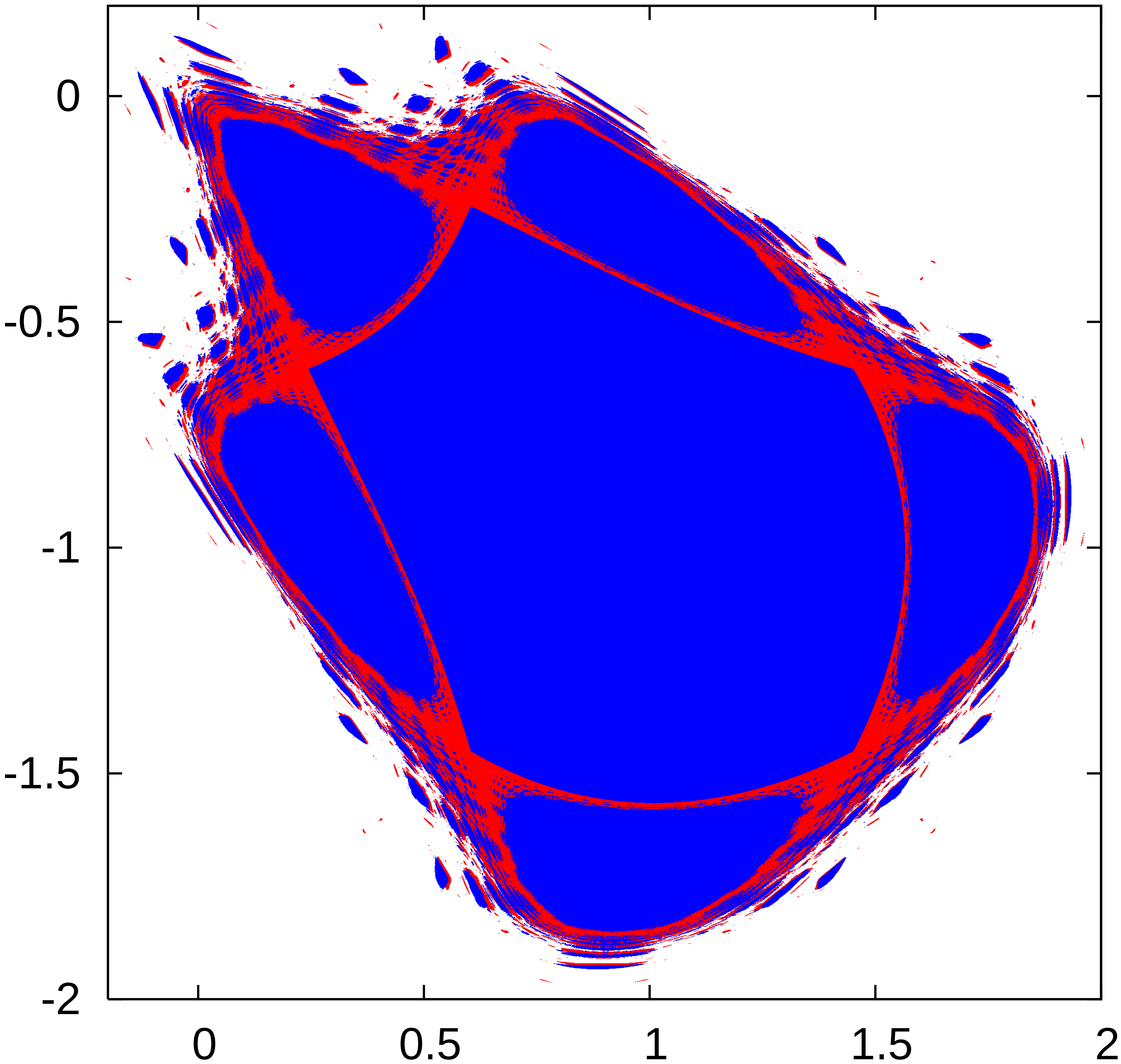
Illustrations and also some examples of **transitions** in the **set of non-escaping points**

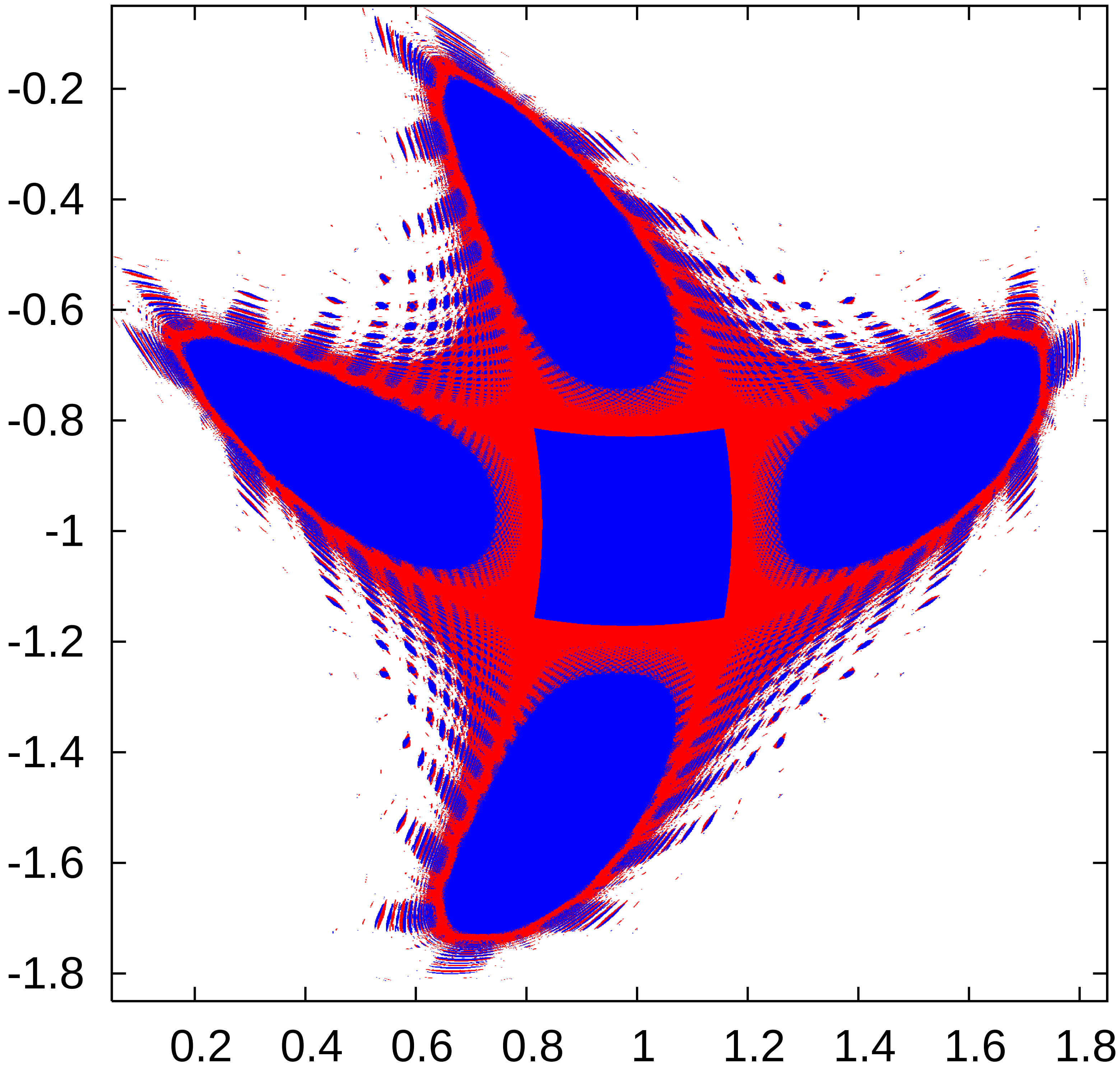
The set of non-escaping points as a function of c



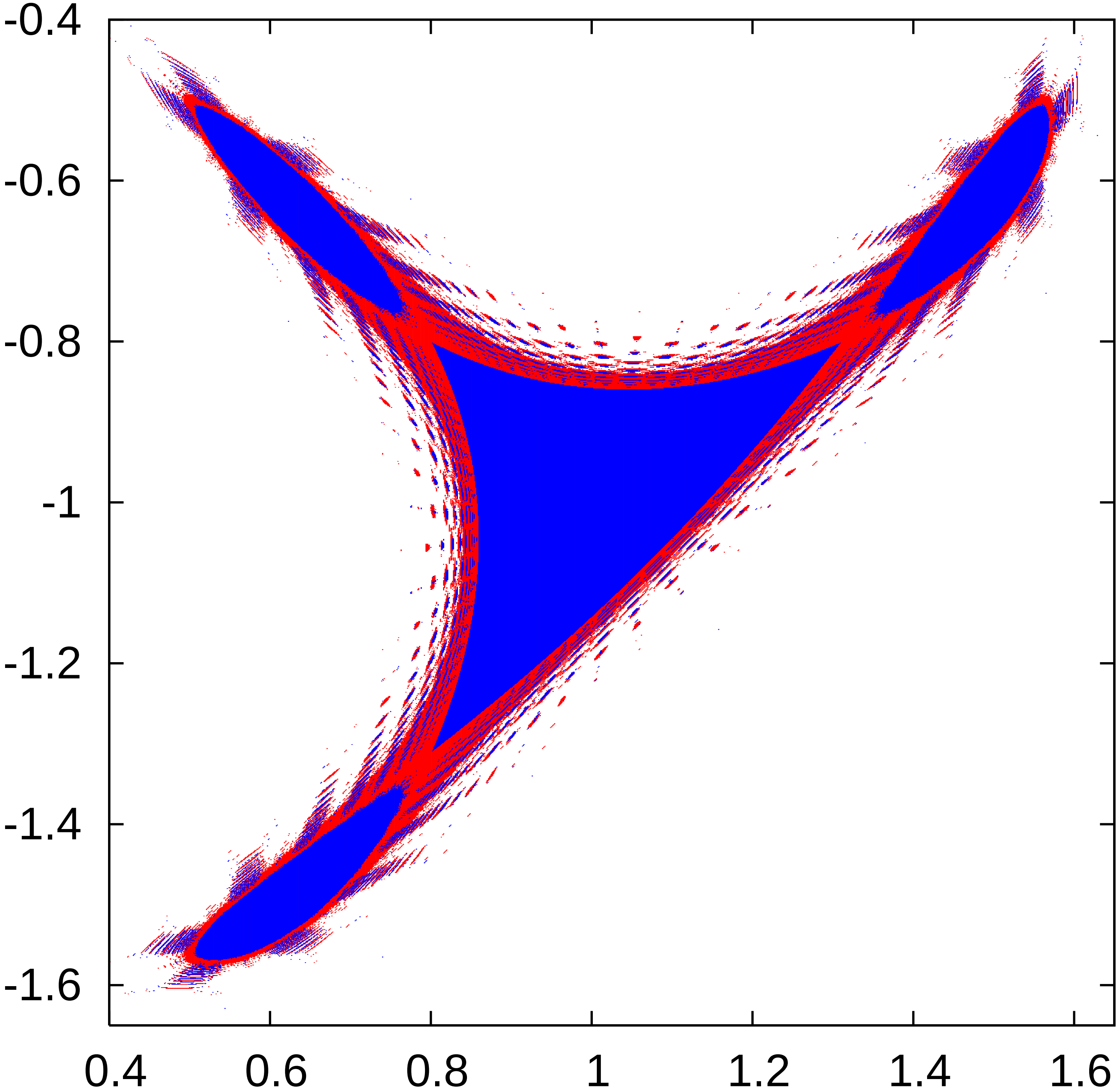
Number of **non-escaping points** on a grid $10^{-3} \times 10^{-3}$. Magnifications on next plots. Computations done by a combination of methods. Observe the **self-similar** character and the **jump discontinuities**. Also the non-escaping points **before and after** the **destruction of rotational i.c. confining islands** of periods 5,4,3.


area
c
area
c
area
c


0
-0.5
-1
-1.5
-2
0
0.5
1
1.5
2

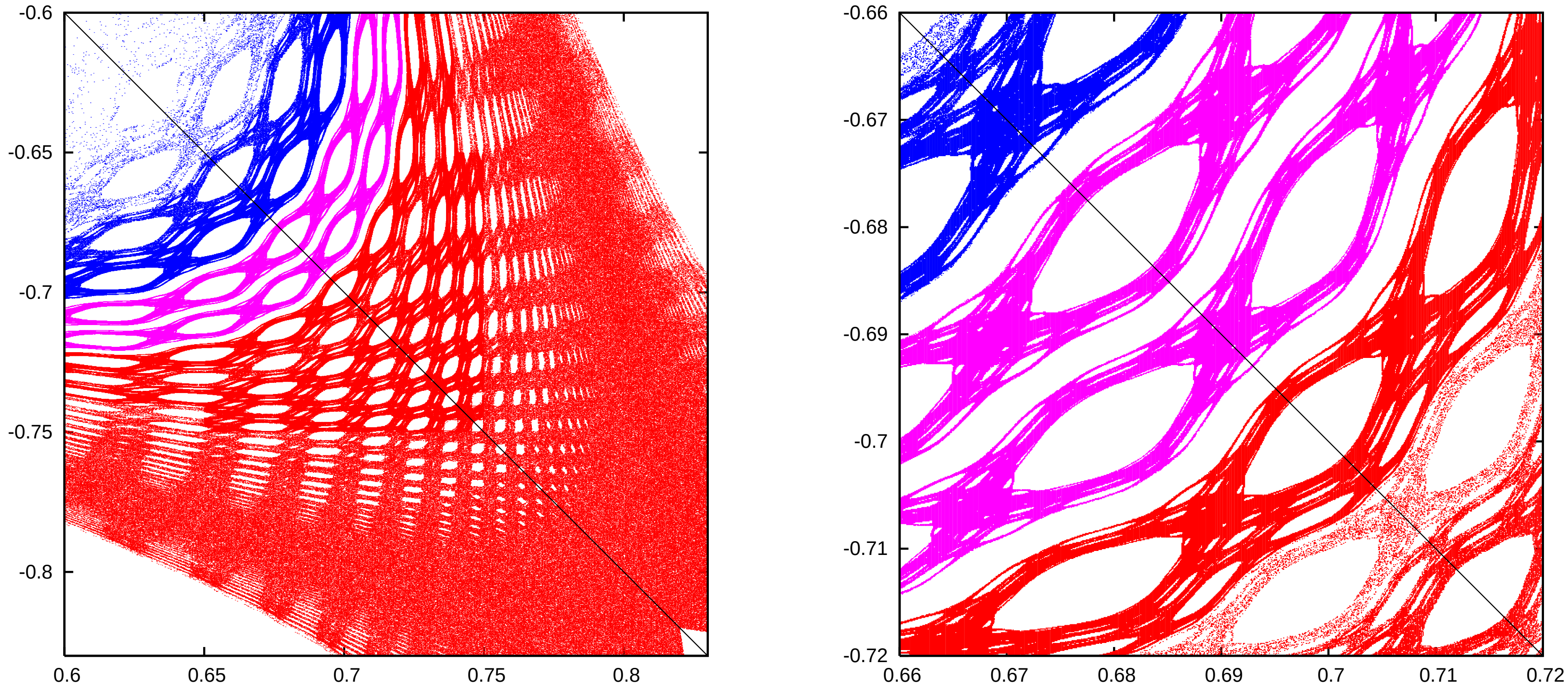


-0.2
-0.4
-0.6
-0.8
-1
-1.2
-1.4
-1.6
-1.8
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8


