Analysis of the stability of a family of singular–limit linear periodic systems in $\mathbb{R}^4$. Applications.

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Abstract

In this paper we consider a 4D periodic linear system depending on a small parameter $\delta > 0$. We assume that the limit system has a singularity at $t = 0$ of the form $\frac{1}{c_1+c_2t^2+\dots}$, with $c_1, c_2 > 0$ and $c_1 \to 0$ as $\delta \to 0$. Using a blow up technique we develop an asymptotic formula for the stability parameters as δ goes to zero. As an example we consider the homographic solutions of the planar three body problem for an homogeneous potential of degree $\alpha \in (0, 2)$. Newtonian three-body problem is obtained for $\alpha = 1$. The parameter δ can be taken as $1 - e^2$ being e the eccentricity (or a generalised eccentricity if $\alpha \neq 1$). The behaviour of the stability parameters predicted by the formula is checked against numerical computations and some results of a global numerical exploration are displayed.

1 Introduction

Given $\alpha \in (0, 2)$ we consider the following linear system

$$\begin{aligned} \ddot{x} &= \frac{\lambda_1}{g^{2-\alpha}}x - 2\dot{y}, \\ \ddot{y} &= \frac{\lambda_2}{g^{2-\alpha}}y + 2\dot{x}, \end{aligned} \tag{1}$$

where λ_1, λ_2 are real parameters different from zero and $g = g(t; \delta)$ is a periodic function on t which depends on α and on a parameter $\delta \in (0, \delta_0]$ with δ_0 small enough. Suppose $g(t; \delta) > 0$ for all t and $g(0; \delta) \to 0$ for $\delta \to 0$. Therefore, the system (1) has a singularity at $t = 0$ for $\delta = 0$. Our purpose is to study the stability parameters of system (1) for small values of $\delta > 0$ under some hypotheses to be specified below.

Let $U(z) = z^\alpha V(z)$ be a real function defined on an open interval $(0, z_b)$ where $V(z)$ is an analytic function for $z > 0$ such that

(A1) *there exists* z_a, $0 < z_a < z_b$, *such that* $V(z_a) = 0$, $V(z) < 0$ *for all* $z \in (0, z_a)$ *and* $V_z(z) > 0$ *for all* $z \in (0, z_b)$. ($V_z(z)$ stands for the derivative of $V(z)$ with respect to z.)

(A2) $V(z) = \gamma + z^s V_1(z)$, *with* $\gamma < 0$, $s > (2 - \alpha)/2$, *and* $V_1(z)$ *is an analytic function on an open set* J, $J \supset [0, z_a]$.

See figure 3 in section 6 for several examples showing the shape of U.

Let us consider the conservative system

$$\ddot{z} = -U_z(z) \tag{2}$$

with $U(z)$ satisfying (A1) and (A2). We denote the energy of (2) by

$$E = \frac{\dot{z}^2}{2} + U(z). \tag{3}$$

We shall assume the following hypothesis for $g(t; \delta)$

(B) *For $\delta > 0$, $g(t;\delta)$ is the periodic solution of (2) on the energy level $E = -\delta$ such that $g(0;\delta) = g_0$, $\dot g(0;\delta) = 0$, being g_0 the minimum of $g(t;\delta)$.*

Note that for $\delta > 0$, if g satisfies property (B) then $g(t;\delta)$ is an even function on t with period $T = T(\delta)$ which tends to a finite value when $\delta \to 0$. Moreover $g_0 = (\delta/|\gamma|)^{1/\alpha}(1 + O(\delta^{s/\alpha}))$.

The motivation to study the system (1) comes from the linear stability analysis of a special kind of solutions of the planar three-body problem, the so called homographic solutions. In section 6 we shall introduce these solutions for a three body problem with some homogeneous potentials. The corresponding variational equations along these solutions can be reduced to a linear system of type (1). In particular the Newtonian case is obtained for $\alpha = 1$ and $U(z) = z(-1 + z/2)$. In this case, $g(t;\delta) = 1 - e\cos t$, where e is the eccentricity of the homographic solution, and $\delta = (1 - e^2)/2$. The time t is the true anomaly. The singularity of the equations is attained for $e = 1$. This means that the corresponding solution goes to collision.

Notation: I_n stands for the identity matrix of order n, $J_{2n} = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$ is the $2n \times 2n$ skew symmetric matrix, and K_{2n} is the $2n \times 2n$ diagonal matrix defined as $K_{2n} = \mathrm{diag}(J_2, \ldots, J_2)$.

In the following we shall write (1) as

$$\dot{\mathbf{x}} = A(t)\mathbf{x}, \qquad A(t) = \begin{pmatrix} 0 & I_2 \\ \tilde A(t) & -2J_2 \end{pmatrix}, \qquad \tilde A(t) = g^{\alpha-2}\mathrm{diag}(\lambda_1, \lambda_2). \tag{4}$$

It is understood that $A(t)$ is a matrix which depends on three parameters: λ_1, λ_2 and δ and we are mainly interested in the system for values of δ small enough. Anyhow, to simplify the notation, the dependence on these parameters will not be explicitly written if there is no confusion. We shall use the same simplification for all linear systems which appear in what follows and for their corresponding monodromy matrices.

The system (4) can be seen as a Hamiltonian system by introducing new variables $\mathbf{y} = (y_1, y_2, y_3, y_4)^T$ defined by $\mathbf{y} = M\mathbf{x}$ with $M = \begin{pmatrix} I_2 & 0 \\ J_2 & I_2 \end{pmatrix}$. The Hamiltonian of the new system is the following one

$$H(\mathbf{y}, t) = \frac{y_3^2 + y_4^2}{2} + y_1y_4 - y_2y_3 - \left(\lambda_1 g^{\alpha-2} - 1\right)\frac{y_1^2}{2} - \left(\lambda_2 g^{\alpha-2} - 1\right)\frac{y_2^2}{2}. \tag{5}$$

Let $\Phi(t)$ be the fundamental matrix of (4) such that $\Phi(0) = I_4$. It is easy to check that

$$\Phi(t) = M^{-1}\Phi_1(t)M, \tag{6}$$

where $\Phi_1(t)$ is the fundamental matrix of the linear Hamiltonian system defined by (5) such that $\Phi_1(0) = I_4$. The symplectic character of $\Phi_1(t)$ implies that if μ is an eigenvalue of $\Phi(T)$ then μ^{-1} is also an eigenvalue. We denote by $\mu_1, \mu_1^{-1}, \mu_2, \mu_2^{-1}$ the eigenvalues of $\Phi(T)$ and define the stability parameters as

$$\mathrm{tr}_i = \mu_i + \mu_i^{-1}, \qquad i = 1, 2.$$

We shall give asymptotic formulae for these stability parameters provided some non degeneracy conditions are satisfied. To do this the main point is to use some kind of blow up technique to see the limit case when δ tends to zero as a linear system on an heteroclinic connection. Two coefficients d_g, e_g (to be introduced in section 4) appear in the computations. These coefficients depend on the particular potential $U(z)$ and on parameters λ_1 and λ_2. We shall assume non degeneracy conditions in the sense that $d_g \neq 0$ and $e_g \neq 0$. The meaning of these hypotheses is that the dominant terms for the stability parameters are then the expected ones, i.e. no unwanted cancellations occur.

Theorem 1. *Let us consider the system (4) where $g(t;\delta)$ satisfies the hypothesis (B) and assume non degeneracy conditions. Let be $\hat{\lambda} = \gamma(2-\alpha)^2/8$ where γ is defined in (A2). We assume that λ_1, λ_2 are different from zero and satisfy $\lambda_1 > \lambda_2 > \hat{\lambda}$ or, $\lambda_1 > \hat{\lambda} > \lambda_2$. Let be $\beta_j = \sqrt{1 - \frac{\lambda_j}{\hat{\lambda}}}$, $j = 1, 2$. Then we have the following asymptotic behaviour for the stability parameters when δ goes to 0*

$$\begin{aligned} \log|tr_1| &= k_1 - \frac{2-\alpha}{2\alpha}\beta_1 \log\delta(1+o(1)) + \dots, \\ \log|tr_2| &= k_2 - \frac{2-\alpha}{2\alpha}\beta_2 \log\delta(1+o(1)) + \dots, && \text{if} \quad \lambda_2 > \hat{\lambda}, \\ tr_2 &= k_3 + k_4\cos[k_5 - \gamma_2(1+o(1))\log\delta] + \dots, && \text{if} \quad \lambda_2 < \hat{\lambda}. \end{aligned} \tag{7}$$

In the last case $\beta_2 = i\gamma_2$. The coefficients k_j, $j = 1, \dots, 5$ are constants and $k_4 \neq 0$.

Let us comment on the hypotheses in theorem 1, where we have $\beta_1 \in \mathbb{R}^+$ and, if $\beta_2 \in \mathbb{R}^+$, then $\beta_1 > \beta_2$. These assumptions will give a dominant term depending on β_1 for the stability parameters. As we will see in section 6, these hypotheses will be satisfied in the case of homographic solutions.

The asymptotic formulae (7) gives $|\mathrm{tr}_1| > 2$ if δ is small enough. Furthermore, if $\beta_2 \in \mathbb{R}^+$ then $|\mathrm{tr}_2| > 2$ and the system is hyperbolic-hyperbolic. In the case $\beta_2 = \gamma_2 i$ with $\gamma_2 \in \mathbb{R}^+$, tr_2 oscillates between the values $k_3 + k_4$ and $k_3 - k_4$ as δ tends to 0. Therefore it can cross the lines $\mathrm{tr}_2 = 2$ and $\mathrm{tr}_2 = -2$ infinitely many times as δ tends to zero depending on the values of $k_3 + k_4$ and $k_3 - k_4$. In particular, if $k_3 - k_4 < -2$ and $k_3 + k_4 > -2$, then $\mathrm{tr}_2 = -2$ for a sequence $\delta_i \to 0$, and we found intervals, for instance $(\delta_{2i}, \delta_{2i-1})$, with $\mathrm{tr}_2 < -2$, that is, hyperbolic–elliptic intervals. This will be the case for the collinear homographic solutions to be studied in section 7. A similar behaviour is found if $k_3 + k_4 > 2$ and $k_3 - k_4 < 2$.

The following remarks concern some trivial extensions of the main result to be used in section 6.

Remark 1. *Let us consider a system $\dot{\mathbf{x}} = A(t)\mathbf{x}$ where $A(t) = \begin{pmatrix} 0 & I_2 \\ \tilde{A}(t) & -cJ_2 \end{pmatrix}$, with $\tilde{A}(t)$ given in (4) and $c \neq 0$ some constant. This system can be reduced to (4) by scaling the variables x_3, x_4 as $X_3 = 2x_3/c$ and $X_4 = 2x_4/c$, and scaling the time by a factor $c/2$. Note that for the transformed system the parameters λ_1, λ_2 as well as the function $U(z)$ should be scaled by a factor $4/c^2$.*

Remark 2. *We consider now the system $\dot{\mathbf{x}} = A(t)\mathbf{x}$ where $A(t) = \begin{pmatrix} 0 & I_2 \\ \tilde{A}(t) & -2J_2 \end{pmatrix}$, $\tilde{A}(t) = g^{\alpha-2}\Lambda$, and Λ not diagonal but symmetric. Let P be a 2×2 orthogonal matrix such that $P^{-1}\Lambda P = diag(\lambda_1, \lambda_2)$, being λ_1 and λ_2 the eigenvalues of Λ. It can be chosen such that $det(P) = 1$. We define $\mathbf{z} = \mathcal{B}^{-1}\mathbf{x}$ where $\mathcal{B} = diag(P, P)$. Then the system for $\mathbf{z}$ reduces to (4).*

The paper is organised in the following way. First some preliminary results are given. In section 3 an auxiliary planar system is studied. In section 4 we give the proof of theorem 1 leaving the proof of the required lemmas to section 5. Section 6 is devoted to the study of homographic solutions for some homogeneous potentials, in particular for the Newtonian case. Finally, in section 7 we give some numerical results for the homographic solutions.

An announcement of some of the results in this paper can be found in [4].

2 Preliminary results

Using the reversibility of the system due to the fact that g is even, we can write the monodromy matrix $\Phi(T)$ in terms of the transition matrix in a half period.

Lemma 1. *The following equality holds*

$$\Phi(T) = \mathcal{F}^{-1}\Phi(T/2)^T\mathcal{F}\Phi(T/2) \tag{8}$$

where $\mathcal{F} = \begin{pmatrix} 0 & -2 & -1 & 0 \\ -2 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$.

Proof Let us denote by $\dot{\mathbf{y}} = B_H(t)\mathbf{y}$ the linear periodic Hamiltonian system with Hamiltonian function (5). As the fundamental matrix $\Phi_1(t)$ is symplectic we get

$$\Phi_1(T) = -J_4\Phi_1(-T/2)^T J_4\Phi_1(T/2).$$

We recall that $g(t)$ is an even function of t. Then $g(-t) = g(t)$ for all t and we get $LB_H(-t)L = -B_H(t)$, where $L = \mathrm{diag}(-1, 1, 1, -1)$. Therefore $\Phi_1(-T/2) = L\Phi_1(T/2)L$ and then (8) follows from (6) with $\mathcal{F} = M^T LJ_4M$. □

In order to prove the theorem we shall work, for $\delta > 0$, with a linear system without any singularity. Let be $q = g^{(2-\alpha)/2}$. The new system is obtained from (4) by introducing new variables $\mathbf{u} = S(t)\mathbf{x}$ where $S(t) = \mathrm{diag}(1, 1, q, q)$ and using time τ defined through $dt = q\ d\tau$. The period of $g(t;\delta)$ in the new time τ will be denoted by $\mathcal{T}(\delta)$ or simply $\mathcal{T}$. In order to simplify the notation, in what follows we shall write $q(t)$ instead of $q(t;\delta)$ if there is no confusion. We remark that for $\delta > 0$, $S(t)$ is non-singular for all t.

We write the new system as

$$\mathbf{u}' = B(\tau)\mathbf{u}, \qquad B(\tau) = q(\dot{S} + SA)S^{-1}. \tag{9}$$

Let $\Psi(\tau)$ the fundamental matrix of (9) such that $\Psi(0) = I_4$. Then $\Phi(t) = S^{-1}(t)\ \Psi(\tau(t))\ S(0)$. As $S(t)$ is $T-$periodic we get for the monodromy matrices $\Phi(T) = S^{-1}(0)\Psi(\mathcal{T})S(0)$ and so, $\Phi(T)$ and $\Psi(\mathcal{T})$ have the same eigenvalues. Furthermore, using lemma 1 it is easy to check that

$$\Psi(\mathcal{T}) = \frac{q_0}{q_a}G_0\Psi(\mathcal{T}/2)^T G_a\Psi(\mathcal{T}/2) \tag{10}$$

where $q_0 = q(0)$ and $q_a = q(T/2)$.

$$G_0 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & -2q_0 \\ 0 & 1 & -2q_0 & 0 \end{pmatrix}, \qquad G_a = \begin{pmatrix} 0 & -2q_a & -1 & 0 \\ -2q_a & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \tag{11}$$

Our purpose now is to obtain an expression for $\Psi(\mathcal{T}/2)$ which allows us to compute the dominant terms of the traces of $\Psi(\mathcal{T})$ for $\delta > 0$ small enough. To do this we shall introduce, in the next section, an artificial planar system for the functions q and $\dot{q}$ involved in $B(\tau)$. This allows us to split $\Psi(\mathcal{T}/2)$ in three matrices following some heteroclinic connections of that planar system.

