A personal view on Dynamical Systems

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Introduction

Dynamical systems (D.S.) study the behaviour and properties of solutions of differential equations and of the sequences obtained by iteration of a given map. They are a key tool in the study of mathematical models of the evolution of phenomena in the physical world and of the simplified models which help to understand them. These models can be given in deterministic form, either by means of ordinary differential equations, partial differential equations or discrete maps, or can include also some noise or random process. They are useful in all domains of science and technology.

In some sense their origin goes back to Poincaré who realised the importance of a qualitative approach. Poincaré himself largely developed the perturbative approach, when the system under study is close to some simple system. At present one cannot dissociate the qualitative and quantitative aspects. Each one of these aspects contributes to a better understanding of the D.S.

In the study of D.S. tools from all areas of mathematics, from algebraic topology to acceleration of convergence in numerical algorithms, or from asymptotic expansions to differential Galois theory, are used. But for systems with some degree of complexity, with some minimal relation to realistic models, it is impossible to produce a fairly complete description of the evolution in the space of states, and its dependence with respect to parameters, without using refined numerical techniques. They are essential for concrete applications and very useful even for theoretical studies. They can be seen as an experimental part of mathematics. This has been powered by the availability of large arrays of processors working in parallel at a reduced cost. But the impact of new algorithms has been even greater, making it possible to address problems of even greater complexity.

A key idea is the representation of the evolution in some state or phase space. Or, better, in some space (SPS) which includes the state and the parameters which a given family of models contains. To understand the possible evolutions of a D.S. and its character (e.g., regular or chaotic) the knowledge of some geometrical objects in this SPS is basic. In some cases the use of a complex time and complex space of states-parameters can be essential to uncover some properties of the real dynamics.

Invariant objects

Let us consider the following fundamental steps to "understand" a system:

- To identify simple objects, such as fixed points and periodic orbits, or other invariant objects (i.e., left invariant by the flow or the map), such as spheres, tori, etc, in the state space.

- To study the stability of these objects. Assume they are unstable and it is possible to escape from their vicinity as time increases. Then we want to describe the set of points (the unstable manifold) which tend to the invariant object when going backwards in time. In a similar way we can study stable manifolds (if any), i.e., the sets of points tending to the invariant object ahead in time.

- To look at the possible connections between unstable and stable manifolds of the same or different invariant objects. These are the so-called homoclinic and heteroclinic orbits. Under some conditions (transversality of the manifolds, at least in some weak sense) they give rise to complicated dynamics, diffusion, strange attractors, etc.

  It became popular to denote this dynamical behaviour as chaos, although it can be very clearly explained in terms of homoclinic orbits. The more relevant feature is the lack of predictability, even in the deterministic case. The invariant objects and their possible connections constitute the "skeleton" of the dynamics. They provide a key tool to understanding the full dynamics.

- All the previous objects and structures can depend on parameters (e.g., physical parameters of the model). Hence, it is interesting to check the stability (in some sense) of the behaviour when the parameters change, which kind of changes or bifurcations are possible and the different patterns of the corresponding dynamics. This can be useful, for instance, if one can act on the parameters to obtain the most desirable effects.

- In many cases a deterministic description of the dynamics is not possible, even for simple systems. It must be done in probabilistic terms. This is a fortiori true for non-deterministic D.S. The "description" of the D.S. must then be done in terms of an evolving measure.

Relevance of different features

Some typical scenarios allow to prove the existence of interesting features of the dynamics like horseshoes, multiplicity of sinks, different kinds of strange attractors, Arnol'd diffusion, exponentially small splitting of separatrices, multiple resonances, travelling waves, etc. To understand the qualitative aspects of the D.S. it is essential to know all these nice results.

But one should also put them in perspective. For a given family of D.S., even if they occur in a generic way, they can play a major or a minor role. In other terms, one should try "to measure" whether they occur in a large fraction of the interesting SPS or if they are confined to extremely narrow domains. It can also occur that its role depends on the size of the parameters. A classical example is the well-known Hénon-Heiles Hamiltonian. For energy below $h = 0.05$ it is hard to recognise that it contains some amount of chaotic behaviour, while this can be seen for $h = 0.10$ and it is predominant for $h = 0.15$.

