ALGEBRAIC CHARACTERIZATION OF AFFINE STRUCTURE ON JET AND WEIL BUNDLES *

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Abstract

We characterize natural transformation between Weil Bundles that are endowed with a canonical affine structure and show several cases. Those transformations are often passed down to Jet Spaces, and we characterize the cases in which the affine structure is also passed down. We prove that the classical situation is an example and give some practical generalization.

Introduction

Theory of Weil Bundles and Jet Spaces is developed in order to understand the geometry of PDE systems. C. Ehresmann formalized contact elements of S. Lie, introducing the spaces of jet of sections; simultaneously A. Weil shown [1] that the theory of S. Lie could be formalized easily by substiting the spaces of contact elements by the more formal spaces of *"point proches"*, known as Weil Bundles. The general theory of jet spaces [3] recovers the classical spaces of contact elements $J^l_m M$ of S. Lie, from the ideas and metodology of A. Weil.

In theory of Weil Bundles, morphisms $A \to B$ of Weil algebras induce *natural transformations* [5] between Weil Bundles. There are well known cases in which these natural transformations are affine bundles that often appear in differential geometry [5]. In [4] I. Kolář showed that this is the behaviour of $M^{A_l} \to M^{A_{l-1}}$. In this paper *we characterize natural transformation that are affine bundles*. It is done easily adopting a diferent point of view on the tangent space on M^A [3]: *there are a canonical affine structure for natural transformation $M^A \to M^B$ induced by surjective morphisms $A \to B$ whose kernel has null square.* It is true for $M^{A_l} \to M^{A_k}$ with $l > k \geq 0$ where $2k + 1 \geq l$.

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In some cases the natural transformations induce maps between Jet Spaces, as in [4]. We will characterize this situation, and moreover, *we will determine when an affine structure on the Weil Bundles is passed down to the Jet Spaces.* Adding to that we will prove that in this case, there also exists an affine structure between the Groups of Automophisms of Weil Algebras. It is true for spaces $J^l_m M \to J^k_m M$ with $l > k > 0$ and $3k + 1 \geq 2l$.

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Notation and conventions

All manifolds and maps are assumed to be infinitely differentiable. All results involving a manifold M assume that it is not empty, and all results involving jet spaces $J^A M$ assume that the jet space $J^A M$ is also not empty (in such case, maybe the algebraic conditions for existance of affine structure are satisfied, but there is not structure at all).

1 Weil bundles

By a *Weil algebra* we shall mean a finite dimensional, local, commutative $\mathbb{R}$-algebra with unit. If A is a Weil algebra, let us denote by $\mathfrak{m}_A$ its maximal ideal. If A, B are Weil algebras, by a morphism $A \to B$ we mean a $\mathbb{R}$-algebra morphism.

Example 1 Let be $\mathbb{R}[[\xi_1, \ldots, \xi_m]]$ the ring of formal series with real coefficients and variables $\xi_1, \ldots, \xi_m$. Let $\mathfrak{m}$ be the maximal ideal generated by those free variables. Then, for all non-negative integer l, the ring

$$\mathbb{R}^l_m = \mathbb{R}[[\xi_1, \ldots, \xi_m]]/\mathfrak{m}^{l+1}$$

is a Weil algebra.

For every Weil algebra A, there is a non-negative integer l such that $\mathfrak{m}^l_A \neq 0$ but $\mathfrak{m}^{l+1}_A = 0$; we will write that l is the *height* of A. The *width* of A is the dimension of the vectorial space $\mathfrak{m}_A/\mathfrak{m}^2_A$. In fact $\mathbb{R}^l_m$ is of height l and width m. If A is of height l and width m, then there exist one surjetive morphism $\mathbb{R}^l_m \to A$ [3, 4].

Definition 1 *Let M be a smooth manifold, and A a Weil algebra. The set M^A of the $\mathbb{R}$-algebra morphisms,*

$$p^A : \mathcal{C}^\infty(M) \to A,$$

is the so-called space of near-points of type A of M (also called A-points of M).

Let us consider a basis $\{a_k\}$ of A. For each $f \in \mathcal{C}^\infty(M)$ we can define real valued functions $\{f_k\}$ on M^A by setting:

$$p^A(f) = \sum_k f_k(p^A) a_k.$$

We shall say that $\{f_k\}$ are the *real components* of f, related to the basis $\{a_k\}$.

Theorem 1 ([3]) *The space M^A is endowed with an unique structure of smooth manifold such that the real components of smooth functions on M are smooth functions on M^A.*

Example 2 It is well known that each morphism $\mathcal{C}^\infty(M) \to \mathbb{R}$ is a point of M. Since the real components on $M^{\mathbb{R}}$ of smooth functions coincide with themselves, then we can write $M^{\mathbb{R}} = M$.

Example 3 For each Weil algebra $\mathbb{R}^l_m$ let us denote M^l_m to the space of near points of type $\mathbb{R}^l_m$. Then M^1_1 is the tangent bundle TM. In general M^l_m is the space of germs at origin of smooth aplications $\mathbb{R}^m \to M$ up to order l [1].

A Weil algebra morphism, $\phi\colon A \to B$ induces, by composition, a smooth map $\hat{\phi}\colon M^A \to M^B$ [4, 3] which is called a *natural transformation*, $\hat{\phi}(p^A) = \phi \circ p^A$.

