Chaos and quasi-periodicity in diffeomorphisms of the solid torus

Henk W. Broer[(1)], Carles Simó[(2)], and Renato Vitolo[(3)]

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[(1] Dept. of Mathematics, University of Groningen, Blauwborgje 3, 9747 AC Groningen, The Netherlands
[(2] Dept. de Matemàtica Aplicada i Anàlisi, Universitat de Barcelona, Gran Via, 585, 08007 Barcelona, Spain
[(3] Dip. di Mat. e Informatica, Università di Camerino, via Madonna delle Carceri, 62032 Camerino, Italy

E-mail: broer@math.rug.nl, carles@maia.ub.es, renato.vitolo@unicam.it

Abstract

The Hénon family of planar maps is considered driven by the Arnol′d family of circle maps. This leads to a five-parameter family of skew product systems on the solid torus. In this paper the dynamics of this skew product family and its perturbations are studied. It is shown that, in certain parameter domains, Hénon-like strange attractors occur. The existence of quasi-periodic Hénon-like attractors is partially discussed, and further supported by numerical evidence.

Contents

1 Introduction

Since the 1990's several mathematical characterisations have been found concerning the structure of strange attractors in families of maps. A basic example is provided by the Hénon attractor [18], occurring in the family of maps

$$H_{a,b} : \mathbb{R}^2 \to \mathbb{R}^2, \quad (x, y) \mapsto (1 - ax^2 + y,\, bx), \tag{1}$$

where a and b are real parameters. Benedicks and Carleson [2, 3] proved that there exists a set of parameter values $\mathfrak{S}$, with positive Lebesgue measure, such that for all $(a, b) \in \mathfrak{S}$ the Hénon map $H_{a,b}$ (1) has a strange attractor coinciding with the closure $\mathrm{Cl}\,(W^u(p))$ of the unstable manifold of a saddle fixed point p. Here $\mathrm{Cl}\,(-)$ denotes the topological closure. Similar techniques were then used to prove occurrence of strange attractors in parametrised families of maps, near homoclinic tangencies in two or higher dimensions [26, 32, 36, 39], and near tangencies in the saddle-node critical case [14]. See [42] for a general set-up to prove existence of strange attractors with one positive Lyapunov exponent in families of two-dimensional maps. The strange attractors considered in these references are called *Hénon-like* [14, 26, 39].

1.1 Setting of the problem

In this paper we study certain model map families, searching these for Hénon-like attractors as well as for so-called *quasi-periodic Hénon-like* attractors. A basic model for this study is the family of maps of the solid torus $\mathbb{R}^2 \times \mathbb{S}^1$, where $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ is the circle, given by

$$\begin{pmatrix} x \\ y \\ \theta \end{pmatrix} \mapsto \begin{pmatrix} 1 - (a + \varepsilon \sin(2\pi\theta))x^2 + y \\ bx \\ \theta + \alpha + \delta \sin(2\pi\theta) \end{pmatrix}, \tag{2}$$

where both (ε, δ) are perturbation parameters. This map is a skew product perturbation of the Hénon map (1) by the Arnol′d family [1]

$$A_{\alpha,\delta} : \mathbb{S}^1 \to \mathbb{S}^1, \quad \theta \mapsto \theta + \alpha + \delta \sin(2\pi\theta). \tag{3}$$

of maps of $\mathbb{S}^1$. First let us consider the uncoupled situation where $\varepsilon = 0$. The dynamics of the Arnol′d family is globally well-known and that of the Hénon family is partially known. They are organised in the respective (α, δ)- and (a, b)-parameter planes, see Figure 1. For the Arnol′d family in the (α, δ)-plane there is a countable union of resonance tongues with non-empty interior, corresponding to hyperbolic periodic dynamics. In the complement, which is of positive measure, we find quasi-periodic dynamics [1, 13]. See Figure 1 (A). Similarly, for the Hénon family in the (a, b)-plane there exists a countable union of strips of non-empty interior corresponding to hyperbolic periodic dynamics. In the complement a set of positive measure corresponds to strange attractors [3]. Most of the strips are extremely narrow and only become visible when they intersect another strip of the same period in such a way that a "crossroad area" is created [4]. See Figure 1 (B).

Remark 1. Figure 1 is mostly obtained by numerical computation of Lyapunov exponents [35]. Figure 1 (B) uses the origin as initial point, which can land either in a periodic sink, or on a strange attractor or can escape 'to infinity'. Notice that, due to multistability other initial points can tend to different attractors. Moreover, some of the periodicity strips are connected to windows of sinks of the Logistic family as this occurs for $b = 0$. The interpretation of the results in Figure 1 (C) is given in Section 4.

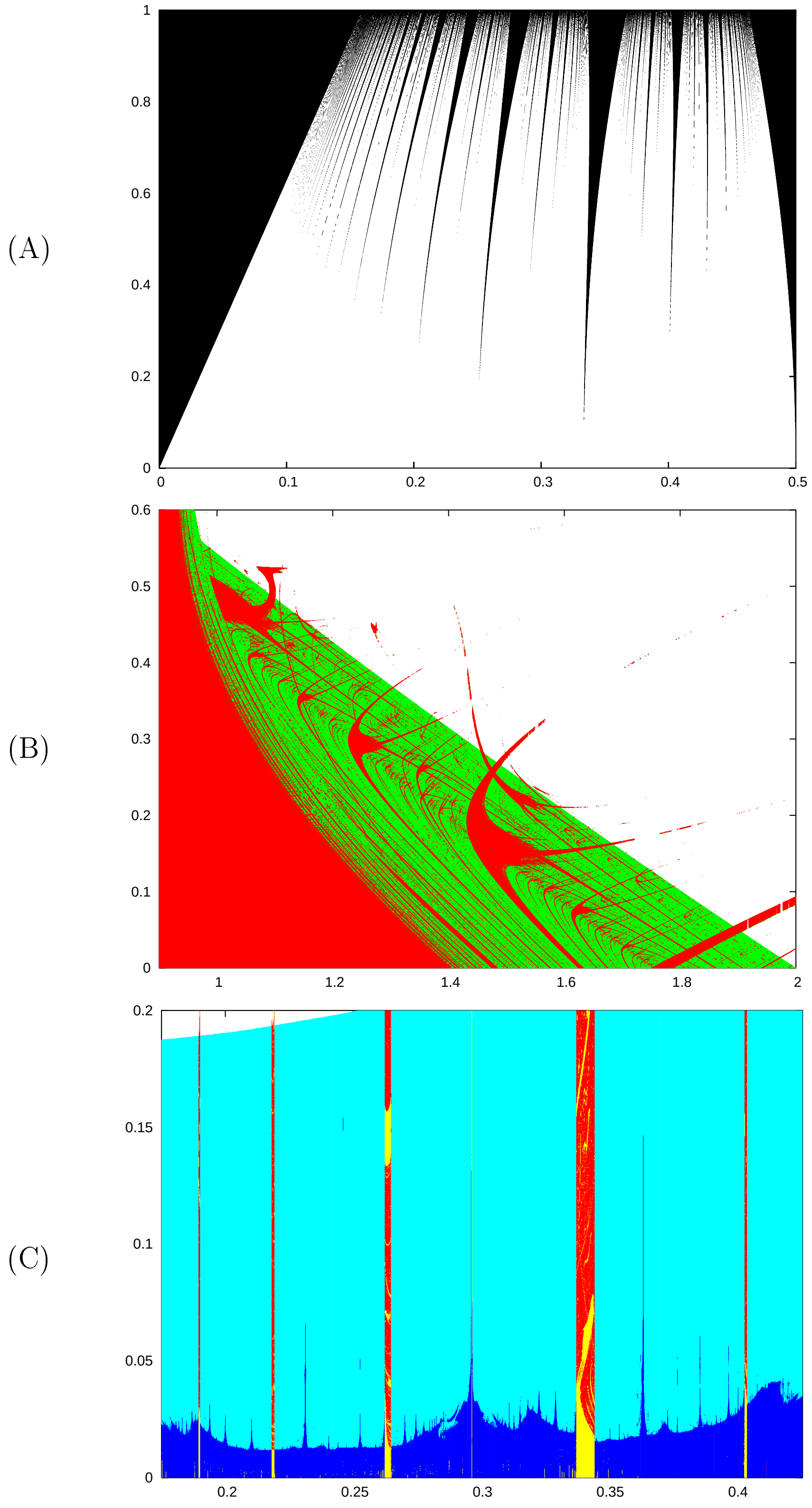

Figure 1: (A) Organisation of the (α, δ)-parameter plane of the Arnol′d family (3) by resonance tongues, containing an open set with periodic dynamics (indicated in black). The remaining parameter values (indicated in white) form a nowhere dense set of positive measure with quasi-periodic dynamics. (B) Organisation of the (a, b)-parameter plane of the Hénon family (1) by strips with periodic dynamics and crossroad areas (in red). A complement of positive measure contains strange attractors (in green). The upper right part of the diagram (in white) corresponds to escape. (C) Diagram of map (2) in the (α, ε)-plane, for $a = 1.25, b = 0.3$ and $\delta = 0.6/(2\pi)$. Visible are: domains which can be interpreted has having periodic attractors (code 1, yellow), quasi-periodic attractors (code 2, blue), Hénon-like attractors (code 3, red) and quasi-periodic Hénon-like attractors (code 4, light blue). For more details see the main text, in particular Sections 1.3, 2.3 and 4.

For map (2) there are at least four combinations of the Arnol′d and Hénon families for the uncoupled case $\varepsilon = 0$ that correspond to parameter domains of positive measure.

1. We start considering the case where the Hénon family is in a periodic attractor, so where the (maximal) Lyapunov exponent $\Lambda_H < 0$.

   (a) In the most simple case, both constituents are in a hyperbolic periodic attractor, compare with Figures 1 (A) and (B). The corresponding (maximal) Lyapunov exponents Λ_A and Λ_H are both negative. In the solid torus $\mathbb{R}^2 \times \mathbb{S}^1$ this also gives a hyperbolic periodic attractor, that is persistent for $|\varepsilon| \ll 1$.

   (b) In a second case, the Arnol′d family is quasi-periodic, while the Hénon family is in a periodic attractor. Now $\Lambda_A = 0$, while $\Lambda_H < 0$. The corresponding uncoupled dynamics in the solid torus again is a normally hyperbolic quasi-periodic attractor, which by centre manifold theory [20] and by KAM theory [5, 6] has certain persistence properties for $|\varepsilon| \ll 1$.

2. In the two remaining cases the Hénon family is in a strange attractor, so with $\Lambda_H > 0$. This attractor is the closure $\mathrm{Cl}\,(W^u(\mathrm{Orb}(p)))$ of the unstable manifold of a periodic saddle point. (Below we shall be more precise.) We have to distinguish two cases.

   (a) In the former of these, the Arnol′d family is in a periodic attractor, so with $\Lambda_A < 0$, and the product system has a Hénon-like attractor. It is the main aim of this paper to show the persistence of this attractor for $|\varepsilon| \ll 1$. For illustrations see Figure 2. Here we shall focus on small values of b, which allows us to rescale our model (2) by ε. In fact we shall consider a sufficiently smooth family of skew-product diffeomorphisms $T_{\alpha,\delta,a,\varepsilon}$ given by

   $$T_{\alpha,\delta,a,\varepsilon} : \mathbb{R}^2 \times \mathbb{S}^1 \to \mathbb{R}^2 \times \mathbb{S}^1, \qquad \begin{pmatrix} x \\ y \\ \theta \end{pmatrix} \mapsto \begin{pmatrix} 1 - ax^2 + \varepsilon f \\ \varepsilon g \\ A_{\alpha,\delta}(\theta) \end{pmatrix}. \tag{4}$$

   Here $(\alpha, \delta, a, \varepsilon)$ are parameters, while f and g are functions of $(a, x, y, \theta, \varepsilon, \alpha, \delta)$. For $0 \leq \delta < (1/2\pi)$ and $\alpha \in [0, 1]$, the map $A_{\alpha,\delta}$ is a diffeomorphism of the circle $\mathbb{S}^1$. We perturb away from cases where (α, δ) is in one of the resonance tongues, see Figure 1 (A).

   (b) In the latter case, where the Arnol′d family is quasi-periodic, so with $\Lambda_A = 0$, the uncoupled product dynamics is quasi-periodic Hénon-like, i.e., on an attractor of the form $\mathrm{Cl}\,(W^u(\mathscr{C}))$, where $\mathscr{C}$ is a quasi-periodic invariant circle of saddle-type, again compare Figure 1 (A). We conjecture that this phenomenon is persistent for $|\varepsilon| \ll 1$, but have only partial results in this direction, supported by numerics. For illustrations see Figures 3 and 4.

The Lyapunov diagram in Figure 1 (C) strongly suggests that all four cases occur in parameter sets of positive measure. More concretely, case 1(a) corresponds to code 1; case 1(b) to code 2; case 2(a) to code 3, and case 2(b) to code 4.

Our interest is with phenomena that are persistent under small perturbations, both within the skew product setting and beyond this. To this end, we also consider a more general class of families defined as follows. First let

$$K = (K_1, K_2) : \mathbb{R}^2 \to \mathbb{R}^2 \tag{5}$$

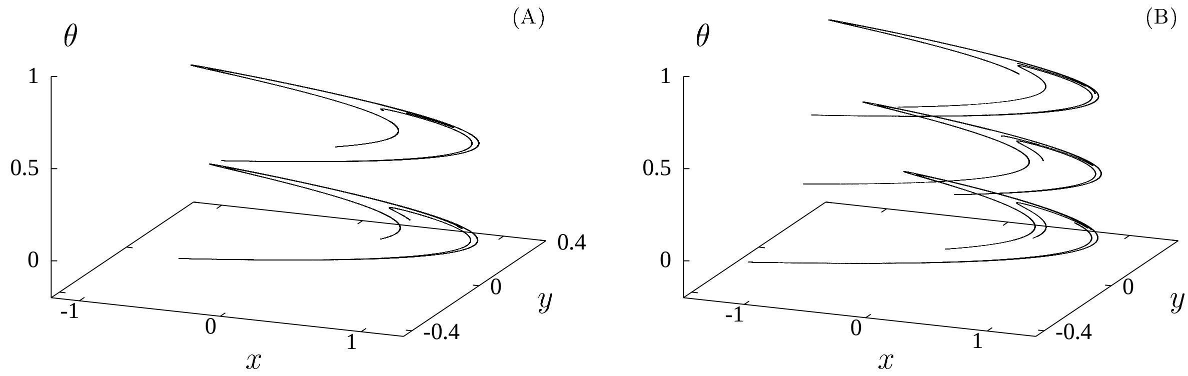

Figure 2: Hénon-like strange attractors of the model family (2) for (α, δ) in Arnol′d tongues of periods two and three. (A) Parameters are fixed at $a = 1.3$, $b = 0.3$, $\varepsilon = 0.2$, $(\alpha, \delta) = (0.51, 0.116)$. (B) Same as (A) for $\alpha = 0.33793$.

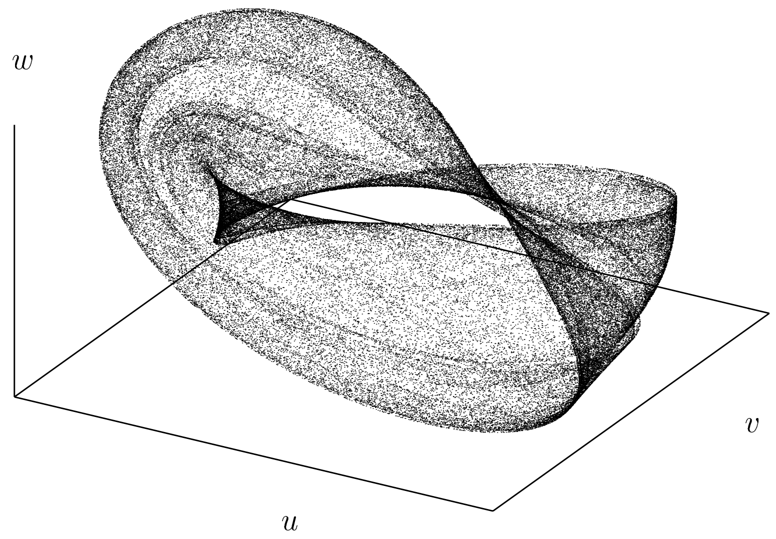

Figure 3: Quasi-periodic Hénon-like strange attractor of the model family (2). Parameter values are fixed at $a = 1.85$, $b = -0.2$, $\delta = 0$, $\alpha = (\sqrt{5}-1)/2$, $\varepsilon = 0.1$. For a better visualisation of the folds, the plot is given in the variables (u, v, w), where $u = (r + 4)\cos(\theta)$, $v = (r + 4)\sin(\theta)$, with $r = x\cos(\theta) + 10y\sin(\theta)$, and $w = -x\sin(\theta) + 10y\cos(\theta)$.

be a dissipative (i.e., area contracting) diffeomorphism, that is sufficiently smooth. Next, denote by $R_\alpha : \mathbb{S}^1 \to \mathbb{S}^1$ the rigid rotation $R_\alpha(\theta) = \theta + \alpha$. Then we define the family

$$P_{\alpha,\varepsilon} : \mathbb{R}^2 \times \mathbb{S}^1 \to \mathbb{R}^2 \times \mathbb{S}^1, \quad (x, y, \theta) \mapsto \big(K_1(x, y) + P_1,\ K_2(x, y) + P_2,\ \theta + \alpha + P_3\big), \tag{6}$$

of diffeomorphisms, where P_j, for $j = 1, \ldots, 3$, is a smooth function of $(x, y, \theta, \alpha, \varepsilon)$ such that $P_j = 0$ for $\varepsilon = 0$. Notice that the model (6) is not a skew product, but that there is full coupling of the two constituents. A hyperbolic fixed point p of K (5) corresponds to a normally hyperbolic invariant circle $\mathscr{C}_{\alpha,0} = \{p\} \times \mathbb{S}^1$ for the map $P_{\alpha,\varepsilon}$ at $\varepsilon = 0$. By normal hyperbolicity the circle $\mathscr{C}_{\alpha,0}$ is persistent under small perturbations [20, Theorem 1.1]. Similar remarks go for the case where p is a hyperbolic periodic point. In the sequel we shall use this both for the case where p is a saddle and where p is a sink.