-0.4
-0.6
-0.8
-1
-1.2
-1.4
-1.6
0.4
0.6
0.8
1
1.2
1.4
1.6

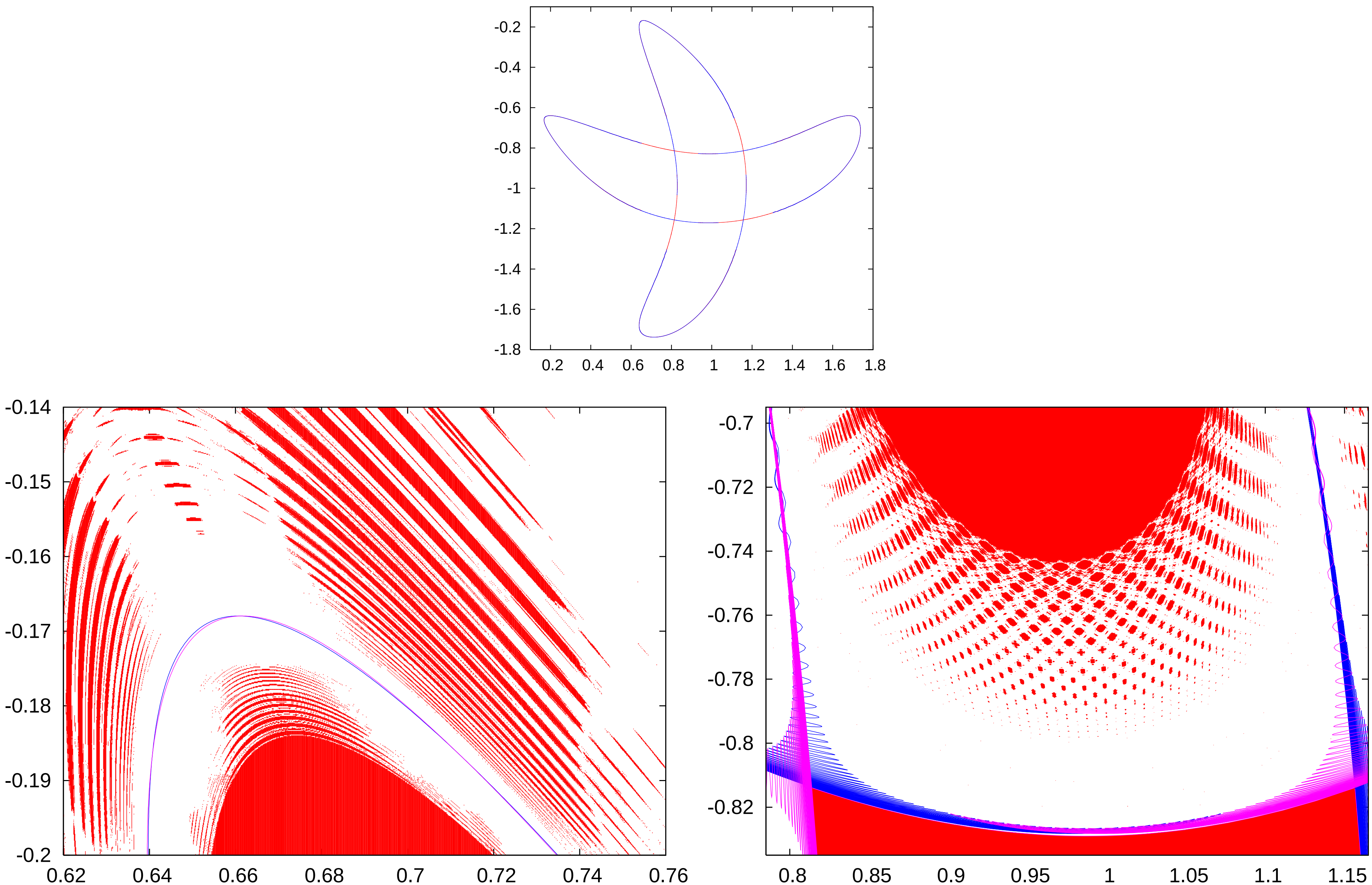
Previous plots correspond to close values of c, separated by 0.001, say $c_1 < c_2$. In **red** we display the set of points which **do not escape** for $c = c_1$; in **blue** the non-escaping points for $c = c_2$.

Next plots also illustrate different regions on the **phase space**. The right lower corner of the first plot is close to the upper left period 4 point in red-blue plot for period 4.



Magnifications of a region close to $y = -x$ for $c = 1.0141$. Points in red, blue, correspond to **single orbits**. Points in magenta correspond to **two orbits**, one in each domain. One can see the **negative** of the **islands**. Only points in blue escape, but **for** $c = 1.0142$ **all these points escape**. For $c = 1.0141$ there are **three gaps** candidate to contain **invariant (rotational) curves**

As an additional example consider the parameter $c = 1.015$, just after the **destruction of the i.c. confining islands of period 4**. One can see a **domain without subsisting points** close to the **invariant manifolds** of the period 4 orbit, see upper plot in next page. It looks as if the manifolds coincide, at this resolution.


-0.2
-0.4
-0.6
-0.8
-1
-1.2
-1.4
-1.6
-1.8
0.2 0.4 0.6 0.8 1 1.2 1.4 1.6 1.8
-0.14
-0.15
-0.16
-0.17
-0.18
-0.19
-0.2
0.62 0.64 0.66 0.68 0.7 0.72 0.74 0.76
-0.7
-0.72
-0.74
-0.76
-0.78
-0.8
-0.82
0.8 0.85 0.9 0.95 1 1.05 1.1 1.15

Phenomenology II: Unfolding the Hopf-saddle-node in $\mathbb{R}^3$. The Michelson system

At a **HSN bifurcation** eigenvalues are $0, \pm\mathrm{i}$. **Normal Forms**, which are formally rotationally invariant can be computed, unfolded under non-degeneracy conditions.

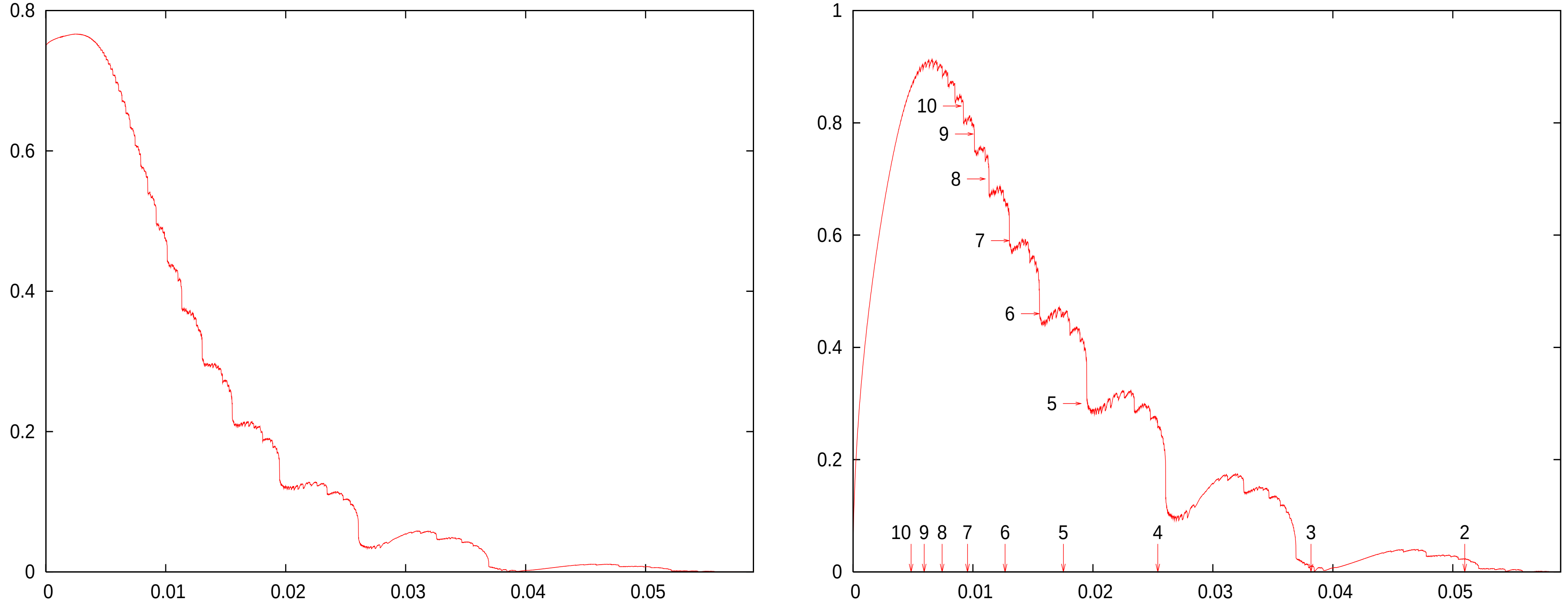
A **critical case** is the **volume preserving** one. It helps to understand other cases.

A simple model, representative enough, is **Michelson system**, appearing also as traveling waves in the **Kuramoto-Shivasinski** PDE.

$$x''' = -1 + ax' + x^2,\; a > 0.$$

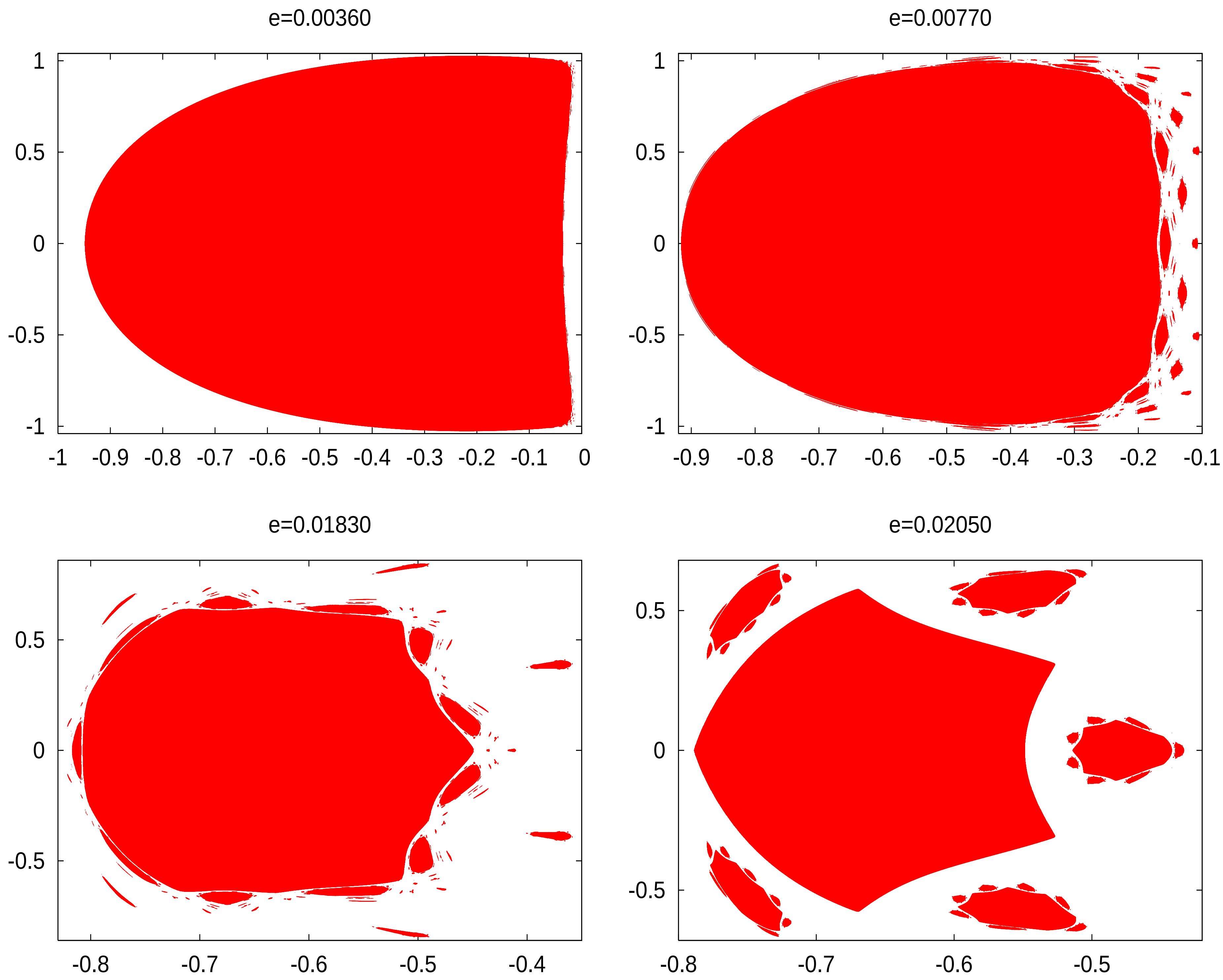
It is convenient to use as parameter $\varepsilon = (-a)^{-3}$. The NF to any order has **connecting 2D and 1D** invariant manifolds of $(\pm 1, 0, 0)$. Real system has **splitting exponentially small in ε**.