Example 4 Let us notice that a Weil algebra A is provided with an unique morphism $A \to \mathbb{R}$. It induces a canonical map : $M^A \to M$, which is a fiber bundle. This bundle is the so-called *Weil Bundle* of type A on M. Let be $p^A \in M^A$ and p its projection in M. Then we say that p^A *is an A-point near* p. For each smooth function f the value $p^A(f)$ depends only on the *germ* at p of f.

A smooth map $f\colon M \to N$ of smooth manifolds, induces by transposition a $\mathbb{R}$-algebra morphism

$$f^*\colon \mathcal{C}^\infty(N) \to \mathcal{C}^\infty(M), \quad f^*(g) = g \circ f$$

we can compose this morphism with A-points of M obtaining A-points on N. Then Weil algebra morphisms and smooth maps transforms near-points by composition, and it suggest a functorial behaviour of *Weil Bundles* with respect those transformations.

We can formalize this situation in the following way. Let us consider the category $\mathcal{M}$ of smooth manifolds, and the category $\mathcal{W}$ of Weil algebras. In the direct product category, $\mathcal{M} \times \mathcal{W}$, objects are pairs (M, A) and morphims are pairs (f, ϕ). We define $w(M, A) = M^A$, and then the natural way of defining the functorial image of the pair

$$f\colon M \to N, \qquad \phi\colon A \to B,$$

is
$$w(f,\phi)\colon M^A \to N^B, \qquad p^A \mapsto \phi\circ p^A \circ f^*.$$

Then, the following result holds:

Proposition 1 *The assignment*
$$w\colon \mathcal{M}\times\mathcal{W} \rightsquigarrow \mathcal{M}, \qquad (M,A)\rightsquigarrow M^A,$$
is a covariant functor.

Remark 1 There are two remarkable cases of induced maps:

- If $X\subset M$ is an embedding, then for all A the induced map $X^A\to M^A$ is an embedding.
- If $A\to B$ is an surjective morphism, then for all M, the induced natural transformation $M^A\to M^B$ is a fiber bundle.

Example 5 Let A be of order l. For each $k\leq l$ let us define $A_k = A/\mathfrak{m}_A^{k+1}$. Then A_k is a Weil algebra of order k, and $A_l = A$. For $k\geq r$ we have a natural proyection $M^{A_k}\to M^{A_r}$ which is a bundle. In particular we have canonical bundles $M_m^k\to M_m^r$.

1.1 Tangent Estructure

Given $p^A\in M^A$, let us denote $\mathrm{Der}_{p^A}(\mathcal{C}^\infty(M),A)$ the space of *derivations* from $\mathcal{C}^\infty(M)$ into A, where the structure of A is induced by p^A. It means, $\mathbb{R}$-linear maps $\delta\colon\mathcal{C}^\infty(M)\to A$ satisfying Leibnitz's formula:
$$\delta(f\cdot g) = p^A(f)\cdot\delta(g) + p^A(g)\cdot\delta(f). \tag{1}$$

If D is a tangent vector to M^A to p^A, then it defines a derivation by setting:
$$\delta(f) = \sum_k (Df_k)a_k,$$
and it is easy to proof that the spaces $\mathrm{Der}_{p^A}(\mathcal{C}^\infty(M),A)$ and $T_{p^A}M^A$ are identified in this way [3]. From now on we will assume this identification. It is not only as vector spaces, it is compatible with proposition 1, in the following way.

Theorem 2 ([3]) *Let us consider a smooth map $f\colon M\to N$, a Weil algebra morphism $\phi\colon A\to B$, and the induced smooth map*
$$w(f,\phi)\colon M^A\to N^B, \qquad p^A\mapsto q^B = \phi\circ p^A\circ f^*.$$
Then the linearizated map $w(f,\phi)'\colon T_{p^A}M^A \to T_{q^B}N^B$ coincides (under the identification assumed above) with the map:
$$\mathrm{Der}_{p^A}(\mathcal{C}^\infty(M),A)\to \mathrm{Der}_{q^B}(\mathcal{C}^\infty(N),B), \qquad \delta\mapsto\phi\circ\delta\circ f^*.$$

1.2 Affine Structure

In this section we will analize the stucture of the fiber bundles induced by a surjective morphism $A \to B$ that has been introduced in remark 1. In some specific cases, [4, 3] it has been proved that those bundles are endowed with a canonical stucture of affine bundles. We will see that this structure has its foundation in the algebraic construction of *near-points spaces*. Indeed, it is an easy task give an algebraic characterization of this fact. A morphism will induce affine structure if and only if its kernel ideal has null square.

The key point is to consider both *near-points* and *tangent vectors to* M^A as $\mathbb{R}$-linear maps from $\mathcal{C}^\infty(M)$ into a Weil algebra. Then they are provided of an addition law as $\mathbb{R}$-linear maps. Under some adequate assumptions we will obtain a new near-point when adding a near-point and a derivation.

Lemma 1 *Let us consider* $p^A \in M^A$ *and* $D \in T_{p^A}M^A$*. Then,* $p^A + D$ *is an* A*-point of* M *if and only if* $(\mathrm{Im}(D))^2 = 0$.

PROOF. Let us define, $\tau = p^A + D$, then for all $f, g \in \mathcal{C}^\infty(M)$,

$$\tau(f \cdot g) = \tau(f) \cdot \tau(g) - D(f) \cdot D(g),$$

since τ is $\mathbb{R}$-linear, it is an algebraic morphism if and only if for all $f, g \in \mathcal{C}^\infty(M)$, $D(f) \cdot D(g) = 0$. □

Lemma 2 *Let us consider* $p^A, q^A \in M^A$*, then* $\delta = q^A - p^A$ *is a derivation and belongs to* $T_{p^A}M$ *if and only only if* $(\mathrm{Im}(\delta))^2 = 0$.

