Taylor method for the integration of ODE

Carles Simó

Dept. Matemàtica Aplicada i Anàlisi, UB

carles@maia.ub.es

LTI07 Advanced School

Barcelona

070903-070907

Contents

- Stability boundaries. The planar RTBP in the outer region. The 3D RTBP around $L_{4,5}$,
- Motion of the giant planets for the age of the solar system,
- Arnol'd diffusion: the classical example and a galactic model,
- Additional comments.

Taylor integration method

Consider an IVP for an ODE like $\dot{x} = f(t,x), \qquad x(t_0) = x_0$, where f is assumed to be analytic in a nbd of $(t_0, x_0) \in \Omega \subset \mathbb{R} \times \mathbb{R}^n$.

We want to produce **in an easy way** the **Taylor expansion** $x(t_0 + h)$ for suitable values of h and use it as a one-step method.

For a very large class of functions the evaluation of f can be split in simple expressions

$$
\begin{aligned}
e_1 &= g_1(t,x), \\
e_2 &= g_2(t,x,e_1), \\
&\vdots \\
e_j &= g_j(t,x,e_1,\ldots,e_{j-1}), \\
&\vdots \\
e_m &= g_m(t,x,e_1,\ldots,e_{m-1}), \\
f_1(t,x) &= e_{k_1}, \\
&\vdots \\
f_n(t,x) &= e_{k_n},
\end{aligned}
$$

where each one of the expressions e_j contains a sum of arguments, a product or quotient of two arguments or an elementary function (like sin, cos, log, exp, $\sqrt{}, \ldots$) of a single argument.

The basic idea is to compute in a **recurrent way** the **power series expansion** of all the e_j. All the g_j have to be seen as **operations with (truncated) power series**. The input are t and the coefficients of order 0 of the components of x_0. At the end of first step we know the terms of order 1 of these components. If the terms up to order s of the arguments of the g_j are known, we obtain easily the coefficient of order s of e_j. Hence the representation of the components x_i of $x(t_0 + h)$ looks like

$$x_i = \sum_{s=0}^{N} a_i^{(s)} h^s.$$

Some **elementary recurrences**

Let $a(t) = \sum_{k\geq 0} a_k t^k, \quad b(t) = \sum_{k\geq 0} b_k t^k$, etc.

$c = a * b: \quad c_n = \sum_{k=0}^{n} a_k b_{n-k}$

$c = a/b,\ b_0 \neq 0, \quad c_0 = a_0/b_0, \quad c_n = \frac{1}{b_0}\left[a_n - \sum_{k=0}^{n-1} c_k b_{n-k}\right],\ n > 0$

$c = a^\alpha,\ \alpha \in \mathbb{R},\ a_0 \neq 0, \quad c_0 = a_0^\alpha,$
$\quad c_n = -\frac{1}{na_0}\sum_{k=0}^{n-1} c_k a_{n-k}[k - \alpha(n-k)],\ n > 0$

$c = \exp(a), \quad c_0 = \exp(a_0), \quad c_n = \frac{1}{n}\sum_{k=0}^{n-1} c_k a_{n-k}(n-k),\ n > 0$

$c = \log(a),\ a_0 \neq 0, \quad c_0 = \log(a_0), \quad c_n = \frac{1}{na_0}\left[na_n - \sum_{k=1}^{n-1} kc_k a_{n-k}\right],$
$\quad n > 0$

$c = \cos(a),\ s = \sin(a), \quad c_0 = \cos(a_0),\ s_0 = \sin(a_0),$
$\quad c_n = -\frac{1}{n}\sum_{k=0}^{n-1} s_k a_{n-k}(n-k), \quad s_n = \frac{1}{n}\sum_{k=0}^{n-1} c_k a_{n-k}(n-k),\ n > 0$

For **many other functions** (e.g. Bessel functions) one can **add** to the list of ODE **the differential equations that these functions satisfy**.

The Hénon-Heiles model

The classical Hamiltonian

$$\frac{1}{2}(x^2 + y^2 + p_x^2 + p_y^2) + \frac{1}{3}x^3 - xy^2,$$

gives rise to

$$\begin{aligned}
\frac{dx}{dt} &= p_x, \\
\frac{dy}{dt} &= p_y, \\
\frac{dp_x}{dt} &= -x - x^2 + y^2, \\
\frac{dp_y}{dt} &= -y + 2xy.
\end{aligned}$$

This is a seminal example displaying regular and chaotic motion. Very convenient to illustrate the method.

At every step, after initializing the terms of order 0 and 1 of the Taylor series, we compute recurrently the last available term of x^2, y^2, xy and produce the next term in the expansions of $x(t+h), y(t+h), p_x(t+h), p_y(t+h)$.

We show also a related movie.

Tolerance, order and step selection

There are several **strategies** to select **order and step** to have, approximately, some **desired truncation error**. Also to save computing time (like multiplying by a precomputed $1/s$ instead of dividing by s), but some **caution** is required to avoid systematic errors.

Some interesting properties

- Under reasonable conditions the **optimal stepsize** (concerning efficiency for fixed truncation error) is almost independent of the number of digits.
- The **optimal order** is approximately linear in the number of digits.
- The **computational cost** (measured in number of elementary operations) is approximately quadratic in the number of digits.

Let us see some details. In particular, that for small enough tolerances, the optimal step size (around a given value of t) is close to $\dfrac{\rho(t)}{\exp(2)}$, where $\rho(t)$ is the radius of convergence of the Taylor series around t.

Stability

For a Taylor method of order n the boundary of the absolute stability domain is obtained from the condition

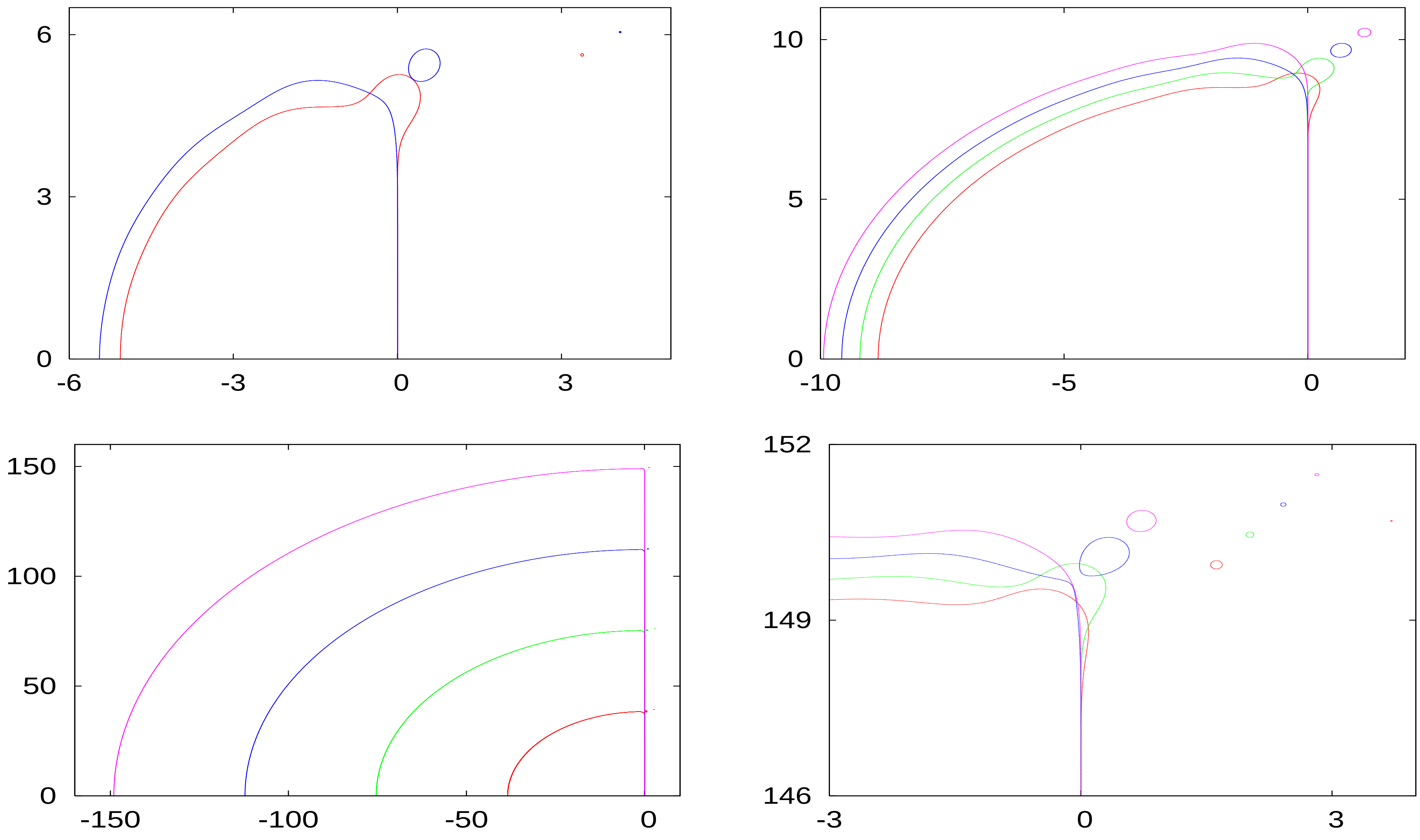
$$|T_n(\exp(z))| = 1,$$

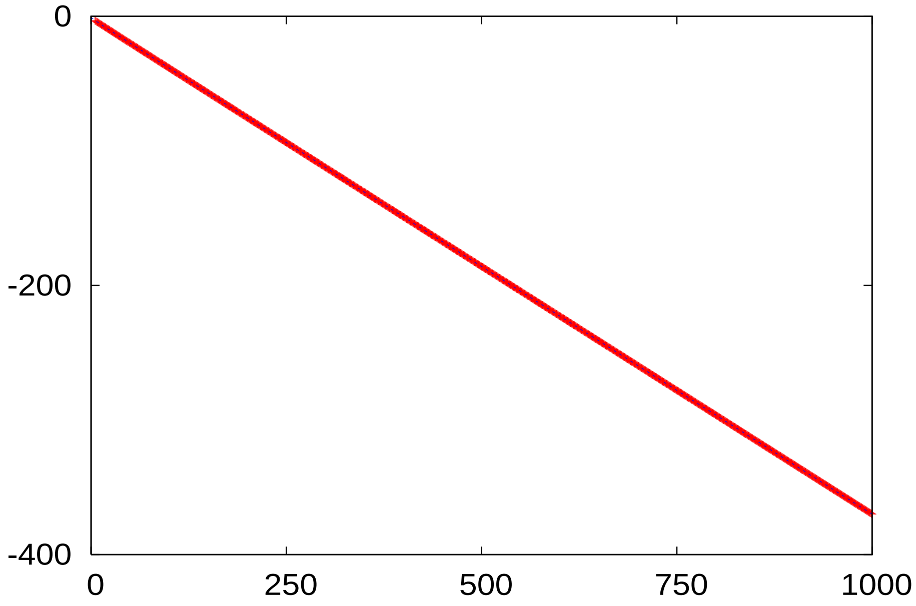
where T_n denotes the truncation to order n of the expansion of exp.

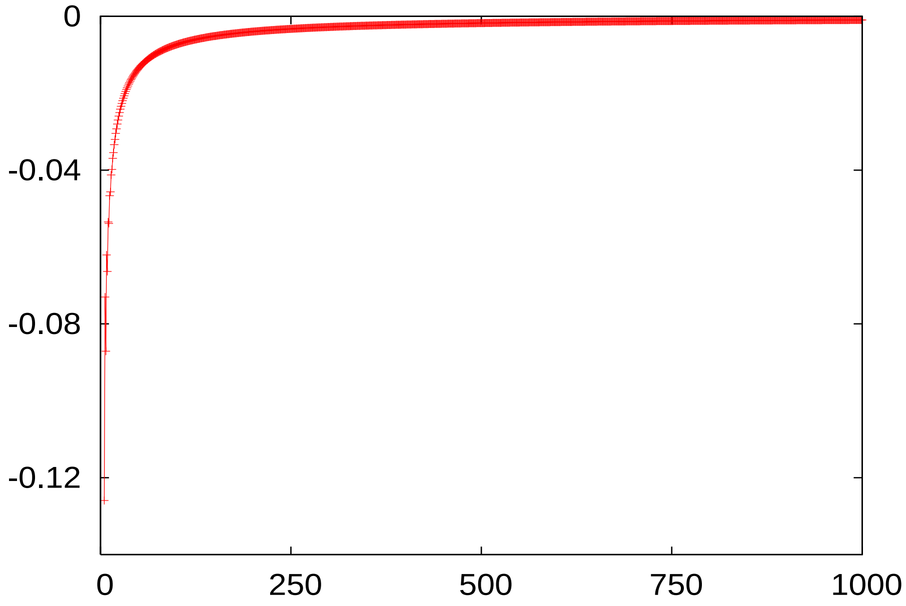
Usually it is convenient to represent $T_n(\exp(z))$ as

$$T_n(\exp(z)) = \exp(z) - \frac{z^{n+1}}{n+1}\left(1 + \frac{z}{n+2}\left(1 + \frac{z}{n+3}(\ldots)\right)\right).$$

Some boundaries of the domain of absolute stability. Top left $n = 10, 11$, right $n = 20 - 23$. Bottom left $n = 100, 200, 300, 400$, right $n = 401 - 404$.