For numerical illustrations and discussion, a concrete version of (6) is used. It consists of

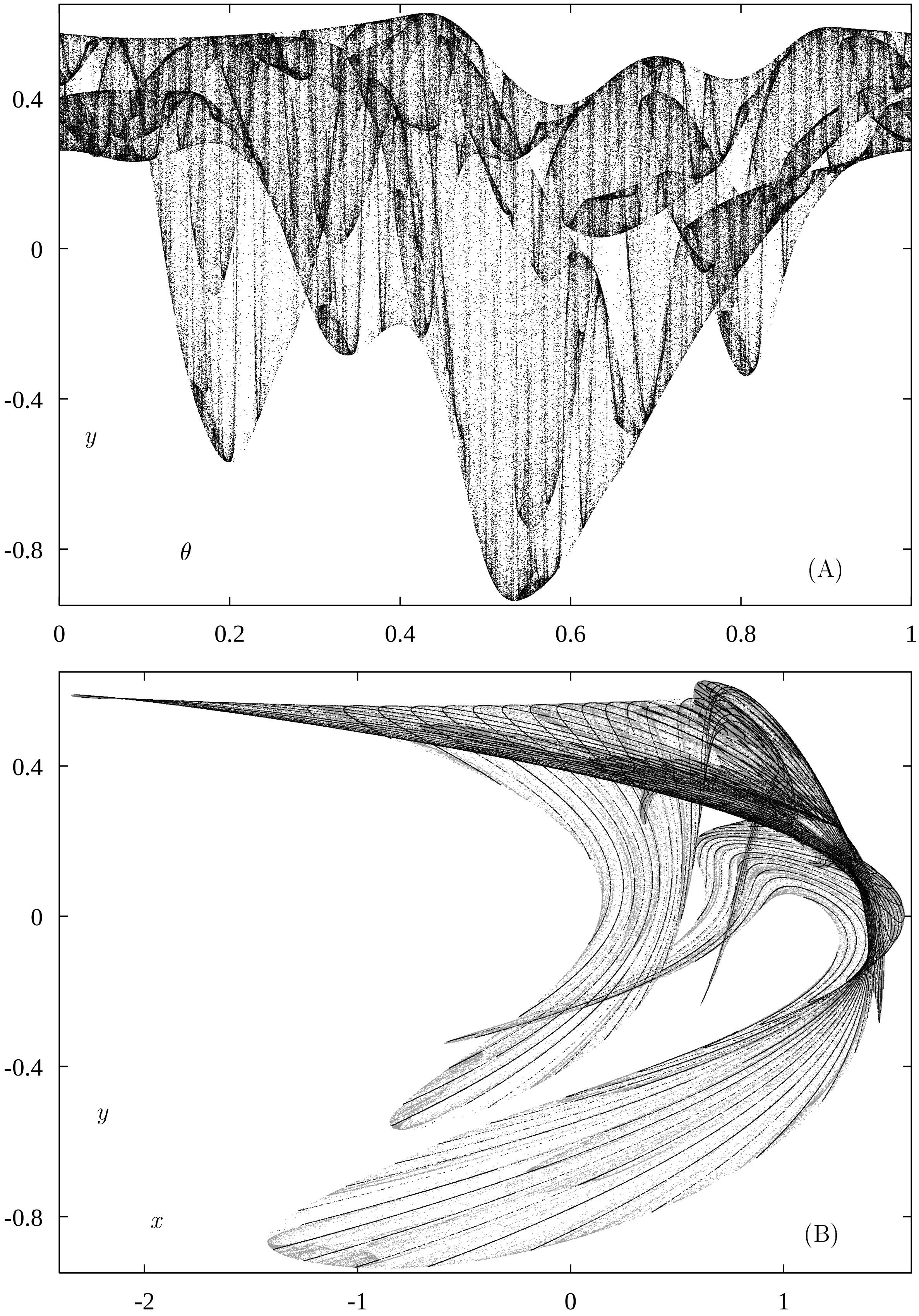

Figure 4:(A) Quasi-periodic Hénon-like attractor of the model family (2), projection on the (θ, y)-plane. Parameter values are fixed at $a = 0.8$, $b = 0.4$, $\delta = 0$, $\alpha = (\sqrt{5} - 1)/2$, $\varepsilon = 0.7$, initial conditions $x_0 = 1.5$, $y_0 = 0$, $\theta_0 = 0$. (B) Same as (A), projection on the (x, y)-plane (in grey, in the background). 'Slices' of the attractor for $2\pi\theta \in [0.1 \times j, 0.1 \times j + 0.001]$, $j = 0, 1, \ldots, 62$, are plotted in black.

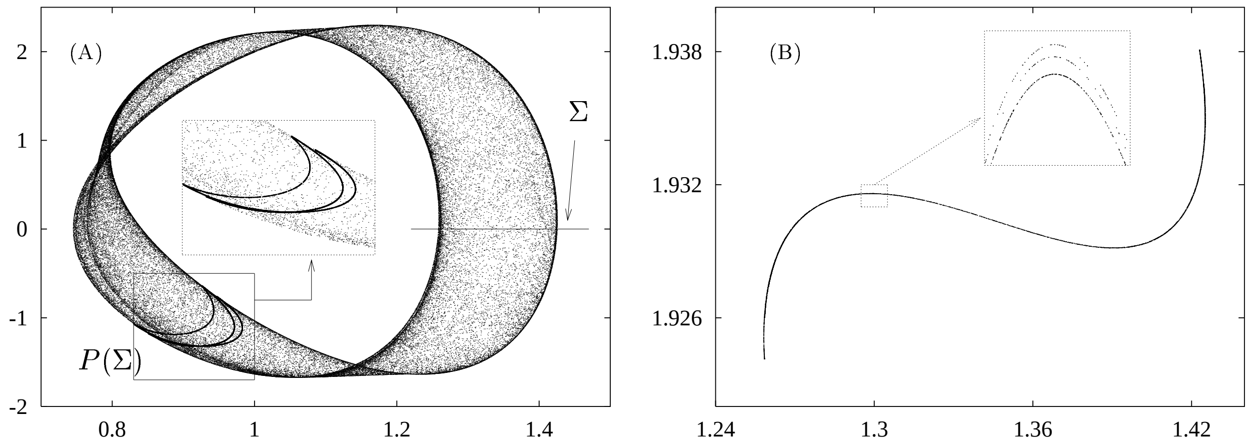

Figure 5: (A) Strange attractor $\mathscr{A}$ of the Poincaré return map of a climatological system [9]. Compare with Figure 3. The attractor $\mathscr{A}$ is plotted with a 'slice' Σ and with the image of Σ under the return map. The slice Σ contains all points with distance less that 0.0001 from the plane $z = 0$. The image of Σ is magnified in the central box. (B) Slice Σ of the attractor $\mathscr{A}$ in (A), projection on the $(x, \tilde{y})$-plane, with $\tilde{y} = y - 0.133 * z$.

a perturbation of (2), where a coupling term in μy is added to the angle dynamics:

$$\mathcal{T} = \mathcal{T}_{\alpha,\delta,a,b,\varepsilon,\mu} : \mathbb{R}^2 \times \mathbb{S}^1 \to \mathbb{R}^2 \times \mathbb{S}^1, \qquad \begin{pmatrix} x \\ y \\ \theta \end{pmatrix} \mapsto \begin{pmatrix} 1 - (a + \varepsilon \sin(2\pi\theta))x^2 + y \\ bx \\ \theta + \alpha + \delta \sin(2\pi\theta) + \mu y \end{pmatrix}, \tag{7}$$

depending on the six parameters $(\alpha, \delta, a, b, \varepsilon, \mu)$.

1.2 Motivation

Quasi-periodic Hénon-like attractors have been conjectured to occur in diffeomorphisms of $\mathbb{R}^3 = \{x, y, z\}$, obtained as Poincaré return maps for a climatological model [9, 10, 41], compare the attractor $\mathscr{A}$ displayed in Figure 5. Examination of a cross-section Σ of the attractor (magnified in Figure 5 (B)) suggests that $\mathscr{A}$ is contained in a two-dimensional manifold which is folded onto itself, in analogy with the structure of the Hénon attractor [18]. This manifold supposedly is the unstable manifold $W^u(\mathscr{C})$ of a quasi-periodic invariant circle $\mathscr{C}$ of saddle type. To illustrate the dynamics inside $\mathscr{A}$ we computed the image of the slice Σ under the return map. This yields a folded curve looking like a planar Hénon attractor.

Remark 2. Also we mention that the occurrence of strange attractors which look similar to Figure 4 (A) is observed in [28, 15, 17, 22, 29]. Although most of these studies deal with endomorphisms of the interval forced by a rigid rotation in a skew product way, and some of them have negative Lyapunov exponents (beyond the one trivially equal to zero), there may be a relationship with the present approach. See Sections 2.3 and 4 for further discussion.

The theoretical knowledge of attractors in higher dimension is limited. As positive exceptions to this we mention Viana [39, 40], Tatjer [36] and Wang & Young [42]. The Hénon-like attractors found in the present paper, to some extent, also belong to this domain. In this sense one may say that the understanding of the quasi-periodic Hénon-like attractor is a next step in this research program. A more detailed discussion and further motivation of the search for quasi-periodic Hénon-like attractors is postponed to Section 2.3.

1.3 Summary and outline

We summarise the remainder of this paper, also explaining its organisation. First in Section 2 the results are presented, where all longer proofs are postponed to Section 3. The contents of Section 2 are related as follows to the subdivision regarding model (2) as given in Section 1.1. Numerical examples beyond the skew product model (2), as well as details on methods and on the interpretation of the results, are given in Section 4.

Normally hyperbolic invariant circles

In Section 2.1 we start considering the more general context of the fully coupled family (6). A hyperbolic periodic point p, for $\varepsilon = 0$ corresponds to a normally hyperbolic invariant circle $\mathscr{C}$.

We start with the case where p is a saddle-point, so where $\mathscr{C}$ normally is of saddle type as well. Theorem 2 asserts that now, under quite general conditions, the closure $\mathrm{Cl}\,(W^u(\mathscr{C}))$ of the unstable manifold of $\mathscr{C}$ attracts an open set. In the families (6) and (2) this corresponds to an open set of parameter values. Note that the attracting set $\mathrm{Cl}\,(W^u(\mathscr{C}))$ is not minimal if $\mathscr{C}$ carries Morse-Smale dynamics. In this situation, the case 2(a) of Section 2 is partially covered.

Next, if p is a hyperbolic periodic point of the family (6), Theorem 3 guarantees that $\mathscr{C}$ is quasi-periodic for a parameter set of positive measure. If p is a saddle-point, in combination with the previous paragraph, we partially cover case 2(b) of Section 1.1 for the family (2). However, if p is a periodic sink, it follows that $\mathscr{C}$ is a quasi-periodic attractor, which completely covers case 1(b) of Section 1.1.

In the case where p is a sink and $\mathscr{C}$ carries Morse-Smale dynamics, for the general setting of (6), the minimal attractors are periodic. For the family (2) this completely covers case 1(a) of Section 1.1.

Persistence of normally hyperbolic invariant circles generally follows by [20, Theorem 1.1]. Theorem 2 is based on a result by Tangerman and Szewc, see [31, Appendix 3]. Our proof of Theorem 3 applies standard KAM Theory, see [6, 5].

The cases where $\Lambda_H > 0$

We now deal with the cases where the Hénon family is in a strange attractor. The main mathematical contents of this paper are formulated in Theorem 4 (in Section 2.2), which deals with the scaled skew product model (4). Here we establish the existence of Hénon-like strange attractors, which completely covers case 2(a) of Section 1.1 in suitable parameter ranges. For completeness, in Section 2.2 technical definitions of 'strange', 'Hénon-like', etc., are included; compare with [14, 26, 39]. The proof of Theorem 4 is based on [14, Theorem 5.2].

Finally, in Section 2.3 we deal with the remaining case, both for the skew product system (4) and for the more general system (6). Here we touch upon the existence of quasi-periodic Hénon-like attractors. Lemma 7 implies that in the product case $\varepsilon = 0$ a topologically transitive attractor occurs, having a dense set of orbits with a positive Lyapunov exponent. Regarding its persistence for $|\varepsilon| \ll 1$, our conjectures and mostly computer suggested.

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2 Statement of the results

As announced before, our results are formulated in the next subsections, while proofs are given in Sections 3.1 and 3.2.

2.1 Invariant circles of saddle-type and basins of attraction

We first consider maps F of the solid torus $\mathbb{R}^2 \times \mathbb{S}^1$ obtained by perturbing the product of a planar map times a rotation on $\mathbb{S}^1$. Assuming that the planar map has a saddle fixed point with a transversal homoclinic point, it is proved that the map F has an attractor contained inside $\mathrm{Cl}\,(W^u(\mathscr{C}))$.

To this end we generalise an unpublished result of Tangerman and Szewc, see [31, Appendix 3], where we are in the general context of the family $P_{\alpha,\varepsilon} : \mathbb{R}^2 \times \mathbb{S}^1 \to \mathbb{R}^2 \times \mathbb{S}^1$, see (6).

Proposition 1. (NORMALLY HYPERBOLIC INVARIANT CIRCLE) *Suppose that K has a hyperbolic fixed point $p = (x_0, y_0)$. Then for all $\alpha \in [0, 1]$ the map $P_{\alpha,0}$ has a normally hyperbolic invariant circle $\mathscr{C}_{\alpha,0} = \{p\} \times \mathbb{S}^1$. The manifold $\mathscr{C}_{\alpha,0}$ is r-normally hyperbolic for all integers r with $1 \le r \le n$. Moreover, for all $r < n$ there exists an $\varepsilon_r > 0$ such that for all $\varepsilon < \varepsilon_r$ and all $\alpha \in [0, 1]$, $P_{\alpha,\varepsilon}$ has a normally hyperbolic invariant circle $\mathscr{C}_{\alpha,\varepsilon}$ of class C^r, which is C^r-close to $\mathscr{C}_{\alpha,0}$.*

Proof: The dynamics of $P_{\alpha,0}$ on $\mathscr{C}_{\alpha,0}$ is parallel with rotation number α. This implies that $\mathscr{C}_{\alpha,0}$ is an r-normally hyperbolic invariant manifold for all $r \le n$ and, therefore, it is of class C^n. So $\mathscr{C}_{\alpha,0}$ (as well as its stable and unstable manifolds), is persistent under C^n-small perturbations. This directly follows from [20, Theorem 1.1]. □

Proposition 1 allows us to construct a basin of attraction with nonempty interior for the invariant set $\mathrm{Cl}\,(W^u(\mathscr{C}_{\alpha,\varepsilon}))$, provided that p is a saddle point, while the one-dimensional unstable manifold $W^u(p) \subset \mathbb{R}^2$ of the map K, see (5), does not escape to infinity. For $(x, y, \theta) \in \mathbb{R}^2 \times \mathbb{S}^1$, denote by $\omega(x, y, \theta)$ the ω-limit set of (x, y, θ) under $P_{\alpha,\varepsilon}$.

Theorem 2. (ATTRACTOR CONTAINED IN $\mathrm{Cl}\,(W^u(\mathscr{C}))$) *Fix integers n and r such that $n \ge 2$ and $1 \le r < n$. Choose $\varepsilon < \varepsilon_r$ as in Proposition 1 and let $\alpha \in [0, 1]$. Suppose that $K : \mathbb{R}^2 \to \mathbb{R}^2$ is of class C^n and satisfies:*

1. *K has a saddle fixed point $p \in \mathbb{R}^2$ and a transversal homoclinic point $q \in W^s(p) \cap W^u(p)$.*
2. *K is uniformly dissipative: there exists $\kappa < 1$ such that $|\det(DK(x, y))| \le \kappa$ for all $(x, y) \in \mathbb{R}^2$.*
3. *$W^u(p)$ is contained in a bounded subset of $\mathbb{R}^2$.*

Then there exists an $\varepsilon^* < \varepsilon_r$ such that for all $\varepsilon < \varepsilon^*$ there exists an open, nonempty bounded set $U \subset \mathbb{R}^2 \times \mathbb{S}^1$ such that for all $(x, y, \theta) \in U$

$$\omega(x, y, \theta) \subset \mathrm{Cl}\,(W^u(\mathscr{C}_{\alpha,\varepsilon})) \,. \tag{8}$$

Remark 3. By taking iterates of the map $P_{\alpha,\varepsilon}$, Theorem 2 can be adapted to the case where p is a saddle periodic point. In this context we have the inclusion (8), where $\mathscr{C}_{\alpha,\varepsilon}$ is a periodically invariant circle, *i.e.* a circle which is invariant under some iterate of $P_{\alpha,\varepsilon}$.

Under the conditions of Theorem 2, the invariant set $\mathrm{Cl}\,(W^u(\mathscr{C}_{\alpha,\varepsilon}))$ attracts all orbits with initial state in an open set U. This holds for an open set of ε-values. In general, however, $\mathrm{Cl}\,(W^u(\mathscr{C}_{\alpha,\varepsilon}))$ is not an attractor, since it might be non-topologically transitive. This occurs, for example, if $\mathrm{Cl}\,(W^u(\mathscr{C}_{\alpha,\varepsilon}))$ contains a periodic attractor.

In the next Theorem we prove that at least the circle $\mathscr{C}_{\alpha,\varepsilon}$ is quasi-periodic (and, hence, topologically transitive) for a set of parameter values having large relative measure.

Theorem 3. (NORMALLY HYPERBOLIC QUASI-PERIODIC CIRCLES) *Let $P_{\alpha,\varepsilon}$ be a C^n-family of diffeomorphisms as in* (6), *where $n \geq 5$. Choose ε_r as in Proposition* 1. *Then there exists an $\varepsilon^{**} < \varepsilon_r$ such that for all $\varepsilon < \varepsilon^{**}$ the following holds.*

1. *There exists a set $D_\varepsilon \subset [0, 1]$ with Lebesgue measure $\mathrm{meas}(D_\varepsilon) > 0$ such that for $\alpha \in D_\varepsilon$ the restriction of $P_{\alpha,\varepsilon}$ to the circle $\mathscr{C}_{\alpha,\varepsilon}$ is smoothly conjugate to an irrational rigid rotation.*

2. $\mathrm{meas}(D_\varepsilon)$ *tends to* 1 *for $\varepsilon \to 0$.*

Proofs of Theorems 2 and 3 are given in Section 3.1.