3 The planar system

We define $Q = -(2-\alpha)q^{-\alpha/(2-\alpha)}\dot{g}$ where $q = g^{(2-\alpha)/2}$ as before. Then, using the time τ, $q(\tau), Q(\tau)$ is a solution of the following system

$$\begin{aligned} q' &= -\frac{1}{2}qQ, \\ Q' &= \frac{\alpha}{2(2-\alpha)}Q^2 + (2-\alpha)\hat{q}^{1-\alpha}U_z(\hat{q}), \end{aligned} \tag{12}$$

where $\hat{q} = q^{2/(2-\alpha)}$. From (3) we get that the system above has a first integral

$$E = \hat{q}^{\alpha}\left(\frac{Q^2}{2(2-\alpha)^2} + V(\hat{q})\right). \tag{13}$$

We note that (12) is well defined on $q = 0$. That system has two equilibria $P_{\pm}$, with $(q, Q) = (0, \pm Q_p)$ where $Q_p = (2-\alpha)\sqrt{-2\gamma}$. $P_{\pm}$ are saddle points lying on the level set $E = 0$. The eigenvalues of the linearised system at $P_{\pm}$ are $\mp Q_p/2,\ \pm Q_p\alpha/(2-\alpha)$. Moreover, we distinguish in the level set $E = 0$ two orbits

$$\begin{aligned} \gamma_0 &= \{(q,Q) \in \mathbb{R}^2 \,|\, q = 0, |Q| < Q_p\} \qquad \text{and} \\ \gamma_+ &= \{(q,Q) \in \mathbb{R}^2 \,|\, q > 0, \frac{Q^2}{2(2-\alpha)^2} + V(\hat{q}) = 0\}. \end{aligned}$$

In a neighbourhood of P_-, γ_+ is given by $Q = G(q) = -(2-\alpha)\sqrt{-2V(\hat{q})}$, so $\dfrac{dG}{dq}(0) = 0$.

On γ_0, (12) reduces to $Q' = \dfrac{\alpha}{2(2-\alpha)}Q^2 + \alpha(2-\alpha)\gamma$. By integrating this equation we get the following solution

$$q_{L_1}(\tau) \equiv 0, \qquad Q_{L_1}(\tau) = -Q_p \tanh\left(\frac{\alpha}{2(2-\alpha)}Q_p\tau\right).$$

The system (12) on γ_+ is

$$q' = -\frac{1}{2}qQ, \qquad Q' = (2-\alpha)\hat{q}V_z(\hat{q}). \tag{14}$$

We shall denote by $q_{L_2}(\tau), Q_{L_2}(\tau)$ the solution of (14) such that $Q_{L_2}(0) = 0$. Notice that $q_{L_2}(0) = z_a^{(2-\alpha)/2} := q_a$. The solution of (14) is also elementary and given by

$$q_{L_2}(\tau) = q_a/\cosh\left(\frac{2-\alpha}{2}q_a\tau\right), \qquad Q_{L_2}(\tau) = Q_p \tanh\left(\frac{2-\alpha}{2}q_a\tau\right).$$

Figure 1 left shows the phase portrait of (12) for $U(z) = z(-1 + z/2)$ which corresponds to the homographic case for the Newtonian potential as it will be proved in section 6. We note that in the general case we are interested in the solutions of (12) near the heteroclinic connection defined by γ_0, γ_+ and the equilibria $P_{\pm}$.

Given $\epsilon, \epsilon_i, i = 0, \ldots, 3$, small enough, we define the following sections (see figure 1 right)

$$\Sigma_0 = \{(q,Q) \,|\, 0 < q < \epsilon_0, Q = 0\}, \quad \Sigma_1 = \{(q,Q) \,|\, 0 < q < \epsilon_1, Q = -Q_p + \epsilon\},$$

$$\Sigma_2 = \{(q,Q) \,|\, q = \epsilon, |Q + Q_p| < \epsilon_2\}, \quad \Sigma_3 = \{(q,Q) \,|\, 0 < q_a - q < \epsilon_3, Q = 0\}.$$

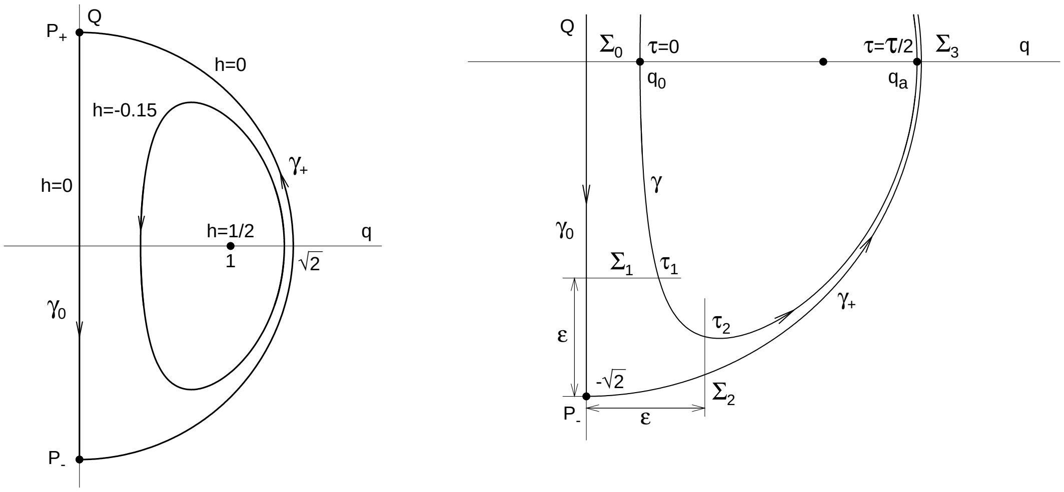

Figure 1: Left: Phase portrait of (12) for $\alpha = 1$ and $U(z) = z(-1+z/2)$. Right: An illustration of the sections used in the proof.

For a fixed value of $\epsilon > 0$, sufficiently small, we can take small enough ϵ_i for $i = 0, \ldots, 3$, such that the Poincaré maps $\mathcal{P}_1 : \Sigma_0 \mapsto \Sigma_1$, $\mathcal{P}_2 : \Sigma_1 \mapsto \Sigma_2$, and $\mathcal{P}_3 : \Sigma_2 \mapsto \Sigma_3$ be well defined.

We denote by $\tau_{L_1} > 0$ the time defined by $Q_{L_1}(\tau_{L_1}) = -Q_p + \epsilon$, and $\tau_{L_2} > 0$ such that $q_{L_2}(-\tau_{L_2}) = \epsilon$. Note that τ_{L_1} and τ_{L_2} are finite and independent of δ once ϵ is fixed.

For a fixed value of $\delta > 0$ small enough, we consider the solution of (12) with $E = -\delta$ such that $(q(0), Q(0)) \in \Sigma_0$. Using the hypothesis (A2) and (3) we get that $q_0 = q(0) = (\delta/|\gamma|)^{(2-\alpha)/(2\alpha)}\,(1+O(\delta^{s/\alpha}))$. Let τ_1 be the smallest positive time such that $(q(\tau_1), Q(\tau_1)) \in \Sigma_1$. In a similar way we define τ_2 such that $(q(\tau_2), Q(\tau_2)) \in \Sigma_2$. It is clear that τ_1 and τ_2 depend on δ. Moreover $\tau_1 \to \tau_{L_1}$ and $\mathcal{T}/2 - \tau_2 \to \tau_{L_2}$ when $\delta \to 0$.

Lemma 2. *Let $\epsilon > 0$ be a fixed small enough value. Then, for any sufficiently small $\delta > 0$ we have*

$$\frac{2}{Q_p + \epsilon} \ln\left(\frac{\epsilon}{q(\tau_1)}\right) \leq \tau_2 - \tau_1 \leq \frac{2}{Q_p - \epsilon} \ln\left(\frac{\epsilon}{q(\tau_1)}\right). \tag{15}$$

Proof Taking $\delta > 0$ small enough, for any $\tau \in [\tau_1, \tau_2]$ the following inequalities hold

$$-Q_p - \epsilon \leq Q(\tau) \leq -Q_p + \epsilon.$$

Multiplying the inequalities above by $-q(\tau)/2$ and using the first equation in (12) we get

$$\tfrac{1}{2}(Q_p - \epsilon)q(\tau) \leq q'(\tau) \leq \tfrac{1}{2}(Q_p + \epsilon)q(\tau). \tag{16}$$

The lemma follows by integration of these inequalities between τ_1 and τ_2. □

The following lemma will be used in the next sections.

Lemma 3. *Let $\epsilon > 0$ small enough. For any $\delta > 0$ sufficiently small we have*

(a) $\int_{\tau_1}^{\tau_2} q(\tau)d\tau \leq \dfrac{2\epsilon}{Q_p - \epsilon}$.

(b) $\int_{\tau_1}^{\tau_2} |Q(\tau) + Q_p| d\tau \leq c_0\epsilon$, *for some constant c_0.*

Proof Using (12) and (16) we get

$$\int_{\tau_1}^{\tau_2} q(\tau)d\tau \leq \frac{2}{Q_p-\epsilon}\int_{\tau_1}^{\tau_2} q'(\tau)d\tau \leq \frac{2}{Q_p-\epsilon}q(\tau_2),$$

and using that $q(\tau_2)=\epsilon$ the inequality (a) follows.

To prove (b), first we introduce $\xi = Q+Q_p$ in order to translate the equilibrium point P_- to the origin $\mathbf{O}$ in the plane (q,ξ). Let $W^{u,+}$ the branch of the unstable invariant manifold of $\mathbf{O}$ with $q>0$. $W^{u,+}$ is given by the graph of the function $F(q) = Q_p - (2-\alpha)\sqrt{-2V(\hat{q})}$. We note that the hypothesis (A2) implies that $F(q) = Q_p - Q_p(1+\hat{q}^s V_1(\hat{q})/\gamma)^{1/2}$. Therefore if $0<q\leq\epsilon$, ϵ small enough,

$$|F(q)| \leq kq^{2s/(2-\alpha)} \leq kq \tag{17}$$

for some constant $k>0$, and using that $s>(2-\alpha)/2$. We define $y=\xi-F(q)$ for $0<q\leq\epsilon$. Then $W^{u,+}$ is on the axis $y=0$ in the plane (q,y) and our region of interest is a neighbourhood of $\mathbf{O}$ with $y\geq 0$. We get the following equation for y

$$y' = -\frac{\alpha}{2-\alpha}Q_p y(1+O_1)$$

where O_1 contains terms of order one in ϵ. Therefore there exists some constant $c_2>0$ such that

$$-\frac{\alpha}{2-\alpha}Q_p y(1+c_2\epsilon) \leq y' \leq -\frac{\alpha}{2-\alpha}Q_p y(1-c_2\epsilon).$$

Then we get

$$\int_{\tau_1}^{\tau_2} y(\tau)d\tau \leq \frac{2-\alpha}{\alpha Q_p(1-c_2\epsilon)}(y(\tau_1)-y(\tau_2)) \leq \frac{2-\alpha}{\alpha Q_p(1-c_2\epsilon)}y(\tau_1), \tag{18}$$

using that $y(\tau_2)\geq 0$. We recall that $y=\xi-F(q)=Q+Q_p-F(q)$. Then, $y(\tau_1)\leq |Q(\tau_1)+Q_p|+|F(q(\tau_1))|\leq \epsilon + kq(\tau_1)\leq (1+k)\epsilon$. Moreover using the part (a) of the lemma and (17)

$$\int_{\tau_1}^{\tau_2} |F(q(\tau))|d\tau \leq \frac{2k}{Q_p-\epsilon}\epsilon.$$

Therefore, from (18) and the inequality above we get

$$\int_{\tau_1}^{\tau_2} |Q(\tau)+Q_p|d\tau \leq \int_{\tau_1}^{\tau_2} (y(\tau)+|F(q(\tau))|)d\tau \leq c_0\epsilon,$$

for some constant $c_0>0$. □

4 Proof of theorem 1

The main idea is to split $\Psi(\mathcal{T}/2)$ in three matrices each one obtained from (9) in a neighbourhood of γ_0, P_- and γ_+ respectively. Then using (10) we shall obtain a suitable expression for $\Psi(\mathcal{T})$ in order to compute the stability parameters.

For a fixed value of $\delta>0$ small enough, let $(q(\tau),Q(\tau))$ be the solution of (12) for $E=-\delta$ such that $q(0)=q_0$, $Q(0)=0$, being q_0 the minimum of $q(\tau)$. Then the matrix $B(\tau)$ in (9) can be written as

$$B(\tau) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \lambda_1 & 0 & -Q(\tau)/2 & -2q(\tau) \\ 0 & \lambda_2 & 2q(\tau) & -Q(\tau)/2 \end{pmatrix} =: B_a(q(\tau),Q(\tau)).$$

We write

$$\Psi(\mathcal{T}/2) = \tilde{\Psi}(\mathcal{T}/2, \tau_2)\tilde{\Psi}(\tau_2, \tau_1)\tilde{\Psi}(\tau_1, 0), \tag{19}$$

where $\tilde{\Psi}(\tau_b, \tau_a)$ stands for the transition matrix of (9) from τ_a to τ_b. We note that τ_1 and τ_2 (as defined in section 3) and also $\mathcal{T}/2$ depend on δ.

Our purpose is to approximate the transition matrices involved in Ψ by simpler ones. First, we shall approximate $\tilde{\Psi}(\tau_1, 0)$ and $\tilde{\Psi}(\mathcal{T}/2, \tau_2)$ in (19) by the transition matrices for the system (9) along γ_0 and γ_+ respectively.

We define

$$B_{L_1}(\tau) := B_a(0, Q_{L_1}(\tau)), \qquad B_{L_2}(\tau) := B_a(q_{L_2}(\tau), Q_{L_2}(\tau)), \qquad B_p := B(0, -Q_p).$$

We note that these matrices do not depend on δ.

Let $Z_1(\tau)$ be the fundamental matrix of

$$\mathbf{u}' = B_{L_1}(\tau)\mathbf{u} \tag{20}$$

such that $Z_1(0) = I_4$.

We denote by $Z_2(\tau)$ the fundamental matrix of

$$\mathbf{u}' = B_{L_2}(\tau)\mathbf{u} \tag{21}$$

such that $Z_2(-\tau_{L_2}) = I_4$.

For a fixed value of $\epsilon > 0$ small enough, $\mathcal{P}_1$ and $\mathcal{P}_3$ are diffeomorphisms. So, we can write

$$\tilde{\Psi}(\tau_1, 0) = Z_1(\tau_{L_1}) + \Delta_1, \qquad \tilde{\Psi}(\mathcal{T}/2, \tau_{L_2}) = Z_2(0) + \Delta_2, \tag{22}$$

for some matrices Δ_1, Δ_2 with $\|\Delta_1\| = O(q_0) = O(\delta^{\frac{2-\alpha}{2\alpha}})$, $\|\Delta_2\| = O(q_{L_2}(0) - q_a) = O(\delta)$.

Now we consider the system (9) in a neighbourhood of the equilibrium point P_-. We write $B(\tau) = B_p + B_1(\tau)$ where

$$B_1(\tau) = \begin{pmatrix} 0 & 0 \\ 0 & B_{11} \end{pmatrix}, \qquad B_{11} = \begin{pmatrix} -\frac{1}{2}(Q + Q_p) & -2q \\ 2q & -\frac{1}{2}(Q + Q_p) \end{pmatrix}. \tag{23}$$

The eigenvalues of B_p are

$$\rho_1^{\pm} = \frac{Q_p}{4}(1 \pm \beta_1), \qquad \rho_2^{\pm} = \frac{Q_p}{4}(1 \pm \beta_2),$$

where $\beta_i = \sqrt{1 - \dfrac{\lambda_i}{\hat{\lambda}}}$, $i = 1, 2$, were introduced in the statement of theorem 1. We recall that if the hypotheses of the theorem are satisfied then $\beta_i \neq 0$, $i = 1, 2$.