The stability boundary as a function of n in $x < 0$. Right: difference with the theoretical function $r(n) = -f(n+1)$ where

$$f(x) = \frac{1}{e}\left[x + \frac{1}{2}\log(2\pi x(1+e^{-1})) + \frac{(\log(2\pi x))^2}{8x} + O(\frac{\log x}{x})\right].$$

A similar expression of $r(n)$ holds for all the negative semiplane, with the same dominant term $\approx n/e$, except on the vicinity of the imaginary axis. If $x = a + ib$ with a small, then, if $b = \frac{n+1}{e}\beta$ and $\beta < 1$ and not too close to 1, then

$$a \approx \frac{\beta^{n+1}}{\sqrt{2\pi(n+1)}} sc(b),$$

where sc denotes either sin or cos. For $n = 4k, 4k+1, 4k+2, 4k+3$ one has, respectively, the functions $\sin(b), -\cos(b), -\sin(b), \cos(b)$.

Complexity

For a given problem, $x' = f(t,x)$, $x(t_0) = x_0$, to compute until a fixed final time t_f, requires an effort of the order

$$K \int_{t_0}^{t_f} \frac{dt}{\rho(t)}.$$

The constant K depends on the problem and it is proportional to N^2, N being the order of the method, which in turn is approximately proportional to $|\log(\epsilon)|$ (ϵ is the desired truncation error), i.e., the number of digits d. That is $K = \hat{K} d^2$, where $\hat{K}$ depends only on the problem.
If we accept that the cost of the operations is proportional to d^2 (FFT methods can do better for large d), the we have

$$\text{Total cost proportional to} \quad d^4,$$

i.e. polynomial complexity.

It is clear that this can not be achieved with methods of fixed order.

Dense output and Poincaré maps

With the Taylor method it is elementary to produce

- Dense output, i.e., values of $x(t)$ for a (large) set of points t_i^*, e.g., equispaced points, different from the points t_j used for the integration. Indeed, we have a representation of the solution in $[t_j, t_j + h_j]$, where h_j (assumed > 0) is the current step size. The solution can be evaluated for all the t_i^* in this interval. It is useful for Frequency Analysis, for instance.

- Poincaré maps. If the Poincaré section is given by $x_{(m)} = 0$, where $x_{(m)}$ denotes the mth coordinate of x, as we have, in particular, $x_{(m)}(t)$ for $t \in [t_j, t_j + h_j]$, if $x_{(m)}(t_j) \times x_{(m)}(t_j + h_j) < 0$, it is enough to solve for a 1D equation.

  If the Poincaré section is given by $g(x) = 0$, the problem is similar. The computation of $g(x(t)), \frac{d}{dt}g(x(t))$ to use a Newton method, do not requires any additional step of integration.

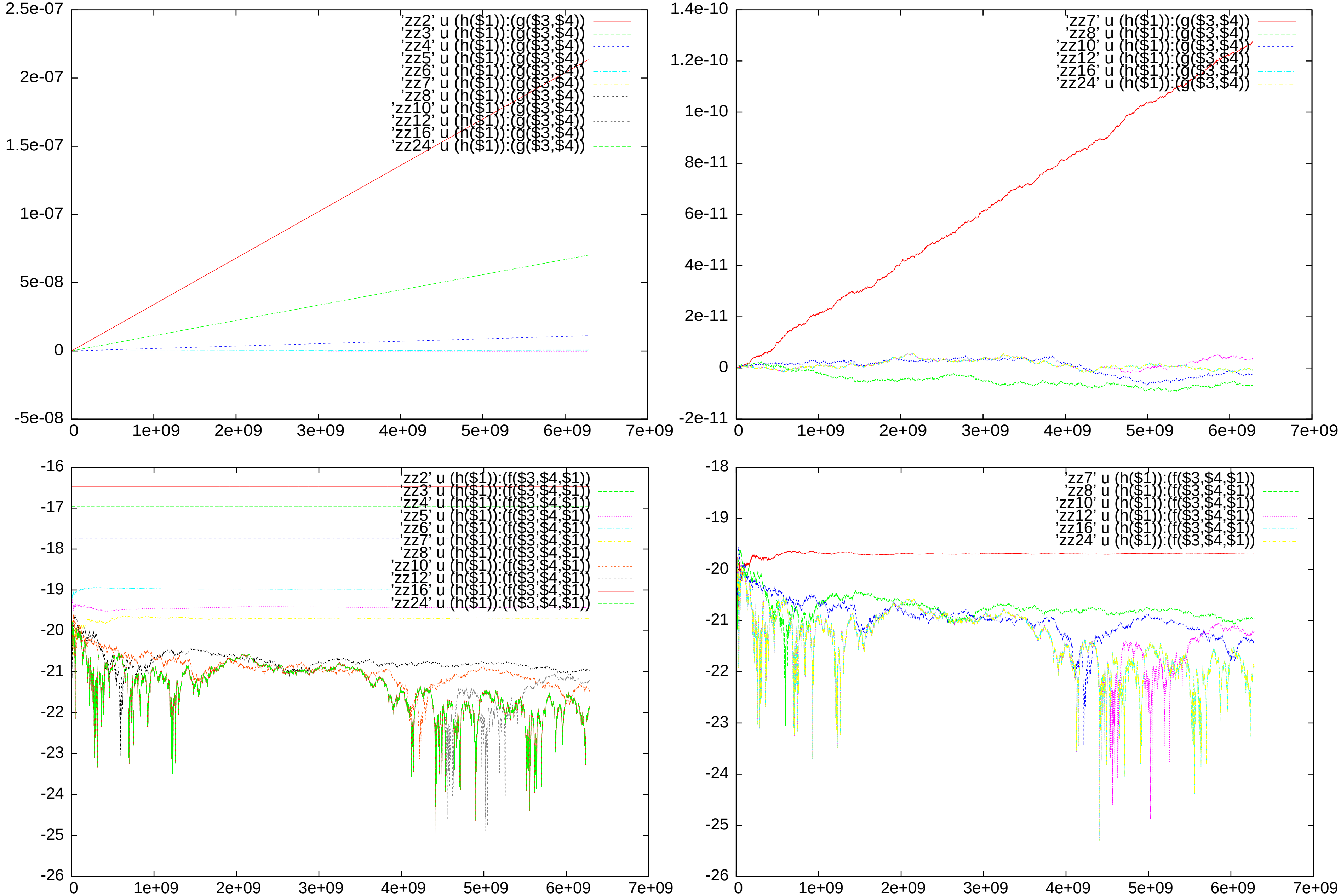
  Warning: If h_j is too large, one can miss the passage through the section.

Possible difficulties I: the harmonic oscillator

Some elementary **pitfalls** are shown in the next figures, for the integration of $\dot{x} = -y$, $\dot{y} = x$. The order used is $N = 24$, the local tolerance is 10^{-21} and the final time $2\pi \times 10^9$. Different runs use **multiplication** by $1/s$ instead of **division** by s to obtain $a_i^{(s)}$ from some s_0 on. The plots show **absolute errors** in energy as a function of t and the **average slope of the energy** as a function of t, both for several values of s_0.

It is also convenient to use **truncation errors small** compared to the ε **of the processor**, to avoid biases, so that these errors **can be neglected**. Only **round off errors** and its **propagation by the dynamics** turns out to be **relevant**.

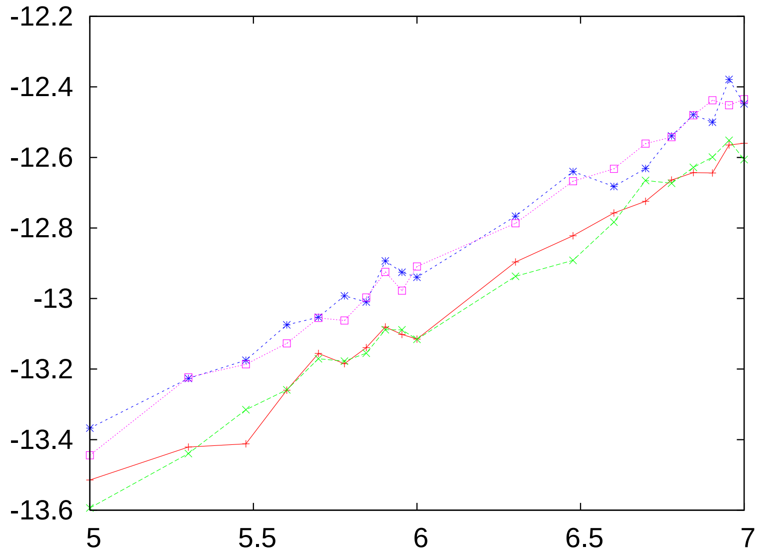
For very long integrations one should be warned that the accumulation of errors in the time updating $t + h \longrightarrow t$ or things like taking modulus 2π can produce relevant errors in timing.



2.5e-07
2e-07
1.5e-07
1e-07
5e-08
0
-5e-08
'zz2' u (h($1)):(g($3,$4))
'zz3' u (h($1)):(g($3,$4))
'zz4' u (h($1)):(g($3,$4))
'zz5' u (h($1)):(g($3,$4))
'zz6' u (h($1)):(g($3,$4))
'zz7' u (h($1)):(g($3,$4))
'zz8' u (h($1)):(g($3,$4))
'zz10' u (h($1)):(g($3,$4))
'zz12' u (h($1)):(g($3,$4))
'zz16' u (h($1)):(g($3,$4))
'zz24' u (h($1)):(g($3,$4))
1.4e-10
1.2e-10
1e-10
8e-11
6e-11
4e-11
2e-11
0
-2e-11
'zz7' u (h($1)):(g($3,$4))
'zz8' u (h($1)):(g($3,$4))
'zz10' u (h($1)):(g($3,$4))
'zz12' u (h($1)):(g($3,$4))
'zz16' u (h($1)):(g($3,$4))
'zz24' u (h($1)):(g($3,$4))
-16
-17
-18
-19
-20
-21
-22
-23
-24
-25
-26
'zz2' u (h($1)):(f($3,$4,$1))
'zz3' u (h($1)):(f($3,$4,$1))
'zz4' u (h($1)):(f($3,$4,$1))
'zz5' u (h($1)):(f($3,$4,$1))
'zz6' u (h($1)):(f($3,$4,$1))
'zz7' u (h($1)):(f($3,$4,$1))
'zz8' u (h($1)):(f($3,$4,$1))
'zz10' u (h($1)):(f($3,$4,$1))
'zz12' u (h($1)):(f($3,$4,$1))
'zz16' u (h($1)):(f($3,$4,$1))
'zz24' u (h($1)):(f($3,$4,$1))
-18
-19
-20
-21
-22
-23
-24
-25
-26
'zz7' u (h($1)):(f($3,$4,$1))
'zz8' u (h($1)):(f($3,$4,$1))
'zz10' u (h($1)):(f($3,$4,$1))
'zz12' u (h($1)):(f($3,$4,$1))
'zz16' u (h($1)):(f($3,$4,$1))
'zz24' u (h($1)):(f($3,$4,$1))
0
1e+09
2e+09
3e+09
4e+09
5e+09
6e+09
7e+09