Use $x'' = 0, x''' > 0$ (in suitable domain) as **Poincaré section**. The **Poincaré map** preserves a **measure** in the (x, x')-plane.

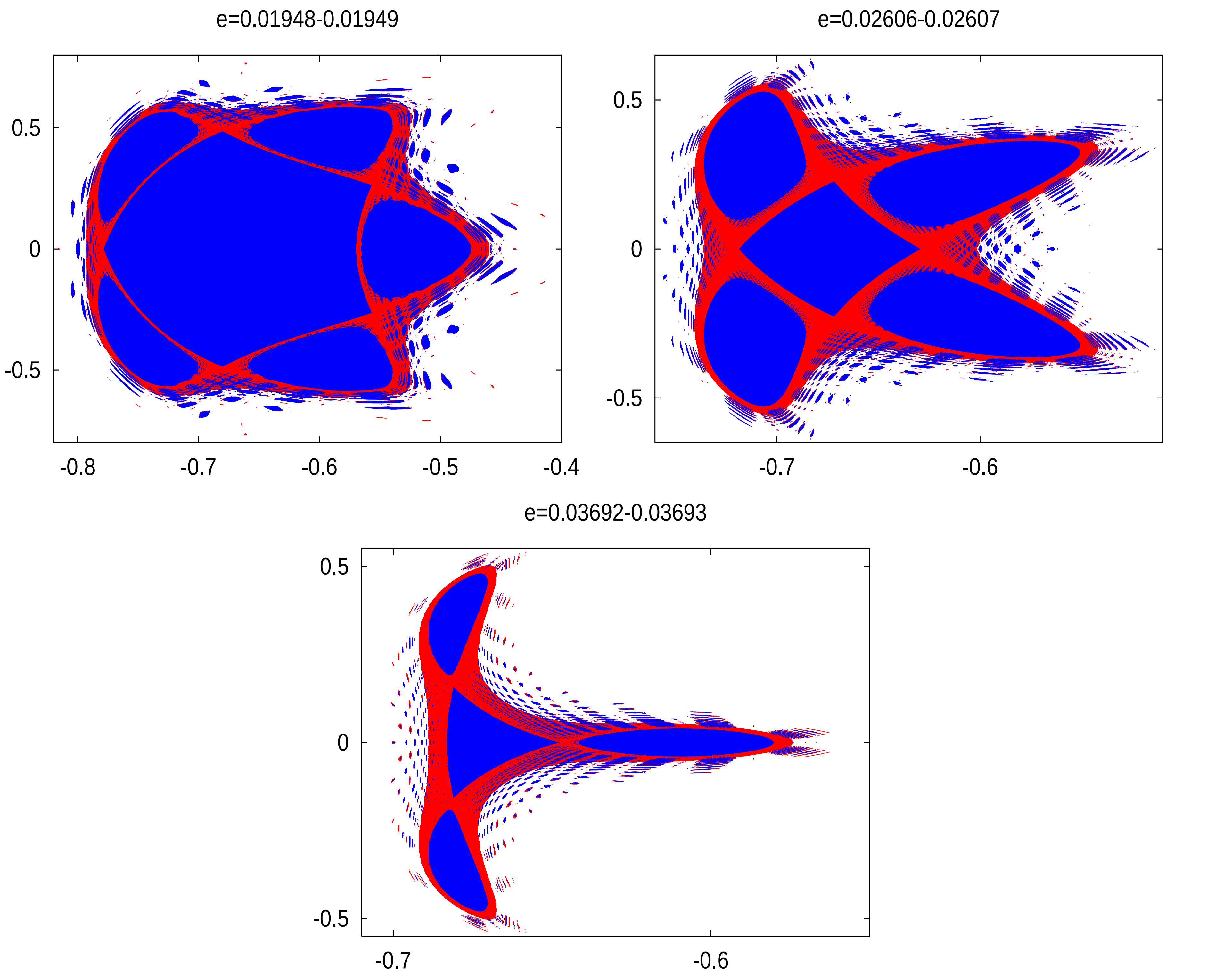


Left: Statistics of the number of points which **do not escape** as a function of ε using a 4000×4200 grid. Right: The values have been multiplied by ε, to better see the details. The location of some **resonances**, both at the **central elliptic fixed point** and at the **breakdown of invariant curves** which disconnects the related islands is shown.

Let us see some **examples**.

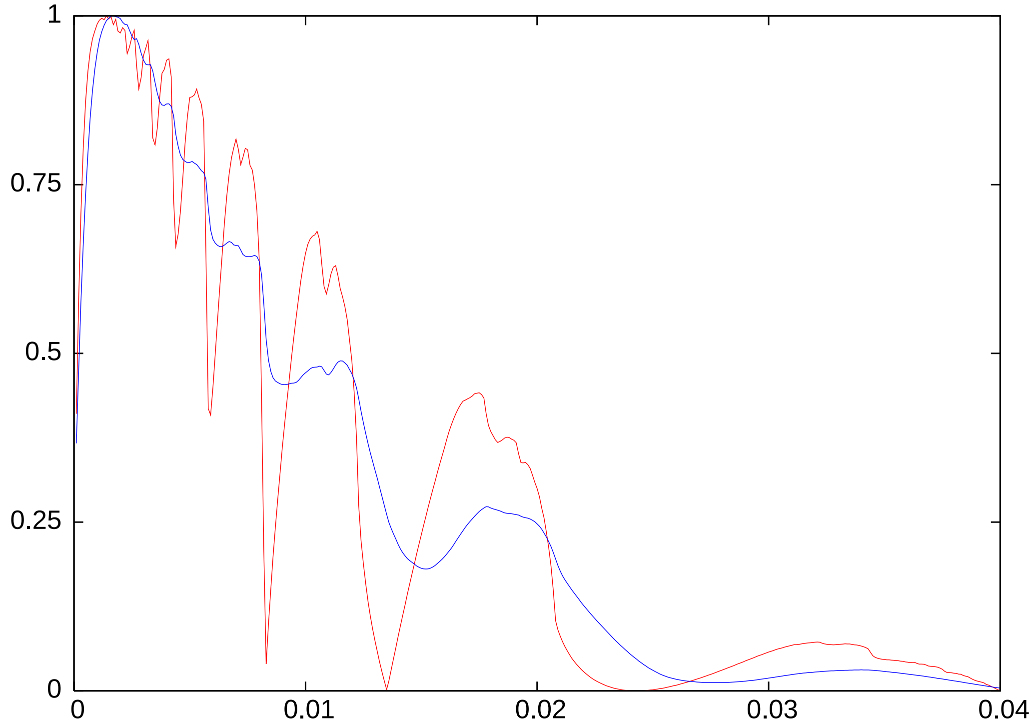


e=0.00360
e=0.00770
e=0.01830
e=0.02050
1
0.5
0
-0.5
-1
-1 -0.9 -0.8 -0.7 -0.6 -0.5 -0.4 -0.3 -0.2 -0.1 0
-0.9 -0.8 -0.7 -0.6 -0.5 -0.4 -0.3 -0.2 -0.1
-0.8 -0.7 -0.6 -0.5 -0.4
-0.8 -0.7 -0.6 -0.5

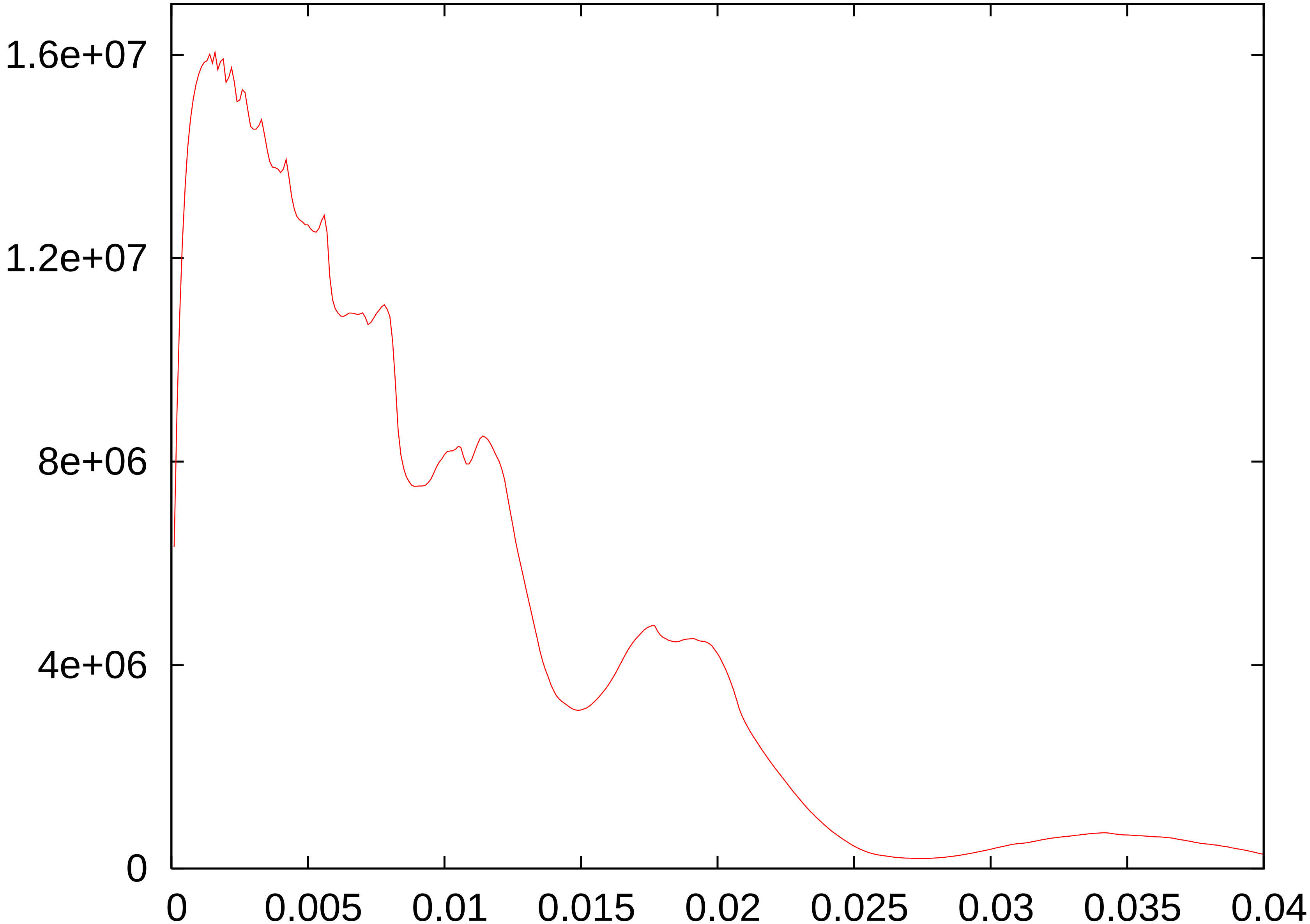


e=0.01948-0.01949
0.5
0
-0.5
-0.8
-0.7
-0.6
-0.5
-0.4
e=0.02606-0.02607
0.5
0
-0.5
-0.7
-0.6
e=0.03692-0.03693
0.5
0
-0.5
-0.7
-0.6

Phenomenology III: The planar RTBP around the Lagrange points



Normalized subsisting points. **Planar case**: $\Delta\rho = 10^{-4}$, $\Delta\alpha = 2 \times 10^{-4}$. Maximal value, for $\mu = 0.0014$, 282757 points. **Spatial case**: $\Delta\rho = 10^{-4}$, $\Delta\alpha = 5 \times 10^{-4}$, $\Delta z = 5 \times 10^{-3}$. Maximal value, for $\mu = 0.0017$, 19014882 points. Total scanned points: 8×10^{10}. Subsisting a first filter 2.17×10^9. **The role of resonances is clearly seen** (**2D, 3D differ**). Spatial case refinement finished recently.



1.6e+07
1.2e+07
8e+06
4e+06
0
0
0.005
0.01
0.015
0.02
0.025
0.03
0.035
0.04

Numerical tools and qualitative explanation

Possible numerical tools

- **Greene's criterion**: Useless because we do not know the desired rotation number of the invariant curve to be destroyed.
- **Lyapunov exponents**: Hard to decide if we can assume a computed value is zero. Takes time to compute on a narrow grid.
- **Frequency analysis**: A tolerance has to be set to decide about constant frequency. Takes time to compute on a narrow grid.
- **The orbit method**: Take a grid of pixels and remove all pixels visited by a point which escapes. Can delete pixels where most of the points subsist. Can be useful combined with
- **The boundary of domain method**: As previous but then points close to the boundary (with some deepness condition) are reexamined for larger number of iterates. Used for previous plots.
- **The heteroclinic connections method**: Invariant curve of rotation number ρ cannot exists if the is a connection between hyperbolic orbits with $\rho_1 < \rho < \rho_2$. Hard to find if the orbits are far away.

A **qualitative explanation** is clear.

