On the Analytical and Numerical Approximation of Invariant Manifolds

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Contents

The study of dynamical systems and, in particular, Celestial Mechanics, requires a combination of analytical and numerical methods. The purely theoretical methods are useful because they give insight on the qualitative behaviour, which kind of phenomena can be expected and the topological obstructions to some types of behaviour. The analytical methods are useful both to study the local qualitative and quantitative patterns of the motion, specially when combined with high order developments with the help of ad hoc algebraic manipulators. But those methods lack the actual detailed behaviour of a given system.

The purely numerical methods, relying only on numerical computations, usually miss interesting behaviour sitting on a small portion of the phase space which is important because it organises the dynamics around it. Furthermore, in many cases, the analytical methods suggest how to overcome numerical difficulties like singularities or too strong dependence to initial conditions. The massive simulations (brute force computations) should be avoided as far as possible and, in general, one should look, numerically, for phenomena which appear as plausible from a theoretical point of view.

The results of numerical computations, taking for granted, of course, the analysis of the methods and the error control, should support the analysis by theoretical means. That is, the pictures obtained should be coherent with simplified models of the system, perhaps localised on a given neighbourhood of a point or of an orbit (or even of a manifold).

When some results are obtained from a numerical simulation the next step is to cook a simplified model, perhaps with many additional hypothesis adapted to the problem at hand, to describe in a coherent and unified way the numerical results. In this way we can rely on our knowledge of the system and the analysis can suggest new numerical experiments to be carried out.

With the preceding considerations in mind we proceed to the computation of invariant manifolds and of homoclinic/heteroclinic points and orbits. However we first need a long series of tools concerning the continuation method and the analysis of bifurcations, the computation of fixed points, periodic orbits and quasiperiodic orbits as well as their stability properties, and therefore we need the variational equations, the Lyapunov exponents, etc. The normal forms are also an useful tool for the analysis of bifurcations and for analytical approximations.

1 The continuation method

This is a key method in many problems: To obtain the evolution of fixed points or periodic orbits when the system depends on parameters, to obtain bifurcation curves or simply to solve systems of nonlinear equations when an initial approximation to start an iterative method is not available. In general we have to obtain the solution $x(\mu)$ of the equation $f(x,\mu)=0$, where $x \in \mathcal{U}$, open set in $\mathbb{R}^n$, $\mu \in I$, real interval, and $f : \mathcal{U} \times I \longrightarrow \mathbb{R}^n$. Of course, μ could be a k-dimensional parameter and then we select a path in the space of parameters, reducing the problem to the one parameter case.

If we look for the solutions of $g(x) = 0$, $g : \mathcal{U} \subset \mathbb{R}^n \longrightarrow \mathbb{R}^n$, one can construct the homotopy $f(x,\mu) = g(x)\mu + h(x)(1-\mu)$, $\mu \in [0,1]$, where h is a function whose zeros are known (or can be found in an easier way) and, hopefully, is close enough to g. When μ ranges from 0 to 1 then $f(x,\mu)$ goes from $h(x)$ to $g(x)$. If the continuation method succeeds we obtain, at the end, the zeros of g. A sufficient condition to succeed is that the Jacobian $D_x f\,(x(\mu),\mu)$ be regular along the solution.

Now we go back to the general case trying to obtain $x(\mu)$ for a general function $f(x,\mu)$. For the moment it is enough to suppose f of class $\mathcal{C}^1$ both with respect to x and μ.

It is better to consider μ as an additional coordinate: $x_0 = \mu$. Then $f : \mathcal{U} \times I \subset \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^n$. Let $y = (x_0, x_1, \ldots, x_n)^\top$. By differentiation of $f(y) \equiv 0$ we obtain $D\,f(y)\Delta y = 0$. If Δs denotes the element of arc length in $\mathbb{R}^{n+1}$ one has $\sum_{j=0}^n (\Delta x_j)^2 = (\Delta s)^2$. The matrix $D\,f(y)$ is a linear map from $\mathbb{R}^{n+1}$ to $\mathbb{R}^n$. Let A_j the determinant of the matrix obtained from $D\,f(y)$ by deleting the j-th column times $(-1)^j$, where, for convenience we number the columns from 0 to n. Then one has

$$\frac{\Delta x_0}{A_0} = \frac{\Delta x_1}{A_1} = \cdots = \frac{\Delta x_n}{A_n}\,,$$

which, together with the arc length relation, gives, by passing to the limit

$$\frac{dx_j}{ds} = \frac{\pm A_j}{\sqrt{\sum_{j=0}^n \; A_j^2}} \quad , \quad j = 0, 1, \ldots, n\,.$$

The $\pm$ sign is associated to the sense taken along the curve of fixed points. The procedure breaks down, in principle, if all the A_j are zero. This is equivalent to $rank\;D\,f(y) < n$. Hence, provided the rank of $D\,f$ is maximal the method works. In particular it works for cases where $A_0 = 0$ if some of the remaining A_j is non zero. This is related to turning points with respect to the parameter μ, showing the advantage of this formulation.

In some cases it is useful to modify slightly the arc length. Suppose, for instance that some interesting function (one of the eigenvalues of $D_x f$ or the trace, etc.) can change quickly in some domain of the variable y and we are interested in a detailed knowledge of this function. Let $h(y)$ this function. Then we modify Δs as follows

$$(\Delta s)^2 = \sum_{j=0}^n (\Delta x_j)^2 + \Big(\sum_{j=0}^n \frac{\partial\, h}{\partial\, x_j}\; \Delta x_j \Big)^2 .$$

The related relations for dx_j/ds are

$$\frac{dx_j}{ds} = \frac{\pm A_j}{\sqrt{\sum_{j=0}^n \; A_j^2 + \Big(\sum_{j=0}^n \; \frac{\partial\, h}{\partial x_j}\; A_j \Big)^2}} \qquad , \quad j = 0, 1, \ldots, n\,.$$

Concerning the numerical integration of the continuation equations dx_j/ds one can use a simple method, such as the Euler method or some low order explicit linear multistep method. Indeed, after obtaining some approximate point on the continuation curve one should refine using the equation $f(y) = 0$ to prevent from the accumulation of errors in the numerical integration method. To find a solution of $f(y) = 0$ we shall use a modified Newton method because the number of variables is higher than the number of equations. Let us present the method in general. Let $g : \mathcal{V} \subset \mathbb{R}^m \longrightarrow \mathbb{R}^n$ with $m > n$. We look for a solution of $g(z) = 0$ starting at some approximate solution $z^{(0)}$. By linearisation around $z^{(k)}$ we have $g(z^{(k)}) + Dg(z^{(k)})\Delta z^{(k)} = 0$ and we impose the additional condition that the Euclidean norm $\|\Delta\, z^{(k)}\|_2$ be minimum. This is a problem of conditioned extrema with Lagrange function

$$L(\Delta z, \psi) = (\Delta z, \Delta z) + \psi^\top \left(g(z) + Dg(z)\cdot \Delta z\right),$$

where ψ is an n-dimensional Lagrange multiplier. It follows easily the recurrence

$$z^{(k+1)} = z^{(k)} - Dg(z^{(k)})^\top \Big(Dg(z^{(k)}) \cdot (Dg(z^{(k)}))^\top\Big)^{-1} g(z^{(k)}),$$

which works provided the rank of $\Delta g(z^{(k)})$ is maximal $(= n)$.

The case where the rank of Df is not maximal will be examined later on in some simple situations, to study the apparition of bifurcations. However it is worthy to mention that in most of the cases (i.e., generically) the continuation equations $dx_j/ds\,,\ j = 0, 1, \ldots, n$ are able "to cross" the bifurcation. Even if all the A_j go to zero when approaching a bifurcation point, it is possible that the limits of $\pm A_j \ / \ \sqrt{\sum_{j=0}^n A_j^2}$ exist along the continuation curve. If the last point computed before the bifurcation is not too close to $A_j = 0$ for all j, the rounding errors can be not very important and we can predict the next point "at the other side" of the bifurcation. Of course, if when we approach the bifurcation the behaviour of the dominant A_j is of the type $0(d^k)$ with k odd (d being the distance at the bifurcation point) then one should reverse the $\pm$ sign to continue along the new branch. In this case we should do a few more steps of the continuation method before refining by using the modified Newton method to prevent from bad conditioning of the matrix $Df \cdot Df^\top$. To guess the behaviour of the A_j's when a bifurcation is approached one can use finite differences if the arc step, Δs, is constant or some other extrapolation method.

Finally we should add a few words concerning the arc step to be used in the continuation curve. In principle the distance between the successive points will be close to Δs. However if the curve has a strong curvature it is desirable to use smaller arc steps. This can be controlled automatically as follows. Let y_m, y_{m+1} and y_{m+2} be 3 consecutive points on the curve. Computing the angles $\alpha = \cos^{-1}\left((z_1, z_2) \,/\, (\|z_1\| \cdot \|z_2\|)\right)$, where $z_1 = y_{m+1} - y_m\,,\ z_2 = y_{m+2} - y_{m+1}$ one should obtain values less than some bound (for instance 0.1 rad). Otherwise one should decrease Δs to keep this bound or one can interpolate additional points. Furthermore it is advisable to prevent from too many Newton iterates to refine the successive points. For instance, if after 5 iterates the required accuracy is not achieved one can reduce the arc step. Successive reductions of the arc step can lead to a too small step, showing some problem in the computation. Then one should stop and look for a detailed analysis of what happens.

2 Analysis of bifurcations in the continuation method

Suppose now $rank\ Df(y^*) = n - k$, with $1 \le k \le n$, for some y^* such that $f(y^*) = 0$. Without loss of generality we assume that the first k rows of $Df(y^*)$ are zero (by substituting the equations $f_1, \dots, f_n$ by suitable linear combinations of them) and that the Taylor expansion of f around y^* is as follows (f should be at least of class $\mathcal{C}^2$):

$$f(y) = \begin{array}{r} \\ k\{ \\ n-k\{ \end{array} \begin{array}{c} \begin{array}{cc} \overbrace{}^{k+1} & \overbrace{}^{n-k} \end{array} \\ \begin{pmatrix} 0 & 0 \\ A & B \end{pmatrix} \end{array} \begin{pmatrix} \Delta u \\ \Delta v \end{pmatrix} + \begin{pmatrix} \Delta y^\top Q_1 \Delta y \\ \vdots \\ \Delta y^\top Q_n \Delta y \end{pmatrix} + O_2 \,,$$

where $\Delta y = y - y^*$, Δu (resp. Δv) denotes the first $k+1$ (resp. the last $n-k$) components of Δy and B is regular. By equating the last $n-k$ components of $f(y)$ to zero one has

$$\Delta v = C \Delta u + O_2(\Delta u) \qquad \text{(by using the implicit function theorem)},$$

where $C = -B^{-1}A$. By substitution in the $k+1$ first equations we obtain a system of the form

$$\Delta u^\top \overline{Q}_1 \, \Delta u + O_2 = 0, \dots, \Delta u^\top \, \overline{Q}_k \, \Delta u + O_2 = 0 \,.$$

First one should study the dominant terms. After obtaining Δu one gets Δv and the linear approximations give the directions of the branches of $f(y) = 0$ at y^*. As we are interested in directions one can look at a normalised Δu (for instance $\Delta u^\top \, \Delta u = 1$) or one can work with $z_j = \Delta u_j \, / \, \Delta u_{k+1}$ (this is due to the fact that Δu are projective coordinates). Using the last method we get a system like

$$\sum_{1 \le i,j \le k} \overline{q}_{ij}^{(p)} \; z_i z_j + 2 \sum_{1 \le i \le k} \overline{q}_{i,k+1} \, z_i + \overline{q}_{k+1,k+1} = 0 \,, \quad p = 1, \dots, k \,.$$

Each solution of this intersection of quadrics gives one bifurcated direction. To solve this system for $z_1, \dots, z_k$ one can use several methods. A general one is the resultant method. Consider two polynomials $R_r(w)$, $S_s(w)$ of degrees r and s respectively. Let $R_r(w) = r_0 w^r + r_1 w^{r-1} + \cdots + r_r$, $S_s(w) = s_0 w^s + s_1 w^{s-1} + \cdots + s_s$, with $r_0 \neq 0$, $s_0 \neq 0$. Then, if they have a common zero, the determinant

$$D = \begin{vmatrix}
r_0 & r_1 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & r_r & 0 & \dots & \dots & 0 \\
0 & r_0 & r_1 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & r_r & 0 & \dots & 0 \\
\vdots & & & & & & & & & & & & & \vdots \\
\vdots & & & & & & & & & & & & & \vdots \\
0 & \dots & \dots & 0 & r_0 & r_1 & \dots & \dots & \dots & \dots & \dots & \dots & r_r & \\
s_0 & s_1 & \dots & \dots & \dots & \dots & \dots & s_s & 0 & \dots & \dots & \dots & \dots & 0 \\
0 & s_0 & s_1 & \dots & \dots & \dots & \dots & \dots & s_s & 0 & \dots & \dots & \dots & 0 \\
\vdots & & & & & & & & & & & & & \vdots \\
\vdots & & & & & & & & & & & & & \vdots \\
\vdots & & & & & & & & & & & & & \vdots \\
0 & \dots & \dots & \dots & 0 & s_0 & s_1 & \dots & \dots & \dots & \dots & \dots & \dots & s_s
\end{vmatrix} ,$$

where the first s rows have been obtained by successive circular shifts to the right of the first one and the last r rows have been obtained in a similar way from the one on the $s+1$-th row, should be zero. If the coefficients r_i, s_j depend on some other variable, t, then $D = D(t)$. In our case computing the resultant between the first equation and any other of the remaining $k-1$ one can obtain $k-1$ resultants (which are polynomials in $z_1, \ldots, z_{k-1}$) by elimination of z_k. Proceeding inductively we obtain finally a single polynomial, T, of degree 2^k in z_1. The procedure is simple but the computation of the resultants is tedious. A symbolic manipulator is quite useful for this method. Some of the zeros of T can be not real. They should be skipped if we are only interested in real branches. At least a couple of branches will be real (associated to some Δy and $-\Delta y$) because in the numerical continuation we have reached y^* following a real branch.

It is now a simple matter, using again the implicit function theorem, to show that every simple zero of T gives rise to a curve solution of $f(y) = 0$ which can be parametrised by Δu_{k+1}. If f is analytical so is this parametrisation.

The difficulties appear in the degenerate cases. For instance when some of the quadratic forms $\overline{Q}_j$ is degenerate or when we obtain multiple zeros of T. Then one should assume more differentiability for f and use higher order terms. To make the exposition easy we restrict to $k = 1$. Then Δv can be expressed in terms of Δu_1 and Δu_2 only. The reduced bifurcation equation is now

$$P_p(\Delta u_1\,,\ \Delta u_2) + o_p = 0\,,$$

where P_p is a polynomial of degree p starting with terms of degree 2. Let

$$P_p(\Delta u_1\,,\ \Delta u_2) = \sum_{2 \le i+j \le p} p_{i,j}\,(\Delta u_1)^i\,(\Delta u_2)^j\,.$$

Then we plot on the lattice $\mathbb{Z} \times \mathbb{Z}$ the couples (i,j) such that $p_{i,j} \neq 0$. We assume that in $P_p + o_p$ the variable Δu_1 (nor Δu_2) does not appear as a factor. If it does this is an easy branch and we proceed with the remaining factor. Then we draw the convex envelope of the marked points (i,j) and we consider the part of this polygonal closer to the origin with endpoints at the axes. It is made of several segments I_t, $t = 1, \ldots, s$. If for a point $z \in \mathbb{R}^2$ we define $Q(z) = \{(x,y) \in \mathbb{R}^2,\ x \ge z_1,\ y \ge z_2\}$, then the set $\cup_{z \in J} Q(z)$, where $J = \cup_{t=1}^{s} I_t$, contains all the market points. We also assume that the terms in o_p do not affect $\cup_{t=1}^{s} I_t$. Let I_t be one of those segments with endpoints (i_{t-1}, j_{t-1}), (i_t, j_t), $i_t > i_{t-1}$, $j_t < j_{t-1}$. Then one should look for solutions of the type $\Delta u_2 = c(\Delta u_1)^{(i_t - i_{t-1})/(j_{t-1} - j_t)}$ with a suitable constant c. For instance consider $p_{1,3} = -3$, $p_{3,2} = 1$, $p_{2,3} = 1$, $p_{0,5} = 1$, $p_{5,1} = 6$, $p_{6,1} = 1$, $p_{8,0} = 4$, $p_{0,9} = -1$. Then $t = 3$ and the endpoints are $(8,0), (5,1), (1,3)$ and $(0,5)$. The dominant terms of the related branches are easily found to be $\Delta u_2 = -\frac{2}{3}\,(\Delta u_1)^3$, $\Delta u_2 = \frac{1 \pm \sqrt{73}}{6}\,(\Delta u_1)^2$, $\Delta u_2 = \pm\sqrt{3}\,(\Delta u_1)^{1/2}$. If there is some segment such that one obtains $\Delta u_2 = c(\Delta u_1)^{\Delta_i/\Delta_j}$ with a multiple value of c then one should introduce the new variable $w = \Delta u_2 - c(\Delta u_1)^{\Delta_i/\Delta_j}$ and proceed with the new polynomial in the variables $(\Delta u_1)^{1/\Delta_j}, w$.

The method described above is the so called Newton-Puiseux method.

3 Fixed points and their stability

Given a differential equation $\dot{x} = f(x)$ the fixed points are obtained by setting $f(x) = 0$. The computation is usually done by Newton's method. If $x^{(0)}$ is an approximate value of the fixed point then the sequence $\{x^{(k)}\}$, where $x^{(k+1)} = x^{(k)} - Df(x^{(k)})^{-1}\ f(x^{(k)})$, converges to a solution, x^*, of $f(x)=0$ if $Df(x^*)$ is invertible and $x^{(0)}$ is close enough to x^*. Furthermore the convergence is quadratic, i.e., if $e^{(k)}=x^{(k)}-x^*$, one has $\|e^{(k+1)}\| = O(\|e^{(k)}\|^2)$. The difficulty related to the election of $x^{(0)}$ can be circumvented by using a continuation method, as described before.