Normal forms are a very useful device to study the vicinity of invariant objects and, in some cases, they are enough to prove the existence of a rich dynamics. However the domain of validity, that is, the region of the SPS in which they are an approximation of the true system which is good enough (for some purpose), has been the subject of less study. I should insist again on the importance of a combined qualitative-quantitative approach and in the direction of the so-called "converse results". That is, when the existence of some interesting object (e.g., KAM tori) is no longer true. The gap between positive and negative results is still extremely large in many problems.

Another important point is the study of transient regimes. In many problems one can describe quite accurately the possible limit states of the D.S. However, depending on the time scales, the model can lose its validity long before the dynamics approaches a limit state. Transients can be more important than the limit.

An elementary example

As a very simple example we can discuss the stability of the triangular points in the spatial restricted three-body problem (3DRTBP). It describes the motion of a massless particle (say a small asteroid A) under the gravitational action of two primaries S, J (say, Sun and Jupiter) which move in circular orbits, in $z = 0$, around their common center of mass. Let $m_S > m_J > 0$ be the masses of the primaries, $\mu = m_J/(m_S + m_J)$ the mass ratio and consider an uniformly rotating system of coordinates (synodical coordinates) in which the primaries remain at rest. Assume the units are normalised in such a way that the total mass is 1, the distance between S and J is 1 and the period of the primaries is 2π. From Euler and Lagrange we know that there are 5 relative equilibrium points, in $z = 0$, in which the centrifugal force cancels the Newtonian attraction. Two of them, the so-called triangular points L_4 and L_5, are linearly stable for $\mu \leq \mu_1 \approx 0.0385$. They are also nonlinearly stable in the planar motion except for $\mu = \mu_2 \approx 0.0243, \mu_3 \approx 0.0135$. The other three relative equilibrium points, L_1, L_2 and L_3, are called collinear.