For instance, in **the Hénon case**, for $c = c_1$ there is still an **invariant rotational curve** which encloses the main islands of period k, $k = 5, 4, 3$, respectively. For $c = c_2$ **the curve has disappeared**.

Before the breakdown of the curve, there were points sitting on a **chaotic, confined region**, moving between families of islands. When the curve disappears, these points are **free to escape**. This explains the **jumps** in the measure of the non-escaping points as a function of c.

Similar things happen for **all the main islands** and also for the **satellite islands of the main islands**, "ad infinitum", giving rise to **self similarity**.

In any case the **bound for the existence of invariant curves** is related to **invariant manifolds**: In Hénon case, of **point** H; in Michelson, of the **two saddles** in the Poincaré map; in the planar RTBP, of the **center manifold around** L_3. The **3D RTBP** is much more complex (related to the case of **4D symplectic maps**, under study now).

Some theoretical aspects I: KAM theory

Some questions appear (beyond **local bifurcations**):

- Why **some invariant curves subsist and others are destroyed**?
- What is the fate of the **separatrix** which before separated the **libration** and **circulation** regimes in the flow approximation (for small c) or in a resonance domain?

Answer to first question is given by **KAM theory** or, in this case, simply **Moser's twist theorem**, the main ideas of proof to be sketched.
It requires **three conditions**

- The unperturbed map must have non-zero **torsion**, i.e., **frequency changes with amplitude**,
- The **rotation number** ρ of the curves which subsist must satisfy a **Diophantine Condition DC,** $|\rho - p/q| > c|q|^{-\tau}$, for some $c > 0, \tau \geq 2$,
- The **perturbation** must be **sufficiently small**.

The three conditions are not independent: **the larger the torsion and the DC, the larger the admissible perturbation can be**.

In a given narrow annulus, where we suspect invariant curves are **going to be destroyed**, both **torsion and perturbation** have mild changes.

A key role is played by the **Diophantine Condition**.

If $\rho = \frac{1}{q_1+}\frac{1}{q_2+}\ldots\frac{1}{q_n+}\ldots$ and $\rho_n = N_n/D_n$ is the rational obtained by truncation,

the **best DC limit properties** correspond to $\tau = 2$ and ρ such that

$$\tilde{c} = \limsup D_n^2|\rho - \rho_n|$$

is maximum, for ρ in the given interval.

Using the fact $D_n^2|\rho - \rho_n| = \frac{1}{\alpha+\beta}$ where

$$\alpha = \frac{1}{q_n+}\frac{1}{q_{n-1}+}\ldots\frac{1}{q_1}, \quad \beta = q_{n+1} + \frac{1}{q_{n+2}+}\frac{1}{q_{n+3}+}\ldots,$$

It is easy to see that the **most robust curves** have **noble rotation numbers**.

Some theoretical aspects II: splitting of separatrices

Concerning separatrices, they **generically split** with an **exponentially small splitting with respect to a suitable parameter**, in some cases in which the dynamics can be reduced to a **perturbation of the identity** and in the **analytic category**.

Typical expressions for the measure of the splitting are of the form

$$|s| \approx A\varepsilon^r \exp(-c\varepsilon^{-d})$$

for suitable $A, r, c, d > 0$, where ε is a relevant small parameter. See, e.g., Fontich-S, *Erg. Th. & Dyn. Systems* **10** (1990), 295–318 and Gelfreich-S, High-precision computations of divergent asymptotic series and homoclinic phenomena, *DCDS*, to appear.

Note that the formula can be interpreted in **both directions**:

- either to **produce an estimate** for given ε, or
- to find **up to which** ε the splitting is less than some value.

Even **small values of the splitting** can have a **relevant impact** in the long term dynamics.

Models for a resonance zone: The standard map, deformations and outer/inner splittings

Present goal is to **explain and predict** the observed behavior.

To this end we introduce several **simple, paradigmatic, models**.

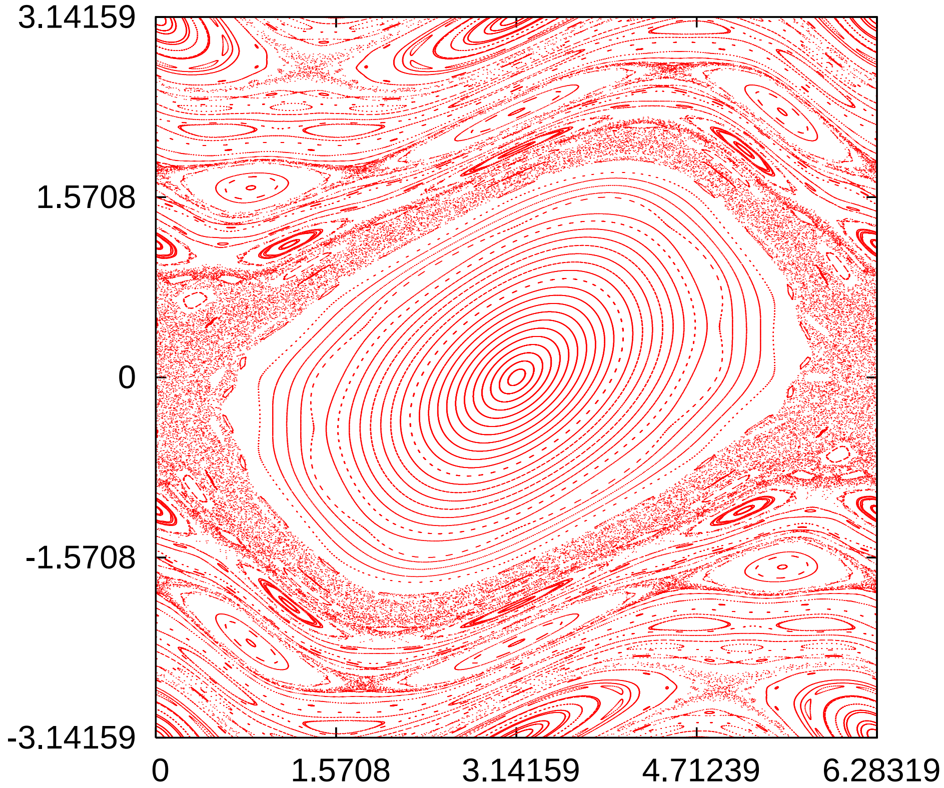
The standard map:

$$SM \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} = \begin{pmatrix} x + \bar{y} \\ y + \varepsilon \sin(2\pi x) \end{pmatrix} \qquad \mathrm{mod}\ 1.$$

SM is a model for **simple resonances**.

Note the symmetry. SM is a discrete version of the **pendulum**. A more realistic model, taking into account **deformation due to the annulus shape** is the discrete version of

$$\mathcal{H}(J, \psi) = \frac{1}{2} J^2 + \frac{c}{3} J^3 - (1 + dJ) \cos(\psi).$$



The plot is done for $\varepsilon = 0.9/(2\pi)$. One can see **i.c.=invariant curves**, either of **rotational** or **librational** type, which give rise to **stability islands**), and also **chaotic zones**. It is **"known"** that **rotational curves** exist up to $\varepsilon = \varepsilon_G \approx 0.971635/(2\pi)$, the so-called **Greene's value**. See Olvera-S, *Physica D* **26** (1987),181–192.

The effect of the **lack of symmetry** is seen in the result (S-Vieiro):

Theorem Let F be an APM. Assume that it has an m-order resonance strip, $m > 4$, located at an average distance $I = I_*$ verifying

(i) $\left.\frac{d\rho}{dI}\right|_{I=I_*} \neq 0$

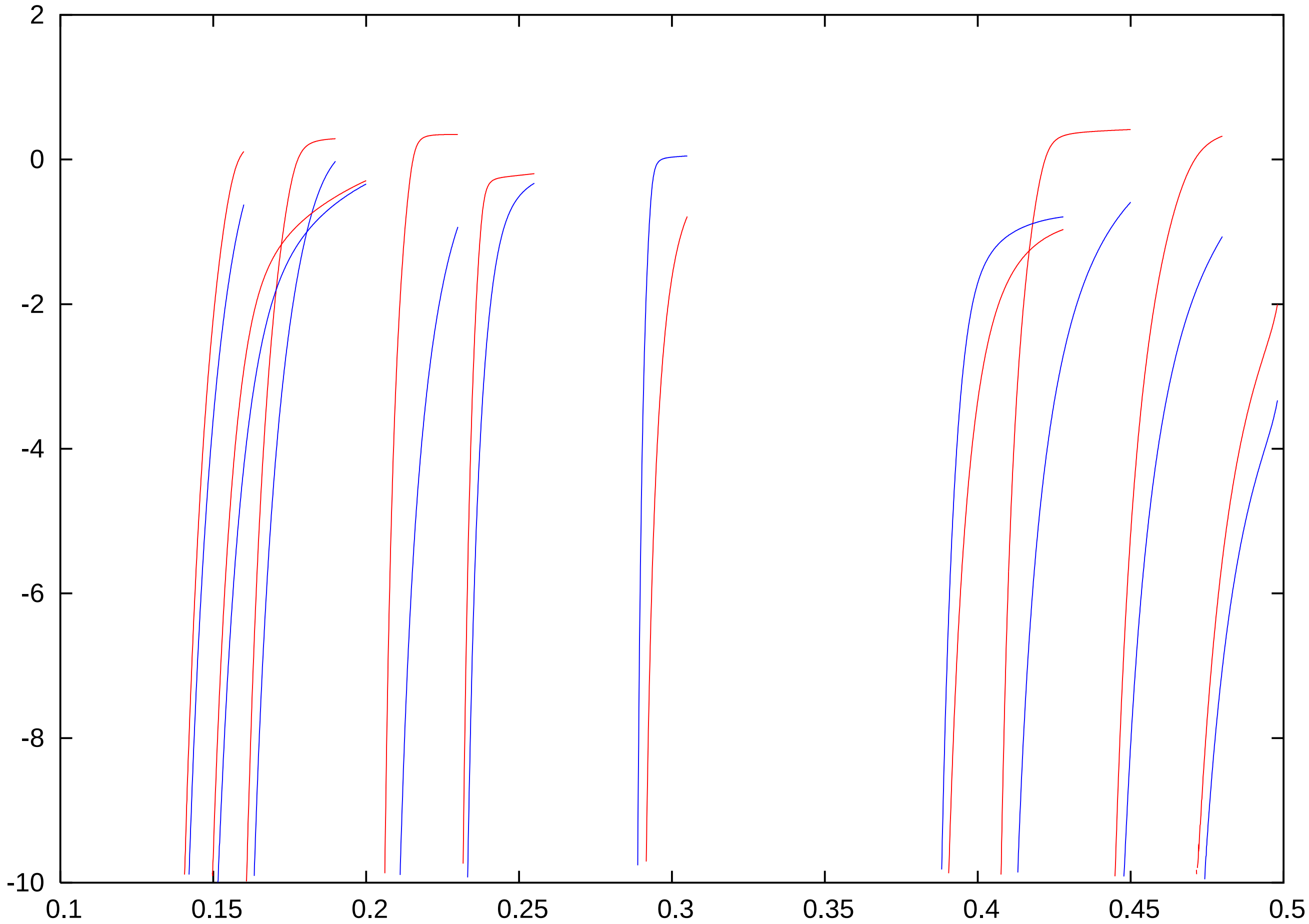
(ii) $\left.\frac{d^{(2)}\rho}{dI^2}\right|_{I=I_*} \neq 0$

Then, the islands of the resonance have, generically, both splittings different. Moreover, if non-exponentially small corrections are ignored, the outer splitting is greater than the inner one provided one of the following conditions

(a) $\left.\frac{d\rho}{dI}\right|_{I=I_*} > 0$ and $\left.\frac{d^{(2)}\rho}{dI^2}\right|_{I=I_*} > 0$ or $\left.\frac{d^{(2)}\rho}{dI^2}\right|_{I=I_*} \lessapprox 0$,

(b) $\left.\frac{d\rho}{dI}\right|_{I=I_*} < 0$ and $\left.\frac{d^{(2)}\rho}{dI^2}\right|_{I=I_*} < 0$ or $\left.\frac{d^{(2)}\rho}{dI^2}\right|_{I=I_*} \gtrapprox 0$.

Otherwise, the inner is greater than the outer. In particular this hold for I_* small, i.e., close to the apparition of the resonance.

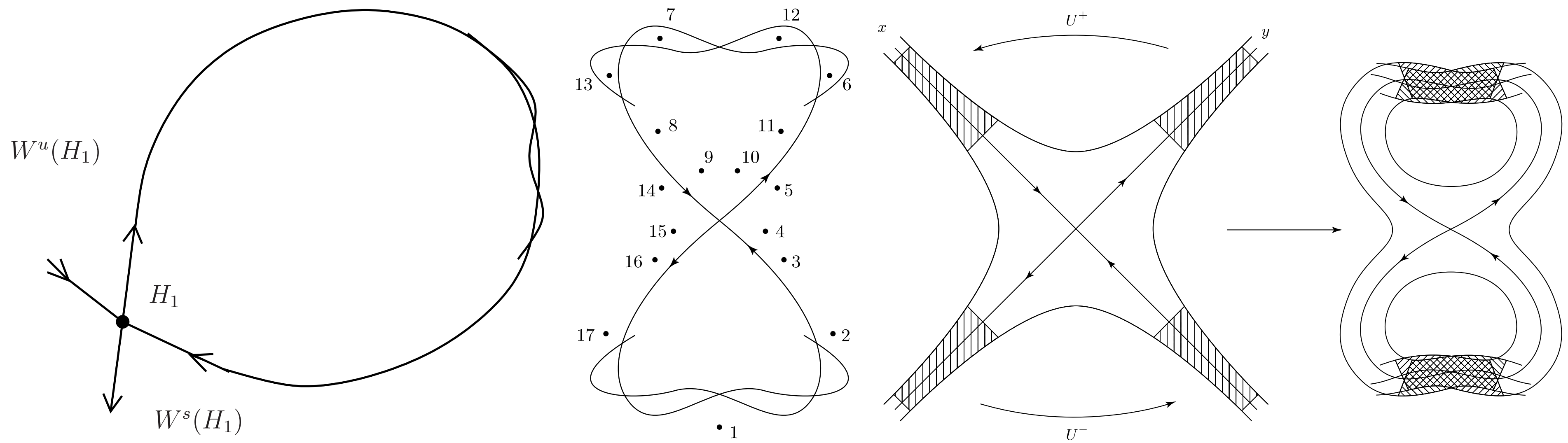


From left to right, it is represented the decimal logarithm of the splitting of the resonances 1:9, 1:8, 1:7, 1:5, 2:9, 2:7, 3:8, 2:5, 3:7 and 4:9, respectively. Each pair of red and blue lines corresponds to the outer splitting an inner splitting, respectively, of a different resonance. In the x-axis it is represented the value of α.

