Boundaries of Stability

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Celestial Mechanics

Celestial Mechanics studies the motion of celestial bodies and, in general, any topic related to the N-body problem and its applications to space science.

One can consider that the origin is the knowledge that Babylonian and Chinese astronomers had of the motion of celestial bodies.

The **fundamental laws** (Newton, etc)

$$\frac{d^2 r_i}{dt^2} = \sum_{j=1, j\neq i}^{N} G m_j \frac{r_j - r_i}{d_{i,j}^3}.$$

or, equivalently,

$$\begin{aligned} \frac{dr_i}{dt} &= v_i, \\ \frac{dv_i}{dt} &= \textstyle\sum_{j=1,j\neq i}^{N} G m_j \frac{r_j - r_i}{d_{i,j}^3}. \end{aligned}$$

Some of its problems

- **Approximate prediction of positions** as given by the **perturbation theories**,
- **The topology** of the **phase space** (or space of positions and velocities, or **space of states**),
- **The collision problem** and, in general, the problem of the **singularities** of the solutions,
- **The central configurations** and the **relative equilibrium solutions**,
- **The periodic solutions** as, for instance, the **halo orbits** and the so called **choreographies**,
- **The quasiperiodic solutions**, as the approximated motion of the **giant planets, around primaries or near some relative equilibria**,
- **The stable and chaotic solutions**, taking into account the effects of **resonances** and the different **time scales**,

- **The mechanisms** which allow to explain/predict **escape and capture**,

and, among the **applications**

- El motion of **asteroids and comets**,
- The study of **asteroids which pass close to the Earth (NEO)**,
- The detection and analysis of **external planetary systems**,
- The motion of **artificial satellites**,
- The design of de **space missions far away from the Earth**.

Dynamical Systems

Generalising, one can say, as a rough idea, that **Dynamical Systems** study **everything which moves**, that is, all the phenomena in which there is a magnitude which **evolves in time**.

Some characteristic features:

- Interest in the **totality of solutions**,
- Dependence with respect to **parameters**,
- **Qualitative** and **quantitative** studies,
- **Analytical** and **numerical** tools,
- Representation of the state of the system: the **states space** $\mathscr{E}$,
- Variation of the parameters of the system: the **parameters space** $\mathscr{P}$,
- A **key point**: **Invariant geometric objects** formed by **solutions**, which integrate the **skeleton** of the system in $\mathscr{E}$ or in $\mathscr{E} \times \mathscr{P}$,
- **Relevance** of different phenomena both in the **theoretical** and **applied** aspects.

At present **Dynamical Systems** are, possibly, the branch of **Mathematics** with more **interactions** with other branches and with more **possibilities of application**.

One represents the state of a system at a given time $\mathbf{t}$ by variables denoted as $\mathbf{w}(\mathbf{t})$ in some **space of states** or **phase space**, let it be in finite dimension or in a suitable functional space.

Equations leading the **dynamics**:

$$
\begin{aligned}
\frac{dw(t)}{dt} &= F(t, w(t)) && \text{(ODE)} \\
\frac{\partial w(t,x)}{\partial t} &= F(t, x, w, \tfrac{\partial w}{\partial x}) && \text{(PDE)} \\
\frac{dw(t)}{dt} &= F(w(t)) && \text{(AODE)}
\end{aligned}
$$

but also **discrete** cases $w \to F(w)$, **stochastic, delay, memory, ...** cases, or with special features (as **conservative, dissipative, ...**). Many **numerical algorithms** can also be included as Dynamical Systems.

Some of its problems

- To identify **simple objects** in the states space,
- **Stability** of these objects, **hyperbolicity** (may be in some weak sense), **stable and unstable manifolds**,
- **Connections** between stable and unstable manifolds: **homoclinic and heteroclinic orbits, complicated dynamics, diffusion, strange attractors, chaos,...**,
- Study of **bifurcations, unfoldings, structural stability, persistence in measure**,
- **Global** description of the dynamics in $\mathscr{E} \times \mathscr{P}$,
- Identification and study of **paradigmatic models**,
- **Numerical, symbolic and graphic tools** suitable for its study. **Design, analysis, implementation, efficiency**.

Boundaries of stability

Consider a **totally elliptic** fixed points of an **autonomous Hamiltonian system or a symplectic map**. It is **linearly stable** if the eigenvalues are simple.

Questions:

a) What about **non linear stability**? In general not true for **3 or more degrees of freedom (DOF)**. There is no way to avoid **Arnol'd diffusion** (except in simple cases when some **Dirichlet-like theorem** applies). But perhaps one can talk about **practical stability** in some sense (mild diffusion in very long time intervals).

b) Up to which distance **there is stability in some sense**? Which **objects** appear at the **stability boundaries** (perhaps in a **weak or practical** sense).

c) How do the results depend on **parameters**?

These are **global** questions. Require use of **analytics** and **numerics**.

A toy problem: the Hénon map

Any quadratic area preserving map can be reduced to

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} c(1-x^2)+2x+y \\ -x \end{pmatrix} \qquad x, y, c \in \mathbb{R}.$$

With this normalisation **fixed points** at H$=(-1,1)$ and E$=(1,-1)$.

Last one **elliptic** if $c \in [0,2]$. A **Normal Form** analysis proves **non linear stability** if $c \in (0,2) \setminus \{1.5\}$.

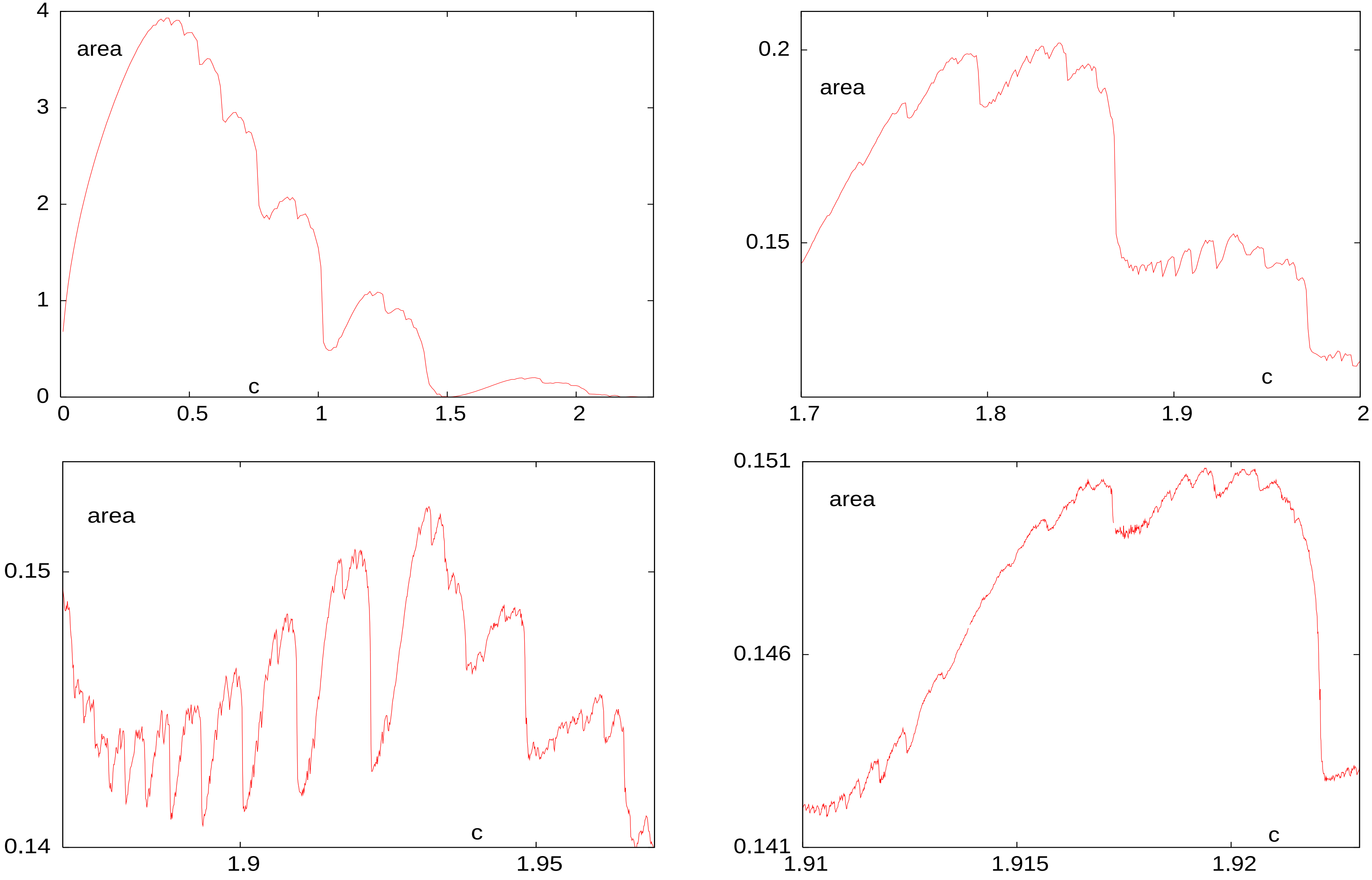
A **global stability** criterion: Points which do not escape to infinity.

Relevant comment: In this toy problem (equivalent to a Poincaré map of a 2DOF Hamiltonian) there are domains of **true stability** due to **KAM theorem**.

Let us scan the domain $[-1,2.5] \times [-1,2.5]$ with step size 0.002 and keep the points non escaping after N iterates.

How to choose N?

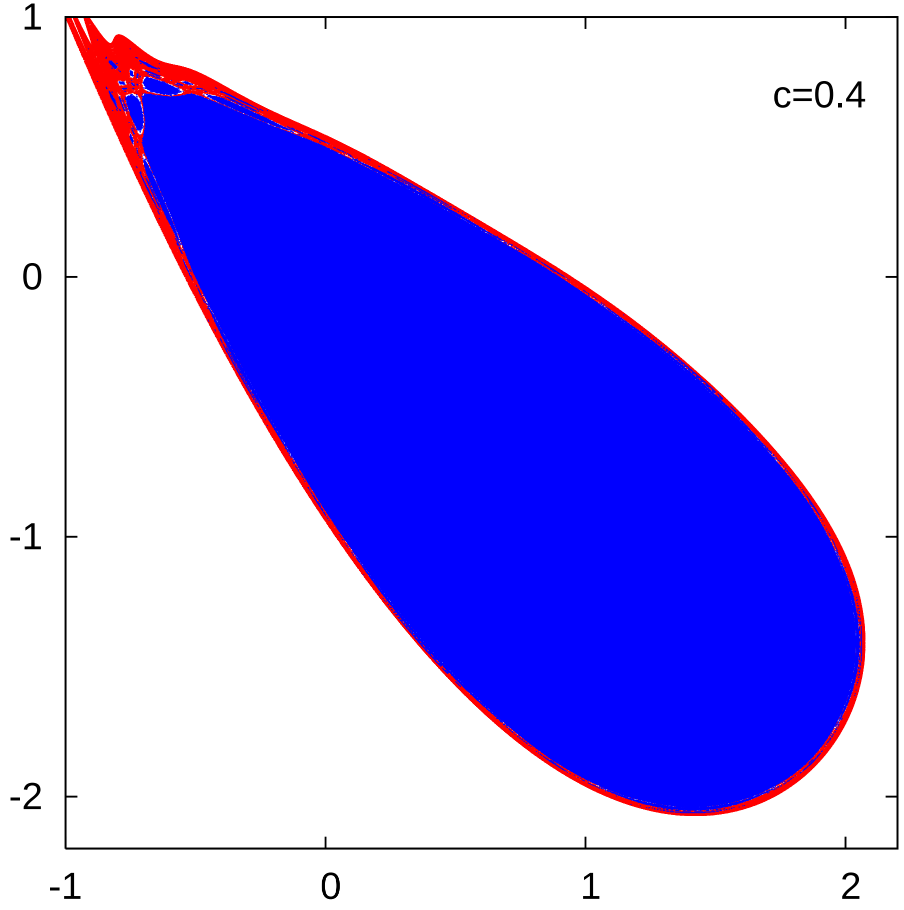
How to interpret the results? The role of $W^{u,s}$ of H.



Non-escaping points using $N = 10^7$ at most iterates.

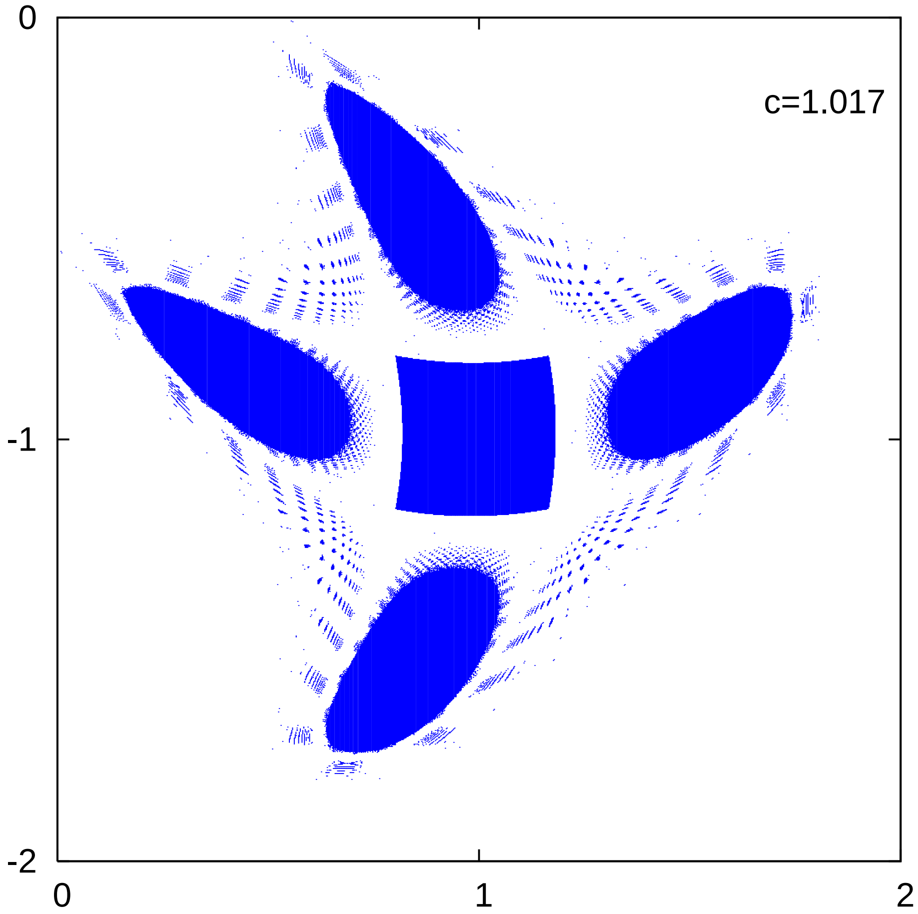
Topics of interest: **Global behaviour, identification of resonances, autosimilarity, continuity properties, the case $c = 1.5$, changing N, the boundaries, ...**

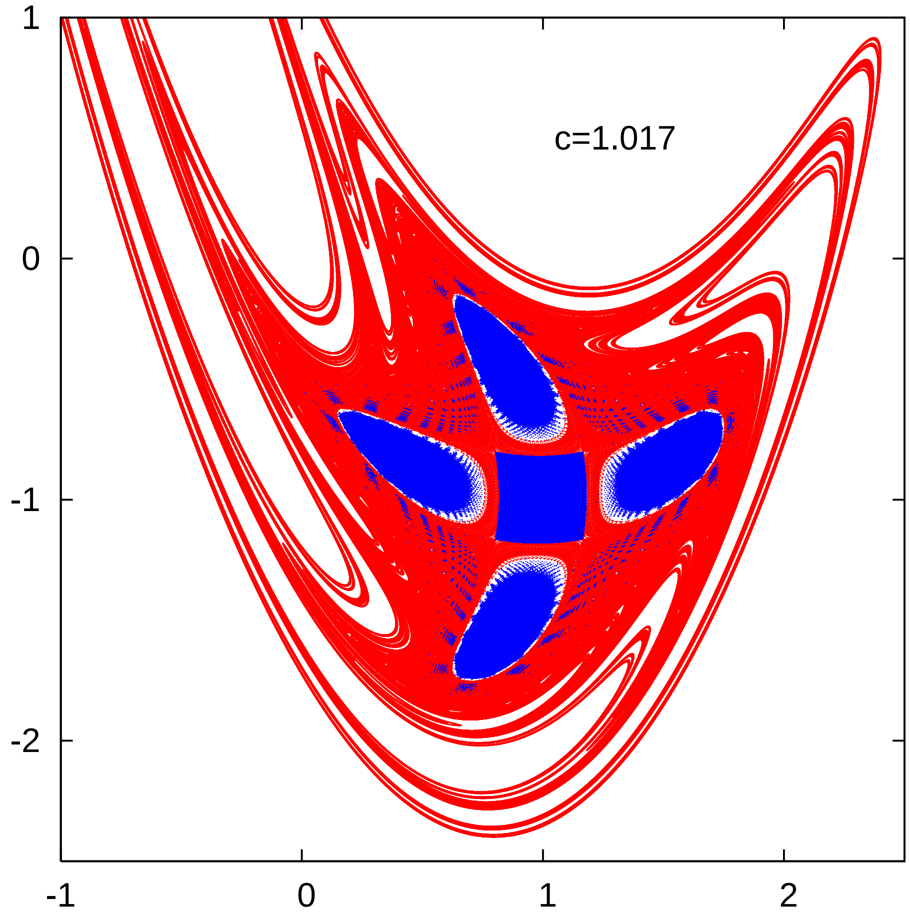
The case $c = 0.4$



Subsisting points with $N = 10^7$ in blue, W^s in red. Close to integrable.

The case $c = 1.017$

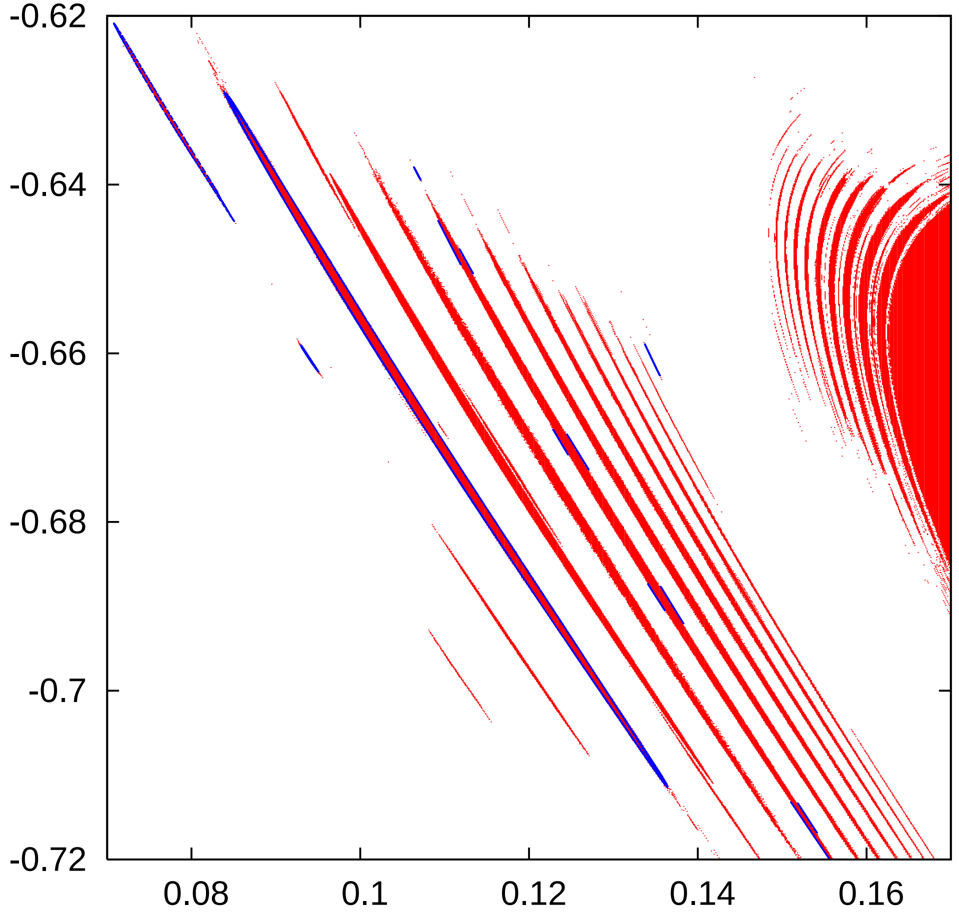




Left: Subsisting points with $N = 10^7$. Right: Also W^s. Far from integrable.

