Studies in Dynamics: From Local to Global Aspects

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Motivation. Domains of validity of theoretical results

We are interested in **dynamical problems**, given either by a **discrete system**, by an **ODE** or a **PDE**.

Applications appear in all **scientific and technical** areas.

As it happens in **many domains of Mathematics** most of the results follow in one of these classes

- They give a very precise description of some phenomena but it has a **very local character**,
- They give a global information but it is **very rough**.

In general there is a **big gap** between the **domain of validity of some result given by a mathematical statement** and the **true domain** where the result is still true.

It is also **highly relevant** to know the precise reasons which **prevent a result to be true** beyond some limit.

Local vs Global methods

Dynamics is very sensible to the dilemmas

Local vs Global methods and results

and also

Pure existential vs Constructive results

An **ultimate goal** would be to have **global precise** results. Probably **beyond the possibilities of Mathematics in the near future**.

But a combination of **partial models** with **experiments** can lead to a **reasonable understanding**, at least of **simple problems** and show also many **open questions**.

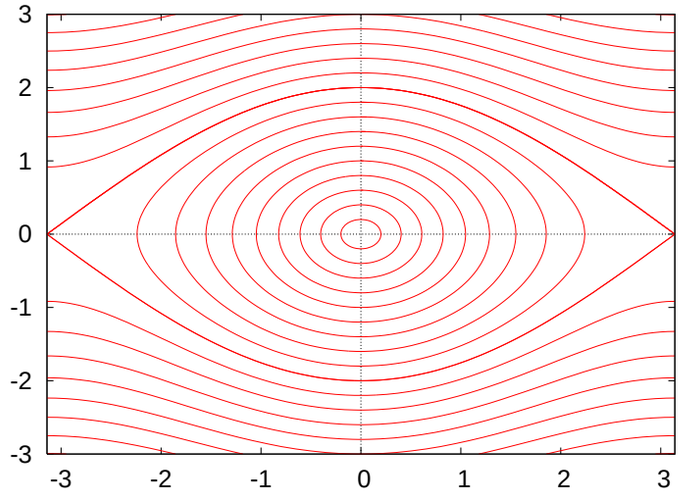
CAP can also be interesting in some cases, when the **validity of a result** can be reduced to check a **finite number of inequalities**.

The mathematical pendulum and its perturbations

One of the simplest problems, with motion described by

$$x' = y, \qquad y' = -\sin(x)$$

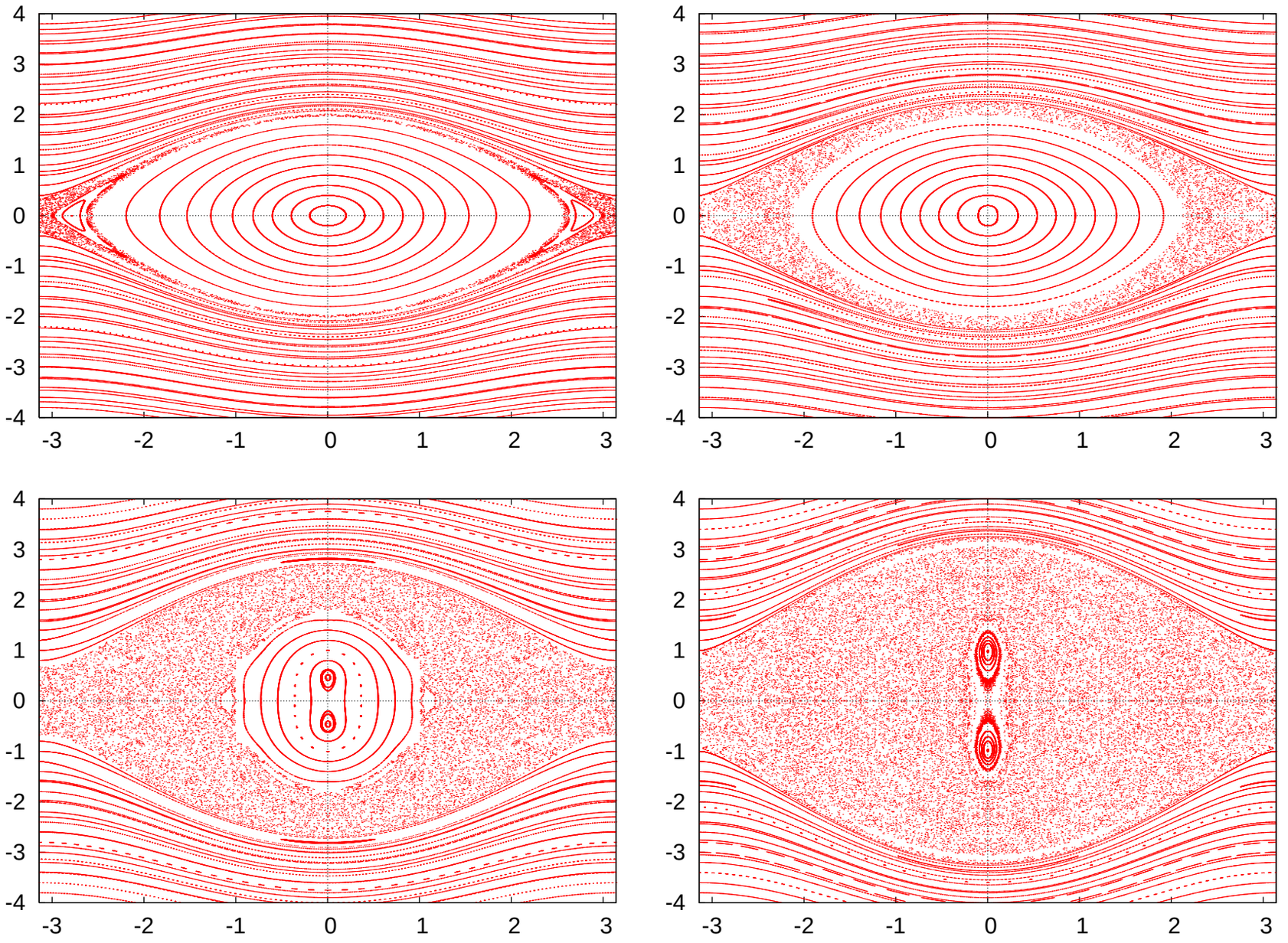
and well known phase portrait.



Let us see how the dynamics changes when we perturb periodically, e.g.

$$x' = y, \qquad y' = -(1 + \beta \sin(t)) \sin(x).$$

The length changes with period 2π and amplitude β. See Broer-Hoveijn-van Noort-S-Vegter, *J. of Dyn. and Diff. Eq.* **16** (2004), 897–947. Illustrations of the **time-2π map** for $\beta = 0.01, 0.1, 0.5, 1.0$.


4
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Some aspects: KAM theory, splitting of separatrices and statistical properties

Some questions appear (beyond **local bifurcations**):

- Why **some invariant curves subsist and others are destroyed**?
- What is the fate of the **separatrix** which before separated the **libration** and **circulation** regimes?

Answer to first question is given by **KAM theory** or, in this case, simply **Moser's twist theorem**, the main ideas of proof to be sketched.
It requires **three conditions**

- The unperturbed map must have non-zero **torsion**, i.e., **frequency changes with amplitude**,
- The **rotation number** ρ of the curves which subsist must satisfy a **Diophantine Condition DC,** $|\rho - p/q| > c|q|^{-\tau}$, for some $c > 0, \tau \geq 2$,
- The **perturbation** must be **sufficiently small**.

The three conditions are not independent: **the larger the torsion and the DC, the larger the admissible perturbation can be**.

Concerning separatrices, they **generically split** with an **exponentially small splitting with respect to a suitable parameter**, in some cases in which the dynamics can be reduced to a **perturbation of the identity** and in the **analytic category**.

Typical expressions for the measure of the splitting are of the form

$$|s| \approx A\varepsilon^r \exp(-c\varepsilon^{-d})$$

for suitable $A, r, c, d > 0$, where ε is a relevant small parameter. See, e.g., Fontich-S, *Erg. Th. & Dyn. Systems* **10** (1990), 295–318 and Gelfreich-S, High-precision computations of divergent asymptotic series and homoclinic phenomena, *DCDS*, to appear.

Note that the formula can be interpreted in **both directions**:

- either to **produce an estimate** for given ε, or
- to find **up to which** ε the splitting is less than some value.

Even **small values of the splitting** can have a **relevant impact** in the long term dynamics, as we shall see.

Another relevant topic are **statistical considerations**.

As an example consider the system

$$\dot{x} = -8xy + \varepsilon z, \quad \dot{y} = 11x^2 + 3y^2 + z^2 - 3, \quad \dot{z} = 2zy - \varepsilon x,$$

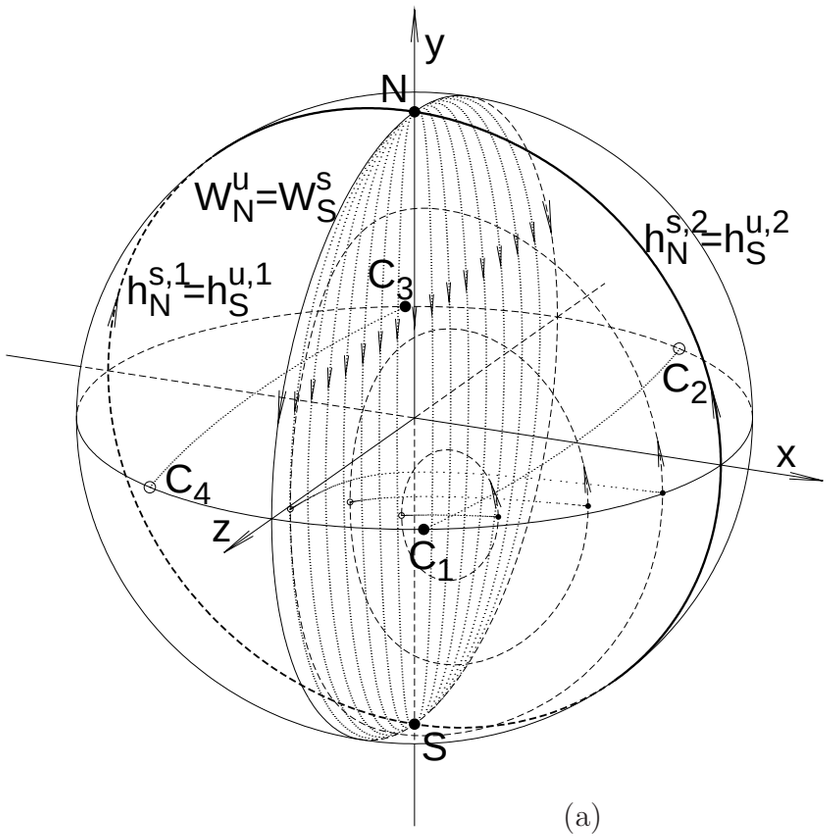
which has $\mathbb{S}^2$ invariant and is **integrable for** $\varepsilon = 0$, i.e., a set of full measure inside $\mathbb{S}^2$ is foliated by periodic solutions. See Neishtadt-S-Vasiliev, *Nonlinearity* **16**(2003), 521-557.

