Equilibria in Three Body Problems: stability, invariant tori and connections

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Certifico que la present memòria ha estat realitzada per l'Anna Samà Camí, i constitueix la seva Tesi per a aspirar al grau de Doctor en Matemàtiques per la Universitat Autònoma de Barcelona.

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Firmat: Dra. Regina Martínez Barchino

Va per tu, bitxo

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Introduction

Celestial Mechanics is devoted to study the motion of planets, asteroids, satellites, ... Its starting point can be considered in the XVII century when Johannes Kepler formulated the laws of the motion of the planets in *Astronomia Nova* (1609). In the year 1687 Newton gave in his work *Philosophiae Naturalis Principia Mathematica* the formulation of the principal object of study of Celestial Mechanics: the n–body problem. This problem studies the motion of n particles system under their mutual attraction, governed by the Newton Gravitational Law. Even though the formulation of the equations that describes the n–body problem is easy, it is difficult to solve them. In fact, the only case completely solved is the 2–body problem. All the efforts to solve explicitly the equations for $n \geq 3$ have failed. Poincaré showed that the main difficulty comes from the existence of small divisors. In his very famous *Méthodes Nouvelles de la Mécanique Céleste* (1899), Poincaré starts the study of the problem from a qualitative point of view. Actually, qualitative methods play a very important role in the study of differential equations.

However, some special solutions of the n–body problem are known: the homographic solutions. For these solutions the configuration of the particles is preserved for any time. This can only be achieved for the so called central configurations. It is well–known that for the Planar Three Body Problem there exists three collinear central configurations, where the masses are located on a straight line, and two triangular ones, in which the masses are on the vertices of an equilateral triangle. Concerning the number and type of central configurations for $n \geq 4$ only partial results are known.

On the other hand for many applications one can make several assumptions which simplify the mathematical problem. The Restricted Three Body Problem (RTBP) is one of the most used models to get a first insight in many applications. In this problem the main assumption is to suppose that one of the bodies has infinitessimal mass, in such a way that it does not influence the motion of the other two bodies, called primaries. Then, one can assume that the primaries move on a solution of the Kepler problem. The Restricted Three Body Problem tries to explain the motion of the infinitessimal mass influenced by the gravitational forces exerted by the primaries. The most interesting cases for the applications correspond to elliptic orbits of the primaries. If the eccentricity, e, of these orbits is zero then we obtain the Circular Restricted Three Body Problem and for $e \in (0,1)$ the Elliptic Restricted Three Body Problem.

In a rotating system of coordinates the Planar Circular Restricted Three Body Problem is described by a Hamiltonian system with two degrees of freedom ([Sz.]). For this problem it is well–known that there exists three collinear equilibrium points $L_{1,2,3}$ and two triangular ones $L_{4,5}$. The collinear equilibrium points are of centre–saddle type. Let be C_{L_i} the value of the Jacobi constant at the equilibrium L_i. The Lyapunov theorem ([S.M.],[Ms.]) ensures the existence of a family of periodic orbits born at the equilibrium. This family can be parametrized by the Jacobi constant in such a way that for a fixed level of the Jacobi constant such that $C_{L_i} - C$ is small enough, the periodic orbit is the unique bounded orbit that for any time it remains in a small neighbourhood of the equilibrium point. Moreover these orbits are hyperbolic. They have stable and unstable 2–dimensional invariant manifolds of codimension 1 once the Jacobi constant has been fixed. Using these invariant manifolds it is possible to give a classification of the orbits passing through a small neighbourhood of the collinear equilibrium points ([Co.2],[McG.1]). The existence of transversal homoclinic orbits to the periodic Lyapunov orbit has been studied in [L.M.S.] for different values of the mass parameter and the Jacobi constant. This allows to introduce a symbolic dynamics ([L.M.S.],[Ms.2]) which gives the existence of orbits passing through different regions of the phase space. The applicability of these orbits to space missions have been studied in [K.L.M.R.].

In this work we distinguish mainly three parts. In the first one we study some questions related to the stability of homographic solutions. The second part is devoted to the Spatial RTBP. For that problem we study the existence of heteroclinic/homoclinic connections to the invariant tori contained in the centre manifold of the Spatial RTBP. Finally we study the applicability of KAM theorem to the centre manifold of the collinear equilibrium points in the Planar Three Body Problem. Next we introduce these three topics.

Homographic solutions

We consider the Planar Three Body Problem with homogeneous potential of degree $-\alpha$, $0 < \alpha < 2$, of the following type

$$U(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3) = \frac{m_1 m_2}{\|\mathbf{q}_1 - \mathbf{q}_2\|^\alpha} + \frac{m_1 m_3}{\|\mathbf{q}_1 - \mathbf{q}_3\|^\alpha} + \frac{m_2 m_3}{\|\mathbf{q}_2 - \mathbf{q}_3\|^\alpha}.$$

Notice that if $\alpha = 1$ we get the Newtonian potential. One can generalize the homographic solutions introduced for the Newtonian potential to the general case $(0 < \alpha < 2)$. These solutions can be written as equilibrium points of a periodic Hamiltonian system with 6 degrees of freedom. To this end one should introduce a change of variable which depends quasiperiodically (periodically if $\alpha = 1$) on time.

As we are interested in the stability of these solutions it will be necessary to compute the eigenvalues of the monodromy matrix. To reduce 2 degrees of freedom we use first the integrals of the centre of masses. At this point the linearized system for homographic solutions has order 8. Then we show that one can write this system as two four dimensional uncoupled system. The monodromy matrix of one of these systems has 1 as eigenvalue with multiplicity four. So, in order to obtain the non trivial characteristic multipliers we need to study the other four dimensional system,

$$\dot{\mathbf{x}} = A(t)\mathbf{x}, \tag{1}$$

where t is the true anomaly in the Newtonian case. Beyond the degree of homogeneity $-\alpha$, this system depends on two parameters: β, that depends on the masses, and e, some generalized eccentricity. We note that the parameter β is different for collinear and triangular cases.

When e equals zero, system (1) has constant coefficients and the characteristic exponents, or equivalently the stability parameters, are trivially computed. As e increases some bifurcation can appear. Furthermore as e goes to 1, we get in the limit case a matrix $A(t)$ in (1) which has a singularity at $t = 0$.

Our purpose is to study the stability of systems which generalize in some sense the behaviour of the linearized homographic case for e near 0 and e near 1. So, we consider linear systems of the following type

$$\dot{\mathbf{x}} = A(t,e)\mathbf{x}, \qquad A(t,e) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \lambda_1 G_1(t,e) & 0 & 0 & -2 \\ 0 & \lambda_2 G_2(t,e) & 2 & 0 \end{pmatrix}, \tag{2}$$

where $\mathbf{x} \in \mathbb{R}^4$, λ_1, λ_2 are real parameters, $e \in [0,1)$, and G_1, G_2 are periodic functions in t, depending on e. We shall study the stability for $e \gtrsim 0$ and $e \lesssim 1$. In any case, we shall make different hypothesis on G_1 and G_2 that will be satisfied in particular in the homographic case.

A system like (2) has several applications. One of them is the study of the stability for the equilibria of some Mechanical systems. Moreover, system (2) can be obtained as a first variational system on a periodic solution of an autonomous system. Chapters 2 and 3 are devoted to study the stability parameters of (2) for $e > 0$ small enough and for $e \lesssim 1$, respectively.

In chapter 2 we study the stability of system (2) for $e \gtrsim 0$ in the case that G_1, G_2 are even periodic functions of t and analytic in e. First of all, in section 2.2 we study the trivial case $e = 0$ where some resonant points are found. As e increases some bifurations can appear giving rise to some regions in the parameters spaces λ_1, λ_2, e with different stability character. To study the boundary surfaces of that regions we use the Normal Form technique. In [B.S.1] this method was used to study the resonant tongues for Hill's quasiperiodic equation depending on two parameters, and in [B.S.2] to the unfolding of Mathieu–like equations in the periodic case.

In this work to study the boundary surfaces we concentrate mainly in the d'Alembert case, that is we assume that for G_1 and G_2 the kth harmonic has an amplitude which is at least of order k in e. This is in fact a very common situation in mechanical systems. For example, it occurs in the study of the stability of families of periodic orbits which born at an equilibrium point from purely imaginary eigenvalues. Assuming d'Alembert property we distinguish between single and double resonances. The most interesting case is the second one. Under non degeneracy conditions in a neighbourhood of a double resonance, by changing the parameters one can get regions of any type for $e \gtrsim 0$.

In chapter 3 we study the stability of system (2) for $e \lesssim 1$. We assume $G_1 = G_2$ with some singularity for $e = 1$ at $t = 0$. The main result in this chapter is an asymptotic formula for the stability parameters. We use a kind of blow up technique to see the limit case as an heteroclinic connection.

In chapter 4 we use the results on chapters 2 and 3 in order to study the stability of the homographic solutions of the Planar Three Body Problem. In this case, the parameters λ_1, λ_2 depend on a unique mass parameter β. So, the bifurcation diagram is represented in the plane (β, e) for fixed α. We compute the resonant parameters at $e = 0$ for any α. However we concentrate mainly in the Newtonian case. Using the Normal Form Method developed in chapter 2, we get the resonant tongues born at $e = 0$ up to a given order.

In the collinear case, $\beta \in (0, 7)$ for the physical problem. However, mathematically we can consider $\beta > 0$. Resonances are obtained at $e = 0$ for frequencies $\dfrac{k}{2}$, $k \geq 3$. If $k = 2n$ no bifurcation takes place for $e > 0$. If $k = 2n + 1$ resonant tongues $\mathcal{T}_{\frac{2n+1}{2}}$ are born at $e = 0$. Despite only $\mathcal{T}_{\frac{3}{2}}$, $\mathcal{T}_{\frac{5}{2}}$ are the unique tongues which emanate from $e = 0$ for $\beta \in (0, 7)$, all the other tongues $\mathcal{T}_{\frac{2n+1}{2}}$ enter this range of β for values of e in $(0, 1)$. The width of $\mathcal{T}_{\frac{3}{2}}$, $\mathcal{T}_{\frac{5}{2}}$ is of order 3 and 5 in e respectively. Moreover the asymptotic formula for e near 1 predicts that all these tongues accumulate at $\beta = \dfrac{1}{8}$ as e goes to 1. This behaviour agrees with the numerical computations done for any $e \in (0,1)$.

Concerning the triangular case, for $0 < \beta < 1$ and $e = 0$ the system is elliptic–elliptic and only one resonant tongue $\mathcal{T}$ born at $\beta = \dfrac{3}{4}$ is found. It defines an elliptic–hyperbolic region in the plane (β, e). The width is of order 1 in e. The behaviour for $\beta = \dfrac{3}{4}$ and $e \gtrsim 0$ was studied by G. Roberts (see [R.]). By expanding the monodromy matrix in series on e, he proves the existence of an elliptic–hyperbolic region for this value of β and for e small enough. The method used in [R.] is not useful in the collinear case because the computations are hard. This is due to the fact that in the collinear case the width of tongues is of bigger order in e and so, one needs to compute at least the third order terms in e of the monodromy matrix of the linearized system on the collinear solution.

Spatial RTBP

Chapter 5 is devoted to the study of homoclinic orbits to the centre manifold of L_2 in the Spatial Restricted Three Body Problem. It is well–known that L_2 is a centre–centre–saddle equilibrium point. Then, it has one–dimensional stable and unstable invariant manifolds, and a four–dimensional centre manifold. In a neighbourhood of L_2 there exist the well–known families of planar and vertical periodic orbits of Lyapunov. These families of periodic orbits have two–dimensional stable and unstable manifolds. Moreover, in the centre manifold there exist invariant tori, with three dimensional stable and unstable manifolds. For the dynamics on the centre manifold see [J.M.], [G.M.]. The intersection of the unstable manifold of one torus in the centre manifold and the stable manifold of another torus give heteroclinic orbits from the first torus to the other one. If we consider the stable and the unstable manifold of the same torus, then we obtain homoclinic orbits to this torus. All these homoclinic and heteroclinic orbits are homoclinic orbits to the centre manifold of L_2. In order to get heteroclinic (or homoclinic) orbits we follow the main ideas developed in [L.M.S.] for the Planar RTBP. We compute up to a given order the intersection of the unstable invariant manifold of a given torus with the section $y = 0$ at the other side of the bigger primary. To do that we consider the Spatial RTBP as a perturbation of the 3–dimensional Hill's problem in a neighbourhood of the equilibrium point and then as a perturbation of the Spatial Synodic Two Body Problem. The stable manifold is obtained from the unstable one using the symmetries of the problem.

We give also some estimates on the difference in action space for two tori in order to have an heteroclinic connection. This allows us to construct heteroclinic chains. In particular from invariant tori close to the planar periodic orbit to invariant tori close to the vertical one in a neighbourhood of the L_2 point.

Planar Three Body Problem

Finally, in chapter 6 we study the existence of invariant tori on the centre manifold of the collinear equilibrium points in the Planar Three Body Problem with Newtonian potential. To this end we do the following steps. First, we perform some canonical transformations to write the Hamiltonian in normal form. Then we reduce the Hamiltonian to the centre manifold. After that, we check, by numerical evaluation of the coefficients of the normal form up to order 4, the nondegeneracy conditions of KAM theorem. The results presented in section 6.4 show that both conditions (either isoenergetic or not) are satisfied for any values of the masses in the triangle of masses.

The linearized system on a collinear equilibrium point has eigenvalues $\pm\lambda$, $\pm\mathrm{i}$, $\pm\mathrm{i}\omega$, $\lambda, \omega \in \mathbb{R}^+$. Then the collinear points are of centre–centre–saddle type. Up to order 4 it is proved that we only need to take into account the resonance $2:1$. The corresponding resonant masses describe a curve in the triangle of masses. Therefore, for resonant masses it is expected to get resonant monomials of order three in the normal form of the Hamiltonian. We prove in section 6.3 that this is not the case. In fact, we prove that the coefficients of these monomials are different from zero for general masses but they become zero for resonant masses, and also in the symmetrical case $m_1 = m_3$. The existence of the homographic solutions allows us to compute analytically, in an easy way, the coefficients of the resonant monomials of order three. These coefficients have $(\omega - 2)$ as a factor. The results given in chapter 6 are published in [M.S.].

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Chapter 1

Homographic solutions in the Planar Three Body Problem

1.1 Equations of motion

Consider three positive point masses m_1, m_2, m_3 moving in an inertial coordinate system in $\mathbb{R}^2$. Let us denote by $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3 \in \mathbb{R}^2$ the position vector of the jth particle of mass m_j, $j = 1, 2, 3$ (see figure 1.1).

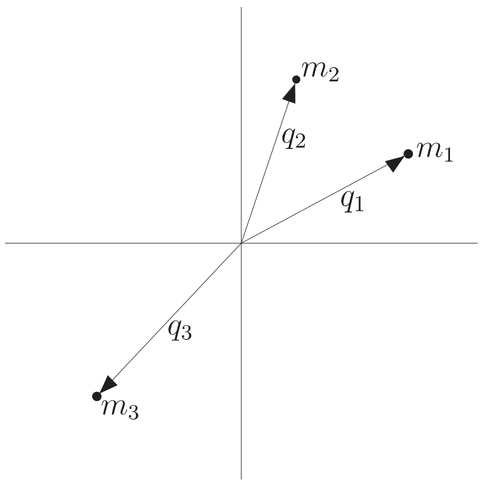


Figure 1.1: Masses m_1, m_2, m_3 and the respective position vectors $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3$.

Assume that the only forces acting on the masses are described by an homogeneous potential of the form

$$U(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3) = \frac{m_1 m_2}{\|\mathbf{q}_1 - \mathbf{q}_2\|^\alpha} + \frac{m_1 m_3}{\|\mathbf{q}_1 - \mathbf{q}_3\|^\alpha} + \frac{m_2 m_3}{\|\mathbf{q}_2 - \mathbf{q}_3\|^\alpha}, \tag{1.1}$$

where $0 < \alpha < 2$. The purpose of the Planar Three Body Problem with homogeneous potential U is to describe the motion of these three masses.

Applying Newton's second law yields the equations of motion

$$m_i \mathbf{q}_i'' = \sum_{j=1, j\neq i}^{3} \frac{m_i m_j (\mathbf{q}_j - \mathbf{q}_i)}{\|\mathbf{q}_i - \mathbf{q}_j\|^{\alpha+2}} = \frac{\partial U}{\partial \mathbf{q}_i}, \qquad i = 1, 2, 3 \tag{1.2}$$

where $U(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3)$ is defined in (1.1) and $' = \dfrac{d}{dt}$.

If $\alpha = 1$ we obtain the equations of the Planar Three Body Problem with Newtonian potential, that is, the only forces acting on the three masses are their mutual gravitational attraction.

Let us define $\Delta_{ij} = \{\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3) \in \mathbb{R}^6 | \mathbf{q}_i = \mathbf{q}_j\}$ for $1 \leq i < j \leq 3$, and $\Delta = \cup_{1\leq i<j\leq 3} \Delta_{ij}$. Δ is said to be the collision set. It is clear that U is not defined for $\mathbf{q} \in \Delta$.

Let $\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3) \in \mathbb{R}^6 \setminus \Delta$ and $M = \mathrm{diag}(m_1, m_1, m_2, m_2, m_3, m_3)$. Then, equations (1.2) can be written as

$$M\mathbf{q}'' = \nabla U(\mathbf{q}).$$

If we define $\mathbf{p} = (\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3) \in \mathbb{R}^6$ by $\mathbf{p} = M\mathbf{q}'$, then $\mathbf{p}_j = m_j \mathbf{q}_j'$ and $\mathbf{p}_j$ is the momentum of the jth particle. In variables $(\mathbf{q}, \mathbf{p})$ the equations of motion

$$\begin{aligned} \mathbf{q}' &= M^{-1}\mathbf{p}, \\ \mathbf{p}' &= \nabla U(\mathbf{q}), \end{aligned} \tag{1.3}$$

form a Hamiltonian system with 6 degrees of freedom, with Hamiltonian function

$$\mathcal{H}(\mathbf{q}, \mathbf{p}) = \frac{1}{2}\mathbf{p}^T M^{-1} \mathbf{p} - U(\mathbf{q}). \tag{1.4}$$

It is well known that there exist six first integrals for the Planar Three Body Problem. These integrals can be used to reduce the order of the system.

First, we can assume that the centre of masses is fixed at the origin. Then

$$\begin{aligned} m_1\mathbf{q}_1 + m_2\mathbf{q}_2 + m_3\mathbf{q}_3 &= 0, \\ \mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 &= 0, \end{aligned} \tag{1.5}$$

obtaining 4 integrals for the system. Therefore, we can reduce the system in 4 equations. We also have that the angular momentum is a first integral for system (1.3). Then, if we fix the angular momentum

$$\mathbf{q}_1 \wedge \mathbf{p}_1 + \mathbf{q}_2 \wedge \mathbf{p}_2 + \mathbf{q}_3 \wedge \mathbf{p}_3 = \mathbf{c},$$

one can reduce in one the dimension of the system.

Moreover, the Hamiltonian function (1.4) is a first integral.

Remark 1.1.1. Let be $m := m_1 + m_2 + m_3$ and $\mu_j = \dfrac{m_j}{m}$, $j = 1, 2, 3$. After the change of variables $\mathbf{Q} = m^{-\frac{1}{\alpha+2}}\mathbf{q}$, $\mathbf{P} = m^{-\frac{\alpha+3}{\alpha+2}}\mathbf{p}$, we get the same equations (1.3) for $\mathbf{Q}$, $\mathbf{P}$ with μ_j instead of m_j, $j = 1, 2, 3$. So, we can assume $m_1 + m_2 + m_3 = 1$.

As $m_1 + m_2 + m_3 = 1$, we can represent the set of admissible masses as points in a triangle defined by

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} m_1 + \begin{pmatrix} 1 \\ 0 \end{pmatrix} m_2 + \begin{pmatrix} \dfrac{1}{2} \\ \dfrac{\sqrt{3}}{2} \end{pmatrix} m_3,$$

where (x, y) denote the usual coordinates in $\mathbb{R}^2$. This relation represents a triangle in which the vertex $(0, 0)$, $(1, 0)$ and $\left(\dfrac{1}{2}, \dfrac{\sqrt{3}}{2}\right)$ correspond to $m_1 = 1$, $m_2 = 1$ and $m_3 = 1$, respectively (see figure 1.2).

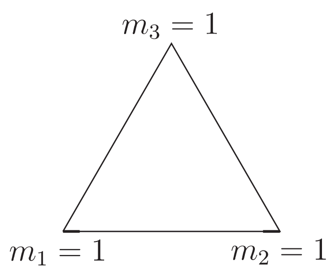


Figure 1.2: Triangle of masses

Notation 1.1.2. I_n stands for the identity matrix of order n, $J_{2n} = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$ is the $2n \times 2n$ skew symmetric matrix, and K_{2n} is the $2n \times 2n$ diagonal matrix defined as $K_{2n} = \mathrm{diag}(J_2, \ldots, J_2)$.

1.2 Homographic solutions

In this section we describe briefly central configurations and homographic solutions for the Planar Three Body Problem with homogeneous potential of degree $-\alpha$, $0 < \alpha < 2$. For details, see [M.H.]. Homographic solutions for a n–body problem are remarkable solutions for the simplicity of motions that describe. This kind of solutions are obtained from central configurations. A central configuration is a configuration in the position space such that for any mass m_j its acceleration, $\ddot{\mathbf{q}}_j$, is proportional to the position, $\mathbf{q}_j$.

We identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$ by considering $\mathbf{q}_j$ as complex numbers. We seek for solutions of (1.2) of the form

$$\mathbf{q}_j(t) \quad = \quad z(t)\mathbf{q}_{c_j}, \qquad j = 1, 2, 3, \tag{1.6}$$

where $\mathbf{q}_{c_j}$ are complex constant numbers and $z(t)$ is a complex valued function (the complex product is considered in the right hand part of (1.6)). Notice that the product by a complex number is a rotation followed by an homotecy, that is, an homography. So, a solution like (1.6) is called **homographic solution**.

If we substitute the expression of these special solutions into (1.2) and we rearrange the terms, yields

$$\|z(t)\|^{\alpha+2} z''(t) z(t)^{-1} m_i \mathbf{q}_{c_i} \quad = \quad \alpha \sum_{j=1, j\neq i}^{3} \frac{m_i m_j (\mathbf{q}_{c_j} - \mathbf{q}_{c_i})}{\|\mathbf{q}_{c_i} - \mathbf{q}_{c_j}\|^{\alpha+2}}.$$

As the right–hand part of last expression is constant, left–hand part needs also to be constant. Then, we obtain

$$z''(t) \quad = \quad -\lambda \frac{z(t)}{\|z(t)\|^{\alpha+2}}, \tag{1.7}$$

where λ is a constant such that

$$-\lambda m_i \mathbf{q}_{c_i} \quad = \quad \alpha \sum_{j=1, j\neq i}^{3} \frac{m_i m_j (\mathbf{q}_{c_j} - \mathbf{q}_{c_i})}{\|\mathbf{q}_{c_i} - \mathbf{q}_{c_j}\|^{\alpha+2}}, \qquad i = 1, 2, 3. \tag{1.8}$$

Therefore, (1.6) is a solution of the Planar Three Body Problem (1.2) if and only if $z(t)$ is a solution of (1.7) and $\mathbf{q}_c = (\mathbf{q}_{c_1}, \mathbf{q}_{c_2}, \mathbf{q}_{c_3})$ satisfies (1.8). Equation (1.7) is the Kepler problem with homogeneous potential of degree $-\alpha$, $0 < \alpha < 2$. The solutions of this equation are discussed in the appendix A.

A solution $\mathbf{q}_c \in \mathbb{R}^6 \setminus \Delta$ of (1.8) is called a **central configuration**.

In the special case when $z(t)$ is a circular solution of (1.7) and $\mathbf{q}_c$ is a central configuration, the solution is also called a **relative equilibrium** because it becomes an equilibrium solution in a rotating coordinate system. For a relative equilibrium the three bodies rotate as a rigid body.

Equations (1.8) is a nontrivial system of nonlinear equations. This system can be written as

$$-\lambda M\mathbf{q}_c \quad = \quad \nabla U(\mathbf{q}_c). \tag{1.9}$$

Then, on a central configuration, the acceleration is proportional to the position.

If we consider the scalar product with $\mathbf{q}_c$ in the equation above, using Euler's Theorem about homogeneous functions (that stays that if $F:\mathbb{R}^m \longrightarrow \mathbb{R}$ is a $\mathcal{C}^1$ homogeneous function of degree β then the equality $\mathbf{x}^T\nabla F(\mathbf{x}) = \beta F(\mathbf{x})$ holds) we obtain $\lambda = \alpha\dfrac{U(\mathbf{q}_c)}{\mathbf{q}_c^T M\mathbf{q}_c} > 0$.

(1.9) is invariant under rotations and under any uniform scaling. That is, if $\mathbf{q}_c$ is a central configuration and we introduce $\mathbf{s}\in\mathbb{R}^6$ as $\mathbf{q}_c = kR(\theta)\mathbf{s}$ where

$$R(\theta) = \mathrm{diag}(R_1(\theta), R_1(\theta), R_1(\theta))$$

is a 6×6 matrix, $\theta\in[0,2\pi]$, $R_1(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ and $k\in\mathbb{R}$, then

$$-\tilde{\lambda}M\mathbf{s} \quad = \quad \nabla U(\mathbf{s}),$$

with $\tilde{\lambda} = \lambda k^{\alpha+2}$. That is, $\mathbf{s}$ is a central configuration.

We will say that two central configurations are similar if they differ in a rotation or in an uniform scaling. Then, when counting central configurations one only counts similarity classes.

Remark 1.2.1. It is not restrictive to assume that $\lambda = 1$. This is due to the fact that if we introduce new variables $\mathbf{Q} = \lambda^{\frac{1}{\alpha+2}}\mathbf{q}$, $\mathbf{P} = \lambda^{-\frac{\alpha}{2(2+\alpha)}}\mathbf{p}$ and scale the time by $\lambda^{\frac{1}{2}}t$, the equations of motion are (1.3) and equation (1.9) holds with $\lambda = 1$.

Summarizing, assume that $\mathbf{q}_c$ is a central configuration. Then for any $z(t)$ solution of the Kepler Problem with homogeneous potential (1.7), an homographic solution of (1.2) is obtained as

$$\mathbf{q}(t) \quad = \quad z(t)\mathbf{q}_c.$$

We shall consider bounded homographic solutions. So, let $z(t)$ be a bounded solution of (1.7). It can be written (see appendix A) as $z(t) = r(t)e^{\mathrm{i}f(t)}$ where r satisfies

$$r'' \quad = \quad -\frac{dV}{dr}(r), \tag{1.10}$$

being $V(r) = -\dfrac{1}{\alpha r^\alpha} + \dfrac{\omega^2}{2r^2}$ the effective potential of the associated Kepler problem and

$$f(t) \quad = \quad \int_0^t \frac{\omega}{r(s)^2} ds. \tag{1.11}$$

We note that ω is the angular momentum for the Kepler problem (1.7). In the Newtonian case, f is the true anomaly.

We shall denote the energy of (1.10) by

$$E_K \quad = \quad \frac{(r')^2}{2} + V(r). \tag{1.12}$$

$V(r)$ has a minimum at $r^* = \omega^{\frac{2}{2-\alpha}}$ and $V(r^*) = -\left(\dfrac{2-\alpha}{2\alpha}\right)\omega^{-\frac{2\alpha}{2-\alpha}}$. For a fixed value $\omega > 0$, if E_K is such that $V(r^*) < E_K < 0$, we get a periodic solution of (1.10) and then a bounded solution of (1.7) as $z(t) = r(t)e^{\mathrm{i}f(t)}$. On the level energy $E_K = V(r^*)$, (1.10) has an equilibrium point at $r = r^*$. Then, $z(t) = r^* e^{\mathrm{i}f(t)}$ is a circular solution of (1.7) and the corresponding homographic solution is a relative equilibrium.

It is well known from Euler and Lagrange that in the Newtonian case there exist three collinear central configurations with the three masses on a line and two triangular ones with the masses on the vertex of an equilateral triangle. From these central configuration, and by (1.6), we obtain the collinear and triangular homographic solutions, respectively. Figure 1.3 shows an elliptic and a circular collinear homographic solutions for the Newtonian Planar Three Body Problem.

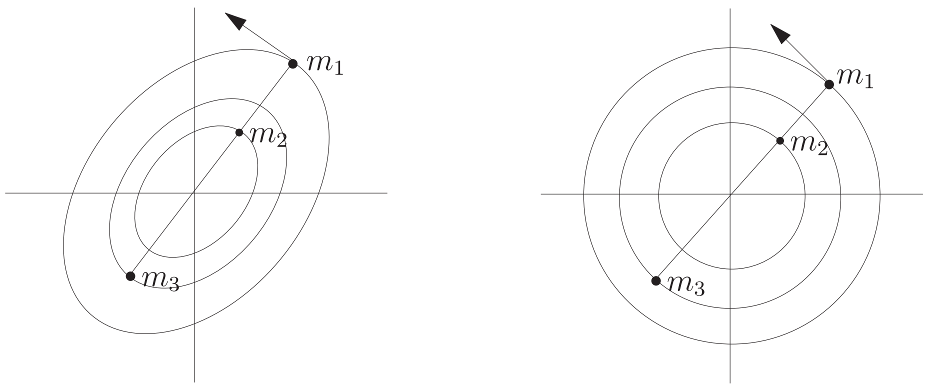


Figure 1.3: An elliptic and a circular collinear homographic solution for the Newtonian Planar Three Body Problem

In the homogeneous case there also exist three collinear homographic solutions and two triangular ones, as we shall see in section 1.5.

1.3 The homographic solutions as equilibrium points

Homographic solutions satisfy the equations of the Planar Three Body Problem with homogeneous potential, that is, a Hamiltonian system with six degrees of freedom. In this section we perform a suitable change of coordinates in such a way that the homographic solutions become equilibrium points of a periodic Hamiltonian system with six degrees of freedom. Moreover, we shall use the integrals of the centre of masses in order to reduce the order of the system by four.

We introduce a rotating and pulsating coordinate system through

$$\mathbf{q}(t) \;=\; r(t)\Omega(f(t))\boldsymbol{\zeta}(t),$$

where $\Omega = \mathrm{diag}(\Omega_1, \Omega_1, \Omega_1)$, $\Omega_1(f) = \begin{pmatrix} \cos f & -\sin f \\ \sin f & \cos f \end{pmatrix}$, and $r(t)$, $f(t)$ as defined in section 1.2, that is, for a given $\omega > 0$, $r(t)$ is a bounded solution of (1.10) and $f(t)$ is defined in (1.11). Using f as independent variable the new system can be written as

$$\begin{aligned} \dot{\boldsymbol{\zeta}} &= K_6\boldsymbol{\zeta} + M^{-1}\boldsymbol{\eta}, \\ \dot{\boldsymbol{\eta}} &= \nabla\mathcal{V}(\boldsymbol{\zeta}) + K_6\boldsymbol{\eta}, \end{aligned} \tag{1.13}$$

where $\dot{}$ stands for the derivative with respect to f, $\boldsymbol{\eta}$ is the conjugate variable of $\boldsymbol{\zeta}$ and

$$\mathcal{V}(\boldsymbol{\zeta}) \;=\; \frac{r^{2-\alpha}}{\omega^2}U(\boldsymbol{\zeta}) + \frac{1}{2}\left(\frac{r^{2-\alpha}}{\omega^2} - 1\right)\boldsymbol{\zeta}^T M\boldsymbol{\zeta}.$$

In these variables, the homographic solutions are equilibrium points of system (1.13).

The Hamiltonian in the new variables becomes

$$H(\boldsymbol{\zeta}, \boldsymbol{\eta}) \;=\; \frac{1}{2}\boldsymbol{\eta}^T M^{-1}\boldsymbol{\eta} - \boldsymbol{\zeta}^T K_6\boldsymbol{\eta} - \mathcal{V}(\boldsymbol{\zeta}).$$

We note that $(\boldsymbol{\zeta}^*, \boldsymbol{\eta}^*)$ is an equilibrium of (1.13) if and only if

$$\begin{aligned} \nabla U(\boldsymbol{\zeta}^*) &= -M\boldsymbol{\zeta}^*, \\ \boldsymbol{\eta}^* &= -MK_6\boldsymbol{\zeta}^*, \end{aligned}$$

that is, $\boldsymbol{\zeta}^*$ is a central configuration. If we recover the initial variables we get an homographic solution $\mathbf{q}(t) = r(t)\Omega(f(t))\boldsymbol{\zeta}^*$.

The system (1.13) is a periodic (due to the presence of $r(f)$) Hamiltonian system with six degrees of freedom. A first reduction of (1.13) is done using the integrals of the centre of masses (1.5). We introduce new variables

$$\begin{array}{ll} \mathbf{u}_i = \boldsymbol{\zeta}_i - \boldsymbol{\zeta}_3, & \mathbf{v}_i = \boldsymbol{\eta}_i, \quad i = 1,2, \\ \mathbf{u}_3 = \boldsymbol{\zeta}_3, & \mathbf{v}_3 = \boldsymbol{\eta}_1 + \boldsymbol{\eta}_2 + \boldsymbol{\eta}_3. \end{array}$$

The equations of motion become

$$\begin{aligned} \dot{\mathbf{u}}_1 &= J_2\mathbf{u}_1 + \alpha_1\mathbf{v}_1 + \frac{1}{m_3}\mathbf{v}_2 - \frac{1}{m_3}\mathbf{v}_3, \\ \dot{\mathbf{u}}_2 &= J_2\mathbf{u}_2 + \frac{1}{m_3}\mathbf{v}_1 + \alpha_2\mathbf{v}_2 - \frac{1}{m_3}\mathbf{v}_3, \\ \dot{\mathbf{u}}_3 &= J_2\mathbf{u}_3 + \frac{1}{m_3}(\mathbf{v}_3 - \mathbf{v}_1 - \mathbf{v}_2), \\ \dot{\mathbf{v}}_1 &= \frac{\partial}{\partial \mathbf{u}_1}\mathcal{V}(\tilde{\mathbf{u}}) + J_2\mathbf{v}_1, \\ \dot{\mathbf{v}}_2 &= \frac{\partial}{\partial \mathbf{u}_2}\mathcal{V}(\tilde{\mathbf{u}}) + J_2\mathbf{v}_2, \\ \dot{\mathbf{v}}_3 &= J_2\mathbf{v}_3, \end{aligned}$$

where $\tilde{\mathbf{u}} = (\mathbf{u}_1^T, \mathbf{u}_2^T, \mathbf{u}_3^T)^T$ and

$$\mathcal{V}(\tilde{\mathbf{u}}) = \frac{r^{2-\alpha}}{\omega^2}\mathcal{U}(\mathbf{u}_1, \mathbf{u}_2) + \frac{1}{2}\left(\frac{r^{2-\alpha}}{\omega^2} - 1\right)\tilde{\mathbf{u}}^T\tilde{C}\tilde{\mathbf{u}},$$

with

$$\mathcal{U}(\mathbf{u}_1, \mathbf{u}_2) = \frac{m_1m_2}{\|\mathbf{u}_1 - \mathbf{u}_2\|^\alpha} + \frac{m_1m_3}{\|\mathbf{u}_1\|^\alpha} + \frac{m_2m_3}{\|\mathbf{u}_2\|^\alpha}, \tag{1.14}$$

$$\tilde{C} = \begin{pmatrix} C & 0 \\ 0 & 0 \end{pmatrix}, \quad C = m_1m_2\begin{pmatrix} \alpha_2m_3I_2 & -I_2 \\ -I_2 & \alpha_1m_3I_2 \end{pmatrix}, \tag{1.15}$$

$\alpha_1 = \dfrac{m_1 + m_3}{m_1m_3}$, $\alpha_2 = \dfrac{m_2 + m_3}{m_2m_3}$. From the integral of the centre of masses (1.5) we get $\mathbf{v}_3 = 0$. The equations for $\mathbf{u}_i, \mathbf{v}_i$, $i = 1,2$, do not depend on $\mathbf{u}_3$. So, we can reduce to consider the following system

$$\begin{aligned} \dot{\mathbf{u}} &= K_4\mathbf{u} + C^{-1}\mathbf{v}, \\ \dot{\mathbf{v}} &= \nabla\mathcal{V}(\mathbf{u}) + K_4\mathbf{v}, \end{aligned} \tag{1.16}$$

where $\mathbf{u} = (\mathbf{u}_1^T, \mathbf{u}_2^T)^T$, $\mathbf{v} = (\mathbf{v}_1^T, \mathbf{v}_2^T)^T$ and

$$\mathcal{V}(\mathbf{u}) = \frac{r^{2-\alpha}}{\omega^2}\mathcal{U}(\mathbf{u}) + \frac{1}{2}\left(\frac{r^{2-\alpha}}{\omega^2} - 1\right)\mathbf{u}^T C\mathbf{u},$$

with $\mathcal{U}(\mathbf{u})$ and C given in (1.14) and (1.15), respectively. We note that we can recover $\boldsymbol{\zeta}_1$, $\boldsymbol{\zeta}_2$ from $\mathbf{u}_1$, $\mathbf{u}_2$ using

$$\begin{pmatrix} m_1\boldsymbol{\zeta}_1 \\ m_2\boldsymbol{\zeta}_2 \end{pmatrix} = C\begin{pmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{pmatrix}.$$

Then, $\boldsymbol{\zeta}_3$ is obtained using the integral of the centre of masses (1.5) as $\boldsymbol{\zeta}_3 = -\dfrac{1}{m_3}(m_1\boldsymbol{\zeta}_1 + m_2\boldsymbol{\zeta}_2)$.