Theorem 3, as happens with Theorem 2, has a direct analogue for the case where p is a hyperbolic periodic point. The Theorems will be applied both for the case where p is a periodic saddle and a periodic sink. In the latter case we prove the existence of quasi-periodic attractors for positive measure in parameter space, as described in the case 2(b) of Section 1.1. Again see Figure 1 (C). This situation corresponds to points with code 2, in blue.

A complementary situation regarding Theorem 3 occurs when the dynamics on $\mathscr{C}_{\alpha,\varepsilon}$ is of Morse-Smale type, compare with the resonance tongues of Figure 1 (A). In that case, for $\Lambda_H < 0$, the attracting set $\mathrm{Cl}\,(W^u(\mathscr{C}_{\alpha,\varepsilon}))$ of system (6) contains a hyperbolic periodic attractor as described in case 1(a) of Section 1.1. Again compare with Figure 1 (C), points with code 1, in yellow. In the next section, for the skew product system (4) we show that in the case where $\Lambda_H > 0$, the set $\mathrm{Cl}\,(W^u(\mathscr{C}_{\alpha,\varepsilon}))$ contains a Hénon-like attractor, which covers the case 2(a) of Section 1.1, compare with Figure 1 (C), points with code 3, in red.

2.2 Hénon-like attractors in a family of skew product maps

By Theorem 2, the set $\mathrm{Cl}\,(W^u(\mathscr{C}))$ is attracting under quite general circumstances. As may be clear from the previous paragraph, in general $\mathrm{Cl}\,(W^u(\mathscr{C}))$ does not have to be topologically transitive, in which case it is not considered an attractor. (For precise definitions see below.) However, in the particular case of map (2), we show that $\mathrm{Cl}\,(W^u(\mathscr{C}))$ contains Hénon-like attractors. We first recall a few basic definitions regarding strange attractors, suited to our purposes.

Definition 1. **[14, 26, 39]** *Let $F : M \to M$ be a C^1-diffeomorphism, where M is an m-dimensional smooth manifold.*

1. *An F-invariant set $\mathscr{A} \subset M$ is called* topologically transitive *if there exists a point $z \in \mathscr{A}$ such that the orbit* $\mathrm{Orb}(z) = \{F^j(z)\}_{j \geq 0}$ *of z under F is dense in $\mathscr{A}$.*

2. *A set $\mathscr{A} \subset M$ is called an* attractor *if it is topologically transitive, compact, F-invariant and if the stable set (basin of attraction) $W^s(\mathscr{A})$ has nonempty interior.*

3. *An attractor $\mathscr{A}$ is called* strange *if there exist constants $\kappa > 0$, $\lambda > 1$, a dense orbit* $\mathrm{Orb}(z) \subset \mathscr{A}$ *and a vector $v \in T_z M$ such that*

$$\|DF^n(z)v\| \geq \kappa\lambda^n \qquad \textit{for } n \geq 0.$$

4. *The attractor $\mathscr{A}$ is called* Hénon-like *if there exist a saddle periodic orbit* $\mathrm{Orb}(p) = \{p, F(p), \ldots, F^n(p)\}$*, a point z in the unstable manifold* $W^u(\mathrm{Orb}(p))$*, constants $\kappa > 0$, $\lambda > 1$, and tangent vectors $v, w \in T_z M$, with $w \neq 0$, such that*

$$\begin{aligned} &i) && \mathscr{A} = \mathrm{Cl}\left(W^u(\mathrm{Orb}(p))\right), && (9)\\ &ii) && \mathrm{Orb}(z) \textit{ is dense in } \mathscr{A}, && (10)\\ &iii) && \|DF^n(z)v\| \geq \kappa\lambda^n \qquad \textit{for } n \geq 0, && (11)\\ &iv) && \|DF^n(z)w\| \to 0 \qquad \textit{as } n \to \pm\infty, && (12)\end{aligned}$$

where $\mathrm{Cl}(\cdot)$ *denotes topological closure.*

In particular, Hénon-like attractors are strange, since by conditions (10) and (11) they admit a dense orbit with a positive Lyapunov exponent. Moreover, Hénon-like attractors are *non-uniformly hyperbolic*; indeed, by condition (12) they contain *critical points*, that is, points belonging to a dense orbit for which a nonzero tangent vector w exists, which is contracted both by positive and by negative iteration of the derivative DF.

We now come to the main result of the present paper, regarding the occurrence of Hénon-like strange attractors in the scaled skew product family (4). First we recall that the restriction of (4) to $\mathbb{S}^1$ is the Arnol′d family of circle maps (3). Moreover, the map (4) is a generalisation of the planar Hénon-like families considered in [26, 39]. The latter are families of planar diffeomorphisms, which are C^3-small perturbations of the Logistic family

$$Q_a : \mathbb{R} \to \mathbb{R}, \qquad x \mapsto 1 - ax^2. \tag{13}$$

The x- and y-components of $T_{\alpha,\delta,a,\varepsilon}$ also depend on the circle dynamics by the perturbative terms f and g. The only requirement on f and g is that their C^3-norms are bounded on compact sets. Occurrence of Hénon-like attractors is proved in the family $T_{\alpha,\delta,a,\varepsilon}$ for all parameter values belonging to a set of positive (Lebesgue) measure. For all values in this set, the parameters (α, δ) are such that the dynamics of the Arnol′d family $A_{\alpha,\delta}$ (3) is of Morse-Smale type: there exist periodic points θ^s and θ^r in $\mathbb{S}^1$, such that θ^s is attracting and θ^r repelling for $A_{\alpha,\delta}$. By $\mathfrak{A}^{q/n}$ we denote the open resonance tongue in the (α, δ)-plane where these periodic points have rotation number q/n [1, 13] and the width of the tongue in α behaves as δ^n [8], compare with Figure 1 (A). The parameter space under consideration is the set of all $(\alpha, \delta, a, \varepsilon) \in \mathbb{R}^4$ such that

$$\alpha \in [0, 1],\ \delta \in \big[0, 1/(2\pi)\big),\ a \in [0, 2],\ |\varepsilon| < 1. \tag{14}$$

The attractors $\mathscr{A}$ we obtain, coincide with the closure of the one-dimensional unstable manifold

$$\mathscr{A} = \mathrm{Cl}\left(W^u\left(\mathrm{Orb}(p)\right)\right),$$

where $p = (x_0, y_0, \theta^s) \in \mathbb{R}^2 \times \mathbb{S}^1$ belongs to a hyperbolic periodic orbit of saddle type. For the statement of the result we need a few definitions and notations.

Definition 2. *1. A map $M : J \to J$, where $J \subset \mathbb{R}$ is an interval, is called* topologically mixing *if for any open intervals J_1, $J_2 \subset J$ there exists n_0 such that*

$$M^n(J_1) \cap J_2 \neq \emptyset \quad \textit{for all } n \geq n_0.$$

2. The interval $K_a = [Q_a^2(0), Q_a(0)]$ is called the core *or the* restrictive interval *of the Logistic family Q_a (13).*

It is well-known that $Q_a([0,1]) = Q_a(K_a) = K_a$ for all a, where K_a is the core of Q_a (13), see *e.g.* [24, Section II.5]. For a given integer $n > 1$, denote by $\Phi(n)$ the set of all integers q such that q and n are relatively prime, where $1 \leq q < n$. For $n = 1$ we put $\Phi(n) = \{1\}$.

Theorem 4. (Hénon-like attractors in (4)) *Choose $a^* \in (1,2)$ such that the quadratic map Q_{a^*} in (13) is topologically mixing on its core $K = [1 - a^*, 1]$ and its critical point $c = 0$ is preperiodic. Let $n \geq 1$ be an integer. There exist a periodic point p_0 of the n-th iterate $Q_{a^*}^n$ and positive constants $\bar{\varepsilon}_n$, $\bar{a}_n$ and χ_n such that the following holds.*

1. *For all $(\alpha, \delta, a, \varepsilon)$ as in (14), with*

$$(\alpha, \delta) \in \cup_{q \in \Phi(n)} \mathrm{Cl}\left(\mathfrak{A}^{q/n}\right), \quad |a - a^*| < \bar{a}_n, \quad |\varepsilon| < \bar{\varepsilon}_n \tag{15}$$

the map $T_{\alpha,\delta,a,\varepsilon}$ has a saddle periodic point p, which is the analytic continuation of p_0 and such that the unstable manifold $W^u(\mathrm{Orb}(p))$ is one-dimensional.

2. *For all $(\alpha, \delta, \varepsilon)$ as in (15) there exists a set $\mathfrak{S}_{\alpha,\delta,\varepsilon}$ with*

$$\mathfrak{S}_{\alpha,\delta,\varepsilon} \subset [a^* - \bar{a}_n, a^* + \bar{a}_n], \qquad \mathrm{meas}(\mathfrak{S}) > \chi_n$$

such that for all $a \in \mathfrak{S}_{\alpha,\delta,\varepsilon}$ the closure $\mathrm{Cl}\,(W^u(\mathrm{Orb}(p)))$ is a Hénon-like attractor of $T_{\alpha,\delta,a,\varepsilon}$.

Corollary 5. *The set of parameter values for which $T_{\alpha,\delta,a,\varepsilon}$ has a Hénon-like attractor contains the set*

$$\mathfrak{S} = \bigcup_{n \in \mathbb{N}} \left\{(\alpha, \delta, a, \varepsilon) \;\middle|\; (\alpha, \delta) \in \cup_{q \in \Phi(n)} \mathrm{Cl}\left(\mathfrak{A}^{q/n}\right), \quad |\varepsilon| < \bar{\varepsilon}_n, \quad a \in \mathfrak{S}_{\alpha,\delta,\varepsilon}\right\},$$

and the set $\mathfrak{S}$ has positive Lebesgue measure

$$\mathrm{meas}(\mathfrak{S}) \geq 2 \sum_{n=1}^{\infty} \bar{\varepsilon}_n \chi_n \sum_{q \in \Phi(n)} \mathrm{meas}\, \mathfrak{A}^{q/n}.$$

Our proof of Theorem 4 is given in Section 3.2. It is based on a result of Díaz-Rocha-Viana [14, Theorem 5.2], and relies on the following facts:

1. For (α, δ) inside any tongue $\mathfrak{A}^{q/n}$, the asymptotic dynamics of $T_{\alpha,\delta,a,\varepsilon}$ is described by an $\mathcal{O}(\varepsilon)$-perturbation of the n-th iterate Q_a^n.
2. For all n the map Q_a^n is a generic n-modal family, in the sense of [14, Section 5.2], also see the definition given in Section 3.2. To show this, we use that Q_{a^*} is a Misiurewicz map [25], and, therefore, it is Collet-Eckmann (see *e.g.* [24, Section V.4]). See Section 3.2 for details.

Notice that the family (2) takes the form (4) after a rescaling $y \mapsto \sqrt{|b|}y$ and by choosing $b = \mathcal{O}(\varepsilon)$. Therefore, by restricting the parameter δ to sufficiently small values, both Theorem 4 and Theorem 2 may be applied to (2).

Corollary 6. *Let a^* and p_0 satisfy the hypotheses of Theorem 4. Then there exists a positive measure set of parameter values such that the family (2) has Hénon-like attractors, contained in the closure of the unstable manifold of a periodically invariant circle.*

Proof: Take a^* and p_0 as in the hypotheses of Theorem 4. Then for all δ and for ε and b sufficiently small, Theorem 4 applies. Moreover, for $(\varepsilon, \delta) = (0, 0)$ the circle $\mathscr{C} = \{p_0\} \times \mathbb{S}^1$ is periodically invariant under map (2). In particular, the conditions of Theorem 2 are satisfied for b sufficiently small, since:

1. The periodic point $(p_0, 0)$ of $H_{a,0}$ has an analytic continuation $\bar{p}(b)$ for all b sufficiently small, and p_0 is chosen such that $\bar{p}(b)$ has transversal homoclinic points, see Proposition 12.
2. $\det(DH_{a,b}(x, y)) = b$;
3. The unstable manifold of all periodic points of $H_{a,b}$ is bounded for b sufficiently small, since the invariant manifolds depend continuously on the map [26, Prop. 7.1].

So for (ε, δ, b) sufficiently small, the conclusions of Theorem 2 hold. □

Two attractors occurring in the family (2) are shown in Figure 4 (A) and (B), for (α, δ) in an Arnol′d resonance tongue of period two and three, respectively. Also compare with Figure 1 (C), points with code 3, in red. It is to be noted that the Hénon-like character of these attractors for larger values of b and ε remains conjectural.

2.3 Quasi-periodic Hénon-like attractors

We start with a further setting of the problems regarding quasi-periodic Hénon-like attractors, in the skew product model family (2).

2.3.1 Further setting of the problem

The present paper has been partially motivated by the problem to find a diffeomorphism F with a strange attractor $\mathscr{A}$ such that

$$\mathscr{A} = \mathrm{Cl}\,(W^u(\mathscr{C}))\,, \tag{16}$$

where $\mathscr{C}$ is an F-invariant circle of saddle type with irrational rotation number, so with quasi-periodic dynamics. In this context, the role of the saddle periodic orbit in (9) is played by a quasi-periodic invariant circle of saddle type. By analogy with the definition of Hénon-like strange attractors (see Section 2.2), we are led to the following definition.

Definition 3. *Let $F : M \to M$ be a C^1-diffeomorphism, where M is an m-dimensional smooth manifold. We say that the attractor $\mathscr{A}$ is* quasi-periodic Hénon-like *if there exist*

1. *A quasi-periodic invariant circle $\mathscr{C}$ of saddle type such that $\mathscr{A} = \mathrm{Cl}\,(W^u(\mathscr{C}))$.*
2. *A point $x \in \mathscr{A}$ such that $\mathrm{Orb}(x)$ is dense in $\mathscr{A}$ and*
3. *a dense set $Z \subset \mathscr{A}$ and constants $\kappa > 0$, $\lambda > 1$ such that for all $z \in Z$ there exist vectors $v, w \in T_zM$ such that conditions (11) and (12) hold.*

The definition mimics the positive Lyapunov exponents and non-uniform hyperbolicity requirements in the definition of Hénon-like attractors and also asks for transitivity. As usual similar definitions can be given with F replaced by a power F^k.

Returning to the skew product context of the model family (2), in the Arnol′d family $A_{\alpha,\delta}$ we fix parameter values (α, δ) such that the dynamics of $A_{\alpha,\delta}$ is quasi-periodic. Recall that the set of all such (α, δ) has positive measure and is nowhere dense [5, Chap. 1]. Next choose parameter values a and b such that the Hénon map (1) has a Hénon-like strange attractor $\mathscr{A}'$, coinciding with the closure of the unstable manifold of a saddle fixed point p. Also recall that, according to [2, 3, 26], such (a, b) form a set of positive measure. Then, at $\varepsilon = 0$ the map (2) has an attractor $\mathscr{A} = \mathscr{A}' \times \mathbb{S}^1$ coinciding with the closure of the unstable manifold of the quasi-periodic saddle-type invariant circle $\{p\} \times \mathbb{S}^1$. It may be clear that requirement 3 of Definition 3 is satisfied by taking $Z = \mathrm{Orb}(z) \times \mathbb{S}^1$, where z is a point satisfying properties 4 *ii), iii)* and *iv)* in Definition 1 of Hénon-like attractors.

Next, to prove that in the product case we obtain a quasi-periodic Hénon-like attractors only item 2 in Definition 3 has to be verified. This is done in the following lemma.

Lemma 7. (Transitivity of (2), uncoupled) *Let T be a dissipative C^1-diffeomorphism in an open subset $\mathcal{U} \subset \mathbb{R}^2$ such that*

1. *T has a hyperbolic fixed point p of saddle-type.*

2. *The closure of the unstable manifold of p is an Hénon-like strange attractor $\mathscr{A}'$.*

Let $R_\alpha : x \mapsto x + \alpha \mod 1$ a rotation over angle $\alpha \in (0, 1) \setminus \mathbb{Q}$. Then the product $F = T \times R$ has a dense orbit in $\mathscr{A} = \mathscr{A}' \times \mathbb{S}^1$.

Proof: We claim that it is sufficient to prove the following:

$(*)$ Given two open sets U, V in $\mathscr{A}$, there exists $k \in \mathbb{N}$ such that $F^k(U) \cap V \neq \emptyset$.

Indeed, given $\varepsilon > 0$ there is a finite number of open sets $V_j, j \in J$, of the form $V_j = V_j' \times (s_j - \varepsilon, s_j + \varepsilon)$ that cover $\mathscr{A}$, where $V_j' \subset \mathbb{R}^2$ is an open ball of radius ε. Let $U_0 = U$. Assuming $(*)$, it follows that $F^{k_1}(U_0)$ intersects V_1 for some $k_1 \in \mathbb{N}$. Define U_1 as the image under F^{-k_1} of this intersection. This process can be repeated for all $j \in J$. After this we restart the whole process with ε replaced by $\varepsilon/2, \varepsilon/4, \varepsilon/8, \ldots, \varepsilon/2^m, \ldots$, each time obtaining an open set U_m such that $U_m \subset U_{m+1}$. The intersection $\cap_{m\in\mathbb{N}} U_m$ gives an initial point for a dense orbit as desired.

Next, let us prove $(*)$. Without loss of generality, assume that $U = U' \times (r - \delta, r + \delta)$ and $V = V' \times (s - \varepsilon, s + \varepsilon)$ for some $\delta, \varepsilon > 0$, where U', V' are open sets in $\mathscr{A}'$. First, for fixed $\varepsilon > 0$ we note that given $r, s \in \mathbb{S}^1$ there exists an increasing sequence $\{n_1, n_2, \ldots\}$ such that $R_\alpha^{n_j}(r) \in (s - \varepsilon, s + \varepsilon)$, where $0 < n_1 < N$ and $n_{j+1} - n_j < N$ for all j, with N independent of r and s. As $W^u(p)$ is dense in $\mathscr{A}'$, there exists a point $q' \in W^u(p) \cap V'$. Consider a preimage $u = T^{-l}(q')$ such that u and its first N iterates are close to p. By continuity, there are open sets $Z_0, Z_1, \ldots, Z_N$ around $u, T(u), \ldots, T^N(u)$ whose images under $T^l, T^{l-1}, \ldots, T^{l-N}$ are contained in V'.