For ε arbitrarily small numerical experiments suggest the system to be **ergodic**. If this is accepted, one can derive several **probabilistic distributions** of different observables.

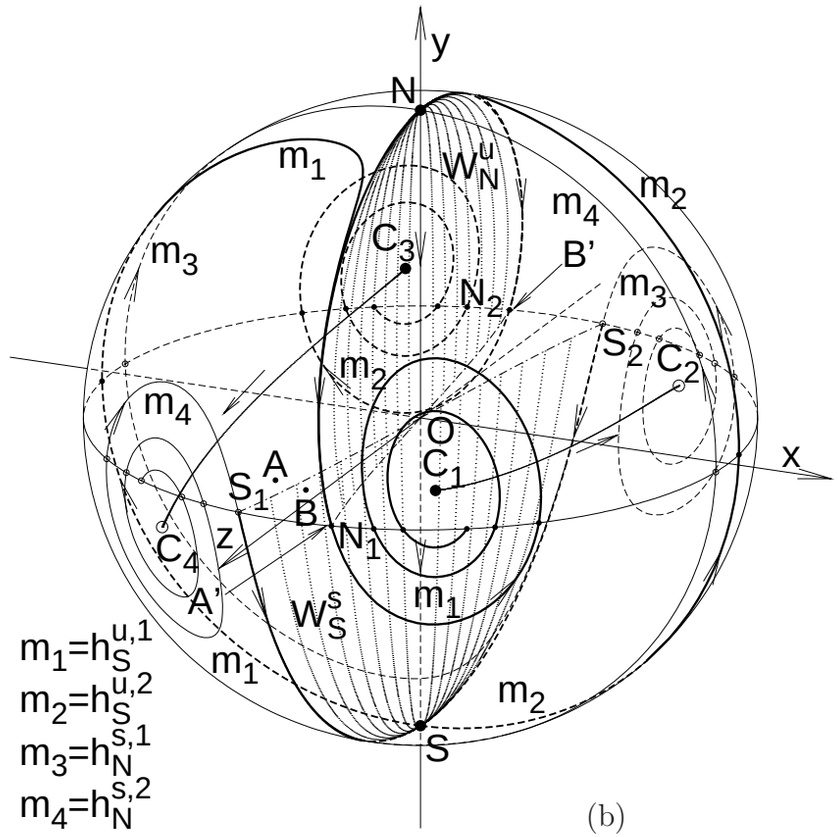
Predictions agree with **numerical estimates of the observables**. However it is **proved that the existing islands prevent the system to be ergodic**. But its **measure is negligible**.

Left: the invariant manifolds of north and south poles and several periodic orbits for the **unperturbed problem** $\varepsilon = 0$. Between C_1 and C_2 and between C_3 and C_4 there are lines of **fixed points**.

Right: The manifolds for $\varepsilon \neq 0$. Observe that the **2D one is splitted** by a quantity proportional to ε. Also it seems that the C points, which become **saddle-foci**, are **connected by 1D manifolds**. This is **not true**, but this splitting is **exponentially small**.

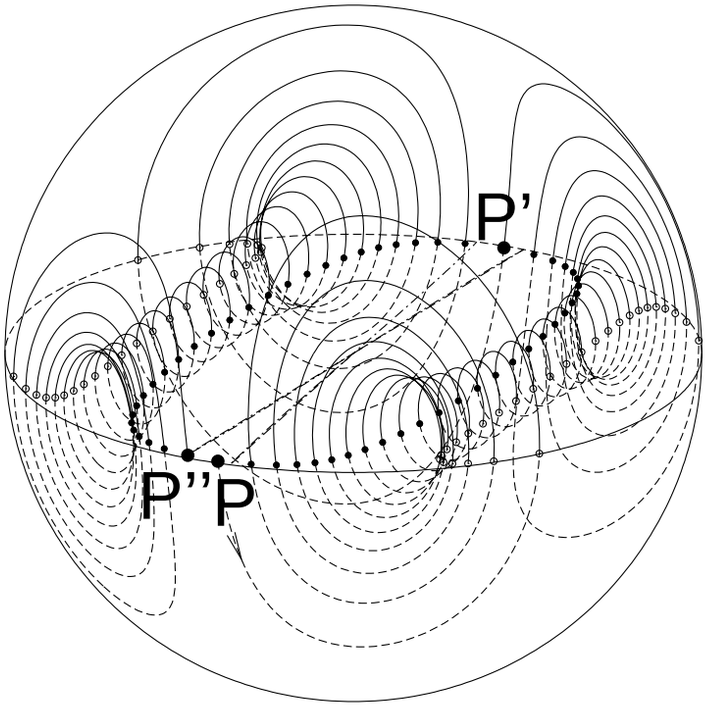


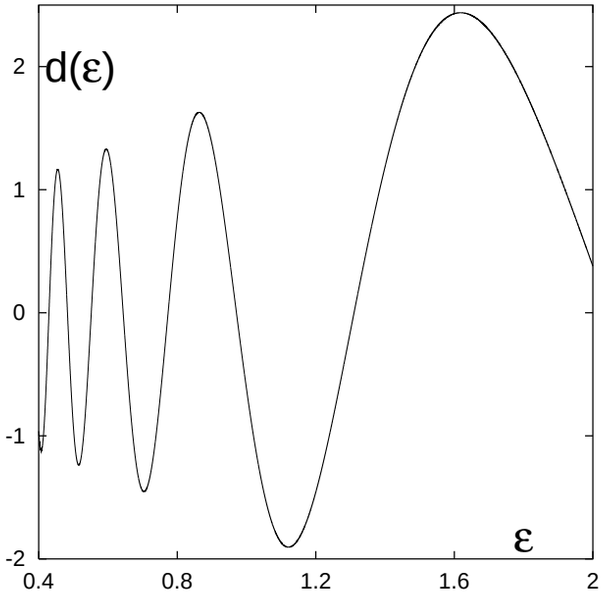
(a)



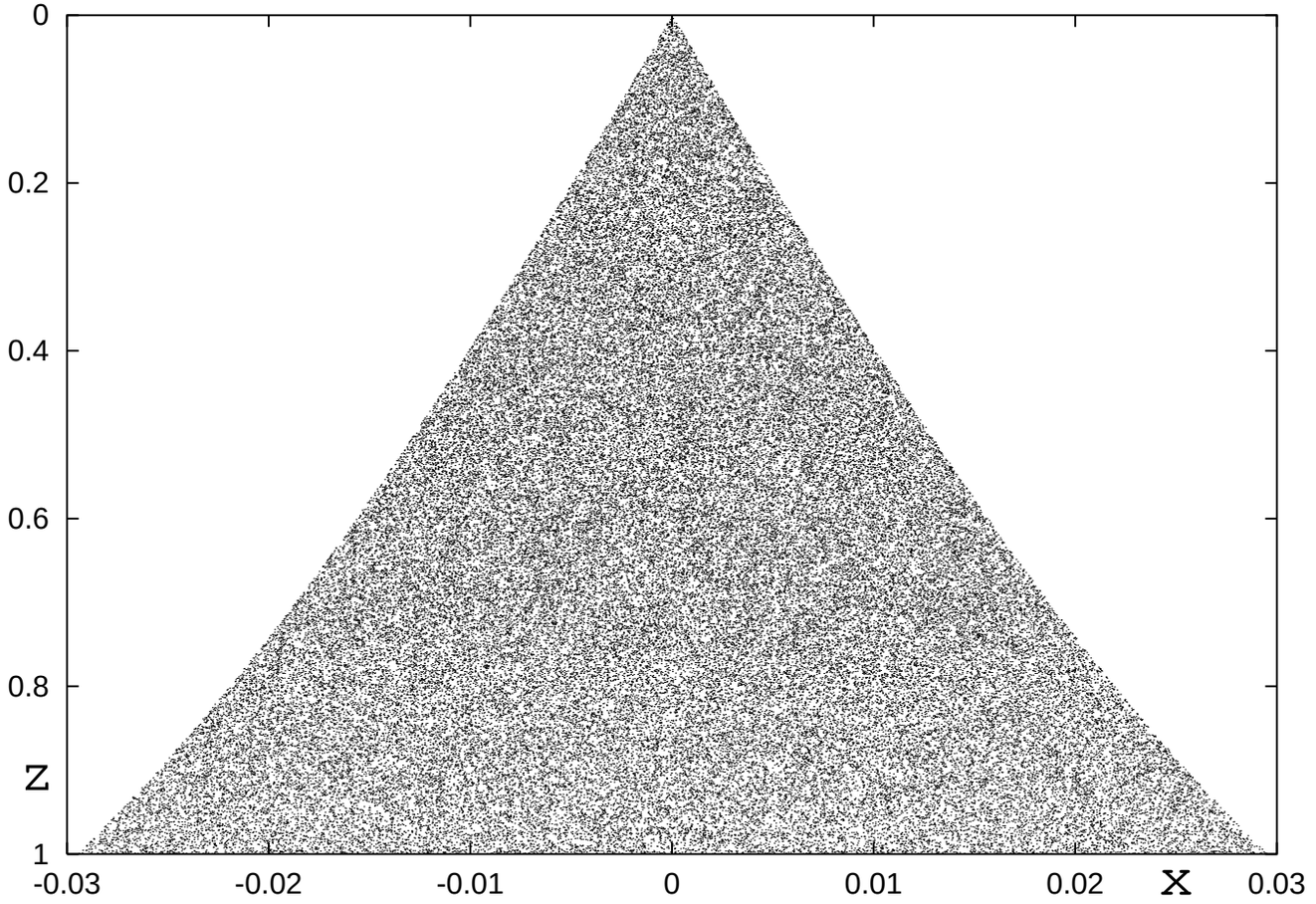
(b)

Left: A **typical orbit** of the perturbed system. Observe the **two sectors** in the equatorial plane.
Right: $d(\varepsilon) = \exp(11.9/\varepsilon)\, y^*(\varepsilon)$ vs ε, where $y^*(\varepsilon)$ is a **measure of the splitting** between the 1D manifolds emanating form C_1 and C_2.