PROOF. For all $f, g \in \mathcal{C}^\infty(M)$,

$$\delta(f \cdot g) = p^A(f) \cdot \delta(g) + p^A(g) \cdot \delta(f) + \delta(f) \cdot \delta(g),$$

then δ satifies Leibnitz's formula if and only if for all $f, g \in \mathcal{C}^\infty(M)$, $\delta(f) \cdot \delta(g)$ is null. □

Lemma 3 *Let* I *be an ideal of a ring* R*. The following conditions are equivalent:*

1. $I^2 = 0$
2. *for all* $x \in I$*,* $x^2 = 0$.

PROOF. Let us assume *2.*, then for $x, y \in I$,

$$0 = (x + y)^2 = x^2 + y^2 + 2xy = 2xy.$$

□

Let $\phi\colon A \to B$ be a surjective morphism of Weil algebras, and let I be its kernel ideal. Let us consider a smooth manifold M, and the induced bundle $\hat{\phi}\colon M^A \to M^B$. The linearization $\hat{\phi}'$, gives rise to the exact sequence,

$$0 \to TV^{\hat{\phi}}_{p^A} M^A \to T_{p^A} M^A \xrightarrow{\hat{\phi}'} T_{p^B} M \to 0$$

that defines the tangent vertical sub-bundle $TVM^A \subset TM^A$. Assuming that tangent vectors are derivations from $\mathcal{C}^\infty(M)$ into A, we will notice that $D \in T_{p^A} M$ belongs to $TV^{\hat{\phi}}_{p^A} M^A$ if and only if $\mathrm{Im}(D) \subset I$.

If $I^2 = 0$, then for all $D \in TV^{\hat{\phi}}_{p^A} M$, we have $(\mathrm{Im}D)^2 \subset I^2 = 0$, and we have a map,

$$\tau_{p^A} : TV^{\hat{\phi}}_{p^A} M^A \to \hat{\phi}^{-1}(p^B), \qquad D \mapsto p^A + D,$$

wich attending to lemma 1, and lemma 2 is an isomophism (because for all $q^A \in \hat{\phi}^{-1}(p^B)$, $q^A - p^A$ takes values in I). Adding to that, for $p^A, q^A \in \hat{\phi}^{-1}(p^B)$, the spaces $TV^{\hat{\phi}}_{p^A} M^A$ and $TV_{q^A} M^A$ are identified in a canonical way. For each $a, b \in A$ such that $\phi(a) = \phi(b)$, $c \in I$, the product $a \cdot c = b \cdot c$, so that p^A and q^A define the same structure on I, so the space of derivations $\mathcal{C}^\infty(M) \to I$ concides. Then, we can define the vector bundle

$$TV^{\hat{\phi}} \to M^B,$$

whose fiber $TV^{\hat{\phi}}_{p^B}$ is the space of derivations $\mathcal{C}^\infty(M) \to I$, where the structure of I if given by any $p^A \in \hat{\phi}^{-1}(p^B)$. Then, $TV^{\hat{\phi}}$ is the vector bundle that modelizes the affine bundle $\hat{\phi}\colon M^A \to M^B$,

$$M^A \times_{M^B} TV^{\hat{\phi}} \to M^A, \qquad (p^A, D) \mapsto p^A + D.$$

On the other hand, if $I^2 \neq 0$, then aplying lemma 3 we can find a derivation $D\colon \mathcal{C}^\infty(M) \to I$ such that $(\mathrm{Im}(D))^2 \neq 0$ and then $p^A + D$ does not belong to M^A. We have proved:

Theorem 3 *Let $\phi\colon A \to B$ be a surjective morphism of Weil algebras, and let I be its kernel ideal. For all manifold M, the addition of derivations and morphisms induces an affine structure on the fiber bundle $\hat{\phi}\colon M^A \to M^B$, if and only if $I^2 = 0$.*

Corolary 1 *Let A of height l, then the natural projection $M^A \to M^{A_k}$ is endowed with a canonical structure of affine bundle if and only if $2k + 1 \geq l$.*

Corolary 2 *The natural projection $M^l_m \to M^k_m$ is endowed with a canonical estructure of affine bundle if and only if $2k + 1 \geq l$.*

2 Jet Spaces

Definition 2 *A jet of M is an ideal $\mathfrak{p} \subset \mathcal{C}^\infty(M)$ such that the quotient algebra $A_\mathfrak{p} = \mathcal{C}^\infty(M)/\mathfrak{p}$ is a Weil algebra. $\mathfrak{p}$ is said of type A, or A-jet, if $A_\mathfrak{p}$ is isomorphic to A. The set $J^A M$ of A-jet is de so-called A-jet space of M.*

An A-point $p^A \in M^A$ is said *regular* if it is a surjective morphism. The set of *regular* A-points is denoted $\check{M}^A$. It is a dense open subset of M^A. It is obvious that an A-point is regular if and only if its kernel is an A-jet. Thus, we have a surjective map:

$$\ker\colon \check{M}^A \to J^A M \tag{2}$$

Let us consider $\mathrm{Aut}(A)$ the group of *automorphisms* of A; it is an algebraic Lie Group. Thus, $\mathrm{Aut}(A)$ acts on $\check{M}^A$ by composition. Two A points related by an automorphism have the same kernel, moreover two A-points with the same kernel are related by an automorphism. Then $J^A M$ is identified the space of orbits $\check{M}^A/\mathrm{Aut}(A)$, and its manifold structure is determined in this way.

