Stability islands in domains of separatrix crossings in slow-fast Hamiltonian systems

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Abstract

We consider a 2 d.o.f. Hamiltonian system with one degree of freedom corresponding to fast motion and the other corresponding to slow motion. The ratio of time derivatives of slow and fast variables is of order $0 < \varepsilon \ll 1$. At frozen values of the slow variables there is a separatrix on the phase plane of the fast variables and there is a region in the phase space (the domain of separatrix crossings) where projections of phase points onto the plane of the fast variables repeatedly cross the separatrix in the process of evolution of the slow variables. Under a certain symmetry condition we prove existence of many, of order $1/\varepsilon$, stable periodic trajectories in the domain of the separatrix crossings. Each of these trajectories is surrounded by a stability island whose measure is estimated from below by a value of order ε. Thus, the total measure of the stability islands is estimated from below by a value independent of ε. The proof is based on analysis of the asymptotic formulas for the corresponding Poincaré map.

1 Introduction

Many problems in theory of charged particles' motion, theory of propagation of short-wave excitations, and celestial mechanics can be reduced to analysis of 2 d.o.f. Hamiltonian systems with fast and slow variables [1, 2, 3]. One degree of freedom corresponds to fast variables, and the other corresponds to slow variables. The ratio of time derivatives of slow and fast variables is of order $0 < \varepsilon \ll 1$. To describe dynamics in such systems one can use the adiabatic approximation constructed as follows.

Consider the fast system, i.e. the system for the fast variables at frozen values of the slow variables. This is a 1 d.o.f. Hamiltonian system involving the slow variables as parameters. Assume that for a range of values of the slow variables there is a region filled with closed trajectories on the phase portrait of the fast system. Then one can introduce "action-angle" variables in the fast system [4]. The "action" variable of the fast system is an adiabatic invariant (i.e. an approximate first integral) of the complete system: its value oscillates with amplitude $\sim \varepsilon$ on time periods of order $\sim 1/\varepsilon$ along a phase trajectory. To describe approximately variation of the slow variables, one should average the rates of their variation over the "angle" variable of the fast motion. This approximation of the real motion is called the adiabatic approximation [5]. The obtained 1 d.o.f. Hamiltonian system for the slow variables, involving the "action" variable of fast motion as a parameter, is called the slow system.

Let in a range of values of the slow variables and the "action" variable phase trajectories of the slow system be closed. Then in this range the motion can be described with V.I.Arnold's theorem on perpetual adiabatic invariance [6]: "action" value along a trajectory perpetually undergoes only small, of order ε, oscillations (under a certain generality condition). The considered region of the phase space is filled up to a small residual set with invariant tori, which are $O(\varepsilon)$-close to the invariant tori of the system of the adiabatic approximation.

We shall consider the situation, when there are separatrices on the fast system's phase portrait (see Figure 1), and there is a region in the phase space (the domain of separatrix crossings) where projections of phase points onto the plane of the fast variables repeatedly cross the separatrix in the process of evolution of the slow variables. In this case the described above construction of the adiabatic approximation needs a certain modification, and the formulated above assertions on the adiabatic approximation accuracy are not applicable.

We shall assume that a certain symmetry condition is valid: the areas inside the separatrix loops in Figure 1 are equal.

The main result of the present paper is as follows. Under certain generality conditions in the domain of separatrix crossings on every energy level there exist many, of order $1/\varepsilon$, stable periodic trajectories of period $\sim 1/\varepsilon$. Each of these trajectories is surrounded with a stability island, and the measure of this island is estimated from below by a value of order ε. Thus, the total measure of the stability islands is estimated from below by a quantity that is independent of ε. A stability island is a domain on an energy level bounded with a two-dimensional invariant torus. A stability island contains a discrete family of invariant tori contractible to the periodic trajectory. Introduce "modified action" equal to the "action" for points inside the separatrix loops in Figure 1, and equal to one half of the "action" for the other points. This "modified action" is a perpetual adiabatic invariant of motion inside a stability island: its value along a phase trajectory perpetually undergoes only oscillations with amplitude of order ε. Therefore, the stability islands are also islands of perpetual adiabatic invariance.

We note that existence of stability islands with total measure that is not small with ε in the domain of separatrix crossings is quite unexpected. Visually, in many problems this domain looks like a region of dynamical chaos (see [7, 8]). It was shown in [9] that a single stability island cannot have a measure larger than $O(\varepsilon)$.

Existence of stability islands with total measure not small with ε in the domain of separatrix crossings was established in [10, 11] in the case of a Hamiltonian system with one degree of freedom and the Hamiltonian function slowly periodically depending on time. Here we generalize this result to 2 d.o.f. systems. Like in [10, 11], the proofs are based on study of asymptotic formulas for the corresponding Poincaré map. In [10, 11] these formulas were constructed with the use of asymptotic expressions for the jump of the adiabatic invariant at a separatrix crossing in systems with one and a half d.o.f. [12, 13, 14] and for variation of the "angle" variable between separatrix crossings in such systems [15]. Below we use analogous formulas for systems with two d.o.f. [16, 17]. To find linearly stable periodic trajectories we look for linearly stable fixed points of the Poincaré map. The results on Lyapunov stability of the periodic trajectories (the fixed points) and on the existence of invariant tori surrounding the periodic trajectories (invariant curves around the fixed points) are provided by the Kolmogorov-Arnold-Moser (KAM) theory.

2 Formulation of the results

Consider a two degrees of freedom Hamiltonian system with Hamiltonian $H = H(p, q, y, x)$, where $q, \varepsilon^{-1}x$ are coordinates, and p, y are canonically conjugated momenta, $\varepsilon > 0$ is a small parameter, $H \in C^\infty$. The corresponding equations of motion are:

$$\dot{p} = -\frac{\partial H}{\partial q}, \quad \dot{q} = \frac{\partial H}{\partial p}, \quad \dot{y} = -\varepsilon\frac{\partial H}{\partial x}, \quad \dot{x} = \varepsilon\frac{\partial H}{\partial y}. \tag{1}$$

Variables p, q are called fast variables, while y, x are called slow variables. Hamiltonian system for p, q at $(y, x) = \text{const}$ is called fast (or unperturbed) system.