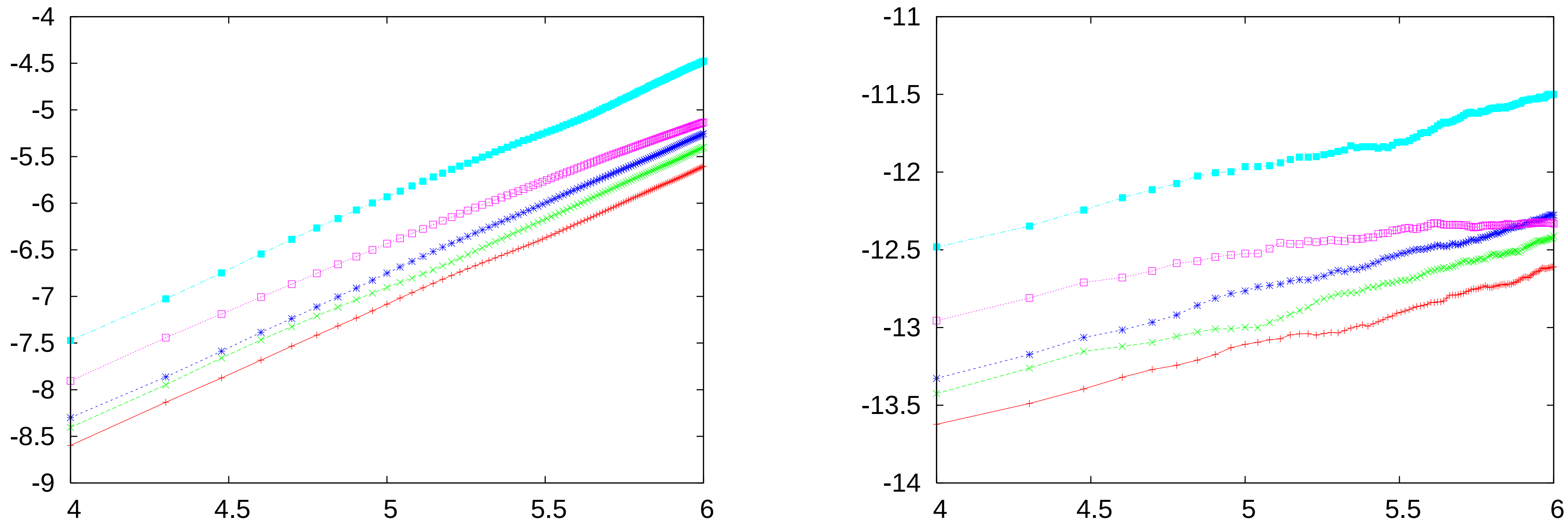
The effect of round off errors

Assuming **iid random round off errors** at each step the behaviour has strong relation with **the effects** of a **random walk**.
In the **conserved quantities** the **standard deviation** σ should increase as $\sqrt{N}$ (N =number of steps) and, hence, $\approx \sqrt{t}$.



The plot shows the case of the **harmonic oscillator** (i.c. (1,0)). In $\log_{10}$ scales the horizontal axis shows **number of revolutions** and the vertical one σ (for a sample of 100 runs with minor random initial variations) of (x, y) at **the end** and the same when **returning back** to $t = 0$.
Slope $\approx 1/2$.

But the h.o. is **isochronous**. If **angular velocities change when actions do** then under same assumptions σ must change like $\approx t^{3/2}$ for integrable systems.



Kepler's problem for $a = 1$, $T = 2\pi$ and $e = 0.1, 0.3, 0.5, 0.7, 0.9$. Left: σ of errors for the passage **time** through pericentre. Right: σ of errors in **energy**. Both has a function of **number of revolutions**. Everything in $\log_{10}$ scales. **Slopes:** $\approx 3/2$, $\approx 1/2$, respectively.
Taylor used to order 28 and truncation error set to 10^{-22}. 100 runs. Computing time: 140^{m} for $e = 0.1$, 558^{m} for $e = 0.9$. Note the **strange improved behaviour of** $e = 0.7$.

Possible difficulties II: the Kepler problem

For large eccentricity a relevant role is played by the singularities in complex time. Assuming elliptic motion with $a = 1$ we have

$$r = 1 - e\cos E, \qquad x = \cos E - e, \qquad y = \sqrt{1-e^2}\sin E,$$

where the eccentric anomaly E is related to the mean one M (and, hence to time) by Kepler's equation $t = E - e\sin E$.

At the pericenter $t = M = E = 0$ we have

$$x = 1 - e,\ y = 0,\ \dot{x} = 0,\ \dot{y} = \sqrt{(1+e)/(1-e)}.$$

Let $\delta = 1 - e$. Then $r = 0$ for $E = e^* = i\sqrt{2\delta} + O(\delta^{3/2})$, which gives

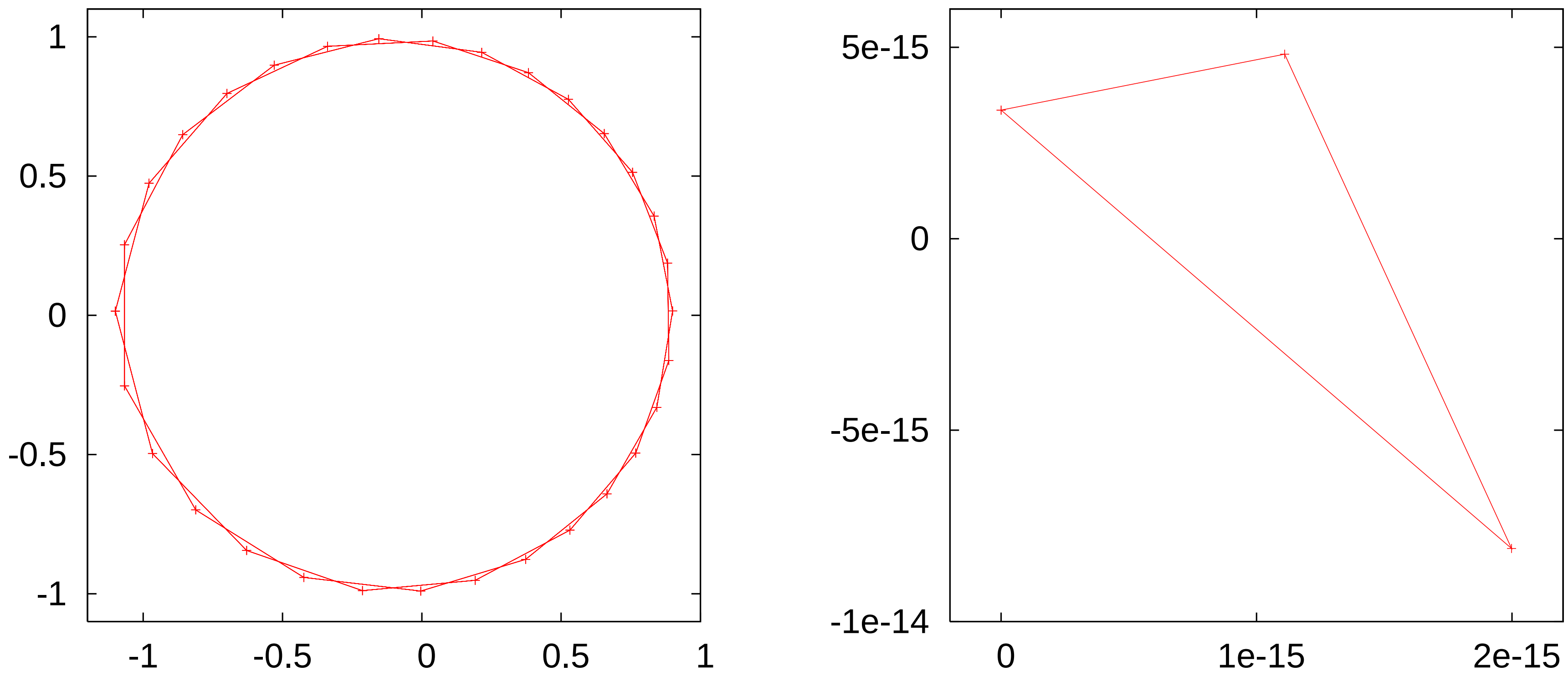
$$t = t^* = i\frac{2\sqrt{2}}{3}\delta^{3/2} + O(\delta^{5/2}).$$

The number of steps in the passage near pericenter is

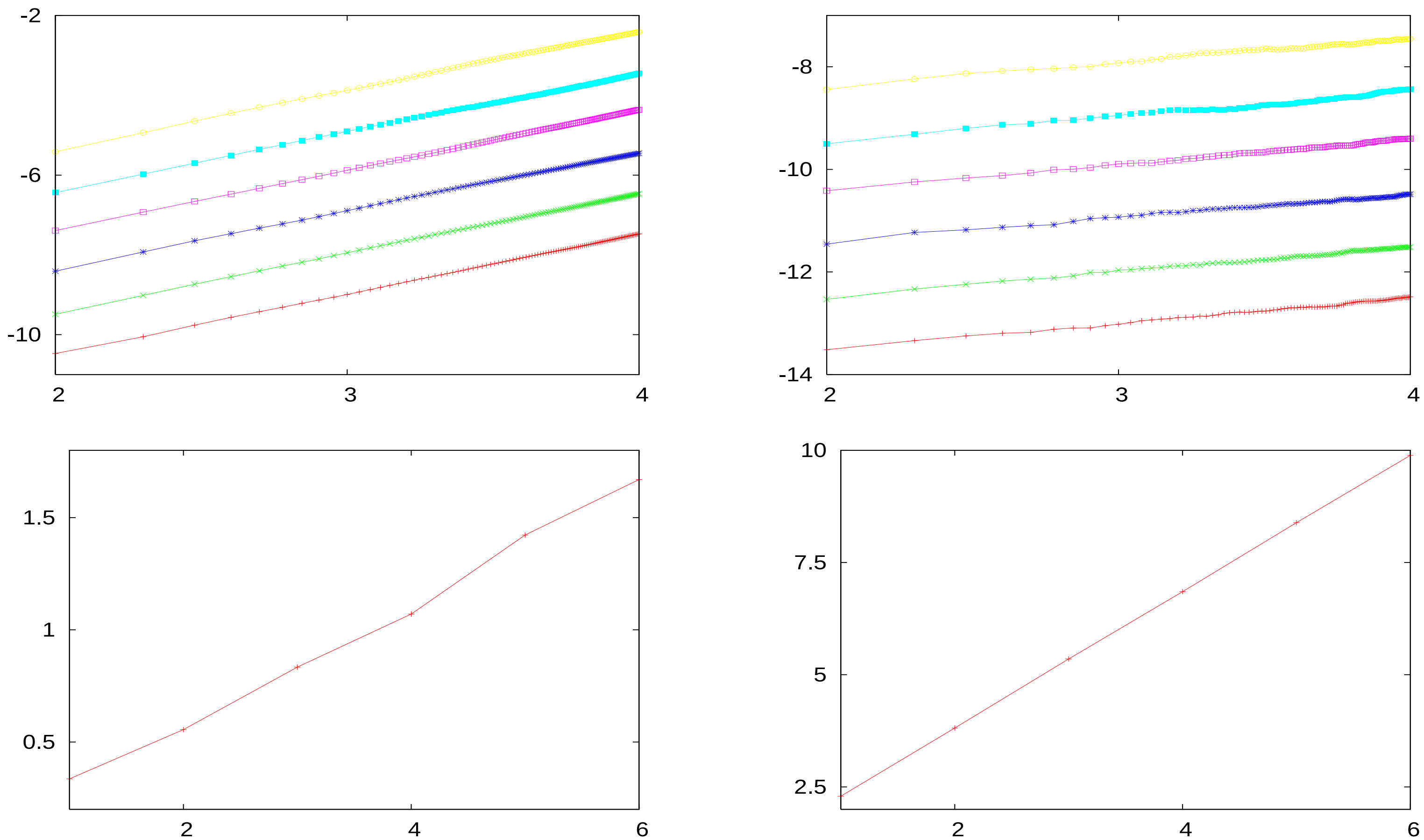
$$N = \int_{-c}^{c} \frac{\exp(2)dt}{\sqrt{t^2 + |t^*|^2}} = 2\exp(2)|\log(|t^*|,$$