Now, there exists a point $x \in U' \cap W^u(p)$ belonging to a dense orbit and also having a positive Lyapunov exponent, such that $T^m(x) \in Z_0$ for some $m \in \mathbb{N}$. It is no restriction to assume that, for some $m \in \mathbb{N}$, the image $T^m(U')$ intersects all $Z_j, j = 0, 1, 2, \ldots, N$. Indeed, in the other case the Lyapunov exponent could not be positive.

Since $T^{l-j}(Z_j \cap T^m(U')) \subset V'$ for $j = 0, \ldots, N$, one has

$$T^{l+m-j}(U') \cap V' \neq \emptyset \tag{17}$$

for all $j = 0, \ldots, N$. To arrange that some of the iterates $R_\alpha^{l+m-j}(r)$ lie inside the interval $(s - \varepsilon, s + \varepsilon)$, observe that $l + m$ is in between two consecutive values n_i and n_{i+1} for some n_i as above. This implies that there exists j with $\leq j \leq N$ such that $l + m - j = n_i$, which, together with (17), yields that $T^{l+m-j}(U) \cap V \neq \emptyset$. $\square$

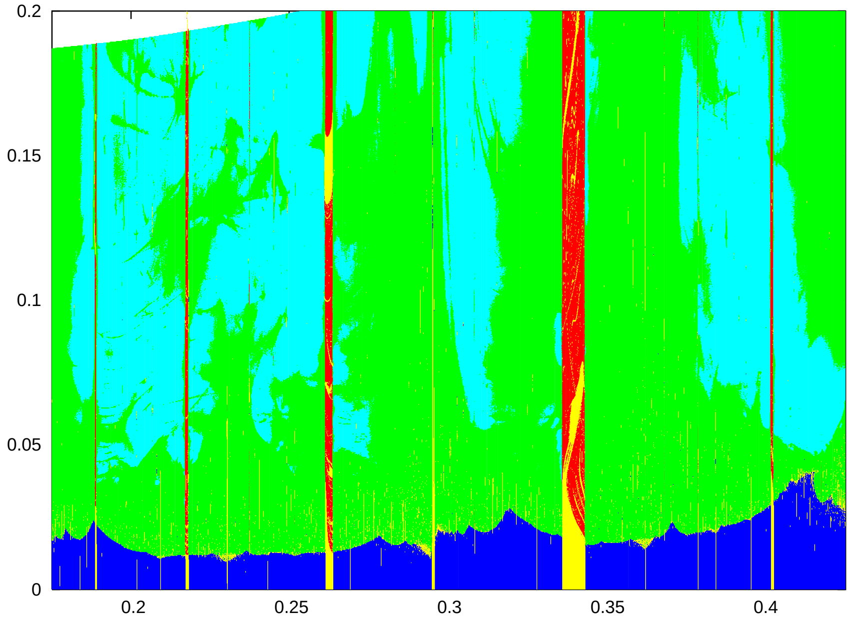

Figure 6: Diagram of the fully coupled system (7) in the (α, ε)-plane, for $a = 1.25, b = 0.3, \mu = 0.01$ and $\delta = 0.6/(2\pi)$. According to the values of the Lyapunov exponents, we interpret as follows: domains of periodic attractors (code 1, yellow), of quasi-periodic attractors (code 2, blue), of Hénon-like attractors (code 3, red) and of 'quasi-periodic Hénon-like' attractors (codes 4, light blue, and 5, green). For more details see Section 2.3 and Section 4.

2.3.2 Conjectural results

Numerical experiments with the map (2) suggest that attractors like $\mathscr{A}$ persist for small (and perhaps, not so small) values of (ε, δ). See figures 3 and 4. Figure 1 (C) gives evidence that quasi-periodic Hénon-like attractors occur in a relatively large part of the parameter plane (code 4, light blue). In this numerical context, quasi-periodic Hénon-like are indicated by the fact that one positive, one negative, and one zero Lyapunov exponent is detected. We assume that the Lyapunov exponents are ordered as

$$\Lambda_1 \geq \Lambda_2 \geq \Lambda_3.$$

A remarkable difference between the skew-product case (2) and the fully coupled case (7) is that a zero Lyapunov exponent practically never occurs, compare Figure 6 and see Section 4.2. Indeed, any small perturbation $\mu \neq 0$ has the effect of shifting the value of Λ_2 away from zero. Its modulus remains small, but the sign may be either positive or negative, where both cases occur. Plots of the attractors visually look the same in the three cases $\Lambda_2 = 0$, $\Lambda_2 < 0$ and $\Lambda_2 > 0$, when μ is small and the other parameters are kept fixed. It remains to be clarified which dynamical and geometrical properties characterise the strange attractors in each of the three cases. In any case, it seems that the way in which the invariant manifolds of an invariant circle are folded, plays an essential role.

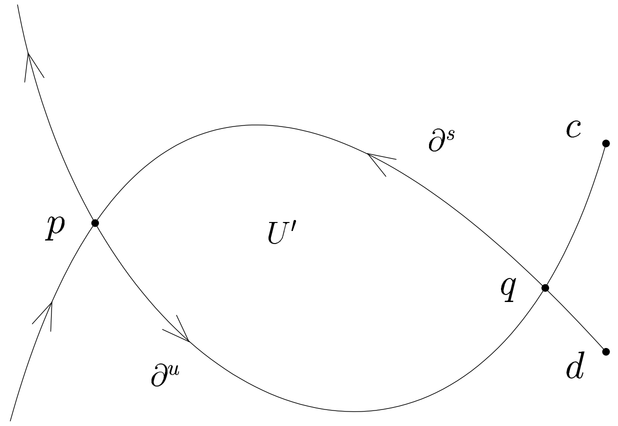

Figure 7: Segments ∂^s and ∂^u of the stable and unstable manifold, respectively, of a saddle fixed point p bound a region U, see text for more explanation.

Remark 4.

1. We used the family (7) for the figures, expecting that it is sufficiently rich for our purposes.
2. When comparing the Figures 1 (C) and 6, the main difference is that in the second case there is no significant occurrence of $\Lambda_2 = 0$. Still we expect that in all cases the attractor is the closure $\mathrm{Cl}\,(W^u(\mathscr{C}))$ of a quasi-periodic invariant circle $\mathscr{C}$ of saddle-type.
3. It seems that in the skew case (2), the phenomenon of an attractor with $\Lambda_1 = 0$ and $\Lambda_2 < 0$, which is not an invariant circle is somewhat related to 'nonchaotic strange attractors', compare with [28, 15, 17, 22, 29]. See also Section 4.3 for further discussion on this topic.

   Interestingly, tiny perturbations away from the skew case seemingly give rise to a quasi-periodic Hénon-like attractor, so with $\Lambda_1 > 0$.

3 Proofs

3.1 Basins of attraction and quasi-periodic invariant circles

In this section we give proofs of Theorem 2 (next section) and Theorem 3 (Section 3.1.2).

3.1.1 The Tangerman-Szewc argument generalised

Let $K : \mathbb{R}^2 \to \mathbb{R}^2$ be a dissipative diffeomorphism having a saddle fixed point $p = (x_0, y_0)$. Suppose the stable and unstable manifolds $W^s(p)$ and $W^u(p)$ intersect transversally at the homoclinic point $q \in W^s(p) \cap W^u(p)$, see Figure 7. Also assume that $W^u(p)$ is bounded as a subset of $\mathbb{R}^2$. The Tangerman-Szewc Theorem (see *e.g.* [31, Appendix 3]) states that the basin of attraction of the closure of $W^u(p)$ contains the open region U' bounded by the two arcs $\partial^s \subset W^s(p)$ and $\partial^u \subset W^u(p)$ with extremes p and q, see Figure 7. This argument is used to prove existence of strange attractors (in particular, with non-trivial basin of attraction) near homoclinic tangencies of a saddle fixed point of a dissipative diffeomorphism, cf. [26, 39, 42].

We first prove Theorem 2 for $\varepsilon = 0$. This is a straightforward generalisation of the above Tangerman-Szewc Theorem. For small ε, the result is obtained by using persistence of normally hyperbolic invariant manifolds [20, Theorem 1.1] and two transversality lemmas.

Proof of Theorem 2. Consider the circle $\mathscr{C}_\alpha = \mathscr{C}_{\alpha,0}$, invariant under map $P_{\alpha,0}$ in (6). The manifolds $W^u(\mathscr{C}_\alpha)$ and $W^s(\mathscr{C}_\alpha)$ are given by $W^u(p) \times \mathbb{S}^1$ and $W^s(p) \times \mathbb{S}^1$, respectively. They intersect transversally at a circle $\mathscr{H} = \{q\} \times \mathbb{S}^1$, consisting of points homoclinic to $\mathscr{C}_\alpha$. Consider the two arcs $\partial^s \subset W^s(p)$ and $\partial^u \subset W^u(p)$ with extremes p and q (Figure 7).They bound an open set $U' \subset \mathbb{R}^2$. Define D^s and D^u to be the portions of stable, and unstable manifold of $\mathscr{C}_\alpha$, respectively, given by

$$D^s = \partial^s \times \mathbb{S}^1 \subset W^s(\mathscr{C}_\alpha) \qquad \text{and} \qquad D^u = \partial^u \times \mathbb{S}^1 \subset W^u(\mathscr{C}_\alpha).$$

Both surfaces D^s and D^u are compact, and their union forms the boundary of the open region $U = U' \times \mathbb{S}^1$, which is topologically a solid torus.

The volume of U decreases under iteration of $P_{\alpha,0}$. Denoting by meas$(\cdot)$ the Lebesgue measure both on $\mathbb{R}^2$ and on $\mathbb{R}^2 \times \mathbb{S}^1$, due to condition 2 in Theorem 2 we have

$$\operatorname{meas}(P^n_{\alpha,0}(U)) = 2\pi \int_{K^n(U')} dxdy = 2\pi \int_{U'} |\det DK^n|\, dxdy \le 2\pi\kappa^n \operatorname{meas}(U').$$

This implies that the forward evolution of every point $(x, y, \theta) \in U$ approaches the boundary of $P^n_{\alpha,0}(U)$:

$$\operatorname{dist}\big(P^n_{\alpha,0}(x, y, \theta), \partial P^n_{\alpha,0}(U)\big) \to 0 \quad \text{as } n \to +\infty.$$

Indeed, suppose that this does not hold. Then there exists a $\varrho > 0$ such that for all n there exists $N > n$ such that the ball with centre $P^N_{\alpha,0}(x, y, \theta)$ and radius $\varrho > 0$ is contained inside $P^N_{\alpha,0}(U)$. But this would contradict the fact that meas$(P^n_{\alpha,0}(U)) \to 0$ as $n \to +\infty$.

The boundary of $P^n_{\alpha,0}(U)$ also consists of two portions of stable and unstable manifold of $\mathscr{C}$:

$$\partial P^n_{\alpha,0}(U) = P^n_{\alpha,0}(D^s) \cup P^n_{\alpha,0}(D^u).$$

The diameter of $P^n_{\alpha,0}(D^s)$ tends to zero as $n \to +\infty$, because all points in D^s are attracted to the circle $\mathscr{C}_\alpha$. Since $W^u(\mathscr{C}_\alpha)$ is bounded, all evolutions starting in U are bounded and approach $W^u(\mathscr{C}_\alpha)$, that is,

$$\operatorname{dist}(P^n_{\alpha,0}(x, y, \theta), P^n_{\alpha,0}(D^u)) \to 0 \quad \text{as } n \to +\infty$$

for all $(x, y, \theta) \in U$. This implies that $\omega(x, y, \theta) \subset \operatorname{Cl}(W^u(\mathscr{C}_\alpha))$ for all $(x, y, \theta) \in U$.

To extend this result to small perturbations $P_{\alpha,\varepsilon}$ of $P_{\alpha,0}$, the following transversality lemmas are used.

Lemma 8. **[33, Chap. 7]** *Consider a map $f : V \to M$, where V and M are C^r-differentiable manifolds and f is C^r. Suppose V is compact, $W \subset M$ is a closed C^r-submanifold and f is transversal to W at V (notation: $f \pitchfork W$). Then $f^{-1}(W)$ is a C^r-submanifold of codimension* $\operatorname{codim}_V(f^{-1}(W)) = \operatorname{codim}_M(W)$. *Further suppose that there is a neighbourhood of $f(\partial_V) \cup \partial_W$ disjoint from $f(V) \cap W$, where ∂_V and ∂_W are the boundaries of V and W. Then any map $g : V \to M$, sufficiently C^r-close to f, is also transversal to W, and the two submanifolds $g^{-1}(W)$ and $f^{-1}(W)$ are diffeomorphic.*

Lemma 9. **[19, Section 3.2]** *Let V_1, V_2, and M be C^r-differentiable manifolds and consider two diffeomorphisms $f_i : V_i \to M$, $i = 1, 2$. Then $f_1 \pitchfork f_2$ if and only if $f_1 \times f_2 \pitchfork \Delta$, where $f_1 \times f_2 : V_1 \times V_2 \to M \times M$ is the product map and $\Delta \subset M \times M$ is the diagonal: $\Delta = \{(y, y) \mid y \in M\}$.*

Fix $r \in \mathbb{N}$ and take $\varepsilon < \varepsilon_r$, where ε_r is given in Proposition 1. Then the map $P_{\alpha,\varepsilon}$ has an r-normally hyperbolic invariant circle $\mathscr{C}_{\alpha,\varepsilon}$ of saddle type. Furthermore, the manifolds $W^u(\mathscr{C}_{\alpha,\varepsilon})$, $W^s(\mathscr{C}_{\alpha,\varepsilon})$, and $\mathscr{C}_{\alpha,\varepsilon}$ are C^r-close to $W^u(\mathscr{C}_\alpha)$, $W^s(\mathscr{C}_\alpha)$, and $\mathscr{C}_\alpha$. We now show that the two manifolds $W^u(\mathscr{C}_{\alpha,\varepsilon})$, $W^s(\mathscr{C}_{\alpha,\varepsilon})$ still intersect transversally. To apply Lemma 8 we restrict to two suitable compact subsets $A^u \subset W^u(\mathscr{C}_\alpha)$ and $A^s \subset W^s(\mathscr{C}_\alpha)$ as follows. Consider the segments $\overline{pc} \subset W^u(p)$ and $\overline{pd} \subset W^s(p)$ in Figure 7. Define

$$A^u = \overline{pc} \times \mathbb{S}^1, \qquad A^s = \overline{pd} \times \mathbb{S}^1.$$

In this way, the circle $\mathscr{H}$ is the intersection of the manifolds A^u and A^s, bounded away from their boundaries. Consider the inclusions $i : A^u \to M$ and $j : A^s \to M$. By the closeness of $W^u(\mathscr{C}_\alpha)$ to $W^u(\mathscr{C}_{\alpha,\varepsilon})$ there exists a C^r-diffeomorphism $h : A^u \to A^u_\varepsilon \subset W^u(\mathscr{C}_{\alpha,\varepsilon})$ such that the map i is C^r-close to $i_\varepsilon \circ h$, where $i_\varepsilon : A^u_\varepsilon \to M$ is the inclusion [30, Section 2.6]. Similarly, there exists a diffeomorphism $k : A^s \to A^s_\varepsilon \subset W^s(\mathscr{C}_{\alpha,\varepsilon})$ such that the map j is C^r-close $j_\varepsilon \circ k$, where $j_\varepsilon : A^s_\varepsilon \to M$ is the inclusion. By Lemma 9 the map $i \times j : A^u \times A^s \to M \times M$ is transversal to the diagonal Δ. For ε small, the map $(i_\varepsilon \circ h) \times (j_\varepsilon \circ k) : A^u \times A^s \to M \times M$ is C^r-close to $i \times j$:

$$\begin{array}{ccc} A^u \times A^s & \xrightarrow{i \times j} & M \times M \\ {\scriptstyle h\times k}\downarrow & & \\ A^u_\varepsilon \times A^s_\varepsilon & \xrightarrow{i_\varepsilon \times j_\varepsilon} & M \times M. \end{array}$$

Since Δ is closed and $A^u \times A^s$ is compact, Lemma 8 implies that there exists an ε^*, with $0 < \varepsilon^* < \varepsilon_r$, such that $(i_\varepsilon \circ h) \times (j_\varepsilon \circ k) \pitchfork \Delta$ for $\varepsilon < \varepsilon^*$. Furthermore, the submanifolds

$$(i \times j)^{-1}(\Delta) \quad \text{and} \quad \big((i_\varepsilon \circ h) \times (j_\varepsilon \circ k)\big)^{-1}(\Delta)$$

are diffeomorphic. We also have that $\big((i_\varepsilon \circ h) \times (j_\varepsilon \circ k)\big)^{-1}(\Delta)$ is diffeomorphic to $A^u_\varepsilon \cap A^s_\varepsilon$, and $(i \times j)^{-1}(\Delta) = A^u \cap A^s = \mathscr{H}$.