Questions: **Islands? Increasing N? Rate of escape? Numerical diffusion?**

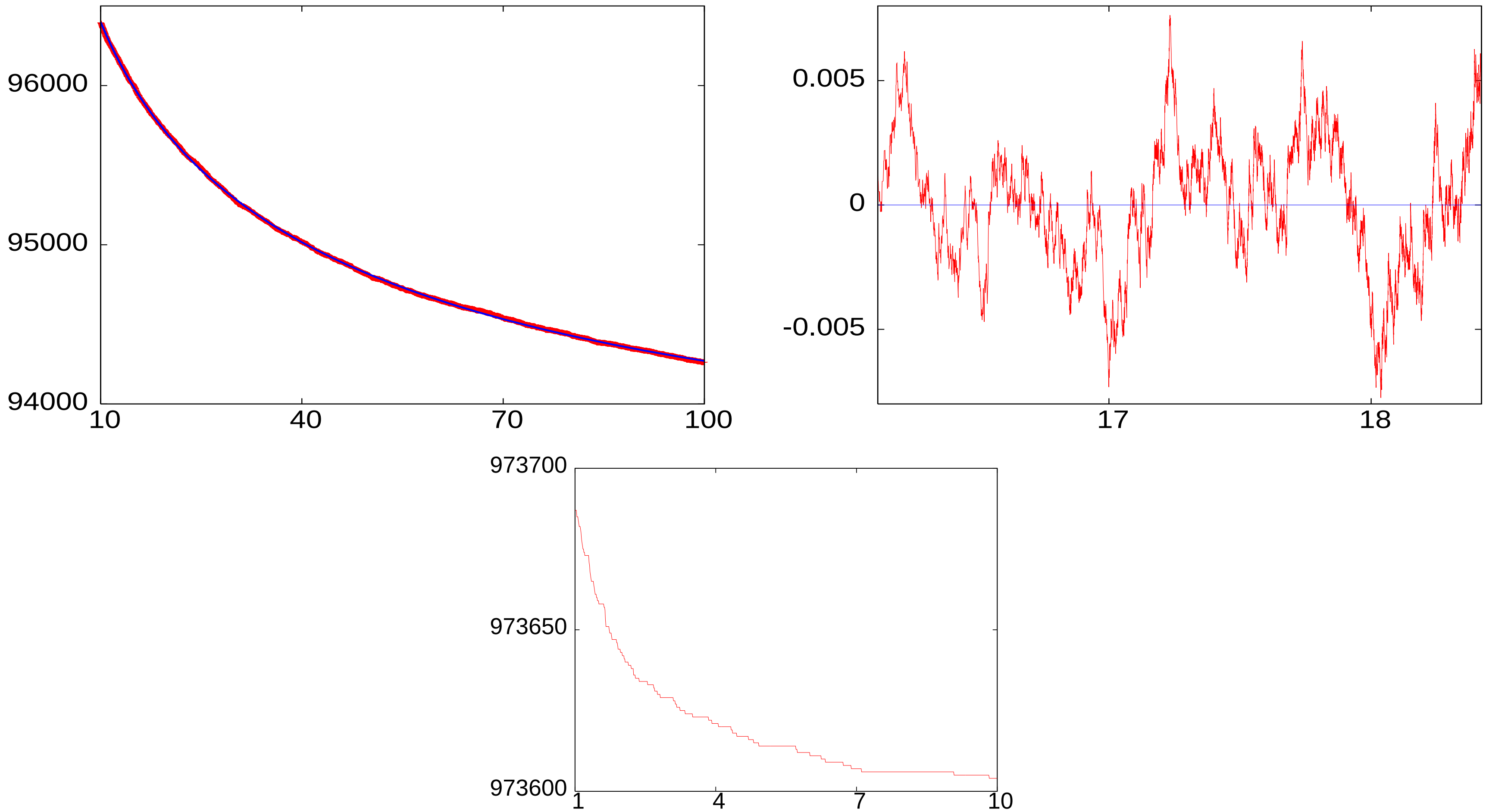
A magnification



Magnification showing some **islands** and orbits in islands of periods 33×2, 33×3, 37, 49×10, 77×2. Number of iterates taken to $N = 10^8$.

Stability boundaries: $W_H^{u,s}$ **Codimension 1 manifolds**

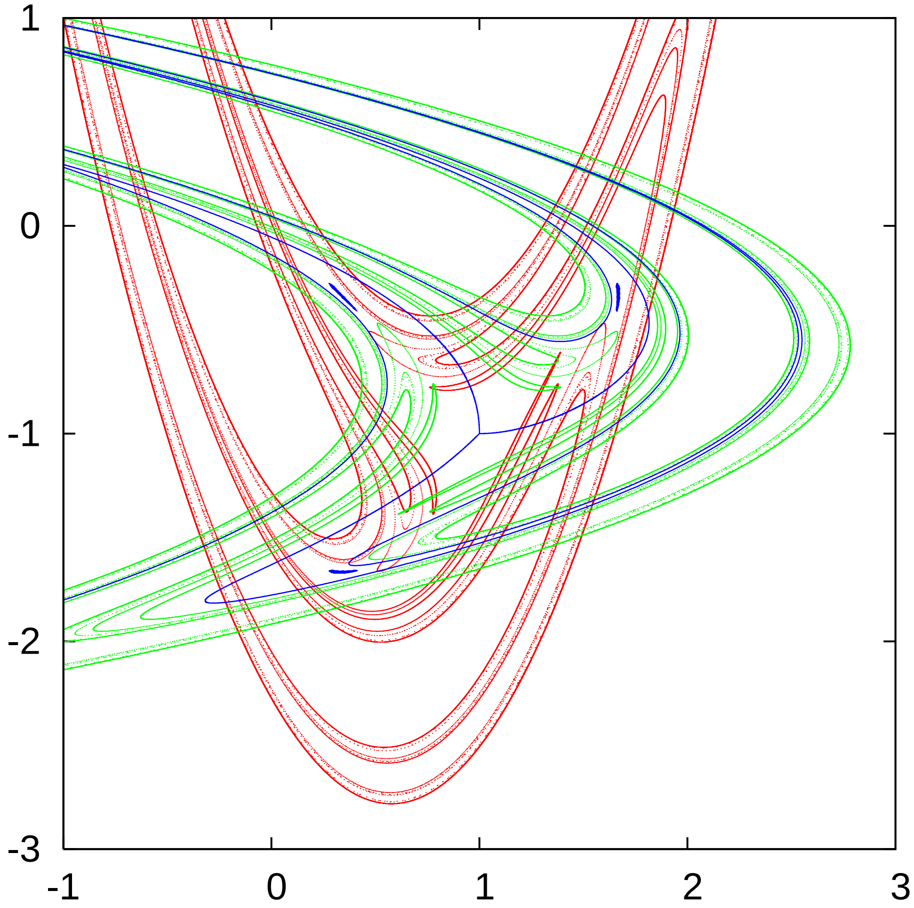
Rates of escape

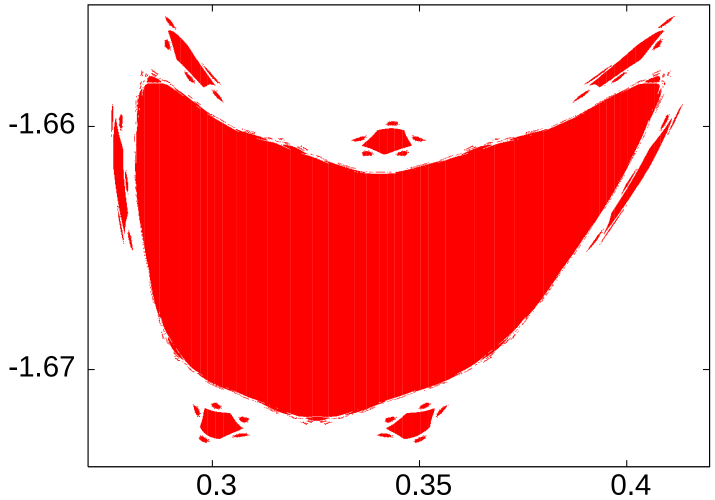


Top: For the previous figure $c = 1.017$, subsisting points $s(N)$ vs N (in millions) and graph of fit $a + b\exp(-cN^d)$, where $d \approx 0.4$, and plot of $\log(-\log((s(N) - a)/b)) - \log(cN^d)$ vs $\log(N)$. Bottom: $s(N)$ vs N for the figure with $c = 0.4$, for comparison.

The case $c = 1.5$

Local instability from **NF**. What about **globally**?

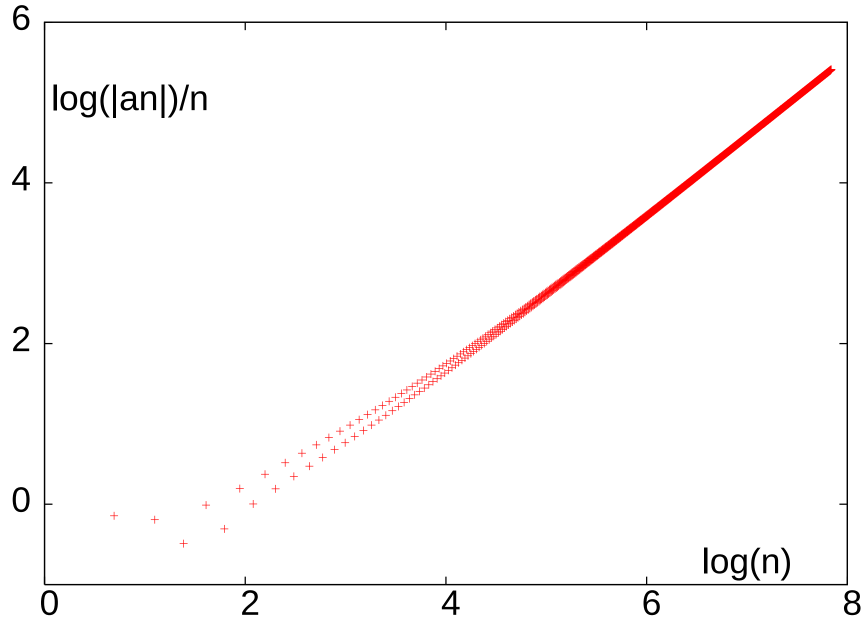


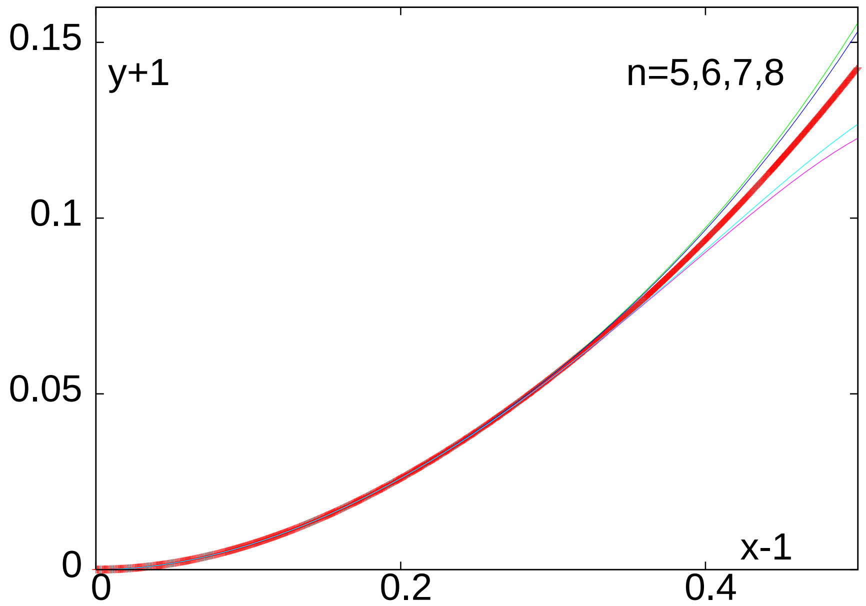


W^s, W^u of hyperbolic point H in red and green, resp. Branches of W^u of "elliptic" point E in blue. Also in blue a period 3 island magnified to the right.

Regularity of invariant manifolds

In all previous cases W^s, W^u of H are **analytic** (in fact **entire**). A **parametric representation** is used. But for E the branches are only **Gevrey**. Computed as a **graph**: $y + 1 = \sum_{k\geq 2} a_k(x-1)^k$.





Left: Values of $\log(|a_n|^{1/n})$ vs $\log(n)$. A fit gives **slope** very close to **1**. Right: Local plot of the manifold and **approximations** of degrees $5, 6, 7, 8$.

$a_n > 0$ (resp. $a_n < 0$) if n is congruent with $0, 1$ (resp. $2, 3$) modulus 4.

Stability of the triangular points in the circular 3D RTBP (Work in progress)

Known results

- They are linearly stable for the mass parameter $\mu \in [0, \mu_1]$, where

$$\mu_s = \omega_{\text{short}}/\omega_{\text{long}} = s, \qquad s \in \mathbb{N}.$$

$\omega_{\text{short}}, \omega_{\text{long}}$ = frequencies at $L_{4,5}$ in the 2D case. $\omega_z = 1$.

$$\left[(1 \pm (1 - 27\mu(1-\mu))^{1/2})/2\right]^{1/2}.$$

- Nonlinear stability for $\mu \in [0, \mu_1] \setminus \{\mu_2, \mu_3\}$ in the 2D case.
- The nonlinear stability of the spatial problem is unknown. Probably there is some amount of Arnol'd diffusion.
- There is a domain of "practical stability", around $L_{4,5}$, which is not too small.

  Let $H = H_2 + H_3 + \ldots$ be the Hamiltonian around $L_{4,5}$.

$$H_2 = \frac{1}{2}\omega_s(q_1^2 + p_1^2) - \frac{1}{2}\omega_l(q_2^2 + p_2^2) + \frac{1}{2}\omega_z(q_3^2 + p_3^2).$$

Compute NF up to a suitable order, assuming no resonance. Bound the remainder and its effect $|\dot{I}_j|$ on actions of the NF. Use linear combinations of the actions which define a convex set around the triangular points.

Given T and δ, there exists an initial radius, ρ_0, such that if $(Q, P)_{t=0} \in B_{\rho_0}$ then $(Q, P)_t \in B_{\rho_0(1+\delta)}$ for all $|t| < T$, where B_ρ denotes the ball of radius ρ centered at the origin. I.e. **practical or effective stability**.

As a result one obtains, in general, **explicit Nekhorosev type estimates** (i.e., for δ fixed, one has $T \approx \exp(c/\rho^d)$, for some positive constants c, d).

- Numerical simulations show a **large "stable domain"**.

The stability domain

Conditions to be satisfied:

a) Projection on (x, y) **not encircling the main primary**,

b) Not **too close approaches to primaries**,

c) Projection on (x, y) **not below some critical** y (e.g. $y_c = -0.5$).

d) Starting on **zero velocity surface**.

On the integration method

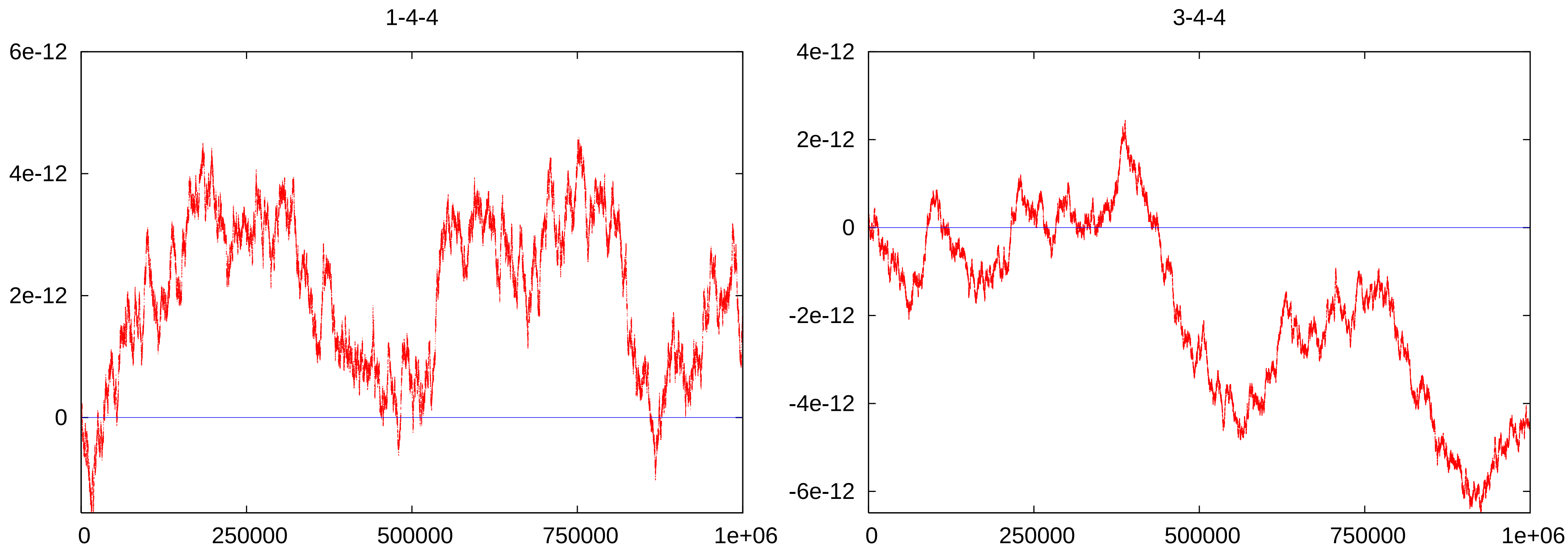
Integration of ODE is done using a **Taylor method**.

Numerical values of the coefficients computed by recurrence (automatic differentiation formulae) up to high order (e.g. order 24) and with time step such that the requested local error is below the ε of the processor (e.g. 10^{-21}).

Question What about **energy conservation**?

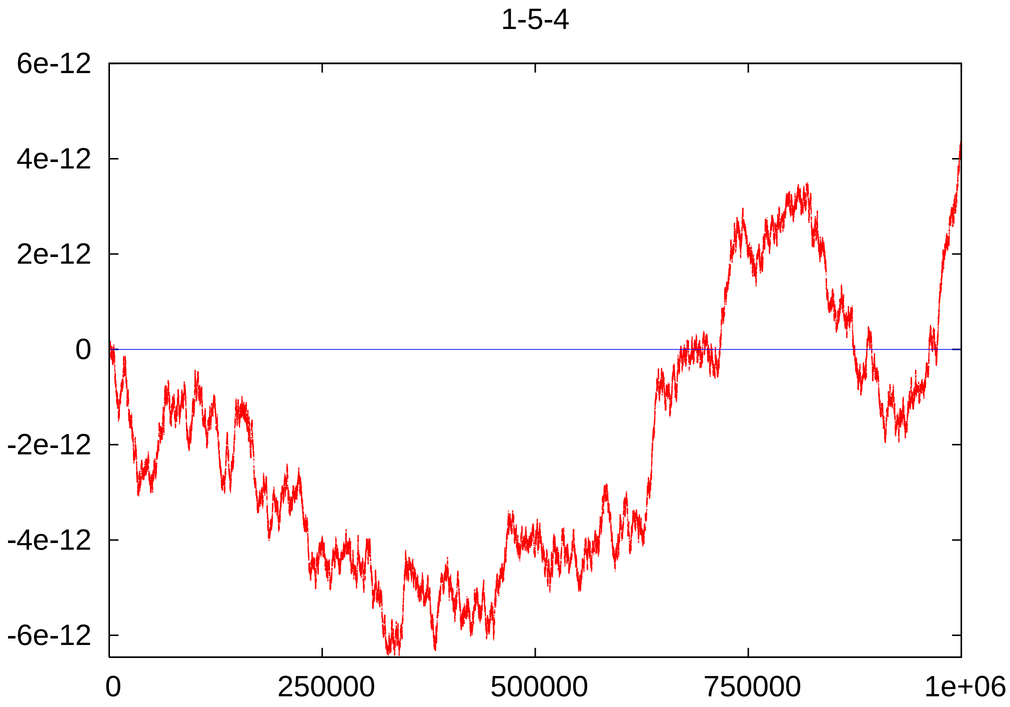
Long range integrations

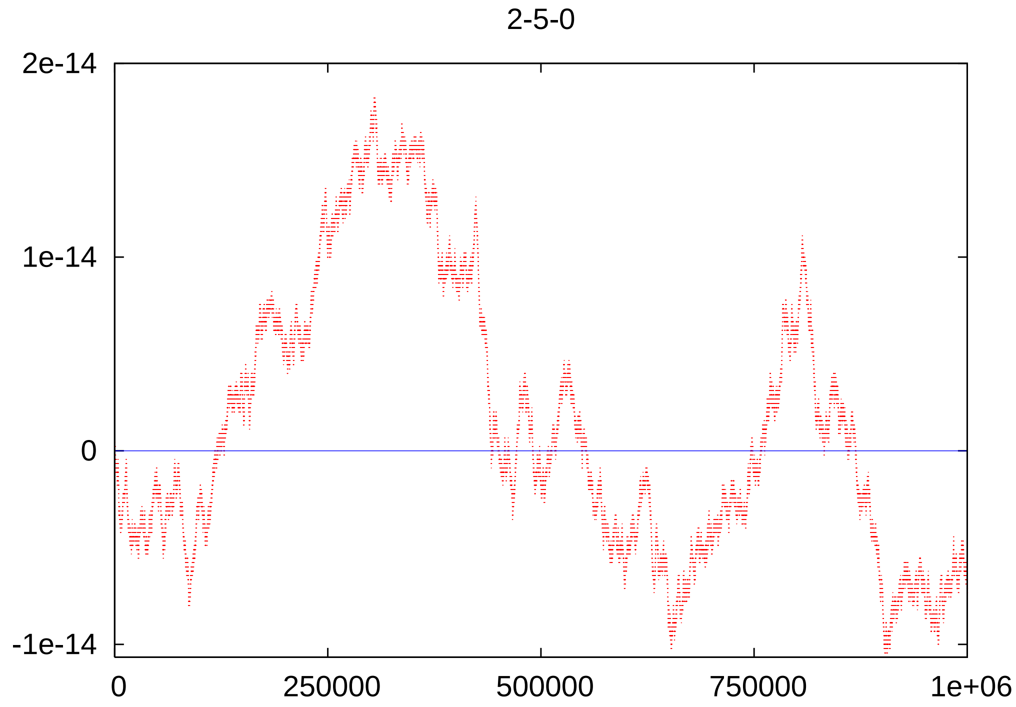
For $\mu = 0.001$ a test on a sample of 175 i. c. as follows: $\Delta\rho = 0.0005$; $\alpha = 0.1\,(0.05)\,0.4$; $z = 0\,(0.2)\,0.8$, heliocentric, from a suitable origin of $\rho = R - 1$, α in revolutions. 75 points subsist for $t_{\rm final} = 10^9$ Jupiter revolutions ($\approx$ **twice the age of Solar System**).