Example 6 The group G^l_m of automorphisms of $\mathbb{R}^l_m$ is called *l-prolongation of the linear group of order m* as can be seen at [6]. In particular G^1_m is the linear group or order m. G^l_m is the group of transformations of $\mathbb{R}^l_m$ around a fix point up to order l.

Theorem 4 ([2]) *There is an unique structure of smooth manifold on $J^A M$ such that $J^A M$ such that* ker *(2) is a principal bundle of structural group* $\mathrm{Aut}(A)$*.*

Example 7 Let us denote $J^l_m M$ to the space of jets of type $\mathbb{R}^l_m$. Then $J^l_m M$ is the spaces of germs m-submanifolds of M up to order l.

$J^A M$ is a bundle over M. We will say that $\mathfrak{p} \in J^A M$ is a jet on $p \in M$ if $\mathfrak{p} \subset \mathfrak{m}_p$,

$$\mathfrak{m}_p = \{f \in \mathcal{C}^\infty(M)\colon f(p) = 0\}.$$

If p^A is an A-point *near* p, then $\ker(p^A)$ is a *jet on* p.

2.1 Functorial behaviour

Let us notice that jet spaces do not show the functorial behaviour that Weil bundles shown. A smooth map $f\colon M \to N$ induces a smooth map in jet spaces, but in general case it is defined only on an open dense subset of $J^A M$, wich depends on f. There is not a natural object associated to a Weil algebra morphism $A \to B$. The natural object associated to the pair (A, B) is some subespace $\Lambda_{A,B} M \subset J^A \times_M J^B M$,

$$\Lambda_{A,B} M = \{(\mathfrak{p}, \bar{\mathfrak{p}}) \in J^A M \times_M J^B M \colon p \subseteq \bar{p}\}$$

that we call the *Lie correspondance*.

This subespace is empty if and only if there is not a surjective morphism from A to B. There is an special case to be analized in wich the Lie correspondance $\Lambda_{A,B}M$ is the graph of a bundle $J^AM \to J^BM$.

Let I be an ideal of A. Then, for each $\sigma \in \mathrm{Aut}(A)$, the space $\sigma(I)$ is another ideal of A. We will say that I is an *invariant ideal* if for all $\sigma \in \mathrm{Aut}(A)$, we have $\sigma(I) = I$. For each k, $\mathfrak{m}_A^k$ is an invariant ideal, and any other ideals obtained from those by general processes of division or derivation are also invariant; some examples are shown at [7]. Let $I \subset A$ be and $\phi\colon A \to A/I = B$ the canonical projection. Then, whe have a commutative diagram:

$$\begin{array}{ccccccc} \check{M}^A & \xrightarrow{\check{\phi}} & \check{M}^B & \qquad & p^A & \longrightarrow & \check{\phi}(p^B) = \phi \circ p^A \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ J^AM & \xrightarrow{\phi^j} & J^BM & & \mathfrak{p} & \longrightarrow & \bar{\mathfrak{p}}\colon\ \mathfrak{p} \subset \bar{\mathfrak{p}} \end{array} \tag{3}$$

Sumarizing, the following result holds:

Theorem 5 *If $I \subset A$ is an invariant ideal, and $B = A/I$ is the quotient algebra, then there is a canonical bundle estructure $J^AM \to J^BM$.*

On the opossite hand it can be shown that if the *Lie correspondance* $\Lambda_{A,B}M$ takes the form of the graph of a bundle $J^AM \to J^BM$ then for each surjective morphism $\phi\colon A \to B$, $\ker(\phi)$ is an invariant ideal.

2.2 Tangent structure

In order to study a linearization of ϕ^j (3), we need some characterization of the tangent space to J^AM on a jet $\mathfrak{p}$.

Theorem 6 ([2, 3]) *$T_{\mathfrak{p}}(J^AM)$ realizes itself canonically as a quotient of the space of derivations $\mathcal{C}^\infty(M) \to A_{\mathfrak{p}}$. A derivation δ defines the null vector if and only if $\delta(\mathfrak{p}) = 0$, thus:*

$$T_{\mathfrak{p}}(J^AM) \simeq \mathrm{Der}(\mathcal{C}^\infty(M), A_{\mathfrak{p}})/\mathrm{Der}(A_{\mathfrak{p}}, A_{\mathfrak{p}}).$$

Let us give some sketch of proof. Let us recall that the Lie algebra of $\mathrm{Aut}(A)$ is the space of derivations $\mathrm{Der}(A, A)$ [3], as can be shown in a matrix representation of the group. Taking $p^A \in \check{M}^A$ such that $\ker(p^A) = \mathfrak{p}$, the representation of $\mathrm{Der}(A, A)$ as fundamental vector fields gives rise to an exact sequence:

$$0 \to Der(A, A) \xrightarrow{*} T_{p^A}(M^A) \to T_{\mathfrak{p}}J^AM \to 0,$$

taking account that $T_{p^A}(M^A) = \mathrm{Der}(\mathcal{C}^\infty(M), A)$, and that p^A induces an unique isomorphism between $A_{\mathfrak{p}}$ and A, we will have isomorphism of the theorem. This isomorphism does not depend on wich p^A we choose for representing $\mathfrak{p}$. It is seen easily using the principal estructure (theorem 4).

3 Affine Structure on Jet Spaces

3.1 Space of regular Points

Let I be an ideal of the Weil algebra A, and $\phi\colon A \to B$ the canonical projection into the quotient algebra B.