Models near a separatrix: Separatrix maps

The simplest version is the classical **separatrix map**:

$$SPM \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} = \begin{pmatrix} x + a + b\log(|\bar{y}|) \\ y + \sin(2\pi x) \end{pmatrix}.$$



SPM is a model for **return close to the separatrix after passing near the saddle**. The y scale has been **normalized** in such a way that the **splitting amplitude** is equal to 1. b is the inverse of the dominant **eigenvalue** at the saddle. So, it is large for weak saddles.

See Broer-S-Tatjer, *Nonlinearity* **11** (1998), 667–770 and S-Treschev, Stability islands in the vicinity of separatrices of near-integrable symplectic maps, *DCDS*, to appear.

For large y can be **approximated by SM with parameter b/y**. This allows to predict **position of islands, existence of i.c., etc.**.

For instance, **invariant curves** start to exist at distance $y \approx d/\varepsilon_G$ from the separatrix, while **elliptic islands** around fixed points can be found starting at $y \approx d/(2/\pi)$.

Near a **resonant zone** the distances at which exist **lower and upper invariant curves** and **curves confining islands** around periodic elliptic points, depend of both **upper and lower splittings**.

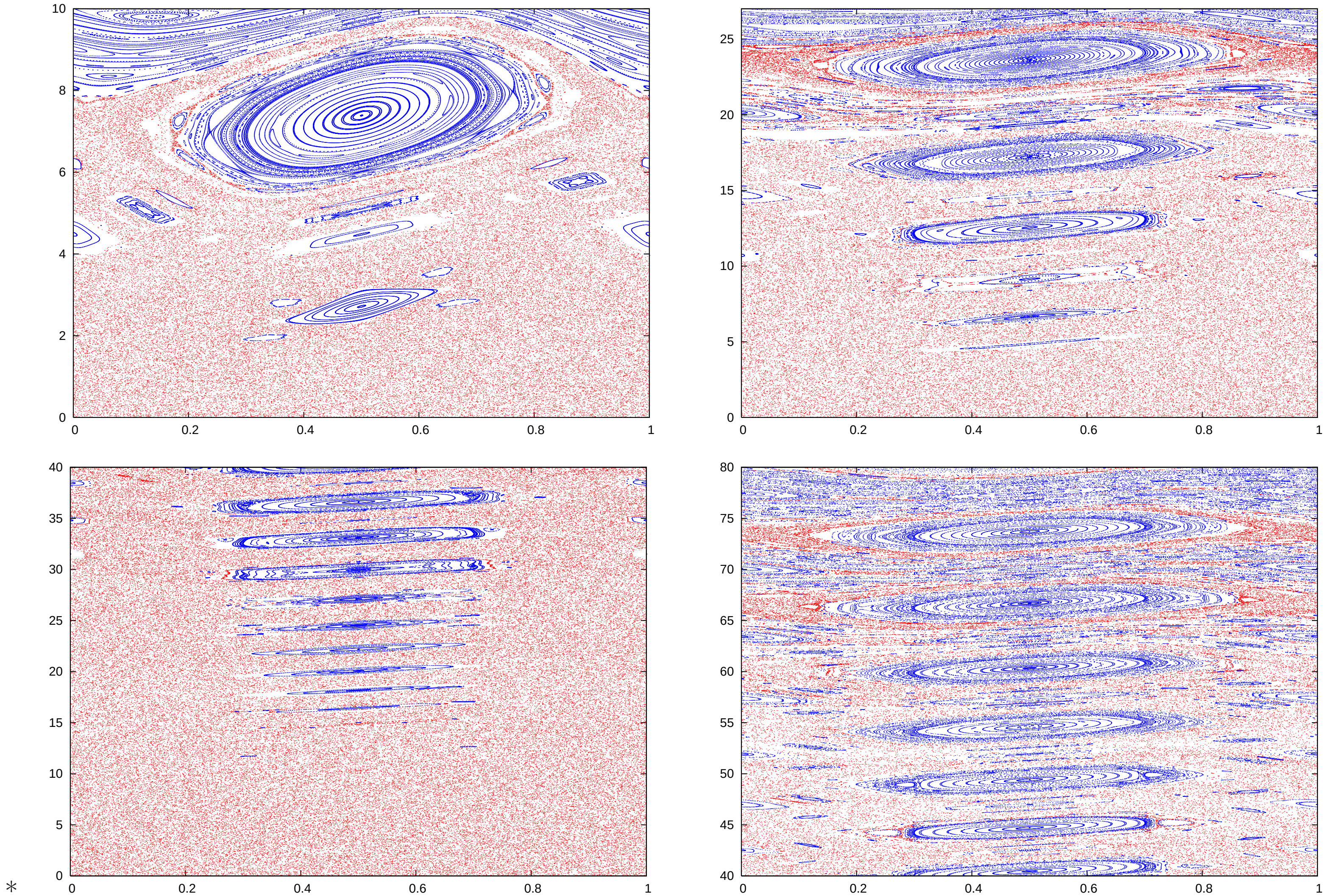
As an example we return to the i.c. confining islands of period 4. (page 16)

A computation of the splitting angle **in the outer and inner homoclinic connections** gives the values $s_o \approx -0.951063 \times 10^{-2}$, $s_i \approx 0.294215 \times 10^{-58}$.

From this one can derive a **composed separatrix map** and predict the distance from the manifolds at which **stable islands and invariant curves** appear in the domain bounded by the initial pieces of the manifolds.

These values agree with the results obtained by **direct simulation**, but one should be careful about **how this depends on the place where we look**. See lower plots.

For the **outer islands** and the **existence or not of i.c.** one requires the use of the $BSPM$. Locally, in what concerns nearby islands, one can approximate again by the SPM.

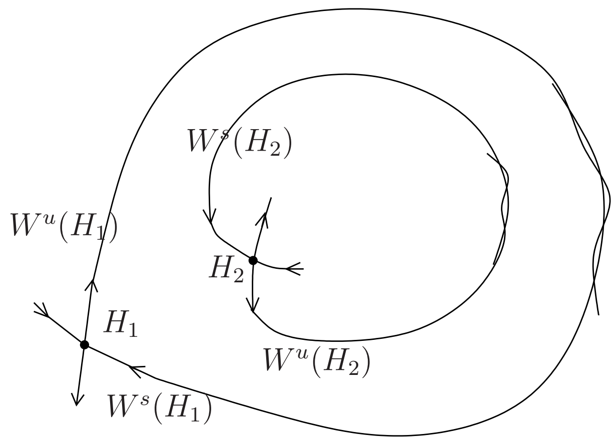


Dynamics of the separatrix map for $a = 0$. 1st row left plot corresponds to $b = 1$ while right one to $b = \sqrt{10}$. 2nd row plots correspond to $b = 10$ in different domains ($y < 40$ and $y > 40$ respectively). Blue (resp. red) domains correspond to regular (resp. chaotic) orbits.

Models in a Birkhoff zone: The biseparatrix maps

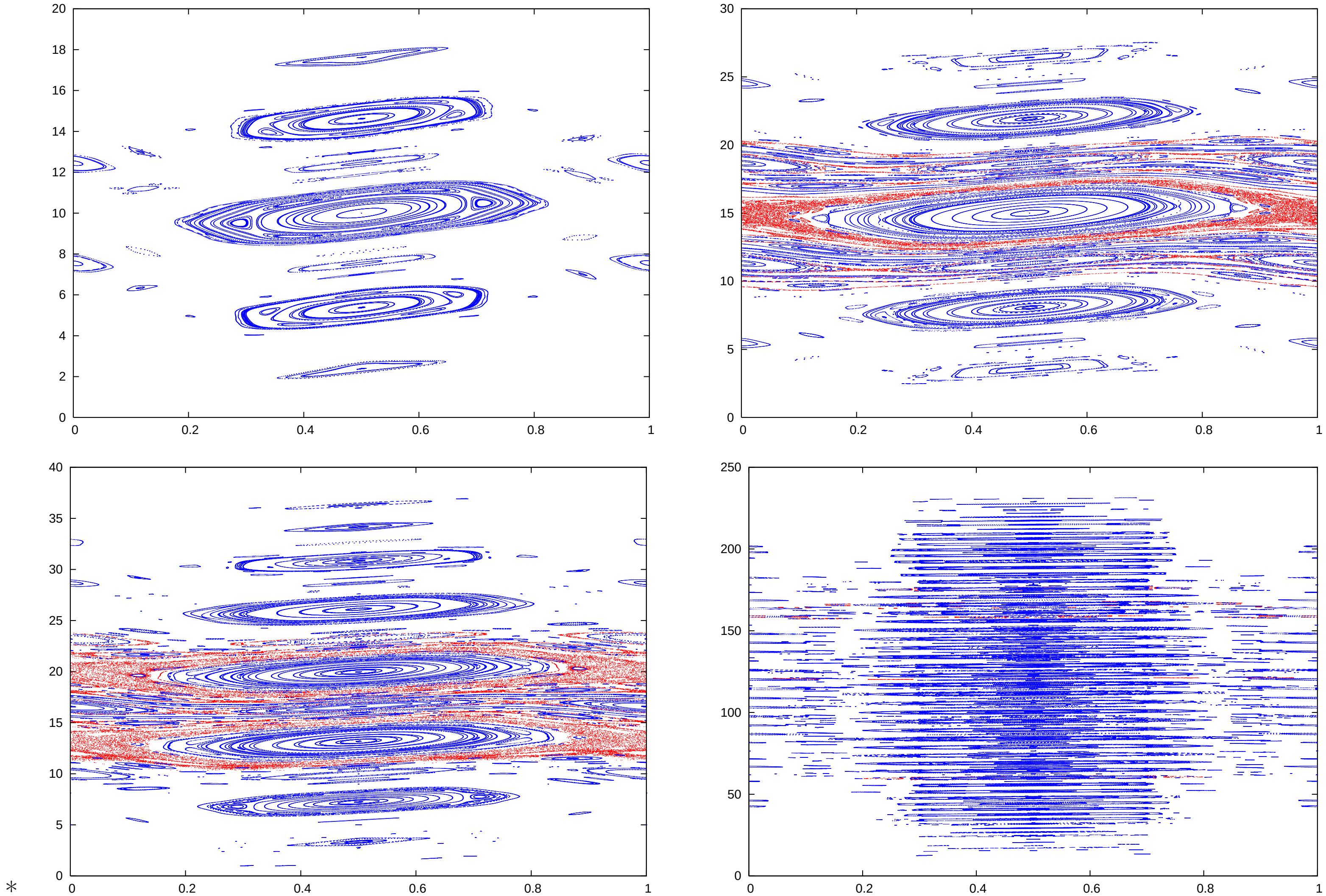
In its simplest presentation is given by

$$BSPM\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}=\begin{pmatrix}x+a+b_1\log|\bar{y}|-b_2\log|d-\bar{y}|\\y+\sin(2\pi x)\end{pmatrix}.$$

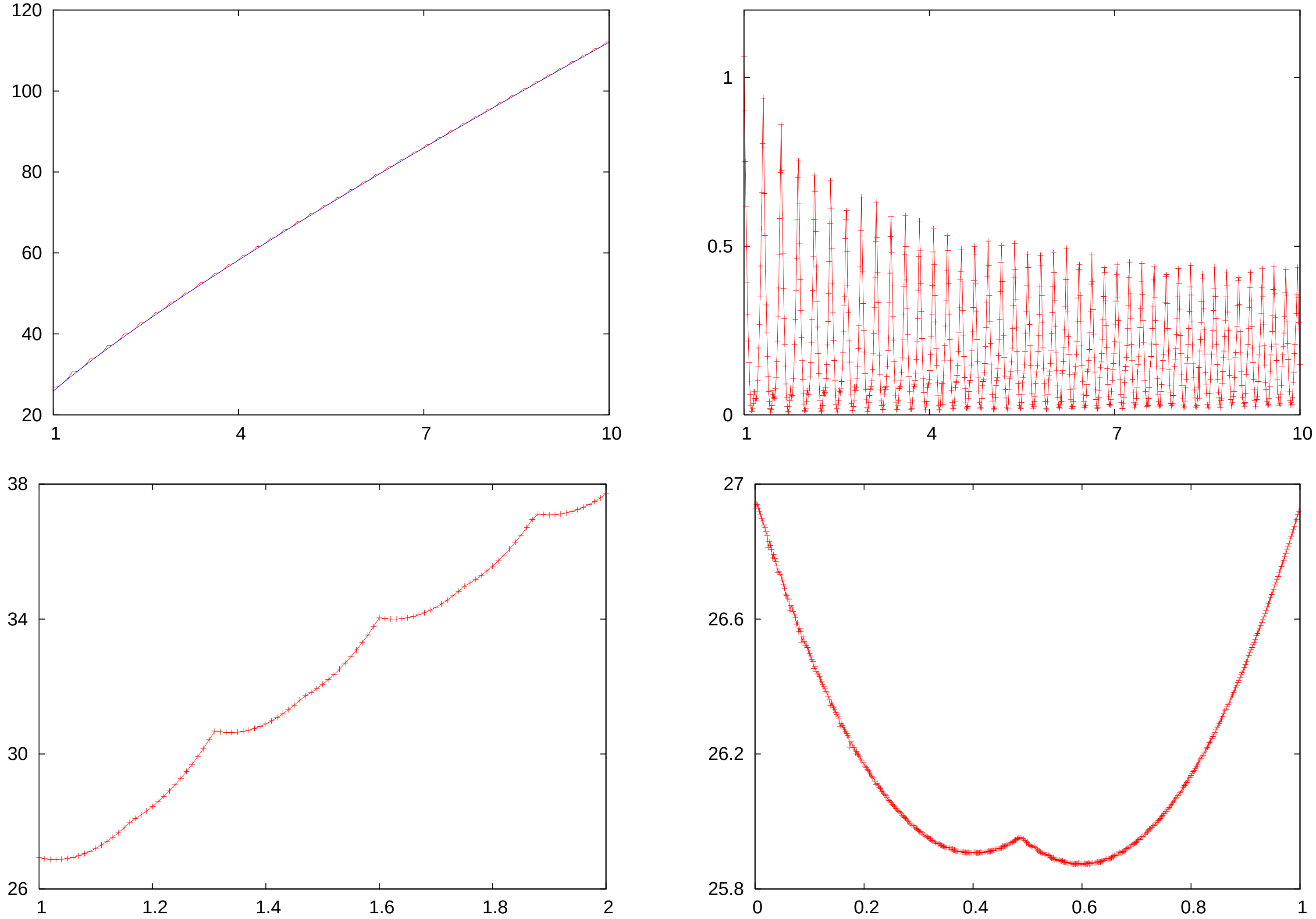


$BSPM$ is a model of the **dynamics between separatrices associated to different fixed or periodic points** (e.g. **"last" invariant curve** or **Birkhoff zones**. Requires a **suitable scale in** y.