Shown: the 3 worst cases and the best one.

Errors w.r.t. initial energy displayed every 1000 Jupiter revolutions (J.r.).





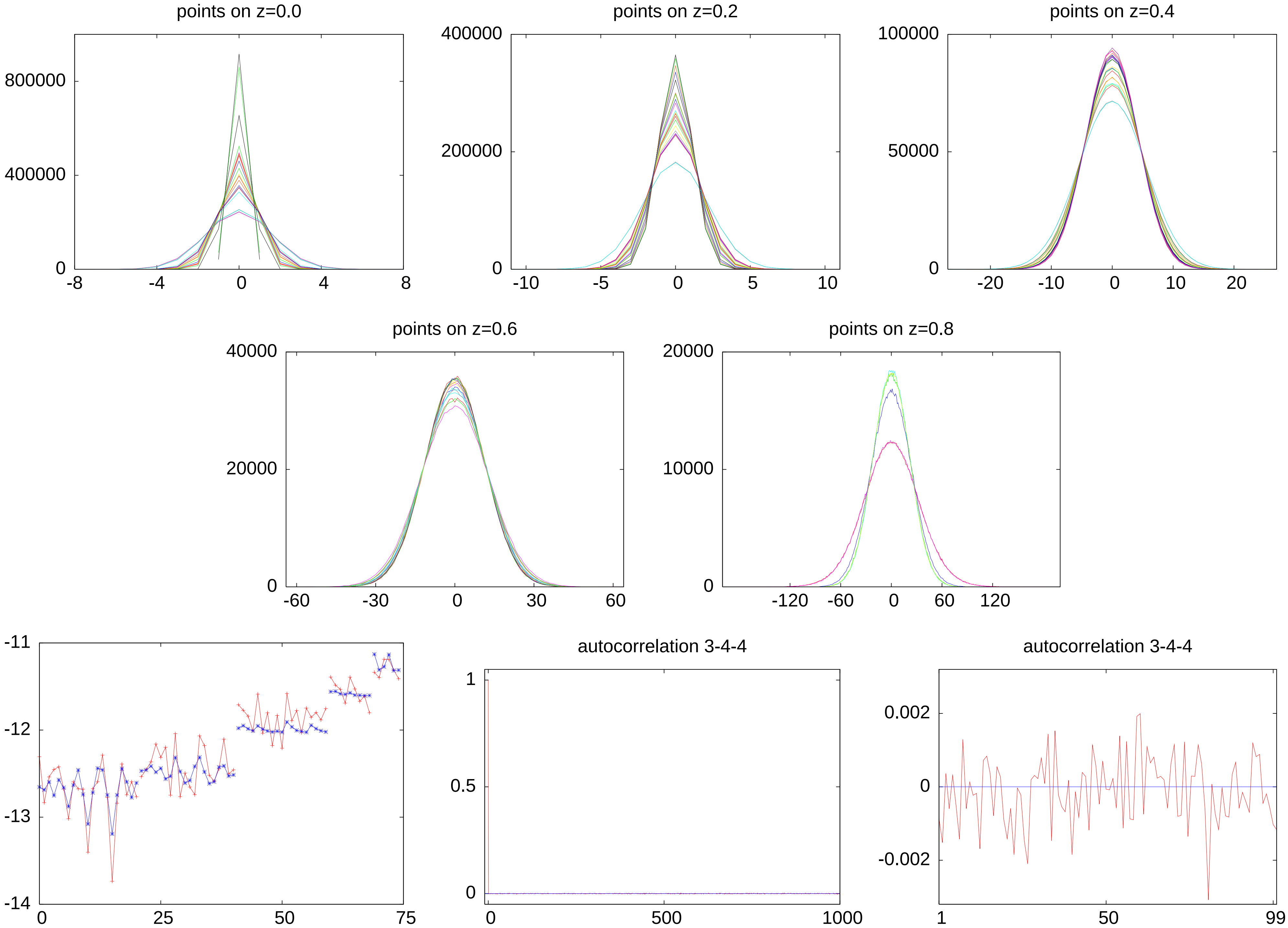
Average **integration step**: Worst case 0.36 years; best case 2.08 years.

Relevant statistics:

a) Histogram of **jumps of energy** every 1000 J.r. Unit: $2^{-52} \approx 2 \times 10^{-16}$,

b) **Maximal errors** and $1\,\sigma$ predicted values from previous plots,

c) **Autocorrelation** for the worst case.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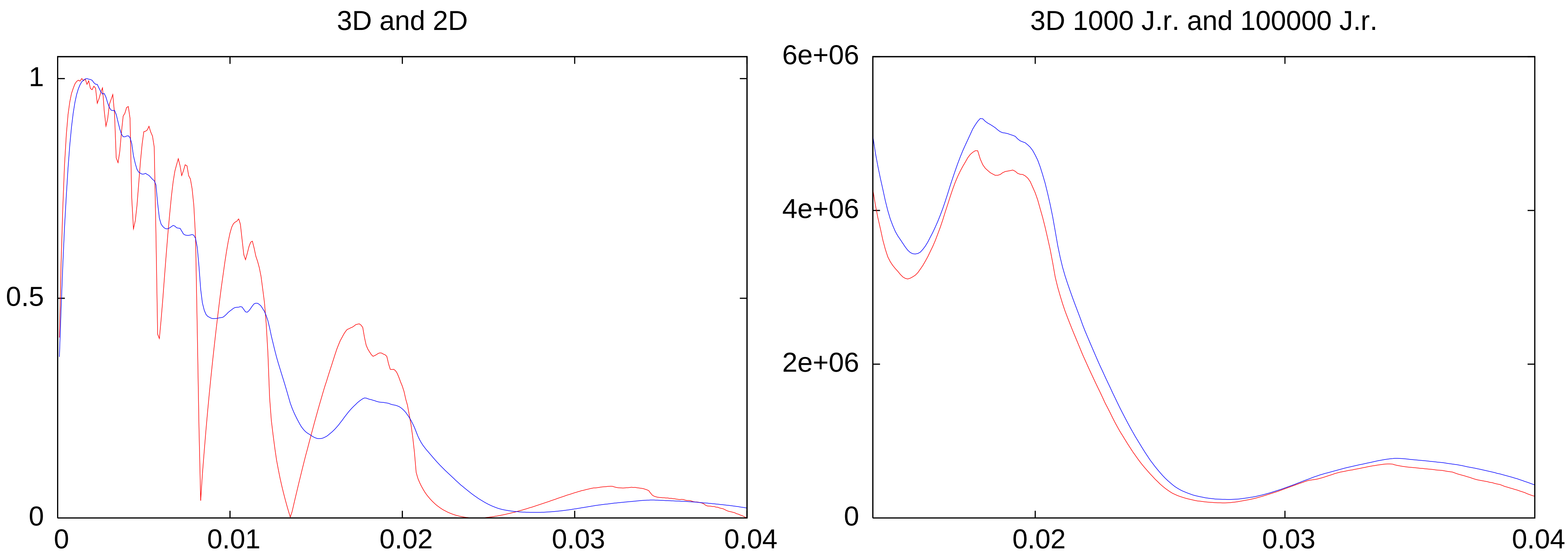
400000
0
-8
-4
0
4
8
points on z=0.2
400000
200000
0
-10
-5
0
5
10
points on z=0.4
100000
50000
0
-20
-10
0
10
20
points on z=0.6
40000
20000
0
-60
-30
0
30
60
points on z=0.8
20000
10000
0
-120
-60
0
60
120
-11
-12
-13
-14
0
25
50
75
autocorrelation 3-4-4
1
0.5
0
0
500
1000
autocorrelation 3-4-4
0.002
0
-0.002
1
50
99

Preliminary results: Statistics



Left: Normalized subsisting points. Planar case: 10^5 secondary revolutions, $\Delta\rho = 10^{-4}$, $\Delta\alpha = 2{\times}10^{-4}$. Maximal value, for $\mu = 0.0014$, 282757 points. Spatial case: 10^3 secondary revolutions, $\Delta\rho = 10^{-4}$, $\Delta\alpha = 5{\times}10^{-4}$, $\Delta z = 5 \times 10^{-3}$. Maximal value, for $\mu = 0.0017$, 19014882 points. Total scanned points: 8×10^{10}. Subsisting 2.17×10^9.

Right: Spatial case: Final time 10^3 and 10^5 secondary revolutions.

The case $\mu = 0$

For $\mu = 0$ one must be in $1-1$ resonance. Hence

$$z = \sqrt{\frac{4}{(1+R^2)^2} - R^2},$$

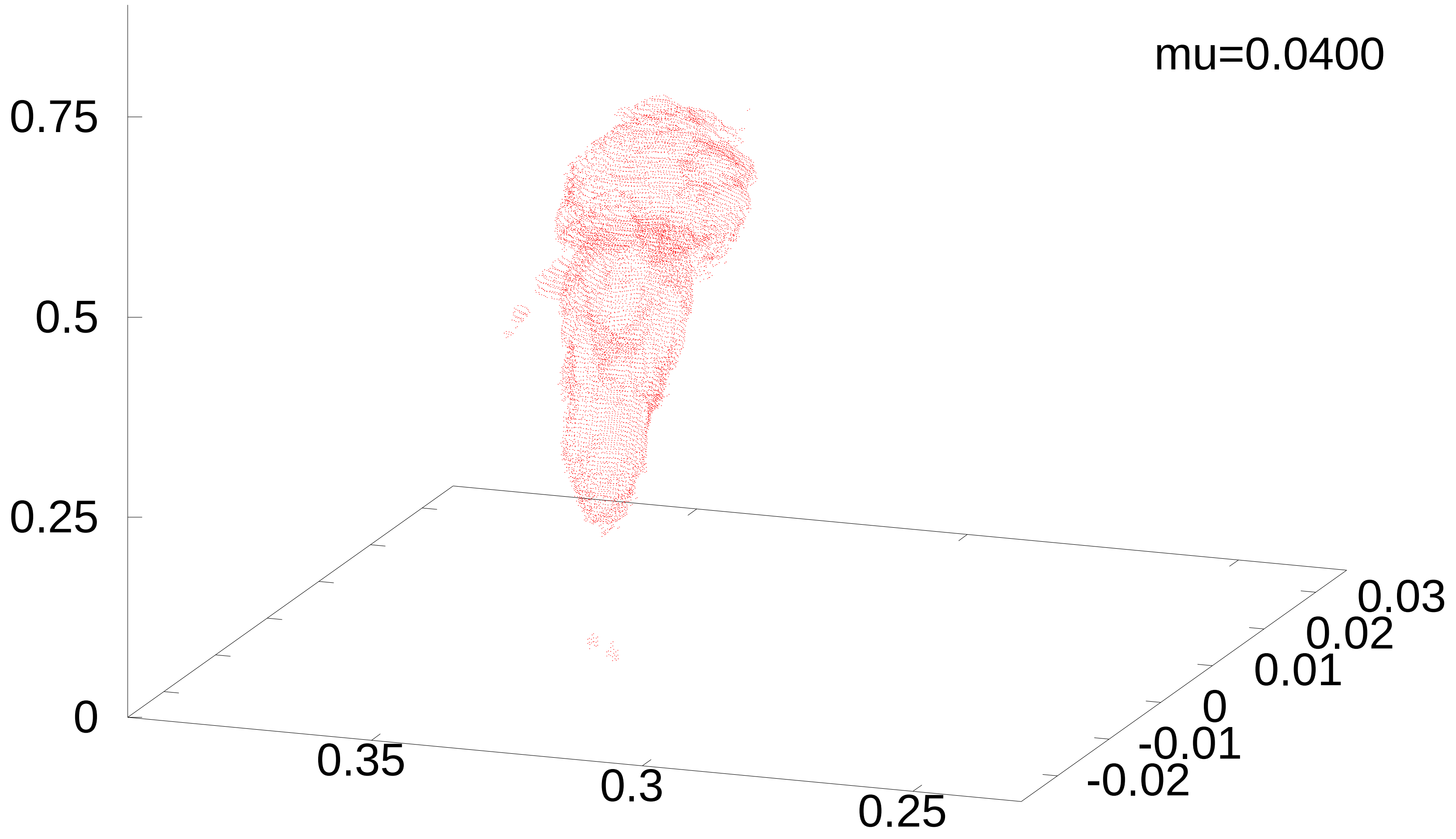
or

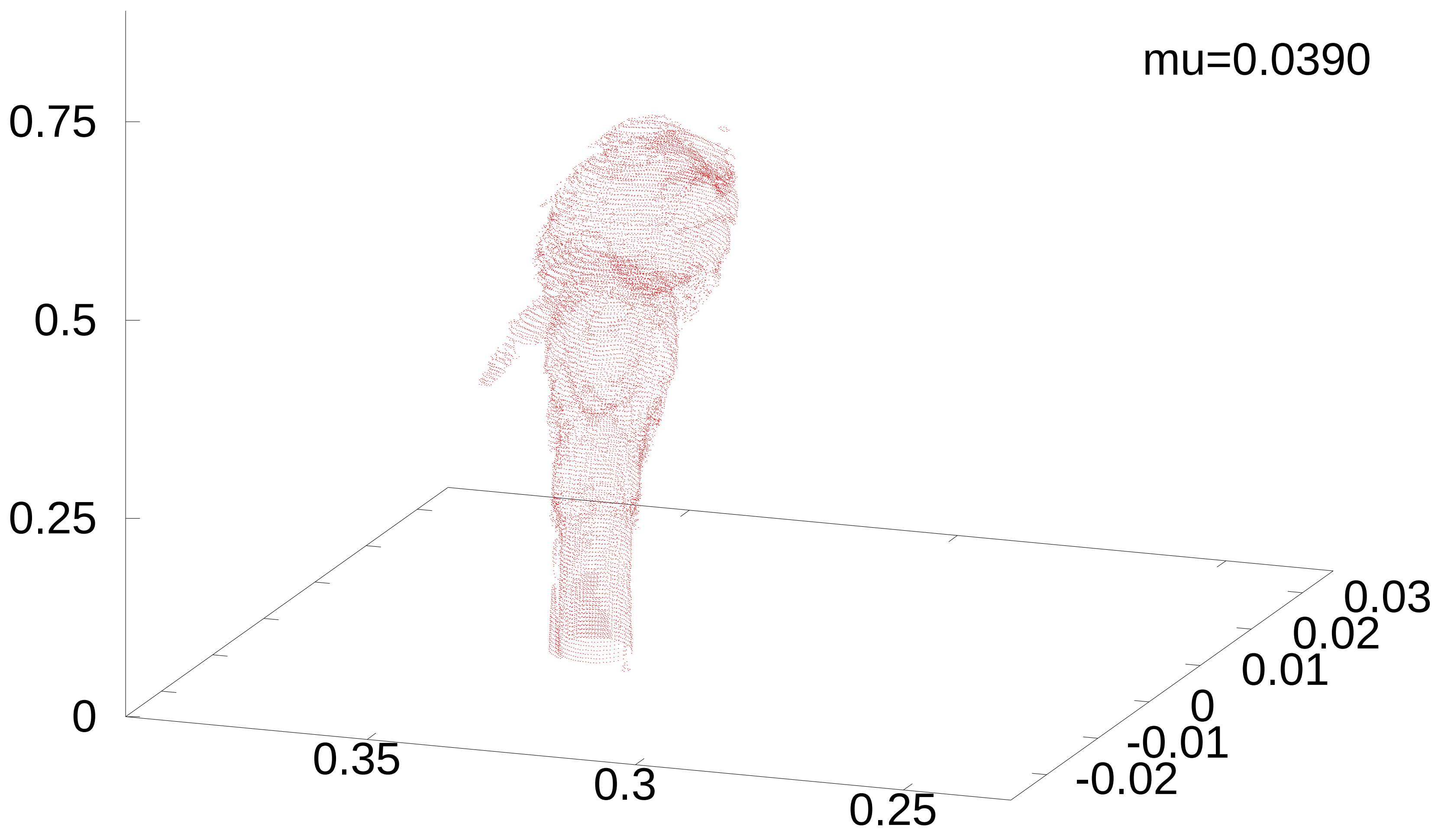
$$\psi = 1 - \frac{1}{2}w + \frac{3}{2^5}w^2 - \frac{1}{2^8}w^3 - \frac{25}{2^{13}}w^4 + \frac{33}{2^{16}}w^5 + \frac{175}{2^{20}}w^6 - \frac{497}{2^{23}}w^7 - \frac{4401}{2^{29}}w^8 + O(w^9),$$

where $w = z^2$, $\psi = \psi(z) = R^2$.

This suggests to plot $(\alpha,\ 1 + \rho - \sqrt{\psi(z)},\ z)$.

A sample of results

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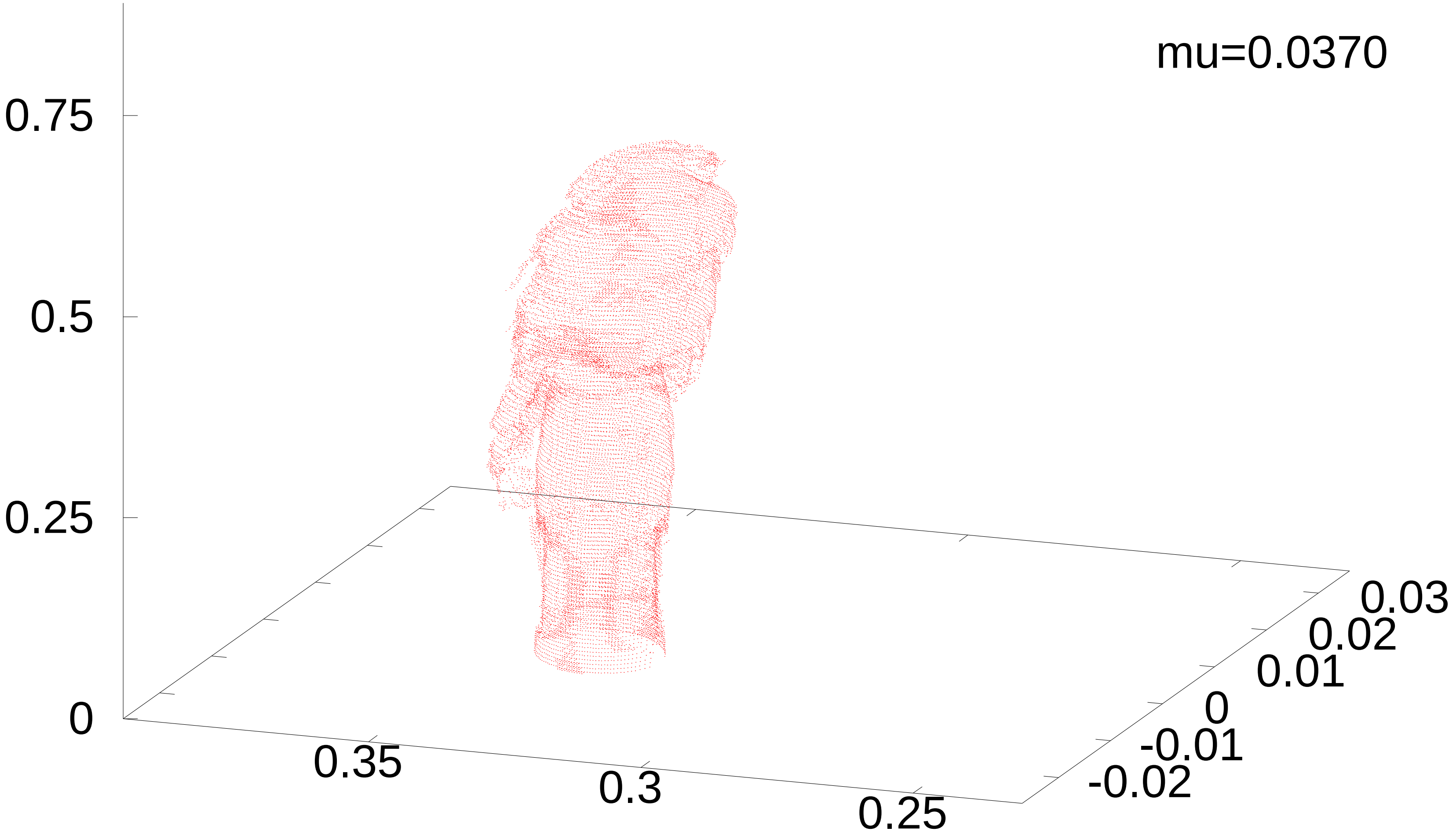