This shows that the intersection $\mathscr{H}_\varepsilon = A^u_\varepsilon \cap A^s_\varepsilon$ is diffeomorphic to $\mathscr{H}$. Define D^u_ε as the part of $W^u(\mathscr{C}_{\alpha,\varepsilon})$ bounded by the invariant circle $\mathscr{C}_{\alpha,\varepsilon}$ and the circle of homoclinic points $\mathscr{H}_\varepsilon$. Define $D^s_\varepsilon = k(D^s)$ similarly. Then the manifolds $D^u_\varepsilon \subset W^u(\mathscr{C}_{\alpha,\varepsilon})$ and $D^s_\varepsilon \subset W^s(\mathscr{C}_{\alpha,\varepsilon})$ form the boundary of an open region $U \subset M$ homeomorphic to a torus. By the closeness of the perturbed manifolds $W^s(\mathscr{C}_{\alpha,\varepsilon})$ and $W^u(\mathscr{C}_{\alpha,\varepsilon})$ to the unperturbed $W^s(\mathscr{C})$ and $W^u(\mathscr{C})$, both U and $W^u(\mathscr{C}_{\alpha,\varepsilon})$ are bounded. Also notice that $P_{\alpha,\varepsilon}$ is dissipative: by taking ε^* small enough, we ensure that $|\det(DF(x,y,\theta))| < \tilde{c} < 1$ for all $\varepsilon < \varepsilon^*$ and (x,y,θ) in U. Therefore, all forward evolutions beginning at points $(x,y,\theta) \in U$ remain bounded. Like in the first part of the proof, one has

$$\omega(x,y,\theta) \subset \mathrm{Cl}\,(W^u(\mathscr{C}_{\alpha,\varepsilon}))$$

for all $(x,y,\theta) \in U$, $\alpha \in [0,1]$ and $\varepsilon < \varepsilon^*$. □

3.1.2 An application of KAM theory

So far, we did not discuss the dynamics in the saddle invariant circle $\mathscr{C}_{\alpha,\varepsilon}$ of map $P_{\alpha,\varepsilon}$ in (6). Generically, the dynamics on $\mathscr{C}_{\alpha,\varepsilon}$ is of Morse-Smale type. In this case, the circle consists of the union of the unstable manifold of some periodic saddle. Theorem 3 describes a complementary case, for which the dynamics is quasi-periodic. Fix $\tau > 2$ and define the set of Diophantine frequencies D_γ by

$$D_\gamma = \left\{ \alpha \in [0,1] \;\middle|\; \left|\alpha - \frac{p}{q}\right| \geq \gamma q^{-\tau} \quad \text{for all } p,\, q \in \mathbb{N},\, q \neq 0 \right\}, \tag{18}$$

where $\gamma > 0$. Since we will apply a version of the KAM Theorem holding for non-conservative, finitely differentiable systems (see [5, Chap. 5] and [6]), a certain amount of smoothness of the circle $\mathscr{C}_{\alpha,\varepsilon}$ is needed, depending on the Diophantine condition specified in (18). Therefore we require that the perturbed family of maps $P_{\alpha,\varepsilon}$ is C^n, for n large enough.

Proof of Theorem 3. Consider map $P_{\alpha,0}$ in (6), and let $p = (x_0, y_0)$ be a saddle fixed point of the diffeomorphism K. The invariant circle $\mathscr{C}_{\alpha,0} = \{p\} \times \mathbb{S}^1$ of $P_{\alpha,0}$ can be trivially seen as a graph over $\mathbb{S}^1$:

$$\mathscr{C}_{\alpha,0} = \left\{(x_0, y_0, \theta) \;\mid\; \theta \in \mathbb{S}^1\right\}.$$

Fix $r \in \mathbb{N}$ large enough and $\varepsilon < \varepsilon_r$, where ε_r is taken as in Proposition 1. By the C^r-closeness of $\mathscr{C}_{\alpha,0}$ and $\mathscr{C}_{\alpha,\varepsilon}$ (Proposition 1), the circle $\mathscr{C}_{\alpha,\varepsilon}$ of $P_{\alpha,\varepsilon}$ can be written as a C^r-graph over $\mathbb{S}^1$:

$$\mathscr{C}_{\alpha,\varepsilon} = \left\{(x_\varepsilon(\theta), y_\varepsilon(\theta), \theta) \in \mathbb{R}^2 \times \mathbb{S}^1 \;\mid\; \theta \in \mathbb{S}^1\right\}, \tag{19}$$

where $x_\varepsilon : \mathbb{S}^1 \to \mathbb{R}$, $x_\varepsilon(\theta) = x_0 + \mathcal{O}(\varepsilon)$, and similarly for $y_\varepsilon(\theta)$. So the restriction of $P_{\alpha,\varepsilon}$ to $\mathscr{C}_{\alpha,\varepsilon}$ has the following form

$$P_{\alpha,\varepsilon}\Big|_{\mathscr{C}_{\alpha,\varepsilon}} : \mathscr{C}_{\alpha,\varepsilon} \to \mathscr{C}_{\alpha,\varepsilon}, \qquad P_{\alpha,\varepsilon}(\theta) = \theta + \alpha + \varepsilon g_\varepsilon(x_0, y_0, \theta, \alpha) + \mathcal{O}(\varepsilon^2).$$

By (19), we may consider $P_{\alpha,\varepsilon}$ as a map on $\mathbb{S}^1$. Fix $\gamma > 0$, $\tau > 3$ and take D_γ as in (18). For $\alpha \in D_\gamma$, the map $P_{\alpha,\varepsilon}$ can be averaged repeatedly over the circle, putting the θ-dependency into terms of higher order in ε, compare [8, Proposition 2.7] and [11, Section 4]. After such changes of variables, $P_{\alpha,\varepsilon}$ is brought into the normal form

$$P_{\alpha,\varepsilon}(\theta) = \theta + \alpha + c(\alpha, \varepsilon) + \mathcal{O}(\varepsilon^{r+1}).$$

In fact, it is convenient to consider α as a variable, and to define the cylinder maps

$$\begin{aligned} P_\varepsilon &: \mathbb{S}^1 \times [0,1] \to \mathbb{S}^1 \times [0,1], \qquad & P_\varepsilon(\theta, \alpha) &= (P_{\alpha,\varepsilon}(\theta), \alpha) \\ R &: \mathbb{S}^1 \times [0,1] \to \mathbb{S}^1 \times [0,1], \qquad & R(\theta, \alpha) &= (R_\alpha(\theta), \alpha), \end{aligned}$$

where $R_\alpha : \mathbb{S}^1 \to \mathbb{S}^1$ is the rigid rotation of an angle α. We now apply a version of the KAM Theorem, holding for non-conservative, finitely differentiable systems (see *e.g.* [5, Chap. 5] and [6]), to the family of diffeomorphisms P_ε. There exists an integer m with $1 \le m < r$ and a C^m-map

$$\Phi_\varepsilon : \mathbb{S}^1 \times [0,1] \to \mathbb{S}^1 \times [0,1], \quad \Phi_\varepsilon(\theta, \alpha) = (\theta + \varepsilon A(\theta, \alpha, \varepsilon), \alpha + \varepsilon B(\alpha, \varepsilon)), \tag{20}$$

such that the restriction of Φ_ε to $\mathbb{S}^1 \times D_\gamma$ makes the following diagram commute:

$$\begin{tikzcd} \mathbb{S}^1 \times D_\gamma \arrow[r, "R"] & \mathbb{S}^1 \times D_\gamma \\ \mathbb{S}^1 \times D_\gamma \arrow[u, "\Phi_\varepsilon"] \arrow[r, "P_\varepsilon"] & \mathbb{S}^1 \times D_\gamma. \arrow[u, "\Phi_\varepsilon"] \end{tikzcd}$$

The differentiability of Φ_ε restricted to $\mathbb{S}^1 \times D_\gamma$ is of Whitney type. Since $P_{\alpha,\varepsilon}\Big|_{\mathscr{C}_{\alpha,\varepsilon}}$ is C^m-conjugate to a rigid rotation on $\mathbb{S}^1$, the circle $\mathscr{C}_{\alpha,\varepsilon}$ is in fact C^m. This proves parts 1 and 2 of the Theorem.

Furthermore, the constant γ in (18) can be taken equal to ε^r. This gives that the measure of the complement of D_γ in $[0,1]$ is of order ε^r as $\varepsilon \to 0$. ☐

3.2 Hénon-like attractors do exist

Our proof of Theorem 4 is based on a result of Díaz-Rocha-Viana [14]. We begin by stating this result.

3.2.1 Perturbations of multimodal families

Two definitions from [14, Section 5.2] are introduced now. For more information about the terminology, we refer to [24, Sections II.5, II.6].

Definition 4. *Let $J \subset \mathbb{R}$ be a compact interval. Fix $d \geq 1$, $k \geq 3$, $a^* \in \mathbb{R}$, and an interval of parameter values $\mathfrak{U} = [a_-, a_+]$, with $a^* \in \operatorname{Int} \mathfrak{U}$. A C^k-family of maps $M_a : J \to J$, with $a \in \mathfrak{U}$, is called a d-family if it satisfies the following conditions:*

1. Invariance: $M_{a^*}(J) \subset \operatorname{Int}(J)$;

2. Nondegenerate critical points: *M_{a^*} has d critical points $\{c_1, \ldots, c_d\} \stackrel{\text{def}}{=} \operatorname{Cr} M_{a^*}$ that satisfy*
$$M''_{a^*}(c_i) \neq 0 \quad \textit{for all } i \qquad \textit{and} \qquad M_{a^*}(c_i) \neq c_j \quad \textit{for all } \ i, j;$$

3. Negative Schwarzian derivative: *$SM_{a^*} < 0$ for all $x \neq c_i$, where*
$$Sf(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2;$$

4. Topological mixing: *There exists an interval $J' \subset \operatorname{Int}(J)$ such that $M_{a^*}(J) = M_{a^*}(J') = J'$ and such that for any open intervals J_1, J_2 in J' there exists n_0 such that*
$$M^n_{a^*}(J_1) \cap J_2 \neq \emptyset \quad \textit{for all } n \geq n_0;$$

5. Preperiodicity: *for each $1 \leq i \leq d$ there exists m_i such that $p_i = M^{m_i}_{a^*}(c_i)$ is a (repelling) periodic point of M_{a^*};*

6. Genericity of unfolding: *For all $c_i \in \operatorname{Cr} M_{a^*}$, denote by $c_i(a)$ and $p_i(a)$ the continuations of c_i and p_i, respectively, for a close to a^*. Then*
$$\frac{\mathrm{d}}{\mathrm{d}a}\big(M^{m_i}_a(c_i(a)) - p_i(a)\big) \neq 0 \qquad \textit{at } a = a^*.$$

Next we introduce the notion of *η-perturbation* of a d-family M_a, with $a \in \mathfrak{U}$ and $d \geq 1$ fixed.

Definition 5. *Fix $\sigma > 0$ and consider the family $\overline{M}_a$ obtained by extending M_a as follows:*
$$\overline{M}_a : J \times I_\sigma \to J \times I_\sigma, \quad \overline{M}_a(x, y) \stackrel{\text{def}}{=} (M_a(x), 0). \tag{21}$$
Also denote by M the map
$$M : \mathfrak{U} \times J \times I_\sigma \to J \times I_\sigma, \quad M(a, x, y) \stackrel{\text{def}}{=} \overline{M}_a(x, y) = (M_a(x), 0).$$
Given a C^k-family of diffeomorphisms
$$G_a : J \times I_\sigma \to J \times I_\sigma, \qquad a \in J,$$
for a $k \geq 3$, denote by G its extension
$$G : \mathfrak{U} \times J \times I_\sigma \to J \times I_\sigma, \quad G(a, x, y) \stackrel{\text{def}}{=} G_a(x, y).$$
*Then G is called a η-*perturbation *of the d-family $\{M_a\}_a$ if*
$$\|M - G\|_{C^k} \leq \eta,$$
where $\|\cdot\|_{C^k}$ denotes the C^k-norm over $\mathfrak{U} \times J \times I_\sigma$.

The following proposition is used in the sequel to prove existence of Hénon-like attractors for the map (2). See [2, 3, 26, 32, 37, 39, 42] for similar results.

Proposition 10. **[14, Theorem 5.2]** *Let $\{M_a\}_a$ be a d-family and p a periodic point of M_{a^*}. Then there exist $\eta > 0$, $\bar{a}$ and $\chi > 0$ such that, given any η-perturbation $\{G_a\}_a$ of $\{M_a\}_a$ the following holds.*

1. *For all a with $|a - a^*| < \bar{a}$ the map G_a has a periodic point p_a which is the continuation of the periodic point $(p, 0)$ of the map $\overline{M}_a$ in (21).*
2. *There exists a set $\mathfrak{S}$, contained in the interval $[a^* - \bar{a}, a^* + \bar{a}] \subset \mathfrak{U}$, with $\operatorname{meas}(\mathfrak{S}) > \chi$, such that for all $a \in \mathfrak{S}$ there exists $z \in W^u(p_a)$ satisfying:*
   (a) *the orbit $\{G_a^n(z) \mid n \geq 0\}$ is dense in $\operatorname{Cl}(W^u(\operatorname{Orb}(p_a)))$;*
   (b) *G_a has a positive Lyapunov exponent at z,* i.e., *there exist $k > 0$, $\lambda > 1$ and $v \neq 0$ such that $\|DG_a^n(z)v\| \geq k\lambda^n$ for all $n \geq 0$;*
   (c) *there exist $w \neq 0$ such that $\|DG_a^n(z)w\| \to 0$ as $n \to \pm\infty$.*

3.2.2 Multimodal families arising from powers of the Logistic map

The proof of Theorem 4, which we present in this section, is based on three facts. First, suppose that $a^* \in [0, 2]$ is such that the quadratic family $Q_a(x) = 1 - ax^2$ in (13) is a d-family in the sense of Definition 4, with $d = 1$. Then for all $n \geq 1$ the family $M_a \stackrel{\text{def}}{=} Q_a^n$ given by the n-th iterate of Q_a is a d-family for some $d \leq 2^n$. Second, for all $\eta_1 > 0$, the composition of an η_1-perturbation of Q_a with an η_1-perturbation of Q_a^n is an η_2-perturbation of Q_a^{n+1}, where $\eta_2 = C(n)\eta_1$ and $C(n)$ is a positive constant depending on n. Third, for each $n > q \geq 1$ and for each $(\alpha, \delta) \in \mathfrak{A}^{q/n}$, the asymptotic dynamics of $T_{\alpha,\delta,a,\varepsilon}$ is described by a map that turns out to be an η-perturbation of the d-family M_a, with $\eta = \mathcal{O}(\varepsilon)$. Moreover, M_a has a periodic point p such that its analytic continuation in the family $T_{\alpha,\delta,a,\varepsilon}$ possesses a transversal homoclinic intersection. Application of Proposition 10 to the point p concludes the proof.

In the next lemma we show that M_a is a d-family. For each $\tilde{a} \in [0, 2)$ there exists a $\beta > 0$ such that for all a with $a \in [0, \tilde{a}]$ the interval $J = [-1-\beta, 1+\beta] \subset \mathbb{R}$ satisfies $Q_a(J) \subset \operatorname{Int}(J)$. In the sequel, it is always assumed that the family Q_a is defined on such an interval J, and that the values of a we consider are such that $Q_a(J) \subset \operatorname{Int}(J)$.

Lemma 11. *Suppose $a^* \in [0, 2) \stackrel{\text{def}}{=} \mathfrak{U}$ is such that the quadratic family*

$$Q_a : J \to J, \quad Q_a(x) = 1 - ax^2$$

satisfies hypotheses 4 and 5 of Definition 4. Then for all $n \geq 1$ there exists $d \geq 1$ such that the family

$$M_a : J \to J, \quad M_a \stackrel{\text{def}}{=} Q_a^n$$

is a d-family with $d \leq 2^n - 1$ critical points.

Proof. Take a^* as above. We first prove the case $n = 1$, that is, $Q_a : J_a \to J_a$ is a 1-family. Conditions 1, 2, 3 of Definition 4 are obviously satisfied by Q_a. Condition 6 will now be proved. By conditions 4 and 5 (assumed by hypothesis), Q_{a^*} is a Misiurewicz map [25], *i.e.*, it has no periodic attractor and $c \notin \omega(c)$, where $c = 0$ is the critical point of Q_{a^*}. Moreover, by [24, Theorem III.6.3] the map Q_{a^*} is Collet-Eckmann (see *e.g.* [24, Section V.4]), that is, there exist constants $\kappa > 0$ and $\lambda > 1$ such that

$$\left|\frac{\mathrm{d}}{\mathrm{d}x} Q_{a^*}^n(Q_{a^*}(c))\right| \geq \kappa\lambda^n \quad \text{for all } n \geq 0. \tag{22}$$

Therefore, by combining [38, Theorem 3] with the Collet-Eckmann condition (22) we get

$$\lim_{n\to\infty} \frac{\frac{\mathrm{d}}{\mathrm{d}a} Q_a^n(c) \mid_{a=a^*}}{\frac{\mathrm{d}}{\mathrm{d}x} Q_{a^*}^{n-1}(Q_{a^*}(c))} > 0. \tag{23}$$

Assume $Q_{a^*}^k(c) = p$, with p periodic (and repelling) under Q_{a^*}. By $p(a)$ denote the continuation of p for a close to a^*. Then, for all n sufficiently large,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}a} Q_a^n(c) \mid_{a=a^*} &= \frac{\partial Q_a^{n-k}}{\partial a}(Q_{a^*}^k(c)) \mid_{a=a^*} + \frac{\partial Q_a^{n-k}}{\partial x}(Q_{a^*}^k(c)) \mid_{a=a^*} \frac{\mathrm{d}}{\mathrm{d}a} Q_a^k(c) \mid_{a=a^*} = \\ &= \frac{\partial}{\partial a} Q_a^{n-k}(p) \mid_{a=a^*} + \frac{\partial}{\partial x} Q_a^{n-k}(p) \mid_{a=a^*} \frac{\mathrm{d}}{\mathrm{d}a}\left[p(a) + Q_a^k(c) - p(a)\right] \mid_{a=a^*} = \\ &= \frac{\mathrm{d}}{\mathrm{d}a}\left(Q_a^{n-k}(p(a))\right) + \frac{\partial}{\partial x} Q_{a^*}^{n-k}(p) \frac{\mathrm{d}}{\mathrm{d}a}\left[Q_a^k(c) - p(a)\right] \mid_{a=a^*}.\end{aligned} \tag{24}$$

The point $Q_a^{n-k}(p(a))$ belongs to a hyperbolic periodic orbit, that varies smoothly with the parameter a. Therefore, its derivative with respect to a (which is the first term in the last equality) is uniformly bounded in n. On the other hand,

$$\frac{\mathrm{d}}{\mathrm{d}x} Q_{a^*}^{n-1}(Q_{a^*}(c)) = \frac{\partial}{\partial x} Q_{a^*}^{n-k}(p) \frac{\mathrm{d}}{\mathrm{d}x} Q_{a^*}^{k-1}(Q_{a^*}(c)).$$

Therefore, by (22), (23), and (24) we conclude that

$$0 < \lim_{n\to\infty} \frac{\frac{\mathrm{d}}{\mathrm{d}a} Q_a^n(c) \mid_{a=a^*}}{\frac{\mathrm{d}}{\mathrm{d}x} Q_{a^*}^{n-1}(Q_{a^*}(c))} = \frac{\frac{\mathrm{d}}{\mathrm{d}a}\left[Q_a^k(c) - p(a)\right]_{a=a^*}}{\frac{\mathrm{d}}{\mathrm{d}x} Q_{a^*}^{k-1}(Q_{a^*}(c))}. \tag{25}$$

This proves that Q_a satisfies condition 6 of Definition 4.