Lemma 4 ([3]) *A finite set $\{a_1, \ldots, a_m\} \subset \mathfrak{m}_A$ is a system of generators of A, if and only if $\{\bar{a}_1, \ldots, \bar{a}_m\} \subset \mathfrak{m}_A/\mathfrak{m}_A^2$ is a basis of $\mathfrak{m}_A/\mathfrak{m}_A^2$.*

Lemma 5 *If $I \not\subset \mathfrak{m}_A^2$ then exist a non trivial sub-algebra $S \subset A$ such that $S/(S \cap I) \simeq B$.*

PROOF. If $I \not\subset \mathfrak{m}_A^2$, then the canonical projection $\mathfrak{m}_A/\mathfrak{m}_A^2 \to m^B/\mathfrak{m}_B^2$ has non trivial kernel. There exist a finite set $\{a_1, \ldots, a_m\} \subset \mathfrak{m}_A$ such that $\{\phi(a_k)\}$ is a basis of $\mathfrak{m}_B/\mathfrak{m}_B^2$, but $\{\bar{a}_k\}$ is not a basis of $\mathfrak{m}_A/\mathfrak{m}_A^2$. Then $S = \mathbb{R}[\{a_k\}]$ verifies $S/(S \cap (I)) = B$. $\square$

Note that each sub-algebra of A is a Weil algebra. For each subset $X \subset \mathfrak{m}_A$, $\mathbb{R}[X]$ is a Weil algebra and its maximal ideal is spanned X.

Lemma 6 *The following conditions are equivalent:*

1. *$I \subset \mathfrak{m}_A^2$.*
2. *$\forall p^A \in M^A$, if $\hat{\phi}(p^A)$ is regular, then p^A is also regular.*

PROOF. Let us assume $I \subset \mathfrak{m}_A$, and consider $p^A \in M^A$ such that $\hat{\phi}(p^A)$ is a regular B-point. Then there are functions $f_1 \ldots, f_m \in \mathcal{C}^\infty(M)$ such that $\{\phi(p^A(f_k))\} \subset \mathfrak{m}_B$ is a system of generators of B. Then $\{\phi(p^{\bar{A}}(f_k)\}$ is a basis of $\mathfrak{m}_B/\mathfrak{m}_B^2$. Since $I \subset \mathfrak{m}_A$ and $\mathfrak{m}_B = \mathfrak{m}_A/I$, we have $\mathfrak{m}_A/\mathfrak{m}_A^2 \simeq \mathfrak{m}_B/\mathfrak{m}_B^2$. Then $\{p^A(\bar{f}_k)\}$ is a basis of $\mathfrak{m}_A/\mathfrak{m}_A^2$, and $\{p^A(f_k)\}$ is a system of generators of A, so that p^A is regular.

Reciprocally, if $I \neq \mathfrak{m}_A^2$ let us consider $S \subset A$ as in the lemma 5. Then $M^S \to M^B$ is a bundle. Let $p^B \in \check{M}^B$ a regular B-point, and p^S any preimage of p^B. p^S is a S-point, thus it is a *non-regular* A-point, but $\hat{\phi}(p^A) = p^B$. $\square$

From now on we will consider the anihilator ideal of I, it is the ideal $\mathrm{Ann}(I)$ of elements of A that anihilate I,

$$\mathrm{Ann}(I) = \{a \in A : \forall b \in I, ab = 0\},$$

let us notice that $I \subseteq \mathrm{Ann}(I)$ if and only if $I^2 = 0$.

We will write $\check{\phi}$ for the restriction of $\hat{\phi}$ to the space of regular A-points $\check{M}^A$.

Theorem 7 *The bundle $\check{\phi}\colon \check{M}^A \to \check{M}^B$ is endowed with a canonical structure of affine bundle (given by addition of morphism and derivations) if and only if $I \subset (\mathfrak{m}_A^2 \cap \mathrm{Ann}(I))$.*

PROOF. Suppose that $I \subset (\mathfrak{m}_A^2 \cap \mathrm{Ann}(I))$. Then the addition of A-points and derivations induces an affine structure on $\check{\phi}$. Let be $p^A \in \check{M}^A$, then

$$\hat{\phi}^{-1}(\check{\phi}(p^A)) = \{p^A + D : D \in TV^{\check{\phi}}\check{M}^A\}$$

and attending to lemma 6, $\hat{\phi}^{-1}(\check{\phi}(p^A)) = \check{\phi}^{-1}(\check{\phi}(p^A))$, so that the addition of a *regular A-point* and a *derivation* is also a *regular A-point*. Then the affine structure of $\hat{\phi}$ induces an affine structure on $\check{M}^A$.