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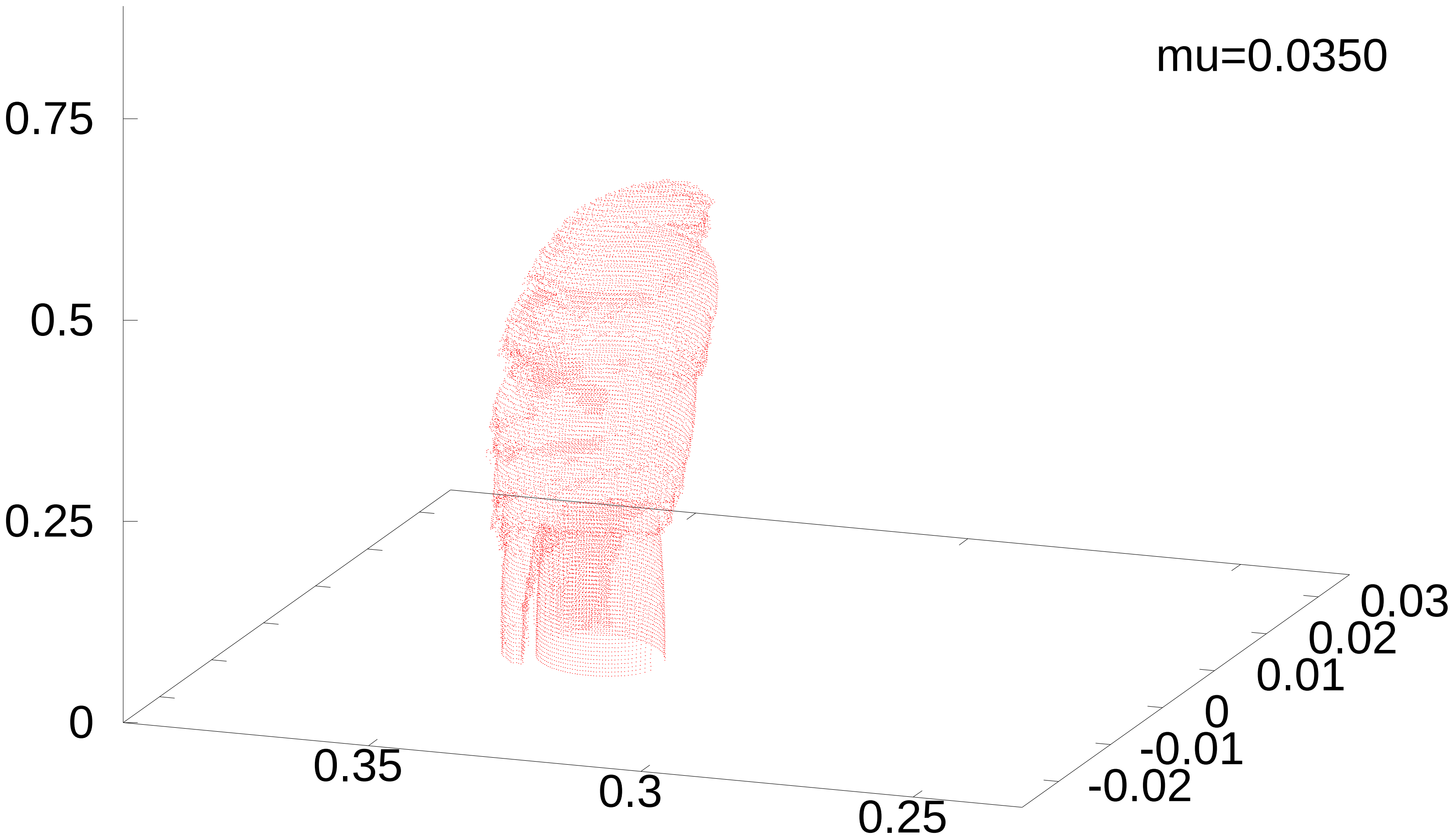
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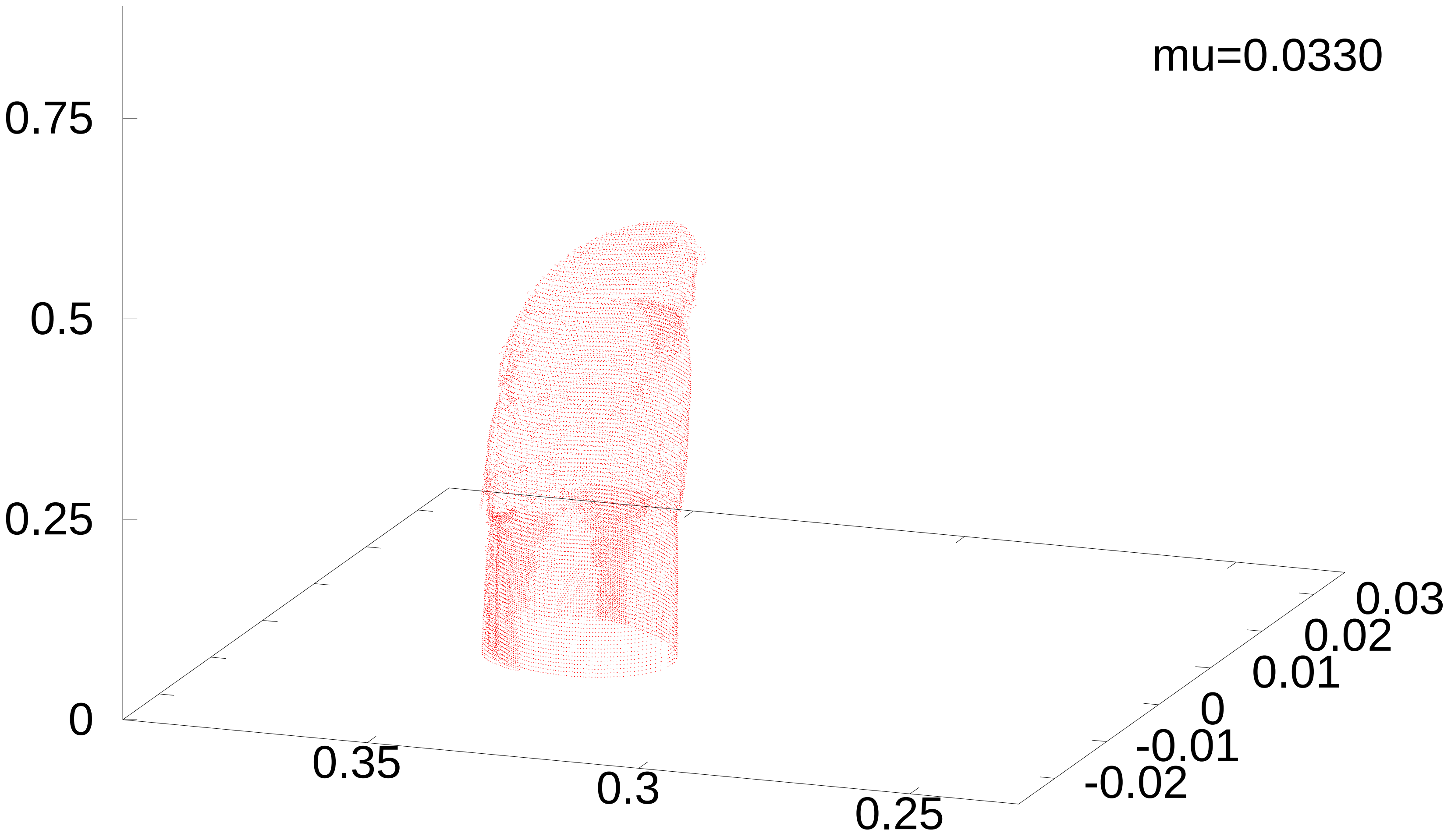

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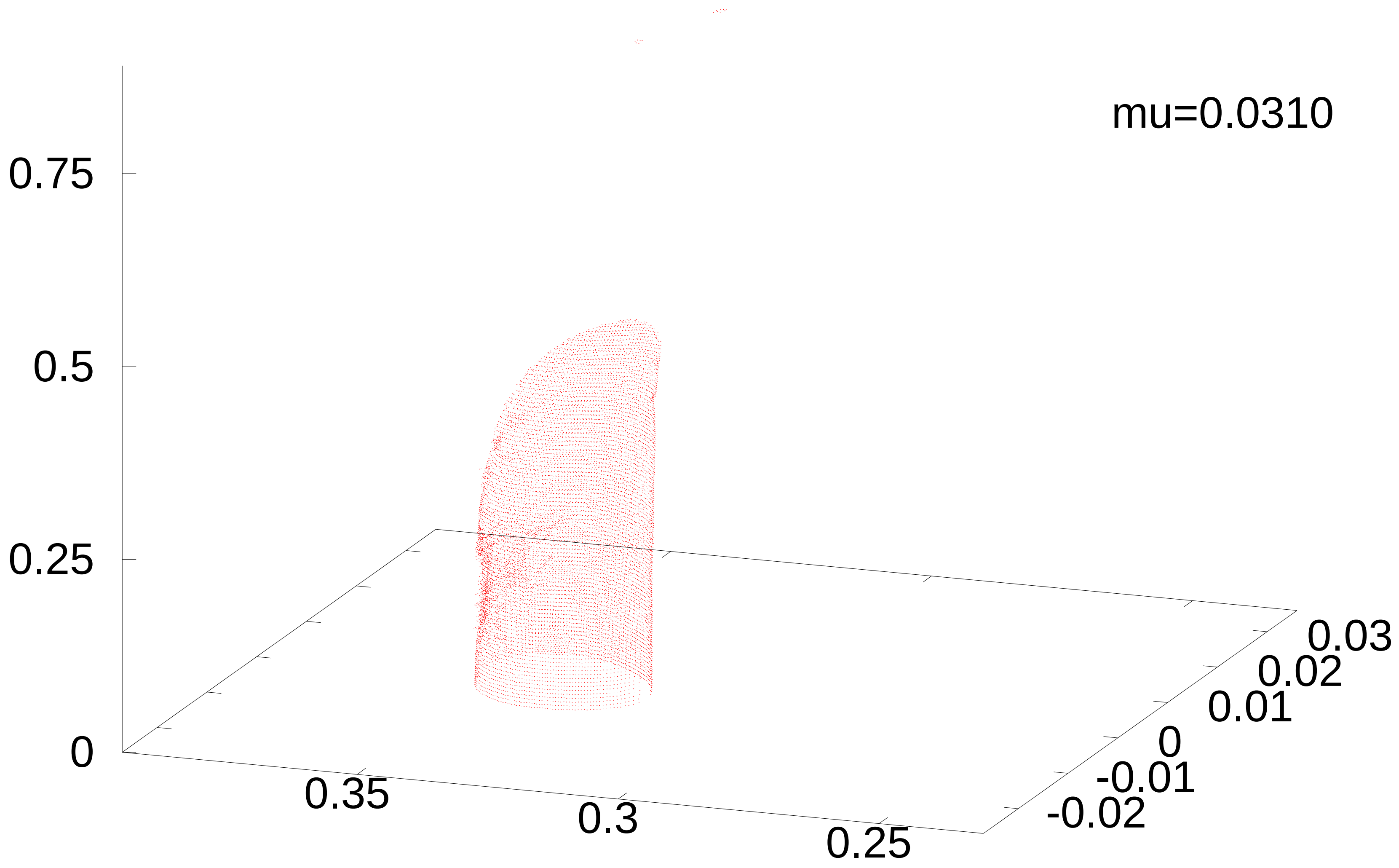

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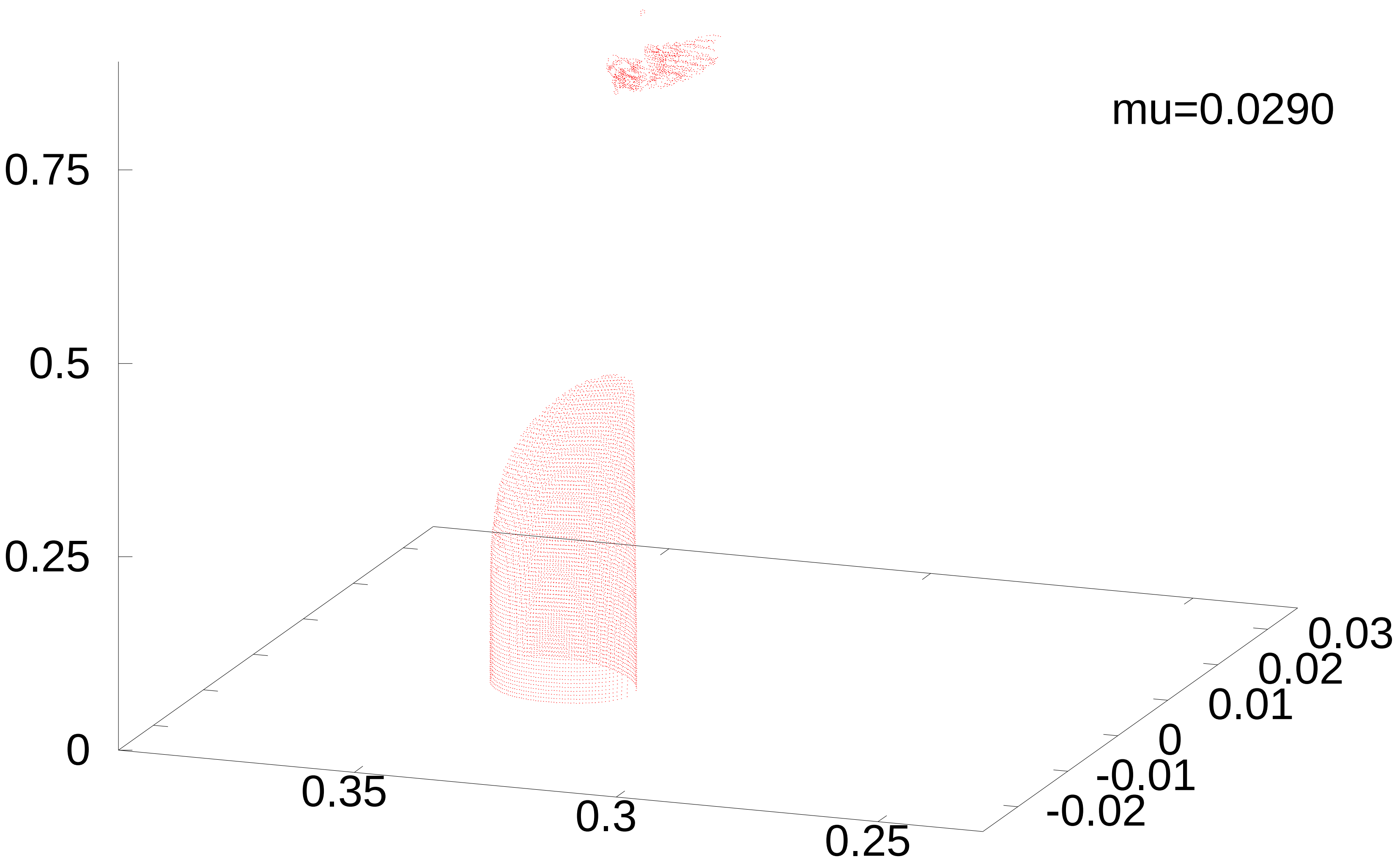