The linear stability of x^* is analysed by means of the linearised equation. If $\lambda_1,\ldots,\lambda_n$ are the eigenvalues of $Df(x^*)$ then linear stability requires $Re\,\lambda_j \le 0$ for $j=1,\ldots,n$. If all the eigenvalues satisfy $Re\,\lambda_j < 0$ then the fixed point is (positively) Lyapunov stable. If some of the eigenvalues satisfies $Re\,\lambda_j > 0$ then the fixed point is unstable. For Hamiltonian systems, with Hamiltonian $H(q,p)$, the differential equations are $\binom{q}{p} = J\nabla H$, where J is the canonical symplectic matrix, $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$. Here I denotes the identity matrix in $\mathbb{R}^n$, n being the number of degrees of freedom of H. In this case if λ is an eigenvalue of the linearised equations also $\overline{\lambda}, -\lambda, -\overline{\lambda}$ are eigenvalues. Then, to have stability for a fixed point of a Hamiltonian system all the eigenvalues should have real part equal to zero. To decide about the nonlinear stability is a hard problem. If the Hamiltonian around the fixed point (q^*,p^*) is of the form

$$H(q,p) = \frac{1}{2}\, D^2H(q^*,\,p^*)\,(q-q^*,\,p-p^*)^2 + O_3\,,$$

then stability is ensured if $D^2H(q^*,p^*)$ is a definite quadratic form (Dirichlet theorem). Otherwise, under mild and generic hypothesis over the higher order terms, the fixed point is stable for two degrees of freedom, as a consequence of the KAM theory. For $n \ge 3$ degrees of freedom there is no way, in general, to prevent from Arnold's diffusion and there is no stability.

For non Hamiltonian planar systems with eigenvalues $\pm i$ the stability can be studied using the method of Lyapunov. Let the differential equations of the form

$$\begin{aligned} \dot{x} &= y + P_2(x,y) + P_3(x,y) + \ldots\ , \\ \dot{y} &= -x + Q_2(x,y) + Q_3(x,y) + \ldots\ , \end{aligned}$$

where $P_k,\ Q_k$ are homogeneous polynomials of degree k. We try to construct a function $V(x,y) = \frac{1}{2}\,(x^2+y^2) + V_3\,(x,y) + V_4\,(x,y) + \ldots,$ V_k being also homogeneous of degree k, such that the relation $\dot{V}(x,y) = \sum_{j\ge 2}\ c_j(x^2+y^2)^j$ holds, at least up to some order.

By direct substitution and assuming $V_k(x,y) = \sum_{j=0}^{k} V_{k-j,j}\,x^{k-j}\,y^j$, one has

$$\begin{aligned} \dot{V}(x,y) = &\Big[x + \sum_{k\ge 3}\Big(\sum_{j=0}^{k-1}(k-j)\,V_{k-j,j}\ x^{k-j-1}\ y^j\Big)\Big]\Big(y + \sum_{\ell\ge 2} P_\ell(x,y)\Big) + \\ &\Big[y + \sum_{k\ge 3}\Big(\sum_{j=1}^{k} j\,V_{k-j,j}\ x^{k-j}\ y^{j-1}\Big)\Big]\Big(-x + \sum_{\ell\ge 2} Q_\ell(x,y)\Big) = \sum_{j\ge 2} c_j(x^2+y^2)^j\,. \end{aligned}$$

The terms of order 2 cancel and the higher order terms allow to determine the coefficients

$V_{k-j,j}$ and c_j. For odd order the system of equations is of the form

$$\begin{pmatrix} 0 & -1 & 0 & & & & \\ k & 0 & -2 & & 0 & & \\ 0 & k-1 & 0 & & & & \\ & & & \ddots & & & \\ & & & & 0 & -k+1 & 0 \\ & 0 & & & 2 & 0 & -k \\ & & & & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} V_{k,0} \\ V_{k-1,1} \\ V_{k-2,2} \\ \vdots \\ V_2, k-2 \\ V_1, k-1 \\ V_0, k \end{pmatrix} + \text{known terms} = 0\,.$$

The determinant of the system is $(k\,!!)^2$ and, hence, the solution is unique.

For even order the system is

$$\begin{pmatrix} 0 & -1 & 0 & & & & & -\binom{k/2}{0} \\ k & 0 & -2 & & 0 & & & 0 \\ 0 & k-1 & 0 & & & & & -\binom{k/2}{1} \\ & & & \ddots & & & & \vdots \\ & & & & 0 & -k+1 & 0 & -\binom{k/2}{k/2-1} \\ & 0 & & & 2 & 0 & -k & 0 \\ & & & & 0 & 1 & 0 & -\binom{k/2}{k/2} \end{pmatrix} \begin{pmatrix} V_{k,0} \\ V_{k-1,1} \\ V_{k-2,2} \\ \vdots \\ V_{2,k-2} \\ V_{1,k-1} \\ V_{0,k} \\ C_{k/2} \end{pmatrix} + \text{known terms} = 0\,.$$

This system has $k+1$ equations and $k+2$ unknowns. It can be split in two systems, one with the unknowns $V_{k-1,1}, V_{k-3,3}, \ldots, V_{3,k-3}, V_{1,k-1}, C_{k/2}$ and the same number of equations and the other with unknowns $V_{k,0}, V_{k-2,2}, \ldots, V_{2,k-2}, V_{0,k}$ and one less equation. The first subsystem has an unique solution because the determinant is $(-1)^{k/2-1}\,k!!$. The second one has a 1-dimensional linear manifold of solutions because it contains an upper triangular minor with determinant $k\,!!$. Furthermore, if in the first subsystem one multiplies the first equation by $\frac{2^{k/2}(k-1)!!}{(k/2)!}$, the second one by $\frac{2^{k/2}(k-3)!!\,1\,!!}{(k/2)!}$, the third one by $\frac{2^{k/2}(k-5)!!\,3\,!!}{(k/2)!}$, and so one, one obtains $-2^k\,c_{k/2} =$ known quantity. Hence, it is possible to compute $c_{k/2}$ without explicit computation of the $V_{k-j,j}$. Of course those values are required for the subsequent c's.

It is clear that for x^2+y^2 small enough $V=$ ctant are closed curves encircling the origin. Then if the first non zero coefficient c_j is positive the origin is unstable. If it is negative the origin is stable. The function V constructed in this way is a local Lyapunov function.

If all the coefficients c_j are zero then we have a nonlinear centre provided the V, constructed in the way described, is convergent on some neighbourhood of the fixed point.

An alternative approach to study the same problem is the method of the normal form. We present it here as one of the simplest NF. Again we start with

$$\dot{x} = y + \sum_{k\geq 2} P_k(x,y) \quad , \quad \dot{y} = -x + \sum_{k\geq 2} Q_k(x,y) \,,$$

or introducing $z = x + iy$, we can write

$$\dot{z} = -i\, z + \sum_{k\geq 2} R_k(z,\bar{z}) \,,$$

where R_k are also homogeneous polynomials of degree k in z and its complex conjugate. We try to perform a change of variables to kill as many terms in the differential equation as possible. Let us consider first the case $\dot{z} = -i\, z + R_k(z,\bar{z}) + O(|z|^{k+1}),\ k \geq 2$. Introducing $z = \zeta + S_k(\zeta,\bar{\zeta})$ one has $\zeta = z - S_k(z,\bar{z}) + O(|z|^{k+1})$. For definiteness we assume $R_k = \sum_{j=0}^{k} R_{k-j,j}\ z^{k-j}\ \bar{z}^j$, $S_k = \sum_{j=0}^{k} S_{k-j,j}\ z^{k-j}\ \bar{z}^j$. By substitution we obtain

$$\begin{aligned}
\zeta = &\Big(1 - \sum_{j\geq 0} S_{k-j,j}\ (k-j)\ z^{k-j-1}\ \bar{z}^j\Big) \Big(-i\, z + \sum_{k\geq 0} R_{k-j,j}\ z^{k-j}\ \bar{z}^j\Big) \\
&- \sum_{k\geq 0} S_{k-j,j}\ j z^{k-j}\ \bar{z}^{j-1}(i\bar{z}) + O(|z|^{2k-1}) = -i\Big(\zeta + \sum_{j\geq 0} S_{k-j,j}\ \zeta^{k-j}\ \bar{\zeta}^j\Big) \\
&+ i\sum_{j\geq 0} S_{k-j,j}(k-j)\, \zeta^{k-j}\ \bar{\zeta}^j + \sum_{j\geq 0} R_{k-j,j}\ \zeta^{k-j}\ \bar{\zeta}^j - i\sum_{j\geq 0} S_{k-j,j}\ j\, \zeta^{k-j}\ \bar{\zeta}^j \\
&+ O(|\zeta|^{2k-1}) \,.
\end{aligned}$$

By collecting terms in $\zeta^{k-j}\ \bar{\zeta}^j$ one has $i\ S_{k-j,j}(k-2\ j-1) + R_{k-j,j}$. This can be made zero by choosing a suitable $S_{k-j,j}$ unless k is odd and $j = \frac{k-1}{2}$. Then we can skip all terms in R_k but the term in $z^{m+1}\ \bar{z}^m$ if $k = 2\,m+1$. Now it is clear that if we start with $\dot{z} = -iz + a_1\, z^2\bar{z} + a_2\, z^3\bar{z}^2 + \cdots + a_{m-1}\ z^m\ \bar{z}^{m-1} + R_{2m}(z,\bar{z}) + 0(|z|^{2m+1})$, the transformations used to skip all the terms in R_{2m} and in the subsequent transformations do not change the coefficients $a_1, \ldots, a_{m-1}$.

As a conclusion we obtain the normal form

$$\dot{z} = -iz(1 + b_1\ r^2 + b_2\, r^4 + \cdots + b_s\, r^{2s}) + O(r^{2s+3}) \quad , \quad r = |z| \,.$$

Let $b_j = c_j + id_j$ for $j = 1, 2, \ldots s$. Then $\dot{r} = -c_1 r^3 - c_2 r^5 \cdots - c_s r^{2s+1} + 0(r^{2s+3})$, and if the first coefficient c_j different from zero is positive the fixed point is stable. It it is negative then the fixed point is stable.

We remark that if one of the eigenvalues of $Df(x^*)$ is equal to zero then the Newton method is not convergent in general. This is due to the fact that in this case it is possible to have not an isolated fixed point but a line of fixed points.

The fixed points of diffeomorphisms are obtained from $x^* = F(x^*)$. Again the fixed point can be obtained by using the Newton method applied to $G(x) = x - F(x) = 0$. As $DG = I - DF$, the difficulties will be associated to $1 \in Spec\, DF(x^*)$.

Linear stability of the fixed point is guaranteed if all the eigenvalues of DF satisfy $|\lambda| \leq 1$ and, if there are λ with $|\lambda| = 1$ they are simple. If all the eigenvalues satisfy $|\lambda| < 1$ then the fixed point is stable. If some of the eigenvalues has modulus larger than 1 then the fixed point is unstable.

If the diffeomorphism is canonical, i.e., $(Q, P) = F(q, p)$ with DF a symplectic transformation $\left((DF)^\top J\ DF = J\right)$ then the existence of an eigenvalue λ implies also the existence of the eigenvalues $\lambda^{-1}, \bar{\lambda}, \bar{\lambda}^{-1}$. For canonical diffeomorphisms a necessary condition for stability is $|\lambda| = 1$ for all the eigenvalues of $DF(x^*)$, the differential at the fixed point. In this case for a planar diffeomorphism (canonical is then equivalent to area preserving) the (nonlinear) stability can be studied using the Moser twist theorem. For $2n$-dimensional canonical transformations, $n \geq 2$, we are faced again to the difficulties of the possible Arnold's diffusion. Even for the case of area preserving diffeomorphisms there are examples such that if we only consider the linear terms the fixed point is unstable (with double eigenvalue equal to 1, and non diagonalizing linear part). Adding the first nonlinear terms it becomes stable, but the addition of further nonlinear terms of higher orders can change the stability properties several times.

4 The numerical integration of ordinary differential equations

In principle every one can use the personal favourite method for the numerical integration of o.d.e. However it is always convenient to use methods with automatic step size control and to have some control at least on the local errors.

The Taylor method is convenient only when the computation of the higher order derivatives $\frac{d^2y}{dx^2} = D_x f + D_y f \cdot f$, $\frac{d^3y}{dx^3} = D_x^2 f + 2D_{xy} f \cdot f + D_y^2 f \cdot (f)^2 + D_y f \cdot (D_x f + D_y f \cdot f)$, etc. is possible due to some recurrence. Otherwise the computation becomes rather cumbersome and time consuming.

Among the one step methods is worth to mention the Runge-Kutta-Fehlberg methods with automatic step size control. The method of orders 7-8 has shown a good performance despite it requires 13 evaluations of the vector field at each step. It is useful if the vector field is easy to compute and one is interested in high precision, with local errors of the order $10^{-13} - 10^{-15}$.

Concerning the linear multistep methods a very efficient method is the one due to Shampine and Gordon. The number of evaluations of the vector field is relatively small and therefore the method is suitable if the computational cost of evaluate the field is high.

Finally if the equations to integrate are stiff one should recommend special methods as the Gear method, some Obrechkoff methods or some implicit Runge-Kutta methods.

A worth of caution should be said about computations asking for very small local truncation errors. It is true that the integration can be carried out in quadruple precision. However a reduction of the local truncation error from 10^{-13} to 10^{-29} even with a method of order 7 requires a reduction of the step size by a factor of 100. Hence, the computing time is increased by a factor of several hundreds. If very high precision is required for a moderate system of o.d.e. then one should consider as a good method the Taylor one.

As a final possibility I would like to mention the use of ad hoc methods for the specific equations at hand. Using adapted methods can reduce the computing time and increase the precision.

5 The variational equations and the differentials of the Poincaré map

For several purposes, like the computation of periodic orbits, their stability, the dependence of orbits with respect to parameters, etc., it is useful to know about the variational equations. Let $\dot{x} = f(t, x, \varepsilon)$ with $t \in \mathbb{R}$ $x \in \mathcal{U}$, open set in $\mathbb{R}^n$, ε a k-dimensional parameter. The exposition is more compact if we consider autonomous systems without parameters. This can always be achieved by introducing $z = (z_0, z_1, \ldots, z_n, z_{n+1}, \ldots, z_{n+k})^\top \in \mathbb{R} \times \mathcal{U} \times \mathbb{R}^k$ whose evolution is described by the system

$$\dot{z}_0 = 1\,, \quad \dot{z}_j = f_j(z) \quad \text{for} \quad j = 1, \ldots, n\,, \quad \dot{z}_j = 0 \quad \text{for} \quad j = n+1, \ldots, n+k\,.$$

Hence we consider directly $\dot{x} = f(x)\,,\ x \in \mathcal{U} \subset \mathbb{R}^n$. Let $\varphi(t, x)$ (or $\varphi_t(x)$ or $\varphi_x(t)$) the solution of $\dot{x} = f(x)$ with initial condition $\varphi(0, x) = x$. If f is of class $\mathcal{C}^r$ then also φ is $\mathcal{C}^r$. Then $\varphi_t(x+h) = \varphi_t(x) + \frac{\partial \varphi_t(x)}{\partial x} h + \frac{1}{2} \frac{\partial^2 \varphi_t(x)}{\partial x^2} (h)^2 + \ldots$. The variational equations are the differential equations satisfied by the linear (or multilinear) functionals $\frac{\partial^j \varphi_t(x)}{\partial x^j}$. We recall that $\frac{\partial^j \varphi_t(x)}{\partial x^j} \in \mathcal{L}(\mathbb{R}^{jn}, \mathbb{R}^n)$.