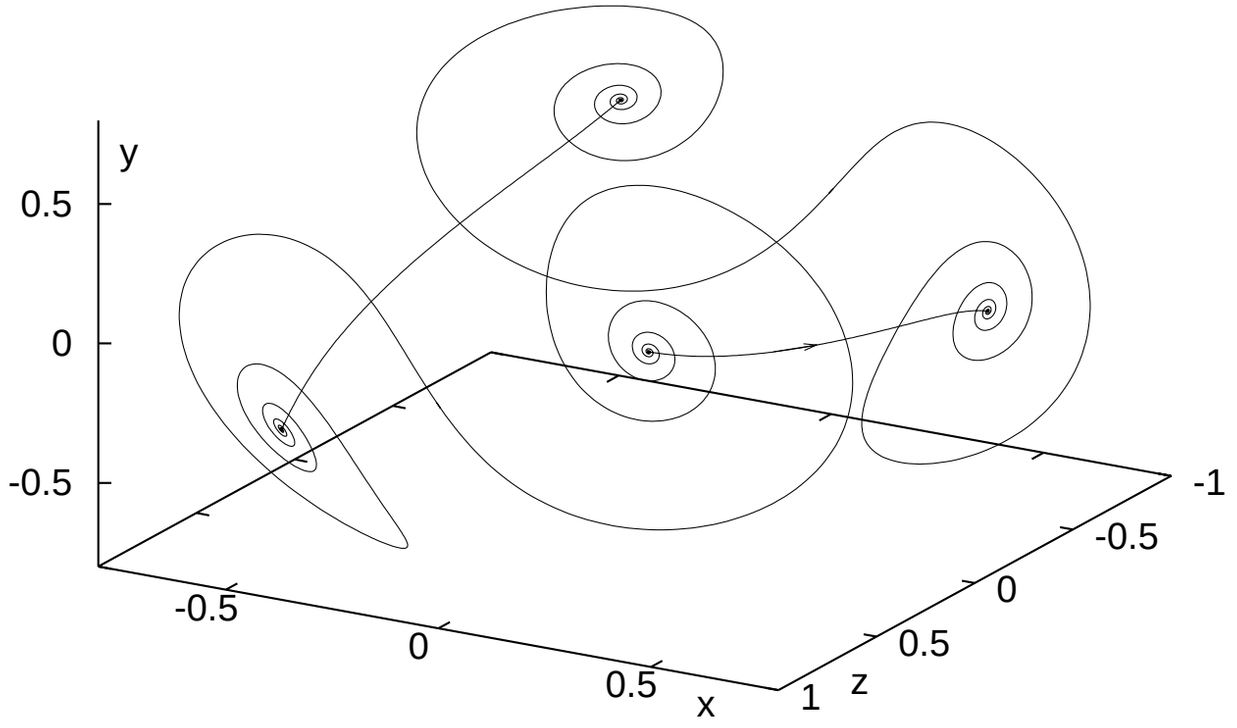




Poincaré iterates of a single orbit through one of the equatorial sectors.



A **stable periodic orbit** which passes **close to the 1D manifolds of the C_j points** and also **close to $\mathbb{S}^2$**. The area of the **stability island** on a section is $\approx 10^{-14}$ and it is located at 10^{-10} from $\mathbb{S}^2$.



A toy but generic problem: the Hénon map

Any quadratic area preserving map can be reduced to

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} = \begin{pmatrix} c(1-x^2)+2x+y \\ -x \end{pmatrix} \qquad x, y, c \in \mathbb{R}.$$

With this normalisation **fixed points** at H$= (-1, 1)$ and E$= (1, -1)$.

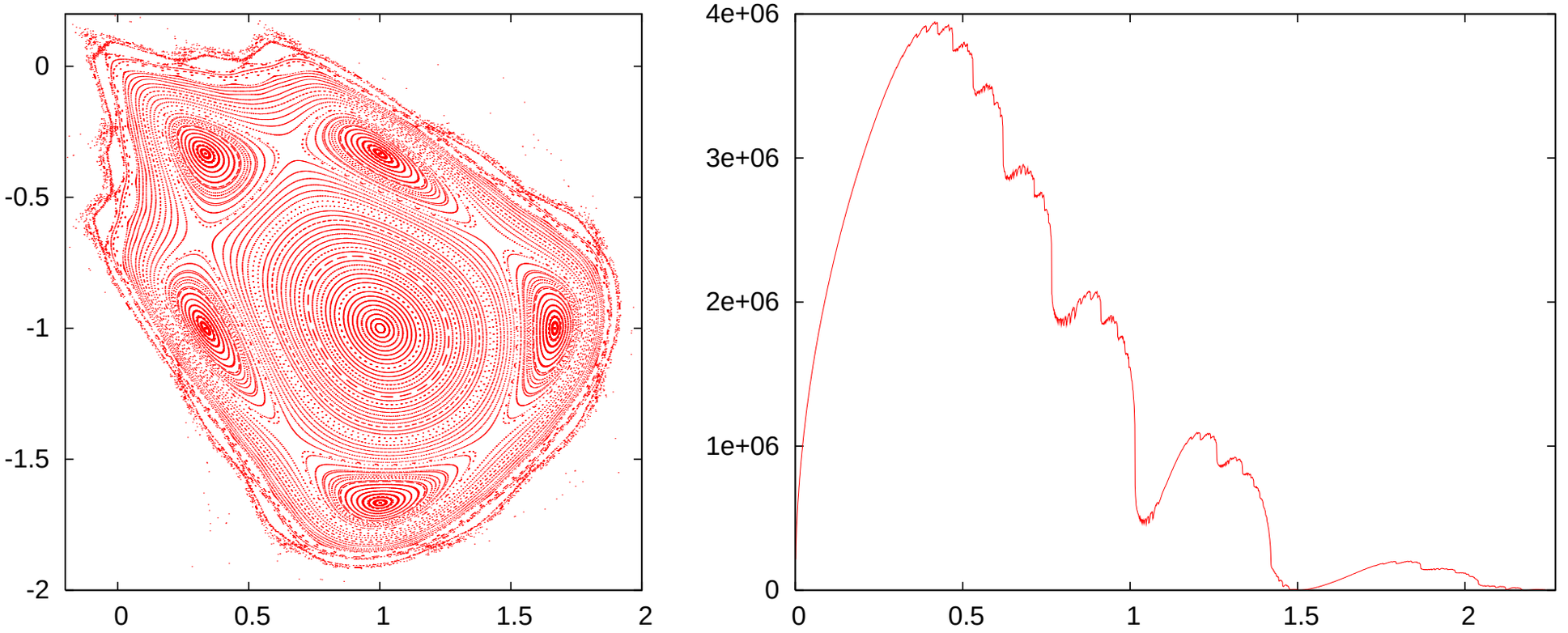
Last one **elliptic** if $c \in [0, 2]$. A **Normal Form** analysis proves **non linear stability** if $c \in (0, 2) \setminus \{1.5\}$.

It appears typically in **generic non-hyperbolic area preserving maps**, e.g., in **Poincaré sections of Hamiltonian systems with two degrees of freedom** in **many parts of the phase space**.

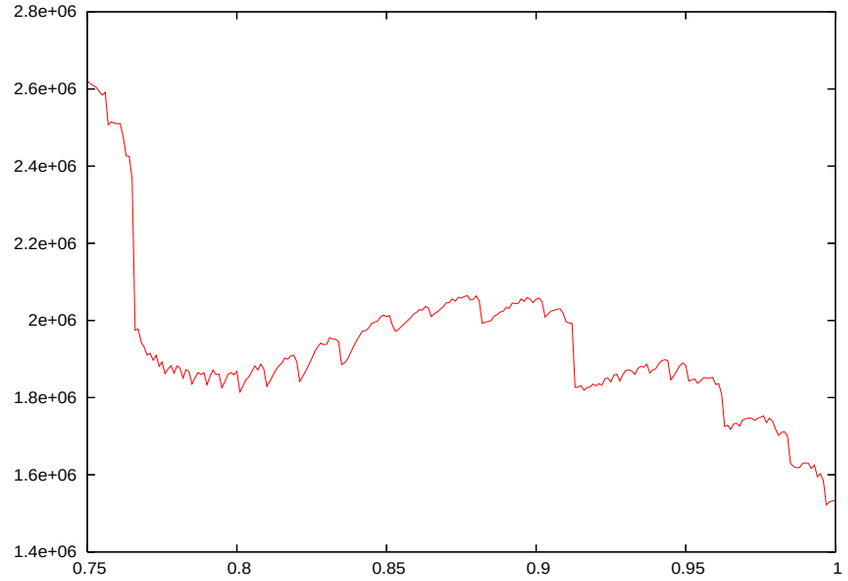
Despite its trivial character still **many of its properties are not know in detail** mainly concerning **global aspects**.

It serves as **paradigm** for many phenomena. See S-Vieiro, Global study of area preserving maps, preprint, 2007 and S-Vieiro, Towards a global study of APM: rates of escape, diffusion and Cantor sets, in preparation.