Assume that at all considered values of the slow variables the phase portrait of the fast system looks as in figure 1. On this phase portrait, there exist a non-degenerate saddle point C and separatrices l_1, l_2 . These separatrices divide the (p, q)-plane into domains $G_i = G_i(y, x)$, $i = 1, 2, 3$. Denote the value of H at the saddle point as $h_s = h_s(y, x)$, and introduce $E = E(p, q, y, x) = H - h_s$. On the separatrices $E = 0$. We suppose that $E > 0$ in G_3 and $E < 0$ in G_1 and G_2. Areas of the domains G_1, G_2 are S_1, S_2 correspondingly, and $S_3 = S_1 + S_2$; $S_i = S_i(y, x)$. We will make the following symmetry assumption:

A) Areas of G_1, G_2 are equal: $S_1(y, x) = S_2(y, x) = S(y, x)$.

Put $\Theta = \Theta(y, x) = \{S, h_s\}$, where $\{\cdot, \cdot\}$ is the Poisson bracket with respect to the variables (y, x) : $\{f, g\} = f'_x g'_y - f'_y g'_x$.

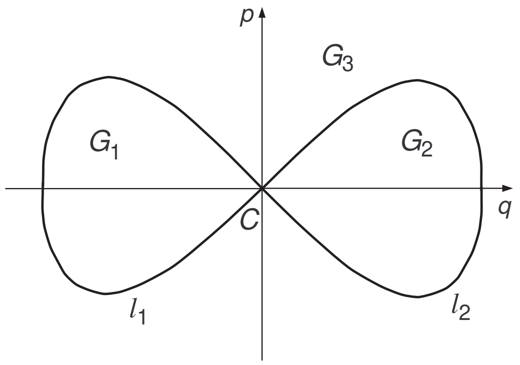

Figure 1: Phase portrait of the fast system.

In the unperturbed system, one can introduce canonical "action-angle" variables (I, φ) separately in each G_i. The "action" variable $I = I(p, q, y, x)$ in the unperturbed system involves y, x as parameters: $I = A/(2\pi)$, where A is the area on the phase portrait of the fast system surrounded by a phase trajectory passing through the point (p, q). For all p, q on the Hamiltonian level line $\{p, q : H(p, q, y, x) = h\}$ in the domain G_i the function I takes the same value $I_i(h, y, x)$.

Introduce the continuous function $\hat{I}(p, q, y, x)$ as follows:

$\hat{I}(p, q, y, x) = I(p, q, y, x)$, if $(p, q) \in G_{1,2}(y, x)$;

$\hat{I}(p, q, y, x) = I(p, q, y, x)/2$, if $(p, q) \in G_3(y, x)$;

$\hat{I}(p, q, y, x) = S(y, x)/(2\pi), \quad \text{if } (p, q) \in l_1 \cup l_2 \cup C.$

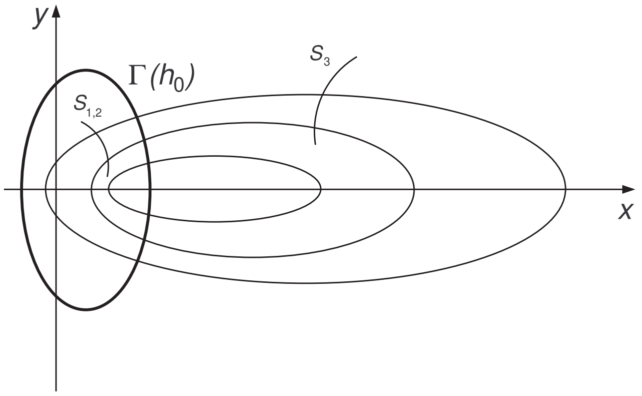

Figure 2: Schematic picture of motion on the slow plane.

Function $\hat{I}$ is an adiabatic invariant of the perturbed system. Far from the separatrix of the unperturbed system, its value along a phase trajectory is preserved with the accuracy of order ε on time intervals of order $1/\varepsilon$ [6]. When a phase point crosses a narrow neighborhood of the separatrix, $\hat{I}$ varies by a value of order $\mathrm{O}(\varepsilon)$ for the majority of initial conditions [16]. Approximation $\hat{I} = \text{const}$ along a phase trajectory is called adiabatic. In this approximation dynamics of slow variables is described by Hamiltonian system with Hamiltonian $H_0(\hat{I}, y, x)$, where H_0 is function H expressed via $\hat{I}, y, x$. Consider the dynamics on an energy level $H = h_0$. On the phase plane of the slow variables (y, x), the separatrix is represented by a curve $\Gamma(h_0) = \{y, x : h_s(y, x) = h_0\}$ (see figure 2). This curve is called the uncertainty curve [3]. It divides the plane into two domains: D_3, corresponding to G_3 on the fast plane, and the other, $D_{1,2}$, corresponding to G_1 and G_2. In the adiabatic approximation, a phase point on the slow plane moves along level lines $I(h_0, y, x) = \text{const}$. As the area inside the separatrix loops varies, a phase trajectory can cross the separatrix. Fix $\nu = 1$ or $\nu = 2$. Let $B_\nu(\hat{I}_0)$ be the adiabatic trajectory with $\hat{I} = \hat{I}_0$ on the plane of slow variables for phase points passing through G_ν:

$$B_\nu(\hat{I}_0) = \{y, x : I_3(h_0, y, x) = 2\hat{I}_0,\ (y, x) \in D_3;\ I_\nu(h_0, y, x) = \hat{I}_0,\ (y, x) \in D_{1,2}\} \qquad (2)$$

Make the following assumption.

B) There exists $\hat{I}_*$ such that curve $B_\nu(\hat{I}_*)$ is closed and intersects with uncertainty curve $\Gamma(h_0)$ at two points; at these points $\Theta(x, y) \neq 0$. On B_ν there are no stationary points of Hamiltonian H_0.

If assumption B) is valid both for $\nu = 1$ and $\nu = 2$ (with the same $\hat{I}_*$), then in the adiabatic approximation, projection of the motion with $\hat{I} = \hat{I}_0$ onto the slow plane looks as follows. In the domain corresponding to G_3, a phase point moves along the curve $I_3(h, y, x) = 2\hat{I}_*$. After crossing $\Gamma(h_0)$, it follows one of the curves $I_\nu(h_0, y, x) = \hat{I}_*$ [in general, these two curves are different]. Then the phase point crosses $\Gamma(h_0)$ again and continues its motion along $I_3(h_0, y, x) = 2\hat{I}_*$.