Knowing that $\varphi_t(x)$ is $\mathcal{C}^r$ the easiest way to derive the variational equations is by successive differentiation of the relation $\dot{\varphi}_t(x) = f(\varphi_t(x))$ with respect to x and exchange of the derivations with respect to t and x. So the first variational equations read

$$\frac{d}{dt} \frac{\partial \varphi_t(x)}{\partial x} = Df(\varphi_t(x)) \frac{\partial \varphi_t(x)}{\partial x} \quad , \quad \left.\frac{\partial \varphi_t(x)}{\partial x}\right|_{t=0} = I\,, \text{ the identity matrix in } \mathbb{R}^n\,.$$

The second order, third order, ... variational equations are obtained in the same way

$$\frac{d}{dt} \frac{\partial^2 \varphi_t(x)}{\partial x^2} = D^2 f(\varphi_t(x)) \Big(\frac{\partial \varphi_t(x)}{\partial x}\Big)^2 + Df(\varphi_t(x)) \frac{\partial^2 \varphi_t(x)}{\partial x^2}\,, \qquad \left.\frac{\partial^2 \varphi_t(x)}{\partial x^2}\right|_{t=0} = 0,$$

$$\frac{d}{dt} \frac{\partial^3 \varphi_t(x)}{\partial x^3} = D^3 f(\varphi_t(x)) \Big(\frac{\partial \varphi_t(x)}{\partial x}\Big)^3 + 3D^2 f(\varphi_t(x)) \Big(\frac{\partial^2 \varphi_t(x)}{\partial x^2}\,, \frac{\partial \varphi_t(x)}{\partial x}\Big)$$
$$+ Df(\varphi_t(r)) \frac{\partial^3 \varphi_t(x)}{\partial x^3}\,, \qquad \left.\frac{\partial^3 \varphi_t(x)}{\partial x^3}\right|_{t=0} = 0, \text{ etc.}$$

The last equation, taking into account that $\frac{\partial^3 \varphi_t(x)}{\partial x^3} \in \mathcal{L}(\mathbb{R}^{3n}, \mathbb{R}^n)$, should be understood as follows:

$$\frac{d}{dt} \frac{\partial^3 \varphi_t(x)}{\partial x^3}(h)^3 = D^3 f(\varphi_t(x)) \Big(\frac{\partial \varphi_t(x)}{\partial x}(h)\Big)^3 + 3D^2 f(\varphi_t(x)) \Big(\frac{\partial^2 \varphi_t(x)}{\partial x^2}(h)^2\,, \frac{\partial \varphi_t(x)}{\partial x}(h)\Big)$$
$$+ Df(\varphi_t(x)) \Big(\frac{\partial^3 \varphi_t(x)}{\partial x^3}(h)^3\Big)\,, \quad \forall h \in \mathbb{R}^n\,,$$

and, working in components, the second equation is as follows:

$$\frac{d}{dt} \frac{\partial \varphi_i(t,x)}{\partial x_j\, \partial x_k} = \sum_{p,q=1}^{n} D_{pq} f_i(\varphi(t,x)) \frac{\partial \varphi_p(t,x)}{\partial x_j} \frac{\partial \varphi_p(t,x)}{\partial x_k} + \sum_{p=1}^{n} D_p f_i(\varphi(t,x)) \frac{\partial^2 \varphi_p(t,x)}{\partial x_j\, \partial x_k}\,.$$

All the variational equations are linear, the first order ones being homogeneous (with non-homogeneous initial conditions) and the remaining ones being nonhomogeneous (with homogeneous initial conditions). This equations should be integrated simultaneously with the initial equation. Despite the number of equations, even at low order as 2 or 3, is very high for moderate values of n, in many problems only a part of the derivatives are required and the symmetries produce a further reduction.

Another important tool is the numerical computation of the Poincaré map and their derivatives. We show how to compute them using the variational equations. Instead of the standard first return map to a surface of section Σ, we consider a more general setting: the initial point is on section Σ_1, given by $g_1(x) = 0$, and the final one on Σ_2, given by $g_2(x) = 0$. Both Σ_1 and Σ_2 are assumed to be transversal to the flow.

Starting at $x^1 \in \Sigma_1$ one defines $P : \Sigma_1 \longrightarrow \Sigma_2$ by $P(x^1) = x^2 = \varphi(t(x^1), x^1)$ such that $\psi(t) = g_2\big(\varphi(t(x^1), x^1)\big) = 0$. The arrival time depends, of course, on the initial point. When $\psi(t)$ changes sign along the integration the intersection point is refined by Newton's method:

$$t^{(m+1)} = t^{(m)} - \frac{g_2(\varphi(t^{(m)}, x^1))}{\Big(\nabla g_2(\varphi(t^{(m)}, x^1)),\ f(\varphi(t^{(m)}, x^1))\Big)} .$$

Sometimes one should use not the first change of sign but the k-th one. This depends on what we are looking for.

Let $x^2 \in \sum_2$ the arrival point. Then, by differentiation w.r.t. x^1 one has

$$\frac{\partial P}{\partial x^1} = f(x^2)\, \frac{\partial t}{\partial x^1} + \frac{\partial \varphi}{\partial x^1}\, (t(x^1), x^1) .$$

The matrix $\frac{\partial \varphi}{\partial x^1}$ is provided by the first variational equations. To compute $\frac{\partial t}{\partial x^1}$ we use $g^2\Big(\varphi\big(t(x^1), x^1\big)\Big) = 0$. By differentiation we get

$$D\, g_2(x^2) \cdot \Big(f(x^2)\, \frac{\partial t}{\partial x^1} + \frac{\partial \varphi}{\partial x^1}\Big) = 0 ,$$

or

$$\frac{\partial t}{\partial x^1} - \frac{1}{(\nabla g_2(x^2),\, f(x^2))}\, Dg_2\, \frac{\partial \varphi}{\partial x^1} ,$$

the denominator being different from zero by the transversality condition.

The initial and final displacements are not arbitrary: they should satisfy the conditions $D\, g_1(x^1) \triangle x^1 = 0$ and $D\, g_2(x^2) \triangle x^2 = 0$. Let $u_1, \ldots, u_{n-1}$, $n-1$ vectors orthogonal to $\nabla g_1(x^1)$ and such that $\{u_1, \ldots, u_{n-1},\ \nabla g_1(x^1)\}$ is a basis of $\mathbb{R}^n$. Similarly $v_1, \ldots, v_{n-1}$ are chosen orthogonal to $\nabla g_2(x^2)$ such that $\{v_1, \ldots, v_{n-1},\ \nabla g_2(x^2)\}$ is a basis of $\mathbb{R}^n$. Then $\frac{\partial P}{\partial x^1}$ is computed as a matrix of the linear map from the space spanned by $\{u_1, \ldots, u_{n-1}\}$ to the one spanned by $\{v_1, \ldots, v_{n-1}\}$ by expressing

$$\frac{\partial P}{\partial x^1}\, u_j = -\frac{f(x^2)}{(\nabla g_2(x^2),\, f(x^2))}\, D\, g_2\, \frac{\partial \varphi}{\partial x^1}\, u_j + \frac{\partial \varphi}{\partial x^1}\, u_j$$

in the $\{v_j\}$ basis.

As an example we suppose that both g_1 and g_2 are given by setting the last x coordinate, x_n, equal to zero. We use $u_j = v_j = e_j$, the j-th vector of the canonical basis. Furthermore we assume that the matrix of the first variational equations is given by $\left(\frac{\partial\varphi}{\partial x^1}\,(t(x^1),x^1)\right)_{ij} = a_{ij}$. Then the $(n-1)\times(n-1)$ matrix $\frac{\partial P}{\partial x^1}\,(x^1)$ restricted to the first $n-1$ components is given by

$$\left(\frac{\partial P}{\partial x^1}\right)_{kj} = a_{kj} - \frac{f_k(x^2)}{f_n(x^2)}\,a_{nj} \quad , \quad 1 \le j,k \le n-1\,.$$

To obtain the second derivative of the Poincaré map we differentiate again:

$$\frac{\partial^2 P}{\partial(x^1)^2} = D\,f(x^2)\cdot f(x^2)\Big(\frac{\partial t}{\partial x^1}\Big)^2 + 2\,Df(x^2)\,\frac{\partial\varphi}{\partial x^1}\,\frac{\partial t}{\partial x^1} + f(x^2)\,\frac{\partial^2 t}{\partial(x^1)^2} + \frac{\partial^2\varphi}{\partial(x^1)^2}\,,$$

where $\frac{\partial^2 t}{\partial(x^1)^2}$ is obtained from

$$\begin{aligned} D^2 g_2(x^2)\Big(f(x^2)\frac{\partial t}{\partial x^1} + \frac{\partial\varphi}{\partial x^1}\Big)^2 + D\,g_2(x^2)\Big[Df(x^2)\cdot f(x^2)\Big(\frac{\partial t}{\partial x^1}\Big)^2 \\ + 2Df(x^2)\,\frac{\partial\varphi}{\partial x^1}\,\frac{\partial t}{\partial x^1} + f(x^2)\,\frac{\partial^2 t}{\partial(x^1)^2} + \frac{\partial^2\varphi}{\partial(x^1)^2}\Big] = 0\,. \end{aligned}$$

To avoid too long expressions we confine the expression in coordinates to the case of the previous example: $g_1 = g_2 = x_n$. Then

$$\begin{aligned} \frac{\partial^2 t}{\partial(x^1)^2}\Big|_{j,k} = -\frac{1}{f_n(x^2)}\,\Big[\frac{1}{\big(f_n(x^2)\big)^2}\,a_{nk}\,a_{nj}\sum_{s=1}^{n} D_n\,f_s(x)^2\cdot f_s(x^2) \\ - \frac{1}{f_n(x^2)}\sum_{s=1}^{n} D_n\,f_s(x^2)(a_{sk}\,a_{nj} + a_{sj}\,a_{nk} + b_{njk}\Big]\,, \end{aligned}$$

where b_{ijk} stands for $\frac{\partial^2\varphi_i}{\partial x_j\,\partial x_k}$. Introducing $f_s = f_s(x^2),\ f_{is} = D_i\,f_s(x^2)$ and substituting

$$\frac{\partial^2 P_i}{\partial x_j\,\partial x_k} = b_{ijk} - \frac{f_i}{f_n}\,b_{njk} + \frac{1}{f_n}\sum_{s=1}^{n}\Big(f_{is} - \frac{f_i}{f_n}\,f_{ns}\Big)\Big(\frac{f_s}{f_n}\,a_{nj}\,a_{nk} - a_{sk}\,a_{nj} - a_{sj}\,a_{nk}\Big).$$

Further examples with higher order variational equations will be presented later in a very simple example.

6 The computation of periodic orbits and their stability

The computation of periodic points of diffeomorphisms $x \longrightarrow F(x),\ x \in \mathcal{U} \subset \mathbb{R}^n$, is done in the same way than the computation of fixed points and also the stability analysis proceeds in a similar way. The only difficulty is the computation of the differential (or successive differentials if required for continuation or stability analysis) of the composite map F^m. This is done by iteration. For instance, let $x_j = F(x_{j-1})$. Then

$$\begin{aligned}
D\,F^j(x_0) &= D\,F(x_{j-1}) \cdot D\,F^{j-1}(x_0)\,,\\
D^2\,F^j(x_0) &= D^2\,F(x_{j-1})\big(D\,F^{j-1}(x_0)\big)^2 + D\,F(x_{j-1}) \cdot D^2\,F^{j-1}(x_0)\,,\\
D^3\,F^j(x_0) &= D^3\,F(x_{j-1})\big(D\,F^{j-1}(x_0)\big)^3 + 3\,D^2\,F(x_{j-1})(D^2\,F^{j-1}(x_0),\ D\,F^{j-1}(x_0))\\
&\quad + D\,F(x_{j-1}) \cdot D^3\,F^{j-1}(x_0)\,,
\end{aligned}$$

where an equality like last one should be understood, when applied to a vector h three times, as

$$\begin{aligned}
D^3\,F^j(x_0)(h)^3 = {}& D^3\,F(x_{j-1})\big(D\,F^{j-1}(x_0)(h)\big)^3\\
&+ 3\,D^2\,F(x_{j-1})(D^2\,F^{j-1}(x_0)(h)^2\,,\ D\,F^{j-1}(x_0)(h))\\
&+ D\,F(x_{j-1}) \cdot (D^3\,F^{j-1}(x_0)(h)^3)\,.
\end{aligned}$$

To compute periodic solutions (non fixed points) of a vector field we consider several cases: Autonomous vector fields, periodic vector fields, weak periodic perturbations of autonomous fields and high dimension systems. The last ones rarely occur in Celestial Mechanics and they are quite common as discretisation of Partial Differential Equations or of some other functional equations (as differential delay equations). The problem with big systems is that the computation of the solutions of the variational equations becomes prohibitive. Hence the variations with respect to initial conditions (and parameters) are not available.

Consider an autonomous system $\dot{x} = f(x)$. Suppose we are asked to compute a T-periodic solution. The simple approach is to look for initial conditions, x_0, such that $G(x_0) = \varphi(T, x_0) - x_0 = 0$. However all the points $\varphi(t, x_0)$, $t \in \mathbb{R}$ are solutions of this equation if x_0 is. Or, equivalently, $DG(x_0) = D\varphi(T, x_0) - Id$ is singular because the tangent vector to the orbit, $f(x_0)$, is an eigenvector of the monodromy matrix $D\varphi(T, x_0)$ with eigenvalue equal to 1. One possible solution to this difficulty is to use some modified Newton method, similar to the one presented in 5.1, but it is more convenient to look for fixed points of the Poincaré map.

Given a section Σ, transversal to the flow, we ask for a point $x_0 \subset \Sigma$ such that $G(x_0) = P(x_0) - x_0 = 0$, where P is the Poincaré map associated to Σ. In this way we do not prefix the period. The zero of G can be obtained by Newton's method using the differential of P as explained in 5.5. The continuation with respect to parameters can be done as in 5.1.

The only question to put is if Newton's method will succeed (except at the bifurcations, related to periodic orbits such that one of the eigenvalues of $D\,P(x_0)$ is equal to 1). If the system has k first integrals then the monodromy matrix has the eigenvalue 1 with multiplicity $k+1$ and it occurs with multiplicity k in $D\,P(x_0)$. Therefore one should look for the Poincaré map restricted to fixed levels of the first integrals to avoid problems in Newton's method. For instance, in the case of Hamiltonian systems one should restrict to a fixed level of energy. In the monodromy matrix it appears a Jordan black of the form $\left(\begin{smallmatrix}1 & \varepsilon\\ 0 & 1\end{smallmatrix}\right)$. The first column is associated to the tangent vector to the orbit and the second one to the variation of the energy. The term ε is different from zero if the period of the orbit changes when the energy does, and equals zero when the period reaches a stationary point.

The analysis of the stability is done as in the study of fixed points of diffeomorphisms.

In systems with symmetries one can take advantage and reduce the set of points in which we look for the periodic orbit to a subset of Σ. Also the stability analysis can be shortened because to obtain the monodromy matrix it can be enough to integrate over a fraction of the period. Those remarks apply if we are interested in symmetric orbits.

Consider now a system like $\dot{x} = f(t,x)$ T-periodic in t. Then the periods of the (possible) periodic orbits should be multiples of T. (There are exceptions, for instance if f does not depend on t on some manifold and this one contains periodic orbits). Now not all the points in the orbit are equivalent, because the origin of time does matter, in contrast with the autonomous case. Then we look for the initial conditions leading to periodic orbit by starting at some point x_0 when $t = 0$ and imposing the condition $G(x_0) = \varphi(T;\ 0, x_0) - x_0 = 0$. Now the tangent vector is not a solution (in general) of the variational equations, because $\frac{d}{dt}\ f(t,x) = D_t f + d_x f \cdot f$ and $D_t f$ is not zero. An equivalent formulation consists in enlarging the phase space using t (mod T) as an additional dependent variable. The enlarged phase space is $S^1 \times \mathcal{U}$, $\mathcal{U}$ open set in $\mathbb{R}^n$, and $S^1 = \mathbb{R}/T$. Then we use the Poincaré section $t = 0$ (mod T).

If our time periodic vector field is a weak periodic perturbation of an autonomous system, $\dot{x} = f(t,x) = f_0(x) + \Sigma_{j\geq 1}\ \varepsilon^j\ J_j(t,x)$, ε small, we shall have numerical troubles. To have T- periodic solutions for ε small enough it is necessary that $\dot{x} = f_0(x)$ have T-periodic solutions (because of the continuous dependence w.r.t. parameters). But then the monodromy matrix associated to f is $M_f = M_{f_0} + O(\varepsilon)$ and so $DG(x_0) = M_{f_0} + O(\varepsilon) - Id$ has eigenvalues (at least 1) which are $O(\varepsilon)$. We can proceed as follows.

Let the solution $x(t)$ be expanded as $x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + \ldots$. Suppose that we know a value $x_0(0)$ giving rise to a periodic orbit of period T of the vector field f_0. To get rid of our lack of knowledge of the right origin of time we consider f_j as $f_j(t-s,x)$, where s is an unknown time shift. The differential equations for $x_0, x_1, x_2, \ldots$ are

$$\dot{x}_0 = f_0(x_0) \quad , \quad x_0(t=0) = x_0(0)\,,$$

$$\dot{x}_1 = D\,f_0(x_0)x_1 + f_1(t-x, x_0) \quad , \quad x_1(t=0) = a\,,$$

$$\dot{x}_2 = D\,f_0(x_0)x_2 + \frac{1}{2}\ D^2 f_0(x_0)(x_1)^2 + D_x f_1(t-s, x_0)x_1 + f_2(t-s, x_0)\,,$$
$$x_2(t=0) = 0\,,$$

....