Typical phase portrait and non-escaping points



Left: **Phase portrait** for $c = 0.75$. Outside that domain, points **escape** to infinity, close to W^u_H. Right: number of **non-escaping points** on a grid $10^{-3} \times 10^{-3}$. Magnifications on next plots. Computations done by a combination of methods. Observe the **self-similar** character and the **jump discontinuities**. Also the non-escaping points **before and after** the **destruction of rotational i.c. confining islands** of periods 5,4,3.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2.6e+06
2.4e+06
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1.6e+06
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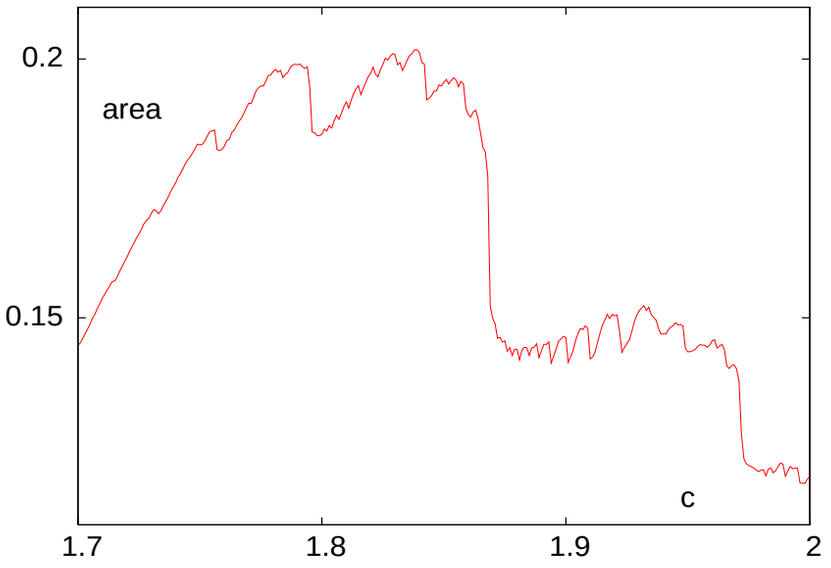
0.8
0.85
0.9
0.95
1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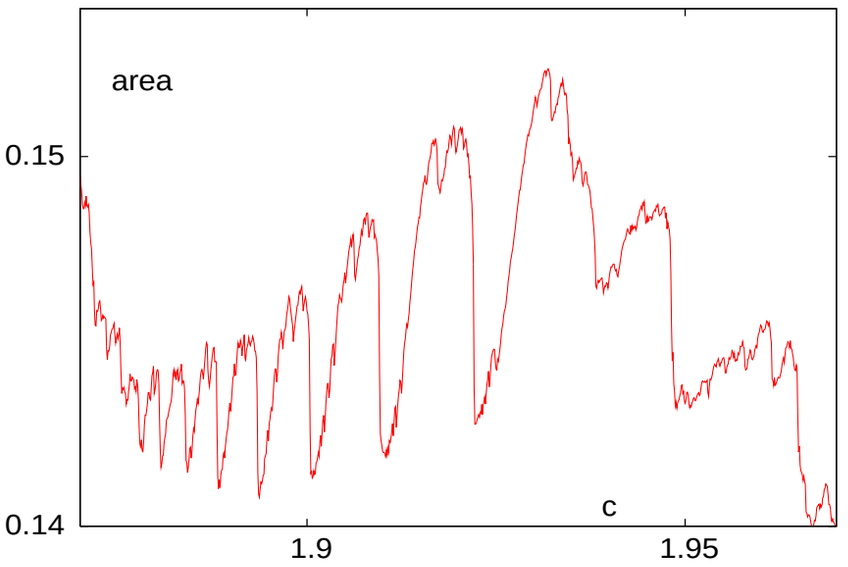
2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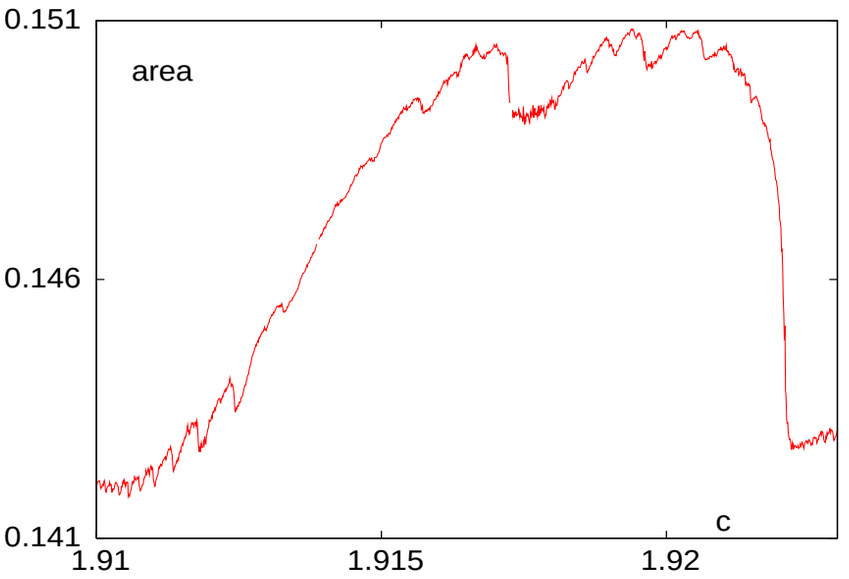
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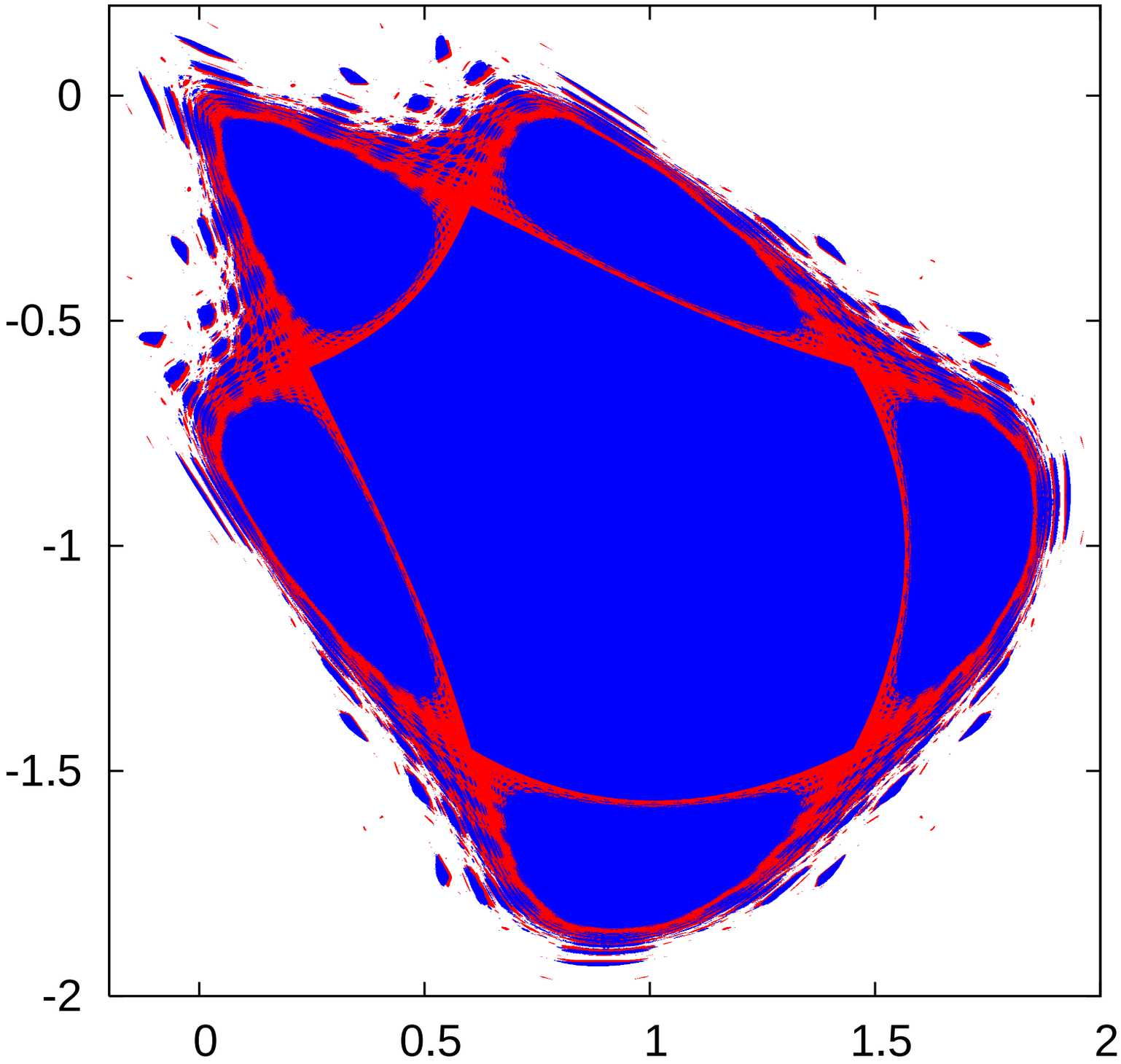
1.92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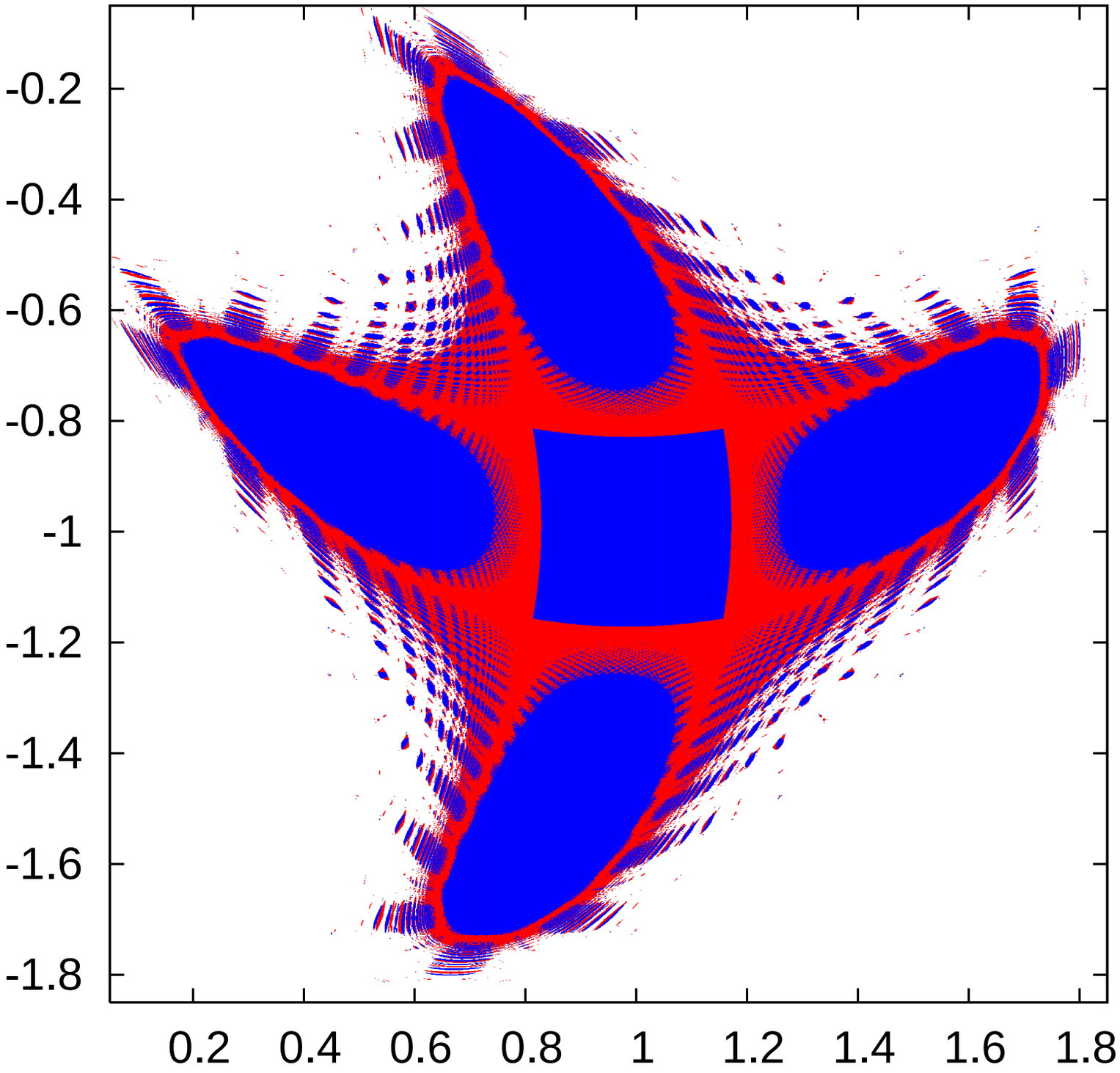
0.5
1
1.5
2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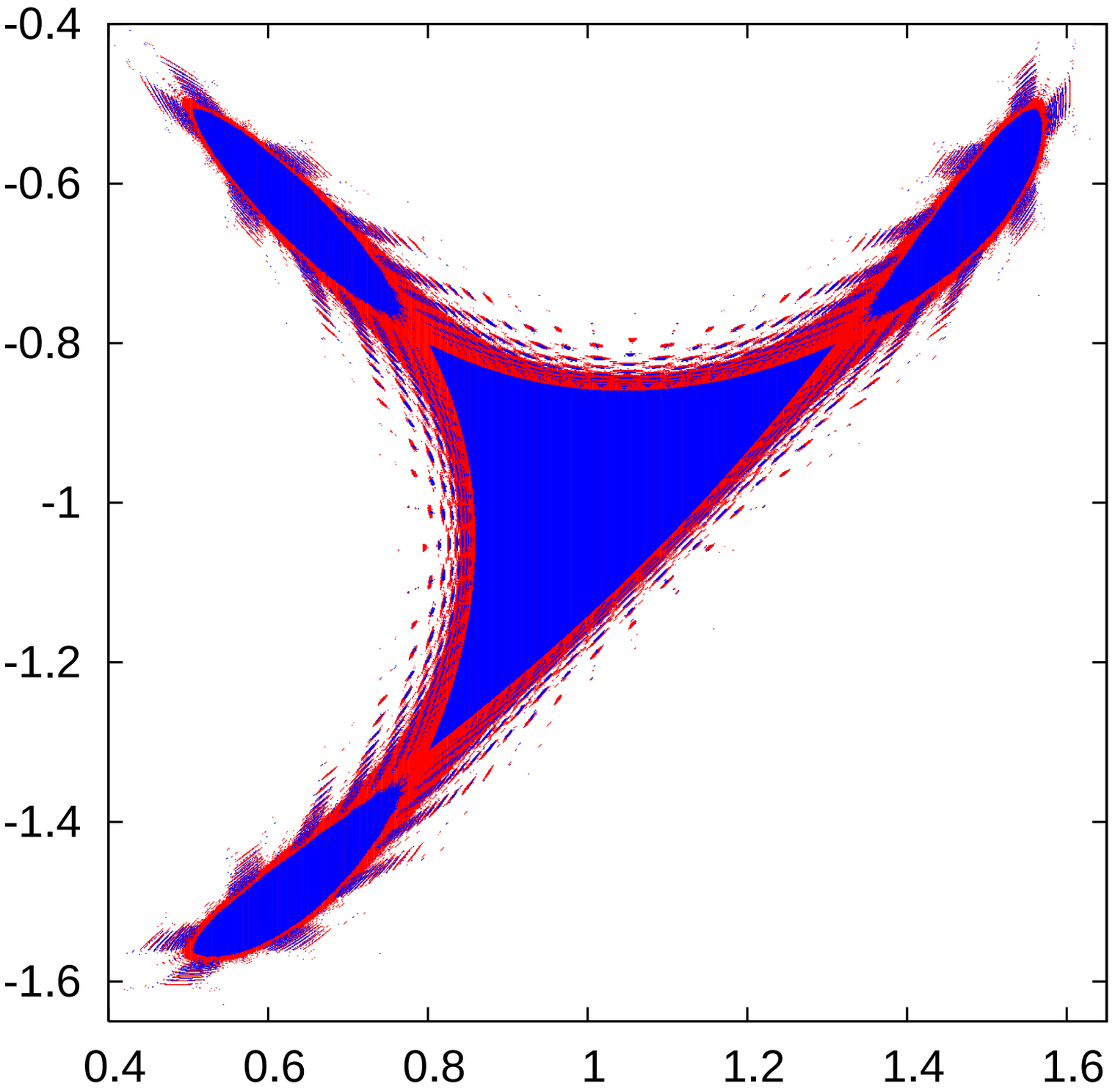
-0.4
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-0.8
-1
-1.2
-1.4
-1.6
0.4
0.6
0.8
1
1.2
1.4
1.6