Assumption B) implies that there exists an interval Ξ of values of $\hat{I}$, such that for all $\hat{I}_0 \in D$ the curve $B_\nu(\hat{I}_0)$ has property B). In the exact system, multiple separatrix crossings occur in this interval of values of $\hat{I}$. Computer simulations demonstrate diffusion of the adiabatic invariant and chaotic dynamics in this region.

The main result of the paper can be formulated as follows.

Theorem. Under additional generality conditions, in the domain $\{p, q, y, x : \hat{I} \in \Xi\}$ on the energy level $H = h_0$ there exist more than C_1^{-1}/ε stable periodic trajectories of system (1). Each of these trajectories is surrounded by an invariant torus bounding a domain of (3-dimensional) phase volume greater than $C_2^{-1}\varepsilon$. Inside each of these domains variation of $\hat{I}$ is smaller than $C_3\varepsilon$.

Corollary. Total measure of these domains is larger than $C_4^{-1} = C_1^{-1}C_2^{-1}$.

Here C_i are positive constants independent of ε. The generality conditions mentioned in the theorem are formulated below, in Sections 6 and 7.

The domains bounded by invariant tori are domains (or islands) of perpetual adiabatic invariance. They are also called stability islands. Projected on the phase plane of fast variables (p, q), the motion along stable periodic trajectories under consideration looks as follows. A phase point starts moving in region G_3, then crosses the separatrix and gets into region G_ν, then crosses the separatrix again, comes back into G_3, and returns to its starting point.

The stable periodic trajectories mentioned in the theorem correspond to stable stationary points of the return map generated by the original system. This map is written in the principal approximation in Section 5.

Remark. The result of the theorem is valid also if phase portrait of the fast system is of the kind shown in figure 1 not for all x, y. It is enough to require that the system has such a phase portrait at the time of separatrix crossing calculated in adiabatic approximation.

3 Adiabatic and improved adiabatic approximations

In the fast system, action-angle variables I, $\varphi \bmod 2\pi$ are introduced separately in each of domains G_i by a canonical transformation of variables. The corresponding generating function $W(I, q, y, x)$ contains y, x as parameters [for brevity, we omit subscripts i]. We take this function in the form

$$W(I, q, y, x) = \int_{q_0(I,y,x)}^{q} \mathcal{P}(I, q', y, x)\mathrm{d}q', \tag{3}$$

where $\mathcal{P}$ is the value of p-variable along the trajectory with the prescribed value of action I, and $q_0(I, y, x)$ defines a curve in G_i transversal to phase trajectories. In the new variables the Hamiltonian has the form $H = H_0(I, y, x)$.

Now make a canonical transformation of variables $(p, q, y, x) \mapsto (\bar{I}, \bar{\varphi}, \bar{y}, \bar{x})$ with the generating function $\bar{y}\varepsilon^{-1}x + W(\bar{I}, q, \bar{y}, x)$. The canonically conjugated pairs of variables are $(\bar{I}, \bar{\varphi})$ and $(\bar{y}, \varepsilon^{-1}\bar{x})$. Formulas for the transformation of variables are:

$$\bar{\varphi} = \partial W/\partial \bar{I}, \;\; p = \partial W/\partial q, \;\; \bar{x} = x + \varepsilon \partial W/\partial \bar{y}, \;\; y = \bar{y} + \varepsilon \partial W/\partial x. \tag{4}$$

In the new variables, Hamiltonian H has the form

$$H = H_0(\bar{I}, \bar{y}, \bar{x}) + \varepsilon H_1(\bar{I}, \bar{\varphi}, \bar{y}, \bar{x}) + \mathrm{O}(\varepsilon^2), \tag{5}$$

where

$$H_1 = \frac{\partial H}{\partial y}\frac{\partial W}{\partial x} - \frac{\partial H_0}{\partial x}\frac{\partial W}{\partial \bar{y}}. \tag{6}$$

In the adiabatic approximation, dynamics is described by Hamiltonian H_0. In this approximation $\bar{I} = \text{const}$ along a phase trajectory.

One can also construct a canonical, close to identical, transformation of variables $(\bar{I}, \bar{\varphi}, \bar{y}, \bar{x}) \mapsto (J, \psi, \hat{y}, \hat{x})$ in order to make the terms of order ε in the Hamiltonian independent of phase (see, for example, [17]). In the new variables, the Hamiltonian takes the form:

$$\mathcal{H} = H_0(J, \hat{y}, \hat{x}) + \varepsilon \mathcal{H}_1(J, \hat{y}, \hat{x}) + \varepsilon^2 \mathcal{H}_2(J, \psi, \hat{y}, \hat{x}, \varepsilon), \quad \mathcal{H}_1 = \langle H_1 \rangle, \tag{7}$$

where the brackets denote averaging over $\bar{\varphi}$.

In improved adiabatic approximation, the dynamics is described by the Hamiltonian $H_0(J, Y, X) + \varepsilon \mathcal{H}_1(J, Y, X)$. In this approximation J is an integral of motion. With the accuracy of order ε^2, the following formula for J is valid (see [16]):

$$J = J(p, q, y, x) = I + \varepsilon u(p, q, y, x), \tag{8}$$

$$u = \frac{1}{4\pi} \left[\int_0^T \left(\frac{\partial E}{\partial y} \int_0^t \frac{\partial E}{\partial x} \mathrm{d}\sigma \right) \mathrm{d}t - \int_0^T \left(\frac{\partial E}{\partial x} \int_0^t \frac{\partial E}{\partial y} \mathrm{d}\sigma \right) \mathrm{d}t \right] + \frac{1}{2\pi} \int_0^T \left(\frac{T}{2} - \sigma \right) \{E, h_s\} \, \mathrm{d}\sigma. \tag{9}$$

The integrals here are calculated along a phase trajectory of the fast system passing through the point (p, q); σ is the time of motion along this trajectory starting from this point, T is the period of motion. Function J is the improved adiabatic invariant. In the complete system far from separatrices its value along a phase trajectory is constant with the accuracy of order ε^2 on time intervals of order ε^{-1}. In what follows, it is convenient to use function $\hat{J}(p, q, y, x)$ equal to J in domains G_1, G_2 and equal to $J/2$ in domain G_3.