We now show that the n-th iterate M_a of the quadratic map is a d-family for all $n > 1$ and for some $d \leq 2^n$. For simplicity, we denote Q_{a^*} by Q for the rest of this proof. Condition 1 holds for M_{a^*} since it holds for Q_{a^*}. Condition 3 follows from the fact that the composition of maps with negative Schwarzian derivative also has negative Schwarzian derivative, see *e.g.* [24, II.6]. Condition 4 is obviously satisfied.

Condition 2 is now proved by induction on n, where the case $n = 1$ is obvious. Obviously, the set $\mathrm{Cr}\, M_{a^*}$ of critical points of M_{a^*} has cardinality $d \leq 2^n - 1$. Moreover,

$$\mathrm{Cr}\, M_{a^*} = Q^{-1}\left(\mathrm{Cr}\, Q^{n-1}\right) \cup \mathrm{Cr}\, Q = \bigcup_{j=0}^{n-1} (Q^{-j})(\mathrm{Cr}\, Q). \tag{26}$$

Suppose that condition 2 holds for a given $n \geq 1$. We first show that

$$(Q^{n+1})''(x) \neq 0 \qquad \text{for all } \; x \in \mathrm{Cr}\, Q^{n+1}. \tag{27}$$

By (26), if $x \in \mathrm{Cr}\, Q^{n+1}$ then either $x = c$, or $Q(x) \in \mathrm{Cr}\, Q^n$. If $x = c$ then

$$(Q^{n+1})''(x) = (Q^n)'\left(Q(c)\right) \cdot (Q)''(c). \tag{28}$$

The second factor is nonzero. If the first factor is zero, then

$$0 = (Q^n)'\,(Q(c)) = Q'(Q^n(c)) \dots Q'(Q(c)).$$

Therefore there exists j such that $Q^j(c) = c$, so that c is an attracting periodic point of Q. But this contradicts the fact that Q is Misiurewicz, so that (28) is nonzero. The other possibility is that $c \neq x$ and $Q(x) \in \operatorname{Cr} Q^n$. In this case,

$$(Q^{n+1})''(x) = (Q^n)''\,(Q(x)) \cdot Q'(x)^2,$$

which is nonzero. Indeed, $Q'(x) \neq 0$, otherwise $x = c$. Moreover $(Q^n)''\,(Q(x)) \neq 0$ by the induction hypotheses since the critical points of Q^n are nondegenerate. This proves (27), from which the first part of condition 2 follows.

We now prove, again arguing by contradiction, that

$$Q^{n+1}(x) \neq y \qquad \text{for all } \ x, y \in \operatorname{Cr} Q^{n+1}.$$

Suppose that there exist $x, y \in \operatorname{Cr} Q^{n+1}$ such that $Q^{n+1}(x) = y$. By (26) there exist i and j such that $Q^i(x) = Q^j(y) = c$, where $0 \leq i, j \leq n$. This would imply that

$$Q^{n+1+j-i}(c) = Q^j(Q^{n+1}(x)) = Q^j(y) = c,$$

with $n + 1 + j - i \geq 1$ and, therefore, c would be an attracting periodic point of Q, which is impossible since Q is Misiurewicz. Condition 2 is proved.

To prove condition 5, fix $y \in \operatorname{Cr} M_{a^*}$ and $j \geq 0$ such that $Q^j(y) = c$. Since c is preperiodic for Q by hypothesis, there exists $k \geq 1$ such that $Q^{j+k}(y) = p$, where p is periodic under Q with period $u \geq 1$. The orbit of y under M_{a^*} is, except for a finite number of initial iterates, a subset of the orbit of p under Q. This shows that y is preperiodic for M_{a^*}.

To prove condition 6, take $y \in \operatorname{Cr} M_{a^*}$, j, u, k and $p \in J$ as in the proof of condition 5. Then there exist integers l and m, with $0 \leq l < u$ and $m \geq 1$, such that

$$M_{a^*}^m(y) = Q^{k+l}(c) = Q^l(p) \in \operatorname{Orb}_Q(p). \tag{29}$$

By condition 5 (assumed by hypothesis) and by (29), the point $z = Q^l(p)$ is periodic (and repelling) under M_{a^*}. Denote by $y(a)$, $z(a)$, and $p(a)$ the continuations of y, z, and p, respectively, for a close to a^*. In particular,

$$Q_a^j(y(a)) = c \quad \text{and} \quad Q_a^l(p(a)) = z(a).$$

We have to show that

$$\frac{\mathrm{d}}{\mathrm{d}a}\left[M_a^m(y(a)) - z(a)\right]\Big|_{a=a^*} \neq 0. \tag{30}$$

By the chain rule we get

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}a}\, Q_a^{l+k}(c)\Big|_{a=a^*} &= \frac{\partial Q_a^l}{\partial a}\,(Q_a^k(c))\Big|_{a=a^*} + \frac{\partial Q_a^l}{\partial x}\,(Q_a^k(c))\Big|_{a=a^*}\,\frac{\mathrm{d}Q_a^k}{\mathrm{d}a}\,(c)\Big|_{a=a^*} = \\
&= \frac{\partial Q_{a^*}^l}{\partial a}(p) + \frac{\partial Q_{a^*}^l}{\partial x}(p)\,\frac{\mathrm{d}Q_{a^*}^k}{\mathrm{d}a}(c), \\
\frac{\mathrm{d}}{\mathrm{d}a}\, Q_a^l(p(a))\Big|_{a=a^*} &= \frac{\partial Q_{a^*}^l}{\partial a}(p) + \frac{\partial Q_{a^*}^l}{\partial x}(p)\,\frac{\mathrm{d}}{\mathrm{d}a}p(a^*),
\end{aligned}$$

where $p = p(a^*) = Q_{a^*}^k(c)$. Therefore,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}a}\left[M_a^m(y(a)) - z(a)\right]\Big|_{a=a^*} &= \frac{\mathrm{d}}{\mathrm{d}a}\left[Q_a^{k+l}(c) - Q_a^l(p(a))\right]\Big|_{a=a^*} = \\
&= \frac{\partial Q_{a^*}^l}{\partial x}(p)\,\frac{\mathrm{d}}{\mathrm{d}a}\left[Q_a^k(c) - p(a)\right]\Big|_{a=a^*}.
\end{aligned}$$

The factor $\frac{\mathrm{d}}{\mathrm{d}a}\left[Q_a^k(c(a)) - p(a)\right]\Big|_{a=a^*}$ is nonzero by (25). The same holds for the other factor, otherwise p would be an attracting periodic point of Q_{a^*}. This proves inequality (30). $\square$

Proposition 10 does not provide a nontrivial basin of attraction for the closure $\mathrm{Cl}\,(W^u(p_a))$. We now show that, under the hypotheses of Theorem 4, there exists a periodic point p_a for which a nontrivial basin of attraction of $\mathrm{Cl}\,(W^u(p_a))$ can be constructed. Therefore, in this case $\mathrm{Cl}\,(W^u(p_a))$ is a Hénon-like attractor.

Proposition 12. *Consider the map $\{M_a^*\}_a = Q_{a^*}^n$, where Q_{a^*} satisfies the hypotheses of Lemma 11. There exist a periodic point p of Q_{a^*}, and positive constants η, $\bar{a}$ and χ such that for any η-perturbation $\{G_a\}_a$ of $\{M_a\}_a = Q_{a^*}^n$ the following holds.*

1. *For all a with $|a - a^*| < \bar{a}$ the map G_a has a periodic point p_a which is the continuation of the periodic point $(p, 0)$ of the map $\overline{M}_a$ in (21). Moreover, p_a has a transversal homoclinic intersection.*

2. *There exists a set $\mathfrak{S}$, contained in the interval $[a^* - \bar{a}, a^* + \bar{a}] \subset \mathfrak{U}$, with $\mathrm{meas}(\mathfrak{S}) > \chi$, such that for all $a \in \mathfrak{S}$ the set $\mathrm{Cl}\,(W^u(\mathrm{Orb}(p_a)))$ is a Hénon-like attractor of the map G_a.*

Proof. To construct a non-trivial basin of attraction for $\mathrm{Cl}\,(W^u(\mathrm{Orb}(p_a)))$, it is sufficient to find a periodic point p_a of $\{G_a\}_a$ that has a transversal homoclinic intersection. Then the basin is provided by the Tangerman-Szewc Theorem (see Theorem 2 and subsequent remark). Indeed, for all η sufficiently small, all η-perturbations of the map Q_a^* are uniformly dissipative. Moreover, the unstable manifold of p_a is bounded, since it is bounded for Q_{a^*} and since the invariant manifolds of a map depend continuously on the map [26, Prop. 7.1]. Therefore, the second part of the proposition follows from the first part, together with the Tangerman-Szewc argument and Proposition 10.

To prove the first part, we claim that the map Q_{a^*} has a periodic point p belonging to a non-degenerate homoclinic orbit. Indeed, if the claim is true, then for η sufficiently small and for a close to a^*, any η-perturbation of M_a possesses a periodic point p_a which is the analytic continuation of p and such that p_a has a transverse homoclinic intersection. The latter property again follows from continuous dependence of the invariant manifolds on the map [26, Prop. 7.1].

To prove the claim that Q_{a^*} has a periodic point p belonging to a non-degenerate homoclinic orbit we first show that there exists a point y_0 belonging to a degenerate homoclinic orbit of Q_{a^*}. Since the critical point c of Q_{a^*} is preperiodic, there exist positive integers k, h such that $Q_{a^*}^k(c) = y_0$ and y_0 is periodic with period h. The unstable manifold of any periodic point of Q_{a^*} is the whole core $[1 - a^*, 1]$, since Q_{a^*} is topologically mixing. Therefore, since the critical point c belongs to $W^u(y_0)$, by taking preimages of c, a point q can be found such that $q \in W^u_{loc}(y_0)$, $Q_{a^*}^l(q) = c$ and $Q_{a^*}^{l+k}(q) = y_0$ for some integer $l > 0$. This means that y_0 belongs to a degenerate homoclinic orbit of Q_{a^*}.

We now prove that there exists a periodic point p of Q_{a^*} having a non-degenerate homoclinic orbit. This is achieved by examining a power of Q_{a^*} for which all points of the orbit of y_0 are fixed. Denote by $\mathrm{Orb}_{Q_{a^*}}(y_0) = \{y_j \mid j = 0, \dots, h-1\}$ the orbit of y_0, under Q_{a^*}, where $y_j = Q_{a^*}^j(y_0)$. Let m be the smallest multiple of h which is larger than k, and denote $f \stackrel{\mathrm{def}}{=} Q_{a^*}^m$. Then, $f(c)$ belongs to $\mathrm{Orb}_{Q_{a^*}}(y_0)$ and all points y_j are fixed for f. We can assume that

$$f'(y_j) > 1 \quad \text{for all} \quad y_j \in \mathrm{Orb}_{Q_{a^*}}(y_0) \tag{31}$$

by taking f^2 instead of f if necessary.

Brouwer's fixed point Theorem and continuity arguments ensure the existence of a fixed point p of f, a critical point c' of f, and an interval $I = (c' - \delta, c')$ such that:

1. $f'(p) < -1$;

2. c' lies in the interval (y, p);

3. f is monotonically increasing in I;

4. p falls in the interval $f(I)$.

The configuration of c, c' and $y = f(c)$ within the graph of f looks like the sketch in Figure 8, in the case $y < c$ and $f''(c) > 0$ (the other combinations of the sign of $y - c$ and $f''(c)$ are treated similarly). Since f is topologically mixing, the interval $f(I)$ is contained in the

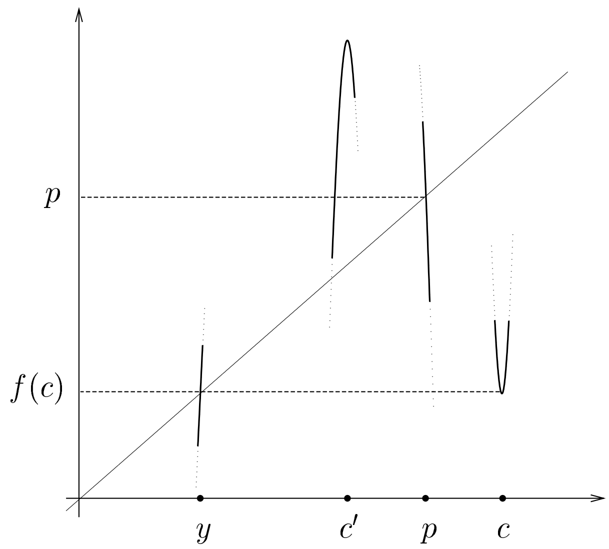

Figure 8: Graph of the map f from the proof of Proposition 12. Only the relevant branches of the graph are plotted, in relation to the fixed points y, p and the critical points c and c'. See the text for details.

unstable manifold of p. Therefore p belongs to a homoclinic orbit $\mathcal{O}$.

Moreover, the homoclinic orbit $\mathcal{O}$ is non-degenerate. Indeed, if this was not the case, then there would exist a critical point c'' of f belonging to $\mathcal{O}$, so that $f^j(c'') = p$ for some $j \in \mathbb{N}$. However, according to (26), and since c is preperiodic, the orbit of c'' under Q_{a^*} eventually lands inside $\mathrm{Orb}_{Q_{a^*}}(y_0)$. It follows that $p \in \mathrm{Orb}_{Q_{a^*}}(y_0)$, which is absurd, since $f'(p) < -1$ whereas (31) holds. $\square$

In the next lemma we show that the composition of a small perturbation of the map $\overline{Q}_a(x, y) = (Q_a(x), 0)$ (we use here the notation of Definition 5) with a small perturbation of $\overline{Q^n_a}(x, y) = (Q^n_a(x), 0)$ yields a small perturbation of $\overline{Q^{n+1}_a}(x, y)$. As in Definition 5, denote by $Q, Q^n : [0, 2] \times J \times I \to J \times I$ the functions $Q(a, x, y) = (Q_a(x), 0)$ and $Q^n(a, x, y) = (Q^n_a(x), 0)$, respectively.

Lemma 13. *For each $\eta > 0$ there exists a $\zeta > 0$ such that for all $F, G : [0, 2] \times J \times I \to J \times I$ such that*

$$\|G - Q\|_{C^3} < \zeta \quad \textit{and} \quad \|F - Q^n\|_{C^3} < \zeta, \tag{32}$$

we have

$$\left\|G \circ F - Q^{n+1}\right\|_{C^3} < \eta. \tag{33}$$

Proof. Write

$$G(a, x, y) = \begin{pmatrix} Q_a(x) + g_1(a, x, y) \\ g_2(a, x, y) \end{pmatrix} \quad \text{and} \quad F(a, x, y) = \begin{pmatrix} Q^n_a(x) + f_1(a, x, y) \\ f_2(a, x, y) \end{pmatrix}.$$

Then

$$G \circ F(a,x,y) - \begin{pmatrix} Q_a^{n+1}(x) \\ 0 \end{pmatrix} =$$
$$\begin{pmatrix} -2a(f_1(a,x,y))^2 - 2af_1(a,x,y)Q_a^n(x) + g_1\big(a, \tilde{f}_1(a,x,y), f_2(a,x,y)\big) \\ g_2\big(a, \tilde{f}_1(a,x,y), f_2(a,x,y)\big) \end{pmatrix},$$

where $\tilde{f}_1(a,x,y) = Q_a^n(x) + f_1(a,x,y)$. The C^3-norm of the terms $-2a(f_1(a,x,y))^2$ and $-2af_1(a,x,y)Q_a^n(x)$ is bounded by a constant times the C^3-norm of f_1. We now estimate the norm of $\tilde{g}_1$, defined by

$$\tilde{g}_1(x_0,x_1,x_2) = g_1(a, \tilde{f}_1(a,x,y), f_2(a,x,y)).$$

Denote $x_0 = a$, $x_1 = x$, and $x_2 = y$. Then any second order derivative of $\tilde{g}_1$ is a sum of terms of the following type:

$$\frac{\partial^2 g_1}{\partial x_j x_k}\frac{\partial \tilde{f}_k}{\partial x_l}, \quad \frac{\partial g_1}{\partial x_k}\frac{\partial^2 \tilde{f}_k}{\partial x_j x_l},$$

where we put $\tilde{f}_2 = f_2$ to simplify the notation. For the third order derivatives a similar property holds. Since the C^3-norm of $\tilde{f}_k$ is bounded, we get that each term in the third order derivative of $\tilde{g}_1$ is bounded by a constant times the C^3-norm of the g_j. This concludes the proof. □

***Proof of Theorem* 4.** The Theorem will be first proved for $a^* < 2$. The case $a^* = 2$ follows by choosing another value $\bar{a}^* < 2$ sufficiently close to 2. Fix $a^* \in [0,2)$ verifying the hypotheses of Lemma 11. To begin with, we consider the case $(\alpha,\delta) \in \operatorname{Int}\mathfrak{A}^1$, the interior of the tongue of period one. Then the Arnol′d family $A_{\alpha,\delta}$ on $\mathbb{S}^1$ has two hyperbolic fixed points θ_1^s (attracting) and θ_1^r (repelling), see [13, Section 1.14]. The θ-coordinate of both points depends on the choice of $(\alpha,\delta) \in \operatorname{Int}\mathfrak{A}^1$. So for all $\theta \in \mathbb{S}^1$ with $\theta \neq \theta_1^r$, the orbit of θ under $A_{\alpha,\delta}$ converges to θ_1^s. This means that the manifold

$$\Theta_1 = \big\{(x,y,\theta) \in \mathbb{R}^2 \times \mathbb{S}^1 \ \mid \ \theta = \theta_1^s\big\} \subset \mathbb{R}^2 \times \mathbb{S}^1$$

is invariant and attracting under $T_{\alpha,\delta,a,\varepsilon}$. Denote by $G_{a,1}$ the restriction of $T_{\alpha,\delta,a,\varepsilon}$ to Θ_1:

$$G_{a,1} : \Theta_1 \to \Theta_1, \qquad (x,y,\theta_1^s) \mapsto (1 - ax^2 + \varepsilon f_1,\ \varepsilon g_1,\ \theta_1^s),$$

where $f_1 = f(a,x,y,\theta_1^s,\alpha,\delta)$ and similarly for g_1. Since $Q_{a^*}(J) \subset \operatorname{Int}(J)$, there exists a constant $\sigma > 0$ such that for all ε sufficiently small and all a close enough to a^*,

$$\begin{gathered} G_{a,1}(J \times I_\sigma \times \{\theta_1^s\}) \subset \operatorname{Int}(J \times I_\sigma \times \{\theta_1^s\}) \quad \text{and} \\ T_{\alpha,\delta,a,\varepsilon}\big(J \times I_\sigma \times (\mathbb{S}^1 \setminus \{\theta_1^r\})\big) \subset \operatorname{Int}\big(J \times I_\sigma \times (\mathbb{S}^1 \setminus \{\theta_1^r\})\big). \end{gathered} \tag{34}$$