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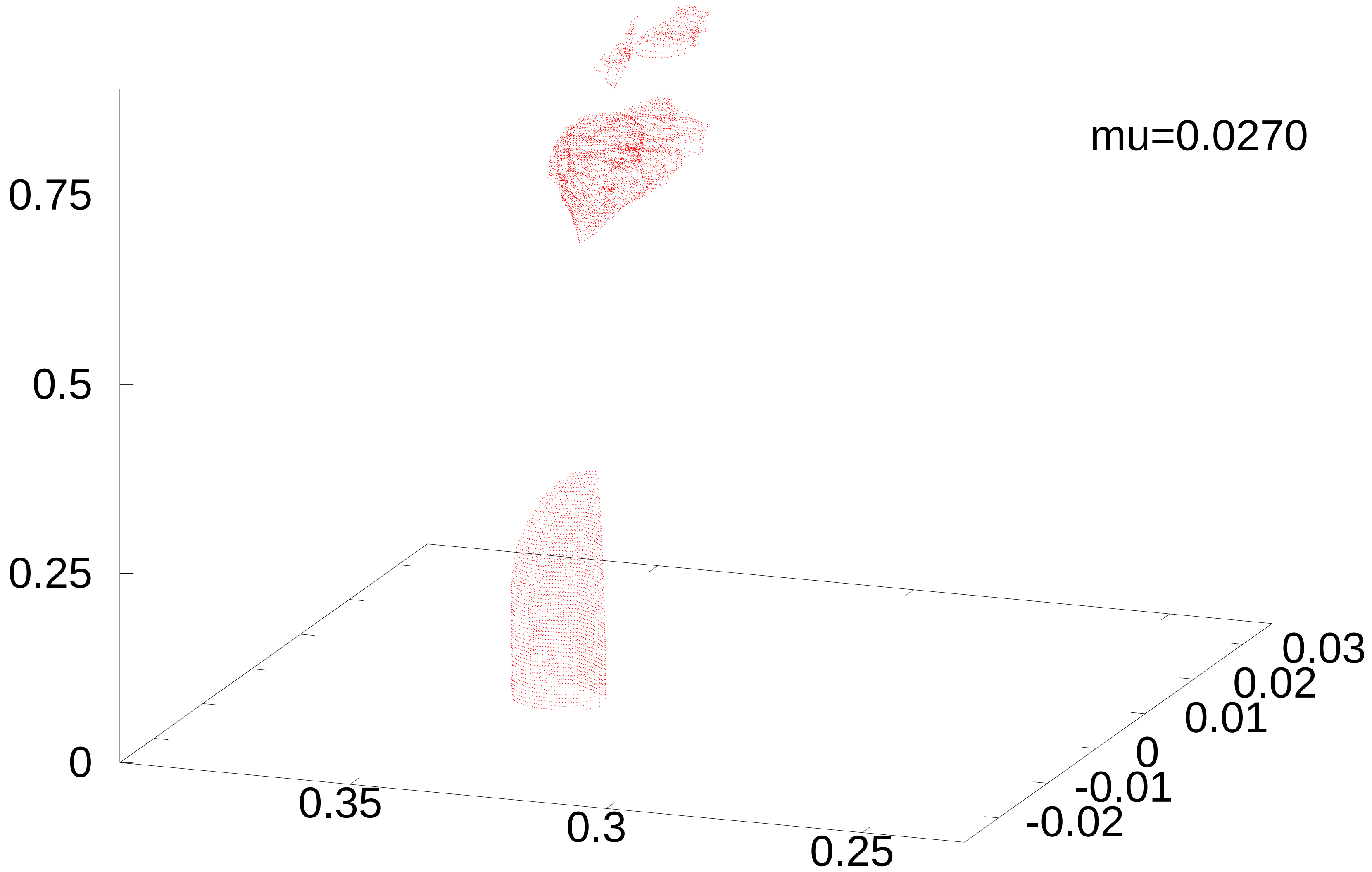
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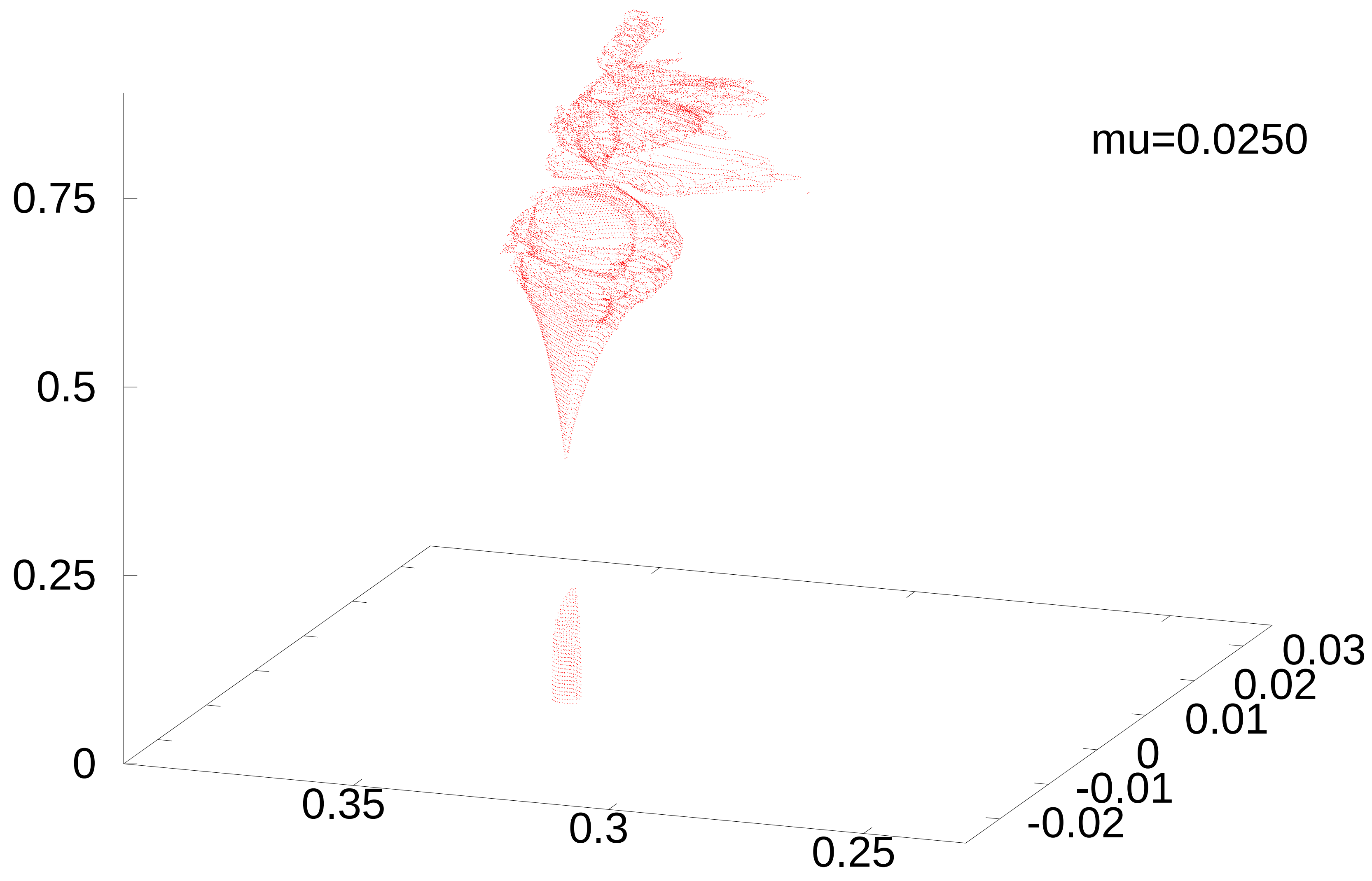
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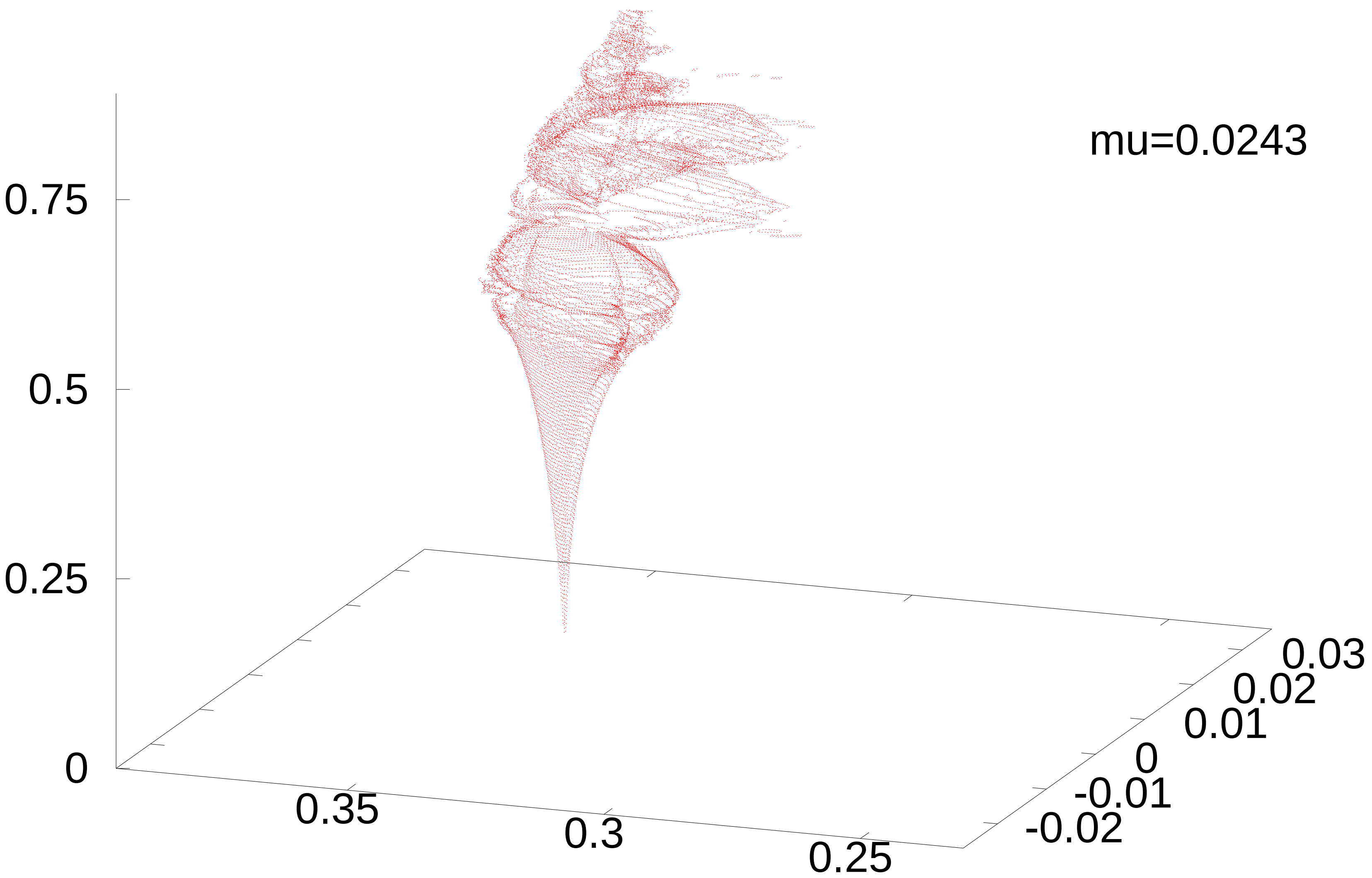
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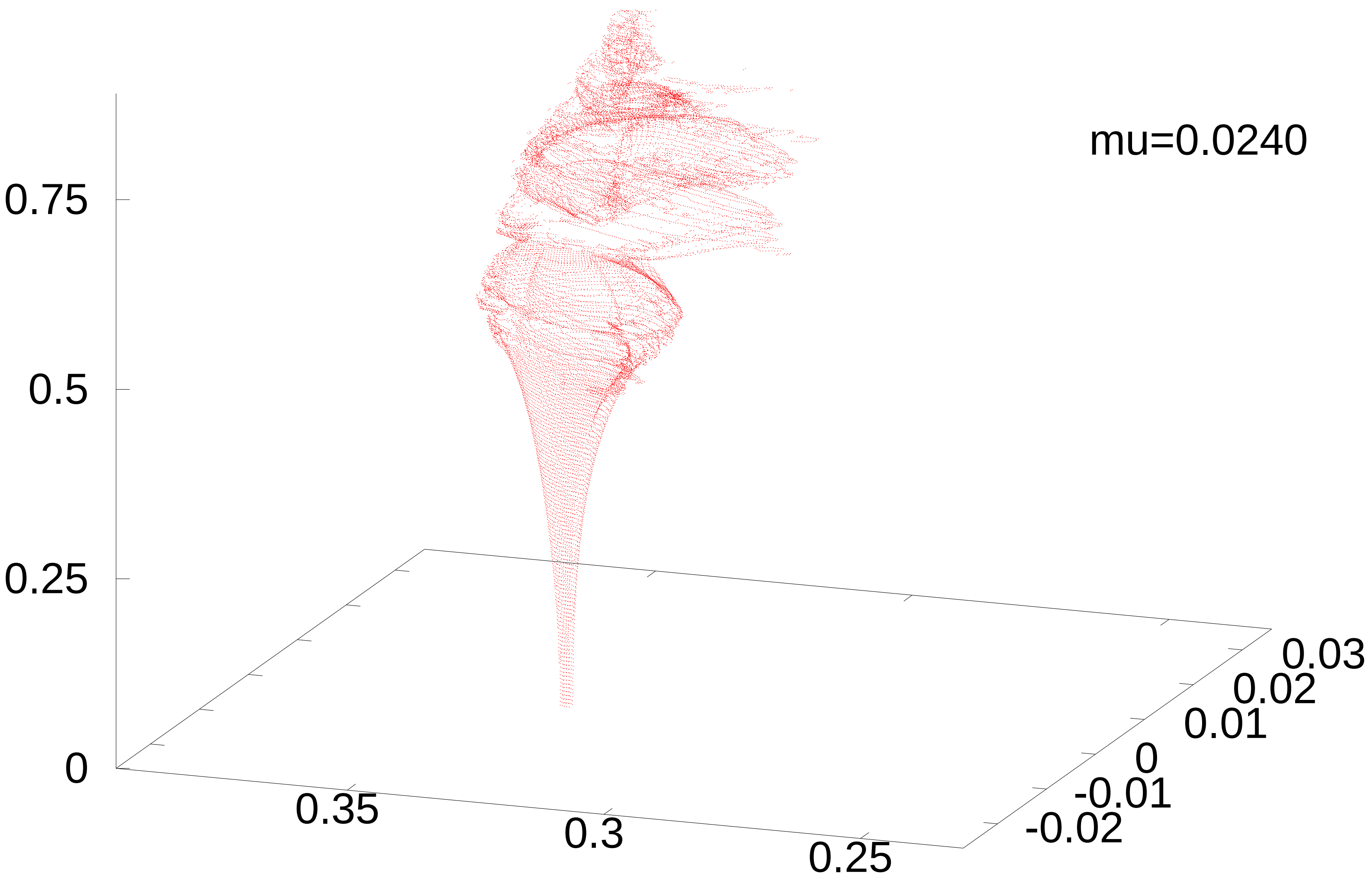
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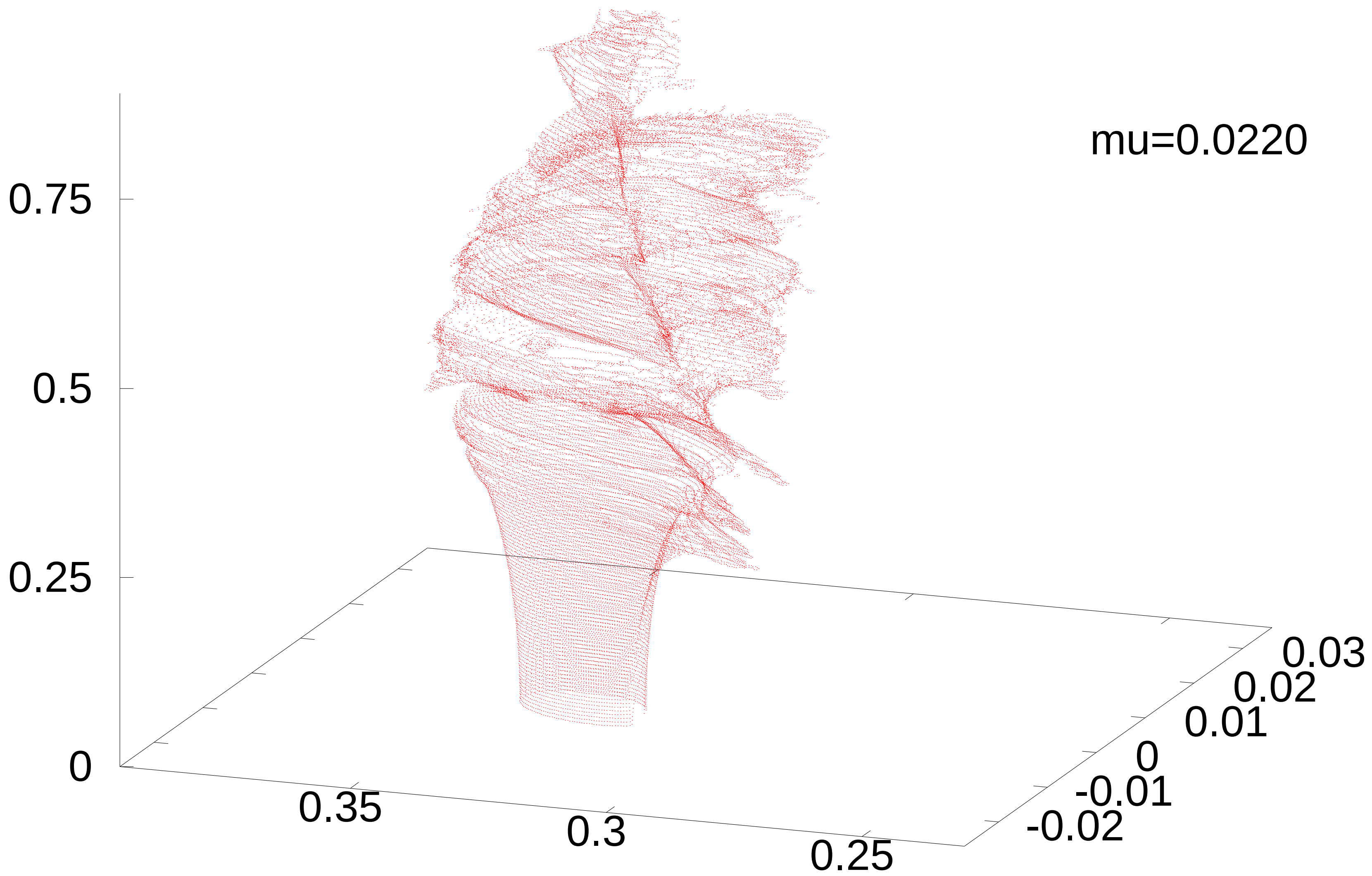

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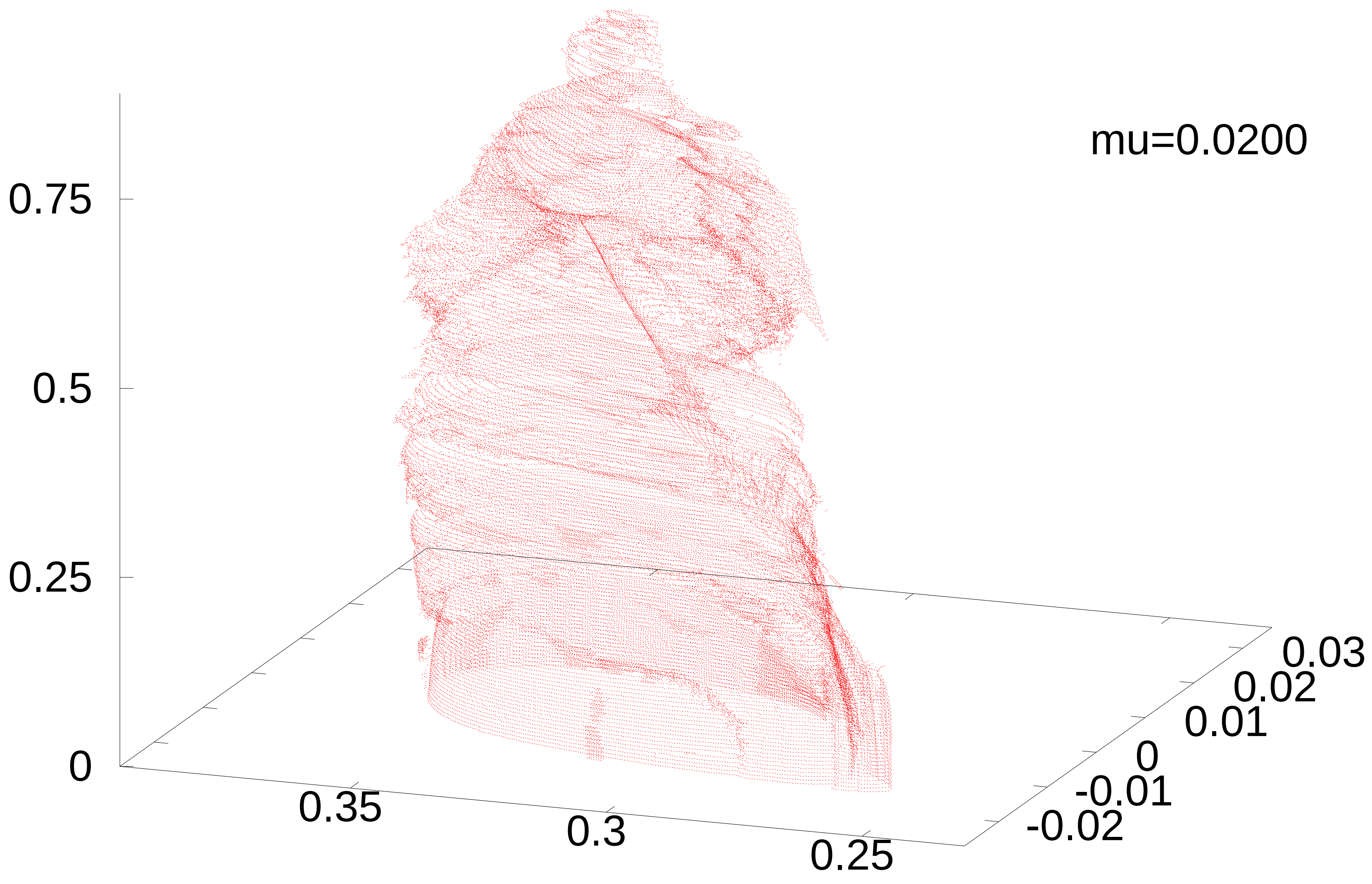

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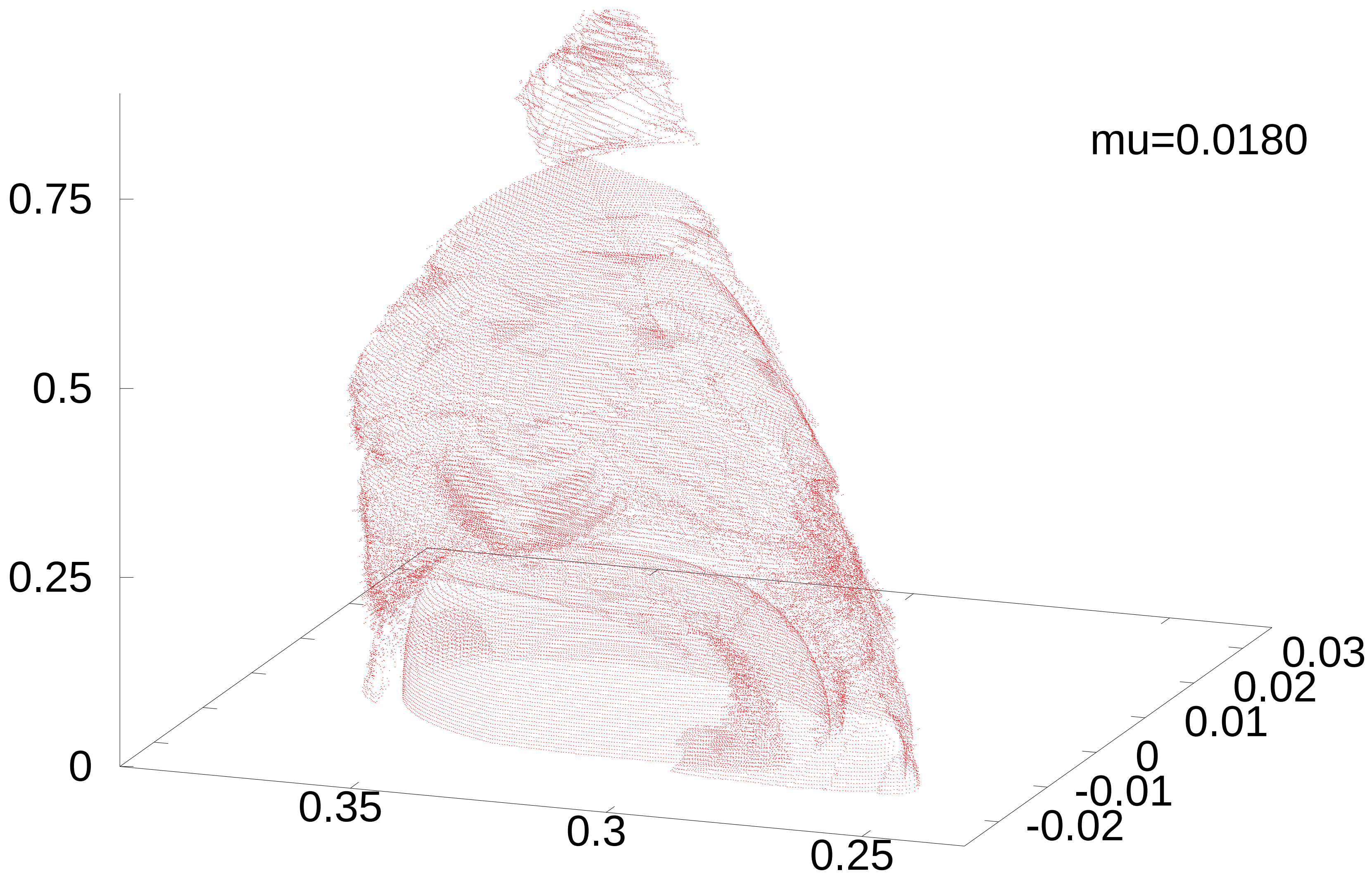