4 Description of separatrix crossing

On the phase plane $\mathbb{P}^{sl}$ of the slow variables (y, x), the separatrix is represented by a curve $\Gamma(h_0)$ (see figure 2). Let $B_\nu(\hat{I}) \in \mathbb{P}^{sl}$ be a curve (see (2)) corresponding to $\hat{I}$ such that $\hat{I} \in \Xi$. According to B), $B_\nu(\hat{I})$ is divided by $\Gamma(h_0)$ into two parts. Consider two segments $\mathcal{S}_3$ and $\mathcal{S}_\nu$ on $\mathbb{P}^{sl}$ separated by $\Gamma(h_0)$, such that each of the segments crosses transversally all the curves $B_\nu(\hat{I})$ with $\hat{I} \in \Xi$ at a distance of order 1 from $\Gamma(h_0)$. Points on the fast plane corresponding to $\mathcal{S}_3$ belong to G_3, and the points corresponding to $\mathcal{S}_\nu$ belong to G_ν. Fix an interval $\Xi_0 \in \Xi$, endpoints of Ξ_0 and Ξ are different.

Let the motion start at $t = 0$ at a point $M^{(0)}(p^{(0)}, q^{(0)}, y^{(0)}, x^{(0)})$, such that $H(p^{(0)}, q^{(0)}, y^{(0)}, x^{(0)}) = h_0$ and on the plane $\mathbb{P}^{sl}$ $(\hat{y}^{(0)}, \hat{x}^{(0)}) \in \mathcal{S}_3$. Hence, $(p^{(0)}, q^{(0)}) \in G_3(y^{(0)}, x^{(0)})$. Let $\hat{I} = \hat{I}^{(0)}$, $\hat{J} = \hat{J}^{(0)}$, $\psi = \psi^{(0)}$ at this point; $\hat{I}^{(0)} \in \Xi_0$. In adiabatic approximation trajectory of slow variables is $B(\hat{I}^{(0)})$, the motion is described by a Hamiltonian system with Hamiltonian $H_0(\hat{I}^{(0)}, y, x)$. Let (x_-, y_-) and (x_+, y_+) denote points of intersections of $B(\hat{I}^{(0)})$ and $\Gamma(h_0)$; sign "$-$" corresponds to passage from G_3 to G_ν, sign "+" corresponds to passage from G_ν to G_3. Below τ_- and τ_+ denote the slow time moments of separatrix crossing in this approximation. In the exact system, in the process of evolution the phase point on $\mathbb{P}^{sl}$ approaches the curve $\Gamma(h_0)$, and accordingly, on the plane (p,q) it approaches the separatrix. We are interested in dynamics of phase points that are being captured into G_ν after the separatrix crossing. For such a phase point, the first crossing of Cq-axis in G_ν (see figure 1) occurs near C at time $\tau = \tau_- + \mathrm{O}(\varepsilon)$. [We assume for simplicity that coordinate axes in figure 1 are the principal axes of the saddle point.] Denote the value of E at the point of this crossing as $h^{(0)}$. We introduce $\eta^{(0)} = 1 - |h^{(0)}/\varepsilon\Theta_-|$. Here $\Theta_- \equiv \Theta(y_-, x_-) > 0$.

After the separatrix crossing the phase point in the plane of slow variables $(\hat{y}, \hat{x})$ moves towards $\mathcal{S}_\nu$. When it crosses $\mathcal{S}_\nu$, the projection of the phase point onto the fast plane is deep inside region G_ν. Denote the value of $\hat{J}$ at this time moment as $\hat{J}^{(1)}$.

Then the phase point again starts approaching the separatrix. At $\tau = \tau_+ + \mathrm{O}(\varepsilon)$ the phase point crosses Cq-axis in G_ν near point C for the last time before entering G_3. Denote the value of E at the point of this crossing as $h^{(1)}$. We introduce $\eta^{(1)} = |h^{(1)}/\varepsilon\Theta_+|$. Here $\Theta_+ \equiv \Theta(y_+, x_+) < 0$.

After crossing $\Gamma(h_0)$, the projection of the phase point onto the plane $(\hat{y}, \hat{x})$ crosses again the segment $\mathcal{S}_3$. Let $\hat{J}^{(2)}$, $\psi^{(2)}$ denote values of $\hat{J}$, ψ at this time moment. Then the phase point approaches the separatrix again, crosses it and gets captured into G_l, $l = 1$ or $l = 2$. Let $E = h^{(2)}$ at the first crossing of Cq-axis in G_l near C. This crossing occurs at time $\tau = \tau_- + T_0 + \mathrm{O}(\varepsilon)$, where T_0 is the slow time period of motion along trajectory $\hat{I} = \hat{I}^{(0)}$ in the adiabatic approximation. We introduce $\eta^{(2)} = 1 - |h^{(2)}/\varepsilon\Theta_-|$. We are interested in dynamics of phase points for which $l = \nu$.

5 The return map

In the energy level $H = h_0$ segment $\mathcal{S}_3$ is represented by a piece of two-dimensional surface $\{p, q, y, x : H(p,q,y,x) = h_0, (\hat{y}, \hat{x}) \in \mathcal{S}_3\}$. This piece can be parameterized by variables $\hat{J}, \psi$. The corresponding Poincaré return map $M : (\hat{J}^{(0)}, \psi^{(0)}) \to (\hat{J}^{(2)}, \psi^{(2)})$ produced by trajectories that pass through G_ν is symplectic. Its stable stationary points correspond to stable periodic orbits of the original problem, of period approximately equal to T_0/ε. It is convenient to study these stationary points using a different set of variables, namely to consider map $\hat{M} : (\hat{J}^{(0)}, \eta^{(0)}, \nu) \to (\hat{J}^{(2)}, \eta^{(2)}, l)$.