Since Θ_1 is diffeomorphic to $\mathbb{R}^2$, we consider $G_{a,1}$ as a map of $\mathbb{R}^2$. Then $G_{a,1}$, is an η-perturbation of the quadratic family $Q_a(x)$, where $\eta = \mathcal{O}(\varepsilon)$. We now apply Proposition 12 to the family $G_{a,1}$. Let p_0 be the periodic point of M_{a^*} as given by Proposition 12. For all ε sufficiently small there exists a constant $\bar{a} > 0$ and a set $\mathfrak{S}$ of positive Lebesgue measure, contained in the interval $[a^* - \bar{a}, a^* + \bar{a}]$, such that the following holds. For all $a \in [a^* - \bar{a}, a^* + \bar{a}]$, $G_{a,1}$ has a saddle periodic point $\bar{p}$ which is the continuation of the point p_0. Furthermore, for all $a \in \mathfrak{S}$ the closure $\widetilde{\mathscr{A}} = \operatorname{Cl}\big(W^u(\operatorname{Orb}_{G_{a,1}}(\bar{p}))\big)$ is a Hénon-like attractor of $G_{a,1}$ contained inside Θ_1. The point $p = (\bar{p},\theta_1^s)$ is a saddle periodic point of the map $T_{\alpha,\delta,a,\varepsilon}$, and $W^u(\operatorname{Orb}_{T_{\alpha,\delta,a,\varepsilon}}(p)) = W^u(\operatorname{Orb}_{G_{a,1}}(\bar{p})) \times \{\theta_1^s\}$. Therefore $\mathscr{A} = \operatorname{Cl}(W^u(p)) = \widetilde{\mathscr{A}} \times \{\theta_1^s\}$ is a

Hénon-like attractor of $T_{\alpha,\delta,a,\varepsilon}$. Moreover, the basin of attraction of $\mathrm{Cl}\,(W^u(p))$ has nonempty interior in $\mathbb{R}^2 \times \mathbb{S}^1$ because of (34). This proves the claim for $(\alpha, \delta) \in \mathrm{Int}\,\mathfrak{A}^1$.

We pass to the case of higher period tongues. Suppose that $(\alpha, \delta) \in \mathrm{Int}\,\mathfrak{A}^{q/n}$, with $n > q \geq 1$. Then $A_{\alpha,\delta}$ has (at least) two hyperbolic periodic orbits

$$\begin{aligned} \mathrm{Orb}(\theta_1^s) &= \{\theta_1^s, \theta_2^s, \ldots, \theta_n^s\} \quad \text{attracting, and} \\ \mathrm{Orb}(\theta_1^r) &= \{\theta_1^r, \theta_2^r, \ldots, \theta_n^r\} \quad \text{repelling.} \end{aligned}$$

For $j = 1, \ldots, n$, denote by Θ_j the manifold

$$\Theta_j = \left\{(x, y, \theta) \in \mathbb{R}^2 \times \mathbb{S}^1) \;\mid\; \theta = \theta_j^s\right\},$$

and define maps G_j as the restriction of $T_{\alpha,\delta,a,\varepsilon}$ to Θ_j:

$$\begin{aligned} G_j : \Theta_j &\to \Theta_{j+1} \qquad \text{for } j = 1, \ldots, n-1 \\ G_n : \Theta_n &\to \Theta_1, \qquad \text{where} \\ (x,\ y,\ \theta_1^s) &\overset{G_j}{\mapsto} (Q_a(x) + \varepsilon f_j,\ \varepsilon g_j,\ \theta_{j+1}^s), \qquad \text{for } j = 1, \ldots, n-1 \\ (x,\ y,\ \theta_n^s) &\overset{G_n}{\mapsto} (Q_a(x) + \varepsilon f_n,\ \varepsilon g_n,\ \theta_1^s). \end{aligned}$$

Here, $f_j = f(a, x, y, \theta_j^s, \alpha, \delta)$. The manifold Θ_1 is invariant and attracting under the n-th iterate of the map $T_{\alpha,\delta,a,\varepsilon}$. For all (x, y, θ) in the complement of the set

$$\{(x, y, \theta) \;\mid\; \theta \in \mathrm{Orb}(\theta_1^r)\},$$

the asymptotic dynamics is given by the map

$$G_{a,1,\ldots n} \overset{\text{def}}{=} G_n \circ G_{n-1} \circ \cdots \circ G_1.$$

Notice that each of the G_j's is an η_j-perturbation of the family Q_a in the sense of Definition 5, where $\eta_j = B\varepsilon$ and B can be chosen uniform on θ_j^s (and, therefore, on (α, δ)).

Let p_0 be the periodic point of M_{a^*} as given by Proposition 12. Then $(p_0, 0)$ is a saddle periodic point for the map $\overline{M}_a$ defined as in (21). Take η, $\bar{a}$, and χ as in Proposition 12. By inductive application of Lemma 13 there exists an $\bar{\varepsilon} > 0$ depending on η and n such that

$$\|G_{a,1,\ldots n} - Q^n\|_{C^3} < \eta,$$

for all $(\alpha, \delta) \in \mathrm{Int}\,\mathfrak{A}^{q/n}$ and all $|\varepsilon| < \bar{\varepsilon}$. That is, $G_{a,1,\ldots n}$ is an η-perturbation of M_a for all q with $1 \leq q < n$ and all $(\alpha, \delta, a, \varepsilon)$ with

$$(\alpha, \delta) \in \mathfrak{A}^{q/n}, \quad \varepsilon \in [-\bar{\varepsilon}, \bar{\varepsilon}].$$

By Proposition 12 there exist an $\bar{a} > 0$ and a set $\mathfrak{S}$ contained in the interval $[a^* - \bar{a}, a^* + \bar{a}]$ such that $\mathrm{meas}(\mathfrak{S}) \geq \chi$ and the following holds. For all $a \in [a^* - \bar{a}, a^* + \bar{a}]$ the map $G_{a,1,\ldots,n}$ has a periodic point $\bar{p}_a$ which is the continuation of the periodic point $(p_0, 0)$ of $\overline{M}_a$. Moreover, for all $a \in \mathfrak{S}$ the closure $\widetilde{\mathscr{A}} = \mathrm{Cl}\left(W^u(\mathrm{Orb}_{G_{a,1,\ldots,n}}(\bar{p}_a))\right)$ is a Hénon-like attractor of $G_{a,1,\ldots,n}$, contained inside Θ_1.

To finish the proof, observe that $p_a = (\bar{p}_a, \theta_1^s)$ is a saddle periodic point of $T_{\alpha,\delta,a,\varepsilon}$. The set $\mathscr{A} = \mathrm{Cl}\left(W^u(\mathrm{Orb}_{T_{\alpha,\delta,a,\varepsilon}}(p_a))\right)$ is compact and invariant under $T_{\alpha,\delta,a,\varepsilon}$. To show that $\mathscr{A}$ has a dense orbit, fix parameter values as provided by Proposition 12 applied to $G_{a,1,\ldots n}$. Let $z \in \Theta_1$ a point having a dense orbit in $\widetilde{\mathscr{A}}$ and satisfying properties (a)–(c) of Proposition 10. Then given $\eta > 0$ and a point

$$q = T^j_{\alpha,\delta,a,\varepsilon}(q') \in T^j_{\alpha,\delta,a,\varepsilon}(\widetilde{\mathscr{A}} \times \{\theta_1^s\}), \quad \text{with } \; 1 \leq j \leq n-1,$$

there exists $m > 0$ such that $\operatorname{dist}(G^m_{a,1,\dots n}(z), q') < \eta$. By continuity of $T^j_{\alpha,\delta,a,\varepsilon}$, for all $\varrho > 0$ there exists $\eta > 0$ such that

$$\operatorname{dist}(T^j_{\alpha,\delta,a,\varepsilon}(q''), T^j_{\alpha,\delta,a,\varepsilon}(q')) < \varrho \quad \text{for all } q'' \text{ with } \quad \operatorname{dist}(q'', q') < \eta.$$

We conclude that for all $\varrho > 0$ there exists $m > 0$ such that

$$\operatorname{dist}(T^j_{\alpha,\delta,a,\varepsilon}(G^m_{a,1,\dots n}(z)), T^j_{\alpha,\delta,a,\varepsilon}(q')) = \operatorname{dist}(T^{j+mn}_{\alpha,\delta,a,\varepsilon}(z), q) < \varrho.$$

This proves that the orbit of z under $T_{\alpha,\delta,a,\varepsilon}$ is dense in $\mathscr{A}$. Properties (11) and (12) will now be proved. Since $G_{a,1,\dots,n} = T^n_{\alpha,\delta,a,\varepsilon}$ on Θ_1, for any $m \in \mathbb{N}$ and any $z \in \mathscr{A}$ we have

$$DT^m_{\alpha,\delta,a,\varepsilon}(z) = DT^r_{\alpha,\delta,a,\varepsilon}(G^s_{a,1,\dots,n}(z))DG^s_{a,1,\dots,n}(z),$$

where $s = m \mod n$ and $r = m - s$. Take z as above and a vector $v = (v_x, v_y, 0) \in T_z\mathscr{A}$ such that $\left\|DG^s_{a,1,\dots,n}(z)v\right\| \geq \kappa\lambda^s$ for all s, where $\kappa > 0$ and $\lambda > 1$ are constants. Since $T^r_{\alpha,\delta,a,\varepsilon}$ is a diffeomorphism for all $r = 1, \dots, s-1$ and $G^s_{a,1,\dots,n}(z)$ belongs to the compact set $\mathscr{A}$ for all $s \in \mathbb{N}$, then there exists a constant $c > 0$ such that

$$\left\|DT^m_{\alpha,\delta,a,\varepsilon}(z)v\right\| = \left\|DT^r_{\alpha,\delta,a,\varepsilon}(G^s_{a,1,\dots,n}(z))DG^s_{a,1,\dots,n}(z)v\right\| \geq c\left\|DG^s_{a,1,\dots,n}(z)v\right\|,$$

where c is uniform in r. This proves property (11). Property (12) is proved similarly. This shows that the closure $\operatorname{Cl}(W^u(p_a))$ is a Hénon-like attractor of $T_{\alpha,\delta,a,\varepsilon}$. $\square$

Remark 5. At the boundary of a tongue $\mathfrak{A}^{q/n}$ the Arnol′d family $A_{\alpha,\delta}$ has a saddle-node periodic point θ_1. However, the basin of attraction of $\operatorname{Orb}\theta_1$ still has nonempty interior, so that the above conclusions hold for all (α, δ) in the closure $\operatorname{Cl}\left(\mathfrak{A}^{q/n}\right)$.

4 Numerical methods, results and interpretation

4.1 Methods and selection of parameters

An important tool in the numerical exploration of dynamical systems consists of the computation of Lyapunov exponents. Let us take a three-dimensional map T as before, with an orbit $\{x_j, j = 0, 1, 2, 3, \dots\}$. Following [35] we start with three independent tangent vectors, of which the successive iterates under the derivative DT are computed, after some transient. At each step (or after a given number of steps to speed up the process) the vectors are orthonormalised.

It is useful to introduce *Lyapunov sums*, as we show now, for simplicity just considering the iterates of one tangent vector. Let v_0 be the initial vector and write $v_n = DT^n_{x_0}(v_0)/\|DT^n_{x_0}(v_0)\|$, i.e., the normalised tangent vector obtained after n iterations. Let $\hat{v}_{n+1} = DT(x_n)v_n$. Then $v_{n+1} = \hat{v}_{n+1}/f_{n+1}$, where $f_{n+1} = \|\hat{v}_{n+1}\|$. The Lyapunov sum then is defined as

$$LS_n = \sum_{j=1}^{n} \log(f_j). \tag{35}$$

The maximal Lyapunov exponent then is the average slope of the Lyapunov sum (35) as $n \to \infty$, that is, the average of the logarithmic rates of increase of the length $\log(f_j)$. The other Lyapunov exponents are estimated as averages of Lyapunov sums in which the coefficients f_j are given by the Gram-Schmidt orthonormalisation, see [35] for details.

In the numerical procedure estimates are produced of the average slope of LS_n up to some n for different values of n up to a maximal number of N iterates. The computations are stopped before N iterates in case of escape, or if a periodic orbit is detected, or if different estimates of the average coincide within a prescribed tolerance ρ. Typical values for N and ρ in the present computations are 10^7 and 10^{-6}, respectively.

One of the major problems is to detect values of the Lyapunov exponents very close to zero. To this end several procedures have been proposed for obtaining the limit. Taking into account that in the skew case the driving behaviour is quasi-periodic or periodic, a method of successive filtering and fitting, similar to [7] can be suitable. Another method like MEGNO (see [12] for an exposition and examples and [23] for a problem which requires a massive use of it) is based of weighted averages and is very useful to detect small values of the Lyapunov exponent. However, presently we simply use the Lyapunov sum (35) because this will help to understand the behaviour of the system in some elementary cases, see Section 4.3.

To scan the behaviour of family $\mathcal{T}$ given by (7), several parameters have been fixed. We chose $\delta = 0.6$ such that resonant zones of the Arnol'd family are not too narrow, while still most of the values of α give rise to quasi-periodic dynamics. Concerning the parameters a and b of the Hénon family, we fixed $b = 0.3$ for historical reasons. It is the value used by Hénon [18], and it is a good compromise between dissipation and visibility of the folds of the unstable manifold. It was also used in [34], where the various attractors for this value of b, for several values of a was studied, as well as the role of homoclinic and heteroclinic tangencies (later on in the literature known as 'crises'). The value $a = 1.25$ corresponds to a periodic attractor of period 7 and allows for moderate values of ε in the forcing before escape occurs. Finally we selected the values $\mu = 0$ and $\mu = 0.01$ for the skew and the fully coupled case, respectively. Other values of μ have also been investigated, see Section 4.2.

Let $\Lambda_1 \geq \Lambda_2 \geq \Lambda_3$ be the three Lyapunov exponents. Since the family (7) is dissipative, the role of Λ_3 is not very relevant. It can only help to decide, in case of periodic or quasi-periodic attractors, whether the normal behaviour is of nodal type ($\Lambda_2 > \Lambda_3$) or of focal type ($\Lambda_2 = \Lambda_3$). The major role is played by Λ_1 and Λ_2 and their position with respect to zero.

In some cases it is also interesting to use a complementary tool to help to decide whether the attractor is quasi-periodic or has some 'strange' character. This occurs for small values of ε. In the skew case $\mu = 0$ one may expect to have a period 7 invariant curve if (α, δ) is in the quasi-periodic domain and ε is sufficiently small. Similar behaviour can be expected for $\mu > 0$ small, for a large relative measure set in (α, δ).

The following method has been used. Consider iterates of $\mathcal{T}^7$, after some transient, and sort them by the values of θ. If the attractor is an invariant curve $(x(\theta), y(\theta))$, then the variation of the components (x, y) can be estimated from the iterates. This variation has to remain bounded when the number of iterates increases and tend to the true variation (but see Section 4.3 for some wrong interpretations of the results of these computations). To recognise Hénon-like attractors for the fully coupled case $\mu > 0$ a similar device is used. The values of the angular component θ of the iterates, should cluster around the periodic orbit obtained in the case $\mu = 0$.

4.2 Numerical results

The diagrams in Figures 1 (C) and 6 are based on the values of the first two Lyapunov exponents Λ_1 and Λ_2. To be more precise, in Figure 1 (C) code 1 (yellow) corresponds to $0 > \Lambda_1$, code 2 (blue) to $0 = \Lambda_1 > \Lambda_2$, code 3 (red) to $\Lambda_1 > 0 > \Lambda_2$ and code 4 (light blue) to $\Lambda_1 > 0 = \Lambda_2$. Typically we considered a Lyapunov exponent as equal to zero whenever $|\Lambda_j| < 10^{-5}$, $j = 1, 2$.

In Figure 6 a new case appears, namely where $\Lambda_1 > \Lambda_2 > 0$. Here the value of Λ_2 is small but definitely positive. It occurs for some regions of the (α, ε) parameter plane which, in the skew case $\mu = 0$ of Figure 1 (C) seemingly show quasi-periodic Hénon-like attractors. For these parameter values we maintain the code 4 (light blue). Furthermore, inside the case $\Lambda_1 > 0 > \Lambda_2$ one has to distinguish two subcases: one which corresponds to Hénon-like attractors (identified by the clustering of θ, as described before) and for which we keep the code 3 (red), and another with Λ_2 close to zero but definitely negative. The latter can be seen as a perturbation of the quasi-periodic Hénon attractors. We use for these the code 5 (green). The existence of parameter values for which the Lyapunov exponents are positive has been recently also found in a quite different context, related to what can be considered as a discrete version of Lorenz attractor, see [16].