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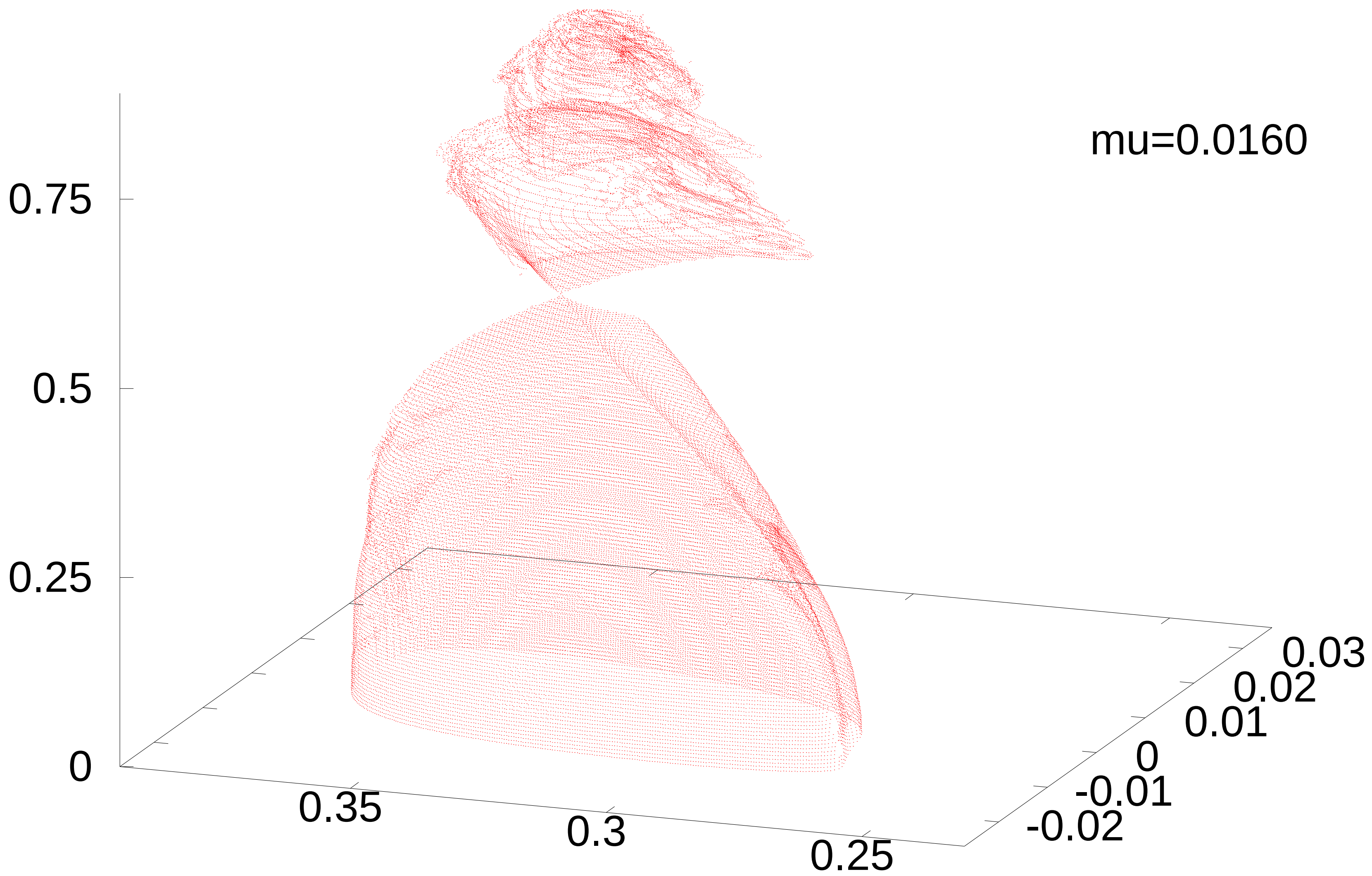
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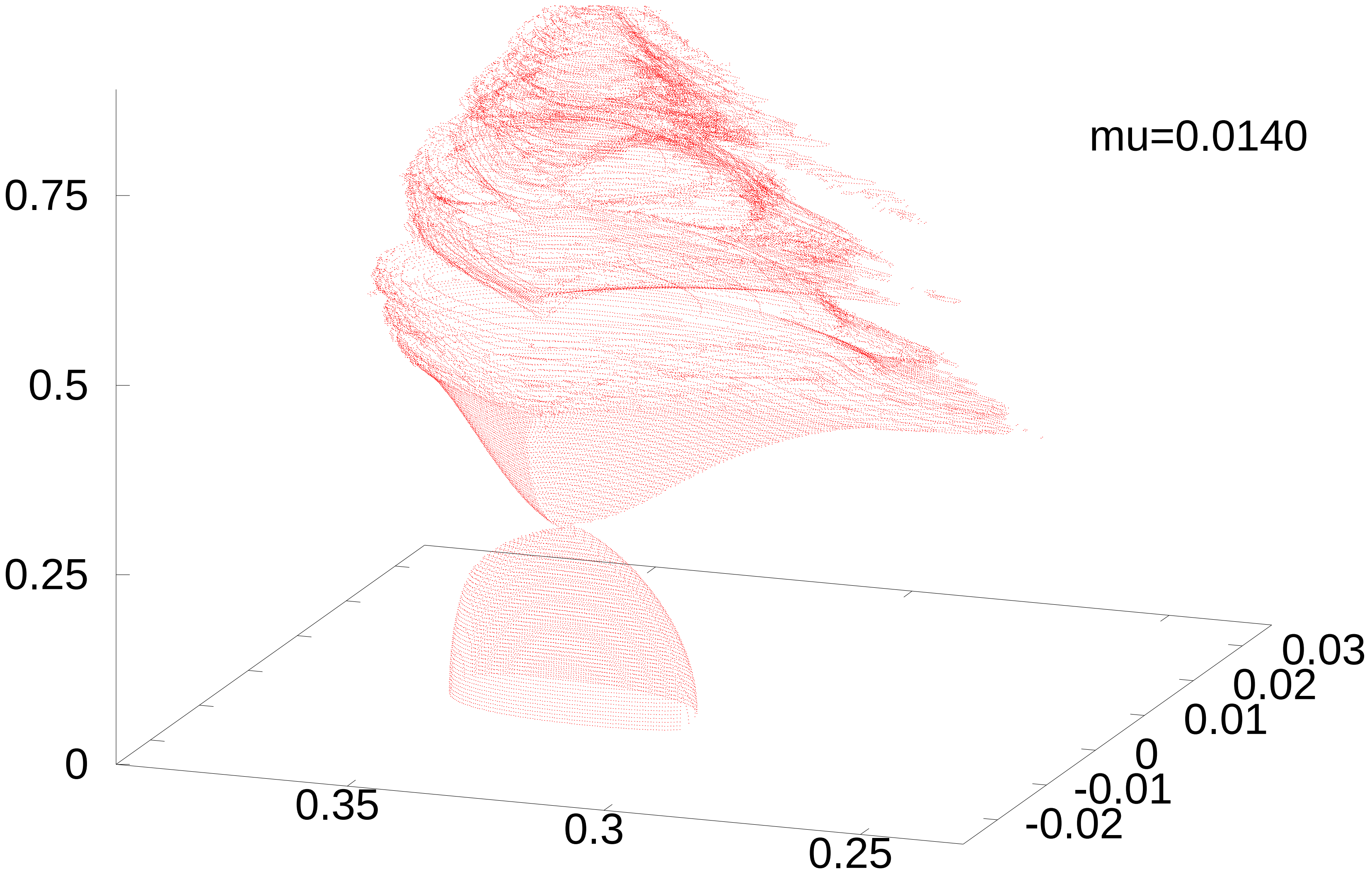
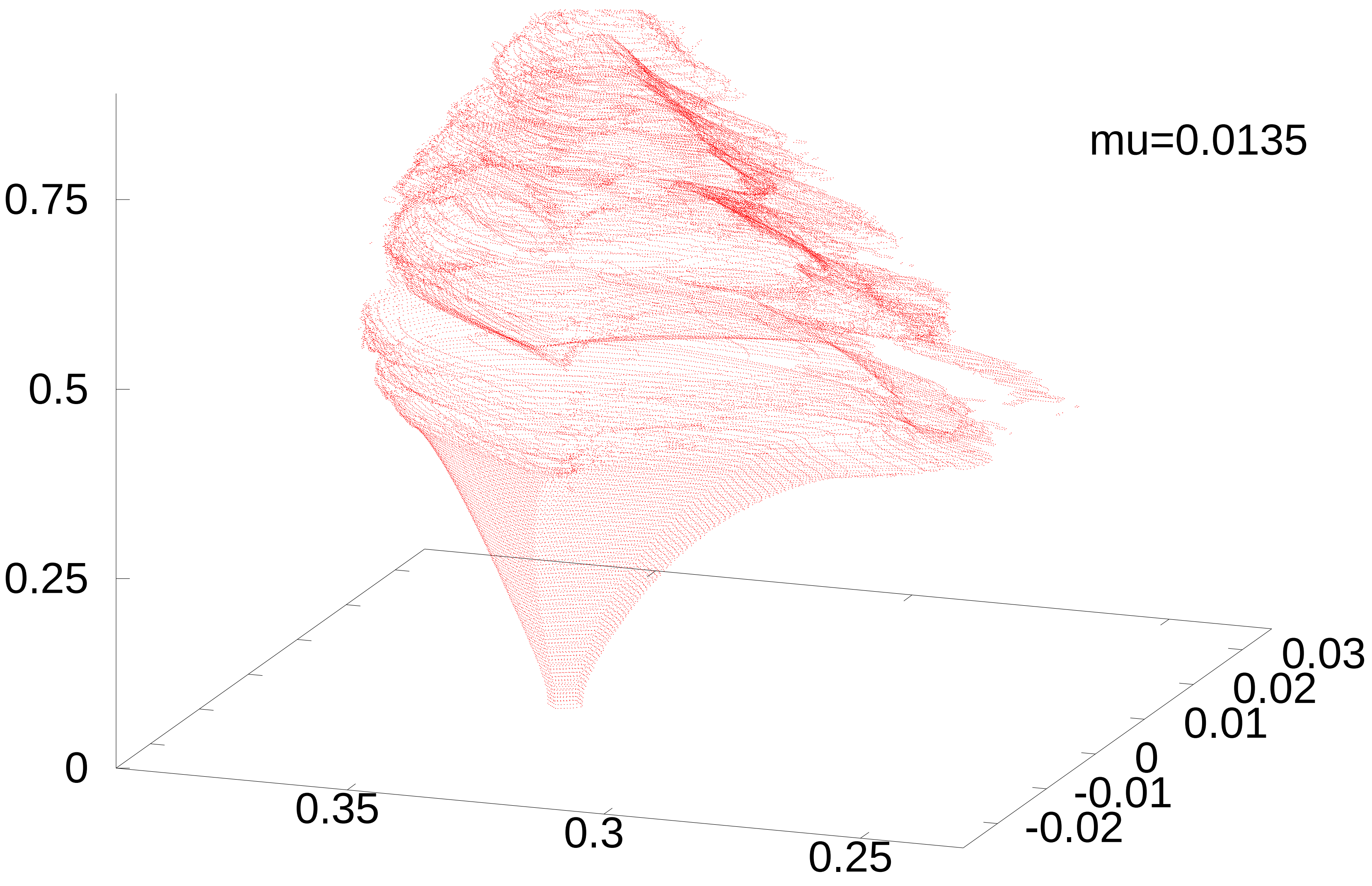
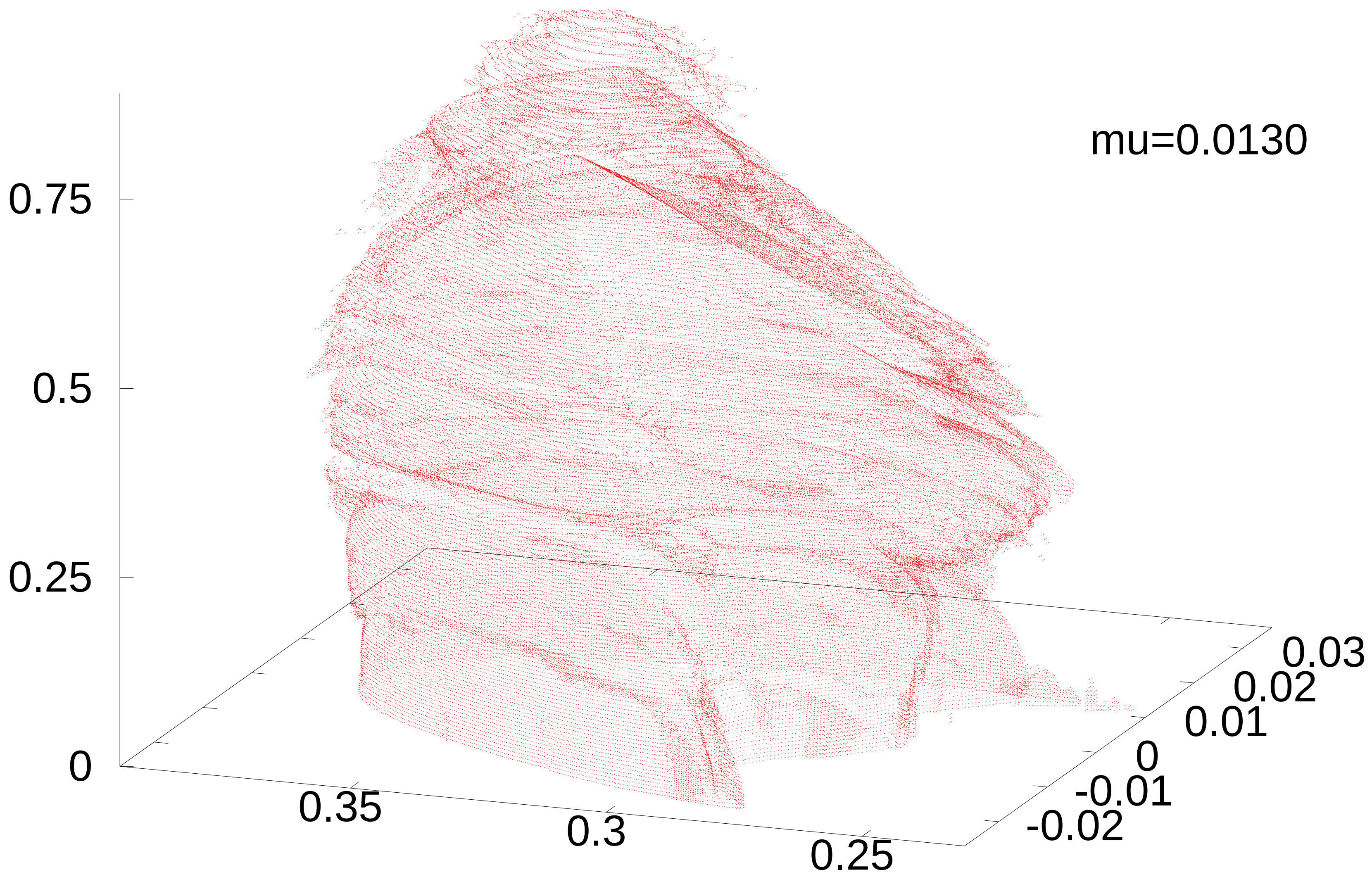
-0.01
-0.02