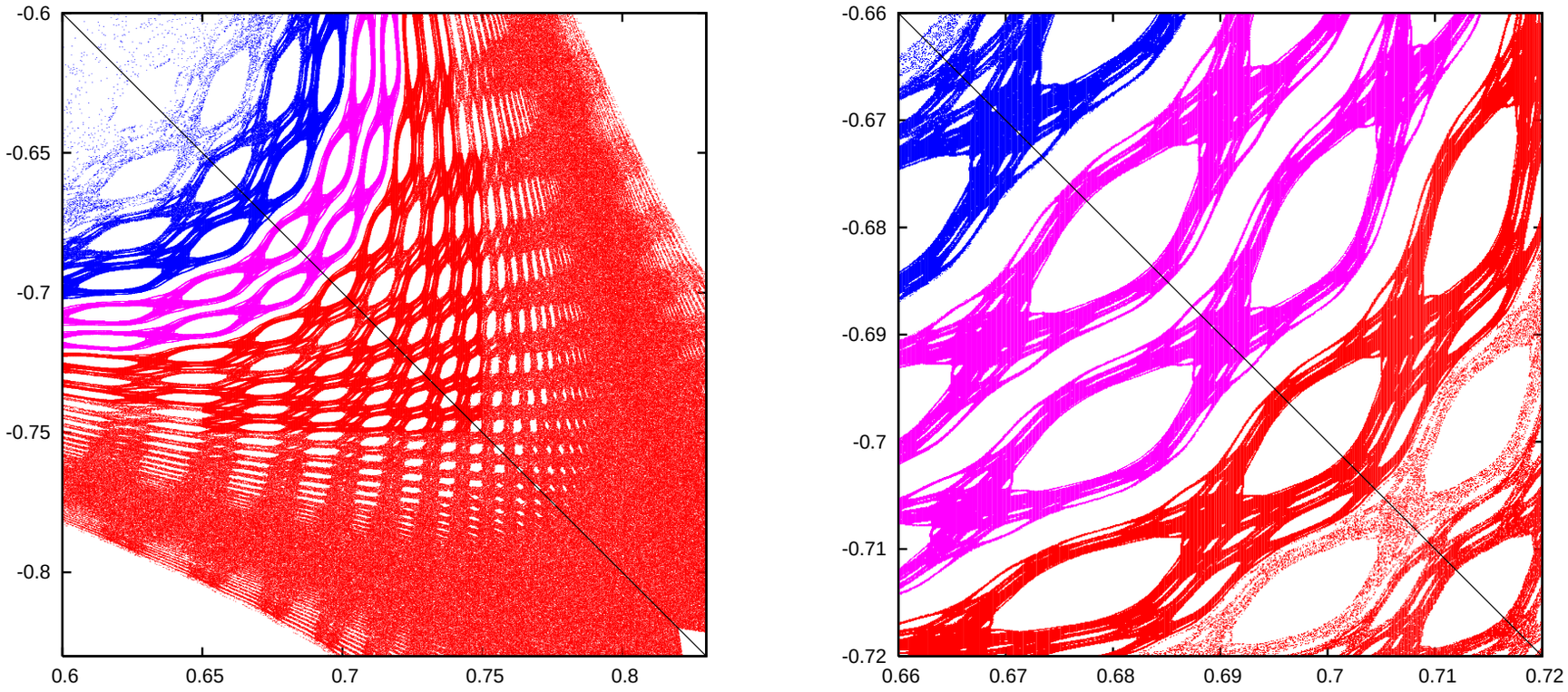
Previous plots require some explanation. Typically they correspond to close values of c, separated by 0.001, say $c_1 < c_2$. In **red** we display the set of points which **do not escape** for $c = c_1$, while in **blue** the non-escaping points for $c = c_2$ are shown.

It is clear what happens. For $c = c_1$ there is still an **invariant rotational curve** which encloses the main islands of period k, $k = 5, 4, 3$, respectively. For $c = c_2$ **the curve has disappeared**.

Before the breakdown of the curve, there were points sitting on a **chaotic, confined region**, moving between families of islands. When the curve disappears, these points are **free to escape**. This explains the **jumps** in the measure of the non-escaping points as a function of c.

Similar things happen for **all the main islands** and also for the **satellite islands of the main islands**, "ad infinitum", giving rise to **self similarity**.

Next plots also illustrate different regions on the **phase space**.



Magnifications of a region close to $y = -x$ for $c = 1.0141$. Points in red, blue, correspond to **single orbits**. Points in magenta correspond to **two orbits**, one in each domain. One can see the **negative** of the **islands**. Only points in blue escape, but **for $c = 1.0142$ all these points escape**. For $c = 1.0141$ there are **three gaps** candidate to contain **invariant (rotational) curves**

Several simple, paradigmatic, models: the standard map, the separatrix map and the biseparatrix map

The present goal is to **explain and predict** the observed behavior.

To this end we introduce several **simple, paradigmatic, models**, which in the simplest version are as follows:

- the **standard map**:

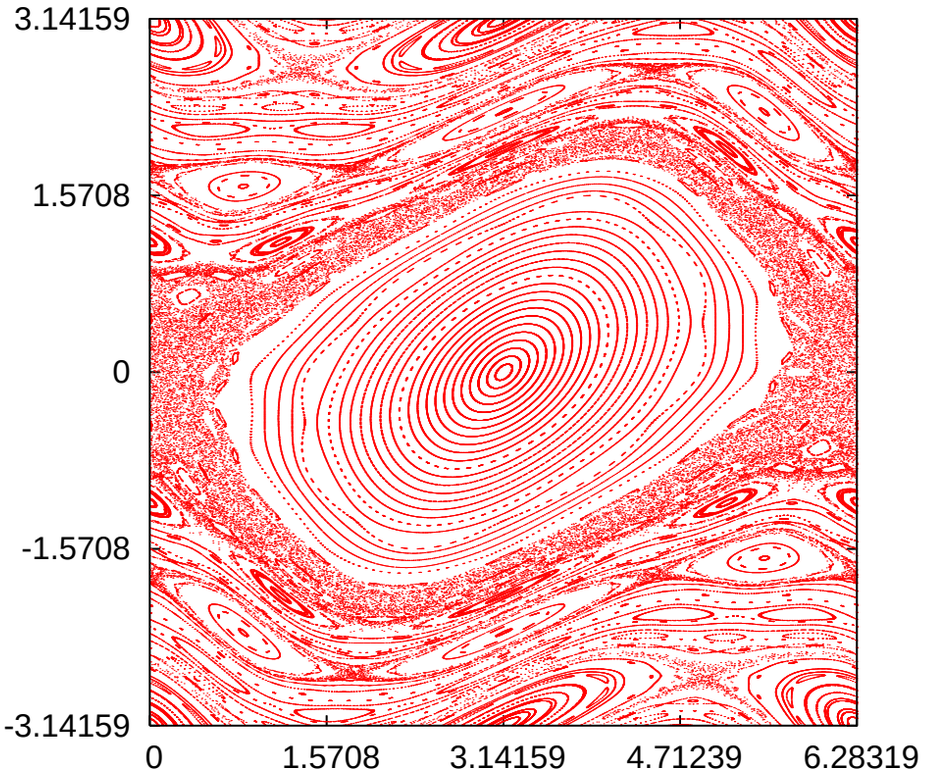
$$SM\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} = \begin{pmatrix} x + \bar{y} \\ y + \varepsilon \sin(2\pi x) \end{pmatrix} \qquad \text{mod } 1.$$

- the **separatrix map**:

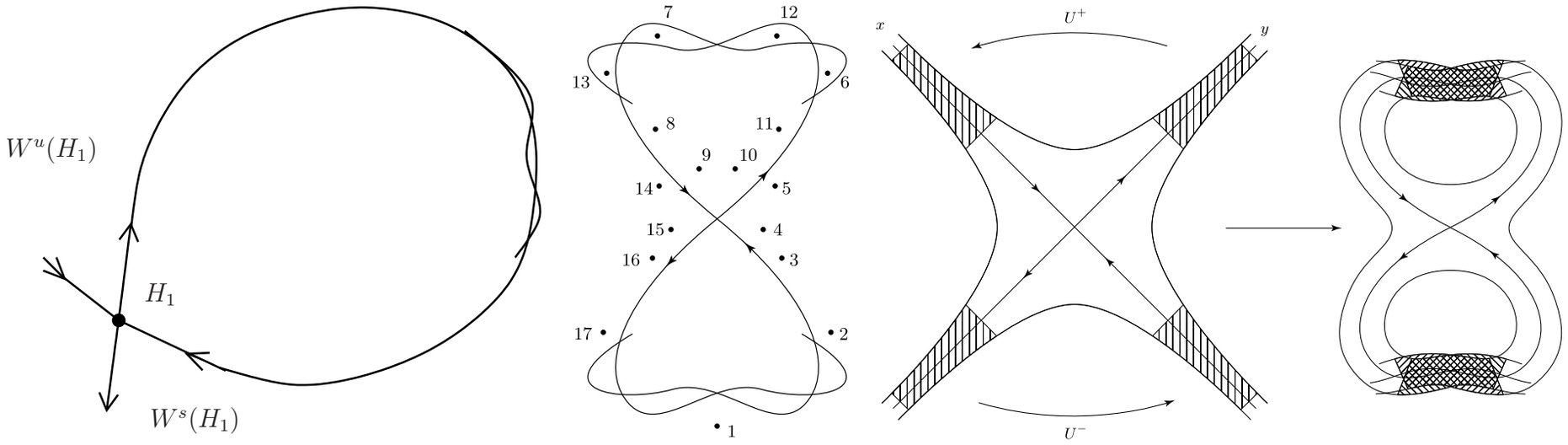
$$SPM\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} = \begin{pmatrix} x + a + b\log(|\bar{y}|) \\ y + \sin(2\pi x) \end{pmatrix}.$$

- the **biseparatrix map**:

$$BSPM\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} = \begin{pmatrix} x + a + b_1\log(\bar{y}) - b_2\log(d - \bar{y}) \\ y + \sin(2\pi x) \end{pmatrix}.$$

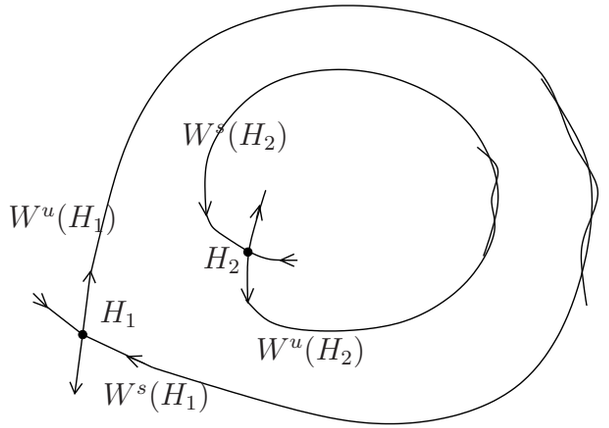


SM is a model for **simple resonances**. The plot is done for $\varepsilon = 0.9/(2\pi)$. One can see **i.c.=invariant curves**, either of **rotational** or **librational** type, which give rise to **stability islands**), and also **chaotic zones**. It is **"known"** that **rotational curves** exist up to $\varepsilon = \varepsilon_G \approx 0.971635/(2\pi)$, the so-called **Greene's value**. See Olvera-S, *Physica D* **26** (1987),181–192.



SPM is a model for **return close to the separatrix after passing near the saddle**. The y scale has been **normalized** in such a way that the **splitting amplitude** is equal to 1. b is the inverse of the **dominant eigenvalue** at the saddle. So, it is large for weak saddles. See Broer-S-Tatjer, *Nonlinearity* **11** (1998), 667–770 and S-Treschev, Stability islands in the vicinity of separatrices of near-integrable symplectic maps, *DCDS*, to appear.

For large y can be **approximated by SM with parameter b/y**. This allows to predict **position of islands, existence of i.c., etc.**.



$BSPM$ is a model of the **dynamics between separatrices associated to different fixed or periodic points**. One has to assume $0 < y < d$ and requires a **variable change of scale in y between the separatrices**.

Again, for $d >> 1$, $y >> 1$, $d - y >> 1$ can be **approximated by SM with parameter** $\dfrac{b_1}{y} + \dfrac{b_2}{d-y}$.

In particular

$$(\sqrt{b_1} + \sqrt{b_2})^2/d > \varepsilon_G$$

prevents the **existence of rotational invariant curves**.

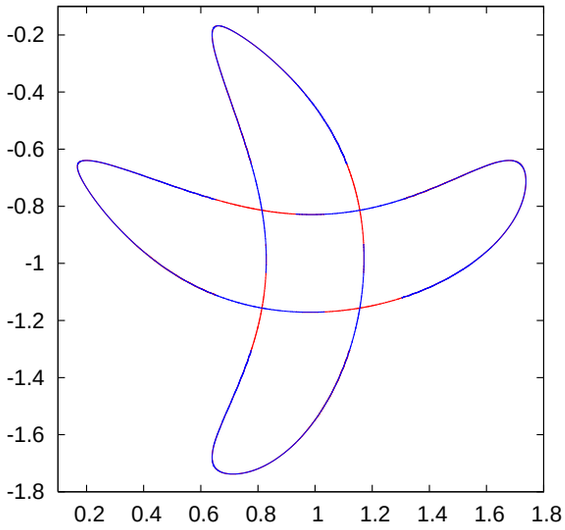
Explaining the main features of the dynamics

As an example consider the parameter $c = 1.015$, just after the **destruction of the i.c. confining islands of period 4**. One can see a **domain without subsisting points** close to the **invariant manifolds** of the period 4 orbit, see upper plot in next page. It looks as if the manifolds coincide, at this resolution.

A computation of the splitting angle **in the outer and inner homoclinic connections** gives the values $s_o \approx -0.951063 \times 10^{-2}$, $s_i \approx 0.294215 \times 10^{-58}$.

From this one can derive a **composed separatrix map** and predict the distance from the manifolds at which **stable islands and invariant curves** appear in the domain bounded by the initial pieces of the manifolds. These values agree with the results obtained by **direct simulation**, but one should be careful about **how this depends on the place where we look**. See lower plots.

For the **outer islands** and the **existence or not of i.c.** one requires the use of the $BSPM$. Locally, in what concerns nearby islands, one can approximate again by the SPM.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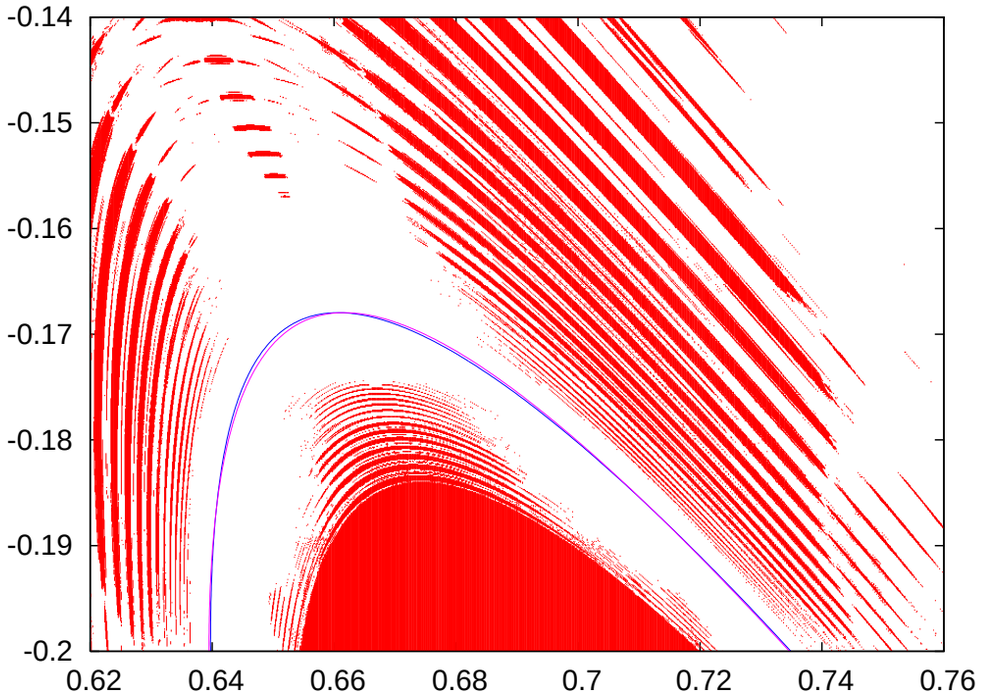
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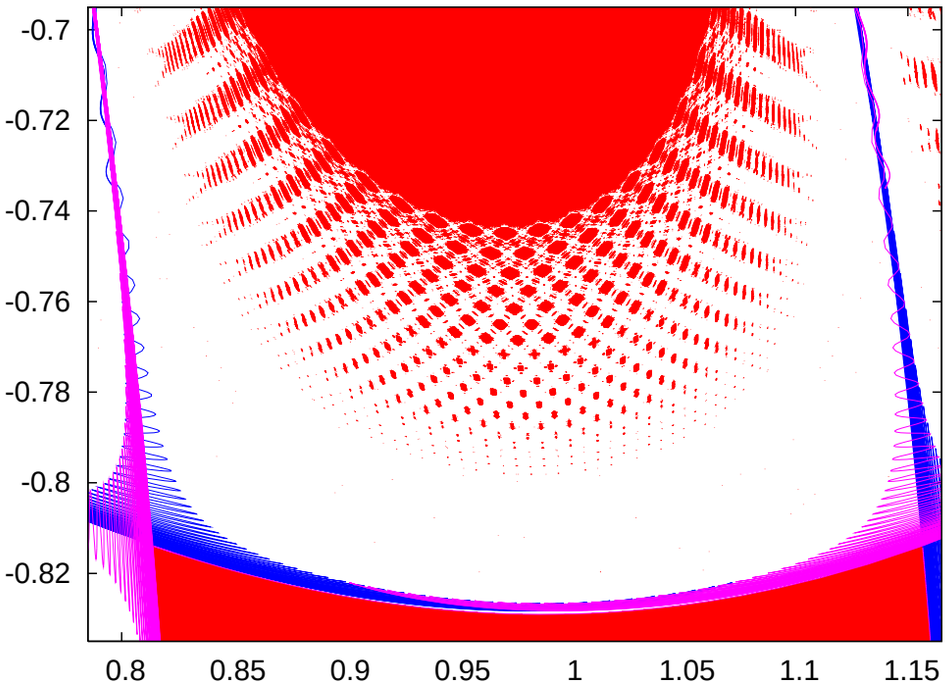
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-0.76
-0.78
-0.8
-0.82
0.8
0.85
0.9
0.95
1
1.05
1.1
1.15

Rates of escape.

It is relevant to know, not only the measure of the set of points which escape, but also the **rate of escape**. Or, equivalently, which **fraction of points subsists as a function of the number of iterates**.