i.e., it increases as $|\log(1-e)|$

Attractors of the algorithm

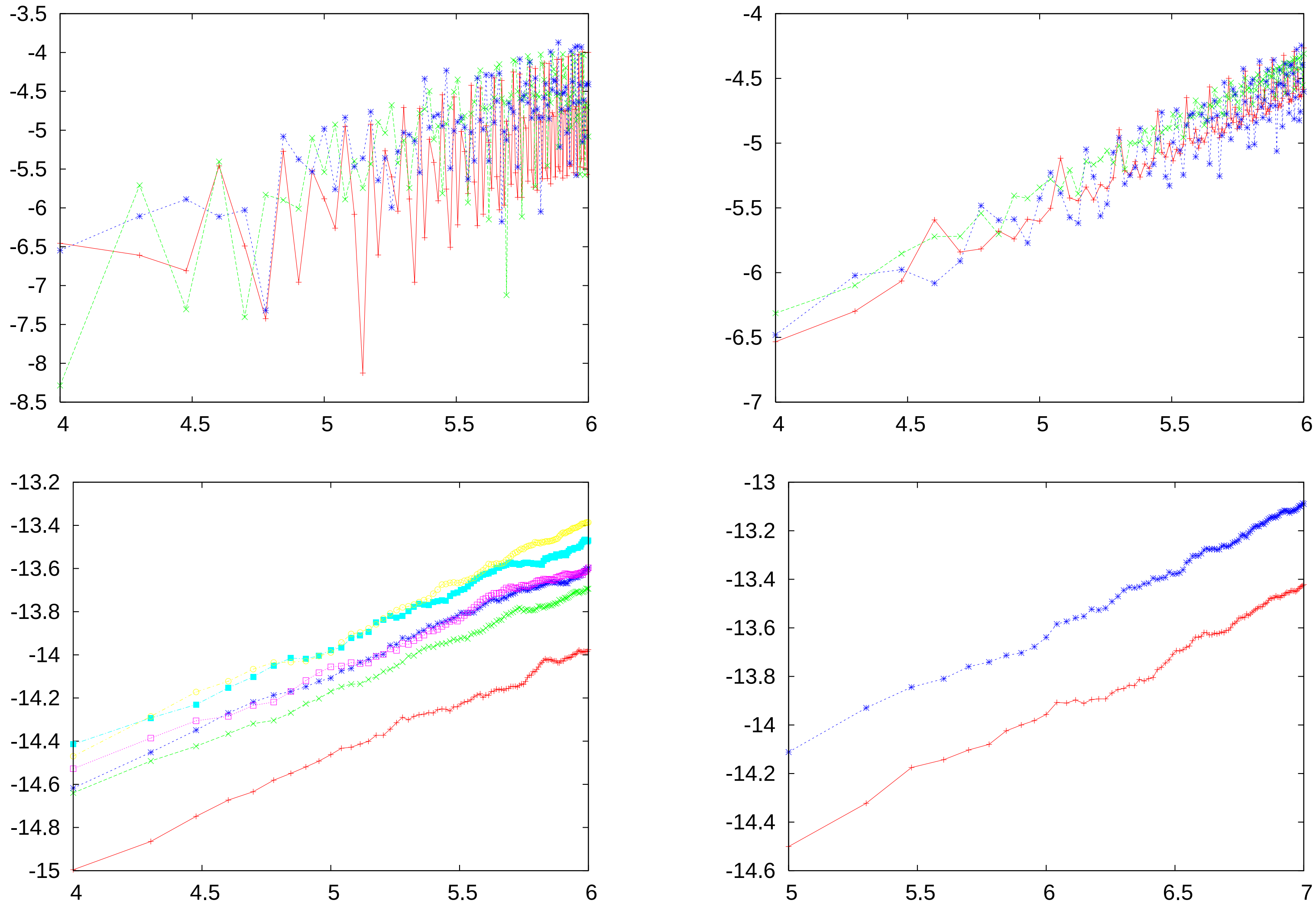


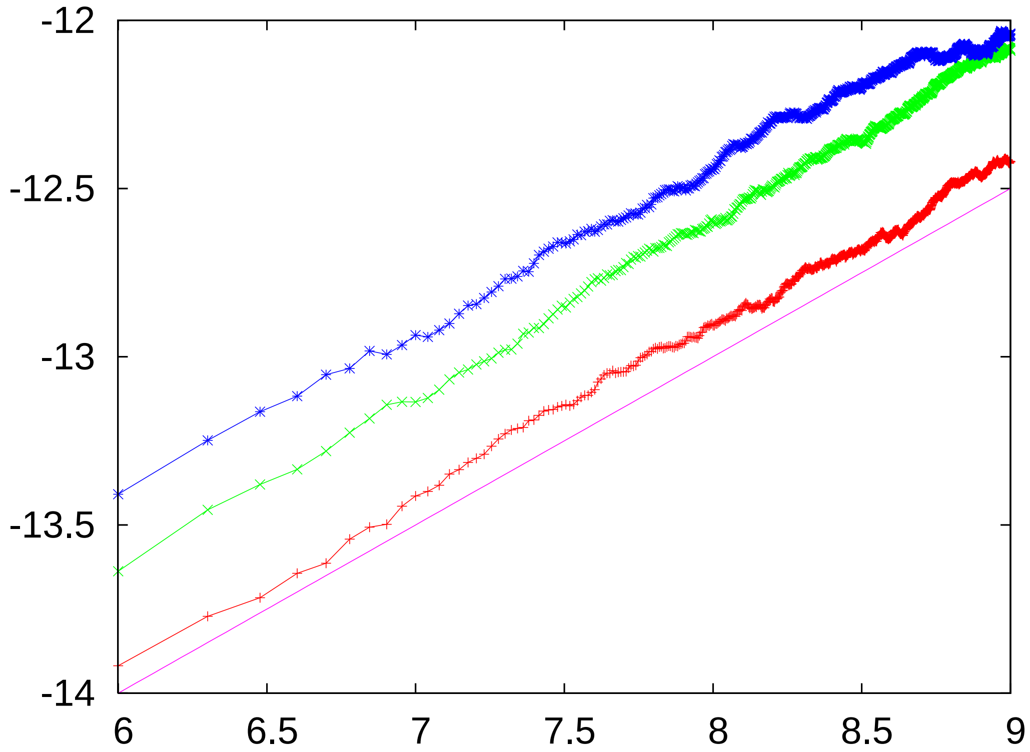
Orbit of Kepler's problem for $e = 0.3$ which is "eventually periodic under the algorithm". Left: 6 revolutions. Right: Computed points are plotted every 31 steps, using $(x - 1.099890709480805, y - 0.01470997998519)$.



Kepler's problem for large e. Top: For $e = 1 - 10^{-k}$, $k = 1$ to 6 from bottom to top we plot in $\log_{10}$ scales σ vs number of revolutions for pericenter time passage (left, slope 3/2) and energy (right, slope 1/2). Bottom: Computing time per revolution (in ms) (left) and $-\log_{10}$ of minimum step size (right), both as a function of k.

Hénon-Heiles system: non-integrable, both **regular and chaotic**





Similar to previous figures for a larger time interval. On top of the reference line $0.5x - 17$, we plot $\log_{10}(\sigma(h))$ for the energies $0.05, 0.125$ (in a regular orbit) and 0.1666666666 (in a caotic orbit). The horizontal scale contains $\log_{10}$ of the number of Poincaré iterates.

Horizontal axis: the number of Poincaré iterates.

Top left: σ for the **return time**.
Top right: σ for the **position errors**.
Energies: 0.050, 0.100, 0.125, **regular orbits**.
Respective **slopes:**$\approx 3/2, \approx 1$.

Bottom: σ for the **energy**. Both **regular orbits** (energies as before) and **chaotic** (energies 0.125, 0.166, 0.1666666666).
Right: larger number of Poincaré iterations.
Slope: $\approx 1/2$.

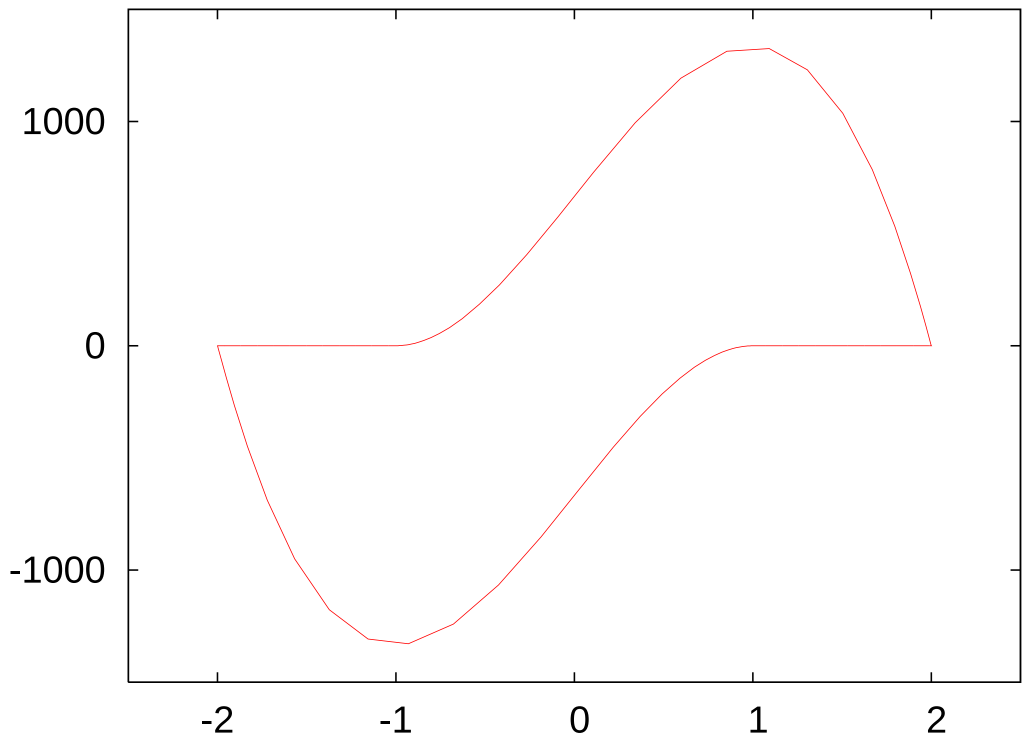
All data in $\log_{10}$ scales

Taylor to order 32 and truncation error 10^{-22}.

Similar behaviour for preservation of **energy and components of angular momentum** for the **Sun-Jupiter-Saturn-Uranus-Neptune** for 4.5×10^9 **years**. **Maximal error** $< 10^{-10}$, or $\approx 10^{-11}$ as relative error. Units: AU, 800 days.

Possible difficulties III: the van del Pol canards

van der Pol equation $\quad x' = y, \quad y' = \mu(1-x^2)y - x \quad$ is elementary, but it poses difficulties for explicit methods if mu large, due to the strong stiffness. Implicit methods behave much better. The figure illustrates the (canard) strongly stable periodic solution for $\mu = 1000$. Despite it looks simple, it requires of the order of 360000 steps to complete one revolution.



Some applications

Just to name a few:

- Computation of Poincaré maps, variational and higher variational equations, periodic orbits, invariant tori, invariant manifolds, homo and heteroclinic connections and transversality in relatively simple models, requiring a good accuracy,
- Analysis of models depending on several parameters, looking for bifurcations, Lyapunov exponents, rotation number, frequency analysis, etc, in massive simulations, where computing time is relevant,
- Problems where multiple precision is essential: splitting of separatrices for systems not far from integrable, detection of the order of regularity of some phenomena (e.g.: problems of regularisation),
- It has been applied to compute space missions, choreographies, etc.
- Combined with results of differential Galois theory, applied to compute monodromy matrices and groups, and higher order extensions, to detect non-integrability in Hamiltonian systems.

Still simple problem: Unfolding the Hopf-SN for flows in $\mathbb{R}^3$

At a **HSN bifurcation** eigenvalues are $0, \pm\mathrm{i}$. **Normal Forms**, which are formally rotationally invariant can be computed, unfolded under non-degeneracy conditions.