The map $\hat{M}$ is a composition of two maps: $\hat{M} = M^{(2)} \circ M^{(1)}$,

$$M^{(1)} : (\hat{J}^{(0)}, \eta^{(0)}, \nu) \to (\hat{J}^{(1)}, \eta^{(1)}, \nu), \quad M^{(2)} : (\hat{J}^{(1)}, \eta^{(1)}, \nu) \to (\hat{J}^{(2)}, \eta^{(2)}, l).$$

The results of [16, 17] give the following formulas for $M^{(k)}$. Suppose that

$$\eta^{(0)}, \eta^{(1)}, \eta^{(2)} \in [c_1^{-1}, 1 - c_1^{-1}]. \tag{10}$$

Then

$$\begin{aligned} \hat{J}^{(1)} &= \hat{J}^{(0)} - (2\pi)^{-1}\varepsilon a_-\Theta_-\ln(2\sin\pi\eta^{(0)}) + (2\pi)^{-1}\varepsilon(d_{\nu,-} - d_{3,-}/2) + \\ &\qquad (2\pi)^{-1}\varepsilon\Theta_-(\eta^{(0)} - 1/2)(b_{\nu,-} - b_{3,-}/2) + O(\varepsilon^{3/2}\ln\varepsilon), && (11) \\ \eta^{(1)} &= \{\eta^{(0)} + \varepsilon^{-1}\Phi_1^{(\nu)}(\hat{J}^{(1)}, \varepsilon) + O(\varepsilon^{1/3}\ln^{-1/3}\varepsilon)\}, && (12) \end{aligned}$$

$$\begin{aligned} \hat{J}^{(2)} &= \hat{J}^{(1)} - (2\pi)^{-1}\varepsilon a_+\Theta_+\ln(2\sin\pi\eta^{(1)}) - \varepsilon(2\pi)^{-1}(d_{\nu,+} - d_{3,+}/2) + \\ &\qquad (2\pi)^{-1}\varepsilon\Theta_+(1/2 - \eta^{(1)})(b_{\nu,+} - b_{3,+}/2) + O(\varepsilon^{3/2}\ln\varepsilon), && (13) \\ \eta^{(2)} &= \{\eta^{(1)} + \varepsilon^{-1}\Phi_2(\hat{J}^{(2)}, \varepsilon) + O(\varepsilon^{1/3}\ln^{-1/3}\varepsilon)\}, && (14) \end{aligned}$$

$$
\begin{aligned}
l = \nu, \qquad & \text{if} \qquad 0 < \{(\varepsilon^{-1}\Phi_2(\hat{J}^{(2)}, \varepsilon) + \eta^{(1)})/2\} < 1/2, & (15)\\
l \neq \nu, \qquad & \text{if} \qquad 1/2 < \{(\varepsilon^{-1}\Phi_2(\hat{J}^{(2)}, \varepsilon) + \eta^{(1)})/2\} < 1. & (16)
\end{aligned}
$$

Here $\{\cdot\}$ denotes the fractional part,

$$
a_\pm = a(y_\pm, x_\pm), \quad b_{j,\pm} = b_j(y_\pm, x_\pm), \quad d_{j,\pm} = d_j(y_\pm, x_\pm),
$$
$$
j = 1,\ 2,\ 3, \quad b_3 = b_1 + b_2, \quad d_3 = d_1 + d_2.
$$

For a one has $a = 1/\sqrt{-g}$, where g is the Hessian of E at the point C. Values $d_{1,2}$ are defined as the main terms of expansions (cf. [16])

$$
2\pi u(p, q, y, x) = d_j + \mathrm{O}(\sqrt{|E|}\ln|E|),
$$

where u is given by (9), and point (p, q) is on Cq-axis (figure 1) in G_j near C. Values b_j come from the expansion of period of motion in $G_j,\ j = 1, 2$ near the separatrix:

$$
T_j = -a \ln|E| + b_j + \mathrm{O}(E\ln|E|).
$$

In (12), (14),

$$
\Phi_1^{(\nu)}(\hat{J}, \varepsilon) = \frac{1}{2\pi}\int_{\tau_-^{(\nu)}}^{\tau_+^{(\nu)}} (\omega_0^{(\nu)}(\hat{J}, Y_\nu(\tau), X_\nu(\tau)) + \varepsilon\omega_1^{(\nu)}(\hat{J}, Y_\nu(\tau), X_\nu(\tau))\,\mathrm{d}\tau\,, \tag{17}
$$

$$
\Phi_2(\hat{J}, \varepsilon) = \frac{1}{\pi}\int_{\tau_+^{(3)}}^{\tau_-^{(3)}} (\omega_0^{(3)}(2\hat{J}, Y_3(\tau), X_3(\tau)) + \varepsilon\omega_1^{(3)}(2\hat{J}, Y_3(\tau), X_3(\tau)))\,\mathrm{d}\tau. \tag{18}
$$

Here $\omega_0^{(j)}$ and $\omega_1^{(j)}$ are defined as $\omega_0^{(j)} = \partial H_0/\partial J$, $\omega_1^{(j)} = \partial\mathcal{H}_1/\partial J$, with $H_0, \mathcal{H}_1$ calculated in region G_j, and (Y_j, X_j) is a solution of Hamiltonian system with Hamiltonian $\hat{H} = H_0(J, y, x) + \varepsilon H_1(J, y, x)$ on the energy level $\hat{H} = h_0$. Values $\tau_\pm^{(j)}$ are the slow time moments when the phase point corresponding to this solution arrives to the separatrix. At these moments $S_j(Y, X) = 2\pi J$. [One can take any of such solutions, they differ by a timeshift which does not change the value of the integrals in (17), (18)]. Conditions (10) are understood inductively: if $\eta^{(0)} \in [c_1^{-1}, 1 - c_1^{-1}]$, and $\eta^{(1)}$, found according to (11), (12) lies in segment $[c_1^{-1}, 1 - c_1^{-1}]$, then (12) is valid, and analogously for $\eta^{(2)}$ and expressions (13), (14).

Equations (11), (13) follow directly from the formula for jump of adiabatic invariant at a separatrix [16] for the case when areas of domains G_1 and G_2 are equal. Equation (12) follows directly from the formula for phase change between separatrix crossings [17]. Equations (14)-(16) follow from the result of [17] in the following way.