The equilibria $(\mathbf{u}^*, \mathbf{v}^*)$ of this system are homographic solutions in these variables. An equilibrium point of system (1.16) must satisfy

$$\mathbf{v}^* = -CK_4\mathbf{u}^*. \tag{1.17}$$

Moreover, as $\nabla\mathcal{V}(\mathbf{u}^*) + K_4\mathbf{v}^* = 0$, we have

$$\frac{r^{2-\alpha}}{\omega^2}\nabla\mathcal{U}(\mathbf{u}) + \left(\frac{r^{2-\alpha}}{\omega^2} - 1\right)C\mathbf{u}^* + K_4\mathbf{v}^* = 0 \quad . \tag{1.18}$$

From (1.17) we obtain $K_4\mathbf{v}^* = C\mathbf{u}^*$ and therefore (1.18) becomes

$$\nabla\mathcal{U}(\mathbf{u}^*) = -C\mathbf{u}^*, \tag{1.19}$$

which together with (1.17) characterize the equilibria of (1.16).

1.4 The linearized system. Reduction of the order

We have seen that the homographic solutions are equilibrium points of a periodic system of differential equations. In this section we shall reduce the linearized system on an equilibrium point to a four dimensional system of differential equations. This reduction will be used in chapter 4 in order to study the stability of the homographic solutions.

The linearized system of (1.16) on an equilibrium $(\mathbf{u}^*, \mathbf{v}^*)$ is

$$\dot{\mathbf{y}} = \mathcal{A}\mathbf{y}, \qquad \mathcal{A} = \begin{pmatrix} K_4 & C^{-1} \\ \mathcal{D} & K_4 \end{pmatrix}, \tag{1.20}$$

where

$$\mathcal{D} = \frac{r^{2-\alpha}}{\omega^2} D\nabla\mathcal{U}(\mathbf{u}^*) + \left(\frac{r^{2-\alpha}}{\omega^2} - 1\right) C, \tag{1.21}$$

and D denotes the differential.

We will see that system (1.20) can be written as two uncoupled systems of order four.

Lemma 1.4.1. *Let $\mathbf{u}^*$ be a solution of (1.19). System (1.20) can be written as two uncoupled linear systems of order four with matrices*

$$\mathcal{B}_1(f) = \begin{pmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 \\ h_1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}, \qquad \mathcal{B}_2(f) = \begin{pmatrix} 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ b_{11} & b_{21} & 0 & -1 \\ b_{12} & b_{22} & 1 & 0 \end{pmatrix}, \tag{1.22}$$

where $h_1 = h_1(f) = (\alpha+2)\dfrac{r^{2-\alpha}}{\omega^2} - 1$, and

$$b_{11} = \frac{r^{2-\alpha}}{\omega^2}(\gamma_{11}+1)-1, \quad b_{12} = \frac{r^{2-\alpha}}{\omega^2}\gamma_{12}, \quad b_{21} = \frac{r^{2-\alpha}}{\omega^2}\gamma_{21}, \quad b_{22} = \frac{r^{2-\alpha}}{\omega^2}(\gamma_{22}+1)-1,$$

being $\gamma_{11}, \gamma_{12}, \gamma_{21}$ and γ_{22} constant coefficients depending on $\mathbf{u}^*$.

Proof

We introduce the following vectors

$$\mathbf{x}_1 = \begin{pmatrix} \mathbf{u}^* \\ \mathbf{0} \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} \mathbf{0} \\ K_4 C\mathbf{u}^* \end{pmatrix}, \quad \mathbf{x}_3 = \begin{pmatrix} \mathbf{0} \\ C\mathbf{u}^* \end{pmatrix}, \quad \mathbf{x}_4 = \begin{pmatrix} K_4\mathbf{u}^* \\ \mathbf{0} \end{pmatrix}.$$

First, we shall show that the subspace X of $\mathbb{R}^8$ spanned by $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4$ is invariant under $\mathcal{A}$. Then, we will introduce also the skew–orthogonal complement in $\mathbb{R}^8$ of X in order to uncouple the system (1.20).

In order to show that X is invariant under $\mathcal{A}$ we need some relations between $D\nabla\mathcal{U}(u)$ and $\nabla\mathcal{U}(u)$. By Euler's Theorem for homogeneous functions we have $\nabla\mathcal{U}(\mathbf{u})^T\mathbf{u} = -\alpha\mathcal{U}(\mathbf{u})$. If we differenciate this equality, we obtain

$$D\nabla\mathcal{U}(\mathbf{u})\mathbf{u} = -(\alpha+1)\nabla\mathcal{U}(\mathbf{u}). \tag{1.23}$$

Moreover, due to the homogeneity of $\mathcal{U}(\mathbf{u})$ we obtain

$$D\nabla\mathcal{U}(\mathbf{u})K_4\mathbf{u} = K_4\nabla\mathcal{U}(\mathbf{u}). \tag{1.24}$$

So, if $\mathbf{u}^*$ is a central configuration from (1.23) and (1.24) and using (1.19) we get

$$D\nabla\mathcal{U}(\mathbf{u}^*)\mathbf{u}^* = (\alpha+1)C\mathbf{u}^*, \qquad D\nabla\mathcal{U}(\mathbf{u}^*)K_4\mathbf{u}^* = -K_4C\mathbf{u}^*. \tag{1.25}$$

Then

$$\mathcal{D}\mathbf{u}^* = \left[(\alpha+2)\frac{r^{2-\alpha}}{\omega^2} - 1\right] C\mathbf{u}^*, \qquad \mathcal{D}K_4\mathbf{u}^* = -CK_4\mathbf{u}^*, \tag{1.26}$$

where the second equality holds due to the fact that K_4 and C commute. Using (1.26) we get easily

$$\mathcal{A}\mathbf{x}_1 = \mathbf{x}_4 + h_1\mathbf{x}_3, \quad \mathcal{A}\mathbf{x}_2 = \mathbf{x}_4 - \mathbf{x}_3, \quad \mathcal{A}\mathbf{x}_3 = \mathbf{x}_1 + \mathbf{x}_2, \quad \mathcal{A}\mathbf{x}_4 = -\mathbf{x}_1 - \mathbf{x}_2,$$

where $h_1 = (\alpha+2)\dfrac{r^{2-\alpha}}{\omega^2} - 1$. So, the four dimensional subspace, X, of $\mathbb{R}^8$ spanned by $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4$ is invariant under $\mathcal{A}$ and the system (1.20) reduced to X is given by the matrix $\mathcal{B}_1(f)$.

Let us denote by W the four dimensional subspace of $\mathbb{R}^8$ spanned by $\mathbf{w}_1$, $\mathbf{w}_2$, $\mathbf{w}_3$, $\mathbf{w}_4$ where

$$\mathbf{w}_1 = \begin{pmatrix} C^{-1}\boldsymbol{\eta}_1 \\ \mathbf{0} \end{pmatrix}, \quad \mathbf{w}_2 = \begin{pmatrix} C^{-1}\boldsymbol{\eta}_2 \\ \mathbf{0} \end{pmatrix}, \quad \mathbf{w}_3 = \begin{pmatrix} \mathbf{0} \\ \boldsymbol{\eta}_1 \end{pmatrix}, \quad \mathbf{w}_4 = \begin{pmatrix} \mathbf{0} \\ \boldsymbol{\eta}_2 \end{pmatrix},$$

and

$$\boldsymbol{\eta}_1 = J_4\mathbf{u}^* + \gamma_1 K_4\mathbf{u}^*, \quad \boldsymbol{\eta}_2 = K_4\boldsymbol{\eta}_1, \qquad \gamma_1 = \frac{(\mathbf{u}^*)^T K_4 J_4 \mathbf{u}^*}{\|\mathbf{u}^*\|^2}. \tag{1.27}$$

We want to see that W is the skew–orthogonal complement in $\mathbb{R}^8$ of X, that is, $W = \{\mathbf{w} \in \mathbb{R}^8 \,|\, \mathbf{w}^T J_8 \mathbf{x}_i = 0, i = 1, \ldots, 4\}$.

We have that

$$\boldsymbol{\eta}_1^T\mathbf{u}^* = -(\mathbf{u}^*)^T J_4\mathbf{u}^* - \gamma_1(\mathbf{u}^*)^T K_4\mathbf{u}^* = 0,$$

$$\boldsymbol{\eta}_2^T\mathbf{u}^* = (\mathbf{u}^*)^T J_4 K_4\mathbf{u}^* + \gamma_1(\mathbf{u}^*)^T K_4^2\mathbf{u}^* = 0.$$

Using these relations we find that W is the subspace that we are looking for.

Next and last step is to find the system reduced to W.

We have that

$$\begin{gathered} \mathcal{A}\mathbf{w}_1 = \begin{pmatrix} K_4C^{-1}\boldsymbol{\eta}_1 \\ \mathcal{D}C^{-1}\boldsymbol{\eta}_1 \end{pmatrix}, \qquad \mathcal{A}\mathbf{w}_2 = \begin{pmatrix} K_4C^{-1}\boldsymbol{\eta}_2 \\ \mathcal{D}C^{-1}\boldsymbol{\eta}_2 \end{pmatrix}, \\ \mathcal{A}\mathbf{w}_3 = \mathbf{w}_1 + \mathbf{w}_4, \qquad \mathcal{A}\mathbf{w}_4 = \mathbf{w}_2 - \mathbf{w}_3, \end{gathered} \tag{1.28}$$

with

$$\mathcal{D}C^{-1}\boldsymbol{\eta}_j = \frac{r^{2-\alpha}}{\omega^2}(D\nabla\mathcal{U}(\mathbf{u}^*)C^{-1}\boldsymbol{\eta}_j + \boldsymbol{\eta}_j) - \boldsymbol{\eta}_j, \quad j = 1, 2.$$

As $\mathbf{u}^*, K_4\mathbf{u}^*, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2$ span $\mathbb{R}^4$, we can write

$$D\nabla\mathcal{U}(\mathbf{u}^*)C^{-1}\boldsymbol{\eta}_j \quad = \quad \gamma_{j1}\boldsymbol{\eta}_1 + \gamma_{j2}\boldsymbol{\eta}_2 + \gamma_{j3}\mathbf{u}^* + \gamma_{j4}K_4\mathbf{u}^*, \quad j = 1, 2, \quad (1.29)$$

for some constants γ_{ji}, $j = 1, 2$, $i = 1, \ldots, 4$.

Due to the symmetry of $D\nabla\mathcal{U}(\mathbf{u}^*)$ and C, from (1.25) we have that

$$(\mathbf{u}^*)^T D\nabla\mathcal{U}(\mathbf{u}^*)C^{-1}\boldsymbol{\eta}_j = (\alpha + 1)(\mathbf{u}^*)^T \eta_j = 0, \quad j = 1, 2$$

$$(K_4\mathbf{u}^*)^T D\nabla\mathcal{U}(\mathbf{u}^*)C^{-1}\boldsymbol{\eta}_1 = (\mathbf{u}^*)^T CK_4C^{-1}\boldsymbol{\eta}_1 = (\mathbf{u}^*)^T\boldsymbol{\eta}_2 = 0,$$

$$(K_4\mathbf{u}^*)^T D\nabla\mathcal{U}(\mathbf{u}^*)C^{-1}\boldsymbol{\eta}_2 = (\mathbf{u}^*)^T CK_4C^{-1}\boldsymbol{\eta}_2 = -(\mathbf{u}^*)^T\boldsymbol{\eta}_1 = 0.$$

Then, $D\nabla\mathcal{U}(\mathbf{u}^*)C^{-1}\boldsymbol{\eta}_j = \gamma_{j1}\boldsymbol{\eta}_1 + \gamma_{j2}\boldsymbol{\eta}_2$, $j = 1, 2$, and

$$\gamma_{11} = \frac{1}{\|\boldsymbol{\eta}_1\|^2}\boldsymbol{\eta}_1^T D\nabla\mathcal{U}(\mathbf{u}^*)C^{-1}\boldsymbol{\eta}_1, \qquad \gamma_{12} = \frac{1}{\|\boldsymbol{\eta}_2\|^2}\boldsymbol{\eta}_2^T D\nabla\mathcal{U}(\mathbf{u}^*)C^{-1}\boldsymbol{\eta}_1,$$
$$\gamma_{21} = \frac{1}{\|\boldsymbol{\eta}_1\|^2}\boldsymbol{\eta}_1^T D\nabla\mathcal{U}(\mathbf{u}^*)C^{-1}\boldsymbol{\eta}_2, \qquad \gamma_{22} = \frac{1}{\|\boldsymbol{\eta}_2\|^2}\boldsymbol{\eta}_2^T D\nabla\mathcal{U}(\mathbf{u}^*)C^{-1}\boldsymbol{\eta}_2. \quad (1.30)$$

A simple computation shows that

$$\begin{aligned} \mathcal{D}C^{-1}\boldsymbol{\eta}_1 &= b_{11}\boldsymbol{\eta}_1 + b_{12}\boldsymbol{\eta}_2, \\ \mathcal{D}C^{-1}\boldsymbol{\eta}_2 &= b_{21}\boldsymbol{\eta}_1 + b_{22}\boldsymbol{\eta}_2. \end{aligned}$$

Then, from (1.28),

$$\begin{aligned} \mathcal{A}\mathbf{w}_1 &= \mathbf{w}_2 + b_{11}\mathbf{w}_3 + b_{12}\mathbf{w}_4, \qquad & \mathcal{A}\mathbf{w}_2 = -\mathbf{w}_1 + b_{21}\mathbf{w}_3 + b_{22}\mathbf{w}_4, \\ \mathcal{A}\mathbf{w}_3 &= \mathbf{w}_1 + \mathbf{w}_4, & \mathcal{A}\mathbf{w}_4 = \mathbf{w}_2 - \mathbf{w}_3, \end{aligned}$$

and the system (1.20) reduced to W is defined by the matrix $\mathcal{B}_2(f)$. □

We shall see that, for any equilibrium $(\mathbf{u}^*, \mathbf{v}^*)$ of (1.16), 1 is an eigenvalue with multiplicity four of the monodromy matrix of the system defined by $\mathcal{B}_1(f)$. Then, the nontrivial characteristic exponents will be given by the system defined by $\mathcal{B}_2(f)$.

$\mathcal{B}_1(f)$ and $\mathcal{B}_2(f)$ depend on f through the function $\dfrac{r^{2-\alpha}}{\omega^2}$ where r is a solution of (1.10). We recall that for the levels of energy of interest one has $r(t)$ bounded. Its minimum tends to zero when ω does. Notice that in the Newtonian case ($\alpha = 1$) $\dfrac{r}{\omega^2} = \dfrac{1}{1 + e\cos f}$, where e denotes the eccentricity of the orbit.

We introduce the function $g = \dfrac{\omega^{\frac{2}{2-\alpha}}}{r}$. Using (1.10) we obtain that

$$\ddot{g} \quad = \quad -g + g^{\alpha-1},$$

where we recall that $\dot{}$ stands for $\dfrac{d}{df}$. Then g is a periodic solution of

$$\ddot{z} \;=\; -\frac{dU}{dz}(z), \quad \text{with} \quad U(z) = \frac{z^2}{2} - \frac{z^\alpha}{\alpha}. \tag{1.31}$$

We denote by T the period of g in time f.

U has a positive zero at $\tilde{z} = \left(\dfrac{2}{\alpha}\right)^{\frac{1}{2-\alpha}}$, and a minimum at $z = 1$ attaining the value $U(1) = \dfrac{\alpha - 2}{2\alpha}$. Figure 1.4 shows some plots of U.

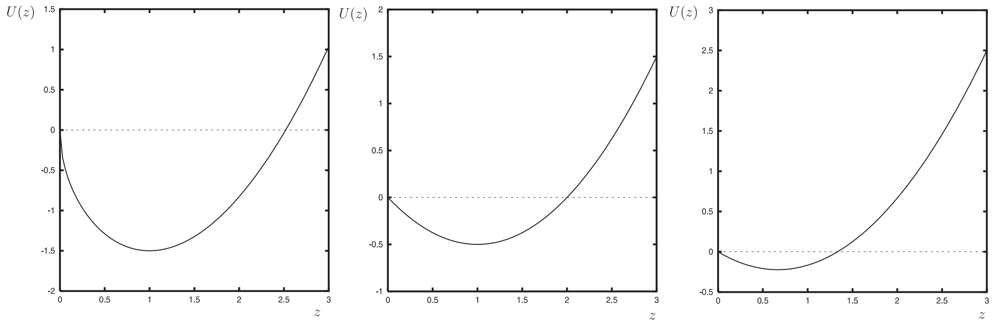


Figure 1.4: Plot of $U(z) = \dfrac{z^2}{2} - \dfrac{z^\alpha}{\alpha}$ for $\alpha = \dfrac{1}{2}, 1, \dfrac{3}{2}$, respectively

The energy of the problem (1.31) is

$$E = \frac{(\dot{z})^2}{2} + U(z). \tag{1.32}$$

Using $\dot{g} = -\omega^{\frac{\alpha}{2-\alpha}} r'$ (where we recall $' = \dfrac{d}{dt}$), we get

$$E \;=\; \omega^{\frac{2\alpha}{2-\alpha}} \left[\frac{1}{2}(r')^2 + \frac{\omega^2}{2r^2} - \frac{1}{\alpha r^\alpha}\right] = \omega^{\frac{2\alpha}{2-\alpha}} E_K, \tag{1.33}$$

where E_K is given in (1.12).

Once a central configuration has been fixed, we get a family of homographic solutions for $E \in \left[-\dfrac{2-\alpha}{2\alpha}, 0\right)$. For the sake of simplicity we shall fix a value of E_K, say $E_K = -\dfrac{1}{2}$, and move ω in the following range

$$0 < \omega \leq \omega_c,$$

where ω_c satisfies

$$-\frac{1}{2}\omega_c^{\frac{2\alpha}{2-\alpha}} \;=\; -\frac{2-\alpha}{2\alpha}, \tag{1.34}$$

which corresponds to a relative equilibrium. In that case, $g \equiv 1$.

Let us define $g_0 = g(0)$ the minimum of g. It is clear that if $0 < \omega < \omega_c$, $0 < g_0 < 1$. Moreover, g_0 tends to zero as ω tends to zero. However, $\dfrac{r^{2-\alpha}}{\omega^2} = g^{\alpha-2}$. If we consider the linear systems defined by $\mathcal{B}_1(f)$ and $\mathcal{B}_2(f)$ as ω tends to 0, the limit systems have a singularity at $f = 0$. In this case we have an homagraphic solution going to collision.

Remark 1.4.2. If $\omega = \omega_c$ then $E = U(1) = \dfrac{\alpha - 2}{2\alpha}$ and (1.31) has an equilibrium point $z^* = 1$. Then, $g(f) \equiv 1$ and the function $h_1(f)$ defined in Lemma 1.4.1 becomes constant. In fact, $h_1 = \alpha + 1$. In this case the linear systems defined by $\mathcal{B}_1(f)$ and $\mathcal{B}_2(f)$ have constant coefficients. Furthermore, $r = \omega^{\frac{2}{2-\alpha}}$ and so, we have a circular solution of the Kepler problem.

Lemma 1.4.3. *For $0 < \omega \leq \omega_c$ the monodromy matrix $\mathcal{C}$ of the linear system*

$$\dot{\mathbf{U}} = \mathcal{B}_1(f)\mathbf{U} \tag{1.35}$$

has the eigenvalue 1 with multiplicity four.

Proof

In order to compute the monodromy matrix $\mathcal{C}$ of (1.35) it is necessary to integrate the variational equations. In this case, we need to solve the same system (1.35) with initial conditions the vectors $\mathbf{e}_j$, $j = 1, \ldots, 4$, of the canonical basis.

Let be $\mathbf{V} = (u_1, u_2, u_3, u_4)^T$. We denote by $\mathbf{V}_j(f)$, $j = 1, \ldots, 4$, the solution of (1.35) with initial condition $\mathbf{V}_j(0) = \mathbf{e}_j$. Then, $\mathcal{C} = (\mathbf{V}_1(T), \mathbf{V}_2(T), \mathbf{V}_3(T), \mathbf{V}_4(T))^T$ where T is the period of g.

System (1.35) can be written as

$$\begin{aligned}
\dot{u}_1 &= u_3 - u_4, \\
\dot{u}_2 &= u_3 - u_4, \\
\dot{u}_3 &= h_1 u_1 - u_2, \\
\dot{u}_4 &= u_1 + u_2,
\end{aligned} \tag{1.36}$$

with $h_1(f) = (\alpha + 2)g^{\alpha-2} - 1$.

First we consider $\omega = \omega_c$. From remark 1.4.2 we know that the matrix $\mathcal{B}_1(f)$ is constant, and the solution of (1.31) for this value of ω corresponds to a circular solution $\mathbf{z}_c(t) = \omega^{\frac{2}{2-\alpha}} e^{if(t)}$ of the planar Kepler problem. Moreover, $h_1(f) = \alpha + 1$. The statement of the Lemma in this particular case follows due to the fact that the constant matrix $\mathcal{B}_1(f)$ has eigenvalues $0, 0, \pm \mathrm{i}\sqrt{2 - \alpha}$ and the circular solution has period $\dfrac{2\pi}{\sqrt{2-\alpha}}$ (see appendix A).

Assume now that $0 < \omega < \omega_c$.

From the two first equations in (1.36) it is easy to see that $u_1 - u_2 = k$ with k a real constant. Then, as $\ddot{u}_1 = \dot{u}_3 - \dot{u}_4$, u_1 satisfies the following equation

$$\ddot{u}_1 \quad = \quad [(\alpha+2)g^{\alpha-2} - 4]u_1 + 2k. \tag{1.37}$$

Once u_1 is obtained, u_2 is recovered from $u_2 = u_1 - k$ and u_3, u_4 are obtained by integration of the following equations

$$\begin{aligned} \dot{u}_3 &= [(\alpha+2)g^{\alpha-2} - 2]u_1 + k, \\ \dot{u}_4 &= 2u_1 - k. \end{aligned}$$

Now, we compute $\mathbf{V}_3, \mathbf{V}_4$. For these vectors, the initial conditions for u_1 and u_2 are $u_1(0) = u_2(0) = 0$. Then, we need to solve equation (1.37) with $k = 0$. It can be seen easily that $u_1(f) = cg\dot{g}$, c a constant, is a solution of this equation.

As for $\mathbf{V}_3$ the initial conditions for u_3 and u_4 are $u_3(0) = 1, u_4(0) = 0$, we have that $\dot{u}_1(0) = 1$. Then

$$c \quad = \quad \frac{1}{g_0^\alpha - g_0^2}. \tag{1.38}$$

We note that c is well defined due to the fact that for $0 < \omega < \omega_c$ one has $0 < g_0 < 1$.

A simple computation shows that

$$\mathbf{V}_3(f) \quad = \quad (cg\dot{g},\, cg\dot{g},\, 1 + c\frac{\alpha+2}{\alpha}(g^\alpha - g_0^\alpha) - c(g^2 - g_0^2),\, c(g^2 - g_0^2))^T.$$

Taking into account that for $\mathbf{V}_4$, $\dot{u}_1(0) = -1$, one can prove that

$$\mathbf{V}_4(f) \quad = \quad (-cg\dot{g},\, -cg\dot{g},\, -c\frac{\alpha+2}{\alpha}(g^\alpha - g_0^\alpha) + c(g^2 - g_0^2),\, 1 - c(g^2 - g_0^2))^T,$$

where c is given in (1.38). As g is T–periodic, $\mathbf{V}_3$ and $\mathbf{V}_4$ are also periodic with period T. Then,

$$\mathbf{V}_3(T) \quad = \quad (0,0,1,0)^T, \qquad \mathbf{V}_4(T) = (0,0,0,1)^T.$$

Now we look for $\mathbf{V}_2(f)$. The initial conditions $u_1(0) = 0$, $u_2(0) = 1$ imply that $k = -1$. Moreover, as $u_3(0) = 0$ and $u_4(0) = 0$, $\dot{u}_1(0) = 0$. So, we have to solve the equation (1.37) for $k = -1$ with initial conditions $u_1(0) = 0$, $\dot{u}_1(0) = 0$. Let us assume, for the moment being, that $u_1(T) = 0$. Therefore $u_2(T) = u_1(T) + 1 = 1$ and

$$\mathbf{V}_2(T) \quad = \quad (0, 1, u_3(T), u_4(T))^T.$$

The monodromy matrix has the following form

$$\mathcal{C} \;=\; \begin{pmatrix} * & 0 & 0 & 0 \\ * & 1 & 0 & 0 \\ * & * & 1 & 0 \\ * & * & 0 & 1 \end{pmatrix},$$

where $*$ denotes some values that are not relevant. However the Liouville Theorem implies that $\det(\mathcal{C}) = 1$. Therefore, 1 is an eigenvalue of $\mathcal{C}$ with multiplicity four.

Our purpose now is to prove that $u_1(T) = 0$.

We introduce variables $x_1 = u_1$, $x_2 = \dot{u}_1$ and write the equation (1.37) for $k = -1$ as a non homogeneous linear system. So, we consider the initial value problem

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= \tilde{h}(f)x_1 - 2, \qquad \tilde{h}(f) = [(\alpha+2)g^{\alpha-2} - 4], \\ x_1(0) &= 0, \quad x_2(0) = 0. \end{aligned} \tag{1.39}$$

We know that $\varphi_2(f) = (b_1(f), b_2(f))^T$ with

$$b_1(f) = cg\dot{g}, \qquad b_2(f) = c(\dot{g}^2 - g^2 + g^\alpha),$$

is a solution of the homogeneous system such that $\varphi_2(0) = (0,1)^T$. Moreover, it is periodic with period T. Then, $\varphi_2(T) = \varphi_2(0)$.

Let $\Phi(f)$ be the fundamental matrix of the homogeneous system such that $\Phi(0) = I_2$. We write

$$\Phi(f) \;=\; (\varphi_1(f), \varphi_2(f)),$$

where $\varphi_1(f) = (a_1(f), a_2(f))^T$ is a solution of the homogeneous system such that $\varphi_1(0) = (1,0)^T$. Using Liouville Theorem, $\det(\Phi(f)) = 1$ for any f. Then, $\det(\Phi(T)) = a_1(T) = 1$.

Let $\mathbf{x}(f) = (x_1(f), x_2(f))^T$ be the solution of the initial value problem (1.39). Using variation of parameters we can write

$$\mathbf{x}(f) \;=\; \Phi(f)\int_0^f \Phi^{-1}(s)\mathbf{r}(s)ds,$$

where $\mathbf{r}(s) = (0,-2)^T$. The first component of $\mathbf{x}(f)$ is

$$x_1(f) \;=\; a_1(f)\int_0^f 2b_1(s)ds - b_1(f)\int_0^f 2a_1(s)ds.$$

We recall that $b_1(T) = 0$. Then,

$$x_1(T) \;=\; 2\int_0^T b_1(s)ds = 2c\int_0^T g\dot{g}ds = c[g^2(T) - g^2(0)] = 0,$$

where the periodicity of $g(f)$ has been used. □

Let us consider now the linear system $\dot{\mathbf{U}} = \mathcal{B}_2(f)\mathbf{U}$. After Lemma 1.4.3 we know that the eigenvalues of the monodromy matrix of this system give us the non trivial characteristic multipliers.

We introduce $\mathbf{w} = M^{-1}\mathbf{U}$, where $M = \begin{pmatrix} I_2 & 0 \\ J_2 & I_2 \end{pmatrix}$, and then we can write our system as

$$\dot{\mathbf{w}} = B(f)\mathbf{w}, \qquad B(f) = \begin{pmatrix} 0 & I_2 \\ \tilde{B} & -2J_2 \end{pmatrix},$$
$$\tilde{B} = g^{\alpha-2}\begin{pmatrix} \gamma_{11}+1 & \gamma_{21} \\ \gamma_{12} & \gamma_{22}+1 \end{pmatrix}. \tag{1.40}$$

We recall that g depends on the angular momentum ω, and γ_{11}, γ_{12}, γ_{21}, γ_{22} are constant depending on the solutions of (1.19), that is, on the central configurations. In the following section we shall solve the equation (1.19), and we will compute these constants γ_{ij}, $i,j = 1,2$, for the collinear and triangular central configurations.

1.5 Central configurations

In this section we study the solutions of the equation (1.19), that is, we seek central configurations. Our purpose is to compute the coefficients $\gamma_{11}, \gamma_{12}, \gamma_{21}, \gamma_{22}$ for any solution $\mathbf{u}^*$ of (1.19). Moreover, we will study some properties of these coefficients that will be useful in chapter 4 in order to study the stability of the homographic solutions.

We write (1.19) as

$$\begin{aligned}
-\alpha\frac{m_1m_2(\mathbf{u}_1-\mathbf{u}_2)}{\|\mathbf{u}_1-\mathbf{u}_2\|^{\alpha+2}} - \alpha\frac{m_1m_3\mathbf{u}_1}{\|\mathbf{u}_1\|^{\alpha+2}} &= -m_1m_2(\alpha_2m_3\mathbf{u}_1-\mathbf{u}_2),\\
\alpha\frac{m_1m_2(\mathbf{u}_1-\mathbf{u}_2)}{\|\mathbf{u}_1-\mathbf{u}_2\|^{\alpha+2}} - \alpha\frac{m_2m_3\mathbf{u}_2}{\|\mathbf{u}_2\|^{\alpha+2}} &= -m_1m_2(-\mathbf{u}_1+\alpha_1m_3\mathbf{u}_2),
\end{aligned} \tag{1.41}$$

where we know from section 1.3 that $\alpha_1 = \dfrac{m_1+m_3}{m_1m_3}$ and $\alpha_2 = \dfrac{m_2+m_3}{m_2m_3}$.

We look for the solutions $\mathbf{u}^*$ of this system of equations.

First we study the case when the three particles lie on a straight line. This kind of solutions are known as **collinear configurations**. We can assume that this line is the abscissa axis and we consider the masses ordered from left to right as m_3, m_2, m_1 (see figure 1.5). Other collinear configurations are obtained from this by a permutation of the masses. So, we can assume $\mathbf{u}^* = ((\mathbf{u}_1^*)^T, (\mathbf{u}_2^*)^T)^T$ with $\mathbf{u}_1^* = (u_1, 0)^T$ and $\mathbf{u}_2^* = (u_2, 0)^T$. We introduce ρ and a by $u_1 = a(\rho+1)$, $u_2 = a$.

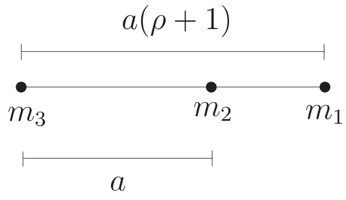


Figure 1.5: Coordinates u_1 and u_2 in a collinear configuration

If we substitute these expressions of $\mathbf{u}_1$ and $\mathbf{u}_2$ in (1.41) we obtain that $\rho > 0$ is the solution of the equation

$$\begin{aligned} & m_1[\rho^{\alpha+2} - (\rho+1)^{\alpha+2}] + m_3\rho^{\alpha+1}[(\rho+1)^{\alpha+2} - 1] + \\ & m_2(\rho+1)^{\alpha+1}(\rho^{\alpha+2} - 1) = 0, \end{aligned} \tag{1.42}$$

and

$$a^{\alpha+2} = \frac{\alpha[m_2(\rho+1)^{\alpha+1} + m_3\rho^{\alpha+1}]}{\rho^{\alpha+1}(\rho+1)^{\alpha+1}[m_3(\rho+1) + m_2\rho]}.$$

We note that $\rho = 1$ is the solution of (1.42) in the symmetric case in which $m_1 = m_3$. In the Newtonian case ($\alpha = 1$) (1.42) is the well known quintic equation for collinear configurations.

Now we look for non collinear solutions. We note that to solve the equation (1.19) is equivalent to compute the critical points of the function

$$\tilde{U}(\mathbf{u}) = \mathcal{U}(\mathbf{u}) + \frac{1}{2}\mathbf{u}^T C\mathbf{u}, \tag{1.43}$$

where $\mathcal{U}$ and C are given in (1.14) and (1.15), respectively. We define $r_1 = \|\mathbf{u}_1 - \mathbf{u}_2\|$, $r_2 = \|\mathbf{u}_1\|$, $r_3 = \|\mathbf{u}_2\|$. Let us see that $\tilde{U}$ can ve written in terms of r_1, r_2 and r_3. A simple computation shows that

$$\mathbf{u}^T C\mathbf{u} = m_1 m_2(\alpha_2 m_3\|\mathbf{u}_1\|^2 - 2\mathbf{u}_1^T\mathbf{u}_2 + \alpha_1 m_3\|\mathbf{u}_2\|^2).$$

As $2\mathbf{u}_1^T\mathbf{u}_2 = \|\mathbf{u}_1\|^2 + \|\mathbf{u}_2\|^2 - \|\mathbf{u}_1 - \mathbf{u}_2\|^2$ we obtain that

$$\tilde{U}(\mathbf{u}) \quad = \quad m_1 m_2 \left(\frac{1}{r_1^\alpha} + \frac{r_1^2}{2} \right) + m_1 m_3 \left(\frac{1}{r_2^\alpha} + \frac{r_2^2}{2} \right) + m_2 m_3 \left(\frac{1}{r_3^\alpha} + \frac{r_3^2}{2} \right).$$

It is easy to see that a non collinear critical point of $\tilde{U}$ satisfies $r_1 = r_2 = r_3 = \alpha^{\frac{1}{\alpha+2}}$. In this case, the three masses are at the vertices of an equilateral triangle and $\|\mathbf{u}_1\| = \|\mathbf{u}_2\| = \|\mathbf{u}_1 - \mathbf{u}_2\| = \varrho$ where $\varrho = \alpha^{\frac{1}{\alpha+2}}$. These solutions are known as **triangular configurations**. There are two triangular configurations (see figure 1.6). We shall work with the first case in this figure. The other triangular configuration is obtained from this by changing the masses m_1 and m_2. We can assume that $\mathbf{u}_1 = \left(\frac{\varrho}{2}, \frac{\sqrt{3}}{2}\varrho \right)^T$ and $\mathbf{u}_2 = \left(-\frac{\varrho}{2}, \frac{\sqrt{3}}{2}\varrho \right)^T$.