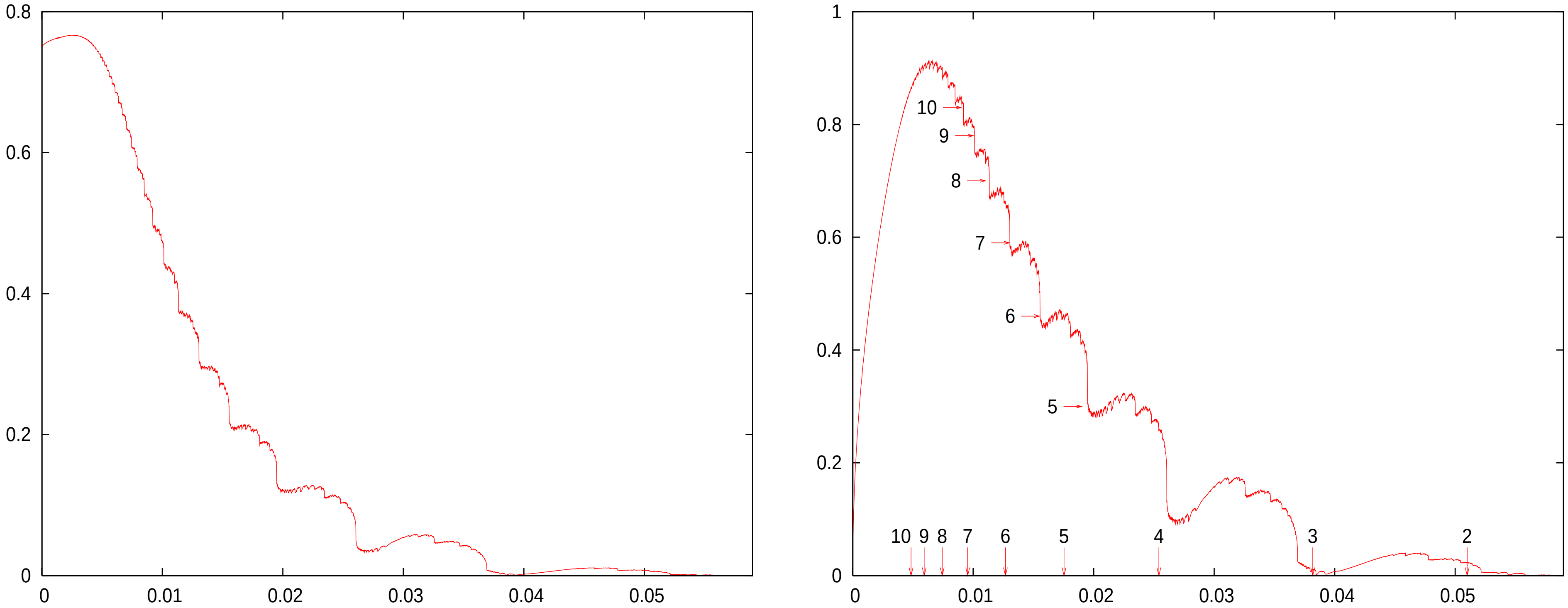
A **critical case** is the **volume preserving** one. It helps to understand other cases.

A simple model, representative enough, is **Michelson system**, appearing also as traveling waves in the **Kuramoto-Shivasinski** PDE.

$$x''' = -1 + ax' + x^2,\ a < 0.$$

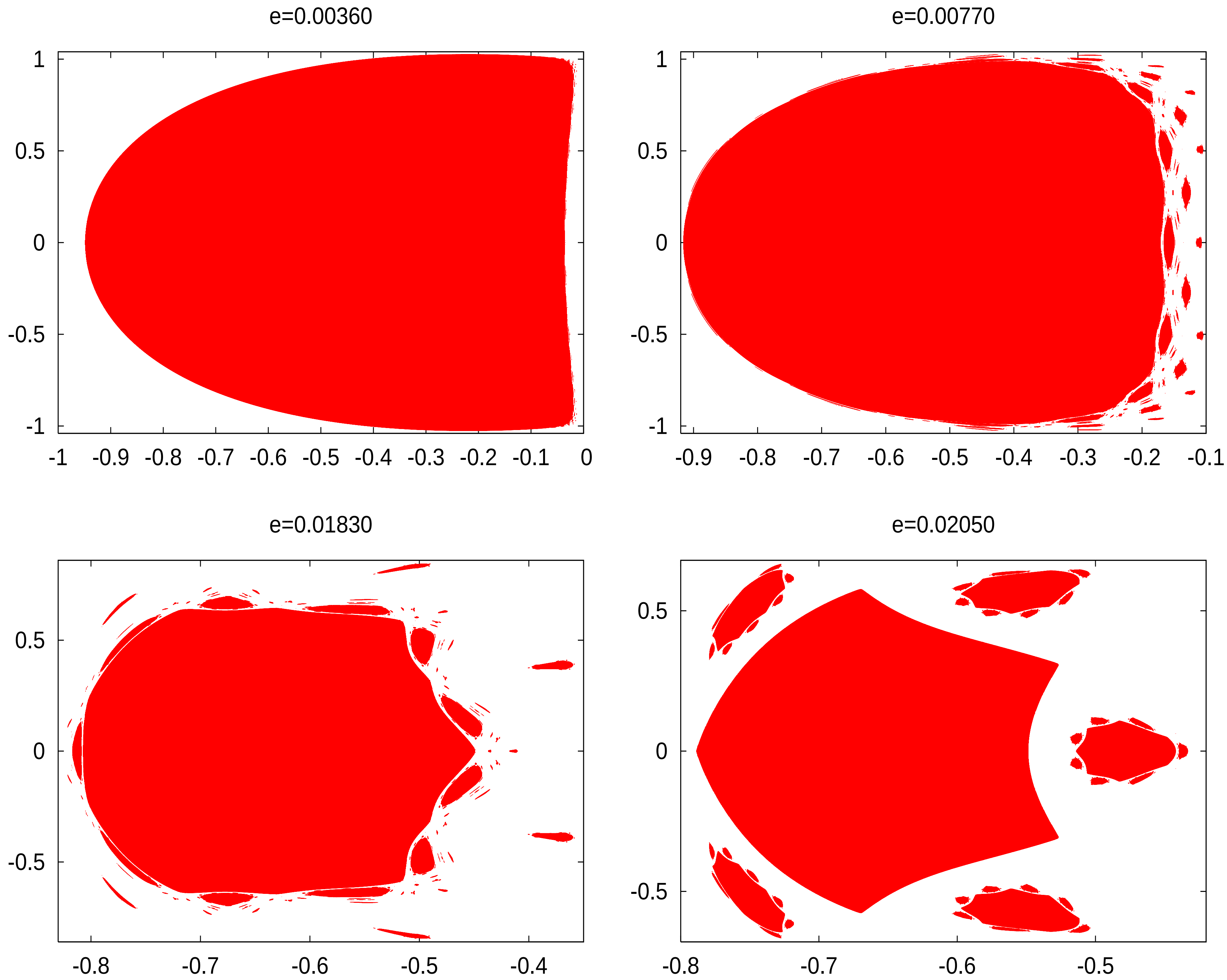
It is convenient to use as parameter $\varepsilon = (-a)^{-3}$. The NF to any order has **connecting 2D and 1D** invariant manifolds of $(\pm 1, 0, 0)$. Real system has **splitting exponentially small in ε**.

Use $x'' = 0, x''' > 0$ (in suitable domain) as **Poincaré section**. The **Poincaré map** preserves a **measure** in the (x, x')-plane.

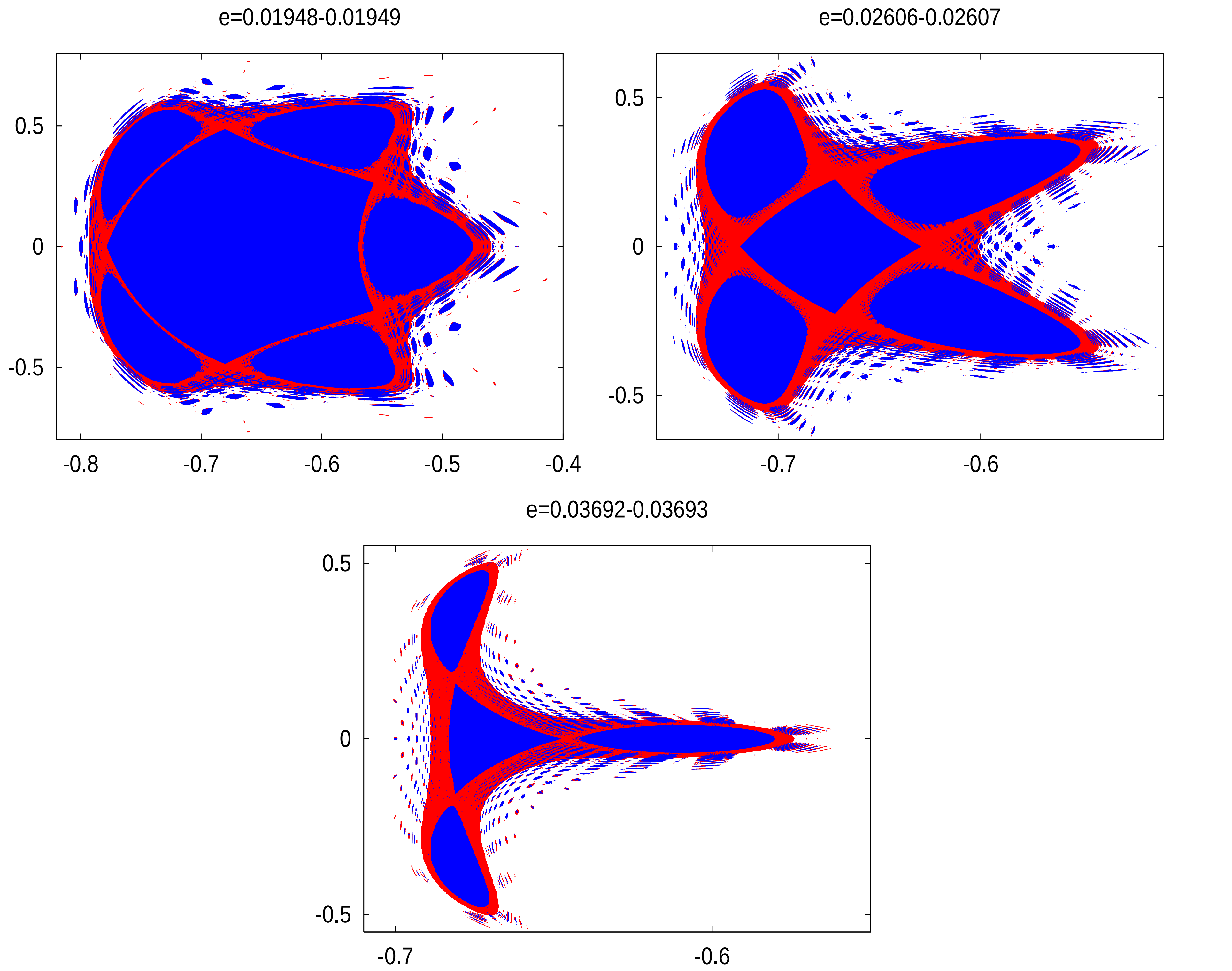


Left: Statistics of the number of points which **do not escape** as a function of ε using a 4000×4200 grid. Right: The values have been multiplied by ε, to better see the details. The location of some **resonances**, both at the **central elliptic fixed point** and at the **breakdown of invariant curves** which disconnects the related islands is shown.

Let us see some **examples** and a **movie**.



e=0.00360
e=0.00770
e=0.01830
e=0.02050


e=0.01948-0.01949
e=0.02606-0.02607
e=0.03692-0.03693
0.5
0
-0.5
-0.8
-0.7
-0.6
-0.5
-0.4

Stability of the planar circular RTBP for outer motion

Next we consider a problem similar to the Hénon map, but requiring numerical integration: The planar **circular RTBP** which describes the motion of a massless particle (think about an **asteroid**) under the gravitational attraction of two primaries (think about **Sun** and **Jupiter**) moving in circular orbit about their center of masses. The **mass ratio** of the system is $\mu = m_J/(m_S + m_J)$.

The system has a **first integral** known as **Jacobi integral** C. For large enough values of C the level of this function has 3 **connected components**: Two close to the primaries and the third one away from them. Motion in this **outer region** corresponds to bodies with perihelion at a distance larger than the Sun-Jupiter distance.