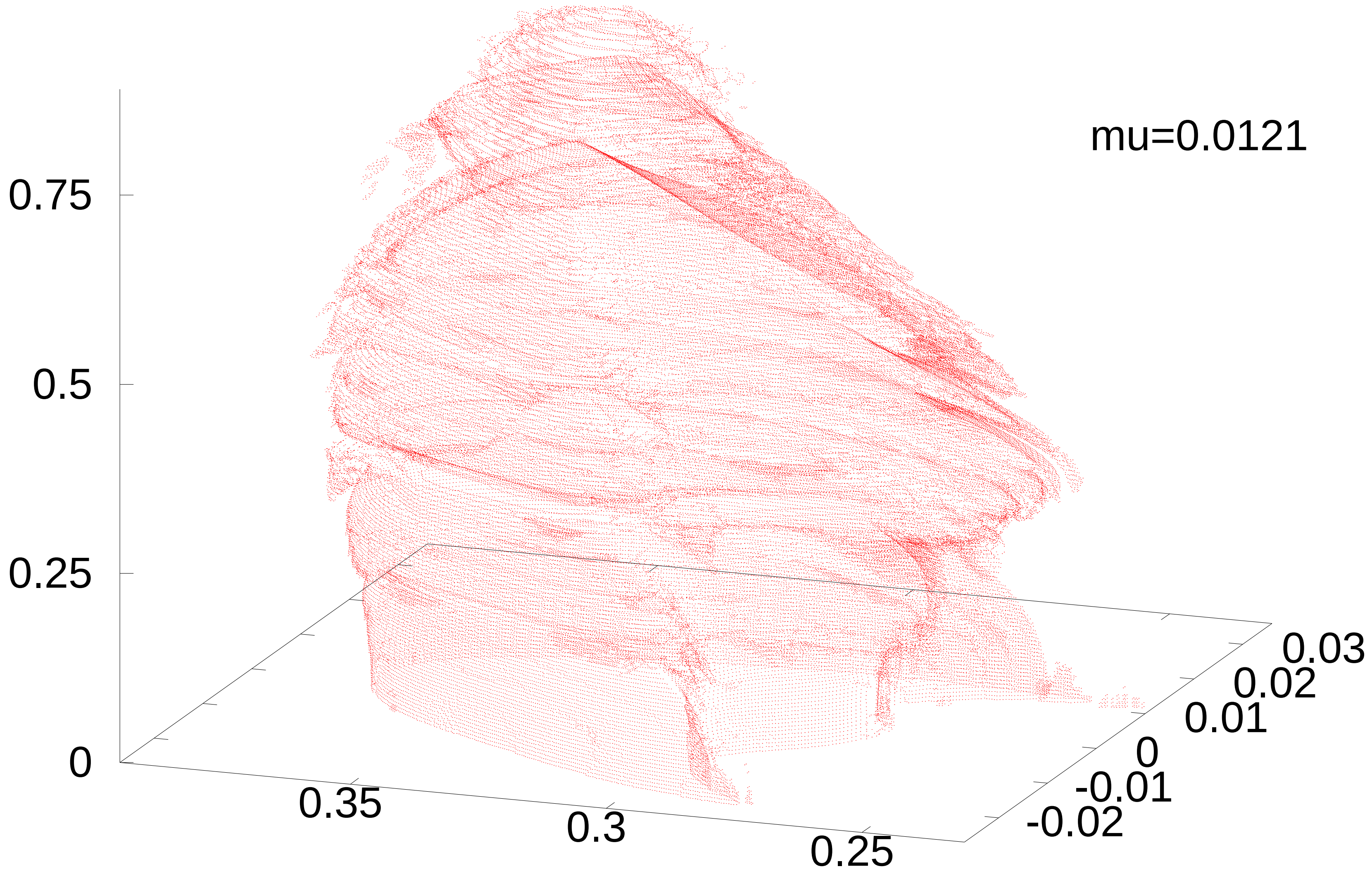

mu=0.0140
0.75
0.5
0.25
0
0.35
0.3
0.25
0.03
0.02
0.01
0
-0.01
-0.02

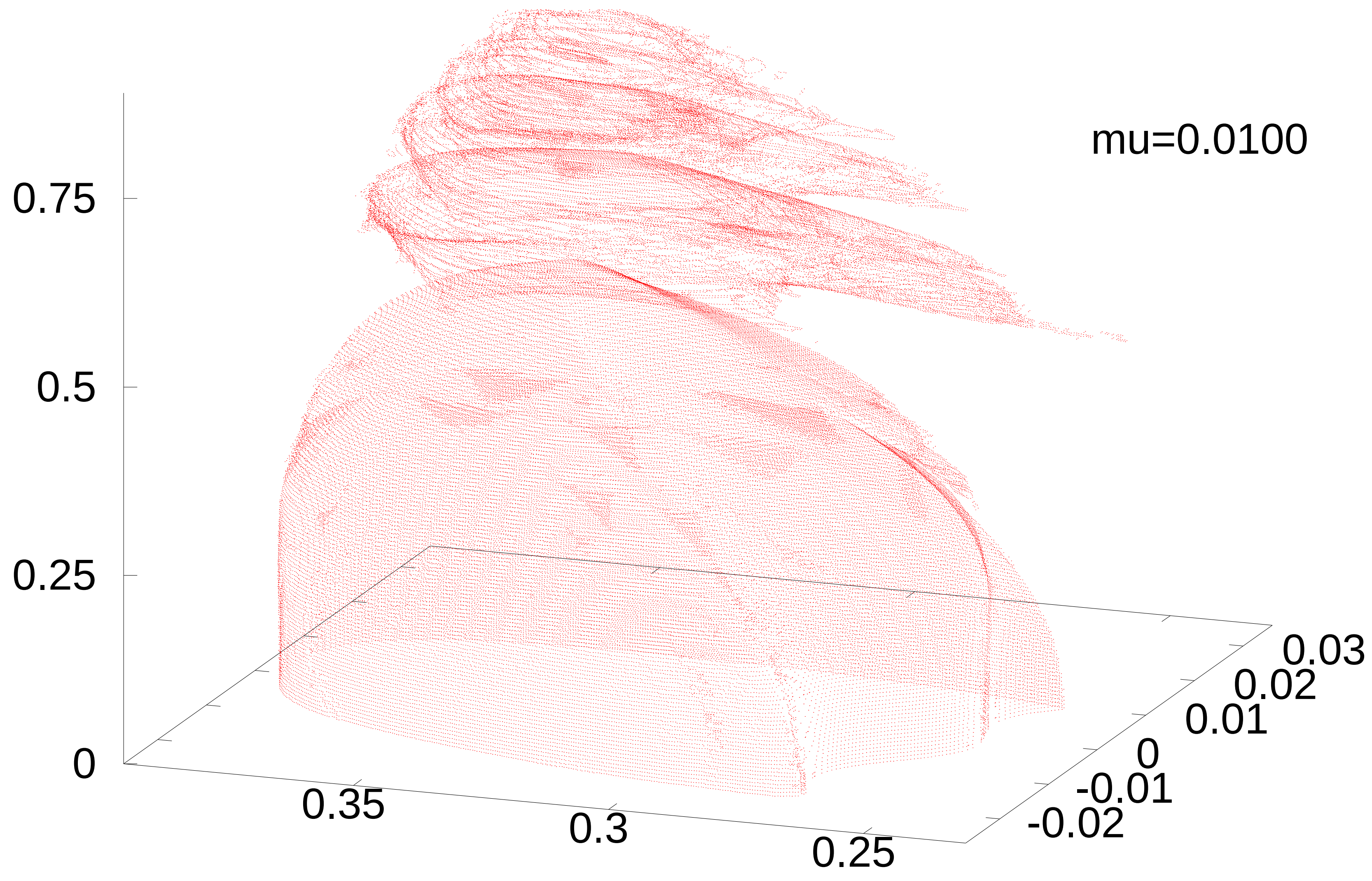


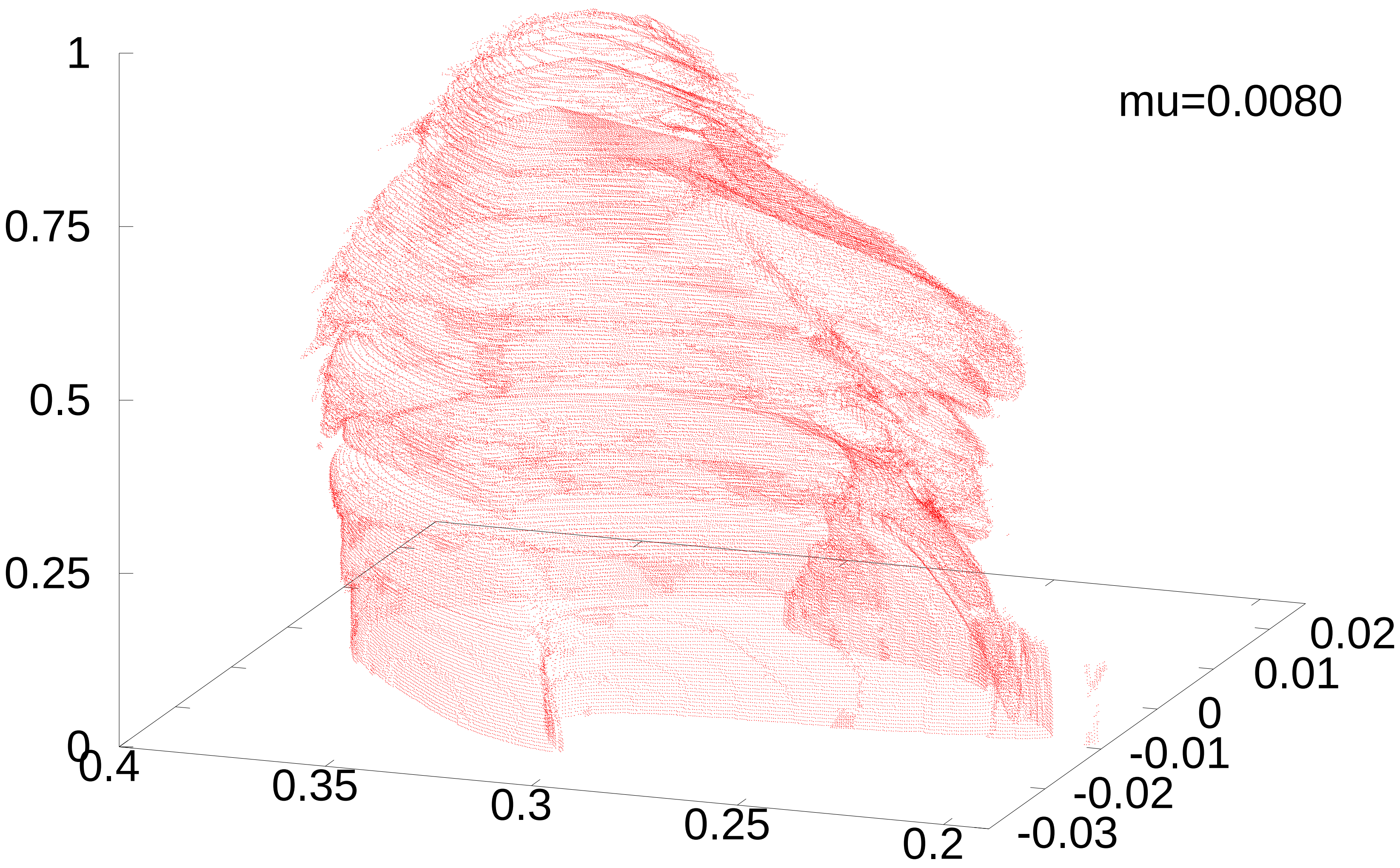
mu=0.0135
0.75
0.5
0.25
0
0.35
0.3
0.25
0.03
0.02
0.01
0
-0.01
-0.02

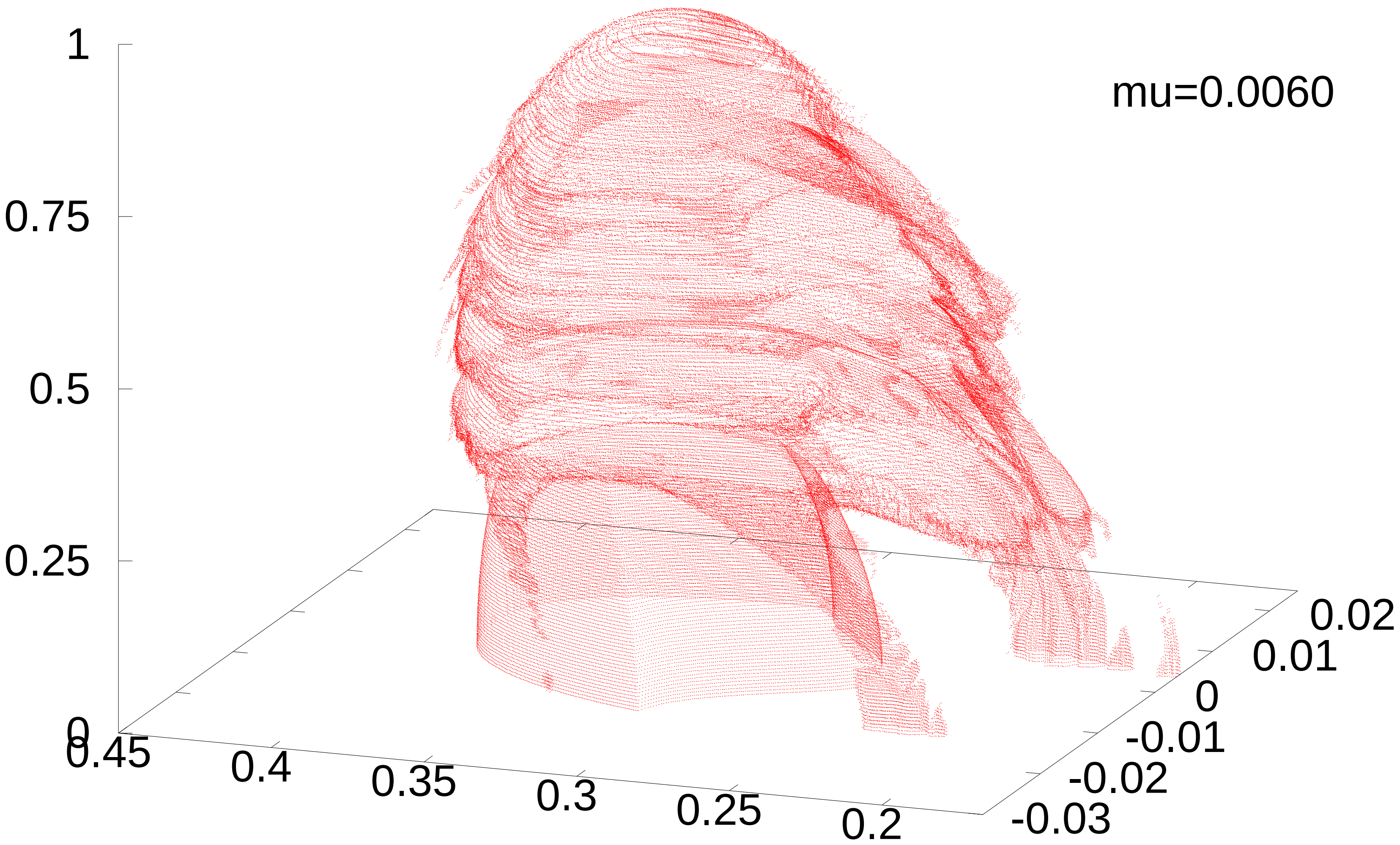


mu=0.0130
0.75
0.5
0.25
0
0.35
0.3
0.25
0.03
0.02
0.01
0
-0.01
-0.02

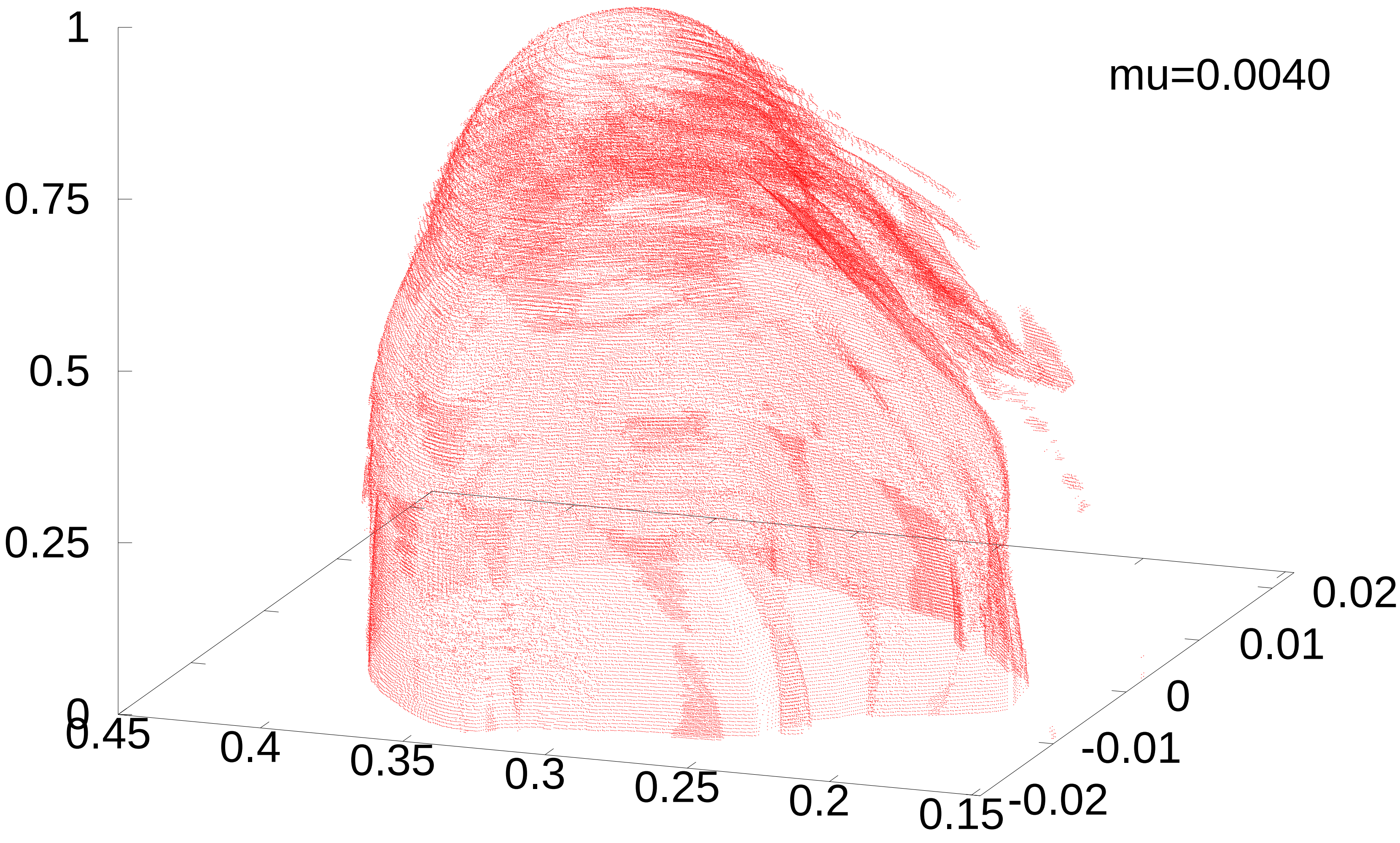

mu=0.0121
0.75
0.5
0.25
0
0.35
0.3
0.25
0.03
0.02
0.01
0
-0.01
-0.02


mu=0.0100
0.75
0.5
0.25
0
0.35
0.3
0.25
0.03
0.02
0.01
0
-0.01
-0.02

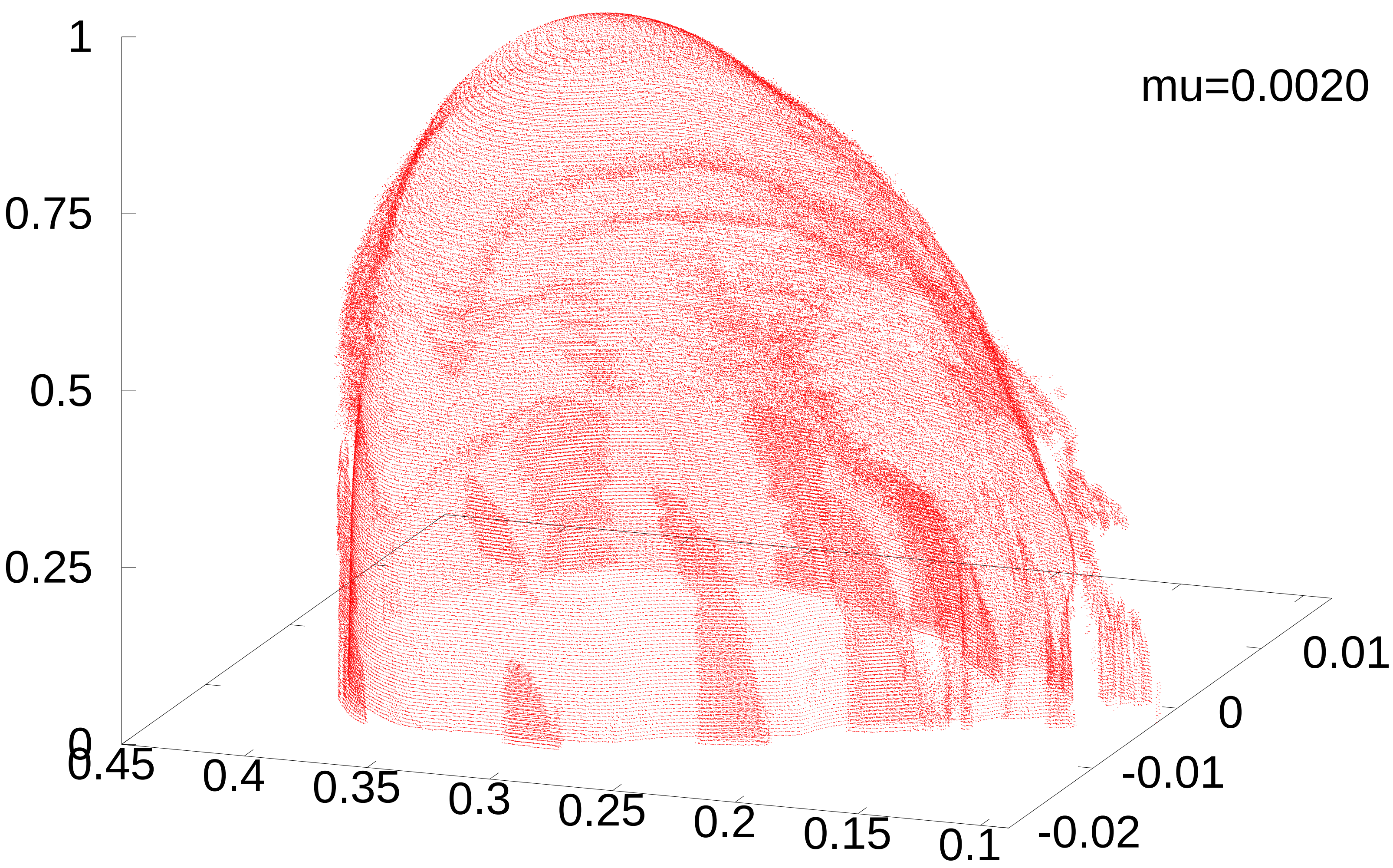

mu=0.0080
1
0.75
0.5
0.25
0
0.4
0.35
0.3
0.25
0.2
0.02
0.01
0
-0.01
-0.02
-0.03



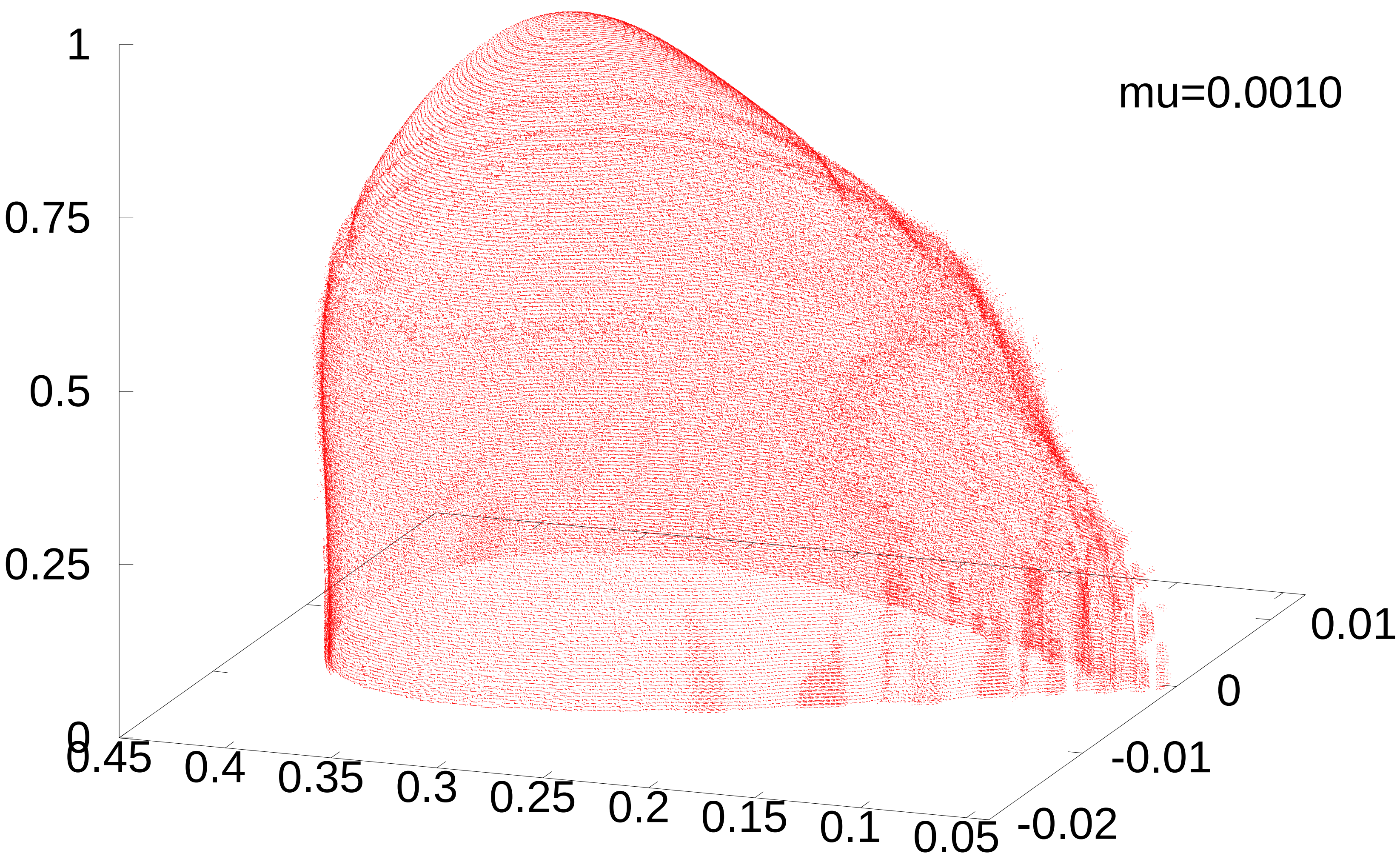
mu=0.0060
1
0.75
0.5
0.25
0
0.45
0.4
0.35
0.3
0.25
0.2
0.02
0.01
0
-0.01
-0.02
-0.03