Let t_+, h_+ be the values of t and E when the phase point crosses the axis Cp near C for the first time after exit from G_ν. At time t_+ the phase point is in G_3 on positive or negative part of axis Cp. Let h_- be the value of E when the phase point crosses the same part of Cp for the last time before exit from G_3. Introduce notations: $\bar{\zeta} = \frac{h_+}{2|\Theta_+|}$, $\zeta = \frac{h_-}{2|\Theta_-|}$. According to [17],

$$
\bar{\zeta} + \zeta = \frac{1}{2\varepsilon}\Phi_2(\hat{J}^{(1)}, \varepsilon) + O(\varepsilon^{1/3}\ln^{-1/3}\varepsilon) \bmod 1, \tag{19}
$$

provided $\bar{\zeta}, \zeta \in [c_2^{-1}, 1-c_2^{-1}]$. Values $h^{(1)}$ (see Section 4) and h_+ are related as $h_+ = h^{(1)} + \varepsilon|\Theta_+| + O(\varepsilon^{3/2})$ [16]. Therefore, $\bar{\zeta} = \frac{1}{2}(1-\eta^{(1)}) + O(\varepsilon^{1/2})$. Values $h^{(2)}$ (see Section 4) and h_- are related as $h_- = h^{(2)} + 2\varepsilon\Theta_- + O(\varepsilon^{3/2})$ if $l = \nu$, and as $h_- = h^{(2)} + \varepsilon\Theta_- + O(\varepsilon^{3/2})$ if $l \neq \nu$. We have $l = \nu$ provided that $1/2 < \zeta < 1$. Consider this case. We obtain that $\zeta = \frac{1}{2}(1+\eta^{(2)}) + O(\varepsilon^{1/2})$. From (19) we find

$$\frac{1}{2}(1-\eta^{(1)}) + \frac{1}{2}(1+\eta^{(2)}) = \frac{1}{2\varepsilon}\Phi_2(\hat{J}^{(2)}, \varepsilon) + O(\varepsilon^{1/3}\ln^{-1/3}\varepsilon) \bmod 1.$$

Therefore,

$$\eta^{(2)} = \eta^{(1)} + \frac{1}{\varepsilon}\Phi_2(\hat{J}^{(2)}, \varepsilon) + O(\varepsilon^{1/3}\ln^{-1/3}\varepsilon) \bmod 2. \tag{20}$$

Then condition $1/2 < \zeta < 1$ implies equation (15).

Now consider the case $l \neq \nu$. In this case $0 < \zeta < 1/2$. We get that ζ and $\eta^{(2)}$ are related as $\zeta = \frac{1}{2}\eta^{(2)} + O(\varepsilon^{1/2})$. From (19) we find

$$\frac{1}{2}(1-\eta^{(1)}) + \frac{1}{2}\eta^{(2)} = \frac{1}{2\varepsilon}\Phi_2(\hat{J}^{(2)}, \varepsilon) + O(\varepsilon^{1/3}\ln^{-1/3}\varepsilon) \bmod 1.$$

Therefore,

$$\eta^{(2)} = \eta^{(1)} + \frac{1}{\varepsilon}\Phi_2(\hat{J}^{(2)}, \varepsilon) - 1 + O(\varepsilon^{1/3}\ln^{-1/3}\varepsilon) \bmod 2. \tag{21}$$

Combining (20) and (21) we obtain equation (14). Condition $0 < \zeta < 1/2$ implies (16).

6 Linearly stable solutions

6.1 Equations for stable stationary points

Linearly stable periodic solutions of the original problem of period approximately equal to T_0/ε correspond to linearly stable fixed points of the map $\hat{M}$. These points are defined by the following set of conditions:

$$\hat{J}^{(2)} = \hat{J}^{(0)}, \quad \eta^{(2)} = \eta^{(0)}, \quad l = \nu, \tag{22}$$

$$|\operatorname{tr} \hat{M}'| < 2, \tag{23}$$

where $\hat{M}'$ is linearization of $\hat{M}$ at the fixed point.

Put $\Phi_1 \equiv \Phi_1^{(\nu)}$. Introduce also the following notations:

$$\begin{gathered}\hat{J}^{(s)} = \varepsilon\xi^{(s)}\,, \quad s = 0,1,2; \\ \varepsilon^{-1}\frac{d\Phi_k(\varepsilon\xi, \varepsilon)}{d\xi} = \gamma_k(\varepsilon\xi, \varepsilon)\,, \qquad \gamma_k(\varepsilon\xi, 0) = \gamma_k^0(\varepsilon\xi)\,, \quad k = 1,2; \\ a_\pm\Theta_\pm = \mp 2\pi\alpha_\pm\,, \quad d_{\nu,\pm} - d_{3,\pm}/2 = -2\pi d_\pm\,, \quad \Theta_\pm(b_{\nu,\pm} - b_{3,\pm}/2) = 2\pi b_\pm.\end{gathered} \tag{24}$$

Suppose that $(\xi^{(0)}, \eta^{(0)}) = (\xi, \eta)$ correspond to a stationary point of map $\hat{M}$. We rewrite equations (22) using formulas (10)-(14), notations (24), and neglecting terms $O(\varepsilon^{1/3}\ln^{-1/3}\varepsilon)$:

$$\alpha_-\ln(2\sin\pi\eta)+d_- -(\eta-1/2)b_- = \alpha_+\ln(2\sin\pi\eta^{(1)})+d_+ +(1/2-\eta^{(1)})b_+, \tag{25}$$

$$\eta^{(1)}-\eta+\gamma_1^0\alpha_-\ln(2\sin\pi\eta)+\gamma_1^0 d_- - \gamma_1^0(\eta-\frac{1}{2})b_- = \varepsilon^{-1}\Phi_1(\varepsilon\xi,\varepsilon)\bmod 1, \tag{26}$$

$$\eta-\eta^{(1)} = \varepsilon^{-1}\Phi_2(\varepsilon\xi,\varepsilon)\bmod 1, \tag{27}$$

$$\eta^{(1)}+\varepsilon^{-1}\Phi_2(\varepsilon\xi,\varepsilon)\in(0,1)\bmod 2, \tag{28}$$

$$\eta,\eta^{(1)}\in(c_1^{-1},1-c_1^{-1}). \tag{29}$$

Stability condition $|\,\mathrm{tr}\,\hat{M}'\,|<2$ can be written using (11)-(14) [we again neglect terms $O(\varepsilon^{1/3}\ln^{-1/3}\varepsilon)$]:

$$-4<Q<0, \tag{30}$$

$$Q=(u_2-u_1)(\gamma_2^0+\gamma_1^0)-u_2u_1\gamma_2^0\gamma_1^0, \tag{31}$$

$$u_1=\alpha_-\pi\cot\pi\eta-b_-,\quad u_2=\alpha_+\pi\cot\pi\eta^{(1)}-b_+.$$

The exact set of conditions for a stable periodic solution differs from (25)-(29) by terms $O(\varepsilon^{1/3}\ln^{-1/3}\varepsilon)$ that can be differentiated in $\varepsilon\xi\,,\eta\,,\eta^{(1)}$ without changing their smallness order. Below, we find non-degenerate solutions of system (25)-(29). According to the implicit function theorem, at small enough ε, each of these solutions corresponds to a solution of the exact system.