On the other hand, let us assume $I \not\subseteq (\mathfrak{m}^2 \cap \mathrm{Ann}(I))$. If $I \not\subseteq \mathrm{Ann}(I)$, then the addition of an A-point and a derivation *is not in general an A-point* and there is not affine structure. Last, let us assume that $I \subseteq \mathrm{Ann}(I)$ but $I \not\subseteq \mathfrak{m}_A^2$. Then there is affine estructure on $\hat{\phi}$, but applying lemma 6, there are non-regular A-points $p^A \in M^A$ such that $\hat{\phi}(p^A)$ is regular. Let us consider $q^A \in \check{\phi}^{-1}(\hat{\phi}(p^A))$, and $D = p^A - q^A \in TV^{\check{\phi}}_{q^A}\check{M}^A$. Then $q^A + D \notin \check{M}^A$. □

Corolary 3 *Let A be of height l. Then for each $l > k > 0$ the natural projection $\check{M}^A \to \check{M}^A_k$ is an affine bundle and only if $2k+1 \geq l$.*

Corolary 4 *For any $l > k > 0$, the natural projection $\check{M}^l_m \to M^k_m$ is an affine bundle if and only if $2k+1 \geq l$.*

3.2 Affine Structure on the Group of automorphisms

Let $I \subset A$ be an invariant ideal of the Weil algebra A, and $\phi \colon A \to B$ the canocical projection into the quotient algebra. Each automorphism $\sigma \in \mathrm{Aut}(A)$ verifies $\sigma(I) = I$, thus it induces an automorphism $\phi_*(\sigma) \in \mathrm{Aut}(B)$.

Definition 3 *We call affine sequence asociated to I the the following sequence:*

$$0 \to K(I) \to \mathrm{Aut}(A) \xrightarrow{\phi_*} \mathrm{Aut}(B) \to 0,$$

where

$$K(I) = \{\sigma \in \mathrm{Aut}(A) \colon \forall a \in A\; \sigma(a) - a \in I, \forall b \in I\; \sigma(b) = b\},$$

is the subgroup of automophisms of A inducing the identity both on B and I.

We will say that the affine sequence is *exact on the left* if $K(I) = \ker(\phi_*)$, in the same way we will say that it is *exact on the right* if ϕ_* is surjective. Note that if it is exact on the right and on the left, then it is an exact sequence.

Let us notice that if $I \subseteq \mathrm{Ann}(I)$, then I is an A-modulus, but it is also a B-modulo. By composition we have a canonical inmersion $\mathrm{Der}(B, I) \subseteq \mathrm{Der}(A, I)$ identifying derivationos from B with derivatios from A wich anihilate I.

Proposition 2 *Let us assume $I \subset \mathrm{Ann}(I)$, then the affine sequence is exact on the left then if and only if $\mathrm{Der}(A, I) \simeq \mathrm{Der}(B, I)$.*

PROOF. Assuming that the affine sequence is exact on the left let us take the sequence of Lie algebras. Note that the Lie algebra of $K(I)$ is the space of derivations which takes values on I, and anhililates I, it is $\mathrm{Der}(B,I)$.

Reciprocally let us assume that $\mathrm{Der}(A,I) = \mathrm{Der}(B,I)$, and let be $\sigma \in \ker(\phi_*)$, then $Id - \sigma$ is a derivation $A \to I$, and by hypothesis, it anihilates I, so that σ induces the identity on I and then $\sigma \in K(I)$. □

Theorem 8 *If $I \subseteq (\mathrm{Ann}(I) \cap \mathfrak{m}_A^2)$ and the affine sequence is exact then ϕ_* is endowed of a natural estructure of affine bundle modeled over the space $\mathrm{Der}(A,I)$, whit the law of addition:*

$$\sigma \oplus D = \sigma + \sigma \circ D$$

PROOF. Let $D \in Der(A,I)$. Then $Id_A + D$ is an automorphism of A. Reciprocally, let σ an automorphism of A such that $\phi_*(\sigma) = Id_B$, then $\sigma - Id_A$ is a derivation, and takes values in I.

$$Der(A,I) \simeq \ker(\phi_*)$$

By definition $\sigma \oplus D = \sigma \circ (Id + D)$, so that $\sigma \oplus Der(A,I) = \sigma \circ \ker(\phi_*)$. We must assure that

$$\sigma \oplus D \oplus D' = \sigma \oplus (D + D').$$

It comes from proposition 2,

$$\sigma \oplus D \oplus D' = \sigma + \sigma D + (\sigma + \sigma \circ D) \circ D' = \sigma \oplus (D + D') + \sigma \circ D \circ D'$$

since each derivation in $\mathrm{Der}(A,I)$ is zero on I, $D \circ D' = 0$. □

Lemma 7 *If $I \subset \mathrm{Ann}(I)^2$ then the affine sequence associated to I is exact on the left.*

PROOF. Let D be a derivation $A \to I$, and $a \in I$, then $a = \sum_k b_k c_k$, $b_k, c_k \in \mathrm{Ann}(I)$, and

$$D(a) = \sum_k b_k D(c_k) + c_k D(b_k) = 0,$$

so D anihilates I. The $\mathrm{Der}(A,I) = \mathrm{Der}(B,I)$ and we conclude by lemma 2. □

Corolary 5 *If numbers $l > r > 0$ verify $3r + 1 \geq 2l$, then the natural projection $G_m^l \to G_m^r$ is an affine bundle.*

PROOF. In general $G_m^l \to G_m^k$ is a surjective morphism. Now we may apply lemma 7 to the case $A \simeq R_m^l$, $I = \mathfrak{m}_A^{k+1}$. Then $\mathrm{Ann}(I) = \mathfrak{m}_A^{l-k}$ and $I \subset \mathrm{Ann}(I)^2$ if and only if $k + 1 \geq 2(l - k)$. □

3.3 Affine structure on Jet Spaces

Let $I \subset A$ an invariant ideal as above, and $I \subset (\mathrm{Ann}(I) \cap \mathfrak{m}_A^2)$.