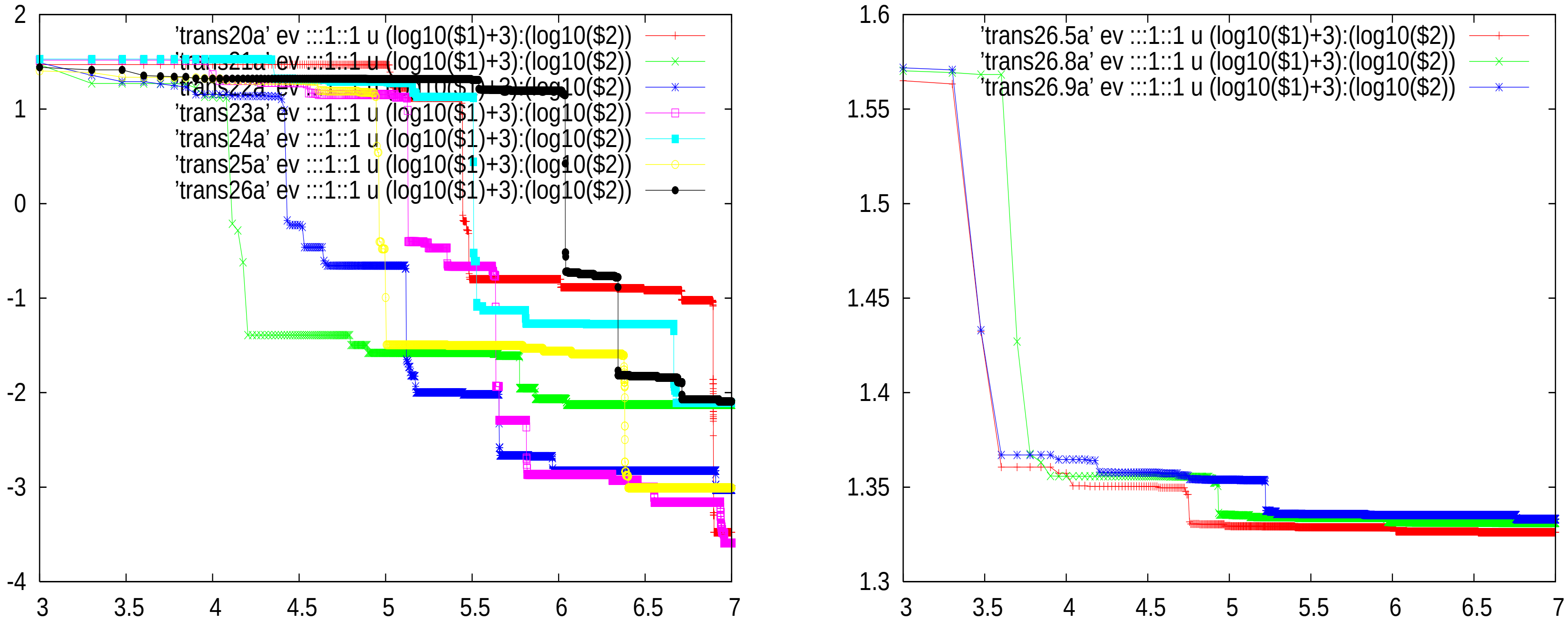
For $d>>1$, $y>>1$, $d-y>>1$ can be **approximated by** SM **with parameter** $b_1/y+b_2/(d-y)$. In particular $(\sqrt{b_1}+\sqrt{b_2})^2/d>\varepsilon_G$ prevents the **existence of rotational invariant curves**.



Dynamics of the biseparatrix map for $a = 0$. From left to right and from top to bottom: $(b_1, b_2, d) = (1, 1, 20)$, $(1, 1, 30)$, $(1, 3, 40)$, and $(10, 12, 250)$. Blue and red denote, respectively, regular and chaotic domains. The white areas correspond to points leaving the domain $0 < y < d$.



Examples of the **goodness of prediction**. Top: for $a = 0, b_1 = 1$ value of d as a function of b_2 to start to have invariant curves $0 < y < d$. Right: value-prediction. Bottom: Nonuniform character. Right: minimal d as a function of a for $b_1 = b_2 = 1$.



Examples of the **transition across a Birkhoff zone**. For $a = 0, b_1 = 1, b_2 = 1$, values of $d = 20, 21, 22, 23, 24, 25, 26$ (left) and $d = 26.5, 26.8, 26.9$ (right). Taking 100 initial points at distance < 0.1 of lower invariant curve we present the evolution of minimal distance to upper curve as a function of iterates in $\log_{10} - \log_{10}$ plot

Critical $d \approx 26.9287$.

Statistical properties: Rates of escape and some difficulties

Relevant to know: not only the measure of the set of points which escape, but also the **rate of escape**. Equivalently: **fraction of points subsists as a function of the number of iterates**.

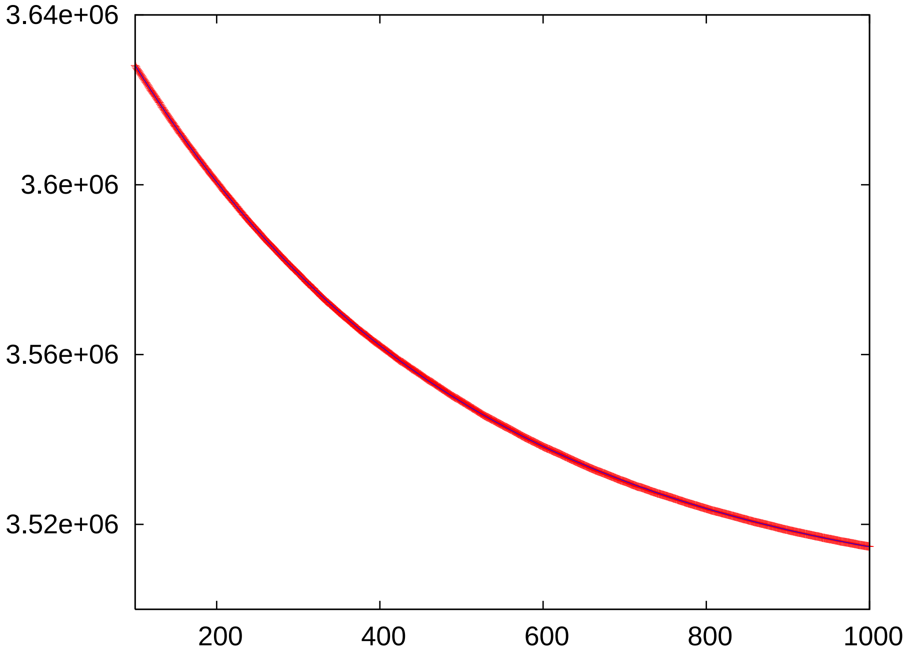
The left upper plot in next page shows a **typical result** after a **transient** of 10^8 iterations. The horizontal axis displays number of iterations, in millions, and the vertical one the number of subsisting points $p(n)$. The value $c = 0.531$ has been used. The results can be **fitted** by

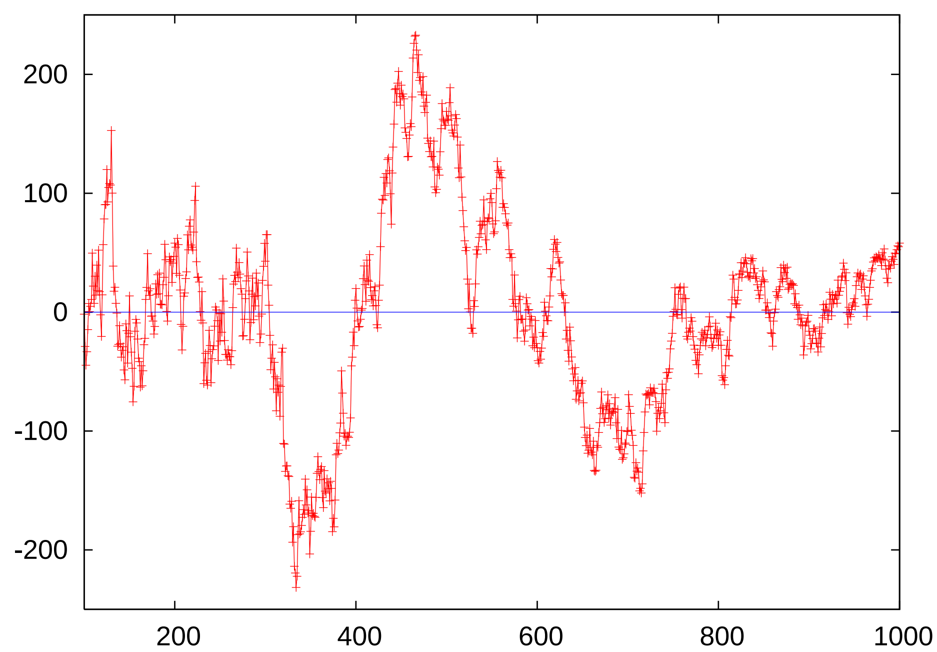
$$q(n)=a+b\exp(-rn),\ a\approx 3.500215\times 10^6,\ b\approx 0.162878\times 10^6,\ r\approx 0.00241827,$$

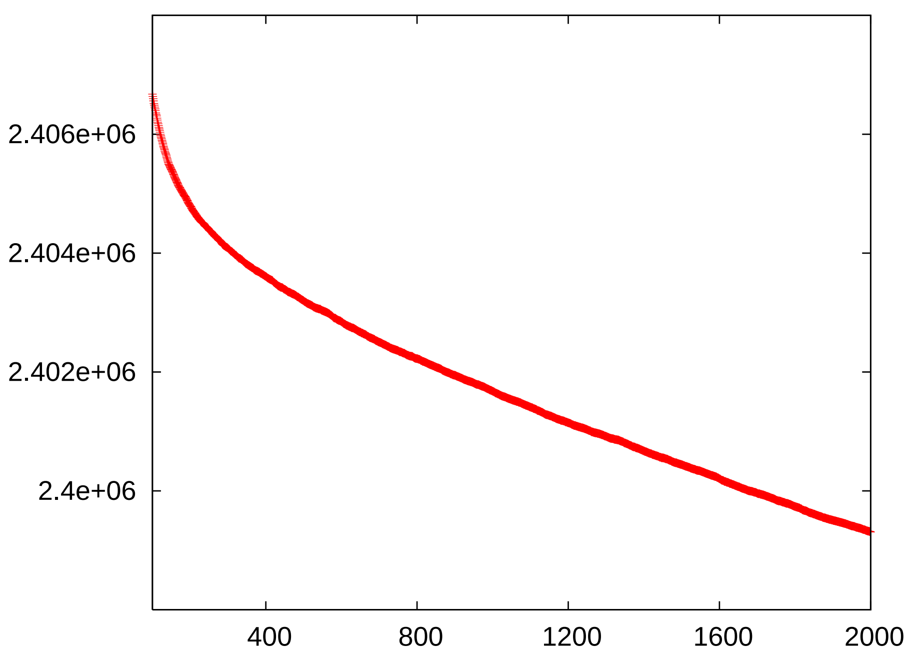
where n is expressed in millions of iterations.