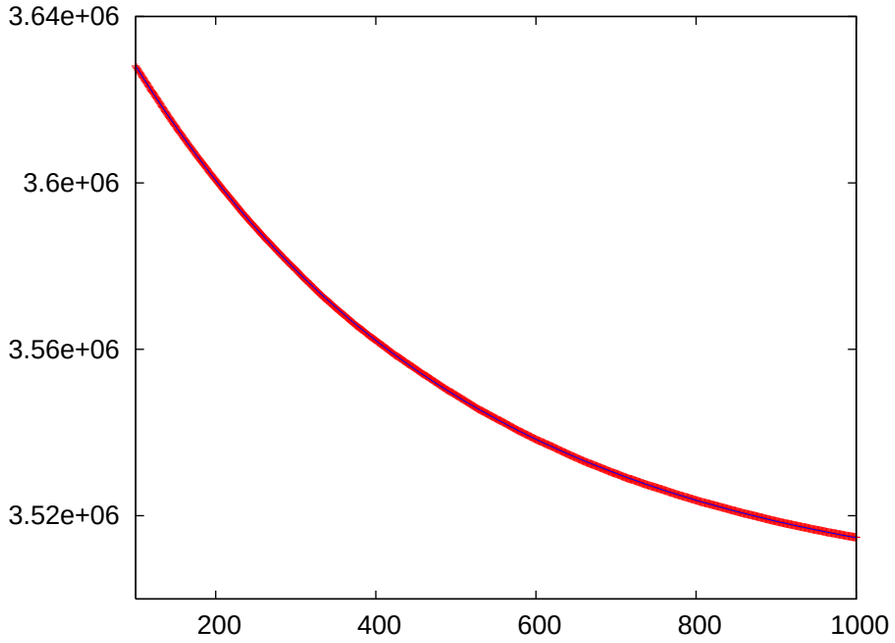
The left upper plot in next page shows a **typical result** after a **transient** of 10^8 iterations. The horizontal axis displays number of iterations, in millions, and the vertical one the number of subsisting points $p(n)$. The value $c = 0.531$ has been used. The results can be **fitted** by

$$q(n) = p + q\exp(-rn),\ p \approx 3.500215\times 10^6,\ q \approx 0.162878\times 10^6,\ r \approx 0.00241827,$$

where n is expressed in millions of iterations.

The right upper plot shows $p(n) - q(n)$, giving an idea of the **accuracy of the fit**. This behaviour is **coherent** with the rate of escape of a **random walk**.

But for values $c = 0.765,\ 0.766$, in lower row, the situation is different. Fits like $q(n)$ for **different intervals of** n show quite **different values of** p, q, r.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3.56e+06
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400
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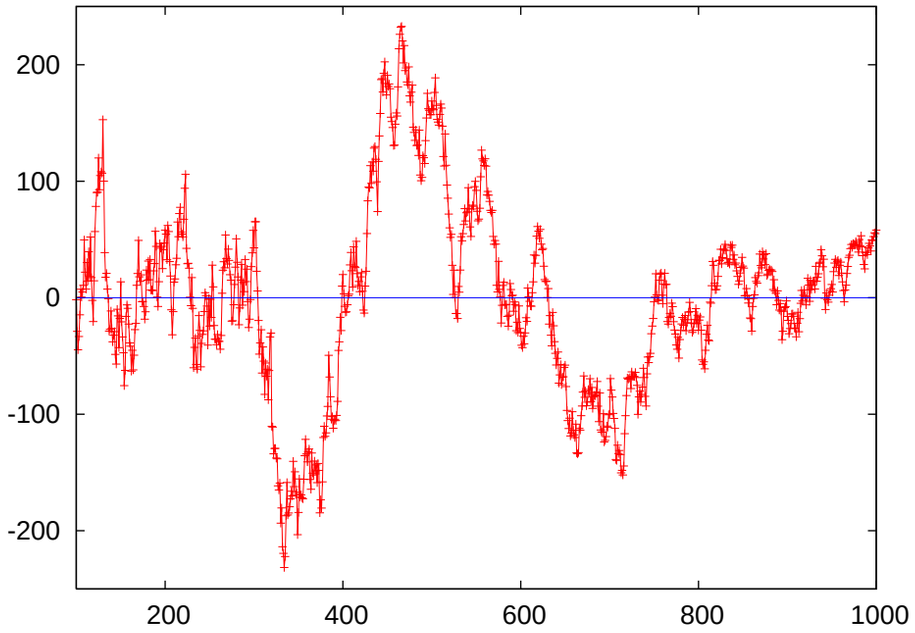
200
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200
400
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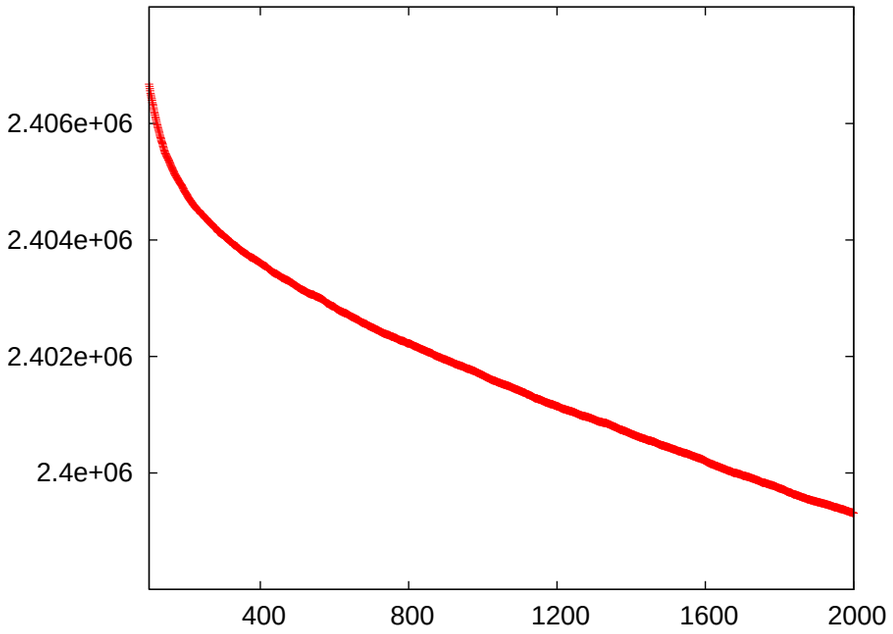
800
1000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2.404e+06
2.402e+06
2.4e+06
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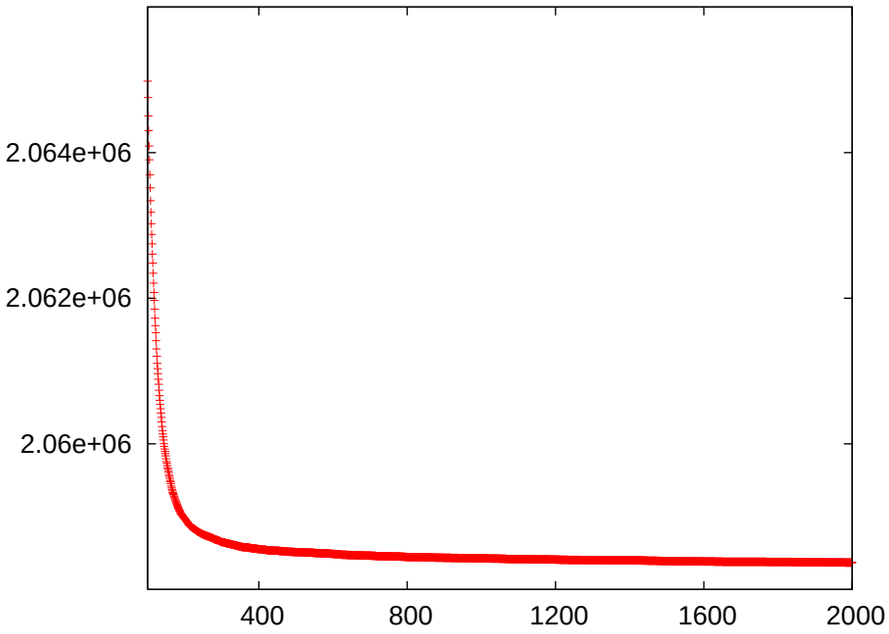
800
1200
1600
2000


2.064e+06
2.062e+06
2.06e+06
400
800
1200
1600
2000

Random walks in the chaotic component

In a chaotic region between **confining i.c.** the dynamics, in principle, looks like a **random walk**. Points seem that tend to **fill a region of positive measure** with some **limit density**. But the **transient can be large**.

A similar behaviour is observed for **escaping points**. The right upper plot in next page shows, for $c = 0.764$ the dynamics of an **escaping point**. For nearly 7×10^{10} iterates the point moves in a **narrow strip in blue**. Later it **moves to the region in red** and finally it escapes.

It has to be remarked that at a distance $\approx 8E - 9$ of the initial point taken, there are points **sitting on a rotational i.c.**, like $x = -y = 0.00545430414$. The plot would make them coincide with the blue one, at plot's resolution.

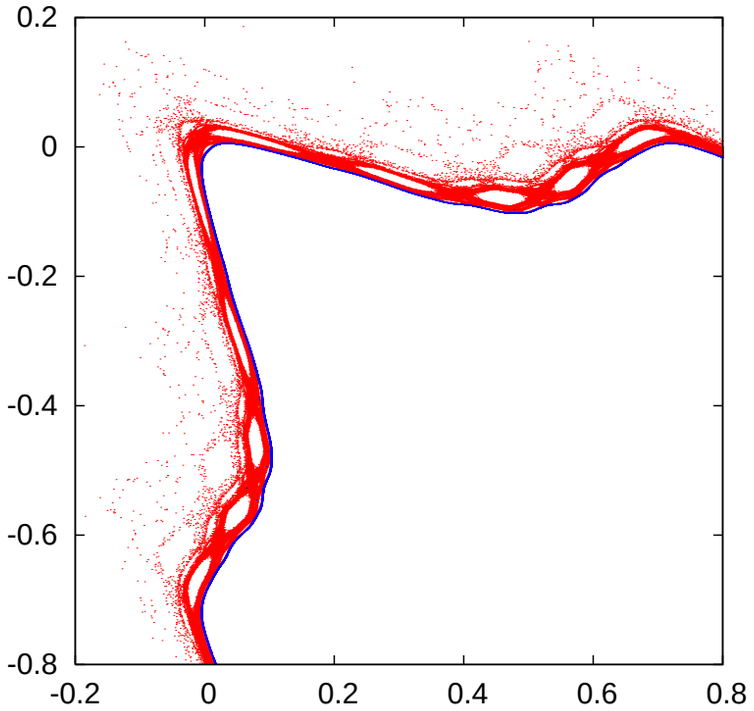
We can ask for the **round off effects** on the simulations. To this end the right upper plot shows the following. Take a **small curve interval**, close to horizontal in suitable coordinates. In present case it has length 10^{-8}. **Keep iterates** having horizontal coordinate on the interval. In case of an invariant curve, we should see it at any resolution level. Due to round

off errors, we see a **cloud of points**. **Fit** the cloud by a polynomial and subtract from data. This gives a measure of the **round off effects**.

In present case the amplitude of the errors for 0.657×10^{12} iterates is below 10^{-13}. The plot shows the **random walk like** behaviour of the differences after every 10^6 iterates.

One of the few values (using 0.001 step in c) for which strong **difficulties occur** is $c = 0.765$. For this value it seems that still **a rotational i.c. exists** around period 5 islands. This is **not true**. The lower plot shows the following. Take points on $y = -x$ between the **"last"** i.c. around the elliptic fixed point and the "last" i.c. around period 5 points. Some points on this line are inside **high period islands**. We have taken 124 equiespaced points in the complement. Even after 10^{11} iterates, only **59 escaped**. The plot shows $\log_{10}$ of the number of iterates vs initial position.