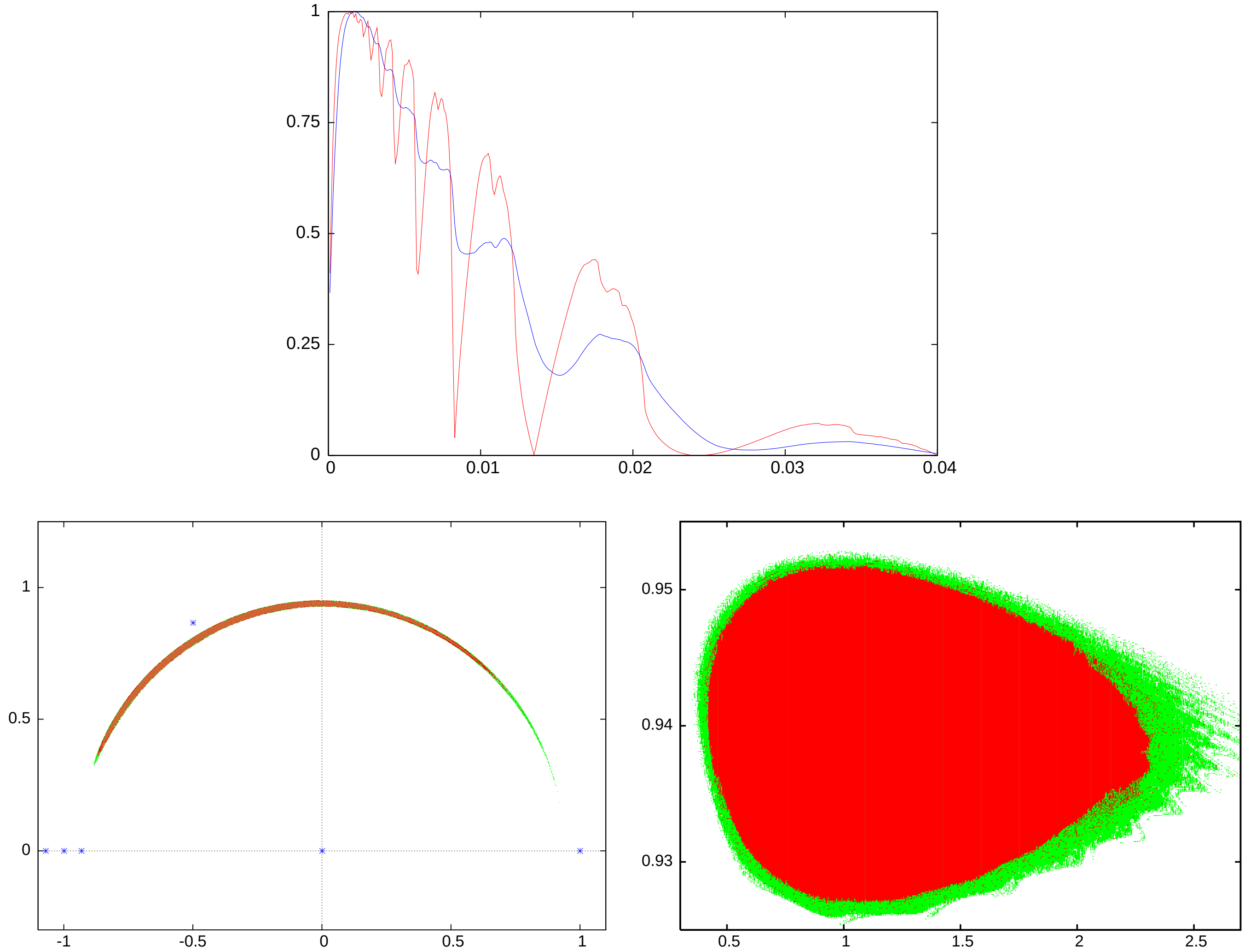

Figure 1: Top: Relative abundance of points in the position space which, starting the motion with zero synodical velocity, do not leave an extended vicinity of L_5 (see text) in 10^4 revolutions of the primaries. In red (resp. blue) the behaviour as a function of μ for the planar (resp. 3D) problem. The plots are normalised to a maximum of 1, attained for $\mu = 0.0014$ (resp. $\mu = 0.0017$) for which the measure of initial conditions is 0.0355 (resp. 0.0298). All main features have a theoretical explanation. Bottom left: Projections in (x, y) of the points subsisting after 10^3 (resp. 10^5) revolutions in green (resp. red) starting on $z = 0.5$ for $\mu = 0.001$. On the x-axis one can see, from left to right, the positions of L_1, J, L_2, S and L_3. The point $L_5 = (\mu - 1/2, \sqrt{3}/2, 0)$ is also shown. Right: the same figure in polar coordinates centred at the origin. The angle (in the horizontal axis) is counted clockwise with origin on the negative x-axis.

In the 3D case these points seem to be unstable, because there is no way to prevent Arnol'd diffusion. Nearby points can escape, but extremely slowly. The existence of "practical stability regions", in which the motion is confined for, say, time intervals of the order of the age of the solar system, was established. A relevant question is which objects sit at what can be considered a "quasi-boundary", which separates regions of very slow diffusion from regions of fast escape. These objects have codimension 1 in the phase space and even if they do not confine motion, escaping from their intricate folds can take a very long time.