The initial and final conditions to be equated are $x_0(0) + \varepsilon a$ and $x_0(T) + \varepsilon x_1(T) + \varepsilon^2 x_2(T) + \ldots$, where $x_0(T) = x_0(0)$. Let $A(t)$ the solution of $\dot{A}(t) = D\,f_0(x_0(t))\ A(t),\ A(0) = I$. Then

$$x_1(t) = A(t)(a + y_1(t,s))\,,$$

where

$$y_1(t,s) = \int_0^t\ A(-\tau)\ f_1\left(\tau - s, x_0(\tau)\right)\ d\tau\,,$$

and

$$x_2(t) = A(t)\ y_2(t,s)\,,$$

with

$$\begin{aligned} y_2(t,s) &= \int_0^t A(-\tau)\ \Big[\frac{1}{2} D^2 f_0(x_0(\tau))\Big(A(\tau)(a+y_1(\tau,s))\Big)^2 \\ &\quad + D_x f_1\,(\tau - s, x_0(\tau))\Big(A(\tau)(a+y_1(\tau,s))\Big) + f_2(\tau - s, x_0(\tau))\Big]\, ds \\ &= B\,(t,s)\,(a)^2 + C\,(t,s)\,(a) + d(t,s)\,, \end{aligned}$$

where C and B are (resp.) suitable linear (resp. bilinear) operators. For instance

$$B\,(t,s) = \int_0^t\ A\,(-\tau)\,\frac{1}{2}\ D^2 f_0(x_0(\tau))\ (A(\tau))^2 d\tau\,.$$

The periodicity conditions are given by

$$a = A(T)(a + y_1(T,s)) + \varepsilon A(T)\,[B(T,s)(a)^2 + C(T,s)(a) + d(T,s)] + 0(\varepsilon)^2\,.$$

Let $z(s) = A(T)\ y_1(T,s)$, $\bar{B}(s) = A(T)\ B(T,s)$, $\bar{C}(s) = A(T)\ C(T,s)$, $\bar{d}(s) = A(T)\ d(T,s)$, and assume for $A(T)$ a structure in Jordan blocks as follows

$$A(T) = \left(\begin{array}{cccc|ccc} 1 & \overbrace{1 \quad\;}^{k} & & 0 & & & \\ & \ddots & \ddots & & & 0 & \\ & & \ddots & \ddots \quad 1 & & & \\ & 0 & & \ddots \quad 1 & & & \\ \hline & & & & \lambda_1 & & 0 \\ & & 0 & & & \ddots & \\ & & & & 0 & & \lambda_{n-k} \end{array}\right),$$

with $\lambda_i \neq 1$, $i = 1, \ldots, n-k$, and the multiplicity, k, of 1 as eigenvector is due to the existence of several (global) first integrals. (Eventually $k = 1$). We denote by a_i (resp. $z_i(s)$) , $i = 1, \ldots, n$, the components of a (resp. $z(s)$) in the Jordan basis, and then

$$\left(\bar{B}(s)\,(a)^2\right)_i = \sum_{j,k} b^i_{j,k}\ a_j\, a_k\ ,\ \left(\bar{C}(s)\,(a)\right)_i = \sum_k c_{i,k}\ a_k\ ,\ \left(\bar{d}(s)\right)_i = d_i\,.$$

The periodicity conditions in components are

$$
\begin{aligned}
-a_{j+1} &= z_j(s) + \varepsilon\Big(\sum_{q,m} b^j_{q,m}\ a_q\, a_m + \sum_q\ c_{j,q}\ a_q + d_j\Big) + 0(\varepsilon^2)\,, \\
& j = 1,\dots,k-1\,, \\
0 &= z_k(s) + \varepsilon\Big(\sum_{q,m} b^k_{q,m}\ a_q\, a_m + \sum_q\ c_{k,q}\ a_q + d_k\Big) + 0(\varepsilon^2)\,, \\
(1-\lambda_j)a_{k+j} &= z_{k+j}(s) + \varepsilon\Big(\sum_{q,m} b^{k+j}_{q,m}\ a_q\, a_m + \sum_q\ c_{k+j,q}\ a_q + d_{k+j}\Big) + 0(\varepsilon^2)\,, \\
& j = 1,\dots,n-k\,.
\end{aligned}
$$

First we look at order 0 in ε. A necessary condition is that the k-th component of $y_1(T,s)$ be zero. This gives a condition on the right value of the time shift s. Let s^* a value such that $y_1(T,s^*)=0$. This is a one dimensional equation and the possible zeroes can be computed easily. Using $s=s^*$ the remaining values a_j can be obtained at order ε^0 except a_1. To obtain a_1 we substitute the zero-th order values of $a_2,\dots,a_n$ in the k-th equation and we obtain a quadratic equation for a_1. Again, if the resulting values of a_1 are simple, a standard argument using the implicit function theorem ensures that we can obtain the components a_j to the required order in ε.

I skip further discussion on this problem but I would like to make a couple of remarks. The first one concerns the existence of symmetries which can reduce the dimensionality of the problem by setting some of the components of a, in a suitable basis, equal to zero. The second one considers problems in which the $\varepsilon=0$ problem has not an isolated T-periodic orbit, but a manifold of them. Then the $A(T)$ matrix has several eigenvectors (and not only one) associated to the eigenvalue 1. We have as many necessary conditions (of the type $z_k(s)=0$ in the previous case) as independent eigenvectors with eigenvalue equal to 1. The determination of the missing a_j coefficients is done by intersection of quadrics.

To study the stability properties of a periodic orbit one can look either at the differential of the Poincaré map or directly at the monodromy matrix. Linear stability requires all the eigenvalues of the monodromy matrix of modulus less than or equal to 1. It it is equal to 1 it should be simple. If the system has several first integrals it is natural to consider the stability on the corresponding energy level of the integrals. It is important to remark that the variational equations $\dot{A}(t) = D_x f(t,x)\ A(t)$ with orbit T-periodic and f also T-periodic in t, are linear with periodic coefficients. Then the Floquet theory applies and the logarithms of the eigenvalues of $A(T)$ divided by T are called the characteristic exponents. Let $e_1,\dots,e_k$ be k vectors which generate an invariant subspace under $A(T)$ (for instance the unstable subspace, associated to eigenvalues, λ, with $|\lambda|>1$). This gives the linear approximation to an invariant manifold at the initial point x_0. To obtain it at some other point $\varphi(t,x_0)$ it is enough to take the linear subspace generated by $\{A(t)\,e_j\ ,\ j=1,\dots,k\}$. The vectors $A(t)\,e_j$ are called the Floquet modes and they tell us the local structure around the p.o.

Finally we consider the case of the computation of periodic orbits when the variational equations are not available (for instance due to a high dimensionality). If the periodic orbit is stable the simple and usual device is to compute he solution numerically for a long time. Then the solution tends to the periodic orbit and we obtain an approximation of the periodic orbit. However this method has several difficulties:

a) If the orbit is weakly attractive the computing time to become close to the periodic orbit can be too large.

b) If the orbit is not attractive (because there are eigenvalues of $A(T)$, other than the tangent vector to the orbit in the autonomous case, with modulus larger than or equal to 1) the method fails.

Hence the method is not able to follow unstable orbits. An alternative method can be used, specially if only a few eigenvalues of $A(T)$ are larger than or equal to 1. Suppose first, to make things easy, that we use $x_n = 0$ as surface of section, that $D\,P$ has only one eigenvalue, λ, with $\lambda > 1$ and all the other eigenvalues have modulus much less than 1. Then, if the initial point, $x^{(0)}$, is close to the fixed point, x^*, of P , we can compute the next two iterates, $x^{(1)} = P(x^{(0)})$, $x^{(2)} = P(x^{(1)})$ and use a generalised version of Aitken's extrapolation. Skipping the effect of contractive eigenvalues we have

$$x^{(1)} - x^* = \lambda(x^{(0)} - x^*), \quad x^{(2)} - x^* = \lambda(x^{(1)} - x^*)\,,$$

and hence

$$x^{(2)} - x^{(1)} = \lambda(x^{(1)} - x^{(0)})\,,$$

allowing to compute λ and then to obtain $x^* = (x^{(1)} - \lambda x^{(0)})/(1-\lambda)$. Of course we do not obtain exactly x^* in this way, but a new value of $x^{(0)}$ and we can iterate the procedure. The method can be applied in the same way if $\lambda < 1$ but all the other eigenvalues, μ, of $D\,P(x^*)$ have modulus with $|\mu| << |\lambda|$. This can produce the stable orbit in an accelerated way.

Now suppose that we have k eigenvalues of $D\,P(x^*)$ which are relevant (with modulus either $|\lambda| > 1\,,\ |\lambda = 1|$ or $|\lambda| < 1$) and all remaining eigenvalues are much less than 1. Let B be the restriction of $D\,P(x^*)$ to the k-dimensional subspace associated to the k-dominant eigenvalues. Starting at some point $x^{(0)}$ close to x^* and computing $x^{(j+1)} = P(x^{(s)}),\ j = 0, 1, \ldots, k$, we can write

$$x^{(j+1)} - x^* \simeq B(x^{(j)} - x^*),\ j = 0, \ldots, k\,,$$

and hence

$$x^{(j+1)} - x^{(j)} = B(x^{(j)} - x^{(j-1)}),\ j = 1, \ldots, k\,,$$

allowing to compute B and then $x^* = (I - B)^{-1}\,(x^{(1)} - B\,x^{(0)})$. The dominant subspace is the one generated by the vectors $x^{(j)} - x^{(j-1)},\ j = 1, \ldots, k$. Again the value of x^* is an improved approximation and can be used as a new $x^{(0)}$ to iterate the procedure. The method is suitable for moderate values of k.

To study the stability properties when the variational equations are not available we can use the previous method. The eigenvalues of B give the dominant eigenvalues of $D\,P(x^*)$ and hence the linear stability. Otherwise, for periodic orbits as well as for general orbits, one can use the method of the Lyapunov exponents.

7 On the computation of Lyapunov exponents

Lyapunov exponents are a generalisation of the characteristic exponents of the periodic orbits to general orbits. They tell us about the local properties near a given orbit. In particular, the existence of positive Lyapunov exponents implies the existence of sensitive dependence w.r.t. initial conditions and, hence, to some amount of unpredictability or chaos. It is possible to have sensitive dependence with zero maximal Lyapunov exponent, but then the distance between nearby orbits increases in a subexponential way.

We present them for flows and the reader can easily translate them to the case of maps.

Let $\dot{x} = f(x)$ an autonomous o.d.e. Consider the starting point x_0, an initial vector ξ_0 and look for $\xi(t) = D_x\,\varphi(t, x_0)\xi_0$. If it exists the limit

$$\lim_{t\to\infty} \frac{1}{t}\,\log\frac{\|\xi(t)\|}{\|\xi_0\|}\,,$$

(of course you can skip the denominator) it is called the Lyapunov exponent associated to ξ_0 and to the orbit through x_0. It is clear that the limit is independent on x_0 and only depends on the orbit of x_0. The same value is obtained starting at $\varphi(t^*, x_0)$ with $D_x\,\varphi(t^*, x_0)\,\xi_0$ instead of ξ_0 , for any real t^*.

For a given point x_0 there exists, generically, a decomposition of the tangent space in linear subspaces $E_0 \subsetneq E_1 \subsetneq \cdots \subsetneq E_m$ such that for all $\xi_0 \in E_j \backslash E_{j-1}$ one obtains the same Lyapunov exponent ℓ_j. Furthermore $\ell_1 < \ell_2 < \ldots < \ell_m$, and ℓ_m is called the maximal Lyapunov exponent. For a periodic orbit some ℓ_j equals zero (just take the tangent vector to the orbit as ξ_0). For an orbit sitting on a k-dimensional torus with irrational flow there is a k-dimensional subspace of the tangent space giving $\ell = 0$. The remaining Lyapunov exponents not associated to the invariant manifold (the p.o. or the torus, say) give the transversal stability properties. Linear stability around an invariant object requires $\ell_m = 0$. It is a simple fact that $\ell_m = 0$ for an integrable Hamiltonian.

Concerning the numerical computations of the Lyapunov exponents there are 2 problems. First, for big systems the variations $D_x\varphi$ are not available. Second is the computation of the limit.

To deal with the first problem we consider x_0 and $x_0 + h\xi_0$ as initial points, with h small. Then we substitute $\xi(t)$ by $(\varphi(t, x_0 + h\xi_0) - \varphi(t, x_0))\,/\,h$. To prevent from numerical overflow and underflow (and to work in a neighbourhood of the orbit of x_0) one should scale the distance from time to time, replacing $\varphi(t, x_0 + h\xi_0)$ by $\varphi(t, x_0) + s(\varphi(t, x_0 + h\,\xi_0) - \varphi(t, x_0))$ with a suitable scale factor s, keeping track of the successive changes of scale. In this way we obtain ℓ_m with probability 1. If we are interested in several Lyapunov exponents and not only the dominant one, just take q initial orthogonal vectors $\xi_1, \ldots, \xi_q$. One can compute the orbits through x_0 and $x_0 + h\,\xi_j$, $j = 1, \ldots, q$, with h small. For a given t let $\bar{\xi}_j(t) = (\varphi(t, x_0 + h\xi_j) - \varphi(t, x_0))\,/\,h$. Then we measure

$$\frac{1}{t}\,\log\frac{\|\bar{\xi}_1\|}{\|\xi_1\|},\ \frac{1}{t}\,\log\frac{\|\bar{\xi}_1\wedge\bar{\xi}_2\|}{\|\xi_1\wedge\xi_2\|}\,,\ldots,\ \frac{1}{t}\,\log\frac{\|\bar{\xi}_1\wedge\bar{\xi}_2\wedge\cdots\wedge\bar{\xi}_q\|}{\|\xi_1\wedge\xi_2\wedge\cdots\wedge\xi_q\|}\,,$$

where $\|\ \|$ denotes the length, the area,..., the q-dimensional volume when applied to a vector, a two-vector, ..., a q-vector. With probability one these quantities tend to

$\ell_m, \ell_m+\ell_{m-1}, \ldots, \ell_m+\ell_{m-1}+\cdots+\ell_{m-q+1}$. Hence it is possible to obtain $\ell_m, \ldots, \ell_{m-q+1}$ by subtraction. To prevent from bad conditioning when computing the j-dimensional volumes, as all the vectors $\bar{\xi}_i$ tend to the maximal expanding direction, one should not only scale the lengths from time to time but also to perform an orthonormalisation. For instance, let $q=2$. We start with ξ_1, ξ_2 orthogonal of modulus 1. At some time t_1 we have $\bar{\xi}_1(t_1),\ \bar{\xi}_2(t_1)$. Let $s_1 = \|\bar{\xi}_1(t_1)\|$ and $a_1 = \|\bar{\xi}_2(t_2) - (\bar{\xi}_1(t_1),\ \bar{\xi}_2(t_2))\ \bar{\xi}_1(t_1) \,/\, s_1^2\|$. We substitute $\bar{\xi}_1(t_1),\ \bar{\xi}_2(t_2)$ by $\bar{\xi}_1(t_1) \,/\, s_1$ and $\left(\bar{\xi}_2(t_2) - (\bar{\xi}_1(t_1),\ \bar{\xi}_2(t_2))\ \bar{\xi}_1(t_1) \,/\, s_1^2\right) / \, a_1$ and start again the computation at $\varphi(t_1, x_0)$ with these two initial vectors up to some new time t_2, where we compute s_2, a_2 and so on. Then

$$\ell_m = \lim_{p\to\infty} \frac{\sum_{j=1}^{p} \log(s_j)}{t_p}, \quad \ell_{m-1} = \lim_{p\to\infty} \frac{\sum_{j=1}^{p} \log(a_j)}{t_p}.$$

This is easy to generalise to any q.

The second difficulty concerns the computation of the limit. As any numerical computation of a limit it is never possible to be sure that if we stop at some final time t_f the quotient will not change dramatically at some greater time (unless we are in the case of periodic orbits). For instance, in the case of a Hamiltonian system we can start in a region mostly dominated by regular motion, with zero maximal Lyapunov exponent, and then to drift to a chaotic region, after some big time interval, by a slow Arnold's diffusion. Perhaps this affects only to a small set of initial conditions. But one can also have motions wandering from some chaotic region to another chaotic region with "local Lyapunov exponents" quite different.

As a conclusion, one should look at the numerical computation of Lyapunov exponents as a local indication of the nearby behaviour both in space and in time.

8 Computation of quasi periodic orbits

The computation of quasi periodic orbits is a much harder question than the computation of periodic solutions. Last ones are characterised by $\varphi(T, x_0) = x_0$ for suitable x_0 and T. This leads to a numerical equation that can be solved by different methods as explained. For quasiperiodic solutions we look for a solution $\varphi(t, x_0)$ which is of the form $F(z),\ z \in \mathbb{R}^k,\ F(z) \in \mathbb{R}^n,\ z = \omega t$, where ω is a vector of incommensurable frequencies, i.e., such that $(m, \omega) = 0$ for some $m \in \mathbb{Z}^k$ implies $m = 0$.

Furthermore, in many cases, the quasiperiodic orbits are not isolated but they belong to an invariant torus and the problem is, in some sense, degenerated because of the manifold of solutions. Before presenting some methods we should remark that the needs are different depending on the applications. For long time spans one should provide a proof of convergence of the procedure (showing also the existence). For short time intervals it can be enough an approximation with control on the errors.

The methods which follow are based on our own experience in several problems:

a) *Methods based on transformations to simplify the system and reduce the problem to a fixed point problem.*

Suppose that we start with an integrable Hamiltonian, $\mathcal{H}_0(I)$, and we perturb by adding $\varepsilon\mathcal{H}_1(I,\theta,t)$, where $\mathcal{H}_1$ depends on the action variables, I, on the angle variables, θ, in a 2π-periodic way, and the dependence with respect to the time is quasi periodic. We look for (canonical) transformations such that the transformed Hamiltonian depends only on the new action variables. We generate the transformation from old to new coordinates $(I,\theta) \to (\bar{I},\bar{\theta})$ as the flow time ε under a suitable (generating) Hamiltonian $S_1(I,\theta,t)$ as follows:

$$\frac{d\theta}{d\varepsilon} = \frac{\partial S_1}{\partial I} \ , \ \frac{dI}{d\varepsilon} = \frac{\partial S_1}{\partial \theta} \ , \text{ initial conditions } (I,\theta), \text{ final conditions } (\bar{I},\bar{\theta}),$$

and hence

$$I = \bar{I} + \varepsilon \frac{\partial S_1}{\partial \bar{\theta}} + O(\varepsilon^2) \ , \ \theta = \bar{\theta} - \varepsilon \frac{\partial S_1}{\partial I} + O(\varepsilon^2),$$

and, by substitution, the transformed Hamiltonian is

$$\mathcal{H}_0(\bar{I}) + D\mathcal{H}_0(\bar{I}) \ \varepsilon \frac{\partial S_1}{\partial \bar{\theta}} + \varepsilon \mathcal{H}_1(\bar{I},\theta,t) + O(\varepsilon^2).$$

The $O(\varepsilon)$ terms are killed by choosing S_1 such that $\left(\omega_0(\bar{I})\, ,\, \frac{\partial S_1}{\partial \bar{\theta}}\right) + \mathcal{H}_1(I,\theta,t) = 0$, where $\omega_0(\bar{I}) = \nabla\mathcal{H}_0(\bar{I})$. The last equation is solved by Fourier expansion w.r.t. $\bar{\theta}$. This leads in a natural way to the problem of small divisors, and the possibility of solving this step relies on the Diophantine properties of $\omega_0(\bar{I})$. If we are able to construct S_1 then we should still kill, successively, the terms in $\varepsilon^2, \varepsilon^3$, etc., and we are also faced to the problem of convergence of the procedure (perhaps on some subset). At the end we have a Hamiltonian, $K(J)$, and the fixed points of it (or the periodic, quasiperiodic solutions) give rise, by back substitution in the chain of transformations, to quasiperiodic solutions of the original system. For short time (in a relative sense) applications it can be enough to carry out a small number of transformations. Furthermore, in each transformation we can skip the terms with coefficient below some bound. Even if they are in resonance this will not be noticed before a long time interval.

b) *Methods based on direct substitution allowing for variation of the frequencies.*

Working with the same problem as in a) assume that we know some quasiperiodic (or periodic) solution of the Hamiltonian $\mathcal{H}_0(I)$. Let this solution be described, in some variables x, as $x_0 = F_0(\nu_0\, t)$, where ν_0 is the k-vector of frequencies for this quasiperiodic solution. Let $\dot{x} = f_0(x) + \varepsilon f_1(x,t)$ be the differential equation for the variables x. We have only f_0 if the Hamiltonian is just $\mathcal{H}_0$, and $f_0 + \varepsilon f_1$ when we consider $\mathcal{H}_0 + \varepsilon\,\mathcal{H}_1$. Furthermore we suppose that the quasiperiodic dependence of $\mathcal{H}_1$ w.r.t. time has a vector of frequencies $\mu \in \mathbb{R}^p$. As $\dot{x}_0 = f_0(x_0)$ we have $D\,F_0(\nu_0\,t)\cdot\nu_0 = f_0(F_0(\nu_0\,t))$.