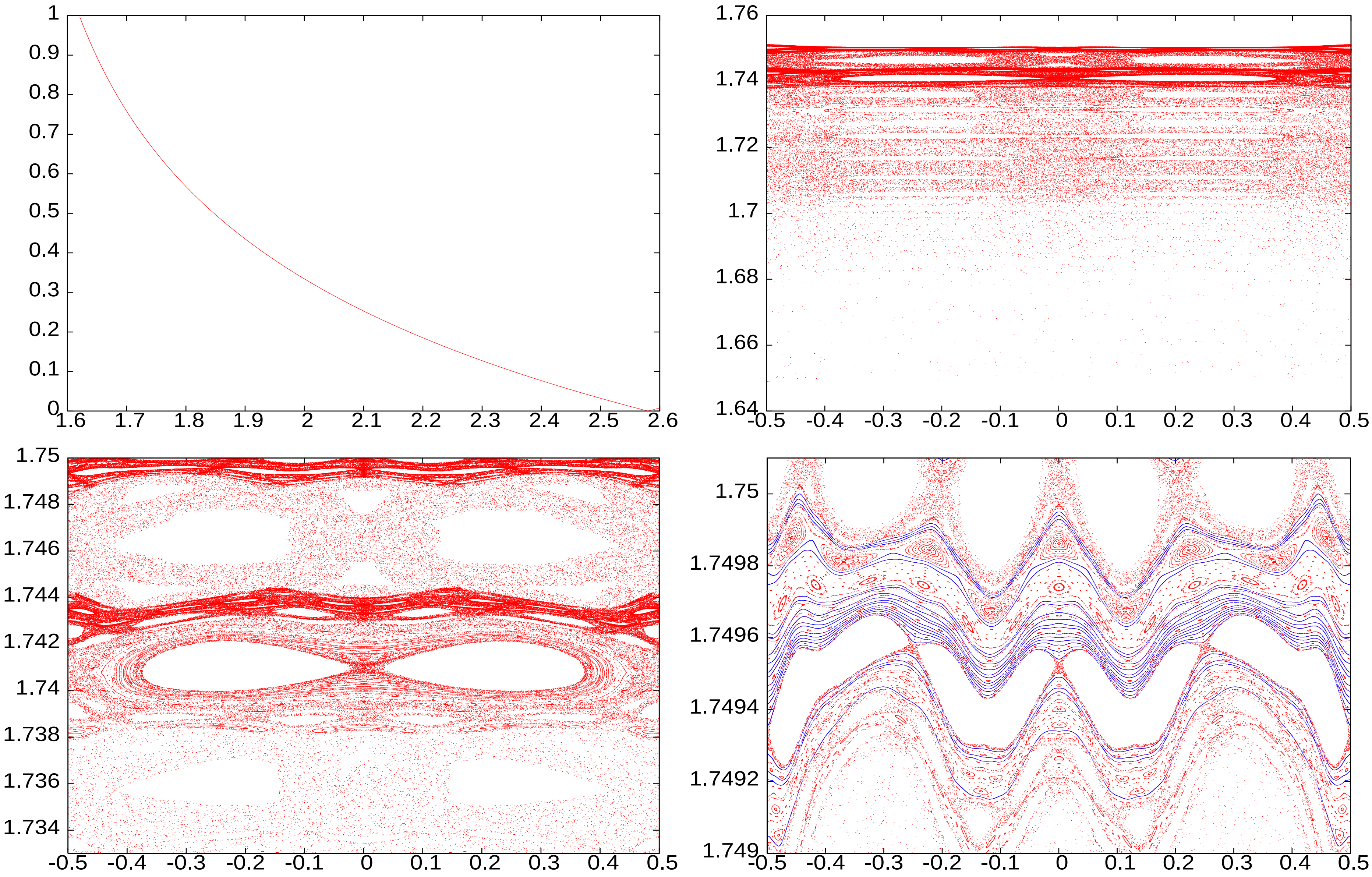
Neglecting Jupiter the motion of the massless body will be a two-body motion around the Sun. If we fix e=**eccentricity** (the so-called **osculating eccentricity**) then C determines the value of the semiaxis a.

Theorem (**V. Kaloshin et al**, in preparation)
Given μ and $C > C_0$ there exist $e_0(\mu, C)$ such that if $e > e_0$ there are bodies with initial osculating eccentricity e which escape to infinity.

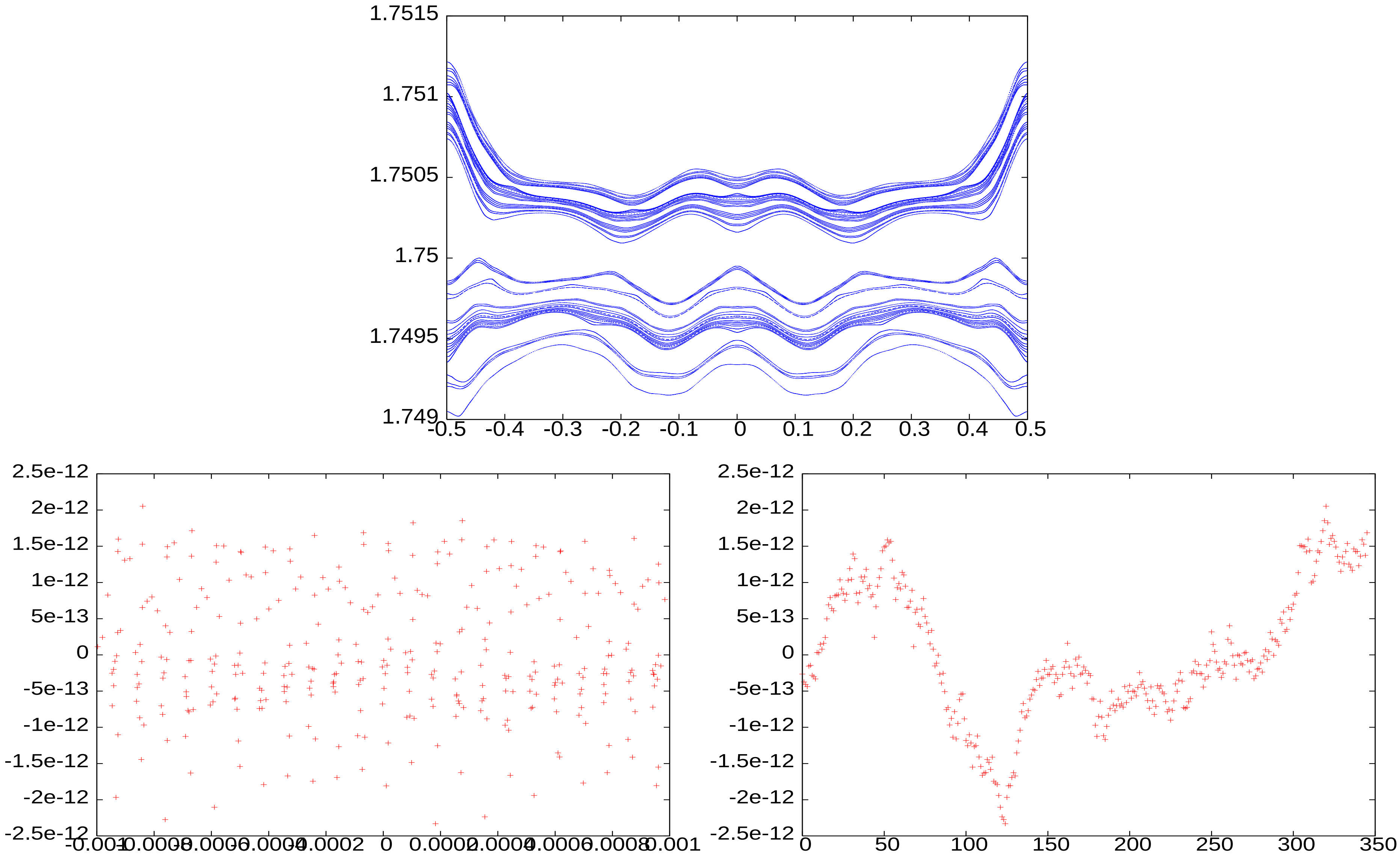
For $\mu = 10^{-3}$, $C = 3.6$ the estimated value of e_0 is 0.75. Proof and application are based on Aubry-Mather theory.

Next figures show orbits of different initial points on the **Poincaré section corresponding to pericentre passages**.

Top left: e as a function of $q = a(1-e)$ for $\mu = 0$. Top right: images of several initial conditions. For $q < 1.742$ (and, probably, $q < 1.748$) there are points in a chaotic zone which can escape. For $q > 1.750$ points seems definitely that do not escape because of the existence of invariant curves. Bottom figures show magnifications. In blue a set of **invariant curves**. In the next page the curves are shown alone. For one located at the bottom (with initial $q = 1.74946$, leaving orthogonally from x axis), the iterates following on a **narrow window** and **how they occur** as a function of the number of iterates in the window are shown. **Final time** 11.25×10^7 **Jupiter revolutions**.



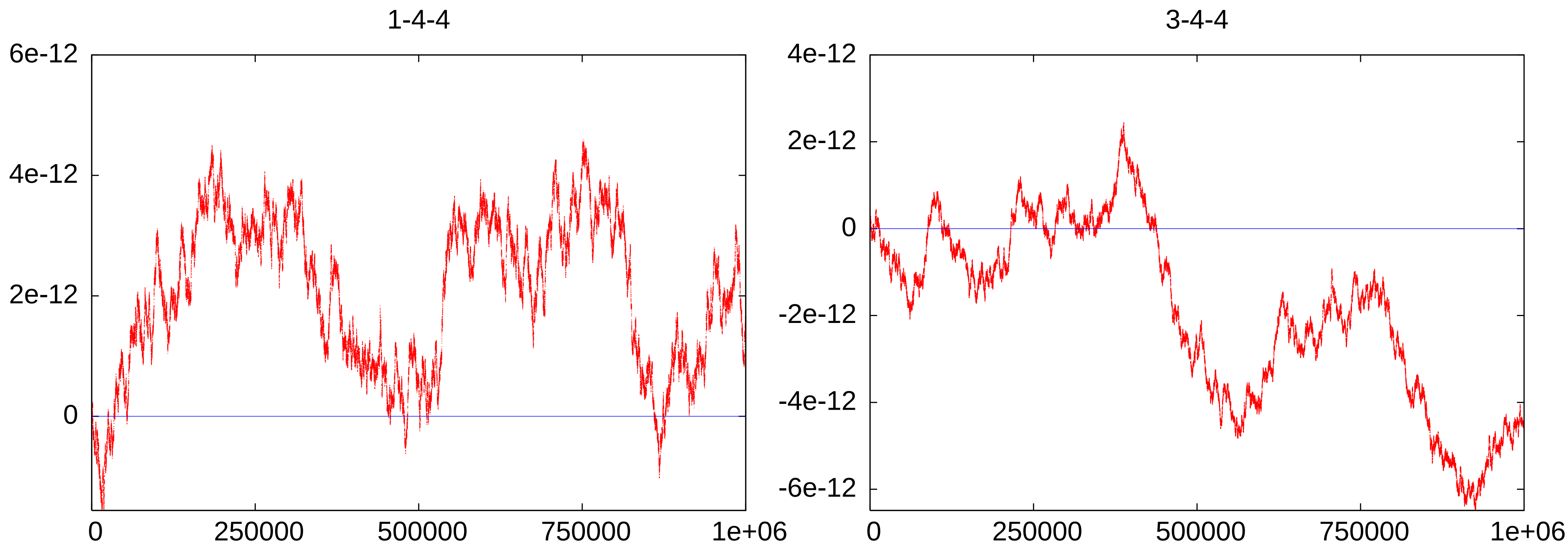
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
1.6
1.7
1.8
1.9
2
2.1
2.2
2.3
2.4
2.5
2.6
1.76
1.74
1.72
1.7
1.68
1.66
1.64
-0.5
-0.4
-0.3
-0.2
-0.1
0
0.1
0.2
0.3
0.4
0.5
1.75
1.748
1.746
1.744
1.742
1.74
1.738
1.736
1.734
-0.5
-0.4
-0.3
-0.2
-0.1
0
0.1
0.2
0.3
0.4
0.5
1.75
1.7498
1.7496
1.7494
1.7492
1.749
-0.5
-0.4
-0.3
-0.2
-0.1
0
0.1
0.2
0.3
0.4
0.5



These results seem to support $0.652 < e_0 < 0.666$ for the chosen μ, C.