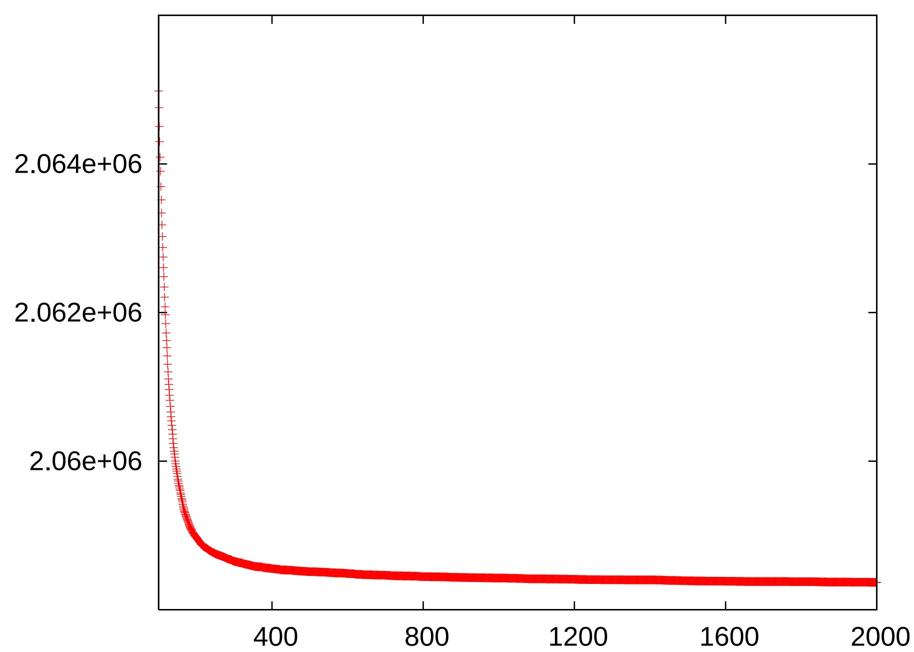
The right upper plot shows $p(n)-q(n)$, giving an idea of the **accuracy of the fit**. This behaviour is **coherent** with the rate of escape of a **random walk**.

But for values $c = 0.765,\ 0.766$, in lower row, the situation is different. Fits like $q(n)$ for **different intervals of** n show quite **different values of** p, q, r.


3.64e+06
3.6e+06
3.56e+06
3.52e+06
200
400
600
800
1000


200
100
0
-100
-200
200
400
600
800
1000

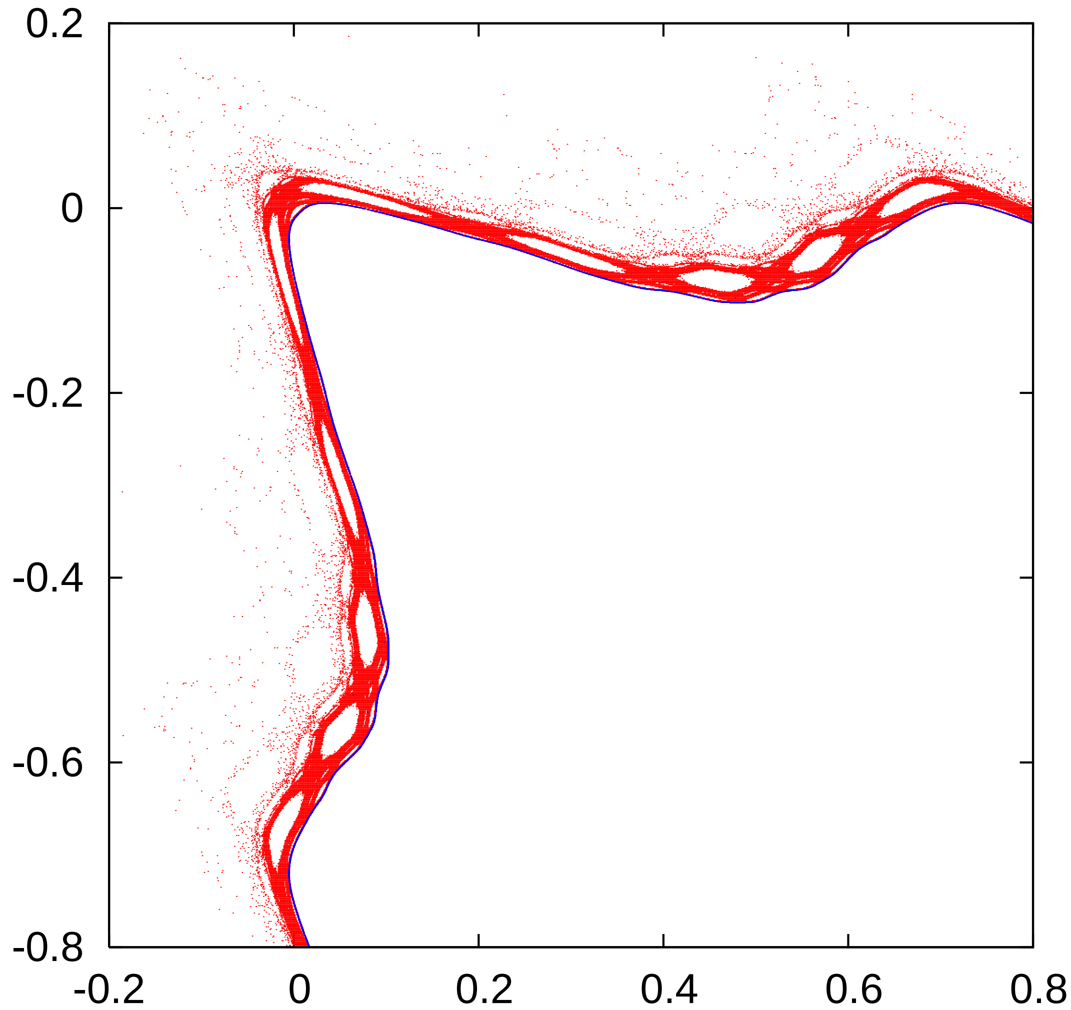

2.406e+06
2.404e+06
2.402e+06
2.4e+06
400
800
1200
1600
2000


2.064e+06
2.062e+06
2.06e+06
400
800
1200
1600
2000

In a chaotic region between **confining i.c.** the dynamics, in principle, looks like a **random walk**. Points seem that tend to **fill a region of positive measure** with some **limit density**. But the **transient can be large**.

A similar behaviour is observed for **escaping points**. The right upper plot in next page shows, for $c = 0.764$ the dynamics of an **escaping point**. For nearly 7×10^{10} iterates the point moves in a **narrow strip in blue**. Later it **moves to the region in red** and finally it escapes.

It has to be remarked that at a distance $\approx 8E-9$ of the initial point taken, there are points **sitting on a rotational i.c.**, like $x=-y=0.00545430414$. The plot would make them coincide with the blue one, at plot's resolution.

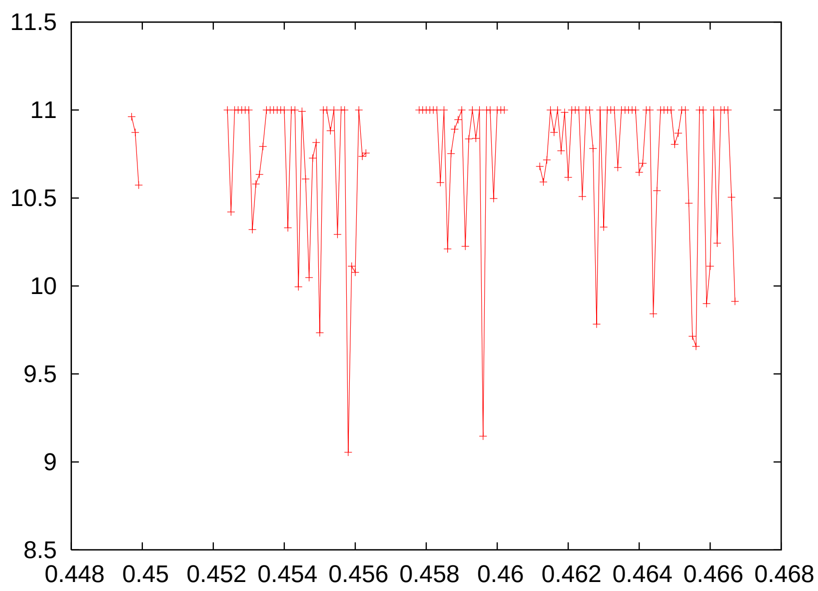

0.2
0
-0.2
-0.4
-0.6
-0.8
-0.2
0
0.2
0.4
0.6
0.8

One of the few values (using 0.001 step in c) for which strong **difficulties occur** is $c = 0.765$.

It seems that still **a rotational i.c. exists** around period 5 islands.

This is **not true!**.

Next plot shows the following. Take points on $y = -x$ between the **"last"** i.c. around the elliptic fixed point and the "last" i.c. around period 5 points. Some of them are inside **high period islands**. Take 124 equiespaced points in the complement. Even after 10^{11} iterates, only **59 escaped**. The plot shows $\log_{10}$ of the number of iterates vs initial position.



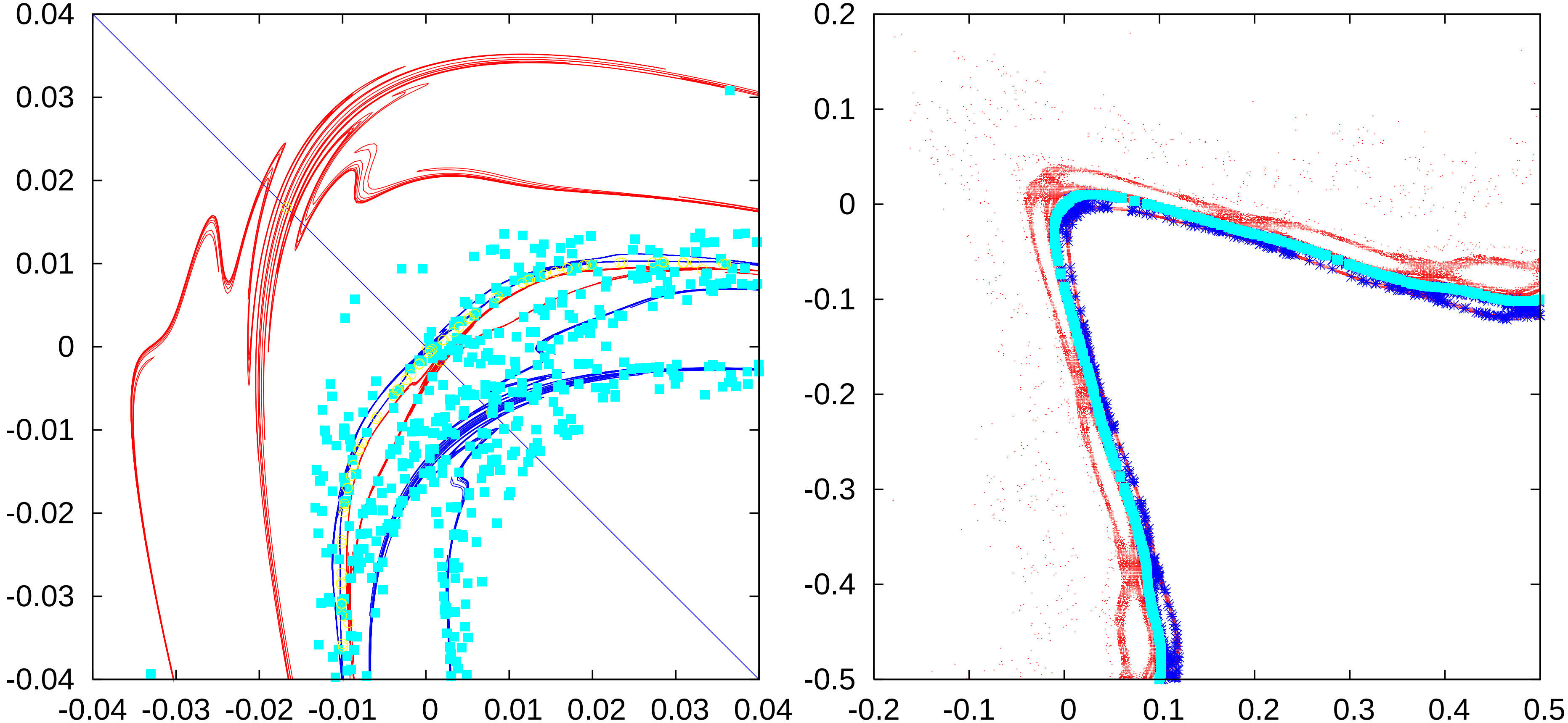
What happens to a curve when it is destroyed?. It remains a **Cantor set** or **Aubry-Mather set**. It is relevant to know **a geometrical interpretation of the gaps** and the **width of the gaps**.

Gaps can be seen to be produced by the **existence of heteroclinic intersections** between invariant manifolds of **periodic orbits** sitting to different sides of the invariant curve, before this one is destroyed.

From an analytic point of view, gaps are related to the **divergence of the solutions of the homological equations**, the linear steps in the construction of the invariant curves.

The answer to last question above is given again by **KAM**. It has been **checked in the simulations** that invariant curves which subsist for longer have **noble rotation numbers**.

Another relevant item when we talk of **last i.c.** in a loose way, is that, **locally, around an i.c., there is also a Cantor set of i.c.** whose relative measure tends to 1 on the curve. There is very little known about **quantitative estimates** in this direction.
Also instructive: **how long it takes to cross different cantori**.



For $c = 0.765$. Left: Invariant manifolds of points of periods $26, 31, 88, 145$ (stable in red, unstable in blue), an approximation of a Cantor set (in yellow) and the orbit of an inner point (in pale blue) which **succeeds in crossing the Cantor sets**. Right: A Cantor set in pale blue, the orbit of an escaping point in red and the first 5×10^6 iterates on the escaping point, showing one point every 1000 iterates, in dark blue.

Refined numerical tools

An efficient method to detect the **existence or not of rotational invariant curves** proceeds as follows (described in terms of action-angle):

- Iterate an initial point. Produce a histogram on how many iterates have the angle in a given interval.
- If there are empty intervals we suspect the point is in an island.
- Otherwise look for the point of minimal action. Select the points which fall in a narrow interval around it.
- Fit by a polynomial (use orthogonal pol., check variance reduction, etc).
- Subtract the fit from data. If invariant curve, we should only see the effect of roundoff.
- Several devices allow to jump over islands, chaotic regions, etc.