For each $\mathfrak{p}$ let us denote by $\pi_{\mathfrak{p}} \colon \mathcal{C}^\infty(M) \to A_{\mathfrak{p}}$ the canonical projection, $\bar{p} = \phi^j(\mathfrak{p})$. Then $A_{\mathfrak{p}} \simeq \mathfrak{p}$ and $\bar{\mathfrak{p}}/\mathfrak{p} \simeq I$. For each $D \in \mathrm{Der}(\mathcal{C}^\infty(M), \bar{\mathfrak{p}}/\mathfrak{p})$ let us define

$$\mathfrak{p} + D = \ker(\pi_{\mathfrak{p}} + D); \tag{4}$$

then $\mathfrak{p} + D \in J^A M$ and $\phi^j(\mathfrak{p} + D) = \bar{\mathfrak{p}}$. Let us notice that this is well defined because $I \subset (\mathrm{Ann}(I) \cap \mathfrak{m}_A^2)$: $\pi_{\mathfrak{p}} + D$ is an $A_{\mathfrak{p}}$-point since $I \subset \mathrm{Ann}(I)$ and it is *regular* because $I \subseteq \mathfrak{m}_A^2$.

Lemma 8 *Each derivation $D \colon \mathcal{C}^\infty(M) \to \bar{\mathfrak{p}}/\mathfrak{p}$ wich is zero on $\mathfrak{p}$ is zero on $\bar{\mathfrak{p}}$ if and only if the affine sequence associated to I is exact on the left.*

PROOF. A derivation $\mathcal{C}^\infty(M) \to \bar{\mathfrak{p}}/\mathfrak{p}$ wich anihilates $\mathfrak{p}$ is a derivation $A_{\mathfrak{p}} \to \bar{\mathfrak{p}}/\mathfrak{p}$. Then the lemma is equivalent to lemma 2. □

Theorem 9 *The law of addition* (4) *defines an affine structure on the bundle $\phi^j \colon J^A M \to J^B M$ for all smooth manifold M if and only if the affine sequence associated to I is exact.*

PROOF. A derivation $\mathcal{C}^\infty(M) \to \bar{\mathfrak{p}}/\mathfrak{p}$ defines a tangent vector $[D] \in T_{\mathfrak{p}} J^A M$, as shown in theorem 6. Moreover $[D] \in TV_{\mathfrak{p}}^{\phi^j} J^A M$ because it takes values on $\bar{\mathfrak{p}}/\mathfrak{p}$.

Let us prove that the following conditions holds if and only if the affine sequence associated to I is exact.

1. *If D and D' define the same tangent vector $[D]$, then $\mathfrak{p} + D = \mathfrak{p} + D'$.*
2. *The natural projecion $\mathrm{Der}(\mathcal{C}^\infty(M), \bar{\mathfrak{p}}/\mathfrak{p}) \to TV_{\mathfrak{p}}^{\phi^j} J^A M$ is surjective.*
3. *For each $\mathfrak{q} \subset \bar{\mathfrak{p}}$ there is an unique $[D] \in TV_{\mathfrak{p}}^{\phi^j}$ such that $\mathfrak{p} + [D] = \mathfrak{q}$.*
4. *For each $\mathfrak{q}$ as above there is a canonical isomorphism $TV_{\mathfrak{p}}^{\phi^j} J^A M \simeq TV_{\mathfrak{q}}^{\phi^j} J^A M$.*

Let D and D' define the same tangent vector $[D] \in TV_{\mathfrak{p}}^{\phi^j} J^A M$. Then, $\delta = D' - D$ is zero on $\mathfrak{p}$. By lemma 8 each δ that anhililates $\mathfrak{p}$, anihilates also $\bar{\mathfrak{p}}$ *if and only if the affine sequence is exact on the left.* Assuming that, since $\mathfrak{p} \subset \bar{\mathfrak{p}}$ and $\mathfrak{p} + D \subset \bar{\mathfrak{p}}$ we have:

$$\ker(\pi_{\mathfrak{p}} + D) = \ker(\pi_{\mathfrak{p}} + D + \delta)$$

so hat if the affine sequence es affine on the left, *1.* holds.

We can prove *2.* as an application of the classical *snake lemma.* We have a natural diagram of exact columns and arrows,

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Der}(A_{\mathfrak{p}}, \bar{\mathfrak{p}}/\mathfrak{p}) & \longrightarrow & \mathrm{Der}(A_{\mathfrak{p}}, A_{\mathfrak{p}}) & \xrightarrow{\psi} & \mathrm{Der}(A_{\bar{\mathfrak{p}}}, A_{\bar{\mathfrak{p}}}) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Der}(\mathcal{C}^\infty(M), \bar{\mathfrak{p}}/\mathfrak{p}) & \longrightarrow & \mathrm{Der}(\mathcal{C}^\infty(M), A_{\mathfrak{p}}) & \longrightarrow & \mathrm{Der}(\mathcal{C}^\infty(M), A_{\bar{\mathfrak{p}}}) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \bar\psi} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & TV^{\phi^j}_{\mathfrak{p}} J^A M & \longrightarrow & T_{\mathfrak{p}} J^A M & \longrightarrow & T_{\bar{\mathfrak{p}}} J^B M & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Attending to the snake lemma, if $\mathrm{coker}(\psi) = 0$ we should have a sequence:

$$\ldots \to 0 \to \mathrm{coker}(\bar\psi) \to 0 \to \ldots$$

and vieceversa. Thus $\mathrm{coker}(\psi) = 0$ if and only if $\mathrm{coker}(\bar\psi) = 0$. Note natural aplication ψ is the linearization of the Lie group morphism $\mathrm{Aut}(A_{\mathfrak{p}}) \to \mathrm{Aut}(A_{\bar{\mathfrak{p}}})$. Since, $A_{\mathfrak{p}} \simeq A$ and $A_{\bar{\mathfrak{p}}} \simeq B$ we conclude that if the affine sequence associated to I is is exact on the right, then *2.* holds.