Let be $P = \begin{pmatrix} I_2 & I_2 \\ P_3 & P_4 \end{pmatrix}$, with $P_3 = \mathrm{diag}(\rho_1^+, \rho_2^+)$, $P_4 = \mathrm{diag}(\rho_1^-, \rho_2^-)$. Then P is nonsingular and

$$P^{-1} B_p P = \frac{Q_p}{4} I_4 + \bar{D}$$

where $\bar{D} = (Q_p/4)\mathrm{diag}(\beta_1, \beta_2, -\beta_1, -\beta_2)$. We introduce a new variable

$$\mathbf{w} = \exp\left(-\frac{Q_p}{4}(\tau - \tau_1)\right) P^{-1}\mathbf{u}$$

and we get the following system for $\mathbf{w}$

$$\mathbf{w}' = (\bar{D} + P^{-1}B_1(\tau)P)\mathbf{w}. \tag{24}$$

Let $W(\tau)$ be the fundamental matrix of (24) such that $W(\tau_1) = I_4$. Then

$$\tilde{\Psi}(\tau,\tau_1) = \exp\left(\frac{Q_p}{4}(\tau-\tau_1)\right) PW(\tau)P^{-1}. \tag{25}$$

From (19) we obtain

$$\Psi(\mathcal{T}/2) = \sigma\tilde{\Psi}(\mathcal{T}/2,\tau_2)PW(\tau_2)P^{-1}\tilde{\Psi}(\tau_1,0) \tag{26}$$

where $\sigma = \exp\left((Q_p/4)(\tau_2-\tau_1)\right)$.

Lemma 4. *Let $\epsilon > 0$ be small enough. If $\delta > 0$ is sufficiently small we have for all $\tau \in [\tau_1,\tau_2]$*

$$W(\tau) = (I_4 + \Delta(\tau))\mathcal{D}(\tau)(I_4 + R),$$

where $\mathcal{D}(\tau) = diag(e^{\nu_1(\tau-\tau_1)}, e^{\nu_2(\tau-\tau_1)}, e^{-\nu_1(\tau-\tau_1)}, e^{-\nu_2(\tau-\tau_1)})$, $\nu_i = \dfrac{Q_p}{4}\beta_i$, $i = 1,2$, $\Delta(\tau)$ is a matrix such that $\|\Delta(\tau)\| \le c_1\varepsilon$ for any $\tau \in [\tau_1,\tau_2]$ and R is a constant matrix such that $\|R\| \le c_2\varepsilon$, for some constants c_1, c_2, uniformly in δ.

The proof of this lemma is given in section 5.

After lemma 4 and using (26), we have that

$$\Psi\left(\mathcal{T}/2\right) \quad = \quad \sigma\left[Z_2(0)P\mathcal{D}(\tau_2)P^{-1}Z_1(\tau_{L_1})\right](I_4 + \Delta_3),$$

where $\|\Delta_3\| = O(\varepsilon, q_0, \delta)$. We remark that we are assuming that $\|Z_2(0)P\mathcal{D}(\tau_2)P^{-1}Z_1(\tau_{L_1})\|$ has the same order of magnitude as the product of norms. We shall see that this is the case if the coefficient d_g, to be introduced later in this section, satisfies $d_g \ne 0$. That is, if the non degeneracy conditions are satisfied. Using (10) we get

$$\Psi(\mathcal{T}) = \frac{q_0}{q_a}\sigma^2\mathcal{M}(I_4 + \mathcal{O}), \qquad \mathcal{M} = A_1\mathcal{D}A_2\mathcal{D}A_3, \tag{27}$$

where $A_1 = G_{00}A_3^T$, $A_2 = P^TZ_2(0)^TG_MZ_2(0)P$, $A_3 = P^{-1}Z_1(\tau_{L_1})$, $\mathcal{D} = \mathcal{D}(\tau_2)$, $\mathcal{O}$ stands for a matrix which contains terms of order ϵ, q_0 and δ, $G_{00} = \begin{pmatrix} 0 & \mathcal{G} \\ \mathcal{G} & 0 \end{pmatrix}$, $\mathcal{G} =$ diag$(-1,1)$, and $G_M = \begin{pmatrix} -2q_{L_2}(0)\mathcal{K} & \mathcal{G} \\ \mathcal{G} & 0 \end{pmatrix}$, $\mathcal{K} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. We note that in (27), A_1, A_2 and A_3 are independent of δ.

Let us denote by $p(x) = x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$ the characteristic polynomial of $\Psi(\mathcal{T})$. Then $a_3 = -(\mathrm{tr}_1 + \mathrm{tr}_2)$, $a_2 = 2 + \mathrm{tr}_1\mathrm{tr}_2$, $a_1 = a_3$, $a_0 = 1$ and the stability parameters can be obtained from a_2 and a_3. To estimate the dominant terms of these coefficients we shall use the matrix $\mathcal{M}$. Let $q(x) = x^4 + b_3x^3 + b_2x^2 + b_1x + b_0$ be the characteristic polynomial of $\mathcal{M}$. Then the stability parameters, up to order 1 in ϵ, are the solutions of the quadratic equation

$$k^2x^2 + kb_3x + b_2 - 2k^2 = 0, \quad \text{where} \quad k = \frac{q_a}{q_0\sigma^2}. \tag{28}$$

Lemma 5. *Let $\epsilon > 0$ be small enough. Assume that λ_1 and λ_2 satisfy the hypotheses of theorem 1. Then*
(a) There exist some constants d_i, $i = 1, \dots 5$ such that

$$-b_3 = d_1\sigma^{2\beta_1} + d_2\sigma^{2\beta_2} + d_3\sigma^{-2\beta_1} + d_4\sigma^{-2\beta_2} + d_5. \tag{29}$$

The coefficient d_1 is the product of two constants, $d_1 = d_n d_g$ with d_n depending on $\lambda_1, \lambda_2, \alpha$ and γ but not on the function V_1 defined in section 1. d_g depends also on V_1. If λ_1 and λ_2 are different from zero, then $d_n \neq 0$.
(b) The coefficient b_2 does not contain terms in $\sigma^{\pm 4\beta_1}$ nor $\sigma^{\pm 4\beta_2}$, that is the dominant terms are

$$b_2 = e_1\sigma^{2\beta_1} + e_2\sigma^{2\beta_2} + e_3\sigma^{2(\beta_1+\beta_2)} + e_4\sigma^{2(\beta_1-\beta_2)} + \dots \tag{30}$$

for some constants $e_1, e_2, e_3, \dots$ The coefficient e_3 is the product of two constants $e_3 = e_n e_g$ where e_n depends on $\lambda_1, \lambda_2, \alpha$ and γ but not on the function V_1 defined in section 1. e_g depends also on V_1. If λ_1, λ_2 are different from zero then $e_n \neq 0$.

Moreover, if $\lambda_2 < \hat{\lambda}$ (see theorem 1), then $d_4 = \bar{d}_2$, and $e_4 = \bar{e}_3$ where the bar stands for the complex conjugate.

The proof of this lemma will be given in the section 5.

Now, the stability parameters are obtained by solving the quadratic equation (28). We recall that if λ_1, λ_2 satisfy the hypotheses of theorem 1 then either $\beta_1 > \beta_2 > 0$ or $\beta_1 > 0$ and $\beta_2 = \gamma_2\mathrm{i}$, $\gamma_2 \in \mathbb{R}$. Let us assume in the next that $d_g \neq 0$, or equivalently $d_1 \neq 0$. So, in any case, the dominant term in $-b_3$ is $d_1\sigma^{2\beta_1}$. For the discriminant of (28) we have $k^2 d_1^2 \sigma^{4\beta_1}(1 + \mathcal{A}_1)$ where using the lemma 5 the dominant terms in $\mathcal{A}_1$ are

$$\begin{aligned}
&2\left(\frac{d_2 d_1 - 2e_3}{d_1^2}\right)\sigma^{-2(\beta_1-\beta_2)}, && \text{if} \quad \beta_2 > 0,\\
&2\left[\left(\frac{d_5 d_1 - 2e_1}{d_1^2}\right) + \left(\frac{d_2 d_1 - 2e_3}{d_1^2}\right)\sigma^{2\beta_2} + \left(\frac{d_4 d_1 - 2e_4}{d_1^2}\right)\sigma^{-2\beta_2}\right]\sigma^{-2\beta_1} && \text{if} \quad \beta_2 = \gamma_2\mathrm{i}.
\end{aligned}$$

Then the stability parameters are obtained as

$$\mathrm{tr}_1 = \frac{q_0\sigma^2}{q_a}\left(d_1\sigma^{2\beta_1} + \dots\right), \tag{31}$$

and

$$\begin{aligned}
\mathrm{tr}_2 &= \frac{q_0\sigma^2}{q_a}\left(\frac{e_3}{d_1}\sigma^{2\beta_2} + \dots\right), && \text{if} \quad \beta_2 > 0, && (32)\\
\mathrm{tr}_2 &= \frac{q_0\sigma^2}{q_a}\left(\frac{e_1}{d_1} + 2\mathrm{Re}\left(\frac{e_3}{d_1}\sigma^{2\beta_2}\right) + \dots\right), && \text{if} \quad \beta_2 = \gamma_2 i, \quad \gamma_2 \in \mathbb{R}. && (33)
\end{aligned}$$

Using lemma 2 and taking logarithms in (31) and (32), the asymptotic formulae given in theorem 1 are obtained.

Remark 3. *We recall that matrices A_1, A_2 and A_3 in (27) do not depend on δ, so their norms are finite. Therefore $\|A_1\|\|\mathcal{D}\|\|A_2\|\|\mathcal{D}\|\|A_3\|$ depends mainly on $\|\mathcal{D}\|^2$ for $\delta > 0$ small enough. Furthermore $\mathcal{D} = \mathcal{D}(\tau_2)$ is a diagonal matrix and so, $\|\mathcal{D}\|^2$ is of the order or $\sigma^{2\beta_1}$. However, if $d_1 \neq 0$ from (31) we have that tr_1 is of order $\sigma^{2\beta_1}$. This gives an estimation of the spectral radius of $\mathcal{M}$. Using that, for any natural norm, $\|\mathcal{M}\|$ is bounded from below by the spectral radius we conclude that it is of the same order of magnitude of the product of norms and then (27) holds.*

Remark 4. *The conditions $d_g \neq 0$ and $e_g \neq 0$ have a simple geometrical interpretation. Let us consider the path γ_+ and its prolongation up to the vicinity of the equilibrium point P_+. The strongest unstable direction near P_-, associated to the eigenvalue ρ_1^+, can be sent, by the variational flow, to the strongest stable direction near P_+. This is a non-generic situation and then $d_g = 0$. If $\beta_2 \in \mathbb{R}$ a similar behaviour can occur for the weakest unstable direction, and then $e_g = 0$.*

5 Proofs of lemmas

In this section we prove lemmas 4 and 5. We start with an auxiliary result.

Lemma 6. *Let us consider the system*

$$\mathbf{x}' = D\mathbf{x} + C(t)\mathbf{x}, \tag{34}$$

where D is a $n \times n$ diagonal matrix and $C(t)$ is a continuous matrix in $t \in [0, \hat{t}]$. Assume that there exists some constant $\hat{\epsilon} < 1/4$ such that

$$\int_0^{\hat{t}} \|C(s)\| ds < \hat{\epsilon}. \tag{35}$$

Let λ be an eigenvalue of D and $\mathbf{v}$ an eigenvector corresponding to λ. Then, there exists a solution, $\varphi(t)$, of (34) such that

$$\|e^{-\lambda t}\varphi(t) - \mathbf{v}\| \leq \|\mathbf{v}\| \frac{3\hat{\epsilon}}{1 - 3\hat{\epsilon}},$$

for all $t \in [0, \hat{t}]$.

Proof It is not restrictive to assume that $D = \mathrm{diag}(D_1, D_2, D_3)$ where D_i, $i = 1, 2, 3$, are diagonal matrices such that the eigenvalues of D_1 (D_2) have real part less (greater) than the real part of λ and D_3 has eigenvalues with real part equal to the real part of λ. We put $e^{tD} = X_1(t) + X_2(t) + X_3(t)$ where $X_1(t) = \mathrm{diag}(e^{tD_1}, 0, 0)$, $X_2(t) = \mathrm{diag}(0, e^{tD_2}, 0)$ and $X_3(t) = \mathrm{diag}(0, 0, e^{tD_3})$. Then there exists a positive constant $a > 0$ such that $\|e^{-t\lambda}X_1(t)\| \leq e^{-ta}$ for all $t \geq 0$, $\|e^{-t\lambda}X_2(t)\| \leq e^{ta}$ for all $t \leq 0$ and $\|e^{-t\lambda}X_3(t)\| = 1$ for all t.

It is easy to check that the solution, $\varphi(t)$, of the integral equation

$$\varphi(t) \quad = \quad e^{t\lambda}\mathbf{v} + \int_0^t X_1(t-s)C(s)\varphi(s)ds - \sum_{j=2}^{3} \int_t^{\hat{t}} X_j(t-s)C(s)\varphi(s)ds \tag{36}$$

is a solution of (34). To obtain the solution of (36) we use an iterative scheme with $\varphi_0(t) \equiv 0$. We define for $m \geq 1$

$$\varphi_m(t) \quad = \quad e^{t\lambda}\mathbf{v} + \int_0^t X_1(t-s)C(s)\varphi_{m-1}(s)ds - \sum_{j=2}^{3} \int_t^{\hat{t}} X_j(t-s)C(s)\varphi_{m-1}(s)ds$$

Then for all $t \in [0, \hat{t}]$ the following inequalities hold

$$\|\varphi_m(t) - \varphi_{m-1}(t)\| \quad \leq \quad |e^{t\lambda}| \, \|\mathbf{v}\| \, (3\hat{\epsilon})^{m-1}, \tag{37}$$

$$\|e^{-t\lambda}\varphi_m(t) - \mathbf{v}\| \quad \leq \quad \|\mathbf{v}\| \sum_{k=1}^{m-1} (3\hat{\epsilon})^k. \tag{38}$$

The inequalities above are proved by induction. For (37) we note that

$$\begin{aligned}\varphi_m(t)-\varphi_{m-1}(t) &= e^{t\lambda}\left\{\int_0^t e^{-\lambda(t-s)}X_1(t-s)C(s)e^{-s\lambda}(\varphi_{m-1}(s)-\varphi_{m-2}(s))ds\right.\\ &\qquad \left.-\sum_{j=2}^{3}\int_t^{\hat t} e^{-\lambda(t-s)}X_j(t-s)C(s)e^{-s\lambda}(\varphi_{m-1}(s)-\varphi_{m-2}(s))ds\right\}.\end{aligned}$$

Then using the bounds for $\|e^{-\lambda t}X_j(t)\|$, $j=1,2,3$ we obtain

$$\begin{aligned}&\|\varphi_m(t)-\varphi_{m-1}(t)\|\le |e^{t\lambda}|\left\{\int_0^t e^{-a(t-s)}\|C(s)\|\|\mathbf{v}\|(3\hat\epsilon)^{m-2}ds+\right.\\ &+\left.\int_t^{\hat t} e^{a(t-s)}\|C(s)\|\|\mathbf{v}\|(3\hat\epsilon)^{m-2}ds+\int_t^{\hat t}\|C(s)\|\|\mathbf{v}\|(3\hat\epsilon)^{m-2}ds\right\}\le |e^{t\lambda}|\|\mathbf{v}\|(3\hat\epsilon)^{m-1},\end{aligned}$$

and inequality (37) holds. To prove (38) first we note that

$$e^{-t\lambda}\varphi_2(t)-\mathbf{v}=\int_0^t e^{-\lambda(t-s)}X_1(t-s)C(s)\mathbf{v}ds-\sum_{j=2}^{3}\int_t^{\hat t}e^{-\lambda(t-s)}X_j(t-s)C(s)\mathbf{v}ds.$$