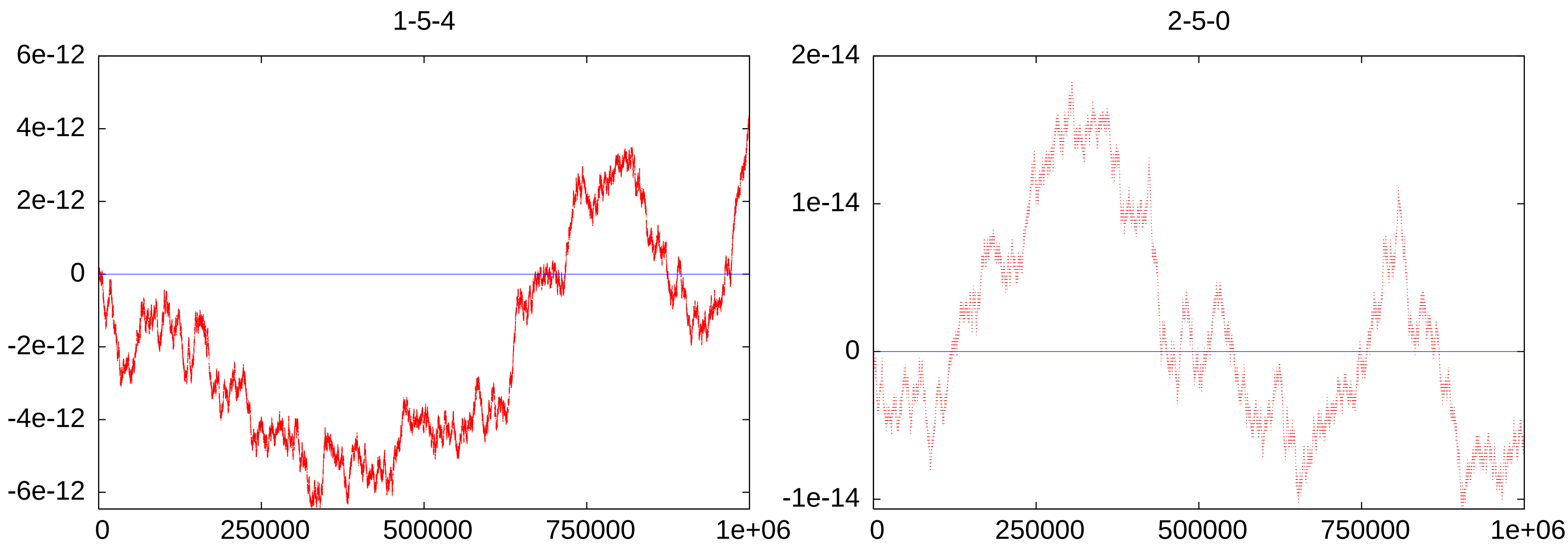
Long range integrations for the 3D RTBP

For $\mu = 0.001$ a test on a sample of 175 i. c. as follows has been done: $\Delta\rho = -0.001\,(0.0005)\,0.001$; $\alpha = 0.1\,(0.05)\,0.4$; $z = 0\,(0.2)\,0.8$, heliocentric, from a suitable origin of $\rho = R - 1$, α in revolutions. 75 points subsist for $t_{\text{final}} = 10^9$ Jupiter revolutions ($\approx$ **twice the age of Solar System**).



Shown: the 3 worst cases and the best one.

Errors w.r.t. initial energy displayed every 1000 Jupiter revolutions (J.r.).



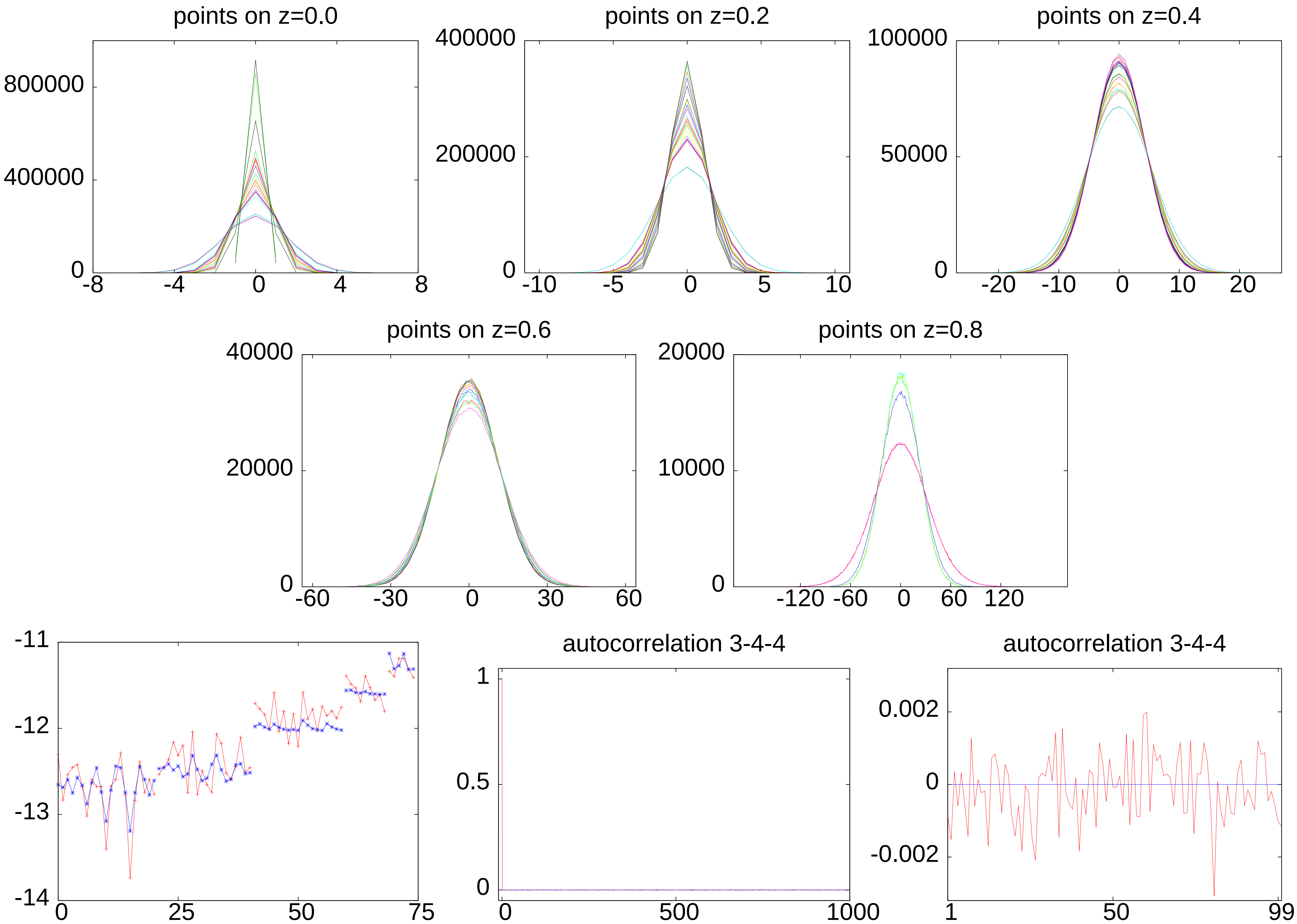
Average **integration step**: Worst case 0.36 years; best case 2.08 years.

Relevant statistics:

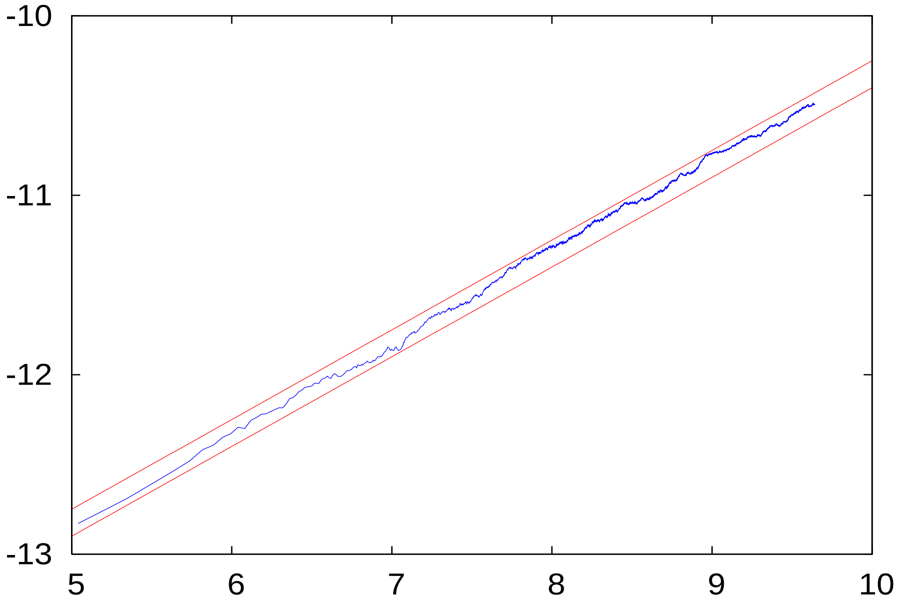
a) Histogram of **jumps of energy** every 1000 J.r. Unit: $2^{-52} \approx 2\times10^{-16}$,

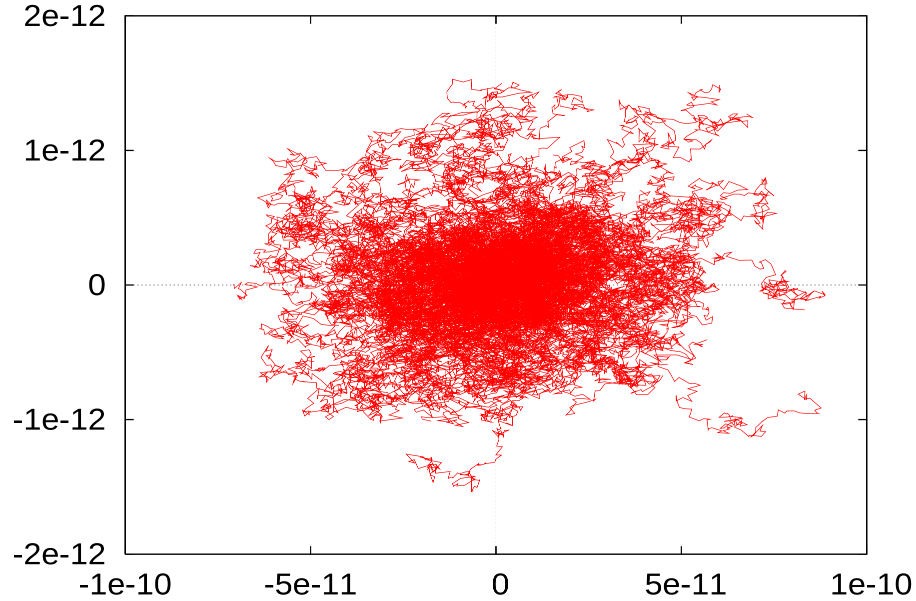
b) **Maximal errors** and $1\,\sigma$ predicted values from previous plots,

c) **Autocorrelation** for the worst case.


points on z=0.0
800000
400000
0
-8
-4
0
4
8
points on z=0.2
400000
200000
0
-10
-5
0
5
10
points on z=0.4
100000
50000
0
-20
-10
0
10
20
points on z=0.6
40000
20000
0
-60
-30
0
30
60
points on z=0.8
20000
10000
0
-120
-60
0
60
120
-11
-12
-13
-14
0
25
50
75
autocorrelation 3-4-4
1
0.5
0
0
500
1000
autocorrelation 3-4-4
0.002
0
-0.002
1
50
99

Motion of the giant planets for the age of the solar system





For the 5 body problem (Sun, Jupiter, Saturn, Uranus, Neptune), integration from 20061104 0^h UT with initial JPL data, for $\approx 4.38 \times 10^9$ years. Unit of distance 1 AU, of time 800 days. Number of runs with minor random initial variations: 100. Left: Standard deviation of variation in energy vs time in years, both in $\log_{10}$ scales. The two companion lines are $0.5x - 15.25$, $0.5x + 15.4$. Right: The random variation of (h, c_y) for the 100 runs. Interval between consecutive points $\approx 10.95 \times 10^6$ years.

The worse case, concerning variation of energy along the 100 integrations and the full time interval, shows a variation of 8×10^{-11}. Taking into account the units used, this means a relative variation below 10^{-11}. For the z component of the total angular motion the results are a smaller but similar. Concerning the x and y components, the variations are much smaller, with errors of the order of 10^{-12}.

The computations have been done using order 28, and a typical average integration step is 9 months.

All the computation were done in parallel, using a cluster of Xeon at 3.2 GHz under the operating system Linux Debian 2.6.12.2. The program was done in FORTRAN and compiled with g77 with the optimization level O3.