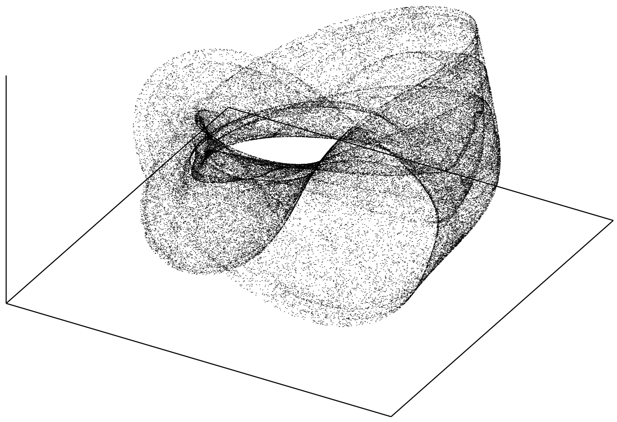

Figure 9: Attractor of $\mathcal{T}$ as in (7) for $(\alpha, \varepsilon, \mu) = (0.31, 0.13, 0.01)$, with two positive Lyapunov exponents: $\Lambda_1 \approx 0.29530$ and $\Lambda_2 \approx 0.00016$ (which is close to zero). We note that no visual difference is observed with attractors having Λ_2 negative close to zero (fully coupled case $\mu > 0$) or $\Lambda_2 = 0$ (skew product case $\mu = 0$). The representation uses variables (u, v, w) similar to Figure 3.

Interestingly, no visual differences can be observed between these attractors in the cases where $\Lambda_1 > 0$ and $\Lambda_2 = 0$ (in the skew case $\mu = 0$) or where $\Lambda_1 > 0$ and Λ_2 is close to zero, and either positive or negative (in the fully coupled model $\mu > 0$). Figure 9 displays the detected attractor for $(\alpha, \varepsilon, \mu) = (0.31, 0.13, 0.01)$ (in the region of code 4 in Figure 6). The plot uses variables (u, v, w) similar to Figure 3. Moving the parameters to $(\alpha, \varepsilon, \mu) = (0.28, 0.13, 0.01)$ (code 5 region in Figure 6) or to $(\alpha, \varepsilon, \mu) = (0.28, 0.13, 0.00)$ (code 4 region in Figure 1), in all these cases the attractor looks quite similar. Further study is needed to clarify the geometric differences, by considering the expected saddle-type invariant circle and its invariant manifolds.

Comparing Figures 1 (C) and 6 we observe:

1) The region code 1 (periodic attractors, yellow) in Figure 1 (C) is essentially preserved in Figure 6, where more periodic attractors were detected near the parameter regions that in the skew case correspond to resonance.

2) The regions with code 3 (Hénon-like attractors, red) are quite similar in both figures.

3) The region with code 4 in Figure 1 (C) (quasi-periodic Hénon attractors, light blue), in Figure 6 gives rise to regions of codes 4 (light blue) and 5 (green) in Figure 6, where the difference is given by the sign of Λ_2 (positive in region 4, negative in region 5, but always close to zero).

4) The region with code 2 (blue) in Figure 1 (C), where $\Lambda_1 = 0 > \Lambda_2$ has grown smaller in Figure 6. There are blue points in Figure 1 (C) (not too close to $\varepsilon = 0$) which have

turned into green in Figure 6 ($\Lambda_1 > 0 > \Lambda_2$). One may expect that the dynamics for these parameter values in the skew case $\mu = 0$ has a quasi-periodic attractor. The numerical evidence, at least working in double precision arithmetics, shows a different kind of attractors in the fully coupled case $\mu > 0$. In the literature these are called 'strange non-chaotic attractors' (SNA). See [17, 22, 15, 28, 29, 21] for examples and partial results in various contexts and also Section 4.3. To illustrate this difference, Figure 10 shows a magnification of both Figures 1 (C) and 6 for $\varepsilon \in [0, 0.05]$.

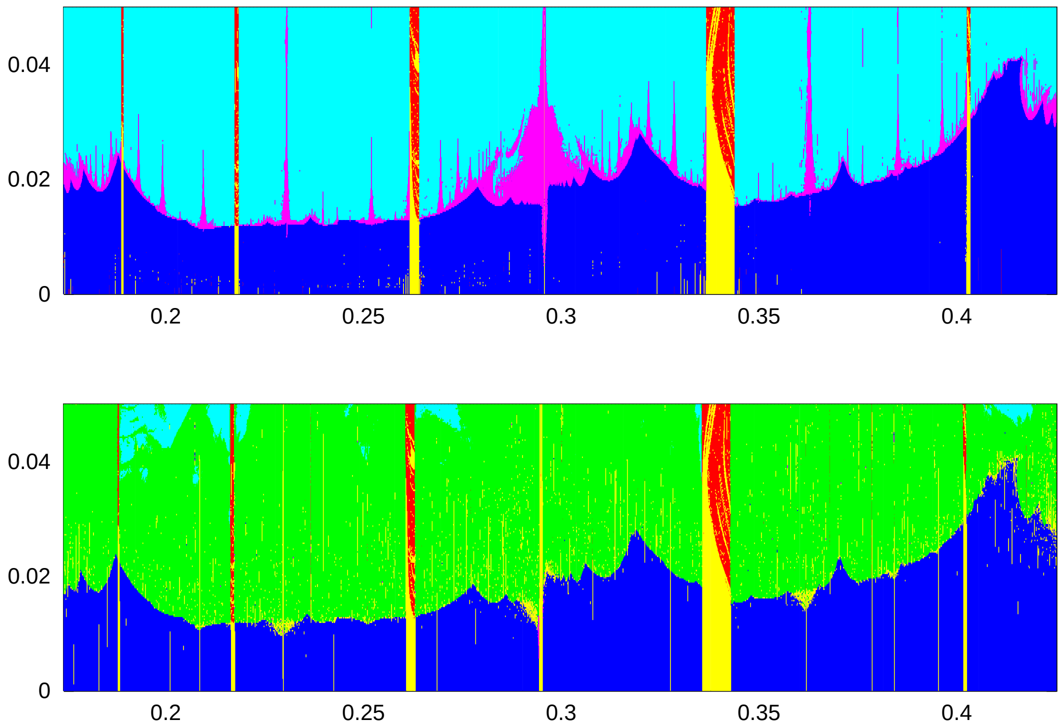

Figure 10: Magnified domain of Figures 1 (C) (top) and 6 (bottom) showing similarities and differences. In the top figure we introduced a new code 6 (in magenta), located on top of the blue region. In Figure 1 (C) this magenta domain was shown in blue. It may correspond to 'strange non-chaotic attractors'. See the text for explanation and discussion.

As explained in Section 4.1 one can use the variation as an indicator to distinguish an invariant circle from invariant sets of other types. In the skew case this reveals a large domain on the upper part of the blue region, in Figure 10 represented in magenta (code 6). This is particularly evident near the region corresponding to the resonance with rotation number 2/7. Narrow domains can be observed near other resonances.

Parameter values at the top part of Figure 10, corresponding to quasi-periodic attractors remain blue (code 2). It is quite striking that they almost exactly coincide with the blue points in the bottom part of the figure. This suggests that the domain of validity of the KAM Theorem 3 is relatively large.

Even more striking is that essentially all parameter values with code 6 (say, candidates for SNA) in the skew case, enter into the 'green region' (code 5), with exactly one positive Lyapunov exponent (where the other two are negative) in the fully coupled case. This behaviour has been checked when varying μ for a sample of values of (α, ε) : even when μ is as small as 10^{-12} this same change has been observed.

4.3 Interpretations of the numerical results

Most of the numerical study is based on the computation of Lyapunov exponents. Knowing the values of these exponents gives some hints on the dynamics, though certain ambiguities can occur. As discussed before, in cases of one positive Lyapunov exponent and two negative ones, one cannot guess whether the attractor is Hénon-like or quasi-periodic Hénon-like.

Next we present a couple of elementary examples which illustrate how careful one must be in the interpretation of the numerical results.

Arithmetic effects

The computed orbit can strongly depend on the kind of arithmetics used in the computations. Let us return for a moment to the Lyapunov sums (35). Assume that they are decreasing on average and, therefore, give evidence of a negative Lyapunov exponent. However the oscillations around a line with slope equal to the Lyapunov exponent can be very large. This means that local errors can be amplified by a big factor. If the amplification is large (before decreasing again) the propagation of the numerical errors can show a completely different dynamics.

To illustrate these numerical effects, let us consider the following toy model

$$(x,\theta) \mapsto (1-(a+\varepsilon\sin(2\pi\theta))x^2, \theta+\alpha), \tag{36}$$

which is the Logistic family, driven by a rigid rotation. Note that this is a particular case of the family (2) for $b=\delta=0$, which, however, is not a diffeomorphism.

First of all we take α small, such that a priori one may expect the system (36) to be not too far from the sequence of attractors corresponding to the 'frozen' values of θ. We took $(a,\varepsilon)=(1,30,0.30)$, thereby ensuring that the frozen values of $a+\varepsilon\sin(2\pi\theta)$ range over the interval $[1,1.6]$. In this domain the attractors of the Logistic family range from the period 2 sink to the chaotic domain, where the support of the invariant measure has a single component (say, a 'one-piece' strange attractor). The value of α has been taken small (in particular $\alpha=\gamma/1000$, where γ is the golden mean) in order to move slowly through the frozen systems, in an adiabatic way.

As a starting point we chose $\theta_0=0.6$. The value of x_0 may be taken arbitrary, presently we picked $x_0=0.123456789$. While computing iterates and the Lyapunov sums $LS_n=\sum_{j=1}^{n}\log(2a|x_j|)$ we observe that LS_n decreases (with some minor oscillations) during the first 600 iterates. It reaches a value close to -665. Then LS_n increases (except for some minor oscillations) for the next 800 iterates, reaching a value close to -460. This implies that in that transient period, along the computed orbit, local errors increase by $\exp(-460+665)\approx 10^{89}$. It may be clear that errors of the order of the computer double precision accuracy will soon produce a departure of the vicinity of the orbit that one would obtain with exact computations.

In Figure 11 we present an example of this phenomenon. If the accuracy is not sufficient, say only 60 decimal digits, we may expect the same to happen, compare with the top right part of the figure. The lower left plot, computed with 150 decimal digits, displays that the attractor is indeed an invariant circle. When the computer accuracy changes, so does the computed orbit and hence also the Lyapunov sums change. With the 'correct' values the first minimum of LS_n (neglecting period 2 small oscillations) roughly is -646 attained at $n=562$. After that we obtain a maximum of -376 for $n=1459$. The magnification $\exp(-376+646)\approx 10^{117}$ shows why a large number of digits is required.

For frozen θ the effective value of the parameter in the Logistic family is $\hat{a}=a+\varepsilon\sin(2\pi\theta)$, see (36). Period two points of the frozen system are born at $\hat{a}=3/4$, as solutions of $\hat{a}^2x^2-$

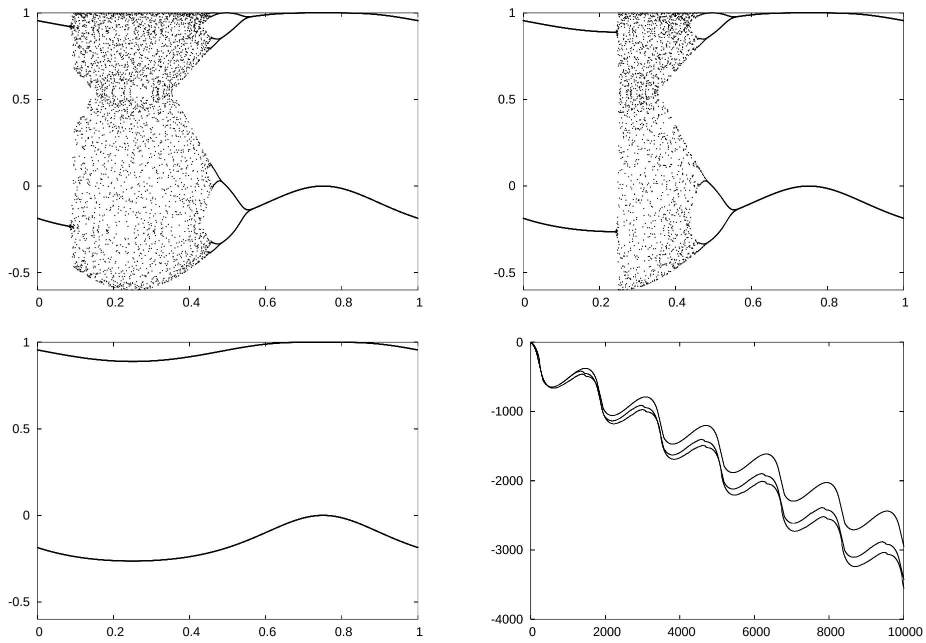

Figure 11: Top and bottom left: attractors observed for the model (36) with $(a, \varepsilon, \alpha) = (1.30, 0.30, \gamma/1000)$, where γ denotes the golden mean. In the top part we use standard double precision arithmetics (left) and 60 decimal digit arithmetics (right). The bottom left figure uses 150 decimal digit arithmetics. The bottom right part depicts the corresponding evolution of the Lyapunov sums (35). In the present example increasing the accuracy also gives an increase of the Lyapunov exponent, but it remains negative in all three the cases. See the text for details.

$\hat{a}x+1-\hat{a}=0$. On the corresponding orbit one has $|D^2T(x)| = 4|(\hat{a}-1)|$. Hence, the Lyapunov sums (35) are Riemann sums of the integral

$$\frac{1}{2\alpha}\int_{\theta_0}^{\theta_0+n\alpha} \log\left(|4(a-1+\varepsilon\sin(2\pi\theta))|\right) d\theta. \tag{37}$$

We note that the integral (37) cannot be distinguished from the upper curve at the bottom right part of Figure 11.

Invariant curves with large oscillations

As mentioned before, one might expect that an invariant curve looks very nice, with moderate oscillation, and decide that a wild oscillation should be a good evidence of the existence of an SNA. The following example shows that this expectation is not justified in all cases.

We consider a forced Logistic family, as studied in [27], given by

$$(x, \theta) \mapsto \left(\mu x(1-x) + \varepsilon\sin(2\pi\theta)), \theta + \alpha(\text{mod } 1)\right), \tag{38}$$

where $\mu = 3$ and $\alpha = \gamma$ (the golden mean). In [27] the authors claim that starting at $\varepsilon = \varepsilon^*$, where $\varepsilon^* \approx 0.1553$, there exists a range of values of ε displaying an SNA. Despite the Lyapunov exponent in the x variable is negative the attractors look like 'strange', having fractal dimension. Their conclusions are based on a too small number of iterates.

To clarify the dynamics we have proceeded as follows: We computed, after some transient, N iterates of (38). These values were sorted with respect to θ and the oscillations were computed of the x variable in the intervals $[0.0, 0.1], \ldots, [0.9, 1.0]$. Let $I^1_{j_1} := [j_1/10, (j_1+1)/10]$ be the interval with largest oscillation. Then we recompute $10N$ iterates considering only the iterates in $I^1_{j_1}$. These iterates are sorted with respect to θ and the oscillations of the x variable are computed in the subintervals of the form $[j_1/10+k/100, j_1/10+(k+1)/100], k = 0, 1, \ldots 9$. Let $I^2_{j_1,j_2}$ be the subinterval with largest oscillation. This process is repeated as many times as needed, until the maximal slope, based on the computed points, is no longer changing in a significant way. Figure 12(left) shows the results for $\varepsilon = 0.1554$. The observed attractor, that with smaller resolution may seem a strange attractor, is in fact a nice curve, certainly with a large oscillation and with large slope. Due to the fact that the method computes oscillations based on a grid, the finally selected interval may depend on the value of N and on the initial values x_0, θ_0, but the results are similar. Of course, due to the large number of iterations performed, there is a loss of digits. To overcome this source of errors we have used between 30 and 40 decimal digits in the computations.

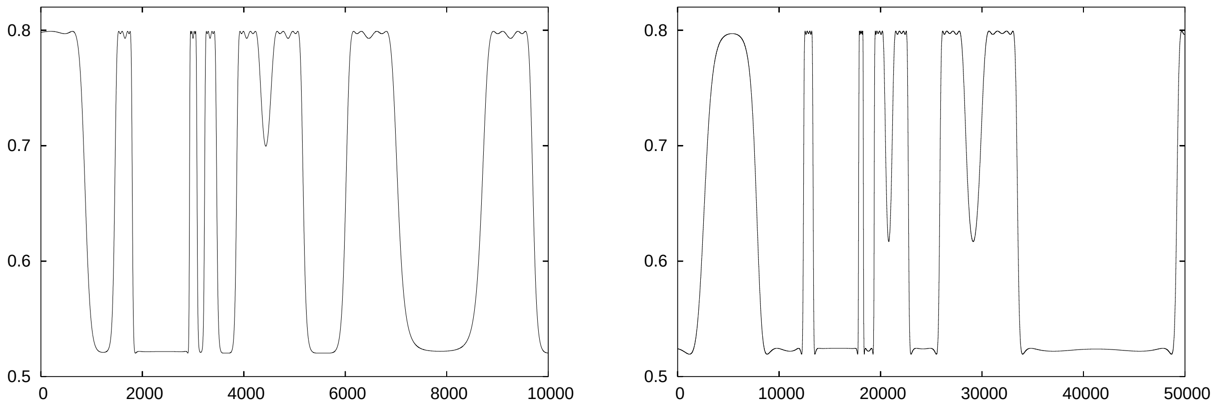

Figure 12: Invariant curves for map (38). Left: $\varepsilon = 0.1554$. On the horizontal axis we plot $(\theta - 0.0070944247) \times 10^{14}$ and on the vertical axis we plot x. Right: $\varepsilon = 0.1555$. On the horizontal axis we plot $(\theta - 0.007235958375) \times 10^{16}$. The maximum slopes in the left and right plots are 1.5×10^{12} and 4.0×10^{14}, respectively.

In the righthand part of Figure 12 similar results are shown for $\varepsilon = 0.1555$, obtained by using a variety of different methods. Preliminary results give evidence that for $\varepsilon = 0.1556$ the largest slope exceeds 10^{18}. Having negative Lyapunov exponent in the x variable implies that the continuation of the invariant curve with respect to ε is still locally possible. For further examples and theoretical discussion we refer to [21].

Summarising, we conclude that certain phenomena which might be attributed to the dynamics can, in fact, be due to a wrong interpretation of the results or to computations done with too few digits. This does not mean that results obtained with a fewer number of digits are not important. Indeed, most of the mathematical models used for concrete applications are approximations and, furthermore, 'real life' problems always contain some amount of noise. The role played in these toy models by the rounding errors can be viewed as noise. So the behaviour of a real system can be closer to the top left of Figure 11 rather than to the bottom left one. But it is always better to known why.

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