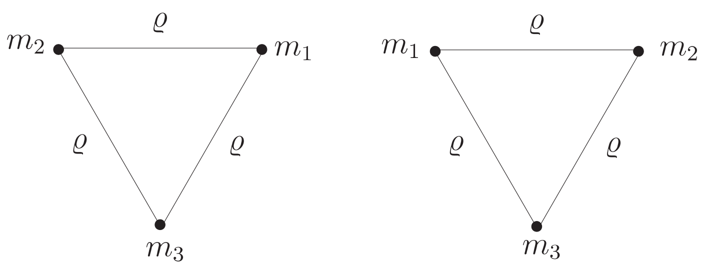


Figure 1.6: The triangular configurations in the Planar Three Body Problem with homogeneous potential of degree $-\alpha$, $0 < \alpha < 2$

We note that the three variables r_1, r_2 and r_3 are local coordinates near a non collinear configuration. Therefore, we have obtained all the non collinear solutions of the equation for central configurations of the Planar Three Body Problem with homogeneous potential of degree $-\alpha$, $0 < \alpha < 2$.

Now we want to compute the coefficients $\gamma_{11}, \gamma_{12}, \gamma_{21}, \gamma_{22}$ in (1.40). To this end we shall distinguish between a collinear and a triangular central configuration.

We fix $\mathbf{u}^* = ((\mathbf{u}_1^*)^T, (\mathbf{u}_2^*)^T)^T$, $\mathbf{u}_1^*, \mathbf{u}_2^* \in \mathbb{R}^2$ a central configuration. It is easy to check that $\|\boldsymbol{\eta}_1\|^2, \|\boldsymbol{\eta}_2\|^2$ and γ_1 in (1.27) can be computed through

$$\begin{aligned} \gamma_1 &= \frac{2(\mathbf{u}_1^*)^T J_2 \mathbf{u}_2^*}{\|\mathbf{u}_1^*\|^2 + \|\mathbf{u}_2^*\|^2}, \\ \|\boldsymbol{\eta}_1\|^2 &= [\|\mathbf{u}_1^*\|^2 + \|\mathbf{u}_2^*\|^2](1+\gamma_1^2) - 4\gamma_1 (\mathbf{u}_1^*)^T J_2 \mathbf{u}_2^*, \\ \|\boldsymbol{\eta}_2\|^2 &= \|\boldsymbol{\eta}_1\|^2. \end{aligned} \tag{1.44}$$

We begin with a triangular configuration. In this case, we have taken $\mathbf{u}_1 = \left(\frac{\varrho}{2}, \frac{\sqrt{3}}{2}\varrho\right)^T$ and $\mathbf{u}_2 = \left(-\frac{\varrho}{2}, \frac{\sqrt{3}}{2}\varrho\right)^T$. Using (1.44) we obtain that

$$\gamma_1 = \frac{\sqrt{3}}{2}, \qquad \|\boldsymbol{\eta}_1\|^2 = \|\boldsymbol{\eta}_2\|^2 = \frac{\varrho^2}{2}.$$

From the definition of $\boldsymbol{\eta}_1$ and $\boldsymbol{\eta}_2$ in (1.27) we have that these vectors can be written as $\boldsymbol{\eta}_1^T = (c_1, c_2, c_1, -c_2)$ and $\boldsymbol{\eta}_2^T = (c_2, -c_1, -c_2, -c_1)$, where $c_1 = \frac{\varrho}{4}$ and $c_2 = \frac{\sqrt{3}\varrho}{4}$. Moreover

$$D\nabla\mathcal{U}(\mathbf{u}^*) \quad = \quad \begin{pmatrix} a_1 & a_2 & a_3 & 0 \\ a_2 & a_4 & 0 & a_5 \\ a_3 & 0 & a_6 & a_7 \\ 0 & a_5 & a_7 & a_8 \end{pmatrix}, \tag{1.45}$$

where a_j, $j = 1, \ldots, 8$ are constants depending on the masses and on ϱ. They are given in appendix B. Then after some trivial computations and using (1.30) we get the constants γ_{ij} for the triangular configurations as

$$\begin{aligned} \gamma_{11} &= -1 + \frac{(\alpha+2)}{4}(m_1 + m_2 + 4m_3), \qquad \gamma_{22} = \alpha - \gamma_{11}, \\ \gamma_{12} &= \gamma_{21} = \frac{\alpha+2}{4}\sqrt{3}(m_2 - m_1). \end{aligned}$$

So, $\tilde{B}$ in (1.40) is a symmetric matrix.

Our purpose now it to write the linearized system on a triangular configuration as the system

$$\dot{\mathbf{x}} \quad = \quad A(f)\mathbf{x}, \tag{1.46}$$

with

$$A(f) = \begin{pmatrix} 0 & I_2 \\ \tilde{A}(f) & -2J_2 \end{pmatrix}, \qquad \tilde{A} = g^{\alpha-2}\Lambda \tag{1.47}$$

where Λ is a 2×2 diagonal constant matrix. To this end, it will be useful the following remark.

Remark 1.5.1. Let us consider the system

$$\dot{\mathbf{w}} \quad = \quad B(f)\mathbf{w}, \tag{1.48}$$

with $B(f) = \begin{pmatrix} 0 & I_2 \\ \tilde{B}(f) & -2J_2 \end{pmatrix}$ and $\tilde{B} = g^{\alpha-2}\tilde{\Lambda}$ where $\tilde{\Lambda}$ is a 2×2 symmetric but not diagonal constant matrix. Let P be an orthogonal matrix such that $P^{-1}\tilde{\Lambda}P$ is diagonal. We introduce new variables as $\mathbf{X} = A_1^{-1}\mathbf{w}$ with $A_1 = \mathrm{diag}(P,P)$. Then, (1.48) becomes

$$\dot{\mathbf{X}} = B_1(f)\mathbf{X},$$

with $B_1(f) = A_1^{-1}B(f)A_1$. We have that

$$B_1(f) = \begin{pmatrix} 0 & I_2 \\ \tilde{A}(f) & -2P^{-1}J_2P \end{pmatrix}$$

where $\tilde{A}(f) = g^{\alpha-2}P^{-1}\tilde{\Lambda}P$ is like in (1.47). Due to the orthogonality of P it is easy to check that $P^{-1}J_2P = \pm J_2$, where the sign $+$ stands for $\det(P) = 1$ and $-$ for $\det(P) = -1$. Then, $B_1(f) = \begin{pmatrix} 0 & I_2 \\ \tilde{A}(f) & -2cJ_2 \end{pmatrix}$ with $c = \pm 1$.

We define $\mathbf{x} = (x_1, x_2, x_3, x_4)^T$ by $x_1 = X_1$, $x_2 = X_2$, $X_3 = \dfrac{x_3}{c}$, $X_4 = \dfrac{x_4}{c}$. Scaling the time by a factor of c the system in variables $\mathbf{x}$ is written as (1.46).

After this remark, we can write the linearized system on a triangular configuration as (1.46) by taking $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2)$ being λ_1, λ_2 the zeroes of

$$p(\lambda) = \lambda^2 - (\alpha+2)\lambda + \frac{\beta_t}{4}, \qquad \beta_t = 3\kappa(\alpha+2)^2, \tag{1.49}$$

where $\kappa = m_1m_2 + m_1m_3 + m_2m_3$. Then, we have that

$$\lambda_1 + \lambda_2 = \alpha + 2 \quad \text{and} \quad \lambda_1\lambda_2 = \frac{\beta_t}{4}. \tag{1.50}$$

Remark 1.5.2. We have that $3\kappa \leq 1$ is satisfied for all positive values of the masses in the triangle of masses. Moreover, the equality holds for $m_1 = m_2 = m_3$.

After this remark we have that the the zeroes of (1.49) are real. Moreover, it is clear that $\beta_t \in (0, (\alpha+2)^2]$.

Now we study the collinear central configurations. For $\mathbf{u}^*$ a collinear configuration we have that $(\mathbf{u}_1^*)^T J_2 \mathbf{u}_2^* = 0$. Then, from (1.44) we have $\gamma_1 = 0$ and from (1.27), $\boldsymbol{\eta}_1 = J_4\mathbf{u}^*$, $\boldsymbol{\eta}_2 = K_4\boldsymbol{\eta}_1$. In order to compute the coefficients γ_{ij}, $i,j = 1,2$, we need to know the matrix $D\nabla\mathcal{U}(\mathbf{u}^*)$. It turns out that

$$D\nabla\mathcal{U}(\mathbf{u}^*) = \begin{pmatrix} a_1 & 0 & a_2 & 0 \\ 0 & a_3 & 0 & a_4 \\ a_2 & 0 & a_5 & 0 \\ 0 & a_4 & 0 & a_6 \end{pmatrix},$$

for some constants a_j, $j = 1, \dots, 6$, that depend on m_1, m_2, m_3 and ρ. From this expression it is easy to prove that $\gamma_{12} = \gamma_{21} = 0$. Therefore, the 2×2 matrix $\tilde{B}$ in (1.40) is diagonal. Moreover, after some computations we get the following expressions

$$\begin{aligned}\gamma_{11} &= -(\alpha+1)\gamma_{22},\\ \gamma_{22} &= -1-\beta_c,\end{aligned}$$

where

$$\begin{aligned}\beta_c &= -1 - \frac{\alpha}{a^{\alpha+2}[1+(\rho+1)^2]}[(\rho+2)[(\rho+1)m_1+m_2]\rho^{-\alpha-2}+\\ &\quad (\rho+1)[m_2\rho+m_3(\rho+1)]+(m_3-m_1\rho)(\rho+1)^{-\alpha-2}]. \qquad (1.51)\end{aligned}$$

Therefore,

$$\tilde{B} = g^{\alpha-2}\begin{pmatrix}(\alpha+1)\beta_c+\alpha+2 & 0\\ 0 & -\beta_c\end{pmatrix}.$$

Now we study some properties of the function β_c. We note that it depends on the three masses and on the solution of (1.42).

We consider the Newtonian case, that is, $\alpha = 1$. Then, equation (1.42) is the well–known Euler's quintic equation and can be written as

$$\begin{aligned}q(\rho) &:= \rho^5(m_2+m_3)+\rho^4(2m_2+3m_3)+\rho^3(m_2+3m_3)-\rho^2(3m_1+m_2)-\\ &\quad -\rho(3m_1+2m_2)-(m_1+m_2)=0. \qquad (1.52)\end{aligned}$$

Moreover, $a^3 = -\dfrac{m_1}{\rho^2}+\dfrac{m_1}{(\rho+1)^2}+m_2+m_3$ and

$$\beta_c = \frac{m_1\left(1+\rho_1^{-1}+\rho_1^{-2}\right)+m_3\left(1+\rho_2^{-1}+\rho_2^{-2}\right)}{m_1+m_2\left(\rho_1^{-2}+\rho_2^{-2}\right)+m_3}, \qquad (1.53)$$

where

$$\rho_1 = \frac{\rho}{\rho+1} \quad \text{and} \quad \rho_2 = \frac{1}{\rho+1}, \qquad (1.54)$$

(see [S.M.]).

Remark 1.5.3. As we have assumed that $m_1+m_2+m_3 = 1$, fixed $m_2 \in (0,1)$ we can take β_c in (1.53) as a function of m_1 and m_3. Then, if we write $\beta_c = \beta_c(m_1, m_3)$ we obtain that $\beta_c(m_1, m_3) = \beta_c(m_3, m_1)$ and then β_c is symmetric with respect to $m_1 = m_3$. To see this, fixed $m_2 \in (0,1)$ it is only necessary to take into account that if $\rho(m_1, m_3)$ denotes the solution of the quintic (1.52) then $\rho(m_3, m_1) = \dfrac{1}{\rho(m_1, m_3)}$ where $\rho(m_3, m_1)$ is obtained from (1.52) by changing

m_1 by m_3 and viceversa. Then, the expressions of ρ_1 and ρ_2 in (1.54) satisfy $\rho_1(m_1, m_3) = \rho_2(m_3, m_1)$ and $\rho_2(m_1, m_3) = \rho_1(m_3, m_1)$. Therefore, $\beta_c(m_1, m_3) = \beta_c(m_3, m_1)$. We note that in the triangle of masses the values such that $m_1 = m_3$ correspond to the height l of the triangle (see figure 1.7).

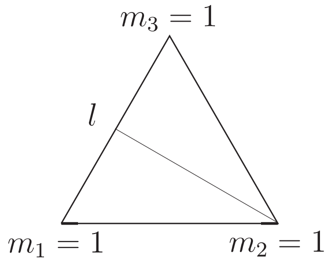


Figure 1.7: Line $m_1 = m_3$ in the triangle of masses

We note that $q(1) = 7(m_3 - m_1)$ and therefore, if $m_1 \leq m_3$ we have that the solution of the quintic equation satisfies $\rho \leq 1$ and the solution is $\rho = 1$ if and only if $m_1 = m_3$. Moreover, $q(0) = -(m_1 + m_2)$ and $\rho = 0$ if and only if $m_1 = m_2 = 0$.

For this problem we know the ranges of β_c (see [M.S.]).

Lemma 1.5.4. *In the Newtonian case, for any positive masses,* $\beta_c \in (0, 7)$. *The values* 0 *and* 7 *are attained in the limit cases* $m_2 = 1$ *and* $m_2 = 0$, $m_1 = m_3 = \dfrac{1}{2}$, *respectively.*

Proof

We fix $m_1 \in (0, 1)$ and we take $m_3 = 1 - m_1 - m_2$. We have that $m_2 \in (0, 1 - m_1)$. From (1.53) we obtain

$$\beta_c = \frac{m_1(3\rho^2 + 3\rho + 1)(1 - m_1 - m_2)\rho^2(\rho^2 + 3\rho + 3)}{\rho^2 + m_2[(\rho + 1)^2(1 + \rho^2) - \rho^2]} \quad .$$

As ρ is a solution of (1.52) we can consider ρ as a function of m_2. Therefore β_c is a function of m_2.

It has been proved that $\dfrac{d\beta_c}{dm_2}(m_2) < 0$ (see [M.S.]). Then, $\beta_c(m_2)$ is a strictly decreasing function of m_2.

Due to the symmetry of β_c with respect to the line $m_1 = m_3$ in the triangle of masses, it is only necessary to consider $m_1 \leq m_3$, that is, $\rho \in (0, 1]$. Therefore, fixed a value of m_1 the monotonicity of β_c implies that its maximum is taken when $m_2 = 0$ and the minimum at $m_1 = m_3$. In the case $m_1 = m_3$ we have that $\rho = 1$ and then $\beta_c = \dfrac{7(1 - m_2)}{1 + 7m_2}$. The minimum of this function is attained when $m_2 = 1$. In this case, $\beta_c = 0$.

For $m_2 = 0$ the quintic equation (1.52) can be written as $m_1(3\rho^2 + 3\rho + 1) = m_3\rho^3(\rho^2 + 3\rho + 3)$, and $\beta_c = (1 - m_1)(1 + \rho)(\rho^2 + 3\rho + 3)$. Using these equalities we obtain that

$$\beta_c = \frac{(\rho^2 + 3\rho + 3)(3\rho^2 + 3\rho + 1)}{\rho^4 + 2\rho^3 + \rho^2 + 2\rho + 1}.$$

It is easy to check that $\beta_c \leq 7$ and $\beta_c = 7$ if and only if $\rho = 1$. Then, the maximum in the triangle of masses is 7. This ends the proof of the lemma. □

Remark 1.5.5. Numerically it can be seen that fixed $m_2 \in (0, 1)$, $\beta_c(m_1, m_3)$ has a maximum at $m_1 = m_3$. Figure 1.8 shows the numeric behaviour of function $\beta_c(m_1, m_3)$ taking into account that $\beta_c(m_1, m_3) = \beta_c(m_1)$.

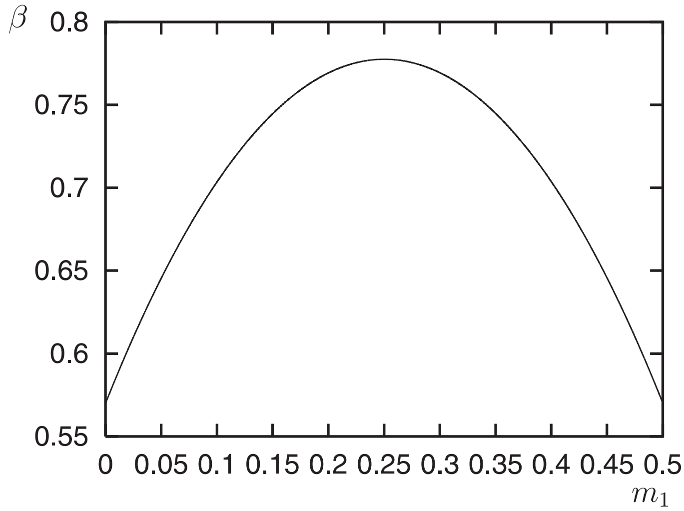


Figure 1.8: Graphic of the function β_c in terms of m_1 for $m_2 = 0.5$

Now we consider the general case. Numerical computations shows that, fixed $m_2 \in (0, 1)$, the maximum of β_c is also attained at $m_1 = m_3$ (see figure 1.9). Moreover, numerically it can be seen that β_c is a decreasing function of m_2. Then, the maximum is attained at $m_1 = m_3$ and $m_2 = 0$. For these values of the masses the solution of (1.42) is $\rho = 1$ and $\beta_c = 2^{2+\alpha} - 1$. Therefore, we shall assume that $\beta_c \in (0, 2^{\alpha+2} - 1)$.

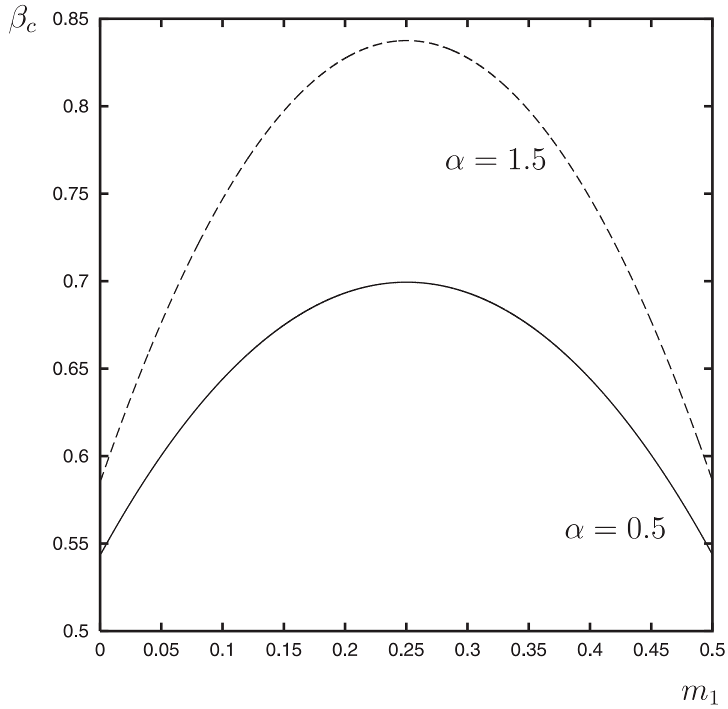


Figure 1.9: Graphic of β_c in terms of m_1 for $m_2 = 0.5$ and $\alpha = 0.5$, $\alpha = 1.5$

1.6 About the study of the stability of the homographic solutions

In this section we shall see that the linearized system on an homographic solution can be included in a three parametric family of four dimensional periodic linear systems. This family can be written in a Hamiltonian formulation. Using the theory of Hamiltonian systems (see [M.H.]), we describe briefly how to study the stability parameters of these differential equations.

We consider the periodic linear system

$$\dot{\mathbf{x}} \quad = \quad A(f,e)\mathbf{x}, \tag{1.55}$$

with $\mathbf{x} \in \mathbb{R}^4$,

$$A(f,e) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \lambda_1 G_1(f,e) & 0 & 0 & -2 \\ 0 & \lambda_2 G_2(f,e) & 2 & 0 \end{pmatrix},$$

where λ_1, λ_2 are real parameters different from zero, $e \in [0,1)$, and $G_1(f,e)$, $G_2(f,e)$ are smooth functions. Let us assume that for $e = 0$, $G_1(f,0) = 1$, $G_2(f,0) = 1$.

We want to see that the linearized system on an homographic solution can be written as (1.55).

First we note that, as we have seen in last section, system (1.40) can be written as

$$\dot{\mathbf{x}} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ g^{\alpha-2}\lambda_1 & 0 & 0 & -2 \\ 0 & g^{\alpha-2}\lambda_2 & 2 & 0 \end{pmatrix} \mathbf{x}, \tag{1.56}$$

where λ_1, λ_2 depends on the central configuration we are considering and g is a periodic solution of (1.31). Table 1.1 shows the expression of λ_1 and λ_2 in both collinear and triangular case. We distinguish between the general and the Newtonian case.

	Newtonian case	General case
Triangular configuration	λ_1, λ_2 zeroes of $p(\lambda) = \lambda^2 - 3\lambda + \frac{\beta_t}{4}$, $\beta_t = 27\kappa$	λ_1, λ_2 zeroes of $p(\lambda) = \lambda^2 - (\alpha+2)\lambda + \frac{\beta_t}{4}$ $\beta_t = 3(\alpha+2)^2\kappa$
Collinear configuration	$\lambda_1 = 2\beta_c + 3$ $\lambda_2 = -\beta_c$ $\beta_c \in (0, 2^{\alpha+2} - 1)$	$\lambda_1 = (\alpha+1)\beta_c + \alpha + 2$ $\lambda_2 = -\beta_c$ $\beta_c \in (0,7)$

Table 1.1: Values of λ_1, λ_2 for a triangular and a collinear central configuration.

We define a generalized eccentricity

$$e := \sqrt{1 + \frac{2\alpha}{2-\alpha} E_K \omega^{\frac{2\alpha}{2-\alpha}}},$$

where E_K is defined in (1.12). We recall that we have normalized scales so that $E_K = -\frac{1}{2}$ and ω is such that $0 < \omega \le \omega_c$ where ω_c is defined in (1.34). Then,

$$e = \sqrt{1 - \frac{\alpha}{2-\alpha}\omega^{\frac{2\alpha}{2-\alpha}}}, \qquad e \in [0,1). \tag{1.57}$$

Given a value of ω, we have that E is fixed by (1.33) and so, g depends on e.

In the Newtonian case ($\alpha = 1$), e is the eccentricity of the orbit. From remark 1.4.2 we know that the case $\omega = \omega_c$ corresponds to the relative equilibrium solution. For this value of ω we have that $e = 0$. On the other hand, we have seen in section 1.4 that when ω tends to 0 then g_0 tends to 0. So, for $e = 1$ system (1.56) has a singularity.

Then, it is clear that system (1.56) can be written as (1.55) by taking $G_1(f, e) = g^{\alpha-2}\lambda_1$ and $G_2(f, e) = g^{\alpha-2}\lambda_2$, where λ_1 and λ_2 are given in table 1.1.

System (1.55) can be written as a linear Hamiltonian system by the change of variables $\mathbf{y} = \tilde{M}\mathbf{x}$ with

$$\tilde{M} = \begin{pmatrix} I_2 & 0 \\ J_2 & I_2 \end{pmatrix}. \tag{1.58}$$

The Hamiltonian function is

$$H(\mathbf{y}, f) = \frac{1}{2}(y_3^2 + y_4^2) + y_1 y_4 - y_2 y_3 - V(y_1, y_2, f, e), \tag{1.59}$$

where $\mathbf{y} = (y_1, y_2, y_3, y_4)^T$ and

$$V(y_1, y_2, f, e) = [\lambda_1 G_1(f, e) - 1]\,\frac{y_1^2}{2} + [\lambda_2 G_2(f, e) - 1]\,\frac{y_2^2}{2}. \tag{1.60}$$

Let $\Phi(f)$ the fundamental matrix of (1.55) such that $\Phi(0) = I_4$. It is easy to check that

$$\Phi(f) = \tilde{M}^{-1}\Phi_1(f)\tilde{M}, \tag{1.61}$$

where $\Phi_1(f)$ is the fundamental matrix of the linear Hamiltonian system defined by (1.59). The symplectic character of $\Phi_1(f)$ implies that if $\tilde{\mu}$ is an eigenvalue of $\Phi(T)$ then $\tilde{\mu}^{-1}$ is also an eigenvalue (see [M.H.]). We denote by $\mu_1, \mu_1^{-1}, \mu_2, \mu_2^{-1}$ the eigenvalues of $\Phi(T)$ and define the stability parameters as

$$tr_i = \mu_i + \mu_i^{-1}, \qquad i = 1, 2.$$

The stability of the system (1.55) can be studied in terms of tr_j, $j = 1, 2$. From the real character of (1.55), if $\mathrm{tr}_j \in \mathbb{C} \setminus \mathbb{R}$ for some $j = 1, 2$ then $\mathrm{tr}_{3-j} = \overline{\mathrm{tr}}_j$ where the bar stands for the complex conjugate. Therefore (1.55) is complex–saddle. We assume that $\mathrm{tr}_j \in \mathbb{R}$ for some $j = 1, 2$. If $|\mathrm{tr}_j| > 2$ then $\mu_j \in \mathbb{R}$ and one of the characteristic multipliers has modulus bigger that 1. In this case, (1.55) is unstable. If $|\mathrm{tr}_j| < 2$ then $\mu_j \in \mathbb{C}$ with $\|\mu_j\| = 1$. Bifurcations can occur when $|\mathrm{tr}_j| = 2$ for some $j = 1, 2$, and when $\mathrm{tr}_1 = \mathrm{tr}_2$.

In chapters 2 and 3 we shall study the stability parameters of the system (1.55) for a family of functions G_j, $j = 1, 2$. In chapter 2 we will analize the case $e \gtrsim 0$ and in chapter 3 the case $e \lesssim 1$ will be considered. In this last case, we shall introduce a parameter δ defined by $\delta = \dfrac{2-\alpha}{2\alpha}(1-e^2)$ and we will study the system for $\delta > 0$ small enough. Using the theory developed in these chapters, in chapter 4 we shall determine the stability parameters of the homographic solutions.

Chapter 2

Stability of a family of periodic linear systems: the perturbative case

2.1 Introduction

Let us consider the family of periodic linear systems

$$\dot{\mathbf{x}} = A(t,\varepsilon)\mathbf{x}, \tag{2.1}$$

with $\mathbf{x} \in \mathbb{R}^4$,

$$A(t,\varepsilon) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \lambda_1 G_1(t,\varepsilon) & 0 & 0 & -2 \\ 0 & \lambda_2 G_2(t,\varepsilon) & 2 & 0 \end{pmatrix},$$

λ_1, λ_2 are real parameters different from zero, ε is a small positive parameter, and $G_i(t,\varepsilon) = 1 - F_i(t,\varepsilon)$, $i = 1,2$, where $F_i(t,\varepsilon)$, $i = 1,2$, are even functions, T–periodic in t and analytic in ε, satisfying $F_i(t,0) = 0$. Then, we can write

$$F_1(t,\varepsilon) = \sum_{j\in\mathbb{N}} \varepsilon^j c_j(t), \quad F_2(t,\varepsilon) = \sum_{j\in\mathbb{N}} \varepsilon^j d_j(t), \tag{2.2}$$

with $c_j(t), d_j(t)$ T–periodic even functions for $j \in \mathbb{N}$.

If $\varepsilon = 0$ then system (2.1) is linear with constant coefficients and one can obtain easily the stability and instability regions in the plane (λ_1, λ_2). These regions are described in section 2.2. The goal of this chapter is to study the bifurcations for ε small and positive.

In chapter 1 we have seen that (2.1) can be written as a linear Hamiltonian system with Hamiltonian function

$$H(\mathbf{y},t) \quad = \quad \frac{1}{2}(y_3^2+y_4^2)+y_1y_4-y_2y_3-V(y_1,y_2,t,\varepsilon), \tag{2.3}$$

where $\mathbf{y}=(y_1,y_2,y_3,y_4)^T$ and

$$V(y_1,y_2,t,\varepsilon) \quad = \quad [\lambda_1 G_1(t,\varepsilon)-1]\,\frac{y_1^2}{2}+[\lambda_2 G_2(t,\varepsilon)-1]\,\frac{y_2^2}{2}. \tag{2.4}$$

The analysis of system (2.1) has several applications. One of them is the study of the stability for the equilibria of mechanical systems defined by a Hamiltonian function of the form (2.3) with a potential $\mathcal{V}(y_1,y_2,t,\varepsilon)$ even in t and such that the quadratic part in y_1 and y_2 has the form (2.4). In this case, the linearized system on the equilibrium point can be written as (2.1).

On the other hand, system (2.1) can be obtained as a first variational system on a periodic solution of an autonomous system. As we have seen in chapter 1, one example is given by the homographic solutions of the planar three body problem with homogeneous potential of order $-\alpha$, with $0<\alpha<2$, since after some reductions the linear stability of these orbits is given by the study of the non–autonomous linear system (1.40), that has the form (2.1). This application is developed in chapter 4.

As we have seen in chapter 1 the fundamental matrices of (2.1) and of the system associated to (2.3) have the same characteristic multipliers. Then, it is equivalent to study the linear stability of the two systems and we can apply the stability theory for Hamiltonian systems (see [M.H.]) to the system (2.1).

Using this theory, in order to study the linear stability of the system (2.1) it is only necessary to compute the stability parameters $\mathrm{tr}_j=\mu_j+\mu_j^{-1}$, $j=1,2$, where μ_j,μ_j^{-1}, $j=1,2$, are the characteristic multipliers of the system.

If $\mathrm{tr}_1,\mathrm{tr}_2\in\mathbb{C}\setminus\mathbb{R}$, then $\mathrm{tr}_2=\overline{\mathrm{tr}_1}$ and (2.1) is a complex–saddle.

Assume that $\mathrm{tr}_1,\mathrm{tr}_2\in\mathbb{R}$. If $|\mathrm{tr}_1|<2, |\mathrm{tr}_2|<2$, then (2.1) is elliptic–elliptic, if $|\mathrm{tr}_1|>2, |\mathrm{tr}_2|>2$, hyperbolic–hyperbolic and if one of the stability parameters has absolute value less than two and the other bigger than two then (2.1) is elliptic–hyperbolic.

We note that in our case the stability parameters depend on $\lambda_1,\lambda_2,\varepsilon$.

Now we explain briefly the methodology used in order to study the bifurcations of (2.1) for $\varepsilon>0$ small enough.

Definition 2.1.1. We say that $(\lambda_1,\lambda_2)=(a_1,a_2)$ is a **resonant point** for $\varepsilon=\varepsilon_0$ of the system (2.1) if for $(\lambda_1,\lambda_2,\varepsilon)=(a_1,a_2,\varepsilon_0)$, $|\mathrm{tr}_j|=2$ for some $j=1,2$ or $\mathrm{tr}_1=\mathrm{tr}_2$.

Let us consider $(\lambda_1, \lambda_2) = (a_1, a_2)$ a resonant point for $\varepsilon = 0$. Our purpose is to study $\mathrm{tr}_1, \mathrm{tr}_2$ in a neighbourhood of (a_1, a_2) for $\varepsilon > 0$ small enough. To this end, we introduce small parameters $\delta_1, \delta_2 \in \mathbb{R}$ and we shall consider $\lambda_j = a_j + \delta_j$, $j = 1, 2$.

We will apply the Normal Form techniques (see [B.S.2]) in order to detect changes in the stability. The idea is to perform some canonical transformations to cancel the time dependence up to high order in $\delta_1, \delta_2, \varepsilon$, if this is possible. In fact, we shall obtain the Floquet matrix up to a fixed order in $\delta_1, \delta_2, \varepsilon$.

The analysis of this Normal Form will give us the linear stability as well as the boundaries of the stability regions.

2.2 The case $\varepsilon = 0$

In this section we study the stability parameters of system (2.1) for $\varepsilon = 0$.

For $\varepsilon = 0$ we obtain the linear system with constant coeficients $\dot{\mathbf{x}} = \tilde{A}_0 \mathbf{x}$ where

$$\tilde{A}_0 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \lambda_1 & 0 & 0 & -2 \\ 0 & \lambda_2 & 2 & 0 \end{pmatrix}.$$

The characteristic polynomial of $\tilde{A}_0$ is

$$p(x) \quad = \quad x^4 - (\lambda_1 + \lambda_2 - 4)x^2 + \lambda_1 \lambda_2. \tag{2.5}$$

We note that a zero ρ of (2.5) satisfies

$$2\rho^2 \quad = \quad \lambda_1 + \lambda_2 - 4 \pm \sqrt{(\lambda_1 + \lambda_2 - 4)^2 - 4\lambda_1\lambda_2}.$$

The stability parameters have different character depending on the region of the plane (λ_1, λ_2) considered. Therefore, we distinguish on this plane the following regions

$$\begin{aligned}
\mathcal{R}_1 &= \{(\lambda_1, \lambda_2) \in \mathbb{R}^2 | \lambda_1\lambda_2 < 0\}, \\
\mathcal{R}_2 &= \{(\lambda_1, \lambda_2) \in \mathbb{R}^2 | \lambda_1\lambda_2 > 0, (\lambda_1 + \lambda_2 - 4)^2 > 4\lambda_1\lambda_2, \lambda_1 + \lambda_2 - 4 < 0\}, \\
\mathcal{R}_3 &= \{(\lambda_1, \lambda_2) \in \mathbb{R}^2 | \lambda_1\lambda_2 > 0, (\lambda_1 + \lambda_2 - 4)^2 < 4\lambda_1\lambda_2\}, \\
\mathcal{R}_4 &= \{(\lambda_1, \lambda_2) \in \mathbb{R}^2 | \lambda_1\lambda_2 > 0, (\lambda_1 + \lambda_2 - 4)^2 > 4\lambda_1\lambda_2, \lambda_1 + \lambda_2 - 4 > 0\}.
\end{aligned}$$

The boundaries of these regions are given by the coordinate axes and for the graphs of the functions $\lambda_2 = (\sqrt{\lambda_1} \pm 2)^2$. Figure 2.1 shows these regions and their boundaries in the plane (λ_1, λ_2).

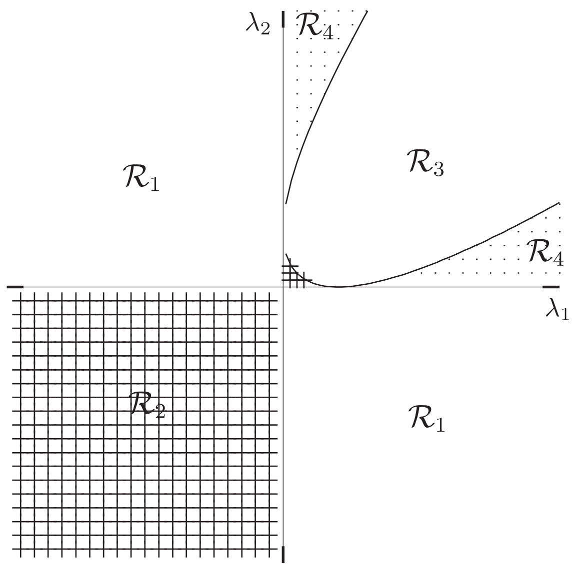


Figure 2.1: Regions $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3, \mathcal{R}_4$ in the plane (λ_1, λ_2).

In $\mathcal{R}_1$, (2.5) has zeroes $\pm\alpha, \pm i\omega$ with $\alpha, \omega \in \mathbb{R}^+$. Therefore, $\mu_1 = e^{\alpha T}$ and $\mu_2 = e^{i\omega T}$. Then we have that $\mathrm{tr}_1 > 2$ and $|\mathrm{tr}_2| \leq 2$. Possible bifurcations for $\varepsilon > 0$ will be for $|\mathrm{tr}_2| = 2$.