Then

$$\|e^{-t\lambda}\varphi_2(t)-\mathbf{v}\|\le\int_0^t e^{-a(t-s)}\|C(s)\|\|\mathbf{v}\|ds+\int_t^{\hat t}e^{a(t-s)}\|C(s)\|\|\mathbf{v}\|ds+\int_t^{\hat t}\|C(s)\|\|\mathbf{v}\|ds\le 3\hat\epsilon\|\mathbf{v}\|.$$

We introduce the following notation $\Delta_m(t):=e^{-t\lambda}\varphi_m(t)-\mathbf{v}$. Then for the general step we get

$$\Delta_m(t)=\int_0^t e^{-\lambda(t-s)}X_1(t-s)C(s)\Delta_{m-1}(s)ds-\sum_{j=2}^{3}\int_t^{\hat t}e^{-\lambda(t-s)}X_j(t-s)C(s)\Delta_{m-1}(s)ds+\Delta_2(t).$$

Therefore

$$\begin{aligned}\|\Delta_m(t)\| &\le \int_0^t e^{-a(t-s)}\|C(s)\|\|\Delta_{m-1}(s)\|ds+\int_t^{\hat t}e^{a(t-s)}\|C(s)\|\|\Delta_{m-1}(s)\|ds\\ &+\int_t^{\hat t}\|C(s)\|\|\Delta_{m-1}(s)\|ds+\|\Delta_2(t)\|\\ &\le\|\mathbf{v}\|\left[\sum_{k=1}^{m-2}(3\hat\epsilon)^k\left(3\int_0^{\hat t}\|C(s)\|ds\right)+3\hat\epsilon\right]\le\|\mathbf{v}\|\sum_{k=1}^{m-1}(3\hat\epsilon)^k\end{aligned}$$

and (38) follows. □

Proof of lemma 4

Now we apply the lemma 6 to the system (24). For a fixed value of $\epsilon>0$ we consider $q_0>0$ small enough and $\tau\in[\tau_1,\tau_2]$ where we recall that τ_1,τ_2 depend on q_0. After a translation of time defined by $s=\tau-\tau_1$ we can restrict to the system (24) for $s\in[0,\hat s]$ with $\hat s=\hat s(q_0)=\tau_2-\tau_1$. In order to apply lemma 6 we consider the system (34) with $D=\bar D=\frac{Q_p}{4}\mathrm{diag}(\beta_1,\beta_2,-\beta_1,-\beta_2)$ and $C(s)=P^{-1}B_1(s+\tau_1)P$ with B_1 defined in (23).

We note that $\|C(s)\|\le\|P\|\,\|P^{-1}\|\,\|B_{11}(s+\tau_1)\|$. After lemma 3, for any $\delta>0$ small enough we have (using $\|\ \|_\infty$)

$$\int_0^{\hat s}\|C(s)\|ds\le\|P\|\,\|P^{-1}\|\left(\frac{4\varepsilon}{Q_p-\varepsilon}+\frac{c_0\varepsilon}{2}\right)=c_1\varepsilon$$

being c_1 a constant. Then, (35) is satisfied with $\hat{\varepsilon} = c_1\varepsilon$ independently of δ, and $\hat{\varepsilon} < 1/4$ if ε is small enough. Then there exist solutions of (24), $\varphi_i(\tau)$, $i = 1, \ldots, 4$ such that

$$\|e^{-\nu_i(\tau-\tau_1)}\varphi_i(\tau) - \mathbf{e}_i\| \leq 3\frac{\hat{\epsilon}}{1-3\hat{\epsilon}}, \qquad i = 1, \ldots, 4, \tag{39}$$

for all $\tau \in [\tau_1, \tau_2]$. In (39) $\mathbf{e}_i$, $i = 1, \ldots, 4$, denotes the canonical basis and $\nu_3 = -\nu_1$, $\nu_4 = -\nu_2$.

Let $Y(\tau)$ be the matrix defined by $\varphi_1, \varphi_2, \varphi_3$ and φ_4 as column vectors. We define $\Delta(\tau) := Y(\tau)\mathcal{D}^{-1}(\tau) - I_4$. Then using (39) it is easy to check that $\|\Delta(\tau)\| \leq O(\varepsilon)$ for all $\tau \in [\tau_1, \tau_2]$ if ε is small enough. Moreover, $Y(\tau)$ is a fundamental matrix of system (24). Then, $W(\tau) = Y(\tau)Y(\tau_1)^{-1} = (I_4 + \Delta(\tau))\mathcal{D}(\tau)(I_4 + \Delta(\tau_1))^{-1}$. We define $R := (I_4 + \Delta(\tau_1))^{-1} - I_4$. Using standard results for natural matrix norms we get $\|R\| < 6c_1\varepsilon$. This ends the proof of the lemma. $\square$

Proof of lemma 5

To prove lemma 5 we need some information about the matrices involved in $\mathcal{M}$. First we note that (20) splits in two uncoupled systems, one for u_1, u_3 and the second for u_2, u_4. So, $Z_1(\tau_{L_1})$ is a 4×4 block diagonal matrix, that is,

$$Z_1(\tau_{L_1}) \;=\; \begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix}, \tag{40}$$

where C_i, $i = 1, \ldots, 4$ are 2×2 diagonal matrices. We write $C_j = \mathrm{diag}(c_{j1}, c_{j2})$, $j = 1, \ldots, 4$. Then, A_1 and A_3 as defined in (27) are also 4×4 block diagonal matrices. So, we write

$$\begin{aligned} A_1 \;&=\; \begin{pmatrix} H_1 & H_2 \\ H_3 & H_4 \end{pmatrix}, \quad \text{with} \quad H_j = \mathrm{diag}(h_{j1}, h_{j2}), \quad j = 1, \ldots, 4, \\ A_3 \;&=\; \begin{pmatrix} E_1 & E_2 \\ E_3 & E_4 \end{pmatrix}, \quad \text{with} \quad E_j = \mathrm{diag}(e_{j1}, e_{j2}), \quad j = 1, \ldots, 4. \end{aligned}$$

Using that $A_1 = G_{00}A_3^T$ we get the following relations

$$\begin{array}{llll} h_{11} = -e_{21}, & h_{12} = e_{22}, & h_{21} = -e_{41}, & h_{22} = e_{42}, \\ h_{31} = -e_{11}, & h_{32} = e_{12}, & h_{41} = -e_{31}, & h_{42} = e_{32}. \end{array} \tag{41}$$

We denote $A_2 = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}$, for some 2×2 matrices X_i, $i = 1, \ldots, 4$. We write also $A_2 = (x_{ij})$.

It is easy to check that $\mathcal{M} = \tilde{D}_1^{-1}\tilde{\mathcal{M}}\tilde{D}_1$ where $\tilde{D}_1 = \mathrm{diag}(\sigma^{\beta_1}, \sigma^{\beta_2}, \sigma^{\beta_1}, \sigma^{\beta_2})$ and

$$\tilde{\mathcal{M}} = \tilde{D}_1^2\tilde{\mathcal{M}}_1 + \tilde{\mathcal{M}}_2 + \tilde{D}_1^2\tilde{\mathcal{M}}_3(\tilde{D}_1^{-1})^2 + \tilde{\mathcal{M}}_4(\tilde{D}_1^{-1})^2, \tag{42}$$

for some matrices $\tilde{\mathcal{M}}_i$, $i = 1, \ldots, 4$ which depend only on A_1, A_2 and A_3 and, hence, they do not depend on δ. So, we can reduce to consider the characteristic polynomial of $\tilde{\mathcal{M}}$. For the elements of these matrices we shall use the following notation $\tilde{\mathcal{M}}_1 = (u_{ij})$, $\tilde{\mathcal{M}}_2 = (v_{ij})$, $\tilde{\mathcal{M}}_3 = (p_{ij})$ and $\tilde{\mathcal{M}}_4 = (w_{ij})$. We get from (42)

$$\begin{aligned} \mathrm{trace}(\tilde{\mathcal{M}}) \;=\;& (u_{11} + u_{33})\sigma^{2\beta_1} + (u_{22} + u_{44})\sigma^{2\beta_2} + (w_{11} + w_{33})\sigma^{-2\beta_1} \\ &+ (w_{22} + w_{44})\sigma^{-2\beta_2} + \mathrm{trace}(\tilde{\mathcal{M}}_2) + \mathrm{trace}(\tilde{\mathcal{M}}_3), \end{aligned}$$

where $\mathrm{trace}(\tilde{\mathcal{M}}_2)$ and $\mathrm{trace}(\tilde{\mathcal{M}}_3)$ do not depend on σ. Then (29) follows by taking into account that $b_3 = -\mathrm{trace}(\tilde{\mathcal{M}})$. Notice that $d_1 = u_{11} + u_{33}$.

For the coefficient b_2 one has $b_2 = (u_{11}u_{33} - u_{13}u_{31})\sigma^{4\beta_1} + (u_{22}u_{44} - u_{24}u_{42})\sigma^{4\beta_2} + \ldots$. However, a simple computation shows that $u_{11} = x_{11}h_{11}e_{11}$, $u_{33} = x_{11}h_{31}e_{21}$, $u_{13} = x_{11}h_{11}e_{21}$ and $u_{31} = x_{11}h_{31}e_{11}$. Then using (41), $u_{11}u_{33} - u_{13}u_{31} = 0$ and b_2 does not contain terms in $\sigma^{4\beta_1}$. In a similar way it can be checked that the coefficient of $\sigma^{4\beta_2}$ in b_2 is $u_{22}u_{44} - u_{24}u_{42} = 0$. Therefore the dominant terms in b_2 are the ones given in (30).

Assume that β_2 is imaginary. As far as the characteristic polynomial of $\mathcal{M}$ has real coefficients we get that d_4 and e_4 are the conjugates of d_2 and e_3 respectively.

Furthermore, $d_1 = u_{11} + u_{33} = -2x_{11}e_{11}e_{21}$ and $e_3 = (u_{11} + u_{33})(u_{22} + u_{44}) - (u_{23}u_{32} + u_{34}u_{43} + u_{21}u_{12} + u_{14}u_{41}) = -4e_{11}e_{12}e_{21}e_{22}\det(X_1)$. Therefore we can take $d_g = x_{11}$, $d_n = -2e_{11}e_{21}$, $e_g = \det(X_1)$ and $e_n = -4e_{11}e_{12}e_{21}e_{22}$. So, d_n and e_n are independent of the function V_1 whereas d_g and e_g depend on V_1.

To finish the proof of the lemma we only need to show that $e_{11}, e_{12}, e_{21}, e_{22} \neq 0$. We recall that $A_3 = P^{-1}Z_1(\tau_{L_1})$ where $Z_1(\tau)$ is the fundamental matrix of (20) such that $Z_1(0) = I_4$. This system can be written as two uncoupled systems of the following type

$$v_1' = v_2, \qquad v_2' = \lambda v_1 - \frac{Q_{L_1}(\tau)}{2} v_2, \tag{43}$$

where $\lambda = \lambda_1$ for the subsystem corresponding to u_1, u_3 and $\lambda = \lambda_2$ for u_2, u_4.

Using $A_3 = P^{-1}Z_1(\tau_{L_1})$ a simple computation shows that

$$\begin{aligned} e_{11} &= \frac{1}{\rho_1^+ - \rho_1^-}(c_{31} - \rho_1^- c_{11}), \qquad & e_{21} &= \frac{1}{\rho_1^+ - \rho_1^-}(c_{41} - \rho_1^- c_{21}), \\ e_{12} &= \frac{1}{\rho_2^+ - \rho_2^-}(c_{32} - \rho_2^- c_{12}), \qquad & e_{22} &= \frac{1}{\rho_2^+ - \rho_2^-}(c_{42} - \rho_2^- c_{22}). \end{aligned} \tag{44}$$

We note that $\begin{pmatrix} c_{11} & c_{21} \\ c_{31} & c_{41} \end{pmatrix}$ and $\begin{pmatrix} c_{12} & c_{22} \\ c_{32} & c_{42} \end{pmatrix}$ are the fundamental matrices of (43) evaluated at $\tau = \tau_{L_1}$ for $\lambda = \lambda_1$ and $\lambda = \lambda_2$, respectively.

Lemma 7. *Assume $\lambda \neq 0$. Let $\mathbf{v}(\tau) = (v_1(\tau), v_2(\tau))^T$ be one of the solutions of (43) with initial conditions $\mathbf{v}(0) = (1,0)^T$ or $\mathbf{v}(0) = (0,1)^T$. Then, for any $\tau > 0$ sufficiently large*

$$v_2(\tau) - \rho^- v_1(\tau) \neq 0, \tag{45}$$

where $\rho^- = \dfrac{Q_p}{4}(1 - \beta)$, $\beta = \sqrt{1 - \dfrac{\lambda}{\hat{\lambda}}}$.

Proof In the case $\lambda < \gamma(2-\alpha)^2/8$, ρ^- is a complex number and then (45) is trivially obtained as far as we consider real solutions of the real system (43).

We assume $\lambda > \hat{\lambda}$. We introduce polar coordinates in (43) as $v_1 = r\cos\phi$, $v_2 = r\sin\phi$. Then

$$\begin{aligned} r' &= r[(1+\lambda)\cos\phi\sin\phi - \frac{Q_{L_1}(\tau)}{2}\sin^2\phi], \\ \phi' &= \lambda\cos^2\phi - \sin^2\phi - \frac{Q_{L_1}(\tau)}{2}\sin\phi\cos\phi. \end{aligned} \tag{46}$$

For the solutions with $\mathbf{v}(0) = (1,0)^T$ and $\mathbf{v}(0) = (0,1)^T$, $r(\tau) \neq 0$ for all τ so, we have to prove that $\sin\phi(\tau) - \rho^- \cos\phi(\tau) \neq 0$, or equivalently, that

$$\tan\phi(\tau) \neq \rho^-, \tag{47}$$

for any $\tau > 0$ sufficiently large. We define new variables $u = \tanh\left(\dfrac{\alpha}{2(2-\alpha)} Q_p \tau\right)$ and $w = \tan(\phi(\tau))$. Then the condition (47) reduces to $w(\tau) \neq \rho^-$, for τ sufficiently large.

We get for w and u the following planar system

$$w' = -w^2 + \frac{Q_p}{2} uw + \lambda, \qquad u' = \frac{\alpha}{2(2-\alpha)} Q_p (1-u^2), \tag{48}$$

which is well defined for any u, w. However, for us, it only makes sense for $|u| \leq 1$. It is also clear that $u(\tau)$ is an increasing function for $|u| < 1$. Moreover, in order to recover the solutions of (46) from (48) we must identify the solutions of (48) with $w(\tau) \to -\infty$ when $\tau \to \tau_*^-$, for some τ_*, with the corresponding ones with $w(\tau) \to \infty$ when $\tau \to \tau_*^+$. Of course we are interested in (48) for $0 \leq u \leq 1$. More precisely, we are interested in the solutions of (48) for $\tau \geq 0$ with initial conditions $w(0) = 0, u(0) = 0$ and $w(\tau) \to \infty$, when $\tau \to 0^+$ and $u(0) = 0$.