Allowing for variation of ν when ε changes: $\nu = \nu_0 + \varepsilon\nu_1 + O(\varepsilon^2)$, and assuming $x = x_0 + \varepsilon\, x_1$, where $x_1 = F_1(\nu t, \mu t)$, we can collect the terms in ε:

$$\begin{aligned}(DF_0(\nu_0 t))(\nu_1) + (D^2F_0(\nu_0 t))(\nu_1,\nu_0)t + \dot{F}_1(\nu_0 t,\mu t) = \\ \Big(Df_0(F_0(\nu_0 t))\, DF_0(\nu_0 t)\Big)(\nu_1)t + Df_0(F_0(\nu_0 t))\, (F_1(\nu_0 t,\mu t)) + f_1(F_0(\nu_0 t),\mu t)\,.\end{aligned}$$

We should select ν_1 and, eventually, the additional free parameters which appear in the expression of x_0 in such a way that the terms containing t and terms giving rise to a non quasiperiodic behaviour in F_1, cancel. For instance, if $k = 1$ by differentiation of $\dot{x}_0 = f_0(x_0)$, again w.r.t. time, one checks that the terms having t as a factor cancel. Whether this choice of ν_1 and additional parameters is possible or not depends on the specific problems. The small divisors problem remains here.

Both in a) and b) the practical construction should be helped by suitable manipulators.

c) *The use of quasi periods.*

Given a quasi periodic function of time, $q(t)$, for every $\delta > 0$ there are associated quasi periods, $T(\delta)$, such that $\|q(t+T(\delta)) - q(t)\| < \delta \ \forall t \in \mathbb{R}$. The value $T(\delta)$ goes to ∞ when δ goes to zero, and good selections of $T(\delta)$ depend on the Diophantine properties of the basic frequency vector of $q(t)$. Suppose we look for quasiperiodic solutions of $\dot{x} = f(x, q(t))$ with q as before. Fixing δ small and using a related quasiperiod, $T(\delta)$, we can ask for some initial value, $x = x_0$, such that the flow transformation from $t = 0$ to $t = T(\delta)$ gives $\varphi(T(\delta); 0, x_0) = x_0$. This would be a (numerical) solution for $0 \leq t \leq T(\delta)$ but not for all $t \in \mathbb{R}$ because the vector field does not repeat exactly after a time interval $T(\delta)$. One can hope that if $\delta \searrow 0$ the initial values x_0 converge to some value x^* such that $\varphi(t;\ 0,\ x^*)$ is a true quasiperiodic solution of $\dot{x} = f(x, q(t))$. But this is an open question, despite practical cases show good behaviour provided the frequency vector of q is well approximated by rationals (in opposition to what we ask usually to cope with the small divisors problem).

d) *Numerical refinement of approximated analytical quasiperiodic solutions.*

Sometimes we can be interested in to have a nice approximation to a true solution of the differential equation and which is close to a quasiperiodic solution restricted to some finite time interval (provided this solutions exists). The method c) already gives this numerical approximation. For the solutions approximated as in a), b) we can refine as follows. Let $\hat{x}(t)$ an analytical expression, quasiperiodic in time, which gives an approximate solution of $\dot{x} = f(x,t)$. Let $t_0,\ t_f$ the initial and final times. We want to refine the solution for $t \in [t_0, t_f]$. It is advisable to take $t_f - t_0$ equal to some quasiperiod $T(\delta)$ for δ not too big. If x is n-dimensional then we select p-values among the components of $\hat{x}(t_0)$ and $n - p$ among the ones of $\hat{x}(t_f)$. Let $x^0_{i_j},\ j = 1, \ldots, p$ and $x^f_{\ell_j},\ j = 1, \ldots, n-p$ their values. The selection of the suitable components depends on the geometrical and numerical properties of the solution we are searching for, or can be given as physical constrains. For instance, if there are unstable direction when we study the local behaviour around the solution we can try to make the related components equal to zero. Then we are faced to a boundary problem: To obtain (numerically) a solution of $\dot{x} = f(x,t),\ \tilde{x}(t)$, such that $\tilde{x}(t_0)|_{i_j} = x^0_{i_j},\ j = 1, \ldots, p,\ \tilde{x}(t_f)|_{\ell_j} = x^f_{\ell_j},\ j = 1, \ldots, n-p$.

If $t_f - t_0$ is big and the variational matrix $\frac{\partial \varphi(t_f; t_0, x_0)}{\partial x_0}$ has big eigenvalues we shall have numerical difficulties. This applies also to periodic orbits with big eigenvalues of the monodromy matrix. Then the suitable approach is the parallel shooting method. Let $t_0 < t_1 < t_2 \ldots < t_s = t_f$ be a splitting of the time interval in s subintervals. We consider as unknowns the following values:

1) The $n - p$ unknown values of $\tilde{x}(t_0)$ at $t = t_0$. We denote them by $\tilde{x}_0$.

2) The $(s-1)\,n$ values of $\tilde{x}$ at $t = t_1, \ldots, t_{s-1}$. We denote them by $\tilde{x}_j$, $j = 1, \ldots, s-1$.

3) The p unknown values of $\tilde{x}(t_s)$. We denote them by $\tilde{x}_s$.

We set out the $s \times n$ equations to obtain $\tilde{x}_0, \tilde{x}_j,\ j = 1, \ldots, s-1,\ \tilde{x}_s$:

$$\varphi(t_j; t_{j-1}, \tilde{x}_{j-1}) = \tilde{x}_j \quad , \quad j = 1, \ldots, s,$$

which can be solved again by Newton's method. As initial values for $\tilde{x}_j$ one can take the analytical approximations $\tilde{x}(t_j),\ j = 0, \ldots, s$. If necessary the following continuation problem can be solved:

$$\dot{x} = \rho\, f(x,t) + (1-\rho)\, \frac{d}{dt}\, \hat{x}(t),$$

where ρ ranges from 0 to 1.

The stability properties of the quasi periodic solution can be estimated, if $t_f - t_0$ is close to $T(\delta)$ with δ small, by looking at the eigenvalues of $\frac{\partial \varphi(t_f; t_0, x_0)}{\partial x_0}$. The related eigenvectors give a hint on the stable/unstable manifolds.

9 Normal forms

First we consider the case of diffeomorphisms. Let 0 be a fixed point and assume that the expansion of a diffeomorphism, F, around 0 is $F(x) = \Lambda x + \sum_{j\geq 2} \sum_{|\alpha|=j} a_\alpha\, x^\alpha$, where Λ is a diagonal matrix, $\Lambda = diag\,(\lambda_1, \ldots, \lambda_n)\,, a_\alpha \in \mathbb{R}^n$ and $\alpha \in (\mathbb{N} \cup \{0\})^n$ is a multiindex. As usual $x^\alpha = \prod_{i=1}^n x_i^{\alpha i}$ and $|\alpha| = \sum_{i=1}^n |\alpha_i|$. We try to kill all the nonlinear terms of F by means of a change of variables $x = y + \sum_{j\geq 2} \sum_{|\alpha|=j} b_\alpha\, y^\alpha$. For the moment being assume we have succeeded up to a given order and we have $F(x) = \Lambda x + \sum_{|\alpha|=j} a_\alpha\, x^\alpha + 0(\,|x|^{j+1})$. Then if $x = y + \sum_{|\alpha|=j} b_\alpha\, y^\alpha$, we have $y = x - \sum_{|\alpha|=j} b_\alpha\, x^\alpha + 0(\,|x|^{2j-1})$ and, by substitution, it is easily obtained that the k-th component of the vector b_α should be chosen as $(b_\alpha)_k = \frac{(a_\alpha)_k}{\lambda_k - \lambda^\alpha}$. This is formally possible if $\lambda_k - \lambda^\alpha \neq 0$, i.e., if the nonresonance condition $\lambda_k \neq \lambda^\alpha\ \ \forall k = 1, \ldots, n,\ \ \forall \alpha$ such that $|\alpha| \geq 2$, is fulfilled. If there are resonances we had to keep the resonant terms in the normal form. It is clear that the transformations we do at order k do not alter the eventual nonlinear terms of lower order.

As an example we take F as an area preserving map around an hyperbolic point. Then $\Lambda = diag\,(\lambda, \lambda^{-1}),\ |\lambda| > 1,\ \lambda \in \mathbb{R}$. Let x_1, x_2 be the local coordinates and F_1, F_2 the components of F. As there are the unavoidable resonances $\lambda = \lambda^{s+1}(\lambda^{-1})^s,\ \lambda^{-1} = \lambda^s(\lambda^{-1})^{s+1},\ \forall s \in \mathbb{N}$, we have a normal form of the type $F_1(x_1,x_2) = p(x_1,x_2)\ x_1$, $F_2(x_1,x_2) = q(x_1,x_2)\ x_2$, where $p(x_1,x_2) = \lambda + \sum_{j\geq 1} c_j(x_1,x_2)^j,\ q(x_1,x_2) = \lambda^{-1} + \sum_{j\geq 1} d_j(x_1,x_2)^j$. It is possible to perform the passage to normal form keeping the area preserving character and then $q(x_1,x_2) = \big(p(x_1,x_2)\big)^{-1}$. Hence the quantity $x_1\, x_2$ is a first integral of the diffeomorphism: $x_1\, x_2 = F_1(x_1,x_2) \cdot F_2(x_1,x_2)$. In general a first integral, G, satisfies $G(x) = G\big(P(x)\big)$. The normal form obtained in this way for area preserving maps is the so called Birkhoff normal form. In the case of a hyperbolic fixed point it is convergent if F is analytical, but for an elliptic fixed point it is generically divergent. In the hyperbolic case the domain of convergence probably goes up to the nearest fixed point (either real or complex), but the full analysis is missing. There are also normal forms in the case of parabolic fixed points, with eigenvalues $+1$ (double) or -1 (double), both in the case of diagonal and nondiagonal Jordan form of the linear part.

Continuing with this example there is an interesting case when F depends on a small parameter, ε, in the following way: $F(x) = \Lambda x + \varepsilon \sum_{j\geq 2}\sum_{|\alpha|=j} a_\alpha x^\alpha$, where $\Lambda = diag\,(\lambda, \lambda^{-1})$, and $\lambda = 1+\mu_1\varepsilon+\mu_2\varepsilon^2+\ldots$, $\mu_1 > 0$. Then we encounter divisors which are $O(\varepsilon)$. However, the nonlinear part of F is also $O(\varepsilon)$ and there is no problem in obtaining (in a convergent way) the normal form, which is $x + O(\varepsilon)$. The transformation has, in general, nonlinear terms which are $O(\varepsilon^0)$. Furthermore, the domain of convergence is uniform with respect to ε for $\varepsilon \in (0, \varepsilon_0)$, ε_0 small enough. From the practical point of view one should compute the coefficients of the transformation and of the normal form as expansions in powers of ε and look for the coefficients of those expansions to avoid numerical problems.

Returning to the general case, if Λ is not diagonal, i.e., contains nilpotent parts, one should look for the natural modifications of the procedure.

In many problems the diffeomorphism can depend on parameters. For specific values of the parameters perhaps we do not have resonances, but when they move in their range they can cross resonances. Then we should keep those resonances in the normal form. Even if for fixed values of the parameters there is a resonance close to the origin (for instance, a "zone of islands" for an elliptic area preserving map) we should keep this resonance in the normal form, producing the so called quasiresonant normal form, if we are interested in the qualitative or quantitative study of the resonant zone. Killing this quasiresonance produces a normal form which, if we stop the procedure at some finite order, can give good approximations to the diffeomorphism but only in a smaller domain which does not contains the resonance zone.

If instead of a fixed point of a diffeomorphism we look at a periodic point then first we should obtain the Taylor expansion of the composed mapping, F^k, around one of the periodic points of period k. This is formally easy, but the computational complexity is big if we are interested in a higher order expansion, the period is long or the dimension is high. In principle there are combinatorial formulae producing the result, but the number of summations can be very large. Recurrent methods can be better. For instance, assume $f(x) = \sum_{j\geq 0} a_j x^j$ and we are interested in computing $g = f^m$ up to some degree n. By logarithmic differentiation we have $m\,f'\,g = f\,g'$ and, if $g(x) = \sum_{j\geq 0} b_j x^j$, we have the recurrent relation

$$b_0 = a_0^m\,, \quad b_{s+1} = \frac{1}{a_0(s+1)} \sum_{j=1}^{s+1} a_j\, b_{s+1-j}\; (j\,m + j - s - 1)\,, \quad s \geq 0\,.$$

Of course, if we need not only f^m, $m \in \mathbb{N}$ but $f^2, f^3, \ldots, f^m$, then it is better to obtain each one of the powers by product of two polynomials up to some order. Problems with several variables are worked out recurrently. If we have p variables we can consider just 1 variable and the coefficients depend on the other $p-1$ ones.

The composition of power series $h(x) = g(f(x))$ can be obtained by computing the powers f^m and adding up. In some cases is better to derive ad hoc formulae. For instance $g = \exp(f)$ can be computed by using $g' = f'\,g$. If f and g as before we have $b_0 = \exp(a_0)$, $b_{k+1} = \frac{1}{k+1}\sum_{j=0}^{k} (k+1-j)\; a_{k+1-j}\; b_j$ for $k \geq 0$. In a similar way one can compute $g = \log f$ from $b_0 = \log a_0$, $g'\cdot f = f'$, or $g = \sin(f)$, $h = \cos(f)$ from $g' = h\cdot f'$, $h' = -g\cdot f'$.

To reverse a power series, $f(x) = \sum_{j\geq 1} a_j x^j$, we should obtain $g(x) = \sum_{j\geq 1} b_j x^j$ such that $f(g(x)) = h(x) = \sum_{j\geq 1} c_j x^j$. One has $b_1 = c_1/a_1$. Let $g^{(p)}(x)$ an approximation to g of degree 2^p. For instance $g^{(0)}(x) = b_1 x$. Then one can use a Newton-like algorithm as follows

$$g^{(p)} = g^{(p-1)} - \frac{(f(g^{(p-1)}) - h)|_{2^p}}{f'(g^{(p-1)})|_{2^{p-1}-1}},$$

where $q|_r$ means the truncation up to order r of the power series q. Then one has $(f(g^{(p)}) - h)|_{2^p} = 0$.

Other advantageous procedures use the Lie-series method, but this is already described in other sections of this book.

If the diffeomorphism is obtained as the Poincaré section of a flow around a periodic orbit then the Taylor expansion around the fixed point can be computed from the variational equations. It can be a major task to reach order exceeding 3 or 4. Then we shall know only a few terms of the normal form. This can be enough for describing the local behaviour or to approximate invariant manifolds. However, in simple cases and making use of symmetries, if any, one can reach high order in a cheap way. Let us consider the following example: $x'' = f(t)\, g(x)$, where t is a T-periodic function with positive mean value and g is an odd function, $g(x) = g_1 x + g_3 x^3 + \ldots, g_1 > 0$. As a system we have $x' = y$, $y' = f(t)\, g(x)$. The origin is a fixed point and we are interested in the diffeomorphism φ_T, given by the time T flow. We introduce $x = \sum_{j\geq 0} \varepsilon^{2j+1} x_{2j+1}(t)$, $y = \sum_{j\geq 0} \varepsilon^{2j+1} y_{2j+1}(t)$ (it is enough to take odd order terms). By substitution we get

$$\begin{aligned}
&x_1' = y_1, \quad y_1' = f(t)\; g_1\; x_1\,, \\
&x_3' = y_3, \quad y_3' = f(t)\; [g_1\; x_3 + g_3\; x_1^3]\,, \\
&x_5' = y_5, \quad y_5' = f(t)\; [g_1\; x_5 + g_3\; 3x_1^2\; x_3 + g_5\; x_1^5]\,, \\
&x_7' = y_7, \quad y_7' = f(t)\; [g_1\; x_7 + g_3\; (3x_1^2\; x_5 + 3x_1\; x_3^2) + g_5\; 5x_1^4\; x_3 + g_7\; x_1^7]\,, \\
&\ldots\ldots\,.
\end{aligned}$$

We take as initial values $x_1(0) = x^0,\ y_1(0) = y^0;\ x_j(0) = y_j(0) = 0$ for $j > 1$.