In $\mathcal{R}_2$ the eigenvalues of $\tilde{A}_0$ have the form $\pm i\omega_1, \pm i\omega_2$ with $\omega_1, \omega_2 \in \mathbb{R}^+$ and $\omega_1 > \omega_2$. Therefore, $\mu_1 = e^{i\omega_1 T}$ and $\mu_2 = e^{i\omega_2 T}$, fact that implies $|\mathrm{tr}_1|, |\mathrm{tr}_2| \leq 2$. If $|\mathrm{tr}_1| = 2$ or $|\mathrm{tr}_2| = 2$ or $\mathrm{tr}_1 = \mathrm{tr}_2$, then bifurcations can be found for $\varepsilon > 0$.

In the region $\mathcal{R}_3$ the zeroes of (2.5) are $\pm\alpha \pm i\beta$ with $\alpha, \beta \in \mathbb{R}^+$. Therefore, the characteristic multipliers are $\mu_1 = e^{(\alpha+i\beta)T}$, $\mu_2 = e^{(\alpha-i\beta)T}$ and their inverses. We note that if $\beta T \neq n\pi$ for all $n \in \mathbb{N}$ then the stability parameters are complex, and in the other case are real. In this last case a bifurcation can occur since $\mathrm{tr}_1 = \mathrm{tr}_2$.

In $\mathcal{R}_4$, (2.5) has zeroes $\pm\alpha_1, \pm\alpha_2$, $\alpha_1, \alpha_2 \in \mathbb{R}^+$, $\alpha_1 \neq \alpha_2$. Therefore, $\mu_1 = e^{\alpha_1 T}$ and $\mu_2 = e^{\alpha_2 T}$. In this case, $|\mathrm{tr}_j| > 2$ for $j = 1, 2$ and no bifurcation can occur for $\varepsilon > 0$ small enough.

Now we study the stability parameters in the boundaries of these regions.

On the axis λ_1 one stability parameter is equal to two, and the other is $2\cos(\sqrt{4-\lambda_1}T)$ if $\lambda_1 < 4$ and bigger than 2 if $\lambda_1 > 4$. We obtain a symmetric behaviour on the axis λ_2.

If $\lambda_2 = (\sqrt{\lambda_1} - 2)^2$ then $\mathrm{tr}_1 = \mathrm{tr}_2$. In this case, if $0 < \lambda_1 < 4$ then $|\mathrm{tr}_1| = |\mathrm{tr}_2| \leq 2$ and $\mathrm{tr}_1 = \mathrm{tr}_2 > 2$ if $\lambda_1 > 4$. On $\lambda_2 = (\sqrt{\lambda_1} + 2)^2$, we obtain $\mathrm{tr}_1 = \mathrm{tr}_2 > 2$ if $\lambda_1 \neq 0$.

The points $(4,0), (0,4)$ in the plane (λ_1, λ_2) correspond to degenerate cases in which 1 is a characteristic multiplier with multiplicity 4. Therefore, on these points we have $\mathrm{tr}_1 = \mathrm{tr}_2 = 2$.

We are interested in the study of the bifurcations for values of (λ_1, λ_2) in the region $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_3$.

Now we study the curves in the plane $(\lambda_1, \lambda_2) \in \mathcal{R}$ formed for resonant points when $\varepsilon = 0$. We will refer to these curves as **resonant curves**.

We know that in $\mathcal{R}_1 \cup \mathcal{R}_2$ one has a resonant curve if $\omega T = n\pi$, $n \in \mathbb{N}$. We note that $p(i\omega) = \omega^4 + (\lambda_1 + \lambda_2 - 4)\omega^2 + \lambda_1\lambda_2$. Then the resonant curve is given by

$$(\lambda_1 + \omega^2)(\lambda_2 + \omega^2) = 4\omega^2, \quad \omega = \frac{n\pi}{T}, \quad n \in \mathbb{N}. \tag{2.6}$$

In $\mathcal{R}_1$ we obtain a uniparametric family of resonant curves indexed by $n \in \mathbb{N}$. However, in $\mathcal{R}_2$ there are two uniparametric families of resonant curves of this type corresponding to ω_1 and ω_2, respectively. For one of them, $n \in \mathbb{N}$. The other family is defined for $n > \dfrac{2T}{\pi}$, $n \in \mathbb{N}$ if $\lambda_1 < 0$, and $n < \dfrac{2T}{\pi}$, $n \in \mathbb{N}$ if $\lambda_1 > 0$.

In $\mathcal{R}_2$ there are also resonant points for $\varepsilon = 0$ when $\omega_1 \pm \omega_2 = \dfrac{2n\pi}{T}$, $n \in \mathbb{N}$, which correspond to the case when $\mathrm{tr}_1 = \mathrm{tr}_2$. The corresponding resonant curves are

$$\lambda_2 \;=\; \lambda_1 + 4\left(1 - \frac{n^2\pi^2}{T^2}\right) \pm 4\sqrt{\lambda_1\left(1 - \frac{n^2\pi^2}{T^2}\right)}. \tag{2.7}$$

If $\lambda_1 > 0$, this uniparametric family of resonant curves is indexed by $n \leq \frac{T}{\pi}$, $n \in \mathbb{N}$, and if $\lambda_1 < 0$, the index are given by $n \geq \frac{T}{\pi}$, $n \in \mathbb{N}$.

Finally, if $(\lambda_1, \lambda_2) \in \mathcal{R}_3$, the resonant points are given when $T\beta = n\pi$ with $n \in \mathbb{N}$. A simple computation shows that the possible bifurcations in $\mathcal{R}_3$ are given on the uniparametric family of curves

$$\lambda_2 \;=\; (\sqrt{\lambda_1} \pm 2\sqrt{1 - \beta^2})^2, \quad \beta = \frac{n\pi}{T}, \tag{2.8}$$

for $n \leq \dfrac{T}{\pi}$, $n \in \mathbb{N}$.

In figure 2.2 there are some examples of resonant curves in the different regions.

The red curves are of the form (2.6) and the green curves corresponds to both curves in (2.7) or (2.8), depending on the region they are. We note that the curves (2.8) in the parameter space (λ_1, λ_2) are exactly the same as (2.7).

In region $\mathcal{R}_1$ there are only curves of the form (2.6) and there are no intersection between these curves. Then, in this region we only have single resonances.

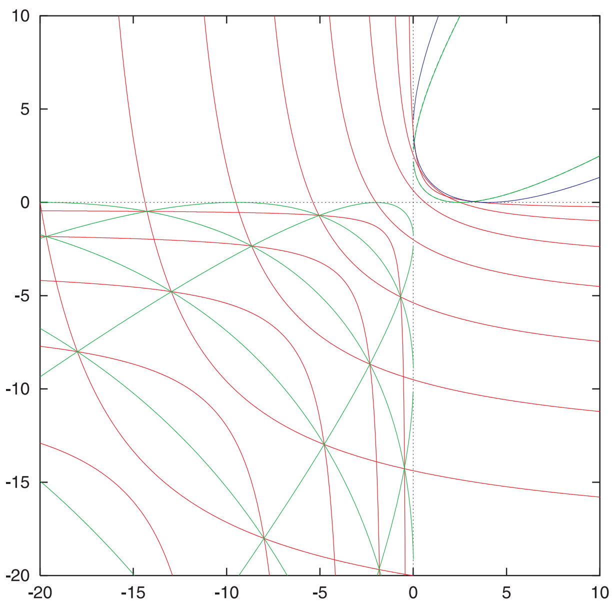


Figure 2.2: Some resonant curves in the plane (λ_1, λ_2) for $T = \dfrac{2\pi}{\sqrt{2-\alpha}}$ with $\alpha = 0.5$. Color codes: Red for $|\mathrm{tr}_j| = 2$ for some $j = 1, 2$, Green for $|\mathrm{tr}_1| = |\mathrm{tr}_2|$, Blue for the boundary of region $\mathcal{R}_3$.

In region $\mathcal{R}_2$ we distinguish different intersections. The intersection between curves of the form (2.6) corresponds to a double resonance in which $\omega_1 = \dfrac{n_1\pi}{T}$ and $\omega_2 = \dfrac{n_2\pi}{T}$ for some $n_1, n_2 \in \mathbb{N}$. We note that if $n_1 \equiv n_2 (\mathrm{mod}\ 2)$ then $\omega_1 \pm \omega_2 = \dfrac{2n\pi}{T}$ for some $n \in \mathbb{N}$. In this case, the intersection point belongs also to a resonant curve (2.7).

In region $\mathcal{R}_3$ there are only single resonances.

Given a point (a_1, a_2) on a resonant curve, that is, (a_1, a_2) is a resonant point for $\varepsilon = 0$, we want to study if there is a bifurcation near (a_1, a_2) for $\varepsilon \gtrsim 0$. To this end we shall introduce small parameters δ_1, δ_2, and we will take $\lambda_1 = a_1 + \delta_1$, $\lambda_2 = a_2 + \delta_2$. In section 2.3 we shall give the Normal Form of the Hamiltonian (2.3) in the regions $\mathcal{R}_1, \mathcal{R}_2$ and $\mathcal{R}_3$ up to a given order in $\delta_1, \delta_2, \varepsilon$. Then in section 2.4 we will study the conditions for bifurcation depending on the region $\mathcal{R}_1$, $\mathcal{R}_2$ or $\mathcal{R}_3$. In section 2.5 we shall study with more detail the boundaries of the resonant regions as well as the bifurcation diagram in the particular case that the periodic functions F_j, $j = 1, 2$, in (2.2) satisfies d'Alembert property. Section 2.6 is devoted to the proof of the Normal Form given in section 2.3.

2.3 Normal Form

In this section we reduce the Hamiltonian system associated to (2.3) to Normal Form.

Let us fix $(a_1, a_2) \in \mathcal{R}$ a point on a resonant curve, that is, $(\lambda_1, \lambda_2) = (a_1, a_2)$ is a resonant point for $\varepsilon = 0$. Let us take $\lambda_j = a_j + \delta_j$, $j = 1, 2$, with $|\delta_j|$, $j = 1, 2$, small enough.

If we take $\varepsilon = 0$ in (2.3) then the system associated to the Hamiltonian function is written as

$$\dot{\mathbf{y}} \;=\; A_0 \mathbf{y}, \tag{2.9}$$

with

$$A_0 = \begin{pmatrix} 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ a_1 - 1 & 0 & 0 & -1 \\ 0 & a_2 - 1 & 1 & 0 \end{pmatrix}. \tag{2.10}$$

We note that A_0 depends on a_1 and a_2.

The Hamiltonian function (2.3) can be written as

$$H(\mathbf{y}, t) \;=\; H_0(\mathbf{y}) + \tilde{H}(\mathbf{y}, t), \tag{2.11}$$

where

$$\begin{aligned} H_0(\mathbf{y}) = -\frac{1}{2}\mathbf{y}^T J_4 A_0 \mathbf{y} = \\ = \frac{1}{2}(y_3^2 + y_4^2) + y_1 y_4 - y_2 y_3 + (1 - a_1)\frac{y_1^2}{2} + (1 - a_2)\frac{y_2^2}{2}, \end{aligned} \tag{2.12}$$

$$\begin{aligned} \tilde{H}(\mathbf{y}, t) = -\frac{\delta_1}{2} y_1^2 - \frac{\delta_2}{2} y_2^2 + \\ +(a_1 + \delta_1)\frac{y_1^2}{2} F_1(t; \varepsilon) + (a_2 + \delta_2)\frac{y_2^2}{2} F_2(t; \varepsilon). \end{aligned} \tag{2.13}$$

The Hamiltonian system associated to H_0 is a linear system with constant coefficients, and depends on a_1 and a_2. Once a_1 and a_2 are fixed the Hamiltonian (2.13) depends on three parameters, δ_1, δ_2 and ε.

Hamiltonian (2.11) admits the following symmetry.

Lemma 2.3.1. *The Hamiltonian (2.11) satisfies* $H(\mathbf{y}, t) = H(\mathbf{y}, -t)$ *and* $H(L\mathbf{y}, t) = H(\mathbf{y}, t)$ *for all* $\mathbf{y} \in \mathbb{R}^4$ *and* $t \in \mathbb{R}$*, where* $L = diag(-1, 1, 1, -1)$.

Proof

The equality $H(\mathbf{y},t) = H(\mathbf{y},-t)$ for all $\mathbf{y} \in \mathbb{R}^4, t \in \mathbb{R}$ is due to the fact that F_j, $j = 1,2$, are even functions.

To show that $H(L\mathbf{y},t) = H(\mathbf{y},t)$ for all $\mathbf{y} \in \mathbb{R}^4, t \in \mathbb{R}$ it is only necessary to take into account that if we change y_1 by $-y_1$ and y_4 by $-y_4$ the Hamiltonian does not change. $\square$

2.3.1 Reduction to diagonal form

In this section we shall diagonalize the system associated to (2.12). In order to keep the Hamiltonian character of the system, it will be necessary to perform a symplectic change of coordinates. To this end, we will take into account the different $\mathcal{R}_j$ regions that defines $\mathcal{R}$.

We denote by $\pm\rho_1$, $\pm\rho_2$, the eigenvalues of A_0. In what follows, we will take $\rho_1 = \lambda, \rho_2 = i\omega$, $\lambda, \omega \in \mathbb{R}^+$ if $(a_1,a_2) \in \mathcal{R}_1$, $\rho_1 = i\omega_1, \rho_2 = i\omega_2$ with $\omega_1, \omega_2 \in \mathbb{R}^+$, $\omega_1 > \omega_2$ if $(a_1,a_2) \in \mathcal{R}_2$, and $\rho_1 = \alpha + i\beta, \rho_2 = \overline{\rho}_1$, $\alpha, \beta \in \mathbb{R}^+$, if $(a_1,a_2) \in \mathcal{R}_3$.

It is easy to check that

$$\mathbf{u}_\rho \;=\; (2\rho, a_1 - \rho^2, a_1 + \rho^2, \rho(a_1 - \rho^2 - 2))^T, \tag{2.14}$$

is an eigenvector of eigenvalue ρ of A_0.

Let us denote by $\mathbf{u}_1, \mathbf{u}_2 \in \mathbb{C}^4$ the eigenvectors corresponding to eigenvalues ρ_1, ρ_2, respectively.

Using the symmetry given by L in lemma 2.3.1, and the fact that $A_0 L = -LA_0$, we obtain that $\mathbf{v}_1 := L\mathbf{u}_1$ and $\mathbf{v}_2 := L\mathbf{u}_2$ are eigenvectors of eigenvalues $-\rho_1, -\rho_2$, respectively.

If we restrict to the generic case, as $\rho_1 \pm \rho_2 \neq 0$, the eigenvectors $\mathbf{u}_1$ and $\mathbf{v}_1$ are J–orthogonal to $\mathbf{u}_2$ and $\mathbf{v}_2$. That is, $\mathbf{u}_1^T J\mathbf{u}_2 = 0$, $\mathbf{u}_1^T J\mathbf{v}_2 = 0$, $\mathbf{v}_1^T J\mathbf{u}_2 = 0$ and $\mathbf{v}_1^T J\mathbf{v}_2 = 0$. Moreover, a vector is J–orthogonal with itself.

We define the matrix M as follows

$$M \;=\; (k_1\mathbf{u}_1, k_2\mathbf{u}_2, k_3\mathbf{v}_1, k_4\mathbf{v}_2), \tag{2.15}$$

with $k_j \in \mathbb{C}$, $j = 1,\dots,4$, satisfying $k_1k_3\mathbf{u}_1^T J\mathbf{v}_1 = 1$, $k_2k_4\mathbf{u}_2^T J\mathbf{v}_2 = 1$. Matrix M is symplectic and defines a canonical change of variables as $\mathbf{y} = M\mathbf{z}$. Moreover, this change of coordinates transforms the system associated to (2.12) in diagonal form, that is, if $\mathcal{H}(\mathbf{z},t)$ denotes the transformed Hamiltonian, then

$$\mathcal{H}(\mathbf{z},t) \;=\; \mathcal{H}_0(\mathbf{z}) + \tilde{\mathcal{H}}(\mathbf{z},t), \tag{2.16}$$

where

$$\mathcal{H}_0(\mathbf{z}) \;=\; \rho_1 z_1 z_3 + \rho_2 z_2 z_4, \tag{2.17}$$

ρ_1 and ρ_2 being the eigenvalues of (2.10) and $\mathbf{z} = (z_1, z_2, z_3, z_4)^T$.

First, our purpose is to choose the constants k_j, $j = 1, \ldots, 4$, in an adequate way in order to have some relations between the new variables. To this end, it will be useful the next lemma.

Lemma 2.3.2. *1. If $(a_1, a_2) \in \mathcal{R}_1$ then $\mathbf{u}_1^T J\mathbf{v}_1 > 0$ if $a_1 > 0$, and $\mathbf{u}_1^T J\mathbf{v}_1 < 0$ if $a_1 < 0$. Moreover, $i\mathbf{u}_2^T J\mathbf{v}_2 > 0$.*

2. If $(a_1, a_2) \in \mathcal{R}_2$ then $i\mathbf{u}_1^T J\mathbf{v}_1 > 0$ and, $i\mathbf{u}_2^T J\mathbf{v}_2 > 0$ if $a_1 < 0$, and $i\mathbf{u}_2^T J\mathbf{v}_2 < 0$ if $a_1 > 0$.

Proof

Let us consider $\mathbf{u}_\rho$ in (2.14) an eigenvector of eigenvalue ρ of A_0.

It is easy to check that

$$\mathbf{u}_\rho^T J\mathbf{v}_\rho = 2\rho\, q(a_1, a_2; \rho^2), \tag{2.18}$$

where

$$q(a_1, a_2; \rho^2) \quad = \quad -\rho^4 + 2a_1\rho^2 + 4a_1 - a_1^2. \tag{2.19}$$

Recall that ρ is a solution of the characteristic polynomial of A_0, that is, $\tilde{p}(\rho) = 0$ where

$$\tilde{p}(x) \quad = \quad x^4 - (a_1 + a_2 - 4)x^2 + a_1 a_2.$$

Then, $-\rho^4 = (4 - a_1 - a_2)\rho^2 + a_1 a_2$, and therefore

$$q(a_1, a_2; \rho^2) \quad = \quad (4 + a_1 - a_2)\rho^2 + a_1 a_2 + 4a_1 - a_1^2.$$

Moreover

$$\rho^2 \quad = \quad \alpha_\pm, \quad \text{where} \quad \alpha_\pm = \frac{a_1 + a_2 - 4 \pm \sqrt{\Delta}}{2}, \quad \text{with} \quad \Delta = (a_1 + a_2 - 4)^2 - 4a_1 a_2.$$

We have that

$$q(a_1, a_2; \alpha_\pm) \quad = \quad -\frac{\sqrt{\Delta}}{2}[\sqrt{\Delta} \mp (4 + a_1 - a_2)],$$

where the sign $-$ stands for α_+ and $+$ for α_-.

If $a_1 > 0$ ($a_1 < 0$) we check that $(4 + a_1 - a_2)^2 > \Delta$ ($(4 + a_1 - a_2)^2 < \Delta$). Therefore, if $a_1 < 0$, one has $\sqrt{\Delta} \mp (4 + a_1 - a_2) > 0$ and then $q(a_1, a_2; \alpha_\pm) < 0$.

Furthermore, if $a_1 > 0$, as far as $(a_1, a_2) \in \mathcal{R}_1 \cup \mathcal{R}_2$, $4 + a_1 - a_2 > 0$ and then $\sqrt{\Delta} - (4 + a_1 - a_2) < 0$ and $\sqrt{\Delta} + (4 + a_1 - a_2) > 0$. So, $q(a_1, a_2; \alpha_+) > 0$ and $q(a_1, a_2; \alpha_-) < 0$.

1. Let us consider $(a_1, a_2) \in \mathcal{R}_1$. Then $\rho_1 = \lambda$, $\rho_2 = i\omega$, $\lambda, \omega \in \mathbb{R}^+$ and $\alpha_+ = \lambda^2$, $\alpha_- = -\omega^2$. Using (2.18), $\mathbf{u}_1^T J\mathbf{v}_1 = 2\lambda\, q(a_1, a_2; \alpha_+)$ and, $\mathbf{u}_1^T J\mathbf{v}_1 > 0$ if $a_1 > 0$ and $\mathbf{u}_1^T J\mathbf{v}_1 < 0$ if $a_1 < 0$. Moreover, $\mathrm{i}\mathbf{u}_2^T J\mathbf{v}_2 = -2\omega^2\, q(a_1, a_2; \alpha_-) > 0$.

2. Assume $(a_1, a_2) \in \mathcal{R}_2$. Then $\rho_1 = i\omega_1$, $\rho_2 = i\omega_2$, $\omega_1, \omega_2 \in \mathbb{R}^+$ with $\omega_1 > \omega_2$. In this case, $\alpha_- = -\omega_1^2$, $\alpha_+ = -\omega_2^2$. We get $\mathrm{i}\mathbf{u}_1^T J\mathbf{v}_1 = -2\omega_1^2\, q(a_1, a_2; \alpha_-) > 0$ and $\mathrm{i}\mathbf{u}_2^T J\mathbf{v}_2 = -2\omega_2^2\, q(a_1, a_2; \alpha_+)$. Then, $\mathrm{i}\mathbf{u}_2^T J\mathbf{v}_2 > 0$ if $a_1 < 0$ and $\mathrm{i}\mathbf{u}_2^T J\mathbf{v}_2 < 0$ if $a_1 > 0$.

□

We note that if $(a_1, a_2) \in \mathcal{R}_3$ then $\mathbf{u}_1^T J\mathbf{v}_1$ and $\mathbf{u}_2^T J\mathbf{v}_2$ are complex.

Then, we can do the following choice for the constants $k_j, j = 1, \dots, 4$.

1. If $(a_1, a_2) \in \mathcal{R}_1$, we take

$$k_1 = \frac{1}{\sqrt{s\mathbf{u}_1^T J\mathbf{v}_1}}, \quad k_3 = sk_1, \quad k_2 = \frac{1}{\sqrt{i\mathbf{u}_2^T J\mathbf{v}_2}}, \quad k_4 = ik_2, \tag{2.20}$$

2. if $(a_1, a_2) \in \mathcal{R}_2$, we take

$$k_1 = \frac{1}{\sqrt{i\mathbf{u}_1^T J\mathbf{v}_1}}, \quad k_3 = ik_1, \quad k_2 = \frac{1}{\sqrt{si\mathbf{v}_2^T J\mathbf{u}_2}}, \quad k_4 = -sik_2, \tag{2.21}$$

3. if $(a_1, a_2) \in \mathcal{R}_3$, we take

$$k_1 = \frac{1}{\sqrt{\mathbf{u}_1^T J\mathbf{v}_1}}, \quad k_3 = k_1, \quad k_2 = \frac{1}{\sqrt{\mathbf{u}_2^T J\mathbf{v}_2}}, \quad k_4 = k_2, \tag{2.22}$$

where $s = \mathrm{sgn}(a_1)$.

After lemma 2.3.2, if $(a_1, a_2) \in \mathcal{R}_1 \cup \mathcal{R}_2$ then $k_1, k_2 \in \mathbb{R}$. If $(a_1, a_2) \in \mathcal{R}_3$ then the real character of A_0 implies that $\mathbf{u}_2 = \overline{\mathbf{u}}_1$, $\mathbf{v}_2 = \overline{\mathbf{v}}_1$ and therefore, $k_2 = \overline{k}_1$ (bar stands for complex conjugate).

From now on, M will be the 4×4 symplectic matrix defined in (2.15) with k_1, k_2, k_3 and k_4 given in (2.20), (2.21) and (2.22) according to the region considered.

Let us define the following matrices

$$S_1 \;=\; M^{-1}LM, \quad S_2 = -JM^TJ\overline{M}. \tag{2.23}$$

Lemma 2.3.3. *The new variable* $\mathbf{z}$ *satisfies* $\overline{\mathbf{z}} = \overline{S}_2\mathbf{z}$, *and the following equalities hold*

$$\mathcal{H}(\mathbf{z}, t) = \mathcal{H}(S_1\mathbf{z}, -t), \quad \mathcal{H}(\mathbf{z}, t) = \overline{\mathcal{H}}(\overline{S}_2\mathbf{z}, t), \tag{2.24}$$

for all $\mathbf{z} \in \mathbb{C}^4$, $t \in \mathbb{R}$.

Moreover

1. *if* $(a_1, a_2) \in \mathcal{R}_1$ *then* $S_1\mathbf{z} = (sz_3, iz_4, sz_1, -iz_2)^T$, $\overline{S}_2\mathbf{z} = (z_1, iz_4, z_3, iz_2)^T$,
2. *if* $(a_1, a_2) \in \mathcal{R}_2$ *then* $S_1\mathbf{z} = (iz_3, -isz_4, -iz_1, isz_2)^T$,
   $\overline{S}_2\mathbf{z} = (iz_3, -isz_4, iz_1, -isz_2)^T$,
3. *if* $(a_1, a_2) \in \mathcal{R}_3$ *then* $S_1\mathbf{z} = (z_3, z_4, z_1, z_2)^T$, $\overline{S}_2\mathbf{z} = (z_2, z_1, z_4, z_3)^T$.

Proof

The new variables $\mathbf{z} \in \mathbb{C}^4$ are defined by $\mathbf{y} = M\mathbf{z}$ where we recall that $\mathbf{y} \in \mathbb{R}^4$. Then

$$\mathbf{z} \;=\; M^{-1}\overline{\mathbf{y}} = -JM^TJ\overline{M}\overline{\mathbf{z}} = S_2\overline{\mathbf{z}}, \tag{2.25}$$

where we have used that M is a symplectic matrix and so, $M^{-1} = -JM^TJ$. Now $\overline{\mathbf{z}} = \overline{S}_2\mathbf{z}$ follows from (2.25).

From lemma 2.3.1,

$$\mathcal{H}(S_1\mathbf{z}, t) = H(MS_1\mathbf{z}, t) = H(MS_1M^{-1}\mathbf{y}, t) = H(L\mathbf{y}, t) = H(\mathbf{y}, t) = \mathcal{H}(\mathbf{z}, t),$$

and using the parity of $\mathcal{H}$ we get the first equality in (2.24).

Furthermore, $\mathcal{H}(\mathbf{z}, t)$ is real. Therefore

$$\mathcal{H}(\mathbf{z}, t) = \overline{\mathcal{H}(\mathbf{z}, t)} = \overline{\mathcal{H}}(\overline{\mathbf{z}}, t) = \overline{\mathcal{H}}(\overline{S}_2\mathbf{z}, t).$$

A simple computation gives

$$S_1 \;=\; \begin{pmatrix} 0 & \tilde{S}_1 \\ \tilde{S}_1^{-1} & 0 \end{pmatrix} \quad \text{with} \quad \tilde{S}_1 = \operatorname{diag}\left(\frac{k_3}{k_1}, \frac{k_4}{k_2}\right).$$

Then,

$$\tilde{S}_1 \;=\; \begin{cases} \operatorname{diag}(s, \mathrm{i}) & \text{if} \quad (a_1, a_2) \in \mathcal{R}_1, \\ \operatorname{diag}(\mathrm{i}, -\mathrm{i}s) & \text{if} \quad (a_1, a_2) \in \mathcal{R}_2, \\ I_2 & \text{if} \quad (a_1, a_2) \in \mathcal{R}_3, \end{cases}$$

where I_2 denotes the 2×2 identity matrix.

Now we compute $\overline{S}_2 = -J\overline{M}^TJM$.

In the region $\mathcal{R}_1$ we have that $\mathbf{u}_1, \mathbf{v}_1 \in \mathbb{R}^4$ and $\overline{\mathbf{u}}_2 = \mathbf{v}_2$. Then, we obtain

$$\overline{M}^TJM = \begin{pmatrix} 0 & 0 & k_1k_3\mathbf{u}_1^TJ\mathbf{v}_1 & 0 \\ 0 & k_2^2\mathbf{v}_2^TJ\mathbf{u}_2 & 0 & 0 \\ k_1\overline{k}_3\mathbf{v}_1^TJ\mathbf{u}_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & k_4\overline{k}_4\mathbf{u}_2^TJ\mathbf{v}_2 \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & \mathrm{i} & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathrm{i} \end{pmatrix} \tag{2.26}$$

In the region $\mathcal{R}_2$ we have $\overline{\mathbf{u}}_j = \mathbf{v}_j$, $j = 1, 2$. Then,

$$\begin{aligned}\overline{M}^T JM = \mathrm{diag}(k_1^2 \mathbf{v}_1^T J\mathbf{u}_1, k_2^2 \mathbf{v}_2^T J\mathbf{u}_2, k_3\overline{k}_3 \mathbf{u}_1^T J\mathbf{v}_1, k_4\overline{k}_4 \mathbf{u}_2^T J\mathbf{v}_2) = \\ = \mathrm{diag}(\mathrm{i}, -\mathrm{i}s, -\mathrm{i}, \mathrm{i}s). \qquad (2.27)\end{aligned}$$

In $\mathcal{R}_3$ we get

$$\begin{aligned}\overline{M}^T JM &= \begin{pmatrix} 0 & 0 & 0 & \overline{k}_1 k_4 \mathbf{u}_2^T J\mathbf{v}_2 \\ 0 & 0 & k_3\overline{k}_3 \mathbf{u}_1^T J\mathbf{v}_1 & 0 \\ 0 & k_2\overline{k}_3 \mathbf{v}_2^T J\mathbf{u}_2 & 0 & 0 \\ k_1\overline{k}_4 \mathbf{v}_1^T J\mathbf{u}_1 & 0 & 0 & 0 \end{pmatrix} = \\ &= \begin{pmatrix} 0 & B_1 \\ -B_1 & 0 \end{pmatrix}, \quad \text{with} \quad B_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \qquad (2.28)\end{aligned}$$

From (2.26), (2.27) and (2.28), $\overline{S}_2$ is easily computed at each region. $\square$

2.3.2 The Normal Form in the different cases

In this section we apply the Normal Form techniques in order to simplify the Hamiltonian (2.16). To do this we need to distinguish the different regions.

In order to get the Normal Form we introduce K as a conjugate variable of time t and we consider the Hamiltonian

$$\mathcal{H}(\mathbf{z}, t, K) \;=\; \mathcal{H}_0(\mathbf{z}, K) + \tilde{\mathcal{H}}(\mathbf{z}, t), \qquad (2.29)$$

where $\mathcal{H}(\mathbf{z}, K) = \mathcal{H}_0(\mathbf{z}) + K$ and $\mathcal{H}_0$ is given in (2.17).

Let be $\nu = \dfrac{T}{\pi}$ and $w = e^{\frac{2it}{\nu}}$. We can write the Hamiltonian as

$$\mathcal{H}(\mathbf{z}, w, K) \;=\; \mathcal{H}_0(\mathbf{z}, K) + \sum_{m=1}^{\infty} \mathcal{H}_m(\mathbf{z}, w), \qquad (2.30)$$

where $\mathcal{H}_m(\mathbf{z}, w)$ contains terms of order m in δ_1, δ_2 and ε. Moreover $\mathcal{H}_m(\mathbf{z}, w)$ is an homogeneous polynomial of degree 2 in $\mathbf{z}$ whose coefficients depend on w and w^{-1}.

We can use Lie series method to perform some canonical transformations in order to cancel the time dependence on the Hamiltonian up to high order. This is done in section 2.6 using the Giorgilli–Galgani algorithm ([G.G.]). Then, if $\mathcal{N} = \mathcal{N}_0 + \mathcal{N}_1 + \mathcal{N}_2 + \ldots$ denotes the transformed Hamiltonian, we obtain

$$\mathcal{N}_m \;=\; \sum_{j=0}^{m} \mathcal{H}_{j,m-j}, \qquad \mathcal{H}_{m,j} = \sum_{l=1}^{j} \frac{l}{j} [G_l, \mathcal{H}_{m,j-l}], \qquad \mathcal{H}_{m,0} = \mathcal{H}_m, \quad (2.31)$$

$\mathcal{N}_j$							
$\mathcal{N}_0$	$\mathcal{H}_{0,0}$						
$\mathcal{N}_1$	$\mathcal{H}_{1,0}$	$\mathcal{H}_{0,1}$					
$\mathcal{N}_2$	$\mathcal{H}_{2,0}$	$\mathcal{H}_{1,1}$	$\mathcal{H}_{0,2}$				
$\mathcal{N}_3$	$\mathcal{H}_{3,0}$	$\mathcal{H}_{2,1}$	$\mathcal{H}_{1,2}$	$\mathcal{H}_{0,3}$			
$\vdots$							
$\mathcal{N}_m$	$\mathcal{H}_{m,0}$	$\mathcal{H}_{m-1,1}$	$\dots$	$\dots$	$\dots$	$\mathcal{H}_{1,m-1}$	$\mathcal{H}_{0,m}$

Table 2.1: Functions involved in the computation of $\mathcal{N}_j$, $j = 0, \dots, m$

and G_m is the solution of the homological equation

$$M_m + [G_m, \mathcal{H}_0] \quad = \quad R_m, \tag{2.32}$$

where

$$M_m = \sum_{j=0}^{m-1} \mathcal{H}_{m-j,j} + \sum_{l=1}^{m-1} \frac{l}{m}[G_l, \mathcal{H}_{0,m-l}],$$

and R_m contains resonant terms of order m in δ_1, δ_2 and ε.

Table 2.1 shows the functions involved in $\mathcal{N}_j$ up to order m, that is, up to $j = m$.

In what follows we shall denote the new variables, say Z_j, $j = 1, \dots, 4$, obtained by the canonical changes of variables involved in the normalization, as z_j, $j = 1, \dots, 4$, again. The next proposition gives the Normal Form depending on the region $\mathcal{R}_1, \mathcal{R}_2$ or $\mathcal{R}_3$. The proof is given in section 2.6.

Proposition 2.3.4. *Let us denote by NF the Normal Form up to a fixed order in the small parameters $\delta_1, \delta_2, \varepsilon$.*

1. *If $(a_1, a_2) \in \mathcal{R}_1$ and $\nu\omega \in \mathbb{N}$, then*

$$\begin{aligned} NF \quad = \quad & K + \lambda z_1 z_3 + i\omega z_2 z_4 + \sigma_1 z_1 z_3 + i\sigma_2 z_2 z_4 + \\ & + \sigma_3 z_2^2 w^{-\nu\omega} - \sigma_3 z_4^2 w^{\nu\omega}, \end{aligned} \tag{2.33}$$

where $\sigma_j \in \mathbb{R}$, $j = 1, \dots, 4$, depend on δ_1, δ_2 and ε.

2. *If $(a_1, a_2) \in \mathcal{R}_2$, then*

$$NF = \begin{cases} N_0 + N_1 & \text{if} \quad \nu\omega_1 \in \mathbb{N}, \quad \nu\omega_2 \notin \mathbb{N}, \\ N_0 + N_2 & \text{if} \quad \nu\omega_1 \notin \mathbb{N}, \quad \nu\omega_2 \in \mathbb{N}, \\ N_0 + N_1 + N_2 & \text{if} \quad \nu\omega_1 \in \mathbb{N}, \quad \nu\omega_2 \in \mathbb{N}, \\ & \quad \text{and} \quad \nu\omega_1 \not\equiv \nu\omega_2 (mod\ 2), \\ N_0 + N_3 & \text{if} \quad \nu\omega_{hs} \in \mathbb{N}, \quad \nu\omega_{hd} \notin \mathbb{N}, \\ & \quad (\nu\omega_1 \notin \mathbb{N}, \ \nu\omega_2 \notin \mathbb{N}), \\ N_0 + N_4 & \text{if} \quad \nu\omega_{hd} \in \mathbb{N}, \quad \nu\omega_{hs} \notin \mathbb{N}, \\ & \quad (\nu\omega_1 \notin \mathbb{N}, \ \nu\omega_2 \notin \mathbb{N}), \\ N_0 + N_1 + N_2 + N_3 + N_4 & \text{if} \quad \nu\omega_1 \in \mathbb{N}, \quad \nu\omega_2 \in \mathbb{N} \\ & \quad \text{and} \quad \nu\omega_1 \equiv \nu\omega_2 (mod\ 2), \end{cases} \tag{2.34}$$

where $\omega_{hs} = \dfrac{\omega_1 + \omega_2}{2}$, $\omega_{hd} = \dfrac{\omega_1 - \omega_2}{2}$, *and*

$$\begin{aligned} N_0 &= K + i\omega_1 z_1 z_3 + i\omega_2 z_2 z_4 + i\sigma_1 z_1 z_3 + i\sigma_2 z_2 z_4, \\ N_1 &= \sigma_3 z_1^2 w^{-\nu\omega_1} - \sigma_3 z_3^2 w^{\nu\omega_1}, \\ N_2 &= \sigma_4 z_2^2 w^{-\nu\omega_2} - \sigma_4 z_4^2 w^{\nu\omega_2}, \\ N_3 &= \sigma_5 z_1 z_2 w^{-\nu\omega_{hs}} + s\sigma_5 z_3 z_4 w^{\nu\omega_{hs}}, \\ N_4 &= i\sigma_6 z_1 z_4 w^{-\nu\omega_{hd}} - is\sigma_6 z_2 z_3 w^{\nu\omega_{hd}}, \end{aligned} \tag{2.35}$$

where $\sigma_j \in \mathbb{R}$, $j = 1, \ldots, 6$ *depend on* $\delta_1, \delta_2, \varepsilon$, *and* $s = sgn(a_1)$.