If $\lambda > \hat{\lambda}$ the system (48) has two equilibria on the line $u = 1$ located at $(w, u) = (\rho^-, 1)$ and $(w, u) = (\rho^+, 1)$ respectively, where

$$\rho^{\pm} = \frac{Q_p}{4}(1 \pm \beta).$$

The first one is a saddle point and the second one is an attractor. For positive λ we get $\rho^- < 0$, $\rho^+ > 0$ and the region $\mathcal{R}_1 = \{(w, u) \,|\, w \geq 0, 0 \leq u \leq 1\}$ is positively invariant for the flow defined by (48). In this region all the orbits tend to the attractor. The orbits we are interested in are contained in $\mathcal{R}_1$ for positive time. So, $w(\tau) \neq \rho^-$ for $\tau > 0$ (see Figure 2 (a)).

If $\hat{\lambda} < \lambda < 0$ then $0 < \rho^- < \rho^+$. Let W^s be the branch of the stable invariant manifold of the point $(\rho^-, 1)$ contained in the strip $\{(w, u) \,|\, |u| \leq 1\}$, and $\mathcal{R}_2 \subset \{(w, u) \,|\, w \geq 0, 0 \leq u \leq 1\}$ the unbounded region with boundaries W^s and $\{(w, u) \,|\, w \geq \rho^-, u = 1\}$. Then $\mathcal{R}_2$ is positively invariant and all the orbits in $\mathcal{R}_2$ tend to $(\rho^+, 1)$ when $\tau \to \infty$. The interesting orbits enter in $\mathcal{R}_2$ for some τ large enough and tend to the point $(\rho^+, 1)$ when $\tau \to \infty$. Then $w(\tau) \neq \rho^-$ if τ is sufficiently large (see Figure 2 (b)). □

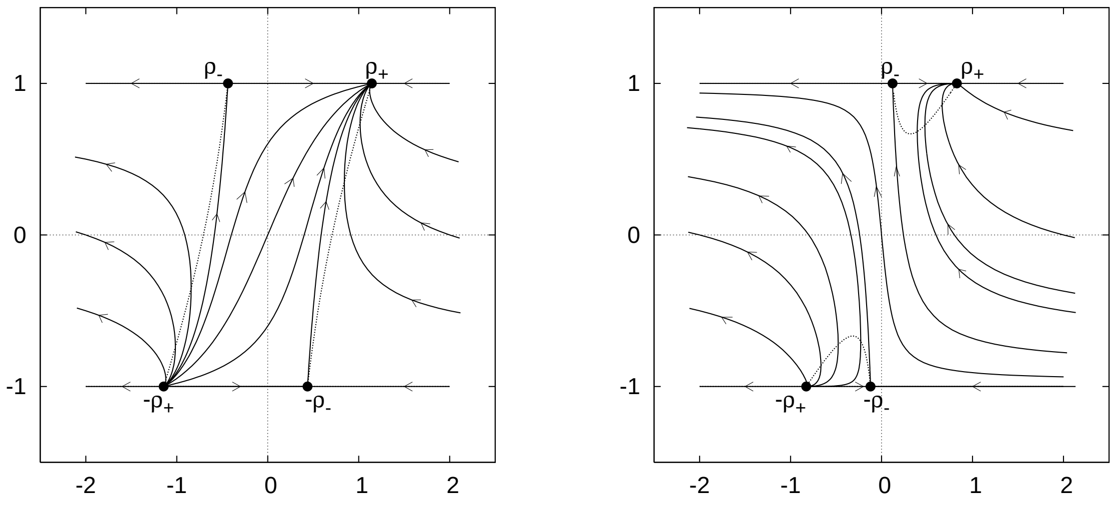

Figure 2: Phase portrait of system (48). Left: $\alpha = 1, \lambda = 0.2$. Right: $\alpha = 0.8, \lambda = -0.1$. The dotted lines joining equilibria are vertical isoclines.

Now, to finish the proof of the lemma 5 we apply lemma 7 for $\lambda = \lambda_1$ and $\mathbf{v}(\tau) = (c_{11}(\tau), c_{31}(\tau))^T$. Note that $\mathbf{v}(0) = (1, 0)^T$. Then, $c_{31}(\tau) - \rho^- c_{11}(\tau) \neq 0$ for τ sufficiently large, in particular for $\tau = \tau_{L_1}$ if $\epsilon > 0$ is small enough. We conclude that $e_{11} \neq 0$. In a similar way one can see that $e_{12}, e_{21}, e_{22} \neq 0$.

We remark that e_{ij}, $i = 1, 2$, $j = 1, 2$, only depend on α and γ but they do not depend on function $V_1(z)$. □

6 Homographic solutions

In this section we discuss an application of theorem 1. We consider the planar three body problem with an homogeneous potential of degree $\alpha \in (0, 2)$

$$U(\mathbf{q}) = \sum_{1 \leq i < j \leq 3} \frac{m_i m_j}{\|\mathbf{q}_i - \mathbf{q}_j\|^\alpha}. \tag{49}$$

In particular, $\alpha = 1$ corresponds to the Newtonian potential. There are some special solutions, called homographic, such that the configuration of the bodies remain constant for all time. In fact the positions of the bodies $\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3) \in \mathbb{R}^6$ are obtained at any time by an homography from a fixed $\mathbf{q}_c \in \mathbb{R}^6$ (see [8] for the Newtonian potential). Two kind of homographic solutions are found, the collinear (the three bodies are aligned) and the triangular ones (the bodies are in the vertices of an equilateral triangle).

To show that there are exactly three collinear central configurations in the studied problem we give next lemma, which covers a wider range of α.

Lemma 8. *Consider a collinear central configuration of the attracting three-body problem with positive masses m_j, $j = 1, 2, 3$ and homogeneous potential of the form $\frac{1}{\alpha} r^{-\alpha}$, $\alpha > -2$ for $\alpha \neq 0$ and $-\log(r)$ for $\alpha = 0$. Then there exist exactly three solutions.*

Proof Let x_j be the coordinate of m_j on the line. It is enough to show that, assuming $x_1 < x_2 < x_3$, there is exactly one solution. Due to the homogeneity of degree $-\alpha - 1$ of the forces, it is not restrictive to assume $x_1 = 0$, $x_3 = 1$ and that

$$m_1 + m_2 + m_3 = 1. \tag{50}$$

The centre of masses then is located at $g = m_2 x + m_3$, where from now on x_2 will be denoted as x.

In a central configuration the actions on one body due to the presence of the others are exactly cancelled by the centrifugal force of a rotation of angular velocity ω around the centre of masses. This leads to the equations

$$\frac{m_2}{x^{\alpha+1}} + \frac{m_3}{1} = \omega^2 (m_2 x + m_3), \tag{51}$$

$$-\frac{m_1}{x^{\alpha+1}} + \frac{m_3}{(1-x)^{\alpha+1}} = \omega^2 (m_2 x + m_3 - x), \tag{52}$$

$$-\frac{m_1}{1} - \frac{m_2}{(1-x)^{\alpha+1}} = \omega^2 (m_2 x + m_3 - 1). \tag{53}$$

It is clear that the three equations are not independent. Multiplying equations (51–53) by m_1, m_2, m_3, respectively, and adding, we obtain a trivial identity. By subtracting (53) from (51) we obtain

$$\omega^2 = \left(\frac{m_2}{x^{\alpha+1}} + \frac{m_2}{(1-x)^{\alpha+1}} + m_1 + m_3 \right). \tag{54}$$

Replacing (54) in (52) we have

$$F(x) := -\frac{m_1}{x^{\alpha+1}} + \frac{m_3}{(1-x)^{\alpha+1}} + \left(\frac{m_2}{x^{\alpha+1}} + \frac{m_2}{(1-x)^{\alpha+1}} + m_1 + m_3 \right) (x - m_2 x - m_3) = 0. \tag{55}$$

Rearranging (55) and using (50) we have

$$F(x) = -\frac{m_1+m_2m_3}{x^{\alpha+1}}+\frac{m_1m_2+m_2m_3}{x^{\alpha}}+\frac{m_3+m_2m_1}{(1-x)^{\alpha+1}}-\frac{m_3m_2+m_2m_1}{(1-x)^{\alpha}}+(1-m_2)^2x-(m_1+m_3)m_3. \tag{56}$$

Consider first the case $\alpha > -1$. It is clear that $F(x) \to -\infty$ (resp. $F(x) \to +\infty$) when $x \to 0^+$ (resp. $x \to 1^-$). Hence, it is enough to check $F'(x) > 0$ for $x \in (0,1)$.

The derivatives of the first two terms in (56) are

$$(\alpha+1)\frac{m_1+m_2m_3}{x^{\alpha+2}} - \alpha\frac{m_1m_2+m_2m_3}{x^{\alpha+1}},$$

which is clearly positive because $\alpha > 0$, $m_2 < 1$ and $x \in (0,1)$. Similar reasoning applies to the third and fourth terms. Last two terms are trivial.

In the elementary case $\alpha = -1$ (constant force) $F(x) = 0$ is a linear equation with positive slope and $F(0) < 0, F(1) > 0$.

In the case $-2 < \alpha < -1$ the dominant terms near 0 and 1 are $-(m_1 + m_2m_3)x^{-\alpha-1}$ and $(m_3 + m_2m_1)(1-x)^{-\alpha-1}$. This implies the existence of one solution. To show unicity it is enough to show $F''' < 0$. From the first two terms in (56) we obtain

$$(\alpha+1)(\alpha+2)\left[(\alpha+3)\frac{m_1+m_2m_3}{x^{\alpha+4}} - \alpha\frac{m_1m_2+m_2m_3}{x^{\alpha+3}}\right],$$

which is clearly negative. A similar formula is obtained for the third and fourth terms. □

Remark 5.

1. *In the trivial case $\alpha = -2$ (harmonic potential) all positions are central configurations. For $\alpha < -2$ the number of solutions to (55) depends on the values of α, m_1, m_2, m_3. It is easy to find no solutions or more than one.*

2. *The non-collinear case is elementary. Writing equations similar to (51)-(53) in $\mathbb{R}^2$, with the bodies located at $(0,0), (1,0), (x,y)$, from the second component of first two equations the relations $\omega^2 = r_{1,3}^{-\alpha-2} = r_{2,3}^{-\alpha-2}$ are obtained. Substitution in the first component of the first equation gives $\omega^2 = 1$. Hence the solutions are two equilateral triangles if $\alpha \neq -2$.*

Using the integrals of the centre of mass the homographic solutions can be seen as equilibria of some 8-D periodic system. To study the linear stability one has to compute the characteristic multipliers of the variational equations which in this case are an 8-D linear periodic system. However, due to the additional integrals of the problem, we know that 1 is a characteristic multiplier of multiplicity 4. A first step is to uncouple the variational equations as two 4-D systems, one of them having all characteristic multipliers equal to one. So, we can reduce to study a 4-D system. We remark that this reduction was done by Roberts ([6]) in the Newtonian case for the triangular homographic solutions by performing successive changes of variables. In this paper we make the reduction in a more general way. Our procedure is inspired in the work of Moeckel [2].

The system for the non trivial characteristic multipliers satisfies the hypotheses of theorem 1 and so, the asymptotic formula holds in this case. We shall discuss the qualitative behaviour predicted by the formula and we shall compare it with numerical computations of the stability parameters.

To carry out the reduction we recall some facts about homographic solutions (see [8] and [7]). Let us consider the Hamiltonian system

$$\mathbf{q}' = M^{-1}\mathbf{p}, \qquad \mathbf{p}' = \nabla U(\mathbf{q}), \tag{57}$$

where $\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3)$, $\mathbf{p} = (\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3)$, $\mathbf{q}_i, \mathbf{p}_i \in \mathbb{R}^2$, $M = \mathrm{diag}(m_1, m_1, m_2, m_2, m_3, m_3)$ and $U(\mathbf{q})$ given in (49). We can assume (50) and the centre of masses fixed at the origin. Then the integrals of the centre of masses are written as

$$m_1\mathbf{q}_1 + m_2\mathbf{q}_2 + m_3\mathbf{q}_3 = 0, \qquad \mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 = 0.$$

After some scalings, a central configuration, $\mathbf{q}_c$, is a solution of the equation $-M\mathbf{q} = \nabla U(\mathbf{q})$. An homographic solution of (57) is a solution of the form

$$\mathbf{q}(t) = r(t)\Omega(f(t))\mathbf{q}_c, \quad \Omega = \mathrm{diag}(\Omega_1, \Omega_1, \Omega_1), \quad \Omega_1(f) = \begin{pmatrix} \cos f & -\sin f \\ \sin f & \cos f \end{pmatrix}, \tag{58}$$

where $r(t)$ is a periodic solution of

$$r'' = -\frac{dV(r)}{dr}, \qquad \text{with} \qquad V(r) = \frac{\omega^2}{2r^2} - \frac{1}{\alpha r^\alpha}, \tag{59}$$

and $f(t) = \displaystyle\int_0^t \frac{\omega}{r(s)^2} ds$. We shall take $r(0) = r_0$ as the maximum of $r(t)$. We note that ω is the angular momentum for the Keplerian problem (59). In the Newtonian case, $\alpha = 1$, f is the true anomaly. We shall denote the energy of (59) as

$$E_K = \frac{(r')^2}{2} + V(r).$$

We introduce a rotating and pulsating coordinate system through $\mathbf{q}(t) = r(t)\Omega(f(t))\boldsymbol{\zeta}(t)$. Using f as independent variable the new system can be written as

$$\dot{\boldsymbol{\zeta}} = K_6\boldsymbol{\zeta} + M^{-1}\boldsymbol{\eta}, \qquad \dot{\boldsymbol{\eta}} = \nabla\mathcal{V}(\boldsymbol{\zeta}) + K_6\boldsymbol{\eta}, \tag{60}$$

where $\dot{}$ stands for the derivative with respect to f, $\boldsymbol{\eta}$ is the conjugate variable of $\boldsymbol{\zeta}$

$$\mathcal{V}(\boldsymbol{\zeta}) = \frac{r^{2-\alpha}}{\omega^2}\hat{U}(\boldsymbol{\zeta}) + \frac{1}{2}\left(\frac{r^{2-\alpha}}{\omega^2} - 1\right)\boldsymbol{\zeta}^T M\boldsymbol{\zeta},$$

and $\hat{U}(\boldsymbol{\zeta}) = r^{-\alpha}U(\mathbf{q})$. The system (60) is Hamiltonian with Hamiltonian function

$$H(\boldsymbol{\zeta}, \boldsymbol{\eta}) = \frac{1}{2}\boldsymbol{\eta}^T M^{-1}\boldsymbol{\eta} - \boldsymbol{\zeta}^T K_6\boldsymbol{\eta} - \mathcal{V}(\boldsymbol{\zeta}).$$

A first reduction of (60) is done by using the integrals of the centre of mass. We introduce new variables

$$\begin{array}{lll} \mathbf{u}_i = \boldsymbol{\zeta}_i - \boldsymbol{\zeta}_3, & \mathbf{v}_i = \boldsymbol{\eta}_i, & i = 1, 2, \\ \mathbf{u}_3 = \boldsymbol{\zeta}_3, & \mathbf{v}_3 = \boldsymbol{\eta}_1 + \boldsymbol{\eta}_2 + \boldsymbol{\eta}_3. & \end{array}$$