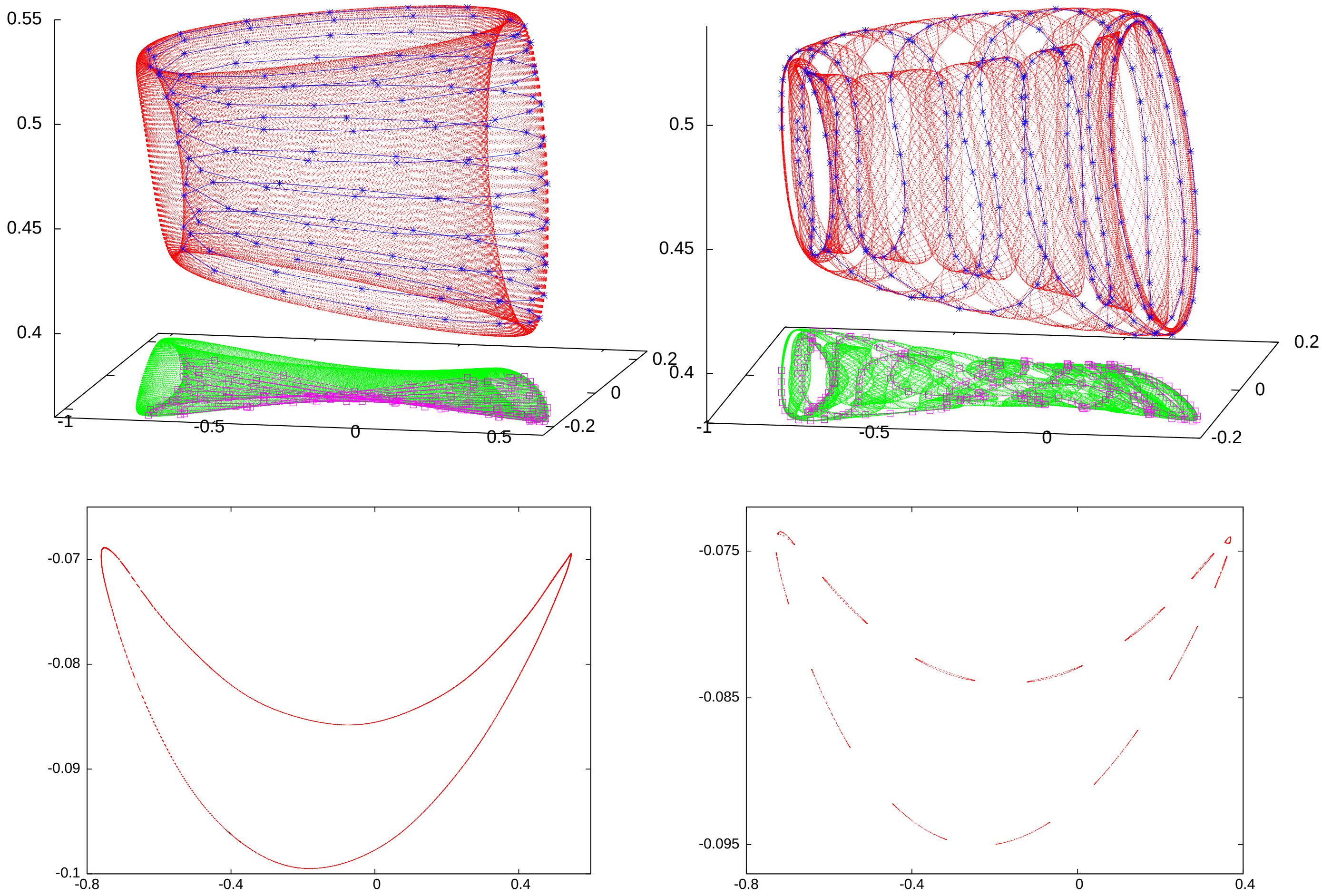

Figure 2: Top: Poincaré sections though $z = 0$ of some 3D tori with initial conditions on $z = 0.5$ near the lower boundary of the red part of figure 1 and angle $\approx 2\pi/3$. They give rise to 2D tori in this section. Variables plotted: $x, \dot{y} - 0.9\dot{z}, \dot{z}$. The projections on a z =ctant plane are also shown. Similar results are obtained for all tori close to the boundary. Left: a torus close to the quasi-boundary. Right: a torus slightly inside the practical stability domain and close (in the Poincaré section) to a 1D invariant torus. Some iterates of the Poincaré map (joined by lines) are shown in blue. In the right plot they are shown every 14 iterates, to give evidence of a nearby resonance. Bottom: Projections on $x, \dot{y} - 0.87\dot{z}$ of a slice of the upper plots. The slice has been taken with $|\dot{z} - 0.47| < 10^{-5}$. They clearly show that, in the phase space, the 3D tori are close to the invariant manifolds of 2D hyperbolic tori.