Worth of mention: **1D and 2D random walks return to initial point** or pass through a prefixed point (in a grid in $\mathbb{Z}^n$) with **probability 1**. For **higher dimensions this is not true**. Approximate values are $p_3 = 0.516386$, $p_4 = 0.239467$, $p_5 = 0.156308$, $p_6 = 0.116963$.
In dimension > 2 not all paths lead to Rome!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0
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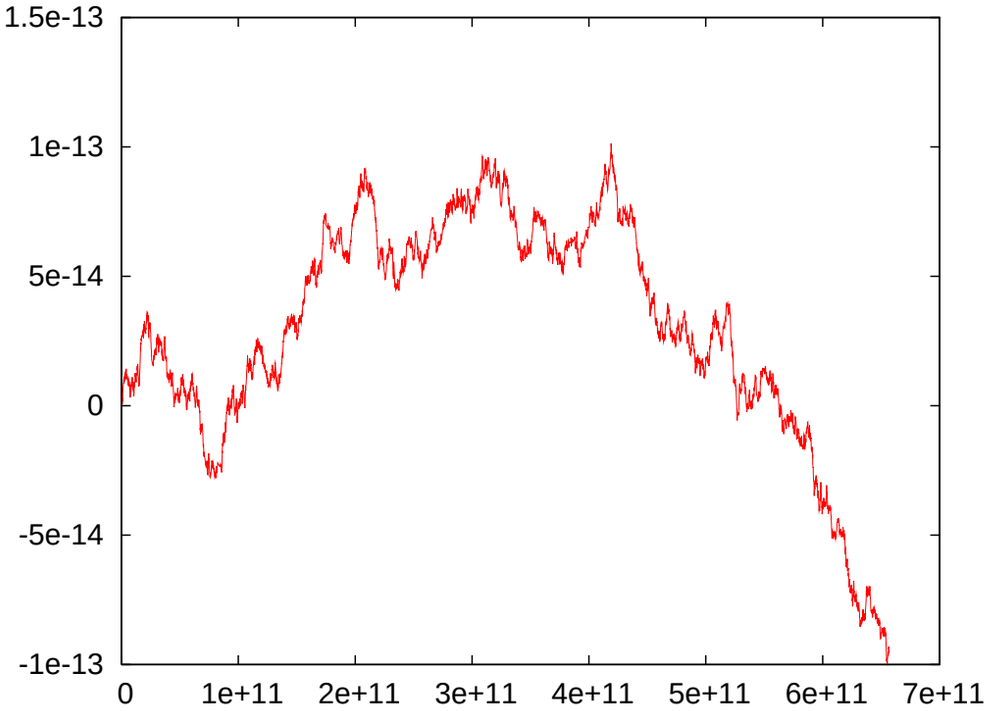
0.4
0.6
0.8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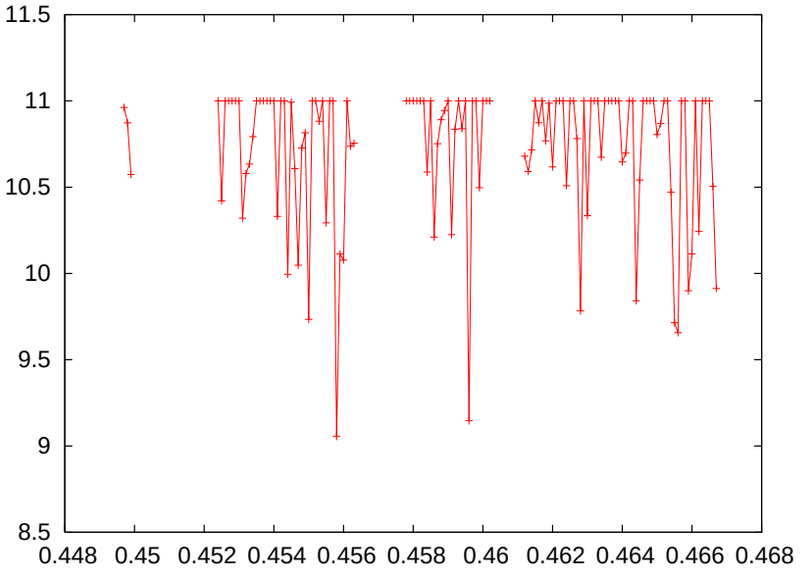
5e+11
6e+11
7e+11


11.5
11
10.5
10
9.5
9
8.5
0.448
0.45
0.452
0.454
0.456
0.458
0.46
0.462
0.464
0.466
0.468

Cantori and their gaps

A fact which has to be stressed is **what happens to a curve when it is destroyed**. What remains is a **Cantor set** or **Aubry-Mather set**. It is relevant to know **a geometrical interpretation of the gaps**, the **width of the gaps** and also **which i.c. are the last ones to be destroyed**.

Gaps can be seen to be produced by the **existence of heteroclinic intersections** between invariant manifolds of **periodic orbits** sitting to different sides of the invariant curve, before this one is destroyed.

It can also be said that it is **the existence of gaps** in the Cantor set, what gives the possibility of **existence of heteroclinics**.

From an analytic point of view, gaps are related to the **divergence of the solutions of the homological equations**, the linear steps in the contruction of the invariant curves.

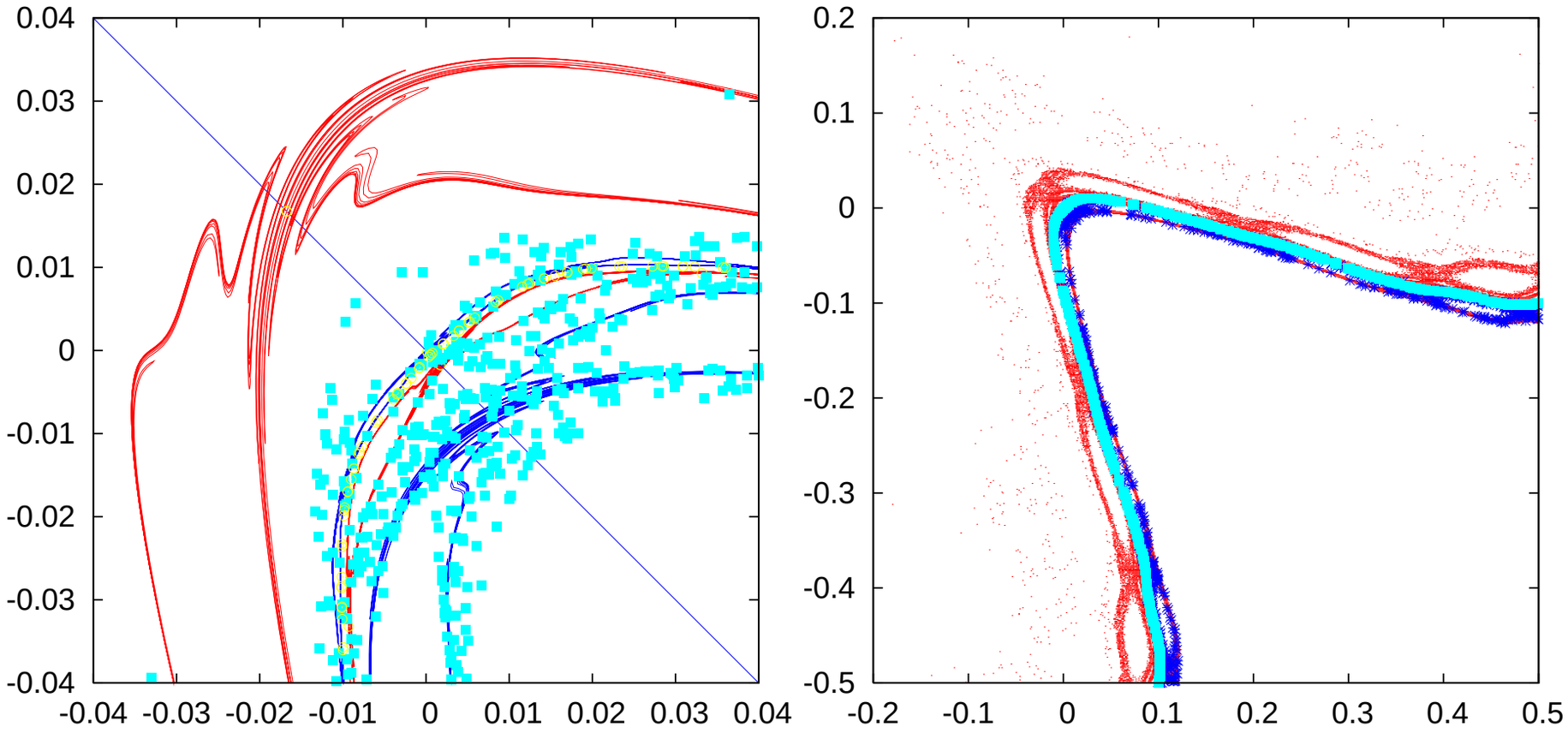
The answer to last question above is given again by **KAM**. Locally both **torsion and perturbation** change by a small amount. The key is the **DC**. In a given interval of ρ the numbers having larger bound in the DC

are **noble numbers**. They have a **c.f.e.** like

$$q_1, q_2, q_3, \ldots, q_m, 1, 1, 1, 1, 1, 1, \ldots$$

This has been **checked on the simulations**.

Another relevant item when we talk of **last i.c.** in a loose way, is that, **locally, around an i.c., there is also a Cantor set of i.c.** whose relative measure tends to 1 on the curve. There is very little known about **quantitative estimates** in this direction.
Also instructive: **how long it takes to cross different cantori**.



For $c = 0.765$. Left: Invariant manifolds of points of periods $26, 31, 88, 145$ (stable in red, unstable in blue), an approximation of a Cantor set (in yellow) and the orbit of an inner point (in pale blue) which **succeeds in crossing the Cantor sets**. Right: A Cantor set in pale blue, the orbit of an escaping point in red and the first 5×10^6 iterates on the escaping point, showing one point every 1000 iterates, in dark blue.

Some open questions

The simple examples shown suggest **infinitely many questions**. Just a sample, covering different aspects:

- Prove the **positivity of the metric entropy** in the 2D case, for generic, analytic, non-uniformly hyperbolic APM.
- Prove the **Poincaré conjecture**: density of periodic points.
- Look for problems of **separatrix splitting** in higher dimension.
- Study **paradigmatic models** similar to Hénon, standard, separatrix and biseparatrix maps (and other maps who would be convenient to introduce) for **symplectic 4D, 6D maps**.
- Analyse the **breakdown of tori** in higher dimensional cases.
- Study the **Cantor sets with a given rotation vector** in higher dimensional cases.
- Find the analogous of **noble numbers in** $\mathbb{R}^n$, $n > 1$.
- Clarify the role of the **no Rome effect** in the **ability to cross gaps** in higher dimensions.

- Find all kinds of **codimension 1 manifolds** which are relevant for the dynamics.
- Produce **quantitative estimates** on the rate of escape and on the rate of diffusion for symplectic 4D, 6D maps.
- Impact of the fact that, in many systems, we are **mainly interested in the transient regime**. The phenomenon is not alive (or the model is no longer valid) after some time.
- Introduce **different kinds of noise**.