Then $\mathbf{v}_3 = 0$. The equations for $\mathbf{u}_i, \mathbf{v}_i$, $i = 1, 2$, do not depend on $\mathbf{u}_3$. So we can reduce to consider the following system

$$\dot{\mathbf{u}} = K_4\mathbf{u} + C^{-1}\mathbf{v}, \qquad \dot{\mathbf{v}} = \nabla\mathcal{V}(\mathbf{u}) + K_4\mathbf{v}, \tag{61}$$

where $\mathbf{u} = \begin{pmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{pmatrix}$, $C = m_1 m_2 \begin{pmatrix} \alpha_2 m_3 I_2 & -I_2 \\ -I_2 & \alpha_1 m_3 I_2 \end{pmatrix}$, $\alpha_1 = \dfrac{m_1 + m_3}{m_1 m_3}$, $\alpha_2 = \dfrac{m_2 + m_3}{m_2 m_3}$ and

$$\mathcal{V}(\mathbf{u}) = \frac{r^{2-\alpha}}{\omega^2}\hat{U}(\mathbf{u}) + \frac{1}{2}\left(\frac{r^{2-\alpha}}{\omega^2} - 1\right)\mathbf{u}^T C\mathbf{u},$$

$$\hat{U}(\mathbf{u}) = \frac{m_1 m_2}{\|\mathbf{u}_1 - \mathbf{u}_2\|^\alpha} + \frac{m_1 m_3}{\|\mathbf{u}_1\|^\alpha} + \frac{m_2 m_3}{\|\mathbf{u}_2\|^\alpha}.$$

The system (61) is periodic on f. The homographic solutions are the equilibria $(\mathbf{u}^*, \mathbf{v}^*)$ of system (61), that is, $\mathbf{v}^* = -CK_4\mathbf{u}^*$ and $\mathbf{u}^*$ is a solution of the equation

$$\nabla \hat{U}(\mathbf{u}) = -C\mathbf{u}. \tag{62}$$

The linearised system of (61) at an equilibrium $(\mathbf{u}^*, \mathbf{v}^*)$ is

$$\dot{\mathbf{y}} = \mathcal{A}\mathbf{y}, \qquad \mathcal{A} = \begin{pmatrix} K_4 & C^{-1} \\ \mathcal{D} & K_4 \end{pmatrix}, \qquad \mathcal{D} = \frac{r^{2-\alpha}}{\omega^2} D\nabla\hat{U}(\mathbf{u}^*) + \left(\frac{r^{2-\alpha}}{\omega^2} - 1\right) C. \tag{63}$$

We will see that (63) can be written as two uncoupled systems or order four.

Lemma 9. *Let $\mathbf{u}^*$ be a solution of (62). The system (63) can be written as two uncoupled linear systems or order four with matrices*

$$\mathcal{B}_1(f) = \begin{pmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 \\ h_1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}, \qquad \mathcal{B}_2(f) = \begin{pmatrix} 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ b_{11} & b_{12} & 0 & -1 \\ b_{21} & b_{22} & 1 & 0 \end{pmatrix}, \tag{64}$$

where $h_1 = h_1(f) = (\alpha+2)\dfrac{r^{2-\alpha}}{\omega^2} - 1$, *and*

$$b_{11} = \frac{r^{2-\alpha}}{\omega^2}(\gamma_{11}+1) - 1, \quad b_{12} = \frac{r^{2-\alpha}}{\omega^2}\gamma_{12}, \quad b_{21} = \frac{r^{2-\alpha}}{\omega^2}\gamma_{21}, \quad b_{22} = \frac{r^{2-\alpha}}{\omega^2}(\gamma_{22}+1) - 1, \tag{65}$$

being $\gamma_{11}, \gamma_{12}, \gamma_{21}$ and γ_{22} some constant coefficients depending on $\mathbf{u}^*$.

Proof We introduce the following vectors

$$\mathbf{x}_1 = \begin{pmatrix} \mathbf{u}^* \\ \mathbf{0} \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} \mathbf{0} \\ K_4 C\mathbf{u}^* \end{pmatrix}, \quad \mathbf{x}_3 = \begin{pmatrix} \mathbf{0} \\ C\mathbf{u}^* \end{pmatrix}, \quad \mathbf{x}_4 = \begin{pmatrix} K_4\mathbf{u}^* \\ \mathbf{0} \end{pmatrix}. \tag{66}$$

First, we shall show that the subspace X of $\mathbb{R}^8$ spanned by $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4$ is invariant under $\mathcal{A}$. Then we will introduce the skew-orthogonal complement of X in $\mathbb{R}^8$, in order to uncouple (63).

The following equalities hold due to the homogeneity of $\hat{U}(\mathbf{u})$

$$D\nabla\hat{U}(\mathbf{u})\mathbf{u} = -(\alpha+1)\nabla\mathcal{U}(\mathbf{u}), \qquad D\nabla\hat{U}(\mathbf{u})K_4\mathbf{u} = K_4\nabla\hat{U}(\mathbf{u}).$$

So, if $\mathbf{u}^*$ is a central configuration we get from the equalities above and (62)

$$D\nabla\hat{U}(\mathbf{u}^*)\mathbf{u}^* = (\alpha+1)C\mathbf{u}^*, \qquad D\nabla\hat{U}(\mathbf{u}^*)K_4\mathbf{u}^* = -K_4 C\mathbf{u}^*. \tag{67}$$

Then, using that K_4 and C commute

$$\mathcal{D}\mathbf{u}^* = \left((\alpha+2)\frac{r^{2-\alpha}}{\omega^2} - 1\right) C\mathbf{u}^*, \qquad \mathcal{D}K_4\mathbf{u}^* = -CK_4\mathbf{u}^*. \tag{68}$$

Using (68) we get easily

$$\mathcal{A}\mathbf{x}_1 = \mathbf{x}_4 + h_1\mathbf{x}_3, \quad \mathcal{A}\mathbf{x}_2 = \mathbf{x}_4 - \mathbf{x}_3, \quad \mathcal{A}\mathbf{x}_3 = \mathbf{x}_1 + \mathbf{x}_2, \quad \mathcal{A}\mathbf{x}_4 = -\mathbf{x}_1 - \mathbf{x}_2.$$

So, X is invariant under $\mathcal{A}$ and the system (63) reduced to X is given by the matrix $\mathcal{B}_1(f)$. In order to uncouple (63) we only need to get the skew–orthogonal complement, W, in $\mathbb{R}^8$, of X, that is $W = \{\mathbf{w} \in \mathbb{R}^8 \,|\, \mathbf{w}^T J_8 \mathbf{x}_i = 0, i = 1, \dots, 4\}$.

We define

$$\mathbf{w}_1 \;=\; \begin{pmatrix} C^{-1}\boldsymbol{\eta}_1 \\ \mathbf{0} \end{pmatrix}, \quad \mathbf{w}_2 = \begin{pmatrix} C^{-1}\boldsymbol{\eta}_2 \\ \mathbf{0} \end{pmatrix}, \qquad \mathbf{w}_3 = \begin{pmatrix} \mathbf{0} \\ \boldsymbol{\eta}_1 \end{pmatrix}, \quad \mathbf{w}_4 = \begin{pmatrix} \mathbf{0} \\ \boldsymbol{\eta}_2 \end{pmatrix},$$

where

$$\boldsymbol{\eta}_1 = J_4\mathbf{u}^* + \gamma_1 K_4 \mathbf{u}^*, \qquad \boldsymbol{\eta}_2 = K_4\boldsymbol{\eta}_1, \qquad \gamma_1 = \frac{(\mathbf{u}^*)^T K_4 J_4 \mathbf{u}^*}{\|\mathbf{u}^*\|^2}. \tag{69}$$

We have that $\boldsymbol{\eta}_1^T\mathbf{u}^* = -(\mathbf{u}^*)^T J_4\mathbf{u}^* - \gamma_1(\mathbf{u}^*)^T K_4\mathbf{u}^* = 0$ and $\boldsymbol{\eta}_2^T\mathbf{u}^* = (\mathbf{u}^*)^T J_4 K_4\mathbf{u}^* + \gamma_1(\mathbf{u}^*)^T K_4^2\mathbf{u}^* = 0$. Using these equalities it is easy to see that W is the subspace spanned by $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4$.

The following step is to find the system reduced to W. We have that

$$\mathcal{A}\mathbf{w}_j = \begin{pmatrix} K_4 C^{-1}\boldsymbol{\eta}_j \\ \mathcal{D}C^{-1}\boldsymbol{\eta}_j \end{pmatrix}, \quad j = 1, 2, \qquad \mathcal{A}\mathbf{w}_3 = \mathbf{w}_1 + \mathbf{w}_4, \qquad \mathcal{A}\mathbf{w}_4 = \mathbf{w}_2 - \mathbf{w}_3$$

with

$$\mathcal{D}C^{-1}\boldsymbol{\eta}_j = \frac{r^{2-\alpha}}{\omega^2}[D\nabla\hat{U}(\mathbf{u}^*)C^{-1}\boldsymbol{\eta}_j + \boldsymbol{\eta}_j] - \boldsymbol{\eta}_j, \qquad j = 1, 2.$$

As $\mathbf{u}^*$, $K_4\mathbf{u}^*$, $\boldsymbol{\eta}_1$, $\boldsymbol{\eta}_2$ span $\mathbb{R}^4$ we can write

$$D\nabla\hat{U}(\mathbf{u}^*)C^{-1}\boldsymbol{\eta}_j = \gamma_{j1}\boldsymbol{\eta}_1 + \gamma_{j2}\boldsymbol{\eta}_2 + \gamma_{j3}\mathbf{u}^* + \gamma_{j4}K_4\mathbf{u}^*, \qquad j = 1, 2,$$

for some constants γ_{ji}, $j = 1, 2$, $i = 1, \dots, 4$. Due to the symmetry of $D\nabla\mathcal{U}(\mathbf{u}^*)$ and C, from (67) we have that

$$(\mathbf{u}^*)^T D\nabla\hat{U}(\mathbf{u}^*)C^{-1}\boldsymbol{\eta}_i = 0, \qquad (K_4\mathbf{u}^*)^T D\nabla\hat{U}(\mathbf{u}^*)C^{-1}\boldsymbol{\eta}_i = 0, \quad i = 1, 2.$$

So, we get

$$D\nabla\hat{U}(\mathbf{u}^*)C^{-1}\boldsymbol{\eta}_1 = \gamma_{11}\boldsymbol{\eta}_1 + \gamma_{12}\boldsymbol{\eta}_2, \qquad D\nabla\hat{U}(\mathbf{u}^*)C^{-1}\boldsymbol{\eta}_2 = \gamma_{21}\boldsymbol{\eta}_1 + \gamma_{22}\boldsymbol{\eta}_2,$$

where

$$\gamma_{ij} \;=\; \frac{1}{\|\boldsymbol{\eta}_j\|^2}\boldsymbol{\eta}_j^T D\nabla\hat{U}(\mathbf{u}^*)C^{-1}\boldsymbol{\eta}_i, \quad i, j = 1, 2. \tag{70}$$

Using (63) we obtain

$$\mathcal{D}C^{-1}\boldsymbol{\eta}_1 = b_{11}\boldsymbol{\eta}_1 + b_{12}\boldsymbol{\eta}_2, \qquad \mathcal{D}C^{-1}\boldsymbol{\eta}_2 = b_{21}\boldsymbol{\eta}_1 + b_{22}\boldsymbol{\eta}_2$$

with b_{ij} defined in (65). Then

$$\begin{aligned} \mathcal{A}\mathbf{w}_1 &= \mathbf{w}_2 + b_{11}\mathbf{w}_3 + b_{12}\mathbf{w}_4, \qquad \mathcal{A}\mathbf{w}_2 = -\mathbf{w}_1 + b_{21}\mathbf{w}_3 + b_{22}\mathbf{w}_4, \\ \mathcal{A}\mathbf{w}_3 &= \mathbf{w}_1 + \mathbf{w}_4, \qquad\qquad\qquad\;\; \mathcal{A}\mathbf{w}_4 = \mathbf{w}_2 - \mathbf{w}_3, \end{aligned}$$

and the system (63) reduced to W is defined by the matrix $\mathcal{B}_2(f)$. □

For any equilibrium $(\mathbf{u}^*, \mathbf{v}^*)$ of (61), we shall see that the non trivial characteristic exponents are given by the system defined by $\mathcal{B}_2(f)$. However, first we introduce the parameters which characterise the family of homographic solutions.

Let us consider the following one-dimensional conservative system

$$\ddot{z} = -\frac{d\mathcal{U}}{dz}(z), \qquad \text{with} \qquad \mathcal{U}(z) = \frac{z^2}{2} - \frac{z^\alpha}{\alpha}, \tag{71}$$

with energy $E = \dfrac{\dot{z}^2}{2} + \mathcal{U}(z)$. If $\alpha \in (0,2)$, $\mathcal{U}(z)$ satisfies the hypotheses (A1) and (A2) with $\gamma = -1/\alpha$ (see figure 3). We are interested in negative values of E, that is, $-(2-\alpha)/(2\alpha) \leq E < 0$. Notice that for $E = -(2-\alpha)/(2\alpha)$, (71) has an equilibrium at $z = 1$.

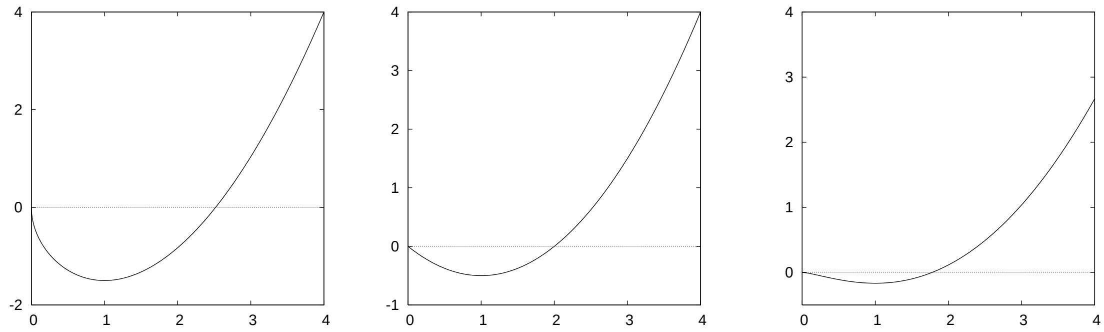

Figure 3: The potential $\mathcal{U}(z)$ for $\alpha = 1/2, 1, 3/2$.

Let $r(t)$ be a bounded solution of (59) for a given value of ω and $E_K < 0$. We introduce

$$g := \omega^{2/(2-\alpha)} r^{-1}. \tag{72}$$

It is easy to check that $g(f)$ is a periodic solution of (71) on the energy level

$$E = \omega^{2\alpha/(2-\alpha)} E_K. \tag{73}$$

Notice that in (71) we take f as independent variable. We shall denote by T the period of g in f. We define a generalised eccentricity as

$$e = \sqrt{1 + \frac{2\alpha}{2-\alpha}\, E_K\, \omega^{2\alpha/(2-\alpha)}}. \tag{74}$$

In the Newtonian case, (74) reduces to the well known relation between eccentricity, energy and angular momentum $e = \sqrt{1 + 2E_K\omega^2}$. For the sake of simplicity we shall fix $E_K = -1/2$. Therefore, the family of homographic solutions is obtained by letting ω range in $(0, \omega_c]$ where $\omega_c = \left(\dfrac{2-\alpha}{\alpha}\right)^{(2-\alpha)/(2\alpha)}$.