3. *If $(a_1, a_2) \in \mathcal{R}_3$ and $\nu\beta \in \mathbb{N}$ then*

$$\begin{aligned} NF = {} & K + (\alpha + i\beta) z_1 z_3 + (\alpha - i\beta) z_2 z_4 + \sigma_1 z_1 z_3 + \overline{\sigma}_1 z_2 z_4 + \sigma_3 z_1 z_4 w^{-\nu\beta} + \\ & \sigma_3 z_2 z_3 w^{\nu\beta}, \end{aligned} \tag{2.36}$$

where $\sigma_1 \in \mathbb{C}$, $\sigma_3 \in \mathbb{R}$ *depend on* $\delta_1, \delta_2, \varepsilon$.

Remark 2.3.5. Proposition 2.3.4 gives the Normal Form up to a given order, say n, when $\lambda_1 = a_1 + \delta_1$, $\lambda_2 = a_2 + \delta_2$ and (a_1, a_2) is a resonant point for $\varepsilon = 0$. The Normal Form can be written as

$$NF = N_0 + \mathcal{N}_n(w),$$

where

$$\begin{aligned} N_0 &= K + (\lambda + \sigma_1) z_1 z_3 + \mathrm{i}(\omega + \sigma_2) z_2 z_4 && \text{if} \quad (a_1, a_2) \in \mathcal{R}_1, \\ N_0 &= K + \mathrm{i}(\omega_1 + \sigma_1) z_1 z_3 + \mathrm{i}(\omega_2 + \sigma_2) z_2 z_4 && \text{if} \quad (a_1, a_2) \in \mathcal{R}_2, \\ N_0 &= K + (\alpha + \mathrm{i}\beta + \sigma_1) z_1 z_3 + (\alpha - \mathrm{i}\beta + \overline{\sigma}_1) z_2 z_4 && \text{if} \quad (a_1, a_2) \in \mathcal{R}_3, \end{aligned}$$

and all the monomials in $\mathcal{N}_n(w)$ depend on w and so, they are time dependent.

However, if $\varepsilon = 0$ the initial Hamiltonian (2.12) is autonomous. In this case, the Normal Form does not depend on w. Therefore, for the coefficients σ_3, σ_4, σ_5, σ_6 in Proposition 2.3.4 we have

$$\sigma_j = O(\varepsilon^k), \quad j = 3, \ldots, 6, \tag{2.37}$$

for some $k \geq 1$ which may depend on the index j.

Furthermore, σ_1 and σ_2 depend on $\delta_1, \delta_2, \varepsilon$. In fact σ_1 and σ_2 have terms of order 1 in δ_1, δ_2. That terms can be easily computed by taking into account the variation of the eigenvalues of the system when $\varepsilon = 0$ and we perturb (a_1, a_2) by (δ_1, δ_2). These terms will be explicitly computed in section 2.5.

2.4 Bifurcations

In this section we study the conditions for bifurcation when $\varepsilon > 0$ is small enough. We shall use the following notation for the different regions in the parameter space $(\lambda_1, \lambda_2, \varepsilon)$.

- EE (elliptic–elliptic) stands for a region such that $|\mathrm{tr}_j| < 2$, $j = 1, 2$,
- EH (elliptic–hyperbolic) corresponds to $|\mathrm{tr}_1| < 2$, $|\mathrm{tr}_2| > 2$ (or vice–versa),
- HH (hyperbolic–hyperbolic) corresponds to $\mathrm{tr}_j > 2$, $j = 1, 2$,
- CS (complex–saddle) stands for complex tr_1, $\mathrm{tr}_2 = \overline{\mathrm{tr}}_1$.

We obtain the equations for the boundaries of the different bifurcation regions in terms of the coefficients of the Normal Form. In next section we shall concentrate in the d'Alembert case, that is, when the functions F_j, $j = 1, 2$, defined in (2.2) satisfy the d'Alembert property.

In order to obtain the boundaries of the bifurcation regions for $\varepsilon > 0$ small enough we study the Hamiltonian system associated to the Normal Form given in Proposition 2.3.4.

Let us take $(a_1, a_2) \in \mathcal{R}_1$ a resonant point for $\varepsilon = 0$.

For $\varepsilon > 0$, bifurcation occurs when a pair of characteristic multipliers on the unit circle collides and become real. In this case, system turns from EH to HH.

Normal Form (2.33) defines the following uncoupled linear system

$$\begin{aligned} \dot{z}_1 &= (\lambda + \sigma_1) z_1, & \dot{z}_2 &= i(\omega + \sigma_2) z_2 - 2\sigma_3 z_4 w^{\nu\omega}, \\ \dot{z}_3 &= -(\lambda + \sigma_1) z_3, & \dot{z}_4 &= -2\sigma_3 z_2 w^{-\nu\omega} - i(\omega + \sigma_2) z_4, \end{aligned} \tag{2.38}$$

where we assume $\nu\omega = n \in \mathbb{N}$.

The system for z_1, z_3 gives real characteristic exponents and, then, a stability parameter is greater than two. This gives an hyperbolic behavior.

In order to study the system for z_2, z_4 we perform the change of variables $u = z_2 w^{-\nu\omega/2}$, $v = z_4 w^{\nu\omega/2}$. Then, this system transforms in the following linear system with constant coefficients

$$\begin{aligned} \dot{u} &= i\sigma_2 u - 2\sigma_3 v, \\ \dot{v} &= -2\sigma_3 u - i\sigma_2 v. \end{aligned} \tag{2.39}$$

The eigenvalues of the above system are $\pm\sqrt{4\sigma_3^2 - \sigma_2^2}$. The bifurcation takes place when these eigenvalues cros zero. In this case, a transition EH$\leftrightarrow$HH occurs. For $\varepsilon > 0$ an instability region HH in the parameter space is created. The boundaries of this region up to a given order in $\delta_1, \delta_2, \varepsilon$ are defined by the equation

$$\sigma_2^2 - 4\sigma_3^2 = 0. \tag{2.40}$$

Now we consider $(a_1, a_2) \in \mathcal{R}_2$ a resonant point for $\varepsilon = 0$.

We study the general case in (2.34), that is, $NF = N_0 + N_1 + N_2 + N_3 + N_4$ where N_i, $i = 0, \ldots, 4$, are given in (2.35). The other cases in (2.34) are obtained by taking the suitable coefficients equal to zero. The linear system defined by NF is the following.

$$\begin{aligned} \dot{z}_1 &= \mathrm{i}(\omega_1 + \sigma_1) z_1 - \mathrm{i}s\sigma_6 z_2 w^{\frac{\nu}{2}(\omega_1 - \omega_2)} - 2\sigma_3 z_3 w^{\nu\omega_1} + s\sigma_5 z_4 w^{-\frac{\nu}{2}(\omega_1 + \omega_2)}, \\ \dot{z}_2 &= \mathrm{i}\sigma_6 z_1 w^{-\frac{\nu}{2}(\omega_1 - \omega_2)} + \mathrm{i}(\omega_2 + \sigma_2) z_2 + s\sigma_5 z_3 w^{\frac{\nu}{2}(\omega_1 + \omega_2)} - 2\sigma_4 z_4 w^{\nu\omega_2}, \\ \dot{z}_3 &= -2\sigma_3 z_1 w^{-\nu\omega_1} - \sigma_5 z_2 w^{-\frac{\nu}{2}(\omega_1 + \omega_2)} - \mathrm{i}(\omega_1 + \sigma_1) z_3 - \mathrm{i}\sigma_6 z_4 w^{-\frac{\nu}{2}(\omega_1 - \omega_2)}, \\ \dot{z}_4 &= -\sigma_5 z_1 w^{-\frac{\nu}{2}(\omega_1 + \omega_2)} - 2\sigma_4 z_2 w^{-\nu\omega_2} + \mathrm{i}s\sigma_6 z_3 w^{\frac{\nu}{2}(\omega_1 - \omega_2)} - \mathrm{i}(\omega_2 + \sigma_2) z_4. \end{aligned} \tag{2.41}$$

We introduce new variables $u_1 = z_1 w^{-\frac{\nu}{2}\omega_1}$, $u_2 = z_2 w^{-\frac{\nu}{2}\omega_2}$, $v_1 = z_3 w^{\frac{\nu}{2}\omega_1}$, $v_2 = z_4 w^{\frac{\nu}{2}\omega_2}$. Then, system (2.41) becomes the following constant coefficients linear system

$$\begin{aligned} \dot{u}_1 &= \mathrm{i}\sigma_1 u_1 - \mathrm{i}s\sigma_6 u_2 - 2\sigma_3 v_1 + s\sigma_5 v_2, \\ \dot{u}_2 &= \mathrm{i}\sigma_6 u_1 + \mathrm{i}\sigma_2 u_2 + s\sigma_5 v_1 - 2\sigma_4 v_2, \\ \dot{v}_1 &= -2\sigma_3 u_1 - \sigma_5 u_2 - \mathrm{i}\sigma_1 v_1 - \mathrm{i}\sigma_6 v_2, \\ \dot{v}_2 &= -\sigma_5 u_1 - 2\sigma_4 u_2 + \mathrm{i}s\sigma_6 v_1 - \mathrm{i}\sigma_2 v_2. \end{aligned} \tag{2.42}$$

To study the bifurcations, we need to distinguish different cases.

1. $\nu\omega_1 \in \mathbb{N}$, $\nu\omega_2 \notin \mathbb{N}$.

In this case, $\sigma_4 = \sigma_5 = \sigma_6 = 0$ and (2.42) is an uncoupled system

$$\begin{aligned} \dot{u}_1 &= i\sigma_1 u_1 - 2\sigma_3 v_1, & \dot{u}_2 &= i\sigma_3 u_2, \\ \dot{v}_1 &= -2\sigma_3 u_1 - i\sigma_1 v_1, & \dot{v}_2 &= -i\sigma_3 v_2. \end{aligned}$$

The system in u_2, v_2 gives us a stability parameter $|\mathrm{tr}_2| < 2$ for $\delta_1, \delta_2, \varepsilon$ small enough. The study of the system in u_1, v_1 is analogous to (2.39) in $\mathcal{R}_1$. Then, we have a transition EE↔EH. The boundaries of the region EH are given by the equation

$$\sigma_1^2 - 4\sigma_3^2 \quad = \quad 0.$$

2. $\nu\omega_1 \notin \mathbb{N}$, $\nu\omega_2 \in \mathbb{N}$.

   This case is analogous to the previous one, obtaining an EH region, where their boundaries are defined by

$$\sigma_2^2 - 4\sigma_4^2 \quad = \quad 0.$$

3. $\nu\omega_1 \notin \mathbb{N}$, $\nu\omega_2 \notin \mathbb{N}$ and $\omega_{hs} \in \mathbb{N}$.

   In this case, $\sigma_3 = \sigma_4 = \sigma_6 = 0$ and (2.42) becomes an uncoupled system

$$\begin{aligned} \dot{u}_1 &= i\sigma_1 u_1 + s\sigma_5 v_2, & \dot{u}_2 &= i\sigma_2 u_2 + s\sigma_5 v_1, \\ \dot{v}_2 &= -\sigma_5 u_1 - i\sigma_2 v_2, & \dot{v}_1 &= -\sigma_5 u_2 - i\sigma_1 v_1. \end{aligned}$$

   The characteristic polynomials of the two uncoupled linear systems above are

$$x^2 \mp \mathrm{i}(\sigma_1 - \sigma_2)x + \sigma_1\sigma_2 + s\sigma_5^2, \tag{2.43}$$

   where $-$ stands for the system in u_1, v_2 and $+$ is for u_2, v_1.

   Let us define $\Delta = -[(\sigma_1 + \sigma_2)^2 + 4s\sigma_5^2]$ the discriminant of (2.43). The sign of Δ determines the character of the region.

   Figure 2.3 shows the character of the characteristic multipliers depending on the discriminant.

   The boundaries of the region CS are given by

$$(\sigma_1 + \sigma_2)^2 + 4s\sigma_5^2 \quad = \quad 0. \tag{2.44}$$

4. $\nu\omega_1 \notin \mathbb{N}$, $\nu\omega_2 \notin \mathbb{N}$ and $\omega_{hd} \in \mathbb{N}$.

   In this case system (2.42) becomes also an uncoupled system,

$$\begin{aligned} \dot{u}_1 &= i\sigma_1 u_1 - is\sigma_6 u_2, & \dot{v}_1 &= -i\sigma_1 v_1 - i\sigma_6 v_2, \\ \dot{u}_2 &= i\sigma_6 u_1 + i\sigma_2 u_2, & \dot{v}_2 &= is\sigma_6 v_1 - i\sigma_2 v_2. \end{aligned} \tag{2.45}$$

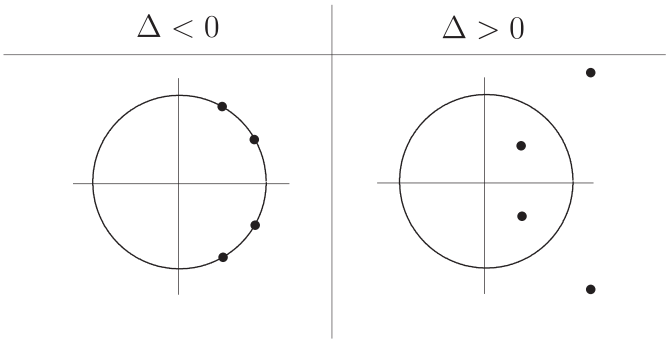


Figure 2.3: Characteristic multipliers in the case 3 according to the sign of Δ

The characteristic polynomials of these uncoupled systems are

$$x^2 \mp \mathrm{i}(\sigma_1 + \sigma_2)x - s\sigma_6^2 - \sigma_1\sigma_2, \tag{2.46}$$

where $-$ stands for the system in u_1, u_2 and $+$ for v_1, v_2.

In this case, the discriminant of (2.46) is $\Delta = -(\sigma_1 - \sigma_2)^2 + 4s\sigma_6^2$. As before we get a transition EE $\leftrightarrow$ CS when the discriminant of (2.46) is zero (see Figure 2.3).

The boundaries of the new region CS are given by the equation

$$-(\sigma_1 - \sigma_2)^2 + 4s\sigma_6^2 \;=\; 0. \tag{2.47}$$

Remark 2.4.1. The equation (2.44) has no real solution if $s = 1$, that is $a_1 > 0$. So, there is no bifurcation for $\nu\omega_{hs} \in \mathbb{N}$ when $a_1 > 0$. In the same way, there is no bifurcation if $\nu\omega_{hd} \in \mathbb{N}$ when $a_1 < 0$. This fact is well known as a consequence of Krein theorem (see [K.]).

5. $\nu\omega_1 \in \mathbb{N}$, $\nu\omega_2 \in \mathbb{N}$ with different parity.

   In this case system (2.42) splits in two uncoupled systems

$$\begin{aligned} \dot u_1 &= i\sigma_1 u_1 - 2\sigma_3 v_1, & \dot u_2 &= i\sigma_2 u_2 - 2\sigma_4 v_2,\\ \dot v_1 &= -2\sigma_3 u_1 - i\sigma_1 v_1, & \dot v_2 &= -2\sigma_4 u_2 - i\sigma_2 v_2. \end{aligned} \tag{2.48}$$

   Let $\Delta_1 = \sigma_1^2 - 4\sigma_3^2$ and $\Delta_2 = \sigma_2^2 - 4\sigma_4^2$ be the discriminants of the characteristic polynomials of the uncoupled systems above. Figure 2.4 shows the behavior for the characteristic multipliers depending on the sign of Δ_1 and Δ_2.

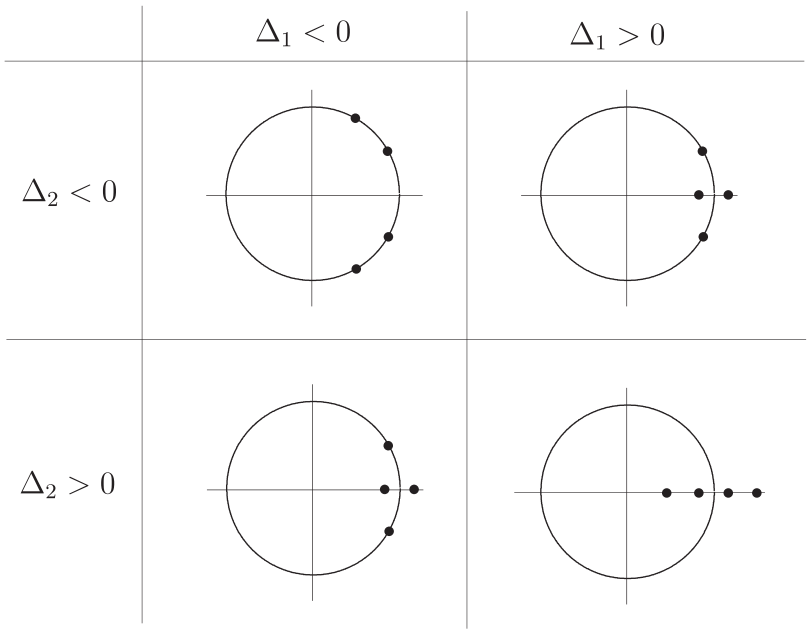


Figure 2.4: Characteristic multipliers according to the sign of $\Delta_1 = \sigma_1^2 - 4\sigma_3^2$ and $\Delta_2 = \sigma_2^2 - 4\sigma_4^2$

The boundaries of the EH region are given by

$$\sigma_1^2 - 4\sigma_3^2 = 0 \quad \text{or} \quad \sigma_2^2 - 4\sigma_4^2 = 0,$$

and the boundaries of the region HH are given by

$$\sigma_1^2 - 4\sigma_3^2 = 0 \quad \text{and} \quad \sigma_2^2 - 4\sigma_4^2 = 0.$$

The cases 1 to 5 are summarized in the following table.

$\nu\omega_1 \in \mathbb{N}, \nu\omega_2 \notin \mathbb{N}$	EE$\leftrightarrow$EH	$\sigma_1^2 - 4\sigma_3^2 = 0$
$\nu\omega_1 \notin \mathbb{N}, \nu\omega_2 \in \mathbb{N}$	EE$\leftrightarrow$EH	$\sigma_2^2 - 4\sigma_4^2 = 0$
$\nu\omega_1 \in \mathbb{N}, \nu\omega_2 \in \mathbb{N}$	EE$\leftrightarrow$EH	$\sigma_1^2 - 4\sigma_3^2 = 0$ or $\sigma_2^2 - 4\sigma_4^2 = 0$
with different parity	EE$\leftrightarrow$HH	$\sigma_1^2 - 4\sigma_3^2 = 0$ and $\sigma_2^2 - 4\sigma_4^2 = 0$
$\nu\omega_1 \notin \mathbb{N}, \nu\omega_2 \notin \mathbb{N}, \frac{\nu}{2}(\omega_1 + \omega_2) \in \mathbb{N}$	EE$\leftrightarrow$CS	$(\sigma_1 + \sigma_2)^2 + 4s\sigma_5 = 0$
$\nu\omega_1 \notin \mathbb{N}, \nu\omega_2 \notin \mathbb{N}, \frac{\nu}{2}(\omega_2 - \omega_2) \in \mathbb{N}$	EE$\leftrightarrow$CS	$(\sigma_1 - \sigma_2)^2 - 4s\sigma_6 = 0$

6. $\nu\omega_1 \in \mathbb{N}$, $\nu\omega_2 \in \mathbb{N}$ with the same parity.

   In this case, generically the coefficients σ_j, $j = 3, 4, 5, 6$ in (2.34) are different from zero. Now the system is not uncoupled.

However, due to his Hamiltonian character, the characteristic polynomial of (2.42) is

$$q(x) \quad = \quad x^4 + d_1 x^2 + d_2, \tag{2.49}$$

where

$$\begin{aligned} d_1 &= \sigma_1^2 + \sigma_2^2 - 4(\sigma_3^2 + \sigma_4^2) + 2s(\sigma_5^2 - \sigma_6^2), && (2.50)\\ d_2 &= D_1 D_2, && (2.51)\\ D_1 &= (\sigma_1 - 2s\sigma_3)(\sigma_2 + 2\sigma_4) + s(\sigma_5 + \sigma_6)^2, \\ D_2 &= (\sigma_1 + 2s\sigma_3)(\sigma_2 - 2\sigma_4) + s(\sigma_5 - \sigma_6)^2. && (2.52) \end{aligned}$$

Let

$$d_3 \quad = \quad d_1^2 - 4d_2, \tag{2.53}$$

the discriminant of (2.49). Then, the different possibilities for the character of the system, excluding boundary values, are

- EH if $d_2 < 0$,
- CS if $d_2 > 0$ and $d_3 < 0$,
- EE if $d_2 > 0$, $d_3 > 0$ and $d_1 > 0$,
- HH if $d_2 > 0$, $d_3 > 0$ and $d_1 < 0$.

Figure 2.5 represents these situations.

Finally we take $(a_1, a_2) \in \mathcal{R}_3$ a resonant point for $\varepsilon = 0$.
We recall that for $\varepsilon = 0$ the system (2.1) is in general saddle-complex.
The linear system defined by the Normal Form (2.36) is

$$\begin{aligned} \dot{z}_1 &= (\rho + \sigma_1) z_1 + \sigma_3 z_2 w^{\nu\beta}, \\ \dot{z}_2 &= \sigma_3 z_1 w^{-\nu\beta} + (\overline{\rho} + \overline{\sigma}_1) z_2, \\ \dot{z}_3 &= -(\rho + \sigma_1) z_3 - \sigma_3 z_4 w^{-\nu\beta}, \\ \dot{z}_4 &= -\sigma_3 z_3 w^{\nu\beta} - (\overline{\rho} + \overline{\sigma}_1) z_4, \end{aligned} \tag{2.54}$$

with $\rho = \alpha + i\beta$.

In order to study the stability parameters of system (2.54) we perform the change of variables $u_1 = z_1 w^{-\frac{\nu\beta}{2}}$, $u_2 = z_2 w^{\frac{\nu\beta}{2}}$, $v_1 = z_3 w^{\frac{\nu\beta}{2}}$, $v_2 = z_4 w^{-\frac{\nu\beta}{2}}$. We obtain the uncoupled linear system with constant coefficients

$$\begin{array}{ll} \dot{u}_1 = (\alpha + \sigma_1) u_1 + \sigma_3 u_2, & \dot{v}_1 = -(\alpha + \sigma_1) v_1 - \sigma_3 v_2, \\ \dot{u}_2 = \sigma_3 u_1 + (\alpha + \overline{\sigma}_1) u_2, & \dot{v}_2 = -\sigma_3 v_1 - (\alpha + \overline{\sigma}_1) v_2, \end{array} .$$

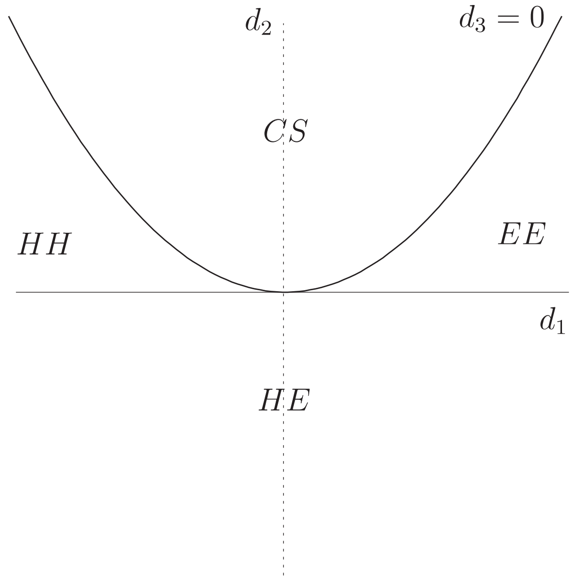


Figure 2.5: Stability regions in the (d_1, d_2) plane

Let be $\Delta = \sigma_3^2 - \mathrm{Im}(\sigma_1)^2$. If $\Delta > 0$ then we have a pair of real eigenvalues and their opposites. Figure 2.6 shows the behavior of the characteristic multipliers in terms of Δ.

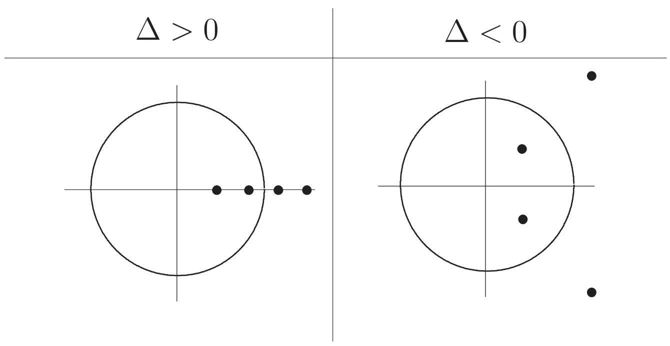


Figure 2.6: Characteristic multipliers according to the sign of $\Delta = \sigma_3^2 - \mathrm{Im}(\sigma_1)^2$

A transition HH↔CS occurs when Δ goes through 0. Then, the equations for the boundaries of the HH region are given by

$$\mathrm{Im}(\sigma_1) = \pm\sigma_3. \tag{2.55}$$

2.5 The d'Alembert case

In this section we study the resonant regions in the particular case that F_j in (2.2) satisfy d'**Alembert property**, that is, functions F_j, $j = 1, 2$, are of the form

$$\sum_{m\geq 1} \varepsilon^m \sum_{l=0}^{m} c_{m,l} \cos\left(l\frac{2\pi t}{T}\right),$$

where $c_{m,l} \in \mathbb{R}$.

After remark 2.3.5 we know that for the coefficients σ_j, $j = 3, 4, 5, 6$, in the Normal Form, (2.37) is satisfied for $k \geq 1$. The d'Alembert property can be used to determine, under non degeneracy conditions, the order of these coefficients as follows.

Let us go back to the initial Hamiltonian (2.11) and consider the time dependent terms (see (2.13))

$$(a_1 + \delta_1)\frac{y_1^2}{2}F_1(t;\varepsilon) + (a_2 + \delta_2)\frac{y_2^2}{2}F_2(t;\varepsilon) \quad .$$

Let us consider a fixed σ_j, $j = 3, 4, 5, 6$, and assume that it is the coefficient of a resonant monomial $\mathbf{z}^{\mathbf{l}}$,

$$\sigma_j w^{\pm n}\mathbf{z}^{\mathbf{l}} \quad \text{with} \quad n \in \mathbb{N}, \tag{2.56}$$

where $\mathbf{z} = (z_1, z_2, z_3, z_4)^T$ and $\mathbf{l} = (l_1, l_2, l_3, l_4)$ with $l_j \in \mathbb{N}$ satisfying $l_1+l_2+l_3+l_4 = 2$. For a fixed $n \in \mathbb{N}$, the d'Alembert property implies that in F_1 and F_2 the terms in $\cos\left(n\frac{2\pi t}{T}\right)$ can be written as

$$\cos\left(n\frac{2\pi t}{T}\right)\varepsilon^n(c_{n,n} + O_1) = \frac{w^n + w^{-n}}{2}\varepsilon^n(c_{n,n} + O_1) \quad ,$$

where $w = e^{\frac{2\mathrm{i}t}{\nu}}$, O_1 contains terms of order at least 1 in ε, and $c_{n,n}$ is a coefficient eventually zero. We recall that in order to get the Normal Form, first we perform the change of variables $\mathbf{y} = M\mathbf{z}$ to (2.11). For the new Hamiltonian as given in (2.29) the terms in w^n, w^{-n} are of order n in ε. So, they only appear in $\mathcal{H}_m(\mathbf{z}, w)$ for $m \geq n$. Now it is not difficult to see that this property is preserved by the Giorgilli–Galgani algorithm. In fact, the terms $\varepsilon^n w^n$, $\varepsilon^n w^{-n}$ appear for the first time in the n–th row of table 1. Therefore, if $j = 3, 4, 5, 6$ for σ_j in (2.56) we get

$$\sigma_j \;=\; c_j\varepsilon^n(1 + O_1),$$

where c_j is a coefficient eventually zero and O_1 contains terms of order 1 in $\delta_1, \delta_2, \varepsilon$. We shall assume in the next as non degeneracy conditions that $c_j \neq 0$, $j = 3, 4, 5, 6$.

2.5.1 Single resonances

We shall consider resonant points (a_1, a_2) which belong to a unique resonant curve. This kind of points are found at every region $\mathcal{R}_1, \mathcal{R}_2$ and $\mathcal{R}_3$.

We begin with $\mathcal{R}_1$ and assume that (a_1, a_2) belongs to a resonant curve (2.6), that is,

$$\gamma_n(a_1, a_2) := (a_1 + \omega^2)(a_2 + \omega^2) - 4\omega^2 = 0, \quad \text{where} \quad \omega = \frac{n\pi}{T}, \tag{2.57}$$

for some $n \in \mathbb{N}$. From now on, n is fixed.

The boundary surfaces which separate the EH and HH regions for $\varepsilon > 0$ are defined by (2.40). Our purpose is to give an estimation of the size of the HH region.

As we have seen in the beginning of this section the d'Alembert property implies that

$$\sigma_3 \quad = \quad c_3\varepsilon^n(1 + O_1), \tag{2.58}$$

where $c_3 \in \mathbb{R}$ and O_1 stands for terms of order 1 in $\delta_1, \delta_2, \varepsilon$. It is assumed that $c_3 \neq 0$. The following lemma gives the terms of σ_2 which are of order 1 in δ_1, δ_2.

Lemma 2.5.1. *Let $(a_1, a_2) \in \mathcal{R}_1$ be such that $\gamma_n(a_1, a_2) = 0$. Then, the dominant terms in the contribution of δ_1 and δ_2 to σ_2 are*

$$-\left[\frac{\omega^2 + a_2}{D(\omega)}\delta_1 + \frac{\omega^2 + a_1}{D(\omega)}\delta_2\right], \tag{2.59}$$

where $D(\omega) = 2\omega[2\omega^2 + a_1 + a_2 - 4] \neq 0$.

Remark 2.5.2. This lemma is also true if F_j, $j = 1, 2$, do not satisfy the d'Alembert property.

Proof

We use the hint given in remark 2.3.5, that is, assume $\varepsilon = 0$ and consider $\sigma_i = \sigma_i(\delta_1, \delta_2)$, $i = 1, 2$.

In the plane of parameters λ_1, λ_2, if $\lambda_1 = a_1 + \delta_1$, $\lambda_2 = a_2 + \delta_2$, the zeroes of the characteristic polynomial $p(x)$ defined in (2.5) are $\pm\Lambda(\delta_1, \delta_2) \in \mathbb{R}$, $\pm\mathrm{i}\Omega(\delta_1, \delta_2)$ where

$$\Lambda(\delta_1, \delta_2) := \lambda + \sigma_1(\delta_2, \delta_2), \qquad \Omega(\delta_1, \delta_2) := \omega + \sigma_2(\delta_1, \delta_2).$$

Here, $\pm\lambda$, $\pm\mathrm{i}\omega$, are the zeroes of $p(x)$ for $(\delta_1, \delta_2) = (0, 0)$. Then, we can write

$$\sigma_2(\delta_1, \delta_2) = \frac{\partial\Omega}{\partial\delta_1}(0, 0)\delta_1 + \frac{\partial\Omega}{\partial\delta_2}(0, 0)\delta_2 + \ldots,$$

and the partial derivatives are easily computed using that $p(\mathrm{i}\Omega) = 0$. $\square$

In order to describe the boundary surfaces we shall consider perturbations of (a_1, a_2) in an orthogonal direction to the resonant curve (2.57), that is, $\lambda_1 = a_1 + \delta_1$, $\lambda_2 = a_2 + \delta_2$ with

$$\begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix} = \delta \nabla \gamma_n(a_1, a_2), \tag{2.60}$$

for some parameter δ, being $|\delta|$ small enough. In this way, σ_2 and σ_3 depend on ε and δ. Moreover, (2.59) becomes

$$-\frac{\|\nabla \gamma_n(a_1, a_2)\|^2}{D(\omega)} \delta.$$

We remark that $\|\nabla \gamma_n(a_1, a_2)\| \neq 0$. Otherwise, we should have $\omega = 0$, but we know that $\omega \neq 0$ if $(a_1, a_2) \in \mathcal{R}_1$.

Therefore we can write

$$\sigma_2 = c\|\nabla \gamma_n(a_1, a_2)\|^2 \delta + \phi_0(\varepsilon) + \delta \phi_1(\varepsilon) + \delta^2 f(\varepsilon, \delta),$$

where $c = -\dfrac{1}{D(\omega)}$, ϕ_0 and ϕ_1 are functions of order 1 in ε and $f(\varepsilon, \delta)$ is of order 1 in ε, δ.

Let us introduce the functions

$$f_1(\varepsilon, \delta) = \sigma_2 - 2\sigma_3, \qquad f_2(\varepsilon, \delta) = \sigma_2 + 2\sigma_3.$$

The boundaries of the HH region are defined by

$$f_1(\varepsilon, \delta) = 0, \qquad f_2(\varepsilon, \delta) = 0.$$

We have that $\sigma_2 = \sigma_3 = 0$ if $(\varepsilon, \delta) = (0, 0)$. Then, $f_1(0, 0) = f_2(0, 0) = 0$. Moreover,

$$\frac{\partial f_1}{\partial \delta}(0, 0) = \frac{\partial f_2}{\partial \delta}(0, 0) = c\|\nabla \gamma_n(a_1, a_2)\|^2 \neq 0.$$

Then, using the Implicit Function Theorem there exist $\delta_+(\varepsilon)$, $\delta_-(\varepsilon)$ analytic functions in $\varepsilon \gtrsim 0$ such that

$$\begin{aligned} f_1(\varepsilon, \delta_+(\varepsilon)) = \sigma_2(\delta_+(\varepsilon), \varepsilon) - 2\sigma_3(\delta_+(\varepsilon), \varepsilon) &= 0, \\ f_2(\varepsilon, \delta_-(\varepsilon)) = \sigma_2(\delta_-(\varepsilon), \varepsilon) + 2\sigma_3(\delta_-(\varepsilon), \varepsilon) &= 0. \end{aligned}$$

Therefore, in the direction of $\nabla \gamma_n(a_1, a_2)$, the boundaries of the HH region are given by

$$\lambda_1 = a_1 + \delta_+(\varepsilon), \qquad \lambda_2 = a_2 + \delta_-(\varepsilon),$$

for $\varepsilon > 0$ small enough.