Figures 1 and 2 illustrate some of the features of the problem. The "practical stability domain" is defined by orbits starting with zero synodical velocity from a given point in the 3D position space. This allows us to sample a set of positive measure in the phase space because most of the points which do not escape give rise to 3D tori. As escaping criteria we take: not to approach S or J at a distance < 0.1 and the projection of (x, y) should not contain points below some critical y (e.g., $y = -0.2$). Notice that the "red" part at the bottom of figure 1 has plenty of narrow resonance zones, not visible with the resolution we used.

The codimension 1 manifolds involved in the quasi-boundaries include the unstable/stable manifolds of the center manifold around L_3 in the planar problem. The 3D problem is more involved, because they also include the invariant manifolds of the center manifolds of some families of periodic orbits of elliptic-hyperbolic type. A systematic computation of these objects offers several non-trivial difficulties, even numerically, in problems as simple as the one in the example. For very small values of μ some analytic information is available, but the value $\mu = 0.001$ used in figure 2, close to the real Sun-Jupiter case, cannot be considered small.

Outlook

Present status in D.S. is interesting and promising. But I believe that new tools and approaches have to be developed. One can expect:

a) The domain of applicability of D.S. to increase. They offer an unique tool to understand complex systems. Qualitative and quantitative aspects will merge.

b) Methodologies typical from "low dimensional" D.S. used in ODE and discrete maps will extend to the much more difficult areas of PDE, delay equations and equations with stochastic parts. For obvious reasons, the state of the art in these domains is far beyond that in ODE.

c) Reliable modelling to extend beyond "classical" domains (some parts of physics, chemistry and technology) to reach new phenomena at the nanometric scale and beyond, including applications to biology, medicine, communications and other areas. This will require a big effort and a change of mentality of the related communities (and not only from the part of mathematicians...).

d) Theoretical results to shift to constructive and efficient methods. A strong emphasis will be put on sharp estimates of domains of validity and in converse results, trying to close the gap between the proved domains of validity and what seem to be the true domains based on physical or numerical evidences.

e) The validation of numerical computations (rigorous and efficient estimates of error, computation of invariant objects far from perturbative cases, computer assisted proofs) to become a routine task in the near future.

f) A big effort to be made to understand transient regimes. We should not forget that our world is permanently in a transient state. Depending on the time scales the influence of changes in the model cannot be neglected.

g) From a theoretical point of view mathematics has tools with a large domain of applicability, but giving less information, and tools which give a detailed description but are quite localised. It would be desirable to find new fields which profit from the advantages of both approaches.

h) Keeping in mind that many descriptions are only possible using probabilistic measures and its evolution, it seems that present digital computers are not the best option. Perhaps computers based on quantum mechanics will be the desirable solution. But it seems this cannot be expected to happen soon.

Conclusions

Dynamical Systems offer a context, a theoretical framework and a methodology, both in abstract and concrete problems, which allows them to be applied in all areas of science and technology. On the other hand they allow the study not only of the behaviour of one solution of a mathematical model but also characteristics of the totality of solutions. It is also possible to study changes with respect to parameters, the robustness of a model and how to control it.

Qualitative and quantitative aspects mutually reinforce and complement each other. There exist paradigmatic models which help to understand more complex models and which appear with some "universality" (i.e., they should necessarily appear under some conditions).

The development of Dynamical Systems, of Mathematics and Science in general (closely followed by Technology) is at present at a very interesting and potentially promising point. Mankind has to face many problems and has tools with which to solve them. It is only a matter of real will to do it. To this end one has to change, possibly, some of the engines which currently drive society.