The first order variational equations can be written as

$$A' = \begin{pmatrix} 0 & 1 \\ g_1 f(t) & 0 \end{pmatrix} A\,, \quad A(0) = Id, \quad A(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{pmatrix}\,.$$

Hence $x_1(t) = \psi_{10}(t)\; x^0 + \psi_{01}(t)\; y^0$, where $\psi_{10} = a_{11},\ \psi_{01} = a_{12}$ and $y_1(t) = \bar\psi_{10}(t)\; x^0 + \bar\psi_{01}(t)\; y^0$, where $\bar\psi_{10} = a_{21},\ \bar\psi_{01} = a_{22}$. Introducing $x_j(t) = \sum_{k=0}^{j} \psi_{j-k,k}(t)(x^0)^{k-j}(y^0)^j$, $y_j(t) = \sum_{k=0}^{j} \bar\psi_{j-k,k}(t)(x^0)^{k-j}(y^0)^j$ we obtain equations for the ψ's, $\bar\psi$'s as $\psi'_{j-k,k} = \bar\psi_{j-k,k}$, and

$$\begin{aligned}
\bar\psi_{30}' &= f(t)\; [g_1\, \psi_{30} + g_3\, \psi_{10}^3]\,, \\
\bar\psi_{21}' &= f(t)\; [g_1\, \psi_{21} + g_3\, 3\psi_{10}^2\, \psi_{01}]\,, \\
\bar\psi_{12}' &= f(t)\; [g_1\, \psi_{12} + g_3\, 3\psi_{10}\, \psi_{01}^2]\,, \\
\bar\psi_{03}' &= f(t)\; [g_1\, \psi_{03} + g_3\, \psi_{01}^3]\,,
\end{aligned}$$

$$\bar{\psi}'_{50} = f\,(t)\ [g_1\,\psi_{50} + 3g_3\,(\psi_{10}^2\,\psi_{30}) + g_5\,\psi_{10}^5]\,,$$
$$\bar{\psi}'_{41} = f\,(t)\ [g_1\,\psi_{41} + 3g_3\,(\psi_{10}^2\,\psi_{21} + 2\psi_{10}\ \psi_{01}\ \psi_{30}) + g_5\,5\psi_{10}^4\,\psi_{01}]\,,$$
$$\bar{\psi}'_{32} = f\,(t)\ [g_1\,\psi_{32} + 3g_3\,(\psi_{10}^2\,\psi_{12} + 2\psi_{10}\ \psi_{01}\ \psi_{21} + \psi_{01}^2\,\psi_{30}) + g_5\,10\psi_{10}^3\,\psi_{01}^2]\,,$$
$$\bar{\psi}'_{23} = f\,(t)\ [g_1\,\psi_{23} + 3g_3\,(\psi_{10}^2\,\psi_{03} + 2\psi_{10}\ \psi_{01}\ \psi_{12} + \psi_{01}^2\,\psi_{21}) + g_5\,10\psi_{10}^2\,\psi_{01}^3]\,,$$
$$\bar{\psi}'_{14} = f\,(t)\ [g_1\,\psi_{14} + 3g_3\,(2\psi_{10}\,\psi_{01} + \psi_{03}\ \psi_{01}^2\ \psi_{12}) + g_5\,5\psi_{10}\,\psi_{01}^4]\,,$$
$$\bar{\psi}'_{05} = f\,(t)\ [g_1\,\psi_{05} + 3g_3\,(\psi_{01}^2\,\psi_{03}) + g_5\,\psi_{01}^5]\,,$$
$$\ldots\ldots\,.$$

Finally we set $t = T$ and $\varepsilon = 1$ obtaining

$$\varphi_T(x^0, y^0) = \Big(\sum_{j\geq 0}\ \sum_{k=0}^{i,j+1}\ \psi_{2j+1-k,k}\ (T)(x^0)^{2j+1-k}\ (y^0)^k\,,$$
$$\sum_{j\geq 0}\ \sum_{k=0}^{2j+1}\ \bar{\psi}_{2j+1-k,k}\ (T)(x^0)^{2j+1-k}\ (y^0)^k\Big).$$

To obtain the normal form of a vector field $\dot{x} = f(x) = \Lambda x + \sum_{j\geq 2}\sum_{|\alpha|=j}\ a_\alpha\, x^\alpha$, where $\Lambda = diag\,(\lambda_1, \ldots, \lambda_n)$, we proceed in a similar way. Assume $f(x) = \Lambda x + \sum_{|\alpha|=j}\ a_\alpha\ x^\alpha + 0\,(\,|x|^{j+1})$ and $x = y + \sum_{|\alpha|=j}\ b_\alpha\ y^\alpha$. Then $y = x - \sum_{|\alpha|=j}\ b_\alpha\ x^\alpha + O\,(\,|x|^{2j-1})$ and, by substitution:

$$\dot{y} = \dot{x} - \sum_{|\alpha|=j}\ b_\alpha\ x^\alpha\Big(\sum_{i=1}^{n}\ \frac{\alpha_i\ \dot{x}_i}{x_i}\Big) + O\,(\,|x|^{2j-1})$$
$$= \Lambda x + \sum_{|\alpha|=j}\ a_\alpha\ x^\alpha - \sum_{|\alpha|=j}\ b_\alpha\ x^\alpha\Big(\sum_{i=1}^{n}\ \alpha_i\,\lambda_i\Big) + O\,(\,|x|^{j+1})$$
$$= \Lambda\Big(y + \sum_{|\alpha|=j}\ b_\alpha\ y^\alpha\Big) + \sum_{|\alpha|=j}\ a_\alpha\ y^\alpha - \sum_{|\alpha|=j}\ b_\alpha\,y^\alpha\Big(\sum_{i=1}^{n}\ \alpha_i\ \lambda_i\Big) + O\,(\,|y|^{j+1})\,.$$

From this the k-th component of b_α is determined by $(b_\alpha)_k = \frac{(a_\alpha)_k}{\sum_{i=1}^{n}\ \alpha_i\,\lambda_i - \lambda_k}$. The procedure works formally provided the nonresonance condition, now $\sum_{i=1}^{n}\ \alpha_i\,\lambda_i\ -\ \lambda_u\ \neq\ 0\quad\forall\alpha\in(\mathbb{N}\cup\{0\})^n,\ |\alpha|\geq 2,\ \forall k\in\{1,\ldots,n\}$, holds. Otherwise the resonant terms should be kept in the normal form.

For Hamiltonian systems it is possible to obtain the normal form by means of canonical transformations, preserving in this way the Hamiltonian character of the equations. This is done nicely and efficiently by means of Lie series, and I refer again to other parts of this book.

10 Analytical approximation of invariant manifolds

Again we consider first the case of fixed (or periodic) points of a diffeomorphism. Let $x = 0$ be a fixed point of $x \to F(x)$. Assume $DF(0)$ can be split in 3 parts, the first one with eigenvalues of modulus larger than 1, the second one with modulus 1 and the last one with modulus less than 1. The related linear subspaces give the linear approximations to the unstable, centre and stable manifolds of the fixed point. We want to compute analytical approximations to those manifolds. Several methods can be used:

1) *The normal form.* We split x in three types of variables u, c, s, associated to the linear approximation of W^u, W^c, W^s, respectively. Then the mapping can be written as

$$\begin{pmatrix} u \\ c \\ s \end{pmatrix} \longrightarrow \begin{pmatrix} \mathcal{U}\, u + \text{resonant terms} \\ \mathcal{C}\, c + \text{resonant terms} \\ \mathcal{S}\, s + \text{resonant terms} \end{pmatrix} = \begin{pmatrix} \bar{u} \\ \bar{c} \\ \bar{s} \end{pmatrix} ,$$

where $\lambda \in Spec\, \mathcal{U} \Rightarrow |\lambda| > 1,\ \lambda \in Spec\, C \Rightarrow |\lambda| = 1,\ \lambda \in Spec\, S \Rightarrow |\lambda| < 1$. From the construction of the normal form it follows that the resonant terms in $\bar{u}$ cannot contain only c or s variables, and similar things happen in $\bar{c}, \bar{s}$. Then the invariant manifolds can be "read off" from the normal form, and $W^u = \{c = 0,\ s = 0\}$, $W^c = \{u = 0,\ s = 0\}$, $W^s = \{u = 0,\ c = 0\}$. We obtain the invariant manifolds in the original coordinates by going back through the transformations.

2) *The graphic transform.* One of the methods used to show the existence of invariant manifolds it is also useful from the computational point of view. In a broad sense we can describe this method by saying that we look for some invariant manifold by just imposing that it is invariant. As before, assume

$$F : \begin{pmatrix} u \\ c \\ s \end{pmatrix} \longrightarrow \begin{pmatrix} \mathcal{U}\, u + G_1(u,c,s) \\ \mathcal{C}\, c + G_2(u,c,s) \\ \mathcal{S}\, s + G_3(u,c,s) \end{pmatrix} ,$$

where $G_1(0,0,0) = 0,\ G_2(0,0,0) = 0,\ G_3(0,0,0) = 0,\ DG_1(0,0,0) = 0,\ DG_2(0,0,0) = 0,$ $DG_3(0,0,0) = 0$. Let $c = g_1(u),\ s = g_2(u)$ the parametrisation of the unstable manifold near $(0,0,0)$. By imposing invariance we obtain

$$\begin{aligned} \mathcal{C}\, g_1(u) + G_2(u, g_1(u), g_2(u)) &= g_1\Big(\mathcal{U}u + G_1(u, g_1(u), g_2(u))\Big) , \\ \mathcal{S}\, g_2(u) + G_3(u, g_1(u), g_2(u)) &= g_2\Big(\mathcal{U}u + G_1(u, g_1(u), g_2(u))\Big) . \end{aligned}$$

From these relations we can compute recurrently the terms of increasing degrees in g_1, g_2. By equating at degree k we have

$$\begin{aligned} \mathcal{C} \sum_{|\alpha|=k} g_{1,\alpha}\, u^\alpha - \sum_{|\alpha|=k} g_{1,\alpha}\, (\mathcal{U}u)^\alpha = \Big[&- G_2(u, g_1(u), g_2(u)) \\ &+ g_1\Big(\mathcal{U}u + G_1(u, g_1(u), g_2(u))\Big) - g_1(\mathcal{U}u)\Big]_k , \end{aligned}$$

$$\begin{aligned} \mathcal{S} \sum_{|\alpha|=k} g_{2,\alpha}\, u^\alpha - \sum_{|\alpha|=k} g_{2,\alpha}\, (\mathcal{U}u)^\alpha = \Big[&- G_3(u, g_1(u), g_2(u)) \\ &+ g_2\Big(\mathcal{U}u + G_1(u, g_1(u), g_2(u))\Big) - g_2(\mathcal{U}u)\Big]_k , \end{aligned}$$

where $g_j(u) = \sum_{k\geq 2} \sum_{|\alpha|=k} g_{j,\alpha}\, u^\alpha,\ j=1,2$ and $[P]_k$ denote the terms of degree k of the power series P. The denominators which appear in the components of $g_{j,\alpha},\ j=1,2$, are clearly different from zero.

A nice problem appears when the fixed point has a weakly unstable manifold, that is, associated to an eigenvalue equal to 1. For instance consider the map in $\mathbb{R}^2$ around $(0,0)$ given by analytical expansions as

$$T : \begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} x + x^3 + 3x^2y + 2y^3 + 0_4 \\ y + x^3 + 2x^2y + 2xy^2 + y^3 + 0_4 \end{pmatrix}.$$

To look for invariant directions entering to or leaving the origin, let $m = y/x$. Then we put $Q_3(1,m) - m\, P_3(1,m)$, where P_3, Q_3 are the cubic terms in the first (resp. second) component of $T\binom{x}{y}$. We obtain $m=1,\ m=-\frac{1}{2}$ as real solutions. Introducing $u = 2y+x$, $s = x-y$, the mapping T is expressed as

$$T : \begin{pmatrix} u \\ s \end{pmatrix} \longrightarrow \begin{pmatrix} u + \frac{1}{3}(2u^3 - 3u^2s + 4us^2) & +0_4 \\ s + \frac{1}{3}(us^2 - s^3) & +0_4 \end{pmatrix}.$$

Skipping the 0_4 terms we have that $s=0$ (resp. $u=0$) is the unstable (resp. stable) manifold. On them we obtain $(u,0) \xrightarrow{T} (u+\frac{2}{3}u^3)$, a slowly unstable axis (resp. $(0,s) \xrightarrow{T} (s-\frac{1}{3}s^3)$, a slowly stable axis).

Now assume that the higher order terms in the first and second components are $\sum_{j+k\geq 4} a_{jk}\, u^j\, s^k$, $\sum_{j+u\geq 4} b_{jk}\, u^j\, s^k$, respectively. It is clear that $b_{j0}=0\ \ \forall j\geq 4$ implies that W^u is given by $s=0$ (and $a_{0k}=0\ \forall k\geq 4$ implies W^s given by $u=0$). Assume it exists $j_0 \geq 4$ such that $b_{j_0,0} \neq 0$ and for any other j with $b_{j0}\neq 0,\ j\geq 4$ one has $j > j_0$. We look for the unstable manifold parametrised by u and given by $(u, g(u))$. Let $g(u) = c\, u^p + o(u^p)$. If $T : (u,s) \longrightarrow (u + G_1(u,s)),\ (s + G_2(u,s))$, the substitution gives $g(u) + G_2(u, g(u)) = g\Big(u + G_1(u, g(u))\Big)$. One should select $j_0 = p+2$ and $c = \frac{3}{2p}\, b_{j_0,0}$. The terms of order $q > p$ in g are obtained recurrently by equating in both sides the terms of degree $q+2$. In a similar way we obtain W^s.

3) *Adapted parametrisation.* If we only consider, for simplicity, unstable and stable manifolds and $F : \binom{u}{s} \to \begin{pmatrix} \mathcal{U}u + G_1(u,s) \\ \mathcal{S}s + G_2(u,s) \end{pmatrix}$, $u \in B_1 \subset \mathbb{R}^k$, $s \in B_2 \subset \mathbb{R}^{n-k}$, G_1, G_2, DG_1, DG_2 null at $(0,0)$, where B_1, B_2 are balls centred at the origin, and $\mathcal{U} = diag\,(\lambda_1,\ldots,\lambda_k)$, $\mathcal{S} = diag\,(\lambda_{k+1},\ldots,\lambda_n)$, with $|\lambda_i| > 1,\ c=1,\ldots,k,\ |\lambda_i| < 1,\ i=k+1,\ldots,n$, we look for a natural parametrisation of the manifolds. Let $p_1,\ldots,p_k$ parameters for W^u such that if W^u is given by $\binom{u}{s} = \hat{\mathcal{H}}(p_1,\ldots,p_k)$, then $F\binom{u}{s} = \hat{\mathcal{H}}(p_1+1,\ldots,p_k+1)$. For points in the unstable manifold tending towards the origin the parameters should tend to $-\infty$. Another parametrisation well adapted to the dynamics is $\binom{u}{s} = \mathcal{H}(q_1,\ldots,q_k)$, such that $F\binom{u}{s} = \mathcal{H}(\lambda_1 q_1,\ldots,\lambda_k q_k)$. As both representations are equivalent through $p_i = \log_{\lambda i}\ q_i$ we restrict to the second one. In the linear case, $G_1 \equiv 0,\ G_2 \equiv 0,\ \mathcal{H}$ is given by $u_i = q_i,\ i=1,\ldots,k,\ s_i = 0,\ i=1,\ldots,n-k$. Hence we look for $\mathcal{H}$ in the form

$$u_i = q_i + \sum_{j\geq 2}\ \sum_{|\alpha|=j} c_{i,\alpha}\, q^\alpha\,,\ i=1,\ldots,k\,,$$

$$s_i = \sum_{j\geq 2}\ \sum_{|\alpha|=j} d_{i,\alpha}\, q^\alpha\,,\ i=1,\ldots,n-k\,,$$

where $\alpha \in (\mathbb{N}\cup\{0\})^k$. Let $G_1(u,s)+\sum_{|\beta|+|\gamma|\geq 2} g^{(1)}_{\beta,\gamma}\, u^\beta\, s^\gamma$, $G_2(u,s)=\sum_{|\beta|+|\gamma|\geq 2} g^{(2)}_{\beta,\gamma}\, u^\beta\, s^\gamma$, with $\beta \in (\mathbb{N}\cup\{0\})^k$, $\gamma \in (\mathbb{N}\cup\{0\})^{n-k}$, and $g^{(j)}_{i,\beta,\gamma}$ $j=1,2$, the i-th component of $g^{(j)}_{\beta,\gamma}$. Also let $\lambda_q, c_\alpha, d_\alpha$ be the vectors with components $\lambda_i\, q_i$, $c_{i,\alpha}$, $d_{i,\alpha}$, respectively.