In a preliminary implementation each one of the runs (and therefore of the full process in parallel) lasted for 203 hours of CPU time. We are working at present on different improvements that should give an important reduction of the computing time.

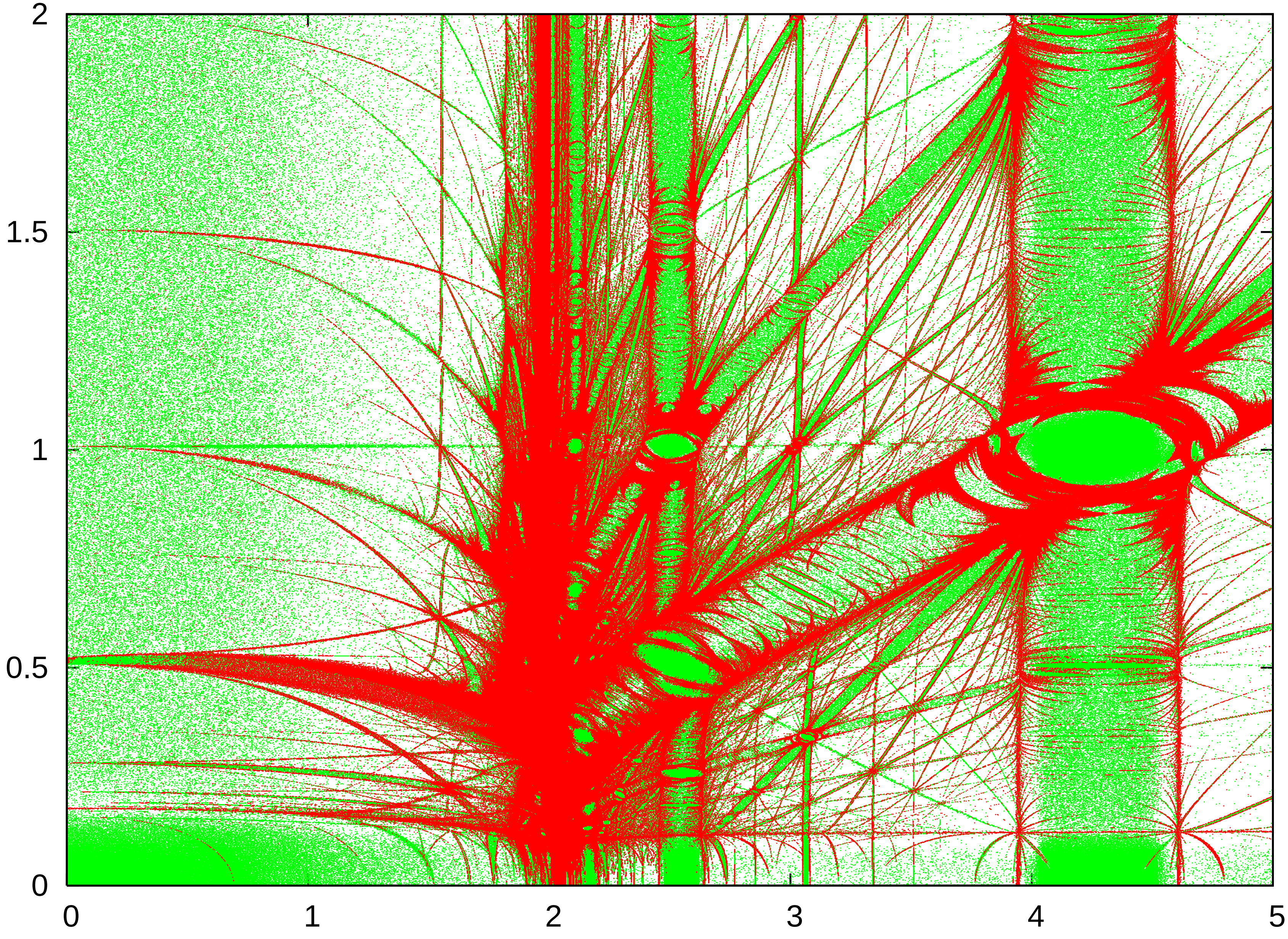
A couple of examples of Arnol'd diffusion

Consider the Hamiltonian

$$H(y, I_2, I_3, x, \theta_2, \theta_3) = \frac{1}{2}(y^2+I_2^2)+I_3+\varepsilon(\cos x-1)\,(1 + \mu(\cos\theta_2 + \cos\theta_3))\,,$$

the **classical Arnol'd example of diffusion**, but **considering both parameters, ε and μ, as equal**.

The plot shows the different degree of chaoticity of the motion. The horizontal (resp. vertical) axis corresponds to the y variable of the pendulum (resp. the initial frequency, ω, of the angle θ_2.) Computed using **MEGNO** (Mean Exponential Growth of Nearby Orbits, a refined Lyapunov indicator).



2
1.5
1
0.5
0
0
1
2
3
4
5

Next example is a **toy model of 3D galaxy**

$$H(\mathbf{p},\mathbf{q}) = \frac{\mathbf{p}^2}{2} + \frac{1}{4}(x^4+y^4+z^4) + \epsilon x^2(y+z),$$

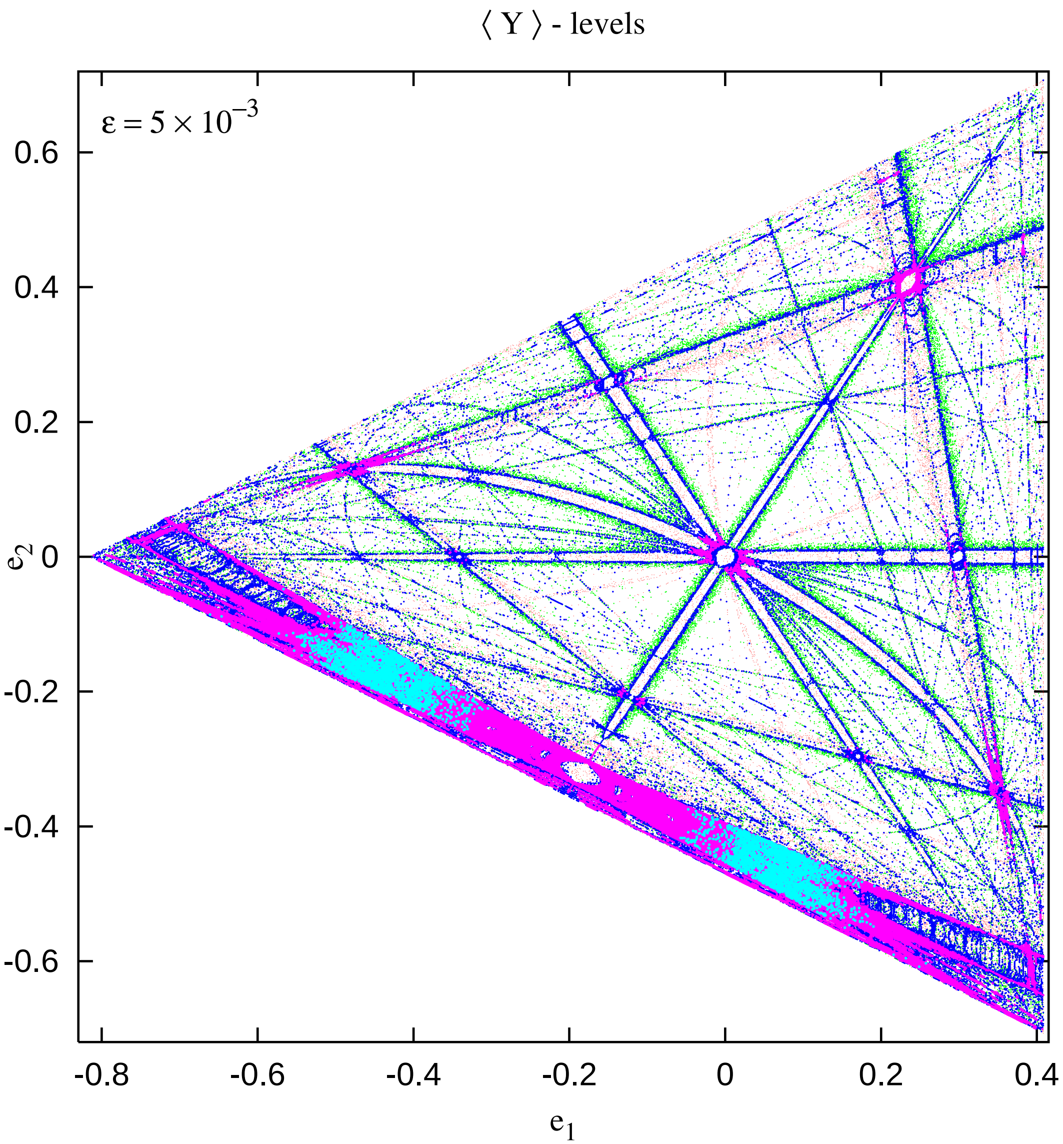
which was investigated on a given level of energy and for different values of ϵ. It can be seen as a **perturbation to the integrable 3D uncoupled quartic oscillator**, as long as $\epsilon \ll 1$.

Let h_i, $i = 1,2,3$ be the energies of each one of the quartic oscillators for $\epsilon = 0$, to be considered as unperturbed actions. We introduce

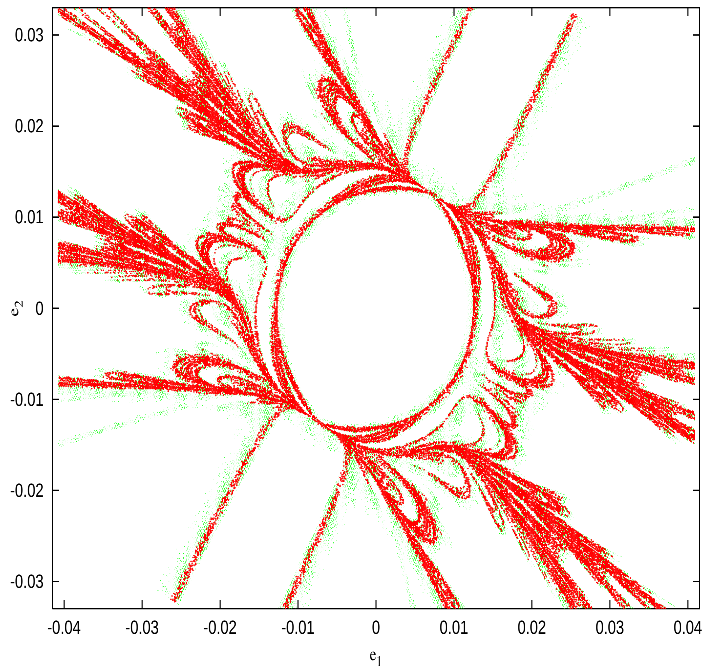
$$e_1 = \frac{1}{\sqrt{6}}(h_1 - 2h_2 + h_3), \qquad e_2 = \frac{1}{\sqrt{2}}(h_1 - h_3),$$

rotated actions for better visualization. The plots show results on the levels of MEGNO for $\epsilon = 5\times 10^{-3}$ with step $10^{-3}h$ in h_1 and h_2. The right plot shows a zoom around the intersection of resonances at the origin, computed with stepsize $10^{-4}h$. The **resonance web** is clearly detected.

Similar results are obtained with **more realistic** models of galaxies (e.g., with logarithmic potential plus corrections).



⟨ Y ⟩ - levels
ε = 5 × 10^−3
0.6
0.4
0.2
e2 0
-0.2
-0.4
-0.6
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
e1



0.03
0.02
0.01
e2 0
-0.01
-0.02
-0.03
-0.04
-0.03
-0.02
-0.01
0
0.01
0.02
0.03
0.04
e1

Additional comments on Taylor method

Taylor method is **suitable and fast provided** the system to be integrated is analytic (or $\mathscr{C}^r$ with r big), with a r.h.s. which is not difficult to evaluate explicitly.

With some **caution** in the selection of order and truncation error can be **as accurate as possible**, the main source of error being round off errors, beyond, of course, the **unavoidable propagation of errors by the dynamics if this one is unstable**.

It seems to be **unbeatable** when faced to **multiprecision**.

It is **not suitable** for **stiff systems, systems for which the evaluation of the vector field has to be done by indirect methods (as in some evolution PDE) or for systems whose solutions pass close to singularities which occur for complex time with small imaginary part**.