6.2 Existence of stable stationary points

We assume that the following generality conditions are valid.

H1. In Ξ_0 there exists an interval of values of I such that

$$\gamma_1^0(I)\cdot\gamma_2^0(I)\neq 0\,,\qquad \frac{\mathrm{d}}{\mathrm{d}I}(\gamma_1^0(I)/\gamma_2^0(I))\neq 0\,. \tag{32}$$

In the following, we consider only those values of $\varepsilon\xi$ that belong to the interval defined by **H1**.

Equation (25) has the form

$$F(\eta;\alpha_-,b_-,d_-)=F(\eta^{(1)};\alpha_+,b_+,d_+)$$

with $F(\eta;\alpha,b,d)=\alpha\ln(2\sin\pi\eta)+d-b(\eta-1/2)$, $\eta\in(0,1)$, $\alpha>0$. We have $F\to-\infty$ as $\eta\to+0$ or $\eta\to 1-0$. Moreover, F has a unique maximum $F_{max}=m(\alpha,b,d)$ on the interval $0<\eta<1$. We put $m_\pm=m(\alpha_\pm,b_\pm,d_\pm)$.

The set of points $(\eta,\eta^{(1)})\in(0,1)\times(0,1)$ satisfying (25) is shown in figure 3. Consider for definiteness the case $m_->m_+$, $\gamma_1^0\gamma_2^0<0$. The other cases can be studied analogously.

Let in figure 3 A and B have coordinates $(0,0)$ and $(0,1)$ respectively. Consider condition (30) on the curve AB. Near point A both u_1 and u_2 are large positive, and therefore Q is large positive. Near point B both u_1 and $-u_2$ are large positive, and therefore $-Q$ is large positive. Therefore, on segment AB there is an interval $L=L(\varepsilon\xi)$ (or several intervals) where condition (30) is valid. If c_1 is sufficiently large, relations (29) are also valid on $L(\varepsilon\xi)$.

Now we look for solutions of system

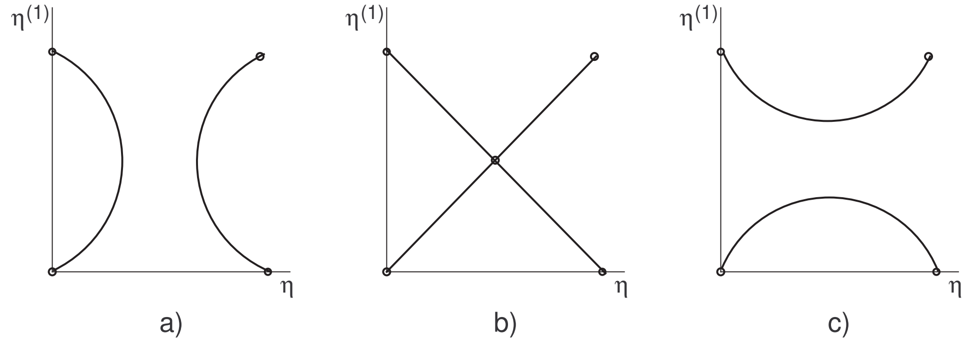

Figure 3: Set of points $(\eta, \eta^{(1)})$ satisfying (25). a) $m_- > m_+$; b) $m_- = m_+$; c) $m_- < m_+$.

$$(\eta, \eta^{(1)}) \in L(\varepsilon\xi), \tag{33}$$
$$\eta^{(1)} - \eta + \gamma_1^0 \alpha_- \ln(2\sin\pi\eta) + \gamma_1^0 d_- - \gamma_1^0 b_-(\eta - 1/2) = \varepsilon^{-1}\Phi_1(\varepsilon\xi, \varepsilon) \mod 1, \tag{34}$$
$$\eta - \eta^{(1)} = \varepsilon^{-1}\Phi_2(\varepsilon\xi, \varepsilon) \mod 2. \tag{35}$$

Relations (33)-(35) are equivalent to (25)-(31). Relation (28) holds because, according to (35), $\eta^{(1)} + \varepsilon^{-1}\Phi_2(\varepsilon\xi, \varepsilon) = \eta \mod 2$ and $\eta \in (0,1)$. Validity of the other relations (25)-(31) follows from the definition of $L(\varepsilon\xi)$ and equations (34), (35).

System (33)-(35) can be interpreted geometrically as follows. Consider the curve $\Lambda(\varepsilon\xi) \subset \mathbb{T}^2 = \{(s_1 \bmod 1, s_2 \bmod 2)\}$, which is the image of $L(\varepsilon\xi)$ under the mapping

$$(\eta, \eta^{(1)}) \mapsto (s_1, s_2) = (\eta^{(1)} - \eta + \gamma_1^0 \alpha_- \ln(2\sin\pi\eta) + \gamma_1^0 d_- - \gamma_1^0(\eta - 1/2)b_-, \quad \eta - \eta^{(1)})$$

defined by left hand sides of (34), (35). Consider also the point

$$\varepsilon^{-1}\Phi(\varepsilon\xi, \varepsilon) = (\varepsilon^{-1}\Phi_1(\varepsilon\xi, \varepsilon) \bmod 1, \quad \varepsilon^{-1}\Phi_2(\varepsilon\xi, \varepsilon) \bmod 2)$$

on $\mathbb{T}^2$. As ξ varies, the curve $\Lambda(\varepsilon\xi)$ is moving slowly on the torus. At the same time, the point $\varepsilon^{-1}\Phi$ is moving fast, with its velocity vector $\gamma(\varepsilon\xi, \varepsilon) = (\gamma_1(\varepsilon\xi, \varepsilon), \gamma_2(\varepsilon\xi, \varepsilon))$ varying slowly. Solutions of system (33)-(35) correspond to values of "time" ξ when point $\varepsilon^{-1}\Phi(\varepsilon\xi, \varepsilon)$ crosses the curve $\Lambda(\varepsilon\xi)$.