If $\omega = \omega_c$ one has in (73) $E = -(2-\alpha)/(2\alpha)$ and $e = 0$. Then $g \equiv 1$ and the corresponding homographic solution is a relative equilibrium. For this kind of solution the three bodies rotate as a rigid body. Moreover, the function $h_1(f)$ defined in lemma 9 becomes constant, $h_1 = \alpha + 1$, and the linear systems defined by $\mathcal{B}_1(f)$ and $\mathcal{B}_2(f)$ have constant coefficients. The linear stability of the homographic solutions for small $e > 0$ (near constant case) is studied in [5] by using a different method (normal forms).

As $\omega > 0$ goes to zero, the homographic solution approaches a collision and $g(0)$ goes to zero. Moreover we can write $\omega^{-2} r^{2-\alpha} = g^{\alpha-2}$. Therefore, the system defined by $\mathcal{B}_2(f)$ has a singularity at $f = 0$ when $\omega = 0$ or equivalently $e = 1$ (singular case). The theorem 1 will be applied here for $1 - e > 0$ small enough.

Remark 6. *$g(f)$ defined in (72) satisfies the hypothesis (B) with* $\delta = \dfrac{2-\alpha}{2\alpha}(1-e^2)$.

Remark 7. *For any $\alpha \in (0,2)$ and δ small, the passage close to triple collision (resp. to homothetic orbit) corresponds to a passage close to γ_+ (resp. γ_0), see figure 1.*

Lemma 10. *For $0 < \omega \leq \omega_c$ the monodromy matrix $\mathcal{C}$ of the linear system*

$$\dot{\mathbf{U}} = \mathcal{B}_1(f)\mathbf{U} \tag{75}$$

has the eigenvalue 1 with multiplicity four.

Proof

For $\omega = \omega_c$, we have $h_1 = h_1(f) = (\alpha+2)g_0^{\alpha-2} - 1 = \alpha + 1$. $\mathcal{B}_1(f)$ is a constant matrix and the characteristic multipliers are easily obtained.

Assume $0 < \omega < \omega_c$. Let be $\mathbf{V} = (u_1, u_2, u_3, u_4)^T$ and denote by $\mathbf{V}_j(f)$, $j = 1, \ldots, 4$ the solution of (75) such that $\mathbf{V}_j(0) = \mathbf{e}_j$.

We note that from (75) $u_1 - u_2 = k$ for some constant k, and

$$\ddot{u}_1 = (h_1 - 3)u_1 + 2k, \tag{76}$$

where $h_1 = h_1(f) = (\alpha+2)g^{\alpha-2} - 1$. Once u_1 is obtained from (76), u_2 is recovered using $u_2 = u_1 - k$ and u_3 and u_4 are obtained by integration from

$$\dot{u}_3 = (h_1 - 1)u_1 + k, \qquad \dot{u}_4 = 2u_1 - k.$$

For $\mathbf{V}_3(f)$ and $\mathbf{V}_4(f)$ we get $k = 0$. In this case, (76) has solutions $u_1(f) = cg\dot{g}$ being c a constant. $\mathbf{V}_3(f)$ is obtained by taking in (76) initial conditions $u_1(0) = 0, \dot{u}_1(0) = 1$. Then $c = 1/(-g_0^2 + g_0^\alpha)$. We note that c is well defined due to the fact that for $0 < \omega < \omega_c$ one has $0 < g_0 < 1$. A simple computation shows that

$$\mathbf{V}_3(f) \;=\; (cg\dot{g},\, cg\dot{g},\, 1 + c(\alpha+2)(g^\alpha - g_0^\alpha)/\alpha - c(g^2 - g_0^2),\, c(g^2 - g_0^2))^T.$$

In a similar way and using the initial condition $\dot{u}_1(0) = -1$ one has

$$\mathbf{V}_4(f) \;=\; (-cg\dot{g},\, -cg\dot{g},\, -c(\alpha+2)(g^\alpha - g_0^\alpha)/\alpha + c(g^2 - g_0^2),\, 1 - c(g^2 - g_0^2))^T.$$

As g is T-periodic, $\mathbf{V}_3(f)$ and $\mathbf{V}_4(f)$ are T-periodic. Then, we have $\mathbf{V}_3(T) = \mathbf{e}_3$, $\mathbf{V}_4(T) = \mathbf{e}_4$.

Now we look for $\mathbf{V}_2(f)$. The initial conditions $u_1(0) = 0$, $u_2(0) = 1$ imply that $k = -1$. Moreover, as $u_3(0) = 0$ and $u_4(0) = 0$ then $\dot{u}_1(0) = 0$. So, we have to solve the equation (76) for $k = -1$ with initial conditions $u_1(0) = 0$, $\dot{u}_1(0) = 0$. Let us assume, for the moment being, that $u_1(T) = 0$. Therefore $\mathbf{V}_2(T) = (0, 1, u_3(T), u_4(T))^T$. In this case the monodromy matrix has the following form

$$\mathcal{C} \;=\; \begin{pmatrix} * & 0 & 0 & 0 \\ * & 1 & 0 & 0 \\ * & * & 1 & 0 \\ * & * & 0 & 1 \end{pmatrix},$$

where $*$ denotes some values that are not relevant. The Liouville Theorem implies that $\det(\mathcal{C}) = 1$. Therefore, 1 is an eigenvalue of $\mathcal{C}$ with multiplicity four.

Our purpose now is to prove that the first component of $\mathbf{V}_2(T)$ is equal to zero, that is, $u_1(T) = 0$. To do that we introduce variables $x_1 = u_1$, $x_2 = \dot{u}_1$ and write the equation (76) for $k = -1$ as a linear system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = [h_1 - 3]x_1 - 2. \tag{77}$$

with initial conditions $x_1(0) = 0$, $x_2(0) = 0$. Let $\Phi(f) = \begin{pmatrix} a_1(f) & b_1(f) \\ a_2(f) & b_2(f) \end{pmatrix}$ be the fundamental matrix of the homogeneous system associated to (77) such that $\Phi(0) = I_2$. We have seen above that (76) with $k = 0$ has solutions like $cg\dot{g}$, being c a constant. Therefore the second column of $\Phi(f)$ is easily obtained as

$$(b_1(f), b_2(f)) = (cg\dot{g}, c(\dot{g}^2 - g^2 + g^\alpha)).$$

Using the periodicity of g and Liouville theorem we have $\det(\Phi(T)) = a_1(T) = 1$.

Let $\mathbf{x}(f) = (x_1(f), x_2(f))^T$ be the solution of the initial value problem (77). The first component of $\mathbf{x}(f)$ can be computed using variation of parameters

$$x_1(f) \quad = \quad a_1(f) \int_0^f 2b_1(s)ds - b_1(f) \int_0^f 2a_1(s)ds.$$

We recall that $b_1(T) = 0$. Then using the periodicity of $g(f)$ we have $x_1(T) = 2c\int_0^T g\dot{g}ds = c(g^2(T) - g^2(0)) = 0$. □

After lemma 10 we can reduce to consider the linear system $\dot{\mathbf{U}} = \mathcal{B}_2(f)\mathbf{U}$. Then if we introduce $\mathbf{w} = M^{-1}\mathbf{U}$, where $M = \begin{pmatrix} I_2 & 0 \\ J_2 & I_2 \end{pmatrix}$, we can write our system as

$$\dot{\mathbf{w}} = A(f)\mathbf{w}, \qquad A(f) = \begin{pmatrix} 0 & I_2 \\ \tilde{A} & -2J_2 \end{pmatrix}, \qquad \tilde{A} = g^{\alpha-2} \begin{pmatrix} \gamma_{11} + 1 & \gamma_{12} \\ \gamma_{21} & \gamma_{22} + 1 \end{pmatrix}. \tag{78}$$

In the appendix we compute the coefficients $\gamma_{11}, \gamma_{12}, \gamma_{21}$ and γ_{22} both in the collinear and the triangular case, and we show that we can consider (78) with $\tilde{A} = g^{\alpha-2} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$, where the values of λ_1, λ_2 are given in the table 1.

Collinear	$\lambda_1 = (\alpha+1)\beta_c + \alpha + 2, \qquad \lambda_2 = -\beta_c,$ $\beta_c \in (0, 2^{\alpha+2} - 1)$
Triangular	λ_1, λ_2 zeroes of $p(\lambda) = \lambda^2 - (\alpha+2)\lambda + \frac{\beta_t}{4},$ $\beta_t = 3(\alpha+2)^2\kappa$

Table 1: Values of λ_1, λ_2 being $\kappa = m_1m_2 + m_1m_3 + m_2m_3$.

Therefore, the linearised system (78) can be written as (4) and using remark 6 the hypotheses (A1), (A2) and (B) are satisfied with $\gamma = -1/\alpha$. In order to apply the theorem 1 we note that $\hat{\lambda} = -\frac{(2-\alpha)^2}{8\alpha} < 0$. The parameters β_1 and β_2 in the theorem are given in the table 2.

Remark 8. *For the Newtonian potential, that is $\alpha = 1$, we get $\beta_1 = \sqrt{25 + 16\beta_c}$, and $\beta_2 = \sqrt{1 - 8\beta_c}$ in the collinear case and, $\beta_{1,2} = \sqrt{13 \pm 12\tilde{\gamma}}$ in the triangular one. These values are related to the eigenvalues at the equilibria on the triple collision manifold (see [1]).*

In the collinear case $\lambda_1 > 0$ and $\lambda_2 < 0$. So, the hypotheses of the theorem are satisfied if and only if $\hat{\lambda} \neq \lambda_2$, that is, $\beta_c \neq (2-\alpha)^2/(8\alpha)$. Therefore, $|\mathrm{tr}_1| > 2$ if $\delta > 0$ (defined in remark 6) is small enough. Let be $\beta^* = (2-\alpha)^2/(8\alpha)$. If $\beta_c < \beta^*$, then $\beta_2 \in \mathbb{R}$ and the second stability parameter is greater than 2. In this case, the system is hyperbolic-hyperbolic (HH) for $\delta > 0$ small enough. If $\beta_c > \beta^*$, then β_2 is pure imaginary. From the corresponding asymptotic formula in (7) we have that tr_2 oscillates as δ tends to zero.

	β_1	β_2
Collinear	$\frac{1}{2-\alpha}\sqrt{8\alpha(\alpha+1)\beta_c+(3\alpha+2)^2}>0,$	$\sqrt{1-\frac{8\alpha\beta_c}{(2-\alpha)^2}},$
Triangular	$\sqrt{1+\frac{4\alpha(\alpha+2)}{(2-\alpha)^2}(1+\tilde{\gamma})}>0$	$\sqrt{1+\frac{4\alpha(\alpha+2)}{(2-\alpha)^2}(1-\tilde{\gamma})}>0$

Table 2: Values of β_1, β_2 being $\tilde{\gamma}=(1-3\kappa)^{1/2}$.

Remark 9. *For the Newtonian potential the critical value of the mass parameter β_c is $\beta^* = 1/8$.*

In the triangular case, we note that $3\kappa \leq 1$ and the equality holds if and only if $m_1 = m_2 = m_3$. Therefore, if $3\kappa < 1$ we have $\lambda_1 > \lambda_2 > 0$. In this case, $\beta_2 > 0$ and the system is hyperbolic-hyperbolic for $\delta > 0$ small enough.

7 Numerical results

We have computed numerically the stability parameters tr_1, tr_2, for the Newtonian potential in both collinear and triangular cases, using β_c and β_t, respectively, as parameters instead of λ_1, λ_2. Also we have used the eccentricity $e \in [0,1)$ instead of $\delta = (1-e^2)/2$. The near constant case, that is, $e > 0$ small enough, has been studied in [5]. We summarise the results in [5] in order to relate them with the near singular case, that is, e close to 1.

For the collinear solutions $\beta_c \in (0,7)$. However it will be useful to consider $\beta_c > 0$ in spite that $\beta_c > 7$ has no physical meaning for the planar three body problem. If $e = 0$ we get a constant system with eigenvalues $\pm\lambda$, $\pm i\omega$ and resonances appear for $\beta_c > 0$. For values of β_c such that $\omega = (2n-1)/2$ for some $n \in \mathbb{N}$, $n \geq 2$, resonant tongues, $\mathcal{T}_\omega$, are born at $e = 0$ giving rise to HH regions in the plane of parameters (β_c, e) (see figure 4). In [5] it is proved that for resonances $\omega = 2n$, $n \in \mathbb{N}$, the two boundaries of resonant tongues coincide. We note that the first tongue, $\mathcal{T}_{3/2}$, is born at $\beta_c = 1.013085794...$ which corresponds to $\omega = 3/2$.

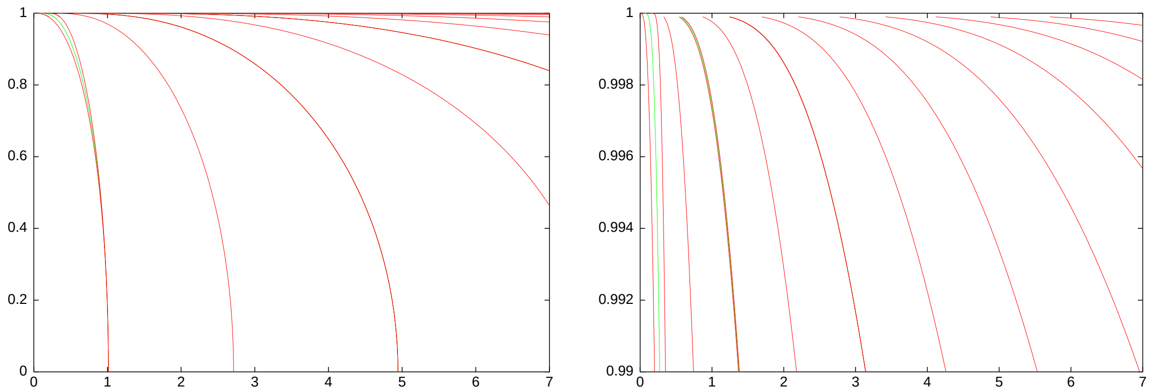

Figure 4: Left: Resonant tongues in the (β_c, e)-plane for the collinear Newtonian homographic solutions. Right: A magnification of the left plot for e close to 1.

Concerning e near 1, we have seen that $\mathrm{tr}_1 > 2$ for any $\beta_c > 0$ and, after the remark 9, the limit behaviour of tr_2 changes at the critical value $\beta_c = 1/8$. We have computed numerically tr_2 as a function of the eccentricity for several values of β_c. The plots are represented in the figure 5 left by taking $-\log_{10}(1-e)$ on the x axis. The computations show that if $\beta_c < 1/8$, tr_2 goes to $-\infty$. Furthermore, if $\beta_c > 1/8$, tr_2 oscillates between 2 and a negative value $k < -2$. Moreover we see numerically that k decreases as $\beta_c \to (1/8)^+$. As tr_2 goes beyond -2, several intervals on e of HH type are created. Therefore for a fixed value $\beta_c = b > 1/8$ we must have in the plane of parameters (β_c, e) a sequence of infinite intervals of type HH which accumulate at $e = 1$. These HH intervals are in fact the intersections of the infinitely many resonant tongues $\mathcal{T}_\omega$ with the line $\beta_c = b$ (see figure 4).