The required condition on $\mathcal{H}$ is written as

$$\lambda_i\Big(q_i+\sum_{|\alpha|\geq 2} c_{i,\alpha}\, q^\alpha\Big)+\sum_{|\beta|+|\gamma|\geq 2} g^{(1)}_{i,\beta,\gamma}\,\Big(q+\sum_{|\alpha|\geq 2} c_\alpha\, q^\alpha\Big)^\beta\Big(\sum_{|\alpha|\geq 2} d_\alpha\, q^\alpha\Big)^\gamma =$$
$$\lambda_i\, q_i+\sum_{|\alpha|\geq 2} c_{i,\alpha}\,(\lambda q)^\alpha\,,$$

$$\lambda_{k+i}\Big(\sum_{|\alpha|\geq 2} d_{i,\alpha}\, q^\alpha\Big)+\sum_{|\beta|+|\gamma|\geq 2} g^{(2)}_{i,\beta,\gamma}\,\Big(q+\sum_{|\alpha|\geq 2} c_\alpha\, q^\alpha\Big)^\beta\Big(\sum_{|\alpha|\geq 2} d_\alpha\, q^\alpha\Big)^\gamma =$$
$$\sum_{|\alpha|\geq 2} d_{i,\alpha}\,(\lambda q)^\alpha\,.$$

By equating the coefficients of q^α in the first (second) equations, we obtain $c_{i,\alpha}(d_{i,\alpha})$. In the determination of $d_{i,\alpha}$ there are not resonances, but they do appear in $c_{i,\alpha}$ if $\lambda_i=\lambda^\alpha$. This is a disadvantage with respect to 2). The advantage is that the domain of validity of the representation is larger. For F an entire function $\mathcal{H}$ is also entire in the absence of resonances. The graphic transform gives a representation of W^u which is no longer valid when the manifold has some fold, i.e., the projection $W^u\to u$ is not injective.

Furthermore the adapted parametrisation allows to compute the invariant manifolds skipping the previous transformation to diagonal (or Jordan) form of the linear part. The price to pay is to solve linear systems at every step to find the $c_{i,\alpha}$, $d_{i,\alpha}$ coefficients. This reduction is elementary but time consuming if n is big ($n\geq 6$, say) and we would like to reach orders $|\alpha|$ somewhat high ($|\alpha|=30$, say). If k is relatively small ($k=1,2$) it can be useful to avoid the previous reduction.

As a very simple example we consider the Hénon map $T:(x,y)\to(1-a\,x^2+y,bx)$. We look for $x=\sum_{j\geq 0} x_j\, t^j$, $y=\sum_{j\geq 0}\, y_j\, t^j$. By the invariance condition of a manifold with eigenvalue λ we obtain

$$1-a\Big(\sum_{j\geq 0} x_j\, t^j\Big)^2+\sum_{j\geq 0} y_j\, t^j=\sum_{j\geq 0} x_j(\lambda t)^j\,,\; b\sum_{j\geq 0} x_j\, t^j=\sum_{j\geq 0} y_j(\lambda t)^j\,.$$

Hence $y_i=b\,x_j\,\lambda^{-j}$ and by substitution in the first equation we obtain

$$\begin{aligned}
&1-a\,x_0^2+b\,x_0=x_0\,,\\
&-a\,2x_0\,x_1+b\,x_1\,\lambda^{-1}=\lambda\,x_1\,,\\
&-a\sum_{i=1}^{j-1} x_i\, x_{j-i}=x_j(\lambda^j-b\lambda^{-j}+2a\,x_0),\quad j\geq 2\,.
\end{aligned}$$

From the first of these equations we select the desired fixed point. The second one gives the characteristic equation for λ. We can normalise x_j (for instance $x_1=1$). The remaining equations allow to determine x_j, $j\geq 2$.

Now we pass to the case of vector fields. The methods are essentially equivalent and we shall consider some complements. The first one is the reduction to the central manifold. Let

$$\dot{x} = \mathcal{C}\,x + f\,(x,y)\,,$$
$$\dot{y} = \mathcal{H}\,y + g(x,y)\,,$$

where $x \in \mathbb{R}^k,\ y \in \mathbb{R}^{n-k},\ \lambda \in Spec\,\mathcal{C} \Rightarrow Re\,\lambda = 0,\ \lambda \in Spec\,\mathcal{H} \Rightarrow Re\,\lambda \neq 0,\ f(0,0) = 0,$ $g(0,0) = 0,\ D\,f(0,0) = 0,\ D\,g(0,0) = 0$. Then $y = 0$ gives the linear approximation to the centre manifold. We try to obtain $y = v(x)$, with $v(0) = 0\,,\ D\,v(0) = 0$ giving the local expression of the centre manifold. Then the flow restricted to this manifold is $\dot{x} = \mathcal{C}\,x + f\,(x, v(x))$. The invariance of $y - v(x) = 0$ is expressed by setting the time derivative equal to zero on this manifold, that is

$$\mathcal{H}v(x) + g\big(x, v(x)\big) = Dv(x)\big(\mathcal{C}x + f(x, v(x))\big) = 0\,.$$

We put $v(x) = \sum_{|\alpha|\geq 2} v_\alpha\, x^\alpha\,,\ \alpha \in (\mathbb{N}\cup\{0\})^k$. Splitting the invariance equation we have

$$D\,v(x)\mathcal{C}x - \mathcal{H}v(x) = g(x, v(x)) - Dv(x)\ f(x, v(x))\,.$$

The left hand side is a linear operator with respect to v, the right hand one being nonlinear. If $v_{i,\alpha},\ i = 1,\dots,k$, are the components of the vector v_α and we assume $\mathcal{H}$ and $\mathcal{C}$ in diagonal form, $\mathcal{H} = diag(\lambda_1,\dots,\lambda_{n-k}), \mathcal{C} = diag(\mu_1,\dots,\mu_k)$ (this is not necessary, but then we have to solve big linear systems) we obtain in the left hand side

$$\begin{pmatrix} \sum_{|\alpha|\geq 2}\ v_{1,\alpha}\ x^\alpha\,\frac{\alpha_1}{x_1} & \dots & \sum_{|\alpha|\geq 2}\ v_{1,\alpha}\ x^\alpha\,\frac{\alpha_k}{x_k} \\ \vdots & & \\ \sum_{|\alpha|\geq 2}\ v_{n-k,\alpha}\ x^\alpha\,\frac{\alpha_1}{x_1} & \dots & \sum_{|\alpha|\geq 2}\ v_{n-k,\alpha}\ x^\alpha\,\frac{\alpha_k}{x_k} \end{pmatrix} \begin{pmatrix} \mu_1\, x_1 \\ \vdots \\ \mu_\alpha\, x_\alpha \end{pmatrix}$$
$$- \begin{pmatrix} \lambda_1\ \sum_{|\alpha|\geq 2}\ v_{1,\alpha}\ x^\alpha \\ \vdots \\ \lambda_{n-k}\ \sum_{|\alpha|\geq 2}\ v_{n-k,\alpha}\ x^\alpha \end{pmatrix}\,,$$

and then the coefficient of x^α in the j-th equation is $v_{j,\alpha}\left(\sum_{i=1}^k\ \mu_i\,\alpha_i - \lambda_j\right)$. The coefficient $v_{j,\alpha}$ can be obtained by equating with the (already known) coefficient of x^α in the j-th equation at the right hand side, provided $\sum_{i=1}^k\ \mu_i\alpha_i - \lambda_j \neq 0$. This is true because the real part is $\neq 0$.

Now we look for the invariant manifolds of a torus. Let $x \in \mathbb{R}^n,\ y \in \mathbb{R}^m,\ \varphi \in \mathbb{T}^p$ and we consider the differential system

$$\dot{x} = A\,x + G_1\,(x,y,\varphi)\ ,$$
$$\dot{y} = B\,y + G_2\,(x,y,\varphi)\ ,$$
$$\dot{\varphi} = \quad \omega + G_3\,(x,y,\varphi)\ ,$$

where $A = diag\,(\lambda_1,\dots,\lambda_n)\,,\ Re\,\lambda_j > 0$ for $j > 1,\dots,n\,,\ B = diag(\mu_1,\dots,\mu_m), Re\,\mu_j \leq 0$ for $j = 1,\dots,m\,,\ (k,\omega) = 0$ for $k \in \mathbb{Z}^p$ implies $k = 0$, and $G_1, G_2, D_1\,G_1, D_2\,G_1, D_1\,G_2,$ $D_2\,G_2, G_3$ are zero at $(0,0,\varphi)\,,\ \forall\,\varphi \in \mathbb{T}^p$. Let

$$G_j(x,y,\varphi) = \sum_{|\alpha|+|\beta|\geq q\,,\,k\in\mathbb{Z}^p} G^{(j)}_{\alpha,\beta,k}\ x^\alpha\, y^\beta\ \exp\ (i(k,\varphi))$$

for $j=1,2,3$, where $q=2$ if $j=1,2$, and $q=1$ if $j=3$. The linear approximation of W^u is given by $y=0$. We look for $y=h\,(x,\varphi)=\sum_{|\alpha|\geq 2,k\in\mathbb{Z}^p} x^\alpha \exp\,(i(k,\varphi))$ for the nonlinear expression. By invariance we have

$$B\,h(x,\varphi)+G_2(x,h(x,\varphi),\varphi)=D_1h(x,\varphi)\Big(Ax+G_1(x,h(x,\varphi),\varphi)\Big)$$
$$+D_2h(x,\varphi)\Big(\omega+G_3(x,h(x,\varphi),\varphi)\Big)\,.$$

One obtains easily the terms in $x^\alpha \exp\,(i(k,\varphi))$ in h with $|\alpha|=2$ by

$$(h_{\alpha,k})_j=\Big(\sum_{s=1}^{n}\ \alpha_s\,\lambda_s-\mu_j+i(k,\omega)\Big)^{-1}\ (G^{(2)}_{\alpha,0,k})_j\,,$$

where $(h_{\alpha,k})_j$ (resp. $(G^{(2)}_{\alpha,0,k})_j$) denotes the j-th component of $h_{\alpha,k}$ (resp. $G^{(2)}_{\alpha,0,k}$). For $|\alpha|>2$ the linear part, which comes from $B\,h(x,\varphi)-D_1\,h(x,\varphi)Ax-D_2h(x,\varphi)\omega$ gives a contribution to $x^\alpha\ \exp\,(i(k,\varphi))$ with j-th component equal to $(\mu_j-\sum_{s=1}^n\alpha_s\lambda_s-i(k,\omega))\,(h_{\alpha,k})_j$ which should be equated to the related component in

$$-G_2(x,h(x,\varphi),\varphi)+D_1\,h(x,\varphi)G_1(x,h(x,\varphi),\varphi)+D_2\,h(x,\varphi)\ G_3(x,h(x,\varphi),\varphi)\,,$$

already known from the previous terms. It is possible to obtain the expansions without obtaining the diagonal (or Jordan) forms of A and B and using trigonometric functions with real coefficients instead of $\exp\,(i(k,\varphi))$. This procedure can be applied, in particular, to the case $p=1$, obtaining the invariant manifolds of a periodic orbit.

Finally we consider the case of a weak normally hyperbolic torus. To avoid difficulties we assume that the unstable dimension is 1, the centre-stable dimension is m and the torus is p-dimensional. The standing equations are

$$\begin{aligned}\dot{x}&=f\,(x,y,\varphi)(A\,x+G_1\,(x,y,\varphi))\,,\\ \dot{y}&=f\,(x,y,\varphi)(B\,y+G_2\,(x,y,\varphi))\,,\\ \dot{\varphi}&=\omega+G_3\,(x,y,\varphi)\,.\end{aligned}$$

Here $A>0$, $\mu\in Spec\,B\Rightarrow Re\,\mu\leq 0$, $G_1,G_2,D_1G_1,D_2G_1,D_1G_2,D_2G_2,G_3$ are zero at $(0,0,\varphi)$, $(k,\omega)=0$ for $k\in\mathbb{Z}^p\Rightarrow k=0$, and f is of order r in x,y, with $k_1\|(x,y)^\top\|^r\leq f\leq k_2\,\|(x,y)^\top\|^r$, $0<k_1<k_2$. We look for the unstable manifold as $y=\sum_{|\alpha|\geq 2} u_\alpha\,(\alpha)x^\alpha$. By substitution

$$\begin{aligned}&f\,(x,u(x,\varphi),\varphi)\Big\{\,[B\,u(x,\varphi)+G_2(x,u(x,\varphi),\varphi)]\\ &[D_1\,u(x,\varphi)(A\,x+G_1(x,u(x,\varphi),\varphi))]\Big\}=\\ &D_2\,u(x,\varphi)(\omega+G_3(x,u(x,\varphi),\varphi)\,.\end{aligned}$$

Let $G_j\,(x,y,\varphi)=\sum_{|\alpha|+|\beta|\geq s}\ G^{(1)}_{\alpha,\beta}(\varphi)\,x^\alpha\,y^\beta$, $j=1,2,3$, $s=2$ for $j=1,2$, $s=1$ for $j=3$, $f\,(x,y,\varphi)=\sum_{|\alpha|+|\beta|\geq r}\ f_{\alpha,\beta}(\varphi)\,x^\alpha\,y^\beta$, and $B=diag\,(\mu_1,\ldots,\mu_m)$. Assume that q is the minimum value of α such that $G^{(2)}_{\alpha,0}(\varphi)\neq 0$. Then, in principle, u starts with terms $0(x^q)$. But $D_2\,u(x,\varphi)$ cannot contain terms of degree less than $r+q$. Therefore all the coefficients $u_\alpha(\varphi)$, $q \le \alpha < q+r$ are constant m-dimensional vectors. Let us see how to determine them and how to obtain the zero average part of $u_\alpha(\varphi)$ for $\alpha \ge q+r$. Equating the terms in x^{q+r} in the invariance relation we obtain, for the j-th component:

$$f_{r,0}(\varphi)\ \left[(\mu j - q\,A)\ u_{q,j} + G^{(2)}_{q,0,j}(\varphi)\right] = D\,u_{q+r,j}(\varphi)\cdot\omega\,.$$

As the right hand part has zero average we obtain $u_{q,j}$ as

$$u_{q,j} = \frac{1}{q\,A - \mu_j}\ \frac{Av(f_{r,0}(\varphi)\cdot G^{(2)}_{q,0,j}(\varphi))}{Av\ (f_{r,0}(\varphi))}\,,$$

where Av is the averaging operator: $A\,v\,(\psi) = \frac{1}{Vol(\mathbb{T}^p)}\ \int_{\mathbb{T}^p}\psi(\varphi)\,d\varphi$. Then $D\,u_{q+r,j}\,(\varphi)\cdot\omega$ is purely periodic and $u_{q+r,j}$ is obtained by Fourier expansion (except the independent term, to be obtained later). At higher order, $\bar q > q$, we have the equation

$$\begin{aligned}&f_{r,0}\,(\varphi)\ \left[(\mu_j - \bar q A)\,u_{\bar q,j}\right] - D\,u_{\bar q+r,j}\,(\varphi)\cdot\omega\\ &= \text{known terms of degree } \bar q + r\,,\ \text{ component } j = K_{\bar q+r,j}\,(\varphi)\,.\end{aligned}$$

Again we obtain $u_{\bar q,s} = \frac{1}{\bar q\,A - pj}\ \frac{Av\left(K_{\bar q+r,j}\,(\varphi)\right)}{Av\left(f_{r,0}(\varphi)\right)}$ and $D\,u_{\bar q+r,j}\,(\varphi)\cdot\omega$ by difference.

If φ is 1-dimensional ($p=1$) it is also possible to use the equations with φ as independent variable:

$$\frac{dx}{d\varphi} = \tilde f\,(x,y,\varphi)[A\,x + G_1(x,y,\varphi)]\,,\quad \frac{dy}{d\varphi} = \tilde f\,(x,y,\varphi)\ [B\,y + G_2(x,y,\varphi)]\,,$$

where $\tilde f = f\,/\,(\omega + G_3)$. Then the invariance relation produces recurrent differential equations; for $q \ge \bar q$, $\frac{d\,u_{\bar q+r,j}}{d\,\varphi}$ = know function. The constant terms in $u_{\bar q,j}$ are determined to avoid secular behaviour.

11 Numerical approximation of Invariant Manifolds

To catch many interesting dynamical properties of a system it is useful to know the behaviour of a significant part of the unstable and stable manifolds of invariant objects as fixed points, periodic orbits, invariant tori and others. The analytical approximations obtained as described in section 10 are enough only in a few simple problems. In general we should extends the manifolds to compute homoclinic points, for instance, and to guess the global behaviour by numerical means. This involves in most of the cases a big computational effort. It can be advisable to use good analytical approximations (high order ones) but this is also time consuming. For every specific problem one can analyse the cheaper option, in terms of computer and personal time.

We start with the simplest example in the case of invariant manifolds of a fixed point of a diffeomorphism. Assume the fixed point $p = (x^*, y^*)$ of $F : \mathcal{U} \subset \mathbb{R}^2 \to \mathbb{R}^2$ has eigenvalues $\lambda > 1$, $\mu \in (0,1)$. We want to compute W^u_p (for W^s_p just use F^{-1}). Let $(x(z),\ y(z))$ a parametric representation around p, with $p = (x(0),\ y(0))$ and $F(x(z),\ y(z),) \approx (x(\lambda z),\ y(\lambda z))$. This can be as simple as the linear approximation

$(x(z),\ y(z)) = (x^\alpha + z\,v_1,\ y^\alpha + z\,v_2)$, with $v = (v_1, v_2)$ a normalised eigenvector of the eigenvalue λ, or a high order approximation. First we check up to which value of z the difference $\|F(x(z),\ y(z)) - (x(\lambda z),\ y(\lambda z))\|$ is less than some tolerance. Let z_0 this value. In general it is not needed to be extremely restrictive concerning the tolerance because the small errors tend to decrease (at least locally) due to the compression in the stable direction. Then we can take a fundamental domain generated by z ranging in $(z_0 / \lambda,\ z_0)$. A simple (and very used) method generates a big piece of W_p^u by taking n points in $(z_0 / \lambda,\ z_0)$ (say $n = 1000$, for instance) and performing iterates of those points. However this is not a good policy because we have, in general, too many points near p and too few far away. A better method is described next.