It follows from assumption **H1**, that there exists I_c such that the ratio $\gamma_1^0(I_c)/\gamma_2^0(I_c)$ is irrational. Then in the interval $\xi \in (I_c - c_3^{-1}/\varepsilon,\, I_c + c_3^{-1}/\varepsilon)$ on every segment of length c_4 of "time" ξ the point $\varepsilon^{-1}\Phi(\varepsilon\xi, \varepsilon)$ crosses transversally the moving curve $\Lambda(\varepsilon\xi)$ at least once. Indeed, the point's trajectory is close to a dense winding of the torus with irrational angular coefficient $\gamma_1^0(I_c)/\gamma_2^0(I_c)$, and the curve $\Lambda(I_c)$ is transversal to the vector $\gamma^0(I_c) = (\gamma_1^0(I_c), \gamma_2^0(I_c))$ at least at one point. [The latter is because $\Lambda(I_c)$ is not straight.]

Therefore, there exist more than $c_3^{-1}c_4^{-1}/\varepsilon - 1 > C_1^{-1}/\varepsilon$ intersection points. Each intersection point corresponds to a linearly stable stationary point of the return map. In Section 8 we obtain an asymptotic expression for the number of the intersection points.

7 Stability islands

Let ξ_r, η_r, $\eta_r^{(1)}$ be one of solutions of system (33)-(35), found in the previous section. One can rewrite map the $\hat{M}$ in variables $\tilde{\xi} = \xi - \xi_r$, η, expand its right hand sides in

ε supposing $\tilde{\xi} \sim 1$, and neglect terms $\mathrm{O}(\varepsilon^{1/3}\ln^{-1/3}\varepsilon)$. Thus, one obtains a map that does not contain ε depending on parameter $I_r = \varepsilon\xi_r$. This map has a stationary linearly stable point $\tilde{\xi} = 0, \eta = \eta_r$. According to Kolmogorov-Arnold-Moser (KAM) theory, under certain conditions [absence of resonances up to the 4-th order, non-zero coefficient in the normal form] the stationary point of the complete map is surrounded with a Cantor family of invariant curves (see, e.g. [5]). These latter conditions of KAM theory in the case under consideration can be written as one condition, namely $f(I_r, \eta_r) \neq 0$, where f is a continuous function. We assume that the following condition is valid.

H2. There exist $I_*, \eta_*, \eta_*^{(1)}, \quad (\eta_*, \eta_*^{(1)}) \in L(I_*)$ such that $f(I_*, \eta_*) \neq 0$.

Then at I close to I_*, $|I - I_*| < c_5^{-1}$, there is a segment of the curve $L(I)$ where $|f(I, \eta, \eta^{(1)}| > c_6^{-1}$. Without loss of generality one can assume that this segment coincides with $L(I)$ [otherwise, we can shorten $L(I)$]. According to KAM theory, at $|I_r - I_*| < c_5^{-1}$ there is an invariant curve at distance bigger than c_6^{-1} from the point $\tilde{\xi} = 0$, $\eta = \eta_r$. We call the domain surrounded by this curve a stability island of the Poincaré map. The area of this island is larger than c_7^{-1}. Returning from $\tilde{\xi}$, η to the original variables ξ, η, we find that the area of the stability island is estimated from below as $c_7^{-1}\varepsilon$.

The boundary of the stability island (an invariant curve) is a section of the invariant torus on the energy level of the original system. The volume of the domain surrounded by this torus is larger than $C_2^{-1}\varepsilon$. Variation of $\hat{I}$ inside this domain is smaller than $C_3\varepsilon$.

8 Distribution of periodic orbits

Let $N(I_c, \Delta, \varepsilon)$ be the number of stable periodic orbits under consideration with $I^{(0)} \in [I_c, I_c + \Delta] \subset \Xi_0$. According to Section 6.2 this number is equal to the number of "time" points $\xi \in [I_c/\varepsilon, (I_c+\Delta)/\varepsilon]$ when the point $\varepsilon^{-1}\Phi(\varepsilon\xi, \varepsilon)$ crosses the curve $\Lambda(\varepsilon\xi)$ on the torus $\mathbb{T}^2 = \{(s_1 \bmod 1, s_2 \bmod 2)\}$. For such values of I_c that $\gamma_1^0(I_c)/\gamma_2^0(I_c)$ is irrational one can calculate $\rho(I_c) = \lim_{\Delta\to 0}\lim_{\varepsilon\to 0}(\varepsilon N(h_0, I_c)/\Delta)$. We give the result omitting the standard argument based on ergodicity of winding of a torus with frequencies $(\gamma_1^0(I_c), \gamma_2^0(I_c))$ (cf. [18]). Let l be a natural parameter on $\Lambda(I_c)$, and $n(l, I_c)$ be a unit normal vector on $\Lambda(I_c)$. Introduce notation

$$\chi(I_c) = \int_{\Lambda(I_c)} |(\gamma^0(I_c), n(l, I_c))| \, \mathrm{d}l, \tag{36}$$

where (γ^0, n) is the Euclidean scalar product of γ^0 and n. Hence, χ is an "absolute" phase flux of the vector field γ^0 across curve Λ: for the phase flux across any element of Λ we take its absolute value. The value of $\chi(I_c)$ equals a product of the length of vector γ^0 and the full length of the projection of curve $\Lambda(I_c)$ onto the normal to vector γ^0. Then

$$\rho(I_c) = \frac{1}{2}\chi(I_c) \tag{37}$$

(coefficient 1/2 in this formula appears because the surface area of the torus equals 2).

Let $\bar{N}(\Xi_0, \varepsilon)$ be the number of stable periodic orbits under consideration with $I^{(0)} \in \Xi_0$. If $\gamma_1^0 \mathrm{d}\gamma_1^0/\mathrm{d}I - \gamma_2^0 \mathrm{d}\gamma_2^0/\mathrm{d}I$ vanishes only on a set of measure 0 on Ξ_0, then

$$\lim_{\varepsilon\to 0} \varepsilon\bar{N}(D_0, \varepsilon) = \frac{1}{2}\int_{\Xi_0} \chi(I)\mathrm{d}I. \tag{38}$$

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