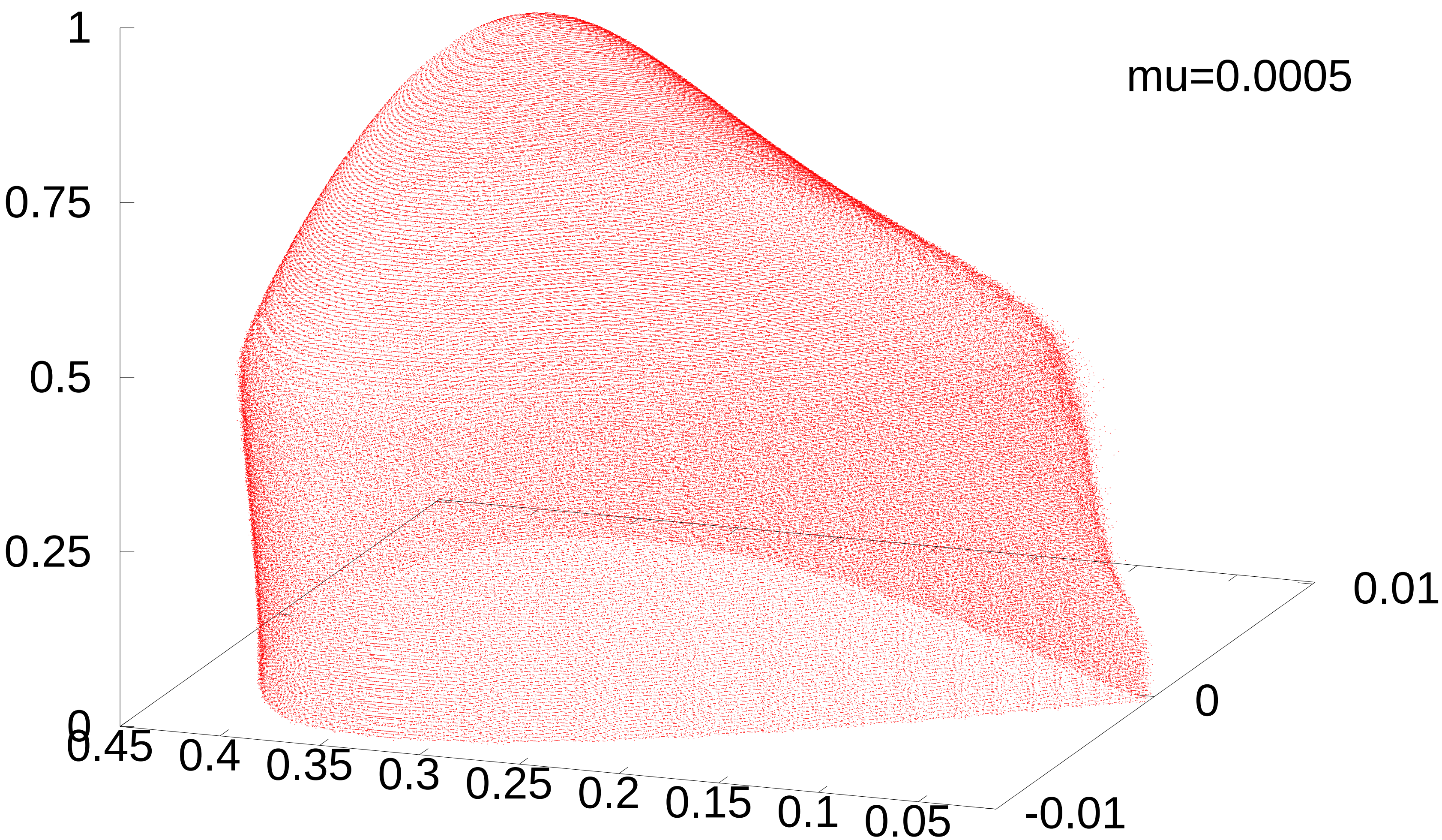
1
0.75
0.5
0.25
0
mu=0.0040
0.45
0.4
0.35
0.3
0.25
0.2
0.15
-0.02
-0.01
0
0.01
0.02



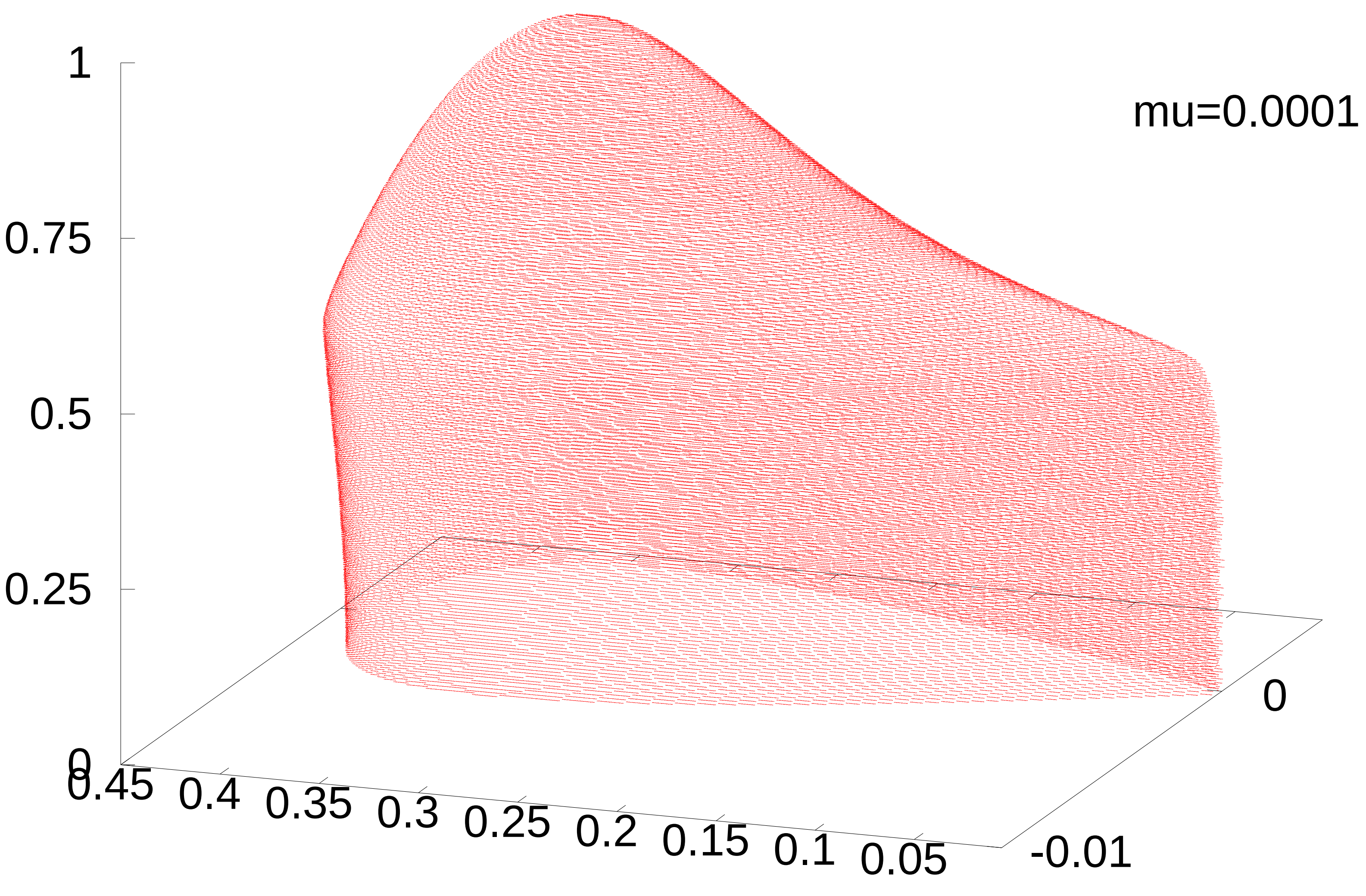
mu=0.0020
1
0.75
0.5
0.25
0
0.45
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.01
0
-0.01
-0.02



mu=0.0010
1
0.75
0.5
0.25
0
0.45 0.4 0.35 0.3 0.25 0.2 0.15 0.1 0.05
0.01
0
-0.01
-0.02

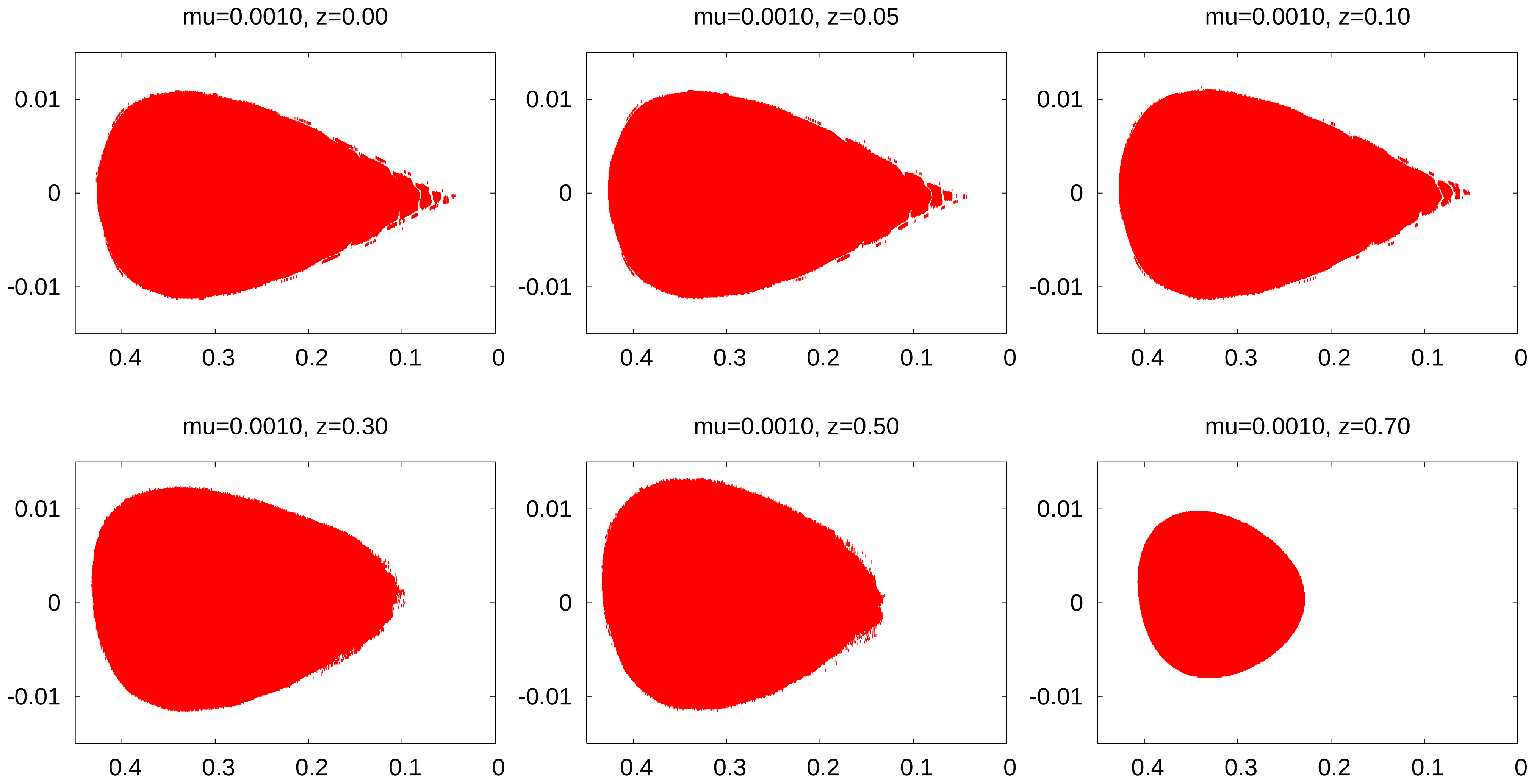


mu=0.0005
1
0.75
0.5
0.25
0
0.45
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0.01
0
-0.01

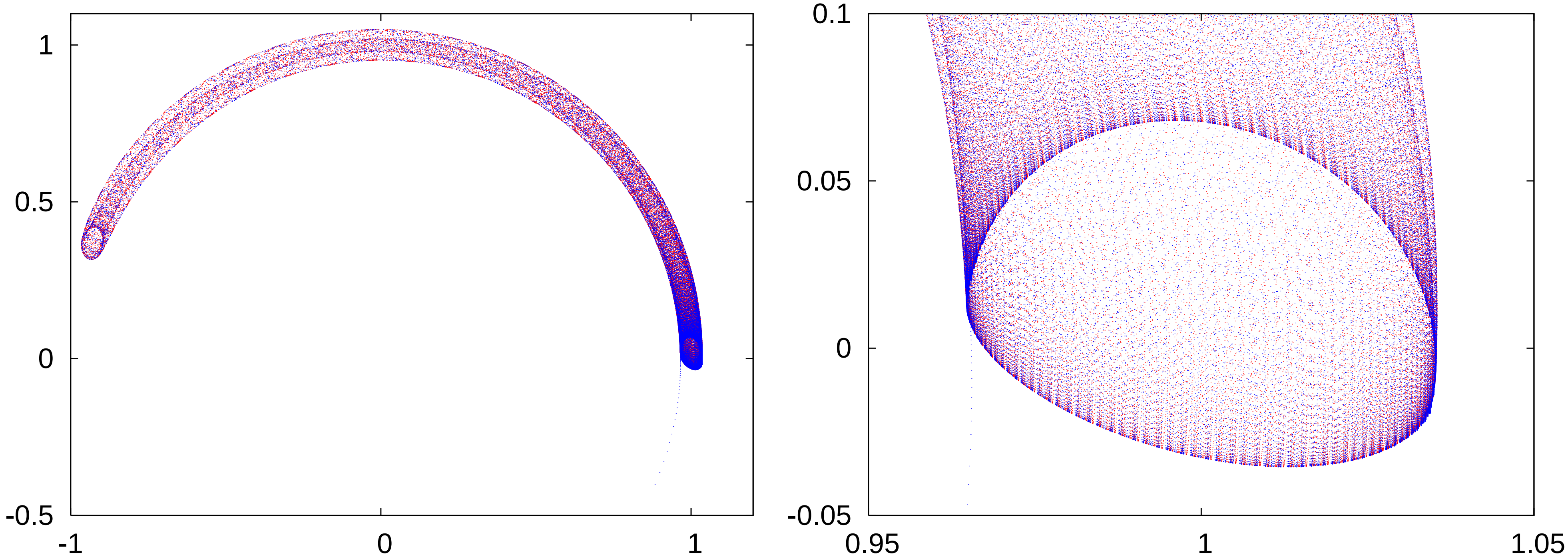


mu=0.0001
1
0.75
0.5
0.25
0
0.45
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
-0.01

Slices for $\mu = 0.001$



At the boundary



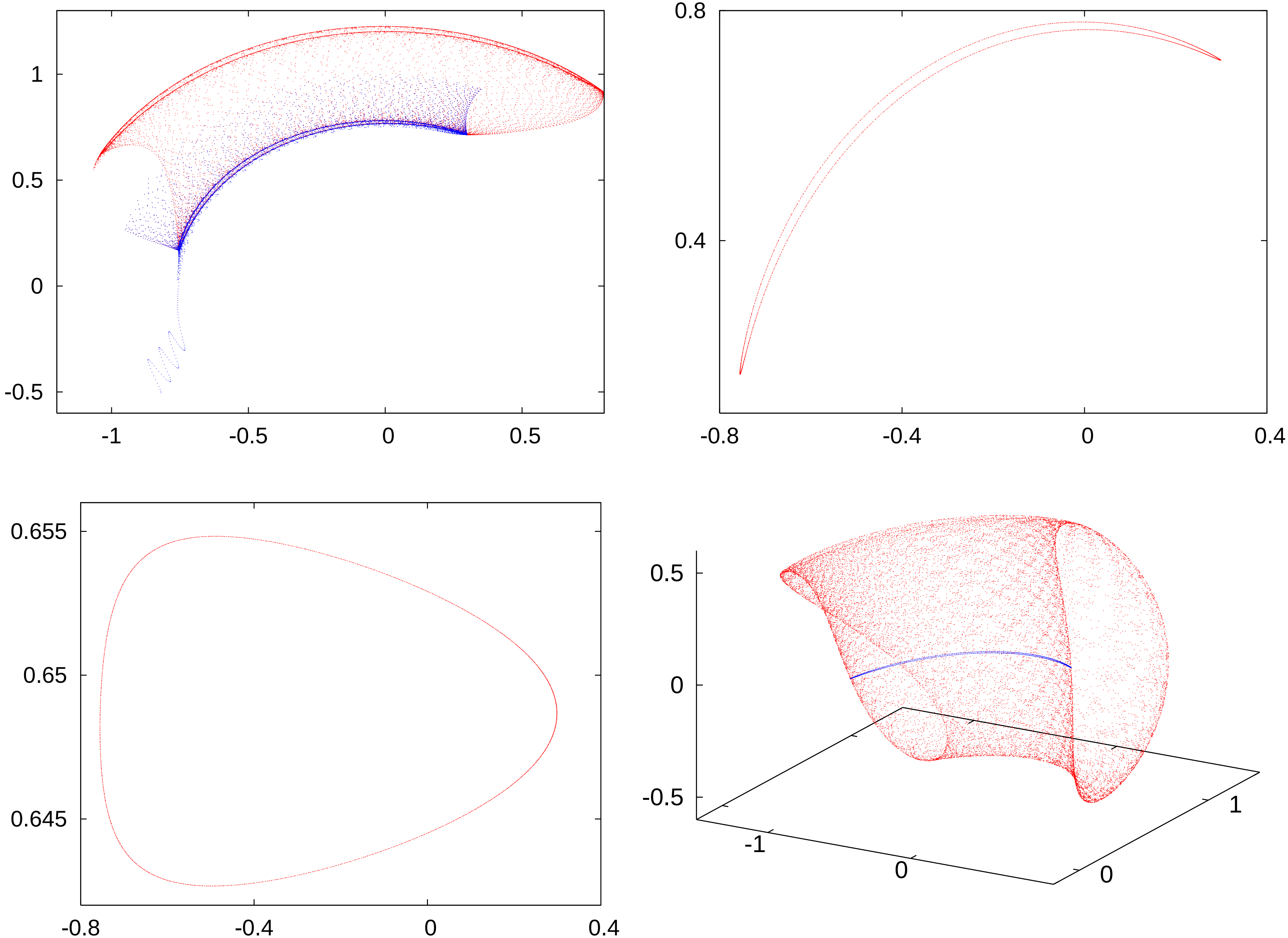
Preliminary comments on the planar case: $W^{u,s}_{L_3}$ at the boundary.

Example of a transition for $\mu = 0.0001$, $\alpha = 0.05$, $z = 0.3$. The two tori (confined, escaping) have ρ which differ in 10^{-10} Shown: the projections on (x, y) of the Poincaré section through $z = 0$. The separating unstable 2D torus (or invariant curve in the section) belongs to $W^{u,s}_{L_3}$.

In the next figure a similar plot for $\alpha = 0.4$, $z = 0.6$. Now the separating unstable 2D torus is not in $W^{u,s}_{L_3}$. It belongs to the center manifold of a family of elliptic-hyperbolic periodic orbits.

Top: initial part of the Poincaré iterates in $z = 0$ of two close initial points projected in (x, y), and the separating unstable invariant curve projected on (x, y).

Bottom: Same curve projected on $(x, \dot{z})$ and the 2D separating unstable torus in the (x, y, z) projection.



1
0.5
0
-0.5
-1
-0.5
0
0.5
0.8
0.4
-0.8
-0.4
0
0.4
0.655
0.65
0.645
-0.8
-0.4
0
0.4
0.5
0
-0.5
-1
0
1
0

Final comments

a) **Different objects** sit at the practical “boundary”,

b) **Different rates of escape** inside and outside the practical “boundary”: very slow diffusion against relatively fast scape,

c) The 3D RTBP is still a toy model. Possible applications to **more realistic problems** (e.g., chemistry),

d) **Cantorian structure smoothed out** by finite time computations. But this is **OK for practical applications**,

e) The effective computation of $W^{u,s}(W^c_{\text{different objects}})$ posses a lot of **computational challenges**. They will be also useful to clarify many more structures,

f) All the relevant objects are **codimension 1 invariant manifolds**,

g) Still **a lot of work** to be done ...