Proposition 2.5.3. *Let be $(a_1, a_2) \in \mathcal{R}_1$ such that $\gamma_n(a_1, a_2) = 0$ for some $n \in \mathbb{N}$. Assume that F_1 and F_2 satisfy the d'Alembert property. If $c_3 \neq 0$ defined in (2.58) then the width $\delta_+(\varepsilon) - \delta_-(\varepsilon)$ of the HH region is of order ε^n being the dominant term*

$$-\frac{8c_3\omega(2\omega^2 + a_1 + a_2 - 4)}{\|\nabla\gamma_n(a_1, a_2)\|^2}\varepsilon^n.$$

Proof

We have that

$$\begin{aligned} f_1(\varepsilon, \delta_+(\varepsilon)) &= c\|\nabla\gamma_n(a_1, a_2)\|^2\delta_+(\varepsilon) + \phi_0(\varepsilon) + \delta_+(\varepsilon)\phi_1(\varepsilon) + \\ &\quad + \delta_+(\varepsilon)^2 f(\varepsilon, \delta_+(\varepsilon)) - 2c_1\varepsilon^n(1 + g(\varepsilon, \delta_+(\varepsilon))) = 0, \end{aligned}$$

with $c = -\dfrac{1}{D(\omega)}$, $g(\varepsilon, \delta)$ a function of order 1 in ε, δ, and

$$\begin{aligned} f_2(\varepsilon, \delta_-(\varepsilon)) &= c\|\nabla\gamma_1(a_1, a_2)\|^2\delta_+(\varepsilon) + \phi_0(\varepsilon) + \delta_-(\varepsilon)\phi_1(\varepsilon) + \\ &\quad + \delta_-(\varepsilon)^2 f(\varepsilon, \delta_+(\varepsilon)) + 2c_1\varepsilon^n(1 + g(\varepsilon, \delta_-(\varepsilon))) = 0. \end{aligned}$$

The difference $f_1(\varepsilon, \delta_-(\varepsilon)) - f_2(\varepsilon, \delta_+(\varepsilon))$ is

$$\begin{aligned} &c\|\nabla\gamma_n(a_1, a_2)\|^2(\delta_+(\varepsilon) - \delta_-(\varepsilon)) + (\delta_+(\varepsilon) - \delta_-(\varepsilon))\phi_1(\varepsilon) + \delta_+(\varepsilon)^2 f(\varepsilon, \delta_+(\varepsilon)) - \\ &\delta_-(\varepsilon)^2 f(\varepsilon, \delta_-(\varepsilon)) - 4c_1\varepsilon^n - 4c_1\varepsilon^n(g(\varepsilon, \delta_+(\varepsilon)) - g(\varepsilon, \delta_-(\varepsilon))) = 0. \end{aligned}$$

From this equation it is easy to obtain the dominant terms of $\delta_+(\varepsilon) - \delta_-(\varepsilon)$. □

A similar analysis can be done in regions $\mathcal{R}_2$ and $\mathcal{R}_3$ in the case of a single resonance, that is, (a_1, a_2) belongs to a unique resonant curve (2.6), (2.7) or (2.8). In any case we shall take (δ_1, δ_2) as (2.60) for the corresponding resonant curve. We given explicitly the case $\mathcal{R}_3$ in the next proposition. We recall that Normal Form in a neighbourhood of (a_1, a_2) is given by (2.36) and the boundaries of the HH region are defined by (2.55).

Proposition 2.5.4. *Let $(a_1, a_2) \in \mathcal{R}_3$, $a_1 \neq a_2$, be a point on a resonant curve (2.8) with $\beta = \dfrac{n\pi}{T}$ for some $n \in \mathbb{N}$. Assume that F_1 and F_2 satisfy d'Alembert property and $\sigma_3 = c_3\varepsilon^n(1 + O_1)$ with $c_3 \neq 0$. Then the width $\delta_+(\varepsilon) - \delta_-(\varepsilon)$ of the HH region is of order ε^n being the dominant term*

$$\frac{2c_3}{c}\varepsilon^n, \quad \textit{where} \quad c = -\frac{\sqrt{a_2} - \sqrt{a_1}}{16\beta\sqrt{a_1 a_2}|\rho|^2}\left(\frac{(\sqrt{a_1} + \sqrt{a_2})^2}{2} + 2(1 - \beta^2)\right), \tag{2.61}$$

$\rho = \alpha + i\beta$.

Proof

Following the same ideas in the proof of lemma 2.5.1 it is not difficult to see that the contribution of δ_1 and δ_2 to σ_1 up to first order is

$$\frac{1}{2\rho W(\rho)}[(\rho^2-a_2)\delta_1+(\rho^2-a_1)\delta_2], \tag{2.62}$$

where $W(\rho)=2\rho^2-(a_1+a_2-4)=4\alpha\beta\mathrm{i}$. Using (2.60) and (2.62) we get that the terms of $\mathrm{Im}(\sigma_1)$ of order one in δ are $c\delta$ where c is given in (2.61). Clearly $c\neq 0$ if $a_1\neq a_2$. There exist two analytic functions $\delta_+(\varepsilon),\delta_-(\varepsilon)$ which satisfy $\mathrm{Im}(\sigma_1)-\sigma_3=0$ and $\mathrm{Im}(\sigma_1)+\sigma_3=0$ respectively. Then, the proposition follows in a similar way as proposition 2.5.3 □

2.5.2 Double resonances

In this section we consider a double resonance, that is, (a_1,a_2) is a resonant point which belongs to two or more resonant curves. Double resonances only occur at $\mathcal{R}_2$. So, we assume $(a_1,a_2)\in\mathcal{R}_2$ and

$$\nu\omega_j \;=\; n_j, \qquad j=1,2, \tag{2.63}$$

for some $n_1>n_2$ natural numbers. We shall consider the case $n_1\equiv n_2(\mathrm{mod}\ 2)$. The Normal Form is $N_0+N_1+N_2+N_3+N_4$ in (2.34). We want to discuss the possibilities for the bifurcations for sufficiently small ε when we perturb a case that for $\varepsilon=0$ is totally elliptic and both frequencies are in resonance.

Resonant points are given by the zeroes of functions d_1, d_2 and $d_3=d_2^2-4d_1$ in (2.50), (2.51), (2.53), respectively. The analysis of the bifurcations amounts to study the composition of the maps

$$\mathcal{N} \;:\; (\lambda_1,\lambda_2,\varepsilon)\mapsto(\sigma_1,\sigma_2,\sigma_3,\sigma_4,\sigma_5,\sigma_6),$$

and

$$\mathcal{P} \;:\; (\sigma_1,\sigma_2,\sigma_3,\sigma_4,\sigma_5,\sigma_6)\mapsto(d_1,d_2),$$

where $\mathcal{N}$ denotes the normalization map and $\mathcal{P}$ the characteristic polynomial of the Floquet matrix.

Lemma 2.5.5. *Let be $(a_1,a_2)\in\mathcal{R}_2$ and $\omega_1>\omega_2$ the frequencies obtained for $\varepsilon=0$. Then, the dominant terms in the contribution of δ_1 and δ_2 to σ_1,σ_2 are*

$$\mathcal{J}\begin{pmatrix}\delta_1\\ \delta_1\end{pmatrix} \qquad \textit{where} \quad \mathcal{J}=\begin{pmatrix}-\frac{\omega_1^2+a_2}{D_1} & -\frac{\omega_1^2+a_1}{D_1}\\ -\frac{\omega_2^2+a_2}{D_2} & -\frac{\omega_2^2+a_1}{D_2}\end{pmatrix},$$

$D_1=2\omega_1[(a_1+a_2-4)+2\omega_1^2]\neq 0$, $D_2=2\omega_2[(a_1+a_2-4)+2\omega_2^2]\neq 0$. Moreover, the matrix $\mathcal{J}$ is regular if $\omega_1\neq\omega_2$ and $a_1\neq a_2$.

Proof

It follows the same idea given in the proof of lemma 2.5.1. In fact, if we denote by $i\Omega(\delta_1, \delta_2) = i[\omega_1 + \sigma_1(\delta_1, \delta_2)]$ and $i\Omega_2(\delta_1, \delta_2) = i[\omega_2 + \sigma_2(\delta_1, \delta_2)]$, the zeroes of the characteristic polynomial $p(x)$, then $\mathcal{J}$ is the Jacobian of Ω_1, Ω_2. It is easy to check that $D_1 D_2 \neq 0$ if $(a_1, a_2) \in \mathcal{R}_2$. Furthermore,

$$|\mathcal{J}| \quad = \quad \frac{(\lambda_1 - \lambda_2)(\omega_1^2 - \omega_2^2)}{D_1 D_2}.$$

□

After lemma 2.5.5 we can use σ_1 and σ_2 as parameters instead of δ_1, δ_2. Then bifurcations will be described in terms of σ_1 and σ_2.

As the functions F_j in (2.2) satisfy d'Alembert property, we have,

$$\begin{aligned}
\sigma_3 &= m_1 \varepsilon^{n_1}(1 + O(\varepsilon, \delta_1, \delta_2)), \\
\sigma_4 &= m_2 \varepsilon^{n_2}(1 + O(\varepsilon, \delta_1, \delta_2)), \\
\sigma_5 &= m_3 \varepsilon^{\frac{n_1+n_2}{2}}(1 + O(\varepsilon, \delta_1, \delta_2)), \\
\sigma_6 &= m_4 \varepsilon^{\frac{n_1-n_2}{2}}(1 + O(\varepsilon, \delta_1, \delta_2)),
\end{aligned}$$

where m_j, $j = 1, \ldots, 4$, are real values. We shall assume non degeneracy conditions in the sense that $m_j \neq 0$, $j = 1, \ldots, 4$.

Our purpose in this section is to prove the following theorem.

Theorem 2.5.6. *Let $(a_1, a_2) \in \mathcal{R}_2$, $a_1 \neq a_2$, and assume $\omega_j = n_j \frac{\pi}{T}$, $j = 1, 2$, $n_1 > n_2$, $n_1 \equiv n_2 (mod\ 2)$. In the d'Alembert case and if non degeneracy conditions are satisfied, one has*

(i) if $n_1 = 3n_2$, then around $(\lambda_1, \lambda_2, \varepsilon) = (a_1, a_2, 0)$, regions EE, EH and CS exist and a region HH has either 0, 1 *or* 2 *connected components.*

(ii) If $n_1 \neq 3n_2$, then the regions EE, EH, HH, CS are always present and no local changes in the topology of these domains occur.

The proof of the theorem will be given at the end of this section as a summary of previous results.

First of all we study the magnitude of σ_j, $j = 3, \ldots, 6$.

We note that $n_1 > \dfrac{n_1 + n_2}{2} > n_2$. Then, $|\sigma_4| >> |\sigma_5| >> |\sigma_3|$ if ε is sufficiently small. Moreover, $\dfrac{n_1 - n_2}{2} < \dfrac{n_1 + n_2}{2}$, fact that implies $|\sigma_6| >> |\sigma_5|$. Now, we distinguish different subcases.

1. If $n_1 > 3n_2$, then $n_1 > \frac{n_1+n_2}{2} > \frac{n_1-n_2}{2} > n_2$ and therefore $|\sigma_3| << |\sigma_5| << |\sigma_6| << |\sigma_4|$.
2. If $n_1 = 3n_2$, then $n_1 > \frac{n_1+n_2}{2} > \frac{n_1-n_2}{2} = n_2$ and therefore $|\sigma_3| << |\sigma_5| << |\sigma_4|$ and σ_6 is of the same order of magnitude of σ_4.
3. If $n_1 < 3n_2$, then $n_1 > \frac{n_1+n_2}{2} > n_2 > \frac{n_1-n_2}{2}$ and therefore $|\sigma_3| << |\sigma_5| << |\sigma_4| << |\sigma_6|$.

We introduce the following scaled parameters

$$\tilde{\sigma}_j = \frac{\sigma_j}{\sigma_4}, \quad j = 1,2,3,5, \quad A = \frac{\sigma_6}{\sigma_4}, \tag{2.64}$$

and we define $\mu := \varepsilon^{\frac{n_1-n_2}{2}}$.

We begin with the second case.

In this case, $\mu = \varepsilon^{n_2}$ and then

$$\tilde{\sigma}_3 = O(\mu^2), \quad \tilde{\sigma}_5 = O(\mu), \quad A = O(1).$$

Using the scalings we introduce new functions (see section 2.4, case 6)

$$\tilde{d}_1 = \frac{d_1}{\sigma_4^2}, \quad \tilde{D}_1 = \frac{D_1}{\sigma_4^2}, \quad \tilde{D}_2 = \frac{D_2}{\sigma_4^2}, \quad \tilde{d}_2 = \tilde{D}_1\tilde{D}_2, \quad \tilde{d}_3 = \tilde{d}_1^2 - 4\tilde{d}_2.$$

Let be $B := sA^2$. We can write these functions in terms of μ like

$$\begin{aligned}
\tilde{d}_1 &= \tilde{\sigma}_1^2 + \tilde{\sigma}_2^2 - (4+2B) + O(\mu^2),\\
\tilde{D}_1 &= \tilde{\sigma}_1(\tilde{\sigma}_2 + 2) + B + O(\mu),\\
\tilde{D}_2 &= \tilde{\sigma}_1(\tilde{\sigma}_2 - 2) + B + O(\mu), \qquad \tilde{d}_2 = \tilde{D}_1\tilde{D}_2,\\
\tilde{d}_3 &= (\tilde{\sigma}_1^2 - \tilde{\sigma}_2^2 + 4)^2 - 4B[(\tilde{\sigma}_1 + \tilde{\sigma}_2)^2 - 4] + O(\mu).
\end{aligned}$$

In order to study resonant points we need to compute the zeroes of $\tilde{d}_j, j = 1,2,3$ functions, as well as the intersections of the curves defined by $\tilde{d}_j = 0$, $j = 1,2,3$. We note that, up to terms of order μ, $\tilde{d}_1, \tilde{D}_1, \tilde{D}_2, \tilde{d}_3$ depends on B. The idea is to study the bifurcation diagram in the plane $(\tilde{\sigma}_1, \tilde{\sigma}_2)$ in terms of B. Notice that $B \neq 0$.

First we will assume that $\mu = 0$. We obtain the following result.

Proposition 2.5.7. *Assume that hypothesis in theorem 2.5.6 are satisfied and $n_1 = 3n_2$. Under the generic assumptions $m_2 \neq 0$, $m_4 \neq 0$ in the Normal Form and neglecting σ_3, σ_5 terms (i.e., setting $\mu = 0$) the unique changes in the bifurcation diagram are produced at $B = -1$ and $B = -\frac{27}{16}$.*

Figure 2.7 shows the bifurcation diagram for $\mu = 0$ in different cases. We note that, in particular, no HH regions exists if $B < -1$.

Proof

In this case

$$\begin{aligned}
\tilde{d}_1 &= \tilde{\sigma}_1^2 + \tilde{\sigma}_2^2 - (4 + 2B), \\
\tilde{D}_1 &= \tilde{\sigma}_1(\tilde{\sigma}_2 + 2) + B, \\
\tilde{D}_2 &= \tilde{\sigma}_1(\tilde{\sigma}_2 - 2) + B, \\
\tilde{d}_3 &= (\tilde{\sigma}_1^2 - \tilde{\sigma}_2^2 + 4)^2 - 4B[(\tilde{\sigma}_1 + \tilde{\sigma}_2)^2 - 4].
\end{aligned}$$

The different stability regions are determined by the intersections of the zero sets of the functions above according to the figure 2.5.

We consider first the set of zeroes of $\tilde{d}_2$. The hyperbolas $\tilde{\sigma}_2 = \mp 2 - \dfrac{B}{\tilde{\sigma}_1}$ defined by $\tilde{D}_1 = 0$ and $\tilde{D}_2 = 0$ respectively have no self intersections. Moreover, $\tilde{d}_2 < 0 \Leftrightarrow -2 - \dfrac{B}{\tilde{\sigma}_1} < \tilde{\sigma}_2 < -2 + \dfrac{B}{\tilde{\sigma}_1}$. Therefore, the region $\tilde{d}_2 < 0$, which corresponds to an EH region, has 2 connected components. Figure 2.8 shows the boundaries of the region EH in the case $B > 0$. For $B < 0$ we get a symmetric picture respect the $\tilde{\sigma}_2$ axis.

In the region $\tilde{d}_2 > 0$ we can have the following behaviors:

- if $\tilde{d}_3 < 0 \rightarrow$ CS,
- if $\tilde{d}_3 > 0 \rightarrow \begin{cases} \text{if} \quad \tilde{d}_1 > 0 & \rightarrow \quad \text{EE} \\ \text{if} \quad \tilde{d}_1 < 0 & \rightarrow \quad \text{HH} \end{cases}$.

Now we consider the curve $\tilde{d}_3 = 0$. We note that the set of zeroes of $\tilde{d}_3$ is symmetric with respect to the origin. Self intersections are determined by the additional conditions $\dfrac{\partial \tilde{d}_3}{\partial \tilde{\sigma}_1} = 0$ and $\dfrac{\partial \tilde{d}_3}{\partial \tilde{\sigma}_2} = 0$. These equations only have common solutions for $B = -1$ and $B = 0$. If $B = -1$ there is a unique solution $(\tilde{\sigma}_1, \tilde{\sigma}_2) = (0, 0)$. If $B = 0$ we get $\tilde{\sigma}_1^2 - \tilde{\sigma}_2^2 + 4 = 0$. However we assume $B \neq 0$. So, only the origin when $B = -1$ gives a real self intersection for us.

In order to study the curve $\tilde{d}_3 = 0$ for any value of $B \neq 0$, it will be useful to introduce $z_1 := \tilde{\sigma}_1 + \tilde{\sigma}_2$ and $z_2 := \tilde{\sigma}_1 - \tilde{\sigma}_2$. Then, $\tilde{d}_3$ can be written as

$$\tilde{d}_3 = (z_1 z_2 + 4)^2 - 4B(z_1^2 - 4).$$

The following claims for the solutions of $\tilde{d}_3 = 0$ are trivially obtained. If $B < -1$, for any real value of z_2 there are two different solutions of z_1 (see figure 2.9 (a)). If $B = -1$, for any real value of z_2, $z_2 \neq 0$, there exist two different solutions of z_1. One of them is $z_1 = 0$, which corresponds to the straight line $\tilde{\sigma}_2 = -\tilde{\sigma}_1$ in the plane $\tilde\sigma_1, \tilde\sigma_2$. If $-1 < B < 0$, for any real value of z_2, $|z_2| > 2\sqrt{1+B}$ there are two different solutions for z_1. If $|z_2| = 2\sqrt{1+B}$, a double solution is obtained. If $B > 0$, for any real value of z_1, $|z_1| > 2$ we get two real solutions for $z_2 = 0$. In the cases $|z_1| = 2$ we get double solutions for z_2, that means, straight lines $\tilde\sigma_1 + \tilde\sigma_2 = \pm 2$ are tangent to $\tilde d_3 = 0$. Figure 2.9 shows the evolution of the region corresponding to $\tilde d_3 < 0$, that is the CS region, in the plane $\tilde\sigma_1, \tilde\sigma_2$.

Now we go to study the intersections of $\tilde d_2 = 0$ and $\tilde d_3 = 0$. This is equivalent to look at the intersections of $\tilde d_1 = 0$ and $\tilde d_2 = 0$. We recall that $\tilde d_2 = \tilde D_1 \tilde D_2$. So, we shall consider the intersections of

$$\tilde d_1 = 0, \qquad \tilde D_1 = 0. \tag{2.65}$$

Using the symmetry, the solutions of $\tilde d_1 = 0$, $\tilde D_2 = 0$ will be easily obtained.

The solutions of (2.65) are the intersection points of a circle of radius $4 + 2B$ and the hyperbola $\tilde\sigma_2 = -2 - \dfrac{B}{\tilde\sigma_1}$. We assume $B > -2$, otherwise (2.65) has no solutions. We shall do the following steps. First we look for the points P_1, P_2 in the hyperbola such that the distance to the origin has a relative minimum. Then, we shall determine the values of B such that points P_1, P_2 are inside the circle of radius $4 + 2B$. We note that for any point P_1 or P_2 which satisfies that condition, there are two solutions of (2.65) , and using the symmetry two additional solutions of $\tilde d_1 = 0$, $\tilde D_2 = 0$ are obtained.

We begin by looking at the points in $\tilde D_1 = 0$ such that the distance to the origin is a relative minimum. To this end, we use a Lagrange multiplier ρ with Lagrangian

$$L = \tilde\sigma_1^2 + \tilde\sigma_2^2 - \rho \tilde D_1.$$

We get a minimum $(\tilde\sigma_{1,m}, \tilde\sigma_{2,m})$ for

$$\tilde\sigma_{1,m} = \frac{4\rho}{4-\rho^2}, \qquad \tilde\sigma_{2,m} = \frac{2\rho^2}{4-\rho^2}, \tag{2.66}$$

where ρ satisfies

$$f_1(\rho, B) := (4-\rho^2)^2 + \frac{32\rho}{B} = 0,$$

or equivalently (note that $f_1(0, B) \neq 0$)

$$\frac{(4-\rho^2)^2}{\rho} = -\frac{32}{B}. \tag{2.67}$$

Figure 2.10 shows the graphic of $g_1(\rho) := \dfrac{(4-\rho^2)^2}{\rho}$. For any value of B, $B \neq 0$ (2.65) has two real solutions ρ_1, ρ_2 giving rise to points P_1, P_2, respectively, in the plane $\tilde{\sigma}_1, \tilde{\sigma}_2$. If $B > 0$ then $\rho_1 < -2 < \rho_2 < 0$ and, $0 < \rho_1 < 2 < \rho_2$ if $B < 0$. Now we study the sign of $\tilde{d}_1$ on P_1, P_2. Using (2.66) for $B \neq 0$ we get

$$\tilde{d}_1(\rho) := \tilde{d}_1(\tilde{\sigma}_{1,m}, \tilde{\sigma}_{2,m}) = -\frac{\rho B}{8}(4+\rho^2) - (4+2B) = -\frac{B}{8}\left[\rho(\rho^2+4) + \frac{8}{B}(4+2B)\right].$$

Let be $g_2(\rho) = \rho(\rho^2+4) + 16$. It is clear that for any value of $B \neq 0$ there is a unique ρ, ρ_3, such that $\tilde{d}_1(\rho_3) = 0$ (see figure 2.10). We are interested in the sign of $\tilde{d}_1(\rho_1)$ and $\tilde{d}_1(\rho_2)$. The figure 2.11 shows the graphics of $g_1(\rho)$ and $g_2(\rho)$ as well as ρ_1, ρ_2, ρ_3 for different values of B. If $B > 0$, $\rho_1 < \rho_3 < -2 < \rho_2 < 0$. Then, $\tilde{d}_1(\rho_1) > 0$ and $\tilde{d}_1(\rho_2) < 0$, that is, only P_2 is inside the circle. If $B < 0$ we distinguish three cases (see figure 2.11).

1. $B < -\dfrac{27}{16}$. Then $\rho_3 < \rho_1 < \rho_2$ and $\tilde{d}_1(\rho_i) > 0$, $i = 1, 2$. P_1 and P_2 are outside the circle, and (2.65) has no real solutions.
2. $-\dfrac{27}{16} < B < 0$. In this case $0 < \rho_1 < \rho_3 < \rho_2$ and then $\tilde{d}_1(\rho_1) < 0$, $\tilde{d}_1(\rho_2) > 0$. Only P_1 is inside the circle and (2.65) has two different solutions.
3. $B = -\dfrac{27}{16}$. Then $\tilde{d}_1(\rho_1) = 0$, $\tilde{d}_1(\rho_2) > 0$. (2.65) has a unique real solution, $(\tilde{\sigma}_1, \tilde{\sigma}_2) = \left(\dfrac{3}{4}, \dfrac{1}{4}\right)$.

We summarize the results above. Consider the set of points in the plane $\tilde{\sigma}_1, \tilde{\sigma}_2$ such that $\tilde{d}_2 > 0$ and $\tilde{d}_3 > 0$. According to the figure 2.5, if $d_1 < 0$ we get an HH region and if $d_1 > 0$ an EE region. We distinguish the following cases.

(a) $B < -\dfrac{27}{16}$. There is no HH region (see figure 2.7 (f)).

(b) $-\dfrac{27}{16} < B < -1$. There are 4 intersection points of $\tilde{d}_3 = 0$ and $\tilde{d}_2 = 0$. This gives an HH region which has two connected components (see figure 2.7 (d)).

(c) $-1 < B$. There are 4 intersection points of $\tilde{d}_3 = 0$ and $\tilde{d}_2 = 0$. However the HH region has one connected component (see figure 2.7 (a), (b)).

□

Now we study the case $\mu \neq 0$, that is, we analize the effect of the neglected terms. We obtain the following result.

Proposition 2.5.8. *Assume that hypothesis in theorem 2.5.6 are satisfied and* $n_1 = 3n_2$. *Under the generic assumptions* $m_j \neq 0$, $j = 1, \ldots, 4$ *in the Normal Form (2.34) the unique changes in the bifurcation diagram are produced at* $B = -(1+\tilde{\sigma}_3)^2$ *and at* $B_\pm = -\dfrac{27}{16} \pm \dfrac{1}{2} sA\tilde{\sigma}_5 + O(\mu^2)$.

Proof

We know from Proposition 2.5.4 that in the case $\mu = 0$, bifurcations are produced at $B = -1$ due to self intersections of $\tilde{d}_3 = 0$ and, $B = -\dfrac{27}{16}$ when $\tilde{d}_1 = 0$ and $\tilde{d}_2 = 0$ have tangencies. We recall that in this case no self intersections of $\tilde{d}_2 = 0$ occurs.

Let us consider $\mu \neq 0$ small enough. In this case, self–intersections of $\tilde{d}_2 = 0$ can occur. Using (2.52) these will occur if

$$\begin{aligned}\tilde{D}_1 &= (\tilde{\sigma}_1 - 2s\tilde{\sigma}_3)(\tilde{\sigma}_2 + 2) + s(\tilde{\sigma}_5 + A)^2 = 0,\\ \tilde{D}_2 &= (\tilde{\sigma}_1 + 2s\tilde{\sigma}_3)(\tilde{\sigma}_2 - 2) + s(\tilde{\sigma}_5 - A)^2 = 0.\end{aligned}$$

Substracting this equations, we obtain that

$$\tilde{\sigma}_1 - s\tilde{\sigma}_2 O(\mu^2) + sAO(\mu) \quad = \quad 0.$$

If we substitute the relation obtained in $\tilde{D}_1 = 0$ it turns that

$$\tilde{\sigma}_2^2 O(\mu^2) + \tilde{\sigma}_2 O(\mu) + B + O(\mu^2) \quad = \quad 0.$$

Then, self–intersections of $\tilde{d}_2 = 0$ can occur, but outside a local neighbourhood of the origin on the $(\tilde{\sigma}_1, \tilde{\sigma}_2)$–plane. Hence, they should not be considered.

Concerning self–intersections of $\tilde{d}_3 = 0$, they are produced if

$$\tilde{d}_3 = 0, \qquad \frac{\partial \tilde{d}_3}{\partial \tilde{\sigma}_1} = 0, \qquad \frac{\partial \tilde{d}_3}{\partial \tilde{\sigma}_2} = 0 \quad .$$

If $\mu = 0$, the system above has the solution $(B, \tilde{\sigma}_1, \tilde{\sigma}_2) = (-1, 0, 0)$. The Jacobian with respect to B, $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$ at that point is different from zero. Then, the Implicit Function Theorem ensures the preservation of the intersection which will occur for a value of B equal, a priori, to $-1 + O(\mu)$ and with values $\tilde{\sigma}_1, \tilde{\sigma}_2 = O(\mu)$.

An elementary computation shows that the self–intersections of $\tilde{d}_3 = 0$ occurs exactly for $B = -(1 + \tilde{\sigma}_3)^2$ at $\tilde{\sigma}_1 = \tilde{\sigma}_2 = \tilde{\sigma}_5$. Furthermore, for that value of B, the line $\tilde{\sigma}_1 + \tilde{\sigma}_2 = 2\tilde{\sigma}_5$ is one of the components of $\tilde{d}_3 = 0$. Figure 2.12 shows an illustration.

It remains to study the modification of the tangencies of the zero sets of $\tilde{d}_1 = 0$ and $\tilde{d}_2 = 0$. We note that symmetry is lost for $\mu \neq 0$. So, one has to consider the cases $\tilde{d}_1 = 0$, $\tilde{D}_1 = 0$ and $\tilde{d}_1 = 0$, $\tilde{D}_2 = 0$ separately. Let us consider the first case. We have

$$\begin{aligned}\tilde{d}_1 &= \tilde{\sigma}_1^2 + \tilde{\sigma}_2^2 - (4 + 2B) + O(\mu^2) = 0,\\ \tilde{D}_1 &= \tilde{\sigma}_1(\tilde{\sigma}_2 + 2) + B + \nu + O(\mu^2) = 0,\end{aligned}$$

where $\nu := 2sA\tilde{\sigma}_5 = O(\mu)$. Up to order μ, $\tilde{d}_1 = 0$ is a cercle. Following the same steps as in the proof of Proposition 2.5.4, we look for the points of $\tilde{D}_1 = 0$ which are at minimum distance to the origin. Using the Lagrangian $L = \tilde{\sigma}_1^2 + \tilde{\sigma}_2^2 - \rho\tilde{D}_1$ we get a minimum $(\tilde{\sigma}_{1,m}, \tilde{\sigma}_{2,m})$ as (2.66) where the Lagrange multiplier ρ satisfies

$$\tilde{D}_1(\tilde{\sigma}_{1,m}, \tilde{\sigma}_{2,m}) = \frac{32\rho}{(4-\rho^2)^2} + B + \nu = 0.$$

However,

$$\tilde{d}_1(\tilde{\sigma}_{1,m}, \tilde{\sigma}_{2,m}) = \frac{16\rho^2}{(4-\rho^2)^2} + \frac{4\rho^4}{(4-\rho^2)^2} - (4+2B).$$

We must solve the following system

$$\begin{aligned} &32\rho + (B+\nu)(4-\rho^2)^2 = 0, \\ &16\rho^2 + 4\rho^4 - (4+2B)(4-\rho^2)^2 = 0. \end{aligned}$$

For $\mu = 0$, we have the solution $\rho = \frac{2}{3}$, $B = -\frac{27}{16}$. One step of Newton's Method around that solution gives the critical value of B

$$B_+ = -\frac{27}{16} + \frac{1}{2}sA\tilde{\sigma}_5 + O(\mu^2),$$

A similar study for $\tilde{d}_1 = 0$, $\tilde{D}_2 = 0$ gives a second critical value

$$B_- = -\frac{27}{16} - \frac{1}{2}sA\tilde{\sigma}_5 + O(\mu^2).$$

□

Remark 2.5.9. The geometrical interpretation is that the two narrow HH domains which in the figure 2.7 (f) disappear on the (b) plot ($B = -\frac{27}{16}$) when going from left to right, disappear for slightly different values of B if $\mu \neq 0$. No further changes occur in the bifurcation diagram for ε small enough in case 2).

Proof of Theorem 2.5.6

The item (i) follows from propositions 2.5.7 and 2.5.8.

To prove (ii) we study the cases 1. and 3.. To this end we use the same scalings as in case 2. We have that A in (2.64) is of order $O(\varepsilon^{\frac{n_1-3n_2}{2}})$. Then, the case 1., $n_1 > 3n_2$, has the same characteristics than a very small value of $|B|$. In case 3., $n_1 < 3n_2$, has the same characteristics than a very large value of $|B|$. Then there are no changes in the bifurcation diagram. □

2.6 Proof of Proposition 2.3.4

In this section we shall prove the proposition 2.3.4.

Let $\mathcal{H}(\mathbf{z}, w, K)$ be the Hamiltonian defined in (2.29). Our purpose is to use the Giorgilli–Galgani algorithm to obtain the Normal Form for this Hamiltonian.

Let be $\mathcal{H}(\mathbf{z}, w) = \mathcal{H}(\mathbf{z}, w, K) - K$. We recall that $\mathcal{H}(\mathbf{z}, w)$ is an homogeneous polynomial of degree 2 in $\mathbf{z}$ whose coefficients depend on w and w^{-1}.

It will be useful to introduce the following functions.

$$\begin{aligned}\mathcal{F}(\mathbf{z}, w) &= f_1 z_1^2 + f_2 z_2^2 + f_3 z_3^2 + f_4 z_4^2 + f_5 z_1 z_2 + f_6 z_1 z_3 + f_7 z_1 z_4 + f_8 z_2 z_3 + \\ &\quad + f_9 z_2 z_4 + f_{10} z_3 z_4, \end{aligned} \tag{2.68}$$

where $f_j = f_j(w)$, $j = 1, \ldots, 10$ can be written as

$$f(w) = \sum_{j\geq 0} (\tilde{c}_j w^j + \tilde{d}_j w^{-j}), \tag{2.69}$$

being the coefficients $\tilde{c}_j$, $\tilde{d}_j$ analytic functions on $\delta_1, \delta_2, \varepsilon$. Let us denote by H_2^T the vector space of functions (2.68).

Given $\mathcal{F}(\mathbf{z}, w)$ in H_2^T, $\overline{\mathcal{F}}(\mathbf{z}, w)$ will be obtained from (2.68) by a substitution of f_j by $\overline{f}_j = \overline{f_j(w)}$, for $j = 1, \ldots, 10$, where the bar stands for the complex conjugate.

From lemma 2.3.3 and taking into account that w has been defined in section 2.3.2 as $w = e^{\frac{2it}{\nu}}$, we get

$$\mathcal{H}(\mathbf{z}, w) = \mathcal{H}(S_1 \mathbf{z}, w^{-1}). \tag{2.70}$$

Moreover, as far as $\mathcal{H}(\mathbf{z}, t)$ in (2.24) is an even function of t, we get

$$\mathcal{H}(\mathbf{z}, w) = \overline{\mathcal{H}}(\overline{S}_2 \mathbf{z}, w). \tag{2.71}$$

We shall see that these two symmetries will be preserved to the Normal Form. To do that we use the Giorgilli–Galgani algorithm (see appendix C) to get the Normal Form.

Let us denote by NF the new Hamiltonian up to order m in $\delta_1, \delta_2, \varepsilon$. Then $NF = \mathcal{N}_0 + \mathcal{N}_1 + \mathcal{N}_2 + \ldots + \mathcal{N}_m$ with $\mathcal{N}_k$ defined at (2.31), that is,

$$\mathcal{N}_k = \sum_{j=0}^{k} \mathcal{H}_{j,k-j}, \qquad \mathcal{H}_{k,j} = \sum_{l=1}^{j} \frac{l}{j} [G_l, \mathcal{H}_{k,j-l}], \qquad \mathcal{H}_{k,0} = \mathcal{H}_k,$$

and G_k is the solution of the homological equation (2.32). In (2.32),

$$\mathcal{H}_0 = \mathcal{H}_0(\mathbf{z}, K) = \rho_1 z_1 z_3 + \rho_2 z_2 z_4 + K.$$

Functions $H_{i,j}, G_k, M_k$ belongs to H_2^T. We denote each term as

$$g = h\mathbf{z}^{\mathbf{l}}w^j, \qquad h = c\ \delta_1^{j_1}\delta_2^{j_2}\varepsilon^{j_3}, \tag{2.72}$$

where $c \in \mathbb{C}$ is a constant, $j_i \in \mathbb{Z}$, $j_i \geq 0$, $i = 1, 2, 3$, $j \in \mathbb{Z}$, and $\mathbf{z}^{\mathbf{l}} = z_1^{l_1}z_2^{l_2}z_3^{l_3}z_4^{l_4}$ with $l_k \in \mathbb{Z}$, $l_k \geq 0$, $k = 1, 2, 3, 4$ satisfying $l_1 + l_2 + l_3 + l_4 = 2$.