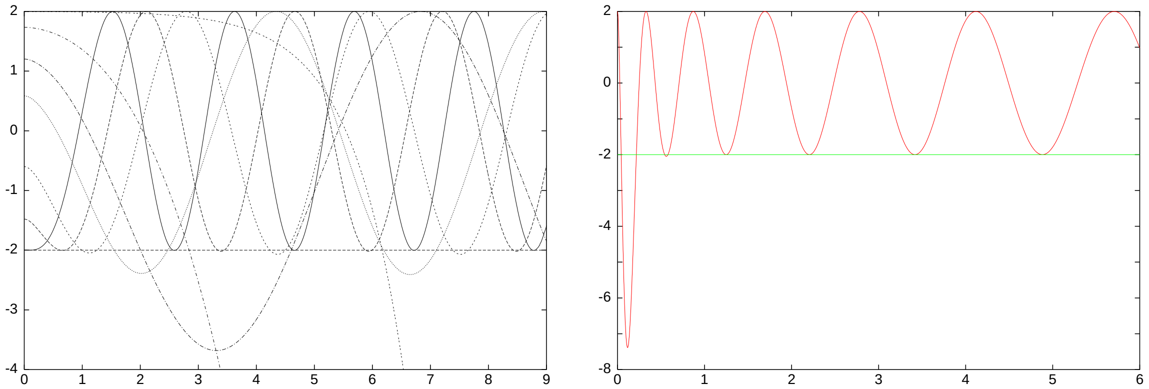

Figure 5: Left: tr_2 as a function of e for several values of β_c. On the horizontal axis we display $-\log_{10}(1-e)$. The different curves can be identified by their intersection with $e = 0$. From top to bottom β_c equals 0.01, 0.1, 0.2, 0.3, 0.5, 0.7 and 1.0. Right: tr_2 as a function of β_c for $e = 0.9999$.

Figure 4 right shows also that the boundaries of $\mathcal{T}_{3/2}$ tend to $\beta_c = 0$ and $\beta_c = 1/8$ respectively, as e goes to 1. This is in agreement with the fact that for $\beta_c < 1/8$ the system is HH for e sufficiently near to 1. However the boundaries of $\mathcal{T}_{(2n-1)/2}$ for $n > 2$ tend to $\beta_c = 1/8$ when e tends to 1. Figure 5 right shows the typical behaviour of the stability parameter tr_2 as a function of β_c when e is near 1. The plot corresponds to $e = 1 - 10^{-4}$. We distinguish clearly the first interval with $\mathrm{tr}_2 < -2$ when β_c is small. This interval corresponds to the first tongue $\mathcal{T}_{\frac{3}{2}}$. In the following oscillations the parameter goes below -2 by a small quantity defining the successive tongues. The numerical computations show that the first minimum goes to infinity as e goes to 1.

It is also interesting to point out that figure 5 right shows that tr_2 does not cross the horizontal line $\mathrm{tr}_2 = 2$, which corresponds to resonances $\omega = n$, $n \in \mathbb{N}$. This means that there is no bifurcation when $\omega = n$ (the two boundaries of $\mathcal{T}_n$ coincide) as it was proved in [5].

Now we consider the triangular case. We have $\beta_1 \in \mathbb{R}$, $\beta_2 \in \mathbb{R}$. Then, if $\delta > 0$ is sufficiently small the system is HH provided that the non degeneracy conditions are satisfied. In this case, this means that the coefficient d_g introduced in lemma 5 is different from zero. We recall that d_g depends on the potential and on λ_1, λ_2.

Figure 6 shows the bifurcation diagram for the triangular homographic solutions for different values of α in the parameter space (β_t, e). In the colour code EE, EH, HH and CS refer, respectively, to elliptic-elliptic, elliptic-hyperbolic, hyperbolic-hyperbolic and complex saddle. In the Newtonian case, we see that for $e \lesssim 1$, the system is HH for any β_t except in a neighbourhood of some critical value $\tilde{\beta}_t$ which, numerically, appears to be equal to 6. Numerical computations of d_g seem to indicate that it is equal to zero for $\beta_t = \tilde{\beta}_t$. Concerning the behaviour for e near 1 in the general case we see numerically that as α increases more critical values $\tilde{\beta}_t$ appear. More concretely, the lower right plot in figure 6 shows values of (α, β_t) for which $d_g = 0$. These values appear along curves and the number of curves seems to tend to ∞ when α approaches 2. To better appreciate the details only values for $\alpha \le 1.9$ have been shown. For reference the maximal admissible value of β_t as a function of α is also displayed.

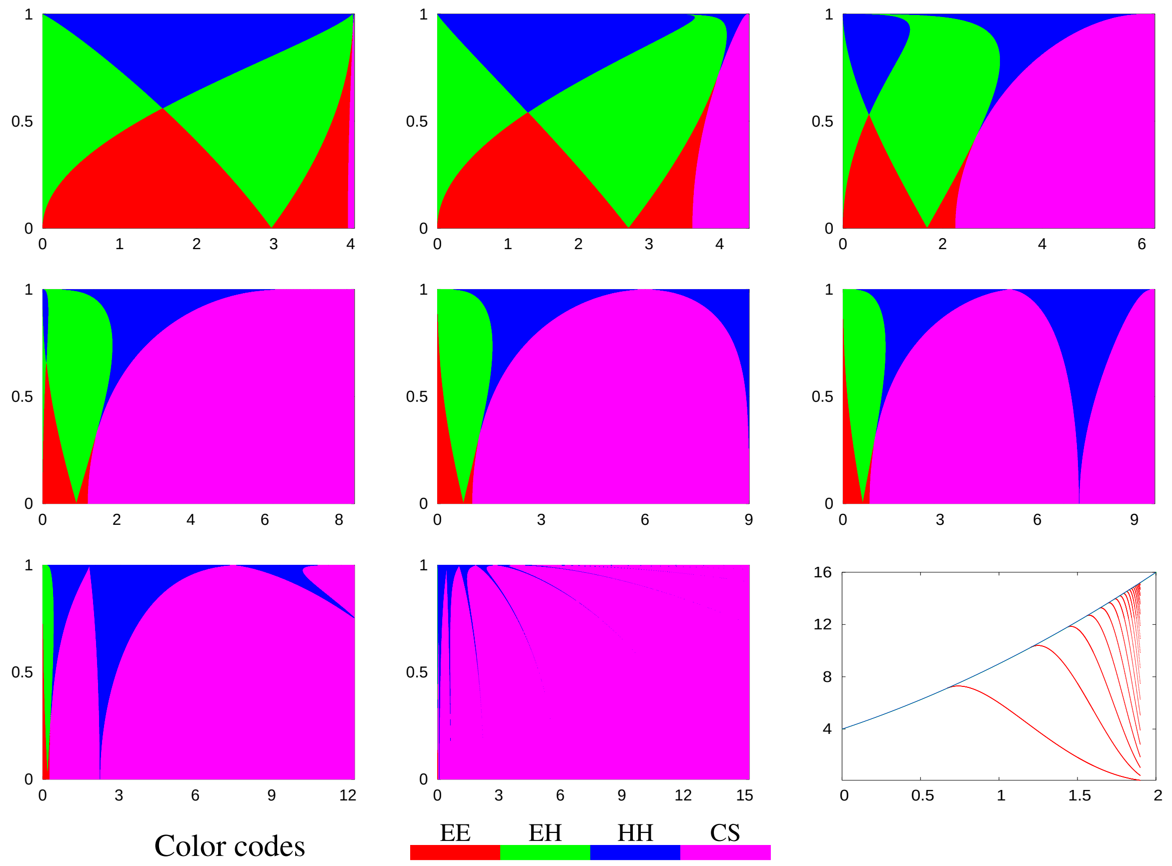

Figure 6: Bifurcation diagram for the triangular homographic solutions. The horizontal (vertical) variable is β_t (e). The values of α are: top file $\alpha = 0.01$, $\alpha = 0.1$, $\alpha = 0.5$; middle file $\alpha = 0.9$, $\alpha = 1$ (Newtonian) $\alpha = 1.1$; bottom file $\alpha = 1.5$, $\alpha = 1.9$. The right plot in the bottom file displays values of (α, β_t) for which $d_g = 0$, i.e., the nondegeneracy condition is not satisfied. The upper parabola $\beta_t = (\alpha + 2)^2$ shows the maximum possible value of β_t.

Finally we return to the Newtonian case. In the corresponding plot in figure 6 one can observe interesting tangencies at the boundary. No analysis of them is carried out in the present work, but we consider worth to mention some data which follow from the numerical computations.

- The tangency at $(0, 1)$ between the vertical axis and the curve which separates the EE and EH domains is of the form $e = 1 - c^*\beta_t^{2/5}$.
- The tangency at $(0, 1)$ between the $e = 1$ line and the curve which separates the EH and HH domains is of the form $e = 1 - c^*\beta_t^4$.
- The tangency at $(6, 1)$ between the $e = 1$ line and the curve which separates the HH and CS domains is of the form $e = 1 - c^*(\beta_t - 6)^2$.

- The tangency at $(9,0)$ between the $\beta_t = 9$ line and the curve which separates the HH and CS domains is of the form $e = c^*(9-\beta_t)^{1/4}$.

In all the above expressions c^* denotes suitable constants. Furthermore there is a point of contact of four different types of domains located at $\approx (1.2091, 0.3145)$.

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8 Appendix

To compute the coefficients γ_{ij}, $i,j=1,2$ in (78) we shall use the definition given in (70).

Let $\mathbf{u}^* = (\mathbf{u}_1^*, \mathbf{u}_2^*)$, $\mathbf{u}_j^* \in \mathbb{R}^2$, $j=1,2$ be a solution of (62) and $\gamma_1, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2$ defined in (69). It is not difficult to get the following relations

$$\gamma_1 = \frac{2(\mathbf{u}_1^*)^T J_2 \mathbf{u}_2^*}{\|\mathbf{u}_1^*\|^2 + \|\mathbf{u}_2^*\|^2}, \quad \|\boldsymbol{\eta}_1\|^2 = [\|\mathbf{u}_1^*\|^2 + \|\mathbf{u}_2^*\|^2](1+\gamma_1^2) - 4\gamma_1(\mathbf{u}_1^*)^T J_2 \mathbf{u}_2^*, \quad \|\boldsymbol{\eta}_2\|^2 = \|\boldsymbol{\eta}_1\|^2. \quad (79)$$

We consider first the collinear case. Assume the masses on a line ordered from left to right as m_3, m_2, m_1. So, we can assume $\mathbf{u}_1^* = (u_1, \mathbf{0})^T$, $\mathbf{u}_2^* = (u_2, \mathbf{0})^T$. The collinear configuration is given by $u_1 = a(1+\rho)$, $u_2 = a$, where $\rho > 0$ is the solution of the following equation, which generalises to any $\alpha \in (0,2)$ the celebrated Euler's quintic equation

$$m_1[\rho^{\alpha+2} - (\rho+1)^{\alpha+2}] + m_3\rho^{\alpha+1}[(\rho+1)^{\alpha+2} - 1] + m_2(\rho+1)^{\alpha+1}(\rho^{\alpha+2} - 1) = 0, \quad (80)$$

and

$$a^{\alpha+2} = \frac{\alpha[m_2(\rho+1)^{\alpha+1} + m_3\rho^{\alpha+1}]}{\rho^{\alpha+1}(\rho+1)^{\alpha+1}[m_3(\rho+1) + m_2\rho]}.$$

If $\mathbf{u}^*$ is a collinear configuration we have $(\mathbf{u}_1^*)^T J_2 \mathbf{u}_2^* = 0$. Then using (79) one has $\gamma_1 = 0$ and from (69), $\boldsymbol{\eta}_1 = J_4\mathbf{u}^*$, $\boldsymbol{\eta}_2 = K_4\mathbf{u}^*$. The coefficients γ_{12}, γ_{21} are defined in (70). It turns out that $D\nabla\hat{U}(\mathbf{u}^*) = \begin{pmatrix} a_1 & 0 & a_2 & 0 \\ 0 & a_3 & 0 & a_4 \\ a_2 & 0 & a_5 & 0 \\ 0 & a_4 & 0 & a_6 \end{pmatrix}$ for some constants a_j, $j = 1,\dots,6$ which depend on m_1, m_2, m_3 and ρ. Now a simple computation shows that $\gamma_{12} = \gamma_{21} = 0$. Therefore the 2×2 matrix $\tilde{A}$ in (78) is diagonal. Moreover, we get the following expressions

$$\gamma_{11} = -(\alpha+1)\gamma_{22}, \qquad \gamma_{22} = -1 - \beta_c,$$

where

$$\begin{aligned}\beta_c \;=\; & -1 + \frac{\alpha}{a^{\alpha+2}[1+(\rho+1)^2]}\{(\rho+2)[(\rho+1)m_1 + m_2]\rho^{-\alpha-2} + \\ & (\rho+1)[m_2\rho + m_3(\rho+1)] + (m_3 - m_1\rho)(\rho+1)^{-\alpha-2}\}. \end{aligned} \quad (81)$$

Therefore

$$\tilde{A} = g^{\alpha-2}\begin{pmatrix} (\alpha+1)\beta_c + \alpha + 2 & 0 \\ 0 & -\beta_c \end{pmatrix}.$$

In [3] it is proved that $\beta_c \in (0,7)$ in the Newtonian case and $\beta_c = 7$ when $m_1 = m_3$, $m_2 = 0$. In the general case, $0 < \alpha < 2$, numerical computations show that the maximum of β_c is also attained when $m_1 = m_3$, $m_2 = 0$. For these values we get $\beta_c = 2^{\alpha+2} - 1$.

For the triangular configurations the three masses are at the vertices of an equilateral triangle. In this case, $\|\mathbf{u}_1\| = \|\mathbf{u}_2\| = \|\mathbf{u}_1 - \mathbf{u}_2\| = \varrho$, where $\varrho = \alpha^{1/(\alpha+2)}$. We can assume that $\mathbf{u}_1 = (\varrho/2, \sqrt{3}\varrho/2)^T$ and $\mathbf{u}_2 = (-\varrho/2, \sqrt{3}\varrho/2)^T$. Using (79) we obtain

$$\gamma_1 = \frac{\sqrt{3}}{2}, \qquad \|\boldsymbol{\eta}_1\|^2 = \|\boldsymbol{\eta}_1\|^2 = \frac{\varrho^2}{2}.$$

From (69) we can write $\boldsymbol{\eta}_1 = (c_1, c_2, c_1, -c_2)^T$ and $\boldsymbol{\eta}_2 = (c_2, -c_1, -c_2, -c_1)^T$, where $c_1 = \varrho/4$ and $c_2 = \sqrt{3}\varrho/4$. Moreover $D\nabla\hat{U}(\mathbf{u}^*) = \begin{pmatrix} a_1 & a_2 & a_3 & 0 \\ a_2 & a_4 & 0 & a_5 \\ a_3 & 0 & a_6 & a_7 \\ 0 & a_5 & a_7 & a_8 \end{pmatrix}$, where a_j, $j = 1, \dots, 8$ are constants depending on the masses and on ϱ. Then after some trivial computations we get

$$\begin{aligned} \gamma_{11} &= -1 + \frac{\alpha+2}{4}(m_1 + m_2 + 4m_3), \qquad \gamma_{22} = \alpha - \gamma_{11}, \\ \gamma_{12} &= \gamma_{21} = \frac{\alpha+2}{4}\sqrt{3}(m_2 - m_1). \end{aligned}$$

So, $g^{2-\alpha}\tilde{A}$ in (78) is a symmetrical matrix. Let P be an orthogonal matrix such that $g^{2-\alpha}\ P^{-1}\tilde{A}P = \mathrm{diag}(\lambda_1, \lambda_2)$. Using remark 2 we can reduce to consider (78) with $\tilde{A} = g^{\alpha-2}\mathrm{diag}(\lambda_1, \lambda_2)$ where λ_1, λ_2 are the zeroes of

$$p(\lambda) = \lambda^2 - (\alpha+2)\lambda + \frac{\beta_t}{4}, \qquad \beta_t = 3\kappa(\alpha+2)^2, \qquad \kappa = m_1m_2 + m_1m_3 + m_2m_3.$$

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