Additional examples I: The outer cometary problem

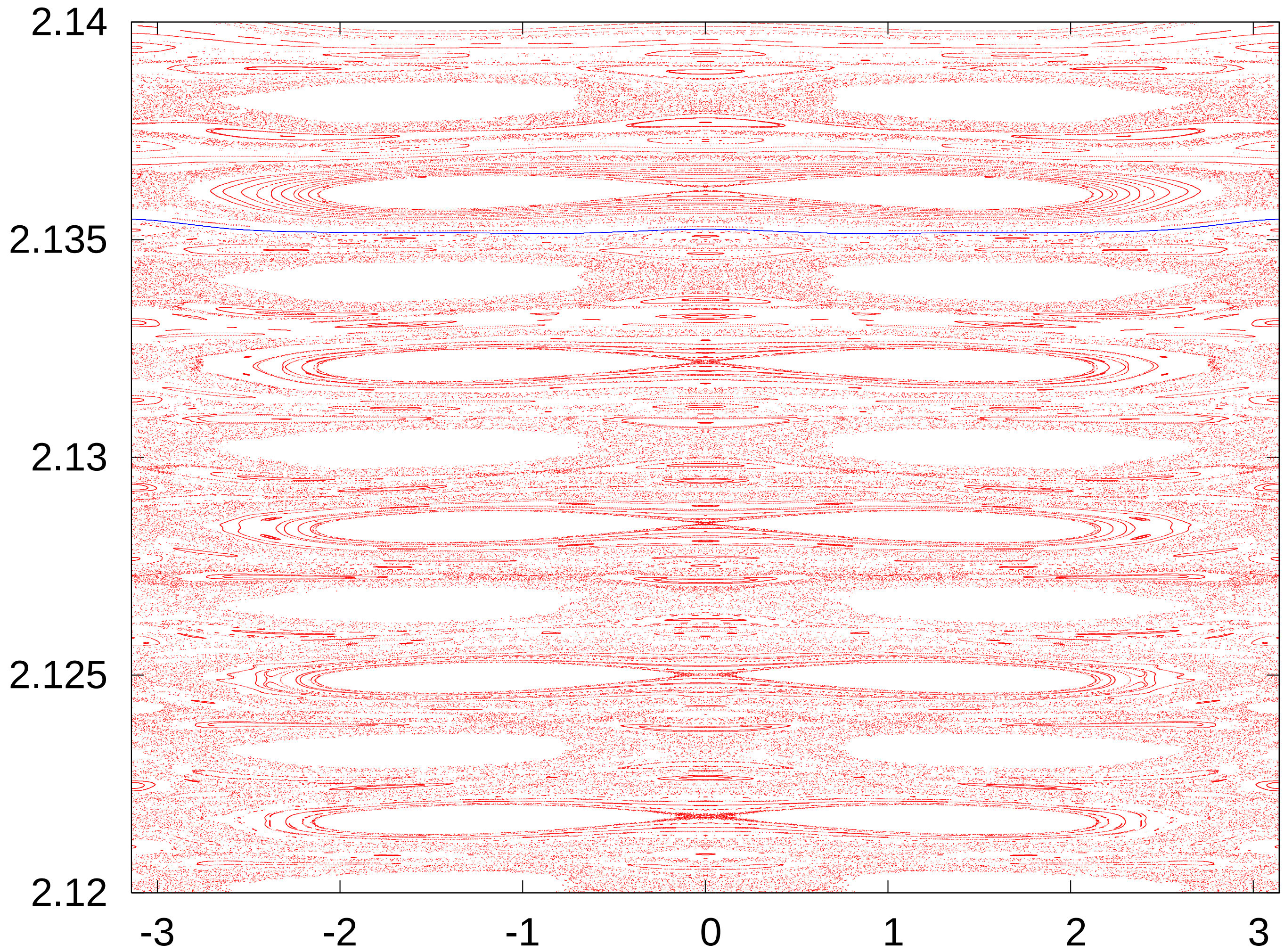
Consider the **planar circular RTBP** with primaries **Sun-Jupiter**, located at $(\mu, 0)$, $(\mu-1, 0)$ in synodic (rotating) coordinates. $\mu \approx 1/1047.3486$.

Question: Given a value of the **Jacobi constant** C and considering motions **external to Jupiter's orbit** where is the **"last" invariant curve** which **prevents from escape**?

How it is evolving with C?

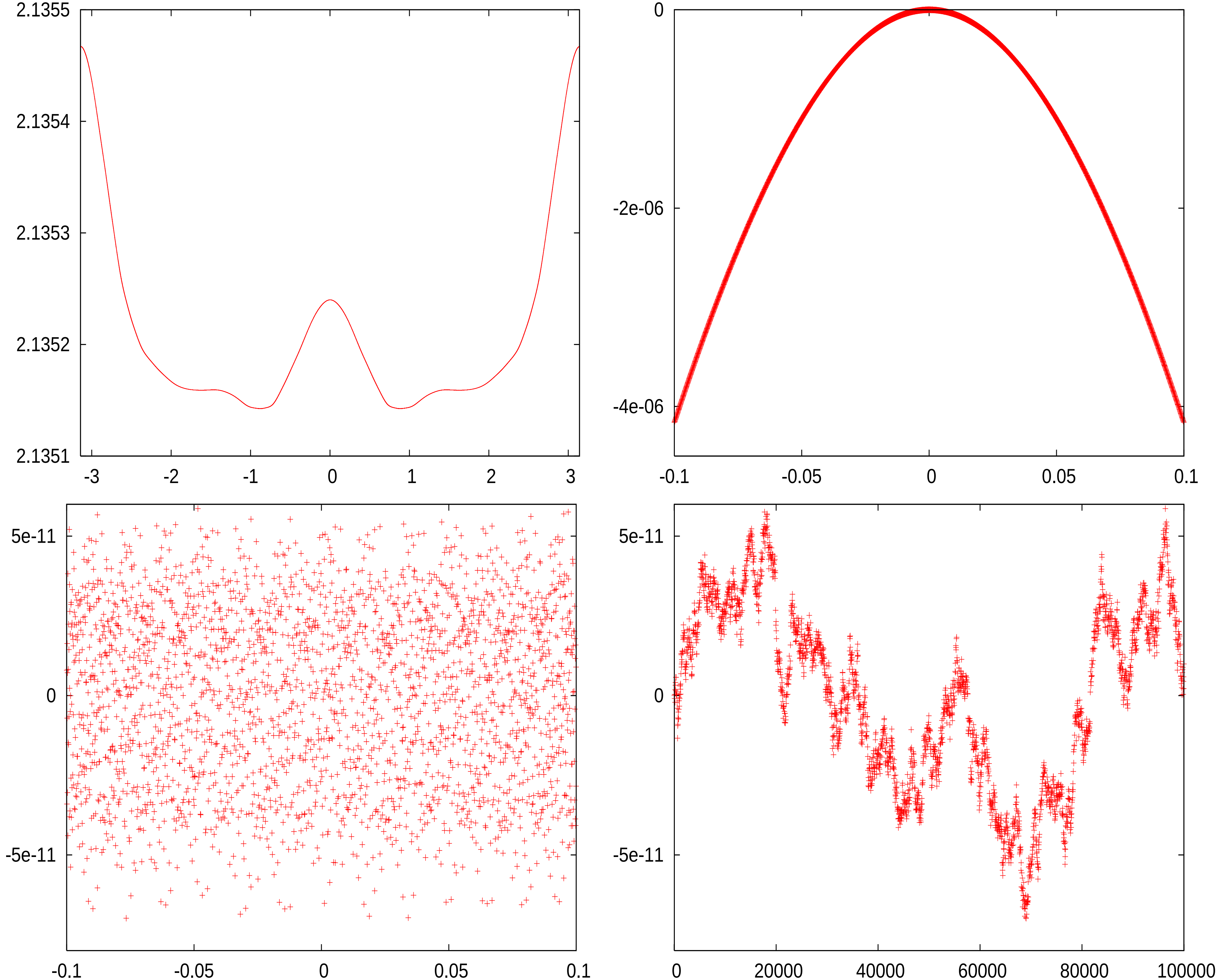
Some work in progress...

Related questions: motions inside the orbit of Jupiter, motions around Jupiter, motion around triangular points, dependence in μ, etc



Poincaré section using **pericentre passages**: radius as a function of angle. Several initial conditions and 1000 iterates for each one. In blue: **approximate last invariant curve**. Jacobi $C = 4$.

Details on curve, magnification, fit errors and evolution with iterates.


2.1355
2.1354
2.1353
2.1352
2.1351
-3
-2
-1
0
1
2
3
0
-2e-06
-4e-06
-0.1
-0.05
0
0.05
0.1
5e-11
0
-5e-11
-0.1
-0.05
0
0.05
0.1
5e-11
0
-5e-11
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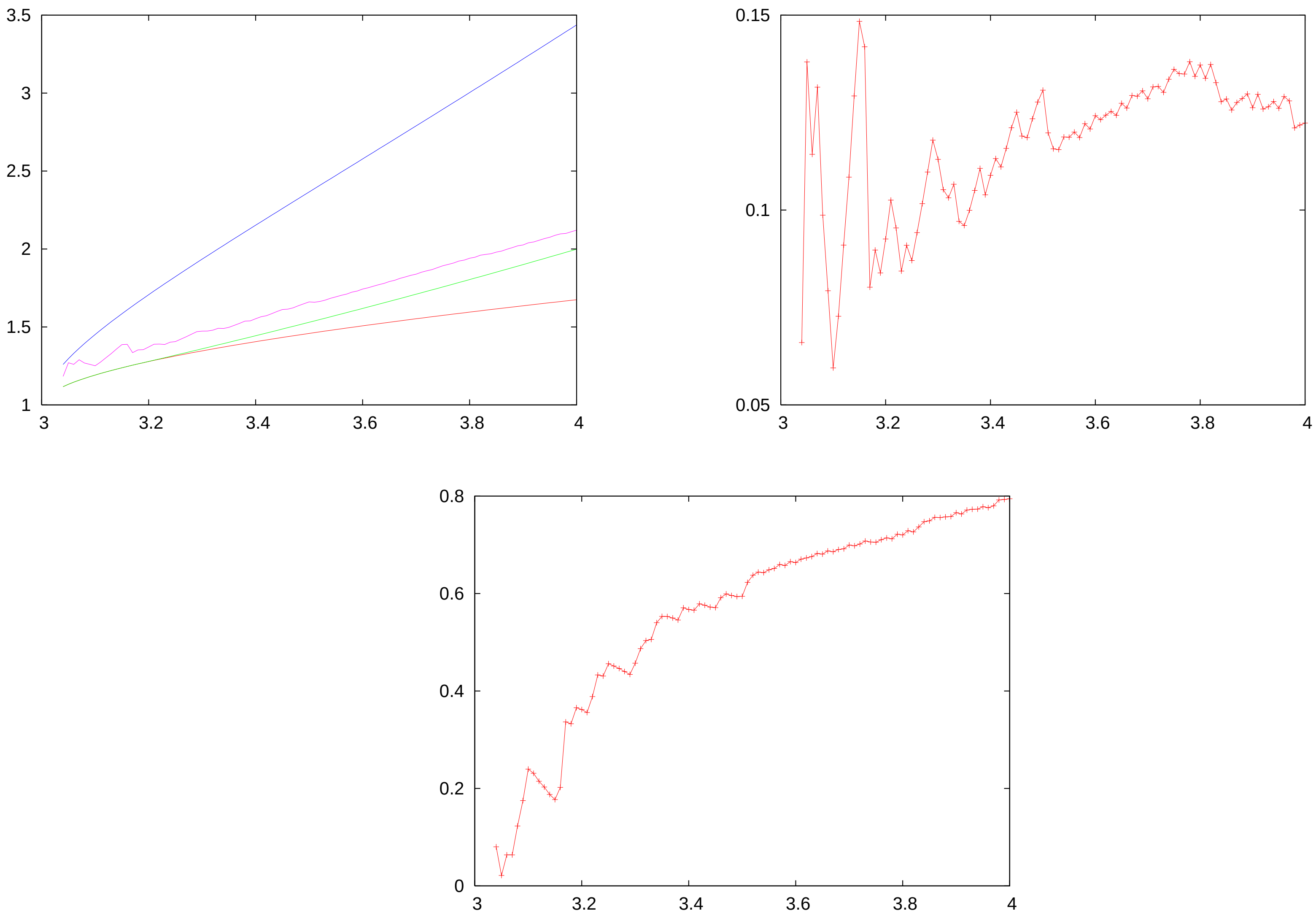
40000
60000
80000
100000



Top left: As a function of C: minimal value of x (red), x with osculating $h_K = 0$ (green); x with osculating $e = 0$ (blue); x for rough location of last invariant curve (magenta). Right: Difference: x last i.c. - x with $h_K = 0$. Bottom: Osculating eccentricity of last invariant curve.

Some open questions

The simple examples shown suggest **infinitely many questions**. Just a sample, covering different aspects:

- Prove the **positivity of the metric entropy** in the 2D case, for generic, analytic, non-uniformly hyperbolic APM.
- Prove the **Poincaré conjecture**: density of periodic points.
- Look for problems of **separatrix splitting** in higher dimension.
- Study **paradigmatic models** similar to Hénon, standard, separatrix and biseparatrix maps (and other maps who would be convenient to introduce) for **symplectic 4D, 6D maps**.
- Analyse the **breakdown of tori** in higher dimensional cases.
- Study the **Cantor sets with a given rotation vector** in higher dimensional cases.
- Find the analogous of **noble numbers in** $\mathbb{R}^n$, $n > 1$.
- Find all kinds of **codimension 1 manifolds** which are relevant for the dynamics.

- Produce **quantitative estimates** on the rate of escape and on the rate of diffusion for symplectic 4D, 6D maps.
- Impact of the fact that, in many systems, we are **mainly interested in the transient regime**. The phenomenon is not alive (or the model is no longer valid) after some time.
- Introduce **different kinds of noise**.