In order to prove *3.* let us consider any other A-jet $\mathfrak{q} \subset \bar{\mathfrak{p}}$, and an isomorphism $\tau\colon A_{\mathfrak{p}} \to A_{\mathfrak{q}}$.

$$\begin{array}{ccccc}
\mathcal{C}^\infty(M) & \xrightarrow{\pi_{\mathfrak{p}}} & A_{\mathfrak{p}} & & \\
 & \searrow{\scriptstyle \pi_{\mathfrak{q}}} & \uparrow{\scriptstyle \tau} & \searrow{\scriptstyle \bar\pi_{\mathfrak{p}}} & \\
 & & A_{\mathfrak{q}} & \xrightarrow[\bar\pi_{\mathfrak{q}}]{} & A_{\bar{\mathfrak{p}}}
\end{array}$$

Let us note that for each $\mathfrak{q}$, we can find τ such $\bar\pi_{\mathfrak{p}} \circ \tau = \bar\pi_{\mathfrak{q}}$ *if and only if the affine sequence is exact on the right.* If it was not, we could find $\mathfrak{q}$ such that there are A-points p^A, q^A representing $\mathfrak{p}, \mathfrak{q}$, and $\check\phi(p^A), \check\phi(q^A)$, wich represent $\bar{\mathfrak{p}}$ are not related by any automorphism of A.

Assuming that there is such τ, $\pi_{\mathfrak{p}}$ and $\tau \circ \pi_{\mathfrak{q}}$ are regular $A_{\mathfrak{p}}$ points that are projected on the same $A_{\bar{\mathfrak{p}}}$ point $\pi_{\bar{\mathfrak{p}}}$. Then, $D = \pi_{\mathfrak{p}} - \tau \circ \pi_{\mathfrak{q}}$ is a derivation of $\mathcal{C}^\infty(M)$ and takes values on $\bar{\mathfrak{p}}/\mathfrak{p}$; It defines a vertical vector of $[D] \in TV^{\phi^j}_{\mathfrak{p}} J^A M$ and it is obvious that

$$\mathfrak{p} + [\delta] = \mathfrak{q}.$$

Let τ be as above, and $\bar\tau\colon A_{\mathfrak{p}} \to A_{\mathfrak{q}}$ under identical assumptions. Then $\sigma = \tau \circ \bar\tau$ is an isomorphism of $A_{\mathfrak{p}}$ that induces the identity on $A_{\bar{\mathfrak{p}}}$ such as the affine sequence is exact on the right, σ induces the identitiy on $\bar{\mathfrak{p}}/\mathfrak{p}$. Then the restriction of τ to the space $\bar{\mathfrak{p}}/\mathfrak{p}$ is canonical and does not depends on τ. This canonical identification $\tau\colon \bar{\mathfrak{p}}/\mathfrak{q} \to \bar{\mathfrak{p}}/\mathfrak{p}$, induces canonical isomorphisms:

$$\begin{tikzcd} \mathrm{Der}(A_{\bar{\mathfrak{p}}}, \bar{\mathfrak{p}}/\mathfrak{q}) \arrow[r] \arrow[d] & \mathrm{Der}(\mathcal{C}^\infty(M), \bar{\mathfrak{p}}/\mathfrak{q}) \arrow[r] \arrow[d] & TV_{\mathfrak{q}}^{\phi^j} J^A M \arrow[d, "\tau_*"] \\ \mathrm{Der}(A_{\bar{\mathfrak{p}}}, \bar{\mathfrak{p}}/\mathfrak{p}) \arrow[r] & \mathrm{Der}(\mathcal{C}^\infty(M), \bar{\mathfrak{p}}/\mathfrak{p}) \arrow[r] & TV_{\mathfrak{p}}^{\phi^j} J^A M \end{tikzcd}$$

Thus, condition *4.* is satisfied assuming that the affine sequence is exact.

We can conclude that if the affine sequence is exact, the spaces $TV_{\mathfrak{p}}^{\phi^j} J^A M$, which depends only on $\bar{\mathfrak{p}}$, define a vectorial bundle $TV^{\phi^j} \to J^B M$, that modelizes the affine structure of ϕ^j

$$J^A M \times_{J^B M} TV^{\phi^j} \to J^A M, \quad (\mathfrak{p}, [D]) \mapsto \mathfrak{p} + [D].$$

On the other hand, if the affine sequence is not exact, then *1.* or *3.* are not satisfied for suitable manifolds (In fact it is enought that the dimension of M to be bigger than the width of A). $\Box$

Corolary 6 *Let A_l be of height l, and $l > k > 0$. Then natural projection $J^{A_l} M \to J^{A_k} M$ is endowed with a canonical structure of affine bundle if $3k+1 \geq 2l$ and $\mathrm{Aut}(A_l) \to \mathrm{Aut}(A_k)$ is surjective.*

Corolary 7 *The natural projection $J^l_m M \to J^r_m M$, for $l > r > 0$, is endowed with a canonical structure of affine bundle if $3r + 1 \geq 2l$.*

Remark 2 Those results generalize the well known affine structure of the spaces of jet of sections. First they show that this estructure is not only for the projection to lower order one-by-one, and second that this is inherent to the spaces $J^l_m M$ as spaces of ideals, and does not depend of its realization as recallment of spaces of jets of sections.

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