$\mathbf{z}^{\mathbf{l}}w^j$ as in (2.72) is a resonant monomial if $[\mathbf{z}^{\mathbf{l}}w^j, \mathcal{H}_0] = 0$, that is,

$$(l_1 - l_3)\rho_1 + (l_2 - l_4)\rho_2 + \frac{2ij}{\nu} \quad = \quad 0.$$

From this equation it is easy to get the following lemma.

Lemma 2.6.1. *z_1z_3, z_2z_4 are resonant terms for all $(a_1, a_2) \in \mathcal{R}$. Moreover,*

1. *if $(a_1, a_2) \in \mathcal{R}_1$ with $\omega\nu \in \mathbb{N}$ then $z_2^2w^{-\nu\omega}$, $z_4^2w^{\nu\omega}$;*
2. *if $(a_1, a_2) \in \mathcal{R}_2$ with*
   - *(a) $\omega_1\nu \in \mathbb{N}$ then $z_1^2w^{-\nu\omega_1}$, $z_3^2w^{\nu\omega_1}$,*
   - *(b) $\omega_2\nu \in \mathbb{N}$ then $z_2^2w^{-\nu\omega_2}$, $z_4^2w^{\nu\omega_2}$,*
   - *(c) $\frac{\nu}{2}(\omega_1 + \omega_2) \in \mathbb{N}$, $z_1z_2w^{-\frac{\nu}{2}(\omega_1+\omega_2)}$, $z_3z_4w^{\frac{\nu}{2}(\omega_1+\omega_2)}$,*
   - *(d) $\frac{\nu}{2}(\omega_1 - \omega_2) \in \mathbb{N}$, $z_1z_4w^{-\frac{\nu}{2}(\omega_1-\omega_2)}$, $z_2z_3w^{\frac{\nu}{2}(\omega_1-\omega_2)}$,*
3. *if $(a_1, a_2) \in \mathcal{R}_3$ and $\nu\beta \in \mathbb{N}$, $z_1z_4w^{-\nu\beta}$, $z_2z_3w^{\nu\beta}$,*

are resonant monomials.

Let $\mathcal{F}(\mathbf{z}, w)$ be in H_2^T.

Definition 2.6.2. $\mathcal{F}(\mathbf{z}, w)$ satisfies the S_2–property if

$$\mathcal{F}(\mathbf{z}, w) \quad = \quad \overline{\mathcal{F}}(\overline{S}_2\mathbf{z}, w), \tag{2.73}$$

for all $\mathbf{z} \in \mathbb{C}^4$, $w \in \mathbb{C}$, $|w| = 1$.

Definition 2.6.3. $\mathcal{F}(\mathbf{z}, w)$ satisfies the S_1^+–property if

$$\mathcal{F}(\mathbf{z}, w) \quad = \quad \mathcal{F}(S_1\mathbf{z}, w^{-1}), \tag{2.74}$$

for all $\mathbf{z} \in \mathbb{C}^4$, $w \in \mathbb{C}$, $|w| = 1$.

Definition 2.6.4. $\mathcal{F}(\mathbf{z}, w)$ satisfies the S_1^-–property if

$$\mathcal{F}(\mathbf{z}, w) \quad = \quad -\mathcal{F}(S_1\mathbf{z}, w^{-1}), \tag{2.75}$$

for all $\mathbf{z} \in \mathbb{C}^4$, $w \in \mathbb{C}$, $|w| = 1$.

Lemma 2.6.5. *The Normal Form up to order m, NF satisfies the S_2–property.*

Proof

We have seen in (2.71) that the initial Hamiltonian satisfies the S_2–property. So, we only need to prove the following statements.

(i) The Poisson bracket preserves the S_2–property.

(ii) Assume that $M \in H_2^T$ satisfies the S_2–property and let G be the solution of the homological equation

$$[G, \mathcal{H}_0] + M = 0.$$

Then, up to resonant terms, G satisfies the S_2–property.

To prove (i) let us consider $\mathcal{F}, \mathcal{G} \in H_2^T$ satisfying the S_2–property. Let be $Q = [\mathcal{G}, \mathcal{F}]$. Using (2.73) we get

$$Q(\mathbf{z}, w) = \nabla \mathcal{G}(\mathbf{z}, w)^T J \nabla \mathcal{F}(\mathbf{z}, w) = \nabla \overline{\mathcal{G}}(\overline{S}_2 \mathbf{z}, w)^T \ \ \overline{S}_2 J \overline{S}_2^T \ \ \nabla \mathcal{F}(\overline{S}_2 \mathbf{z}, w).$$

However, using the definition (2.23) of S_2 and the symplectic character of M we have that $\overline{S}_2 J \overline{S}_2^T = J$. Then,

$$Q(\mathbf{z}, w) = \overline{Q}(\overline{S}_2 \mathbf{z}, w) \quad .$$

Now we prove (ii). Let be $\mathcal{D} = \operatorname{diag}(\rho_1, \rho_2, -\rho_1, -\rho_2)$ and the homological equation

$$M(\mathbf{z}, w) + \frac{\partial G}{\partial t}(\mathbf{z}, w) + \nabla G(\mathbf{z}, w)^T \mathcal{D} \mathbf{z} \ = \ 0. \tag{2.76}$$

We assume $M(\mathbf{z}, w) = \overline{M}(\overline{S}_2 \mathbf{z}, w)$. From (2.76) and using that $\overline{\mathcal{D} S_2} = \overline{S}_2 \mathcal{D}$ we get

$$\frac{\partial G}{\partial t}(\mathbf{z}, w) - \frac{\partial \overline{G}}{\partial t}(\overline{S}_2 \mathbf{z}, w) + [\nabla G(\mathbf{z}, w)^T - \nabla \overline{G}(\overline{S}_2 \mathbf{z}, w)^T \overline{S}_2] \mathcal{D} \mathbf{z} \ = \ 0.$$

Let us define $Y(\mathbf{z}, w) = G(\mathbf{z}, w) - \overline{G}(\overline{S}_2 \mathbf{z}, w)$. Then,

$$\begin{aligned} [Y, \mathcal{H}_0] \ &= \ \frac{\partial Y}{\partial t}(\mathbf{z}, w) + \nabla Y(\mathbf{z}, w)^T \mathcal{D} \mathbf{z} = \\ &= \frac{\partial G}{\partial t}(\mathbf{z}, w) - \frac{\partial \overline{G}}{\partial t}(\overline{S}_2 \mathbf{z}, w) + [\nabla G(\mathbf{z}, w)^T - \nabla \overline{G}(\overline{S}_2 \mathbf{z}, w)^T \overline{S}_2] \mathcal{D} \mathbf{z} = 0. \end{aligned}$$

Therefore $Y(\mathbf{z}, w)$ only has resonant terms. □

Lemma 2.6.6. *The Normal Form NF up to order m of $\mathcal{H}(\mathbf{z}, w, K)$ satisfies the S_1^+–property.*

Proof

From (2.70) the initial Hamiltonian satisfies the S_1^+–property. So, we shall prove the following statements.

(i) If $\mathcal{F} \in H_2^T$ satisfies the S_1^+–property and $\mathcal{G} \in H_2^T$ satisfies the S_1^-–property, then $Q := [\mathcal{G}, \mathcal{F}]$ satisfies the S_1^+–property.

(ii) Assume that $M \in H_2^T$ satisfies the S_1^+–property. Let $G \in H_2^T$ be the solution of the homological equation

$$[G, \mathcal{H}_0] + M = 0.$$

Then, up to resonant terms, G satisfies the S_1^-–property.

The proof of (i) follows the same steps as (i) in the proof of 2.6.5. However using the definition (2.23) of S_1 we get $S_1 J S_1^T = -J$. Then,

$$Q(\mathbf{z}, w) = -\nabla \mathcal{G}(S_1\mathbf{z}, w^{-1})^T \; S_1 J S_1^T \; \nabla \mathcal{F}(S_1\mathbf{z}, w^{-1}) = Q(S_1\mathbf{z}, w^{-1}) \quad .$$

To proof (ii) let us assume that M satisfies the S_1^+ property. Then, $M(\mathbf{z}, w) = M(S_1\mathbf{z}, w^{-1})$. Taking into account the homological equation we get

$$\frac{\partial G}{\partial t}(\mathbf{z}, w) - \frac{\partial G}{\partial t}(S_1\mathbf{z}, w^{-1}) + [\nabla G(\mathbf{z}, w)^T - \nabla G(S_1\mathbf{z}, w^{-1})^T S_1]\mathcal{D}\mathbf{z} \;=\; 0,$$

where the equality $S_1\mathcal{D} = -\mathcal{D}S_1$ has been used.

Let us define $Y(\mathbf{z}, w) = G(\mathbf{z}, w) + G(S_1\mathbf{z}, w^{-1})$. Then,

$$[Y, \mathcal{H}_0] \;=\; \frac{\partial G}{\partial t}(\mathbf{z}, w) - \frac{\partial G}{\partial t}(S_1\mathbf{z}, w^{-1}) + [\nabla G(\mathbf{z}, w)^T + \nabla G(S_1\mathbf{z}, w^{-1})^T S_1]\mathcal{D}\mathbf{z} = 0.$$

Therefore $Y(\mathbf{z}, w)$ only has resonant terms.

As a consequence of (i) and (ii) we have that the functions $H_{i,j}$ which appear in the Giorgilli–Galgani algorithm satisfy the S_1^+–property, while G_k satisfy the S_1^-–property. $\square$

Now we use the S_2 and S_1^+–properties of NF to get the relations between the coefficients of $f_i(w)$ which appear in NF.

Let us consider $(a_1, a_2) \in \mathcal{R}_1$. The resonant terms are given in lema 2.6.1. So, we write the Normal Form as

$$\begin{aligned} NF(\mathbf{z}, w) \;&=\; K + \lambda z_1 z_3 + i\omega z_2 z_4 + a_6 z_1 z_3 + a_9 z_2 z_4 + \\ &\quad + a_2 z_2^2 w^{-\nu\omega} + a_4 z_4^2 w^{\nu\omega}, \end{aligned}$$

for some constants a_2, a_4, a_6, a_9.

Using that $S_1\mathbf{z} = (sz_3, \mathrm{i}z_4, sz_1, -\mathrm{i}z_2)^T$ (lemma 2.3.3),

$$\begin{aligned} NF(S_1\mathbf{z}, w^{-1}) &= K + \lambda z_1 z_3 + i\omega z_2 z_4 + a_6 z_1 z_3 + a_9 z_2 z_4 + \\ &\quad - a_2 z_4^2 w^{\nu\omega} - a_4 z_2^2 w^{-\nu\omega}. \end{aligned}$$

The S_1^+–property for NF implies that

$$a_4 = -a_2. \tag{2.77}$$

Moreover,

$$\begin{aligned} \overline{NF}(\overline{S}_2\mathbf{z}, w^{-1}) &= K + \lambda z_1 z_3 + i\omega z_2 z_4 + \overline{a}_6 z_1 z_3 - \overline{a}_9 z_2 z_4 + \\ &\quad - \overline{a}_2 z_4^2 w^{\nu\omega} - \overline{a}_4 z_2^2 w^{-\nu\omega}. \end{aligned}$$

The S_2–property implies that $a_6 \in \mathbb{R}$, a_9 is imaginary and

$$a_4 = -\overline{a}_2. \tag{2.78}$$

From (2.77) and (2.78), $a_2 \in \mathbb{R}$ and $a_4 = -a_2$. This proves (2.33).

Let be $(a_1, a_2) \in \mathcal{R}_2$. We consider the case for which the Normal Form contains all possible resonant terms and we write it as

$$\begin{gathered} NF(\mathbf{z}, w) = K + \mathrm{i}\omega_1 z_1 z_3 + \mathrm{i}\omega_2 z_2 z_4 + a_6 z_1 z_3 + a_9 z_2 z_4 + a_1 z_1^2 w^{-\nu\omega_1} + a_3 z_3^2 w^{\nu\omega_1} + \\ a_2 z_2^2 w^{-\nu\omega_2} + a_4 z_4^2 w^{\nu\omega_2} + a_5 z_1 z_2 w^{-\nu\omega_{hs}} + a_{10} z_3 z_4 w^{\nu\omega_{hs}} + \\ a_7 z_1 z_4 w^{-\nu\omega_{hd}} + a_8 z_2 z_3 w^{\nu\omega_{hd}}, \end{gathered}$$

where we recall that $\omega_{hs} = \dfrac{1}{2}(\omega_1 + \omega_2)$ and $\omega_{hd} = \dfrac{1}{2}(\omega_1 - \omega_2)$. Then,

$$\begin{gathered} NF(S_1\mathbf{z}, w^{-1}) = K + \mathrm{i}\omega_1 z_1 z_3 + \mathrm{i}\omega_2 z_2 z_4 + a_6 z_1 z_3 + a_9 z_2 z_4 - a_1 z_3^2 w^{\nu\omega_1} - \\ a_3 z_1^2 w^{-\nu\omega_1} - a_2 z_4^2 w^{\nu\omega_2} - a_4 z_2^2 w^{-\nu\omega_2} + s a_5 z_3 z_4 w^{\nu\omega_{hs}} + s a_{10} z_1 z_2 w^{-\nu\omega_{hs}} - \\ s a_7 z_2 z_3 w^{\nu\omega_{hd}} + s a_8 z_1 z_4 w^{-\nu\omega_{hd}}. \end{gathered}$$

Then,

$$a_3 = -a_1, \quad a_4 = -a_2, \quad a_{10} = s a_5, \quad a_8 = -s a_7.$$

In a similar way, using the S_2–property we get

$$a_6 = -\overline{a}_6, \quad a_9 = -\overline{a}_9, \quad a_3 = -\overline{a}_1, \quad a_4 = -\overline{a}_2, \quad a_{10} = s\overline{a}_5, \qquad a_8 = s\overline{a}_7.$$

Therefore $a_1, a_2, a_5 \in \mathbb{R}$, a_6, a_7, a_9 imaginary and $a_3 = -a_1$, $a_4 = -a_2$, $a_{10} = s a_5$, $a_8 = -s a_7$. This proves (2.34).

Finally, we take $(a_1, a_2) \in \mathcal{R}_3$. Then, NF has the form

$$NF(\mathbf{z}, w) = K + (\alpha + \mathrm{i}\beta)z_1 z_3 + (\alpha - \mathrm{i}\beta)z_2 z_4 + a_6 z_1 z_3 + a_9 z_2 z_4 + a_7 z_1 z_4 w^{-\nu\beta} + a_8 z_2 z_3 w^{\nu\beta}.$$

As $S_1\mathbf{z} = (z_3, z_4, z_1, z_2)^T$, we have

$$NF(S_1\mathbf{z}, w^{-1}) = K + (\alpha + \mathrm{i}\beta)z_1 z_3 + (\alpha - \mathrm{i}\beta)z_2 z_4 + a_6 z_1 z_3 + a_9 z_2 z_4 + a_8 z_1 z_4 w^{-\nu\beta} + a_7 z_2 z_3 w^{\nu\beta}.$$

Therefore, due to the S_1^+–property, $a_7 = a_8$. Moreover,

$$\overline{NF}(\overline{S}_2\mathbf{z}, w^{-1}) = K + (\alpha + \mathrm{i}\beta)z_1 z_3 + (\alpha - \mathrm{i}\beta)z_2 z_4 + \overline{a}_9 z_1 z_3 + \overline{a}_6 z_2 z_4 + \overline{a}_8 z_1 z_4 w^{-\nu\beta} + \overline{a}_7 z_2 z_3 w^{\nu\beta}.$$

By the S_2–property, $\overline{a}_9 = a_6$ and $\overline{a}_8 = a_7$.

Then, $a_7 = a_8$ with $a_7 \in \mathbb{R}$ and $\overline{a}_9 = a_6$. This proves (2.36).

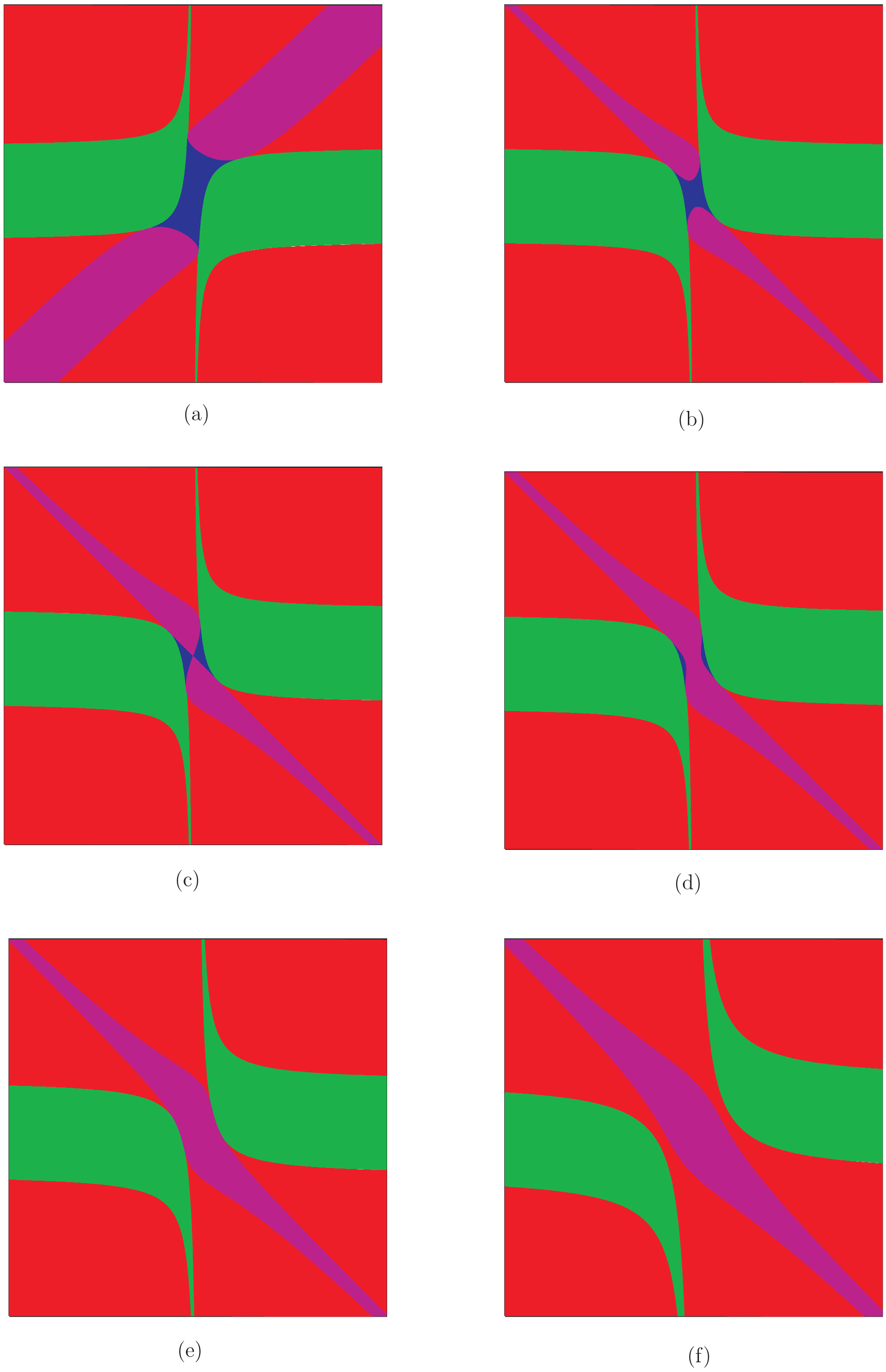


Figure 2.7: A sample of the bifurcation diagrams near double resonance in the d'Alembert case with $n_1 = 3n_2$ and $\mu = 0$. Values of B: (a) 1, (b) -0.9, (c) -1, (d) -1.1, (e) $-27/16$, (f) -4. Color codes: red for EE, green for HE, blue for HH, magenta for CS. The horizontal (resp. vertical) variable is $\tilde{\sigma}_1$ (resp. $\tilde{\sigma}_2$).

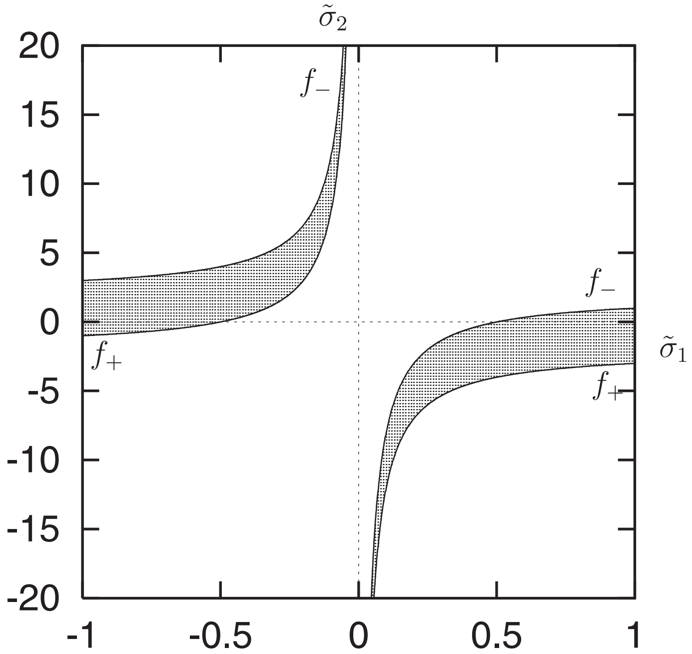


Figure 2.8: Region EH and their boundaries for $B = 1$ where $f_+(\tilde{\sigma}_1) = -2 - \dfrac{B}{\tilde{\sigma}_1}$ and $f_-(\tilde{\sigma}_1) = 2 - \dfrac{B}{\tilde{\sigma}_1}$

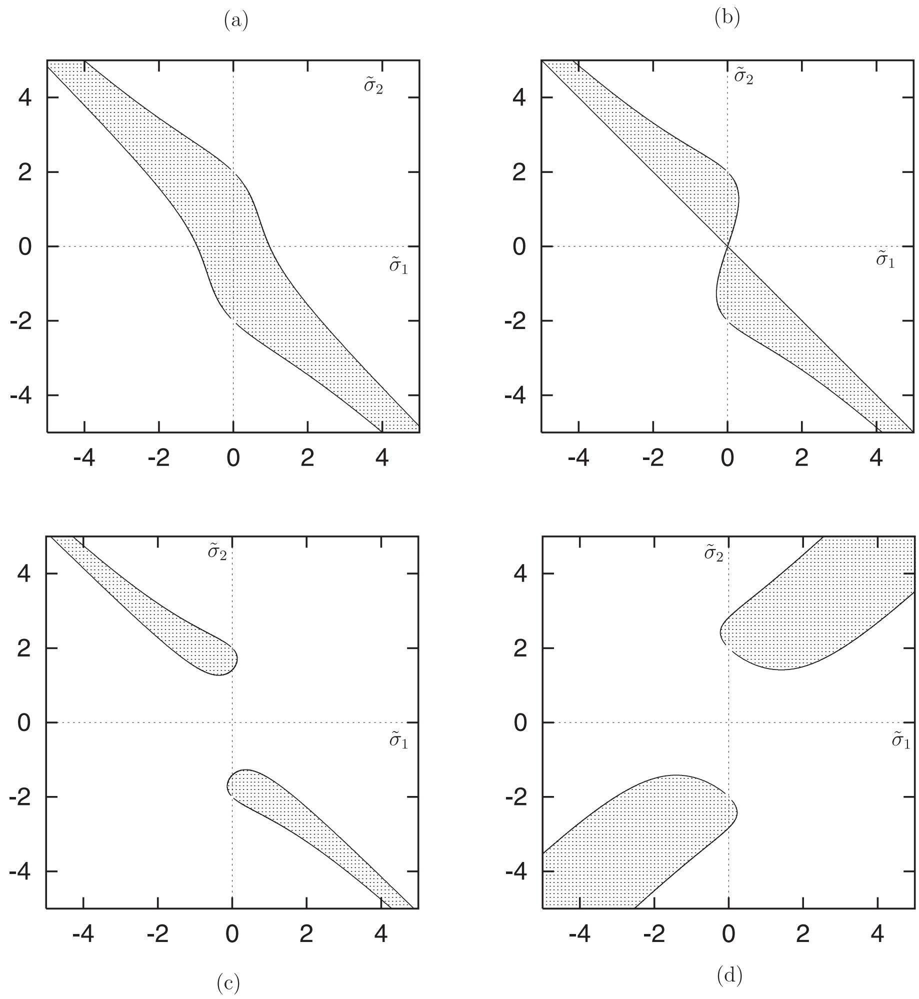


Figure 2.9: Graphic of the curve $\tilde{d}_3 = 0$ for $B = -2$, $B = -1$, $B = -0.5$ and $B = 1$, respectively. The dashed area corresponds to $\tilde{d}_3 < 0$.

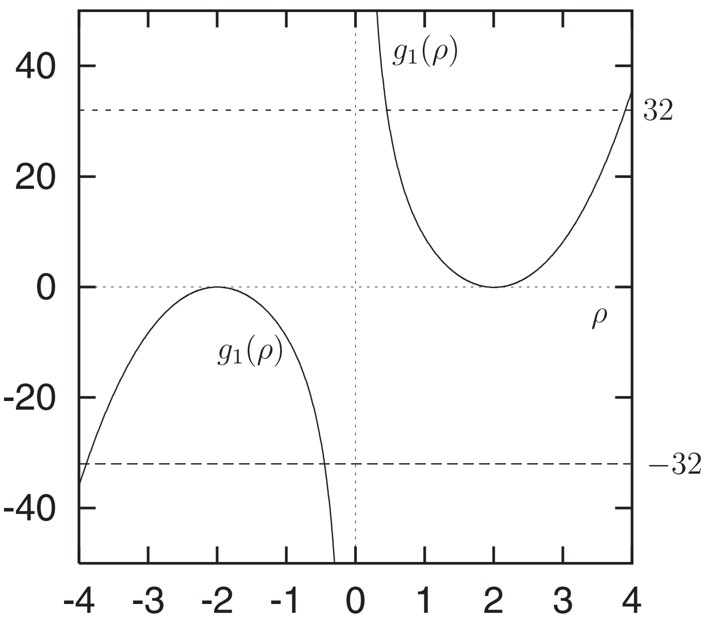


Figure 2.10: Graphic of $g_1(\rho)$ and intersections with $-\dfrac{32}{B}$ for $B = 1$, and $B = -1$

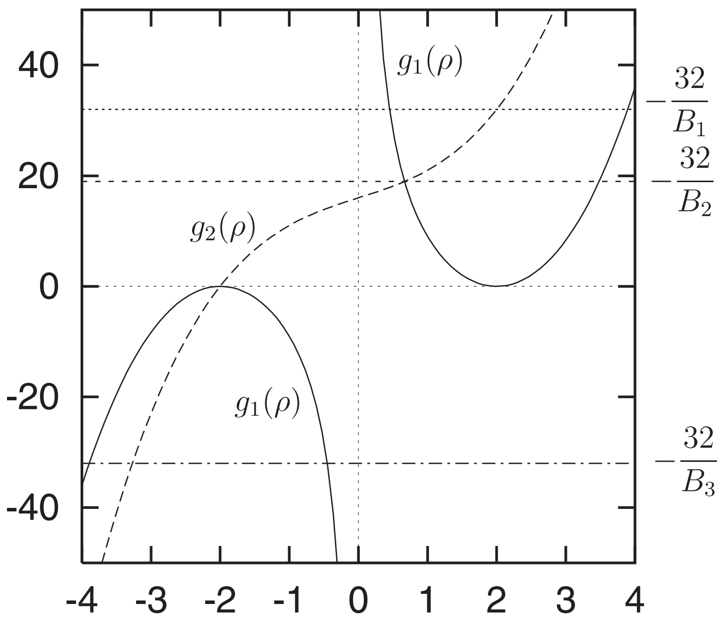
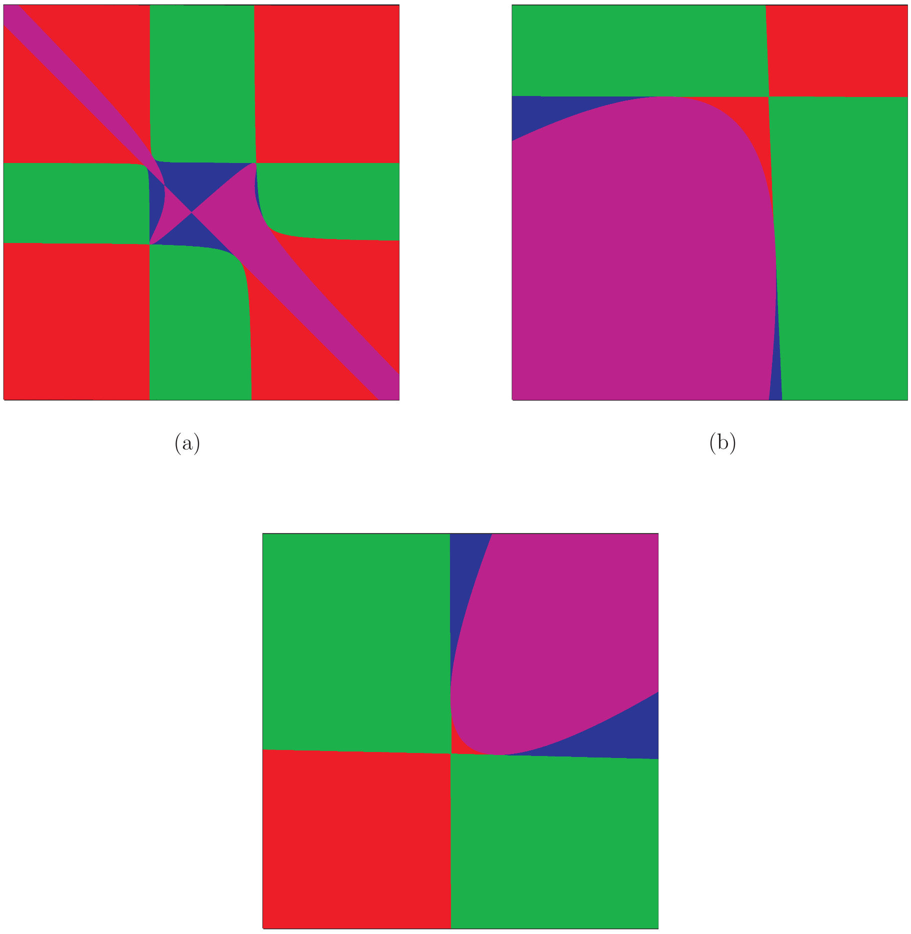


Figure 2.11: Graphic of $g_1(\rho)$, $g_2(\rho)$ and intersections with $-\dfrac{32}{B_j}$ for $B_1 = -1$, $B_2 = -\dfrac{27}{16}$ and $B_3 = 1$



(a) (b)

(c)

Figure 2.12: An example of self–intersections of $\tilde{d}_3 = 0$ in the general case. Scaled parameters used: $\tilde{\sigma}_3 = -1.3$, $\tilde{\sigma}_7 = -0.5$, $A = -0.3$, $s = -1$. Variables plotted as in figure 2.7. (b) plot shows a global view, (a) and (b) are magnifications. Up to 19 connected components can be seen.

Chapter 3

Stability of a family of periodic linear systems: the singular limit case

3.1 Introduction

We consider again a periodic linear system as the one given in (1.55), that is,

$$\dot{\mathbf{x}} \;=\; A(t)\mathbf{x}, \quad A(t) = \begin{pmatrix} 0 & I_2 \\ \tilde{A}(t) & -2J_2 \end{pmatrix}, \quad \tilde{A} = g^{\alpha-2}\mathrm{diag}(\lambda_1, \lambda_2), \quad (3.1)$$

where $0 < \alpha < 2$, λ_1, λ_2 are real parameters different from zero and $g = g(t;\delta)$ is a periodic function on t which depends on a parameter $\delta \in [0, \delta_0]$ with δ_0 small enough. Suppose $g(t,\delta) > 0$ for all t and $g(0,\delta) \to 0$ for $\delta \to 0$. Therefore, the system (3.1) has a singularity at $t = 0$ for $\delta = 0$. Our purpose is to study the stability parameters of system (3.1) for small values of $\delta > 0$ under some hypothesis to be specified below.

Let $U(z) = z^\alpha V(z)$ be a real function defined on an open interval $(0, z_b)$ where $V(z)$ is an analytic function for $z > 0$ such that

(A1) *there exists* z_a, $0 < z_a < z_b$ *such that* $V(z_a) = 0$, $V(z) < 0$ *for all* $z \in (0, z_a)$ *and* $V_z(z) > 0$ *for all* $z \in (0, z_b)$.

(A2) $V(z) = \gamma + z^s V_1(z)$ *with* $\gamma < 0$, $s > \dfrac{2-\alpha}{2}$ *and* $V_1(z)$ *and analytic function on an open set* J, $J \supset [0, z_a]$.

In (A1), $V_z(z)$ stands for the derivative of $V(z)$ with respect to z.

Remark 3.1.1. We have that the only zeroes of $U(z)$ in $[0, z_a]$ are 0 and z_a.

Figure 3.1 shows the plot of $U(z)$ corresponding to the homographic system (1.31) for different values of α.

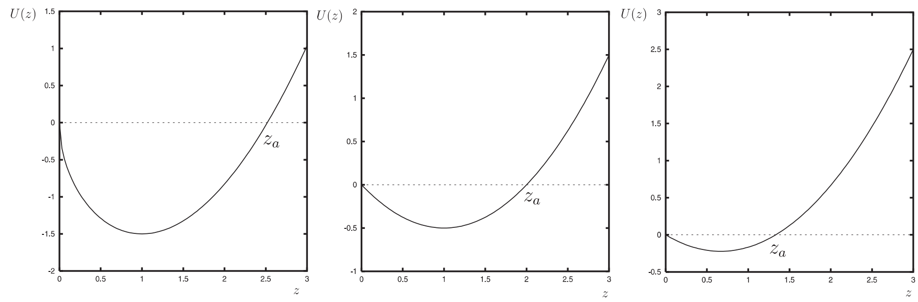


Figure 3.1: Plot of $U(z) = z^{\alpha}\left(-\frac{1}{\alpha} + \frac{z^{2-\alpha}}{2}\right)$ for values of α from left to right: $0 < \alpha < 1$, $\alpha = 1$ (Newtonian case), $1 < \alpha < 2$

Let us consider the conservative system

$$\ddot{z} = -U_z(z) \tag{3.2}$$

with $U(z)$ satisfying (A1) and (A2). $U_z(z)$ stands for the derivative with respect z. We denote the energy of (3.2) by

$$E = \frac{\dot{z}^2}{2} + U(z). \tag{3.3}$$

We shall assume the following hypothesis for $g(t;\delta)$

(B) *For $\delta > 0$, $g(t;\delta)$ is the periodic solution of (3.2) on the energy level $E = -\delta$ such that $g(0;\delta) = g_0$, $\dot{g}(0;\delta) = 0$ being g_0 the minimum of $g(t;\delta)$.*

Figure on the left in 3.2 shows the phase portrait of system (3.2) on the plane $(z, \dot{z})$ for $U(z) = z\left(-1 + \frac{z}{2}\right)$. Fixed $\delta > 0$, the figure on the right shows how we choose $g(t;\delta)$.

Note that for $\delta > 0$, $g(t;\delta)$ is periodic with period $T = T(\delta)$. Moreover, from (3.3) we have $-\delta = U(g_0)$. Then, $g_0 = \left(\frac{\delta}{|\gamma|}\right)^{\frac{1}{\alpha}}(1 + O(\delta^{\frac{s}{\alpha}}))$.

We remark that if g satisfies property (B) then $g(t;\delta)$ is an even function on t.