First we fix some distance Δs and some angle $\Delta\alpha$ such that successive points $(x_m,\ y_m)$ in W_p^u should satisfy

$$\|(x_m\ ,\ y_m) - (x_{m-1}\ ,\ y_{m-1})\| \le \Delta s\,,$$

$$(x_m - y_{m-1})(x_{m-1} - x_{m-2}) + (y_m - y_{m-1})(y_{m-1} - y_{m-2}) \ge$$

$$\cos(\Delta\alpha)\cdot\left[\left((x_m - x_{m-1})^2 + (y_m - y_{m-1})^2\right)\left((x_{m-1} - x_{m-2})^2 + (y_{m-1} - y_{m-2})^2\right)\right]^{1/2}.$$

Assume we have computed up to $(x_m,\ y_m)$ satisfying the previous conditions and we have a current value of z, z_m and a current value of $\Delta\,z_m = z_m - z_{m-1}$. Let $\Delta\,s_m$, $\Delta\,\alpha_m$ be the distance from $(x_{m-1},\ y_{m-1})$ to $(x_m,\ y_m)$ and the angle between the vectors $(x_m - x_{m-1},\ y_m - y_{m-1})$ and $(x_{m-1} - x_{m-2},\ y_{m-1} - y_{m-2})$. Then we select $\Delta\,z_{m+1}$ as $\min\left(\frac{\Delta s}{\Delta\,s_m}\ ,\ \frac{\Delta\alpha}{\Delta\,\alpha_m}\right)\cdot\Delta\,z_m\cdot 0.8$. Here 0.8 can be considered as a security factor. From $z_{m+1} = z_m + \Delta\,z_{m+1}$ as parameter we perform k iterates, the same ones used to obtain $(x_m,\ y_m)$ from z_m, and we get $(x_{m+1},\ y_{m+1})$. If $z_{m+1} > z_0$ we fall outside the allowable range of z. Then we divide both z_{m+1} and $\Delta\,z_{m+1}$ by λ and replace k by $k+1$. In this way we certainly restart all the computations in the fundamental domain but we have a faithful and uniform representation of W_p^u. If despite our choice of $\Delta\,z_{m+1}$ we miss one of the conditions $\Delta\,s_{m+1} \le \Delta s$, $\Delta\,\alpha_{m+1} \le \Delta\alpha$ (usually one can miss last one at the end of a very narrow tongue of W_p^u) we can interpolate additional values of z. The values Δs, $\Delta\alpha$ need not to be extremely small. If necessary one can use some kind of interpolation to plot W_p^u from the list of points $\{(x_m,\ y_m)\}$.

The same method can be applied to obtain unstable invariant manifolds of periodic orbits with one stable direction by using Poincaré sections. For fixed points of vector fields the invariant manifolds are just orbits, and, if $\dim W_p^u = 1$ then we need to compute just one orbit. We remark that for periodic orbits the surface of section where we want to have the invariant manifold needs not to be the same one where we start the computations.

We turn now to a more difficult question: To obtain higher dimensional invariant manifolds. To simplify the exposition we suppose we are dealing with fixed points of diffeomorphisms, but this applies also to fixed points or periodic orbits of flows. The key difficulty is that if we have k unstable directions with related eigenvectors v_j, $j = 1, \ldots, k$ and eigenvalues λ_j, $j = 1, \ldots, k$, $|\lambda_j| > 1$, starting at a point $p + \sum_{j=1}^k z_j\,v_j$ with $\sum_{j=1}^k z_j^2$ small near the fixed point p (if we use the linear approximation) we shall escape essentially in the direction of the dominant eigenvalue. That is, if $|\lambda_1| > |\lambda_2| \ge |\lambda_3| \ge \ldots$ for almost any k-ple of values $(z_1, \ldots, z_k)$ we obtain points very close to the ones generated by iteration starting at $p + z_1\,v_1$. A typical behaviour is as follows: Assume $k = 2$ and we look for points in W_p^u which are in some hypersurface Σ for away from p. We suppose that the local representation of W_p^u is given by $g(z_1, z_2)$ and $F(g(z_1, z_2)) \simeq g(\lambda_1 z_1,\ \lambda_2 z_2)$. Typically we put $z_1 = \rho\ \cos\gamma\,,\ z_2 = \rho\ \sin\gamma\,,\ \rho$ small, $\gamma \in [0, 2\,\pi)$. For a fixed value of γ we look for a value of ρ such that $F^k(g(\rho\ \cos\ \gamma,\ \rho\ \sin\ \gamma)) \in \Sigma$. This is a one dimensional equation and k is chosen in such a way that ρ is kept small to have a good local approximation. Let $q(\gamma)$ the point obtained in Σ. Then changes in γ around 0 and π give very small changes in $q(\gamma)$, but if γ is close to $\pi/2$ or $3\pi/2$ very small changes in γ produce big changes in $q(\gamma)$. If we want a good description of the curve $W_p^u \cap \Sigma = \{q(\gamma),\ \gamma \in [0, 2\pi)\}$ obtaining points $q(\gamma_m),\ m = 1, \ldots,$ which are roughly regularly spaced, the difference $\gamma_m - \gamma_{m-1}$ should be very small near $\gamma = \pi/2$ or $3\pi/2$ and, probably, beyond the capabilities of the computer.

A general rule to overcome this behaviour consists on the use of high order expansions allowing to start the computations at a relatively big distance of p. Then we decrease the number of iterates, k, before reaching Σ and therefore the Lipschitz constant of the map $\gamma \to q(\gamma)$. Another method can be used successfully only in particular cases and proceeds as follows: Assume that the slow unstable direction associated to the eigenvalue λ_2 belongs to some invariant submanifold, N, of F passing through p. Then we can obtain good approximations of $W_p^u \cap N$ by restricting F to N. At the same time that we obtain $F^k(g(0, \rho))$ we can transport under $D\,F^k(g(0, \rho))$ the direction of v_1. In this way we obtain points in $W_p^u \cap N$ and the linear approximation to W_p^u along this curve (or better than the linear approximation if we are allowed to compute $D^j\,F^k$). Let $r = F^k(g(0, \rho))$ and $w = D\,F^k(g(0, \rho))\,v_1$. Starting at $r + \bar{\rho} w$ we can enlarge, by iteration of F, this local strip contained in W_p^u. Of course, these new iterations can go quickly far away from N but we can "travel" previously along N as far as desired. This method is rather easy to implement and to extend to higher dimensional W_p^u provided the slow directions are contained in a chain of previously known submanifolds invariant under F.

To obtain W^u for a normally hyperbolic invariant torus, $\mathbb{T}^m$, parametrised by angles $\theta_j \in [0, 2\pi),\ s = 1, \ldots, m$ is simple in theory but it is a hard computational task due to the multiplicity of parameters. In any case, when we have several parameters to describe the manifold locally, it is advantageous to obtain an approximation of reasonable high order to minimise the total work. This applies specially, even in the case of fixed points, when the hyperbolicity is weak and the linear approximation gives $\lambda = 1$. Then a poor local approximation will require a prohibitive number of iterates before reaching a sizeable distance from the invariant object.

In problems typically dissipative we can have hyperbolic fixed points with, for instance, $\dim\ W_p^u = 1,\ \dim W_p^s \gg 1$. To decide if a point is in W_p^s (this is relevant if this is a homoclinic or heteroclinic point, or to look for boundaries of basins of attraction) we can perform forward iterates of the diffeomorphism F. If the initial point is close to W_p^s the iterates should pass close to p and escape along W_p^u. By studying those escapes, for instance cutting W_p^u by a transversal section of the same dimensionality then W_p^s, we can define the initial point to obtain points rather close to W_p^s.

12 Computation of homoclinic and heteroclinic point. Global bifurcation curves

Now we study the determination of points in $W_p^u \cap W_p^s$, where p, q are invariant objects, either fixed points or periodic orbits or invariant tori, for instance, of a diffeomorphism or a flow. If $p \neq q$ the points are heteroclinic; if $p = q$ then they are named homoclinic. If we obtain a point $h \in W_p^u \cap W_q^s$ for a flow then the full orbit of h is in $W_p^u \cap W_q^s$. So, in the case of flows we restrict to search to some surface of section Σ for determinacy. Let $f\,(z_1,\dots,z_k,\ \theta_1,\dots,\theta_j)$ be a local representation of W_p^u, where k is the dimensionality of the unstable directions at p and $\theta_1,\dots,\theta_j$ the parameters describing the object p (for instance j angles in $[0,2\pi)$ if $p = \mathbb{T}^j$). In a similar way we assume W_q^s represented as $g\,(w_1,\dots,w_\ell,\ \tau_1,\dots,\tau_i)$. If we reduce the problem to the case of diffeomorphisms the general procedure consists in solving the system

$$F^{k_1}\left(f\left(z_1,\dots,\, z_k,\ \theta_1,\dots,\, \theta_j\right)\right) = F^{-k_2}\left(g\left(w_1,\dots,\, w_\ell,\ \tau_1,\dots,\, \tau_i\right)\right)$$

for $z_1,\dots,\, z_k,\ \theta_1,\dots,\, \theta_j,\ w_1,\dots,\, w_\ell,\ \tau_1,\dots,\, \tau_i$. The numbers of iterates, k_1, k_2, are kept fixed and they are established by an initial guess. Alternatively we can put $F^{k_1+k_2}\,(f) = g$ and $k_1 + k_2$ can be obtained by looking when $F^m(f)$ comes close to q when changing m.

The solution, if any, of the preceding equations has dimension d. Generically $d = k + j + \ell + i - n$ if n is the total dimension of the space. However we can have $d \geq 0$ even if $k + j + \ell + i < n$ if some symmetries occur. The system can be solved by Newton's method (suitably modified if $d > 0$). When the homo/heteroclinic points are available it is relevant to check the transversality.To this end, if h is obtained taking $z = z^*,\ \theta = \theta^*,\ w = w^*,\ \tau = \tau^*$ we compute the vectors

$$(D\,F^{k_i})\,(f(z^*,\theta^*))\,u \quad , \quad (D\,F^{-k_2})\,(g(w^*,\tau^*))\,v\,,$$

where u is any of the vectors $\frac{\partial f}{\partial z_m}\ (z^*,\theta^*)$, $\frac{\partial f}{\partial \theta_m}\ (z^*,\theta^*)$ and v any of the vectors $\frac{\partial g}{\partial w_m}\ (w^*,\tau^*)$, $\frac{\partial g}{\partial \tau_m}\ (w^*,\tau^*)$. In this way we obtain at h a total of $k + j + \ell + i$ vectors. If we rank of the matrix formed by all of them is n we say that W_p^u and W_q^s are transversal at $h : W_p^u \ \pitchfork_h\ W_q^s$. This transversality is preserved by small variations of F and the dynamics associated to the existence of h is (qualitatively) robust.

Now suppose F depends on some parameter ε (possibly a vectorial parameter). Changes in the transversality are associated to global bifurcations (as opposite to the local ones which refer to changes in the stability of fixed points, periodic orbits, etc.) As an example if we have a planar diffeomorphism $F\,(x,y,\varepsilon)$ depending on a two-dimensional parameter ε and we look for the bifurcation curve associated to homoclinic tangencies between saddle points p and q we have the following system

$$F^{k_1}\,(f(z),\varepsilon_1,\varepsilon_2) = F^{-k_2}\,(g(w),\varepsilon_1,\varepsilon_2)\ ,$$

$$\left[\,(D\,F^{k_1})\,(f(z),\varepsilon_1,\varepsilon_2)\ f'(z)\right] \wedge \left[\,(D\,F^{-k_2})\,(g(w),\varepsilon_1,\varepsilon_2)\ g'(w)\right] = 0\ ,$$

where $f\,(z)\,,\ g\,(w)$ are the local expressions of $W_p^u,\ W_p^s$ and $f'(z) = \frac{\partial}{\partial z}\,f$, $g'(w) = \frac{\partial}{\partial w}\,g$. We remark that the fixed points, p, q, and therefore the local expansions f, g also depend on ε. In the previous system we have 4 variables, $z, w, \varepsilon_1, \varepsilon_2$, and 3 equations. In general we shall obtain a curve in the ε plane giving heteroclinic tangencies. The continuation method can be used to obtain this curve, to analyse its own bifurcations, etc. This is a simple but long task because if $f\,(z)$ and $g\,(w)$ are expanded just to first order in z and w the computation of $\frac{\partial}{\partial \varepsilon_1}\ \left(\left[(D\,F^{k_1})\,(f(z),\varepsilon_1,\varepsilon_2)\ F'\right] \wedge \left[\,(D\,F^{-k_2})\,(g(w),\varepsilon_1,\varepsilon_2)\ g'\right]\right)$ involves, in particular, the computation of $\frac{\partial}{\partial,\varepsilon_1}\ \lambda(\varepsilon)$, $\frac{\partial}{\partial,\varepsilon_1}\ \mu(\varepsilon)$, where $\lambda(\varepsilon)\,,\,\mu(\varepsilon)$ are the eigenvalues at $p(\varepsilon)$, $q(\varepsilon)$ in the directions of W_p^u, W_q^s, respectively. Working out the equations in extenso we have

$$
\begin{aligned}
&F\,(p(\varepsilon),\varepsilon) = p(\varepsilon)\,, \\
&F\,(q(\varepsilon),\varepsilon) = q(\varepsilon)\,, \\
&D\,F\,(p(\varepsilon),\varepsilon)\ u(\varepsilon) = \lambda(\varepsilon)\ u(\varepsilon)\ ,\ \ \|u(\varepsilon)\|_2 = 1\ , \\
&D\,F\,(q(\varepsilon),\varepsilon)\ v(\varepsilon) = \mu(\varepsilon)\ v(\varepsilon)\ ,\ \ \|v(\varepsilon)\|_2 = 1\ , \\
&f\,(z,\varepsilon) = p(\varepsilon) + z\,u(\varepsilon)\ , \\
&g\,(w,\varepsilon) = q(\varepsilon) + w\,v(\varepsilon)\ , \\
&F^{k_1})\,(f(z,\varepsilon),\varepsilon) - F^{-k_2}\,(g(w,\varepsilon),\varepsilon) = 0\ ,
\end{aligned}
$$

$$
G(z,w,\varepsilon) = \Big[\Big(D\,F^{k_1}(f(z,\varepsilon),\varepsilon)\Big)\ u(\varepsilon)\Big] \wedge \Big[\Big(D\,F^{-k_2}(g(w,\varepsilon),\varepsilon)\Big)\ v(\varepsilon)\Big] = 0\ ,
$$

and therefore, for $i = 1,2$, we have

$$
\begin{aligned}
\frac{\partial}{\partial\,\varepsilon_i}\ G(z,w,\varepsilon) = &\Big\{(D^2\,F^{k_1})(f(z,\varepsilon),\varepsilon)\Big(\frac{\partial\,p}{\partial\,\varepsilon_i} + z\,\frac{\partial\,u}{\partial\,\varepsilon_i},\ u\,(\varepsilon)\Big) + \\
&\Big(\frac{\partial}{\partial\,\varepsilon_i}\ D\,F^{k_1}\Big)\,(f(z,\varepsilon),\varepsilon)\ u\,(\varepsilon) + D\,F^{k_1}\,(f(z,\varepsilon),\varepsilon)\ \frac{\partial\,u}{\partial\,\varepsilon_i}\Big\} \wedge \\
&\Big(D\,F^{-k_2}(g(w,\varepsilon),\varepsilon)\Big)\ v(\varepsilon) + \Big(D\,F^{k_1}(f(z,\varepsilon),\varepsilon)\Big)\ u(\varepsilon) \wedge \\
&\Big\{(D^2\,F^{-k_2})(g(w,\varepsilon),\varepsilon)\Big(\frac{\partial\,q}{\partial\,\varepsilon_i} + w\,\frac{\partial\,v}{\partial\,\varepsilon_i},\ v\,(\varepsilon)\Big) + \\
&\Big(\frac{\partial}{\partial\,\varepsilon_i}\ D\,F^{-k_2}\Big)\,(g(w,\varepsilon),\varepsilon)\ v\,(\varepsilon) + D\,F^{-k_2}\,(g(w,\varepsilon),\varepsilon)\ \frac{\partial\,v}{\partial\,\varepsilon_i}\Big\}\,,
\end{aligned}
$$

where

$$
\begin{aligned}
&D\,F\,(p(\varepsilon),\varepsilon)\ \frac{\partial\,p}{\partial\,\varepsilon_i} + \frac{\partial}{\partial\,\varepsilon_i}\ F\,(p(\varepsilon),\varepsilon) = \frac{\partial\,p}{\partial\,\varepsilon_i}\ , \\
&D^2F\,(p(\varepsilon),\varepsilon)\ \Big(\frac{\partial\,p}{\partial\,\varepsilon_i},\ u(\varepsilon)\Big) + \frac{\partial}{\partial\,\varepsilon_i}\ D\,F\,(p(\varepsilon),\varepsilon)\ u(\varepsilon) + D\,F\,(p(\varepsilon),\varepsilon)\ \frac{\partial\,u}{\partial\,\varepsilon_i} = \\
&\frac{\partial\,\lambda}{\partial\,\varepsilon_i}\ u(\varepsilon) + \lambda(\varepsilon)\ \frac{\partial\,u}{\partial\,\varepsilon_i}\ , \\
&\Big(u\,(\varepsilon)\,,\ \frac{\partial\,u}{\partial\,\varepsilon_i}\Big) = 0\ ,
\end{aligned}
$$

and similar equations for the point $q(\varepsilon)$ and the eigenvector and eigenvalue $v(\varepsilon)$, $\mu(\varepsilon)$, allow us to determine $\frac{\partial\,p}{\partial\,\varepsilon_i}$, $\frac{\partial\,u}{\partial\,\varepsilon_i}$, $\frac{\partial\,q}{\partial\,\varepsilon_i}$, $\frac{\partial\,v}{\partial\,\varepsilon_i}$, $i = 1,2$. If F is given by Poincaré section one should integrate the first and second order variational equations with respect to the spatial variables and the parameters. The analysis of the possible bifurcations inside the bifurcation curve requires the use of third order variational equations in the simplest case.