As we have seen in chapter 1, the motivation to study the system (3.1) comes from the linear stability analysis of the homographic solutions. The corresponding variational equations of the linearized system on these solutions can be reduced

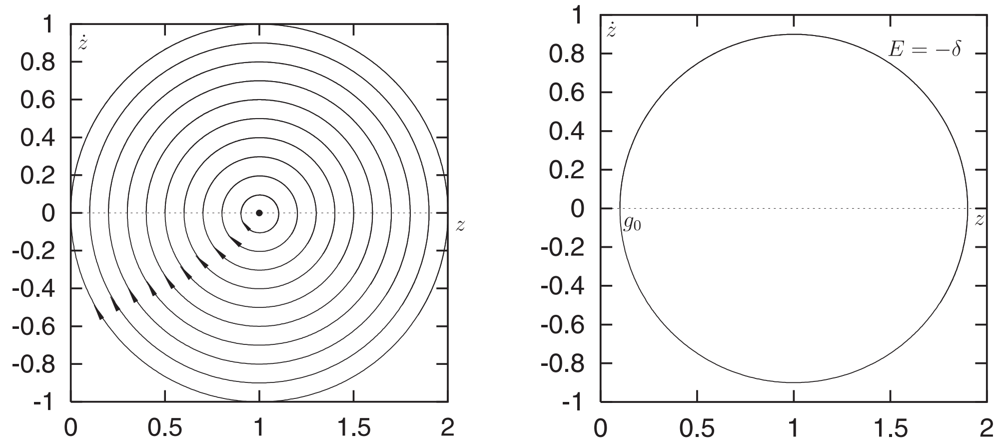


Figure 3.2: Phase portrait of (3.2) for $U(z) = z\left(-1 + \dfrac{z}{2}\right)$

to a linear system of type (3.1). In particular the Newtonian case is obtained for $\alpha = 1$ and $U(z) = z\Big(-1 + \dfrac{z}{2}\Big)$. In this case, $g(t;\delta) = 1 - e\cos t$, where e is the eccentricity of the homographic solution, and $\delta = \dfrac{1-e^2}{2}$. The time t is the true anomaly. The singularity of the equations is attained for $e = 1$.

To simplify the notation the dependence on the parameters of system (3.1) will not be explicitly written if there is no confusion. We shall use the same simplification for all linear systems and their corresponding monodromy matrices which appear in the chapter.

From section 1.6 system (3.1) can be written in Hamiltonian formulation. We recall that the Hamiltonian function is

$$H = \frac{1}{2}(y_3^2 + y_4^2) + (-y_2y_3 + y_1y_4) - \left(\lambda_1 g^{\alpha-2} - 1\right)\frac{y_1^2}{2} - \left(\lambda_2 g^{\alpha-2} - 1\right)\frac{y_2^2}{2}. \quad (3.4)$$

We shall give asymptotic formulae for these stability parameters. To do that the main point is to use some kind of blow up technique to see the limit case when δ tends to zero as a linear system on an heteroclinic connection.

We shall work, for $\delta > 0$, with a linear system without any singularity. So, we consider the change of variables $\mathbf{u} = S(t)\mathbf{x}$ where $S(t) = \mathrm{diag}(1,1,q,q)$ being $q = q(t;\delta)$ defined by $q = g^{\frac{2-\alpha}{2}}$, and we use time τ defined through $dt = qd\tau$. We remark that for $\delta > 0$, $S(t)$ is non singular for all t.

Then, the new system can be written as

$$\mathbf{u}' = B(\tau)\mathbf{u}, \qquad B(\tau) = q(\dot{S} + SA)S^{-1}, \quad (3.5)$$

where $'$ stands for the derivative with respect to τ. The period T of $g(t;\delta)$ in the new time τ will be denoted by $\mathcal{T}(\delta)$ or simply $\mathcal{T}$. In order to simplify the notation, in the following we shall write $q(t)$ instead of $q(t;\delta)$ if there is no confusion.

Let $\Psi(\tau)$ be the fundamental matrix of (3.5). We have that

$$\Phi(t) \quad = \quad S^{-1}(t)\Psi(\tau(t))S(0), \tag{3.6}$$

where Φ denotes the fundamental matrix of (3.1) such that $\Phi(0) = I_4$. Due to the T–periodicity of S we get for the monodromy matrices the equality $\Phi(T) = S^{-1}(0)\Psi(\mathcal{T})S(0)$ and so, $\Phi(T)$ and $\Psi(\mathcal{T})$ have the same eigenvalues. Then, in order to obtain the stability parameters of system (3.1) it is only necessary to study the eigenvalues of $\Psi(\mathcal{T})$.

In order to compute the dominant terms of the traces of $\Psi(\mathcal{T})$ for values of δ near 0 it will be useful to study some properties of q. To this, we shall introduce in the next section an artificial planar system for the functions q and $\dot{q}$ involved in $B(\tau)$.

3.2 An auxiliary planar system

In this section we study an artificial planar system that will be useful in order to study the stability parameters of system (3.1).

We define $Q(\tau) = -(2-\alpha)q^{-\frac{\alpha}{2-\alpha}}\dot{g}$ where $q = g^{\frac{2-\alpha}{2}}$ as before. We shall see that $q(\tau), Q(\tau)$ satisfies a planar system in time τ.

It is easy to see that $q' = -\dfrac{1}{2}qQ$. Moreover,

$$Q' \quad = \quad \frac{\alpha}{2(2-\alpha)}Q^2 - (2-\alpha)q^{\frac{2-2\alpha}{2-\alpha}}\ddot{g}.$$

We recall that g is a solution of the potential equation (3.2). Then, using the time τ, $q(\tau), Q(\tau)$ is a solution of the following system

$$\begin{aligned} q' &= -\frac{1}{2}qQ, \\ Q' &= \frac{\alpha}{2(2-\alpha)}Q^2 + (2-\alpha)q^{\frac{2-2\alpha}{2-\alpha}}U_z(\hat{q}), \end{aligned} \tag{3.7}$$

where $\hat{q} = q^{\frac{2}{2-\alpha}}$, that is, $\hat{q} = g$.

As (3.3) is a first integral of (3.2) and $\hat{q}$ is a solution of this system, we have that

$$E \quad = \quad q^{\frac{2\alpha}{2-\alpha}}\left[\frac{Q^2}{2(2-\alpha)^2} + V(\hat{q})\right], \tag{3.8}$$

is a first integral of system (3.7).

From hypothesis (A2) we have that $U(z) = \gamma z^\alpha + O(z^{\alpha+s})$, with $s > \dfrac{2-\alpha}{2}$. Then,

$$q^{\frac{2-2\alpha}{2-\alpha}} U_z(\hat{q}) = \gamma\alpha + O(q^{\frac{2s}{2-\alpha}}), \quad \text{with} \quad \frac{2s}{2-\alpha} > 1.$$

Therefore the planar system (3.7) is well–defined at $q = 0$.

We study now the behavior of the orbits of system (3.7) for $E = 0$. On this energy level there are two equilibrium points $P_\pm$ with $(q, Q) = (0, \pm Q_p)$ where $Q_p = (2-\alpha)\sqrt{-2\gamma}$.

Taking into account that the Jacobian of the vector field that defines system (3.7) is

$$\begin{pmatrix} -\frac{1}{2}Q & -\frac{1}{2}q \\ (2-\alpha)O(q^{\frac{2s}{2-\alpha}-1}) & \frac{\alpha}{2-\alpha}Q \end{pmatrix}, \quad \text{with} \quad \frac{2s}{2-\alpha} > 1,$$

the eigenvalues of the linearized system at $P_\pm$ are $\mp\dfrac{Q_p}{2}$, $\pm Q_p\dfrac{\alpha}{2-\alpha}$. Then, $P_\pm$ are saddle points.

On the energy level $E = 0$ we distinguish also two orbits

$$\begin{aligned} \gamma_0 &= \{(q, Q) \in \mathbb{R}^2 \mid q = 0, |Q| < Q_p\} \qquad \text{and} \\ \gamma_+ &= \{(q, Q) \in \mathbb{R}^2 \mid q > 0, \frac{Q^2}{2(2-\alpha)^2} + V(\hat{q}) = 0\}. \end{aligned}$$

In a neighbourhood of P_-, γ_+ is given by

$$Q = G(q) = -(2-\alpha)\sqrt{-2V(\hat{q})}.$$

Then, $\dfrac{dG}{dq}(0) = 0$.

On γ_0 (3.7) reduces to

$$Q' = \frac{\alpha}{2(2-\alpha)}Q^2 + \alpha(2-\alpha)\gamma.$$

So, we get the following solution

$$q_{L_1}(\tau) \equiv 0, \quad Q_{L_1}(\tau) = -Q_p \tanh\left(\tfrac{\alpha}{2(2-\alpha)}Q_p\tau\right).$$

On γ_+, $\dfrac{Q^2}{2(2-\alpha)^2} + V(\hat{q}) = 0$. Then, the system on γ_+ is

$$\begin{aligned} q' &= -\frac{1}{2}qQ, \\ Q' &= (2-\alpha)\hat{q}V_z(\hat{q}). \end{aligned} \tag{3.9}$$

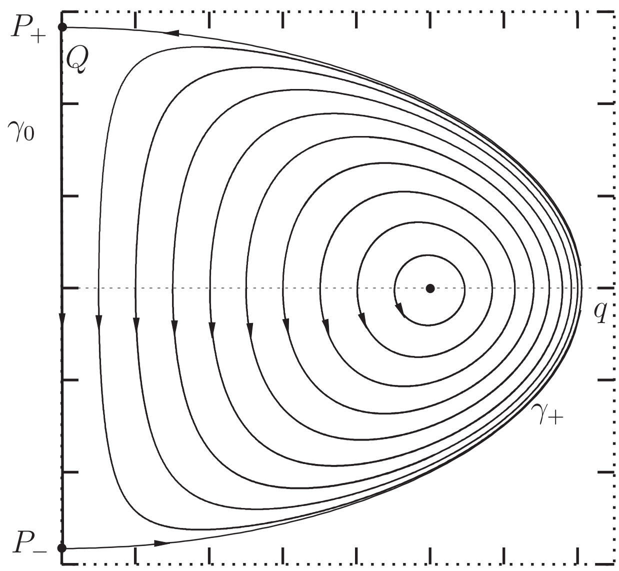


Figure 3.3: Phase portrait of system (3.7) in the newtonian case.

We shall denote by $q_{L_2}(\tau), Q_{L_2}(\tau)$ the solution of (3.9) such that $Q_{L_2}(0) = 0$.

Figure 3.3 shows the phase portrait of (3.7) for $U(z) = z\left(-1 + \dfrac{z}{2}\right)$ which corresponds to the homographic case for the newtonian potential.

As we are interested in $E = -\delta$ with $\delta > 0$ small enough, we need to study the behavior of the solutions of (3.7) near the heteroclinic connection defined by γ_0, γ_+ and the equilibrium points $P_\pm$.

To this end, we take $\varepsilon, \varepsilon_i$, $i = 1, \ldots, 4$, small enough, and we define the following sections

$$\begin{aligned}
\Sigma_0 &= \{(q,Q)|0 < q < \varepsilon_1, Q = 0\},\\
\Sigma_1 &= \{(q,Q)|0 < q < \varepsilon_2, Q = -Q_p + \varepsilon\},\\
\Sigma_2 &= \{(q,Q)|q = \varepsilon, |Q + Q_p| < \varepsilon_3\},\\
\Sigma_3 &= \{(q,Q)|q_a - q < \varepsilon_4, Q = 0\}.
\end{aligned}$$

For a fixed value of $\varepsilon > 0$ sufficiently small we can take small enough ε_i for $i = 1, \ldots, 4$, such that the Poincaré maps $\mathcal{P}_1 : \Sigma_0 \to \Sigma_1$, $\mathcal{P}_2 : \Sigma_1 \to \Sigma_2$ and $\mathcal{P}_3 : \Sigma_2 \to \Sigma_3$ be well defined.

Figure 3.4 shows these situation in the newtonian case.

We denote by $\tau_{L_1} > 0$ the time defined by $Q_{L_1}(\tau_{L_1}) = -Q_p + \varepsilon$, and $\tau_{L_2} > 0$ such that $q_{L_2}(-\tau_{L_2}) = \varepsilon$. That is, τ_{L_1} is the time needed for (q_{L_1}, Q_{L_1}) to go from Σ_0 to Σ_1, and $-\tau_{L_2}$ is the time used for the solution (q_{L_2}, Q_{L_2}) to travel backwards

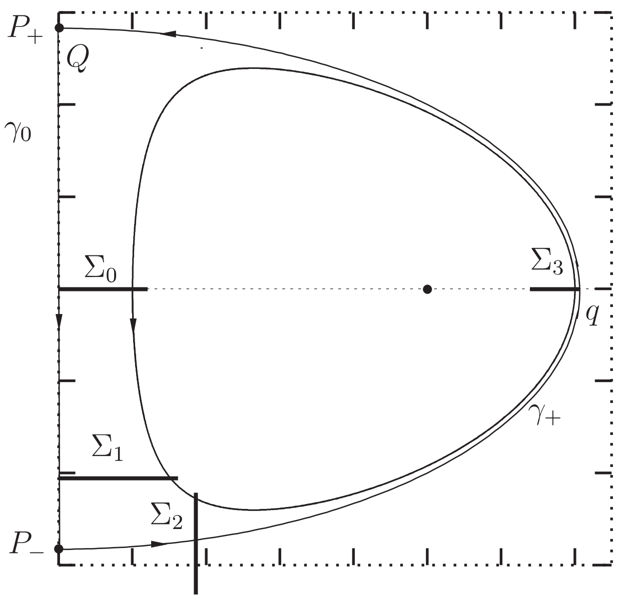


Figure 3.4: Poincaré sections

from Σ_3 to Σ_2. Note that τ_{L_1} and τ_{L_2} are finite and independent of δ once ε is fixed.

Fixed a value of $\delta > 0$ small enough, we consider the solution of (3.7) with $E = -\delta$ such that $(q(0), Q(0)) \in \Sigma_0$. Taking into account that $Q(0) = 0$ and using the hypothesis (A) and the energy (3.8) we get that $q_0 = q(0) = \left(\frac{\delta}{|\gamma|}\right)^{\frac{2-\alpha}{2\alpha}} (1+O(\delta^{\frac{s}{\alpha}}))$.

Let τ_1 be the smallest positive time such that $(q(\tau_1), Q(\tau_1)) \in \Sigma_1$. In a similar way we define τ_2 such that $(q(\tau_2), Q(\tau_2)) \in \Sigma_2$. It is clear that τ_1 and τ_2 depend on δ. Moreover, $\tau_1 \to \tau_{L_1}$ and $\frac{\mathcal{T}}{2} - \tau_2 \to \tau_{L_2}$ when $\delta \to 0$.

The following lemma gives bounds of $\tau_2 - \tau_1$.

Lemma 3.2.1. *Let $\epsilon > 0$ be a fixed value small enough. Then, for any $\delta > 0$ sufficiently small we have*

$$\frac{2}{Q_p + \epsilon} \ln\left(\frac{\epsilon}{q(\tau_1)}\right) \le \tau_2 - \tau_1 \le \frac{2}{Q_p - \epsilon} \ln\left(\frac{\epsilon}{q(\tau_1)}\right).$$

Proof

We have that $(q(\tau_1), Q(\tau_1)) \in \Sigma_1$ and $(q(\tau_2), Q(\tau_2)) \in \Sigma_2$. Taking δ small enough, for any $\tau \in [\tau_1, \tau_2]$ the following inequalities hold

$$-Q_p - \varepsilon \le Q(\tau) \le -Q_p + \varepsilon.$$

Multiplying these inequalities by $-\dfrac{1}{2}q(\tau)$ and using the first equation in (3.7) we get

$$\frac{1}{2}(Q_p - \varepsilon)q(\tau) \leq q'(\tau) \leq \frac{1}{2}(Q_p + \varepsilon)q(\tau). \tag{3.10}$$

We note that $q(\tau)$ is an increasing function in $[0, \tau_2]$. Then, $q(\tau) > 0$ in $[\tau_1, \tau_2]$, and the inequalities above can be written as

$$\frac{1}{2}(Q_p - \varepsilon) \leq \frac{q'(\tau)}{q(\tau)} \leq \frac{1}{2}(Q_p + \varepsilon).$$

By integration of these inequalities one obtains

$$\frac{1}{2}(Q_p - \varepsilon)(\tau_2 - \tau_1) \leq \ln\left(\frac{\varepsilon}{q(\tau_1)}\right) \leq \frac{1}{2}(Q_p + \varepsilon)(\tau_2 - \tau_1).$$

Now, the statement on the lemma follows by a simple computation. □

The following lemma will be used in next sections.

Lemma 3.2.2. *Let $\epsilon > 0$ small enough. For any $\delta > 0$ sufficiently small we have*

(a) $\int_{\tau_1}^{\tau_2} q(\tau)d\tau \leq \dfrac{2\epsilon}{Q_p - \epsilon}$.

(b) $\int_{\tau_1}^{\tau_2} |Q(\tau) + Q_p|d\tau \leq c_0\epsilon$, *for some constant c_0.*

Proof

(a) From (3.10) in the proof of lemma 3.2.1, for $\tau \in [\tau_1, \tau_2]$ the following inequality holds

$$q(\tau) \quad \leq \quad \frac{2}{Q_p - \varepsilon}q'(\tau).$$

We integrate between τ_1 and τ_2 the inequality above, obtaining

$$\int_{\tau_1}^{\tau_2} q(\tau)d\tau \leq \frac{2}{Q_p - \varepsilon}(q(\tau_2) - q(\tau_1)) \leq \frac{2}{Q_p - \varepsilon}q(\tau_2).$$

As $q(\tau_2) = \varepsilon$, we obtain the desired result.

(b) The idea is to study the distance between the component Q of a solution defined in $[\tau_1, \tau_2]$ and the unstable manifold of P_- with $q > 0$.

We consider a neighbourhood of P_- defined by $|Q + Q_p| \leq \varepsilon$, $0 < q \leq \varepsilon$. We introduce $\xi = Q + Q_p$ in order to translate the equilibrium point P_- to the origin in the plane (q, ξ). Let $W^{u,+}$ be the branch of the unstable invariant manifold of the origin with $q > 0$. We know that in the plane (q, Q) the unstable manifold of P_- is given by γ_+, and then in a neighbourhood of P_- is given by $Q = G(q) = -(2-\alpha)\sqrt{-2V(\hat{q})}$. Then, in the plane (q, ξ), $W^{u,+}$ is given by the graphic of the function $F(q) = Q_p - (2-\alpha)\sqrt{-2V(\hat{q})}$, that is, $\xi = F(q)$.

We recall that $Q_p = (2-\alpha)\sqrt{-2\gamma}$ and, by the hypothesis (A2), $V(\hat{q}) = \gamma + z^s V_1(\hat{q})$. Then,

$$F(q) \quad = \quad Q_p - Q_p\sqrt{1 + z^s\frac{V_1(\hat{q})}{\gamma}}. \tag{3.11}$$

Therefore if $0 < q \le \varepsilon$, ε small enough,

$$|F(q)| \quad \le \quad kq^{\frac{2s}{2-\alpha}}, \tag{3.12}$$

for some constant $k > 0$.

We define $y = \xi - F(q)$ for $0 < q \le \varepsilon$. In this way, $W^{u,+}$ lies on the q axis in the plane (q, y) and our region of interest is a neighbourhood of $\mathbf{0}$ with $y \ge 0$. In that region using (3.7) and (3.11) we get the following equation for y

$$y' \quad = \quad -\frac{\alpha}{2-\alpha}Q_p y(1 + O(\varepsilon)).$$

So, there exists a constant $c_2 > 0$ such that

$$-\frac{\alpha}{2-\alpha}Q_p y(1 + c_2\varepsilon) \le y' \le -\frac{\alpha}{2-\alpha}Q_p y(1 - c_2\varepsilon).$$

By integration of the inequality on the right hand we obtain

$$\textstyle\int_{\tau_1}^{\tau_2} y(\tau)d\tau \le \frac{2-\alpha}{\alpha Q_p(1-c_2\varepsilon)}(y(\tau_1) - y(\tau_2)) \le \frac{2-\alpha}{\alpha Q_p(1-c_2\varepsilon)}y(\tau_1),$$

where we have used that $y(\tau) \ge 0$ for all $\tau \in [\tau_1, \tau_2]$.

Using that $\dfrac{2s}{2-\alpha} > 1$, from (3.12) we get

$$|F(q)| \quad \le \quad kq, \tag{3.13}$$

in a small neighbourhood of P_-. Then, $y(\tau_1) = Q(\tau_1) + Q_p - F(q(\tau_1)) \le Q(\tau_1) + Q_p + |F(q(\tau_1))| \le (k+1)\varepsilon$. Moreover, by integration of the inequality (3.13) between τ_1 and τ_2 and using (a),

$$\textstyle\int_{\tau_1}^{\tau_2} |F(q(\tau))|d\tau \le \frac{2k}{Q_p - \varepsilon}\varepsilon.$$

As $Q + Q_p = y + F(q)$ we have

$$\textstyle\int_{\tau_1}^{\tau_2} |Q(\tau) + Q_p|d\tau \le \int_{\tau_1}^{\tau_2}(y(\tau) + |F(q(\tau))|)d\tau \le c_0\varepsilon,$$

for some constant $c_0 > 0$.

□

3.3 Asymptotic formulae for the stability parameters

In this section we will prove the main result of this chapter, that give us asymptotic formulae for the stability parameters of system (3.1). Some auxiliar results are proven in sections 3.4 and 3.5.

Theorem 3.3.1. *Let us consider the system (3.1) where $g(t;\delta)$ satisfies the hypothesis (B). Let be $\hat{\lambda} = \gamma\frac{(2-\alpha)^2}{8}$ where γ is defined in (A2). We assume that λ_1, λ_2 satisfy $\lambda_1 > \lambda_2 > \hat{\lambda}$ or, $\lambda_1 > \hat{\lambda} > \lambda_2$. Let be $\beta_j = \sqrt{1-\frac{\lambda_j}{\hat{\lambda}}}$, $j = 1, 2$. Then we have the following asymptotic behaviour for the stability parameters when δ goes to 0*

$$\begin{aligned}
\log tr_1 &= k_1 - \frac{2-\alpha}{2\alpha}\beta_1 \log\delta(1+o(1)) + \ldots, \\
\log tr_2 &= k_2 - \frac{2-\alpha}{2\alpha}\beta_2 \log\delta(1+o(1)) + \ldots, \qquad \text{if} \quad \lambda_2 > \hat{\lambda}, \\
tr_2 &= k_3 + k_4 \cos[k_5 - \gamma_2(1+o(1))\log(\delta)] + \ldots, \qquad \text{if} \quad \lambda_2 < \hat{\lambda},
\end{aligned} \tag{3.14}$$

provided that some coefficient, d_1, is different from zero. In the last case γ_2 denotes $\frac{\beta_2}{i}$. The coefficients k_j, $j = 1, \ldots, 5$ are constants. The coefficient k_4 is different from zero if some coefficient $e_3 \neq 0$.

The coefficients d_1 and e_3 will be defined in lemma 3.3.5. They depend on the particular potential $U(z)$ as well as on the parameters λ_1 and λ_2.

About the hypothesis in theorem 3.3.1 we have $\beta_1 \in \mathbb{R}^+$ and in the case $\beta_2 \in \mathbb{R}^+$, $\beta_1 > \beta_2$. These assumptions will give a dominant term depending on β_1 in the stability parameters. As we will see in chapter 4, these hyphotesis will be satisfied in the case of homographic solutions.

We note that asymptotic formulae (3.14) gives $\mathrm{tr}_1 > 2$ if δ is small enough. Furthermore, if $\beta_2 > 0$ then $\mathrm{tr}_2 > 2$ and the system is hyperbolic–hyperbolic. In the case $\beta_2 = \gamma_2\mathrm{i}$, tr_2 oscillates between the values $k_3 + k_4$ and $k_3 - k_4$ as δ tends to 0. Therefore it can cros the lines $\mathrm{tr}_2 = 2$ and $\mathrm{tr}_2 = -2$ infinitely many times as δ tends to zero depending on the values of $k_3 + k_4$ and $k_3 - k_4$. In particular, if $k_3 - k_4 < -2$ and $k_3 + k_4 > 2$, $\mathrm{tr}_2 = -2$ for a sequence $\delta_i \to 0$, and we found intervals $(\delta_{2i}, \delta_{2i-1})$ with $\mathrm{tr}_2 < -2$, that is, hyperbolic-elliptic intervals. This will be the case for the collinear homographic solutions to be studied in chapter 1. A similar things occurs if $k_3 + k_4 > 2$ and $k_3 - k_4 < 2$.

In order to prove Theorem 3.3.1 we need to study the stability parameters of (3.1). As we have seen in section 3.1 this is equivalent to study the stability of (3.5).

First we begin by writing the monodromy matrix $\Psi(\mathcal{T})$ in terms of the transition matrix in a half period.

Lemma 3.3.2. *The equality*

$$\Psi(\mathcal{T}) \quad = \quad \frac{q_0}{q_a} G_0 \Psi\left(\frac{\mathcal{T}}{2}\right)^T G_M \Psi\left(\frac{\mathcal{T}}{2}\right), \tag{3.15}$$

holds, where

$$G_0 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & -2q_0 \\ 0 & 1 & -2q_0 & 0 \end{pmatrix}, \quad G_M = \begin{pmatrix} 0 & -2q_a & -1 & 0 \\ -2q_a & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \tag{3.16}$$

$q_0 = q(0)$ *and* $q_a = q(T/2)$.

Proof

Let $\Phi(t)$ be the fundamental matrix of the system (3.1) such that $\Phi(0) = I_4$.We begin showing that the following equality holds

$$\Phi(T) = \mathcal{F}^{-1}\Phi(T/2)^T \mathcal{F}\Phi(T/2) \tag{3.17}$$

where $\mathcal{F} = \begin{pmatrix} 0 & -2 & -1 & 0 \\ -2 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$.

Let

$$\dot{\mathbf{y}} \quad = \quad B_H(t)\mathbf{y} \tag{3.18}$$

the linear periodic Hamiltonian system with Hamiltonian function (3.4).

Now we prove that the fundamental matrix $\Phi_1(t)$ of this system such that $\Phi_1(0) = I_4$ satisfies the relation

$$\Phi_1(T) \quad = \quad -J_4\Phi_1\left(-\frac{T}{2}\right)^T J_4\Phi_1\left(\frac{T}{2}\right).$$

As $\Phi_1(t)$ is a fundamental matrix of a T–periodic system, then $\Phi_1(t+T)$ is also a fundamental matrix. Then, there exists a non–singular constant matrix C such that $\Phi_1(t+T) = \Phi_1(t)C$. If we take $t = t_0$ then

$$\Phi_1(t+T) \quad = \quad \Phi_1(t)\Phi_1(t_0)^{-1}\Phi_1(t_0+T).$$

Then, if $t = 0$ and $t_0 = -\frac{T}{2}$, $\Phi_1(T) = \Phi_1\left(-\frac{T}{2}\right)^{-1}\Phi_1\left(\frac{T}{2}\right)$.

As $\Phi_1(t)$ is a symplectic matrix for all t, $\Phi_1(t)^{-1} = -J_4\Phi_1(t)^T J_4$. Taking $t = -\frac{T}{2}$, one obtain the desired relation.

The hypothesis (B) implies that $g(t)$ is an even function of t. Then, $B_H(t)$ is an even function of t. A simple computation shows that

$$B_H(t) \quad = \quad \begin{pmatrix} -J_2 & I_2 \\ \tilde{A}(t) - I_2 & -J_2 \end{pmatrix}.$$

Then, $LB_H(t)L = -B_H(t)$ where $L = \mathrm{diag}(-1, 1, 1, -1)$, and the parity of B_H yields

$$LB_H(t)L \quad = \quad -B_H(-t). \tag{3.19}$$

Let us take define $\tilde{\Phi}(t) := L\Phi_1(-t)$. Using the property (3.19) it is easy to check have that $\tilde{\Phi}$ is also a fundamental matrix of the Hamiltonian system (3.18) associated to (3.4). Therefore, $\Phi_1(-t) = L\Phi_1(t)L$. In particular, $\Phi_1\left(-\frac{T}{2}\right) = L\Phi_1\left(\frac{T}{2}\right)L$. Then,

$$\Phi_1(T) \quad = \quad -JL\Phi_1\left(\frac{T}{2}\right)^T LJ\Phi_1\left(\frac{T}{2}\right),$$

and taking into account (1.61), that is, $\Phi(t) = \tilde{M}^{-1}\Phi_1(t)\tilde{M}$, where $\tilde{M}$ is defined in (1.58), we obtain

$$\Phi(T) \quad = \quad -(\tilde{M}^{-1})^T J_4 L(\tilde{M}^T)^{-1}\Phi\left(\frac{T}{2}\right)^T \tilde{M}^T LJ_4\tilde{M}\Phi\left(\frac{T}{2}\right).$$

From this equality and denoting $\mathcal{F} = \tilde{M}^T LJ_4\tilde{M}$, we obtain (3.17). From this relation and using (3.6) we have that

$$\Psi(\mathcal{T}) \quad = \quad S(0)\mathcal{F}^{-1}S(0)\Psi\left(\frac{\mathcal{T}}{2}\right)^T S^{-1}\left(\frac{T}{2}\right)\mathcal{F}S^{-1}\left(\frac{T}{2}\right)\Psi\left(\frac{\mathcal{T}}{2}\right).$$

The statement of the lemma holds taking $G_0 = \frac{1}{q_0}S(0)\mathcal{F}^{-1}S(0)$ and $G_M = q_a S^{-1}\left(\frac{T}{2}\right)\mathcal{F}S^{-1}\left(\frac{T}{2}\right)$. $\square$

Our purpose now is to get an expression for $\Psi\left(\frac{\mathcal{T}}{2}\right)$ which allows us to compute the dominant terms of the traces of $\Psi(\mathcal{T})$ for values of δ near 0. To do that we shall use the planar system (3.7) in order to split $\Psi\left(\frac{\mathcal{T}}{2}\right)$ in three matrices each obtained from (3.5) following the heteroclinic connections of that planar system, that is, in a neighbourhood of γ_0, P_- and γ_+, respectively.

For a fixed value of $\delta > 0$ small enough, let $(q(\tau), Q(\tau))$ be the solution of (3.7) for $E = -\delta$ such that $q(0) = q_0$, $Q(0) = 0$, being q_0 the minimum of $q(\tau)$. Then the matrix $B(\tau)$ in (3.5) can be written as

$$B(\tau) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \lambda_1 & 0 & -Q(\tau)/2 & -2q(\tau) \\ 0 & \lambda_2 & 2q(\tau) & -Q(\tau)/2 \end{pmatrix} =: B_a(q, Q). \tag{3.20}$$

From now on we consider

$$\dot{\mathbf{y}} \;=\; B(\tau)\mathbf{y}.$$

Let $\tilde{\Psi}(\tau_b, \tau_a)$ be the transition matrix of (3.5) from τ_a to τ_b, that is, $\tilde{\Psi}(\tau_b, \tau_a)$ is the fundamental matrix of (3.5) satisfying $\tilde{\Psi}(\tau_a, \tau_a) = I_4$ evaluated at $\tau = \tau_b$.

Then, we can write

$$\Psi\left(\frac{\mathcal{T}}{2}\right) = \tilde{\Psi}\left(\frac{\mathcal{T}}{2}, \tau_2\right)\tilde{\Psi}(\tau_2, \tau_1)\tilde{\Psi}(\tau_1, 0). \tag{3.21}$$

We note that τ_1 and τ_2 (as defined in section 3.2) and also $\dfrac{\mathcal{T}}{2}$ depend on δ.

Our purpose is to approximate the transition matrices involved in (3.21) by simpler ones. First, we shall give approximations of $\tilde{\Psi}(\tau_1, 0)$ and $\tilde{\Psi}\left(\dfrac{\mathcal{T}}{2}, \tau_2\right)$ in (3.21) by the transition matrices for the system (3.5) along γ_0 and γ_+, respectively.

Using (3.20) we define

$$B_{L_1}(\tau) = B_a(0, Q_{L_1}(\tau)), \qquad B_{L_2}(\tau) = B_{L_2}(q_{L_2}(\tau), Q_{L_2}(\tau)), \tag{3.22}$$

where we recall that $(0, Q_{L_1}(\tau))$, $(q_{L_2}(\tau), Q_{L_2}(\tau))$ are the solutions of (3.7) corresponding to γ_0, γ_+, respectively.

Let $Z_1(\tau)$ be the fundamental matrix of

$$\mathbf{u}' \;=\; B_{L_1}(\tau)\mathbf{u}, \tag{3.23}$$

such that $Z_1(0) = I_4$.

Remark 3.3.3. The system (3.23) only depends on λ_1, λ_2, α and λ. It does not depend on the particular form of the function $V_1(z)$ defined in (A2).

We denote by $Z_2(\tau)$ the fundamental matrix of

$$\mathbf{u}' \;=\; B_{L_2}(\tau)\mathbf{u}, \tag{3.24}$$

such that $Z_2(-\tau_{L_2}) = I_4$.

For a fixed value of $\varepsilon > 0$ small enough $\mathcal{P}_1$ and $\mathcal{P}_3$ are diffeomorphisms. So, we can write

$$\tilde{\Psi}(\tau_1, 0) = Z_1(\tau_{L_1}) + \Delta_1, \qquad \tilde{\Psi}\left(\frac{\mathcal{T}}{2}, \tau_2\right) = Z_2(0) + \Delta_2,$$

for some matrices Δ_1, Δ_2 with $\|\Delta_1\| = O(q_0)$, $\|\Delta_2\| = O(q_{L_2}(0) - q_0) = O(\delta)$. We recall that $q_0 = O(\delta^{\frac{2-\alpha}{2\alpha}})$.

This relations give us approximation of the matrices $\tilde{\Psi}(\tau_1, 0)$ and $\tilde{\Psi}\left(\frac{\mathcal{T}}{2}, \tau_2\right)$ by the fundamental matrices of the system (3.5) on γ_0 and γ_+.

Now we want to obtain an approximation of the transition matrix $\tilde{\Psi}(\tau_2, \tau_1)$

We consider the system (3.5) in a neighbourhood of the equilibrium point P_-, and we write

$$B(\tau) \;=\; B_p + B_1(\tau),$$

$$B_1(\tau) = \begin{pmatrix} 0 & 0 \\ 0 & B_{11} \end{pmatrix} \quad \text{with} \quad B_{11} = \begin{pmatrix} -\frac{1}{2}(Q+Q_p) & -2q \\ 2q & -\frac{1}{2}(Q+Q_p) \end{pmatrix},$$

where

$$B_p \;=\; B_a(0, -Q_p),$$

that is the matrix of (3.20) evaluated on P_-. We note that B_p does not depend on δ.

The characteristic polynomial of B_p is

$$p(x) \;=\; x^2\left(\frac{Q_p}{2} - x\right)^2 + (\lambda_1 + \lambda_2)x\left(\frac{Q_p}{2} - x\right) + \lambda_1\lambda_2.$$

Performing the change of variables $y = \dfrac{Q_p}{4} - x$, the equation for the eigenvalues transforms in a biquadratic equation. Then, it is easy to compute the roots of p and one obtains that the eigenvalues of B_p are

$$\rho_1^{\pm} = \frac{Q_p}{4}(1 \pm \beta_1), \qquad \rho_2^{\pm} = \frac{Q_p}{4}(1 \pm \beta_2), \tag{3.25}$$

where $\beta_i = \sqrt{1 - \dfrac{8\lambda_i}{\gamma(2-\alpha)^2}}$, $i = 1, 2$. We note that under the hypothesis of Theorem 3.3.1, $\beta_1 \neq 0$, $\beta_2 \neq 0$ and $\beta_1 \neq \beta_2$. Hence the eigenvalues of B_p are also differents.

The associated eigenvectors are $(1, 0, \rho_1^{\pm}, 0)^T$ and $(0, 1, 0, \rho_2^{\pm})^T$, respectively. Let P be the matrix

$$P = \begin{pmatrix} I_2 & I_2 \\ P_3 & P_4 \end{pmatrix}, \qquad P_3 = \mathrm{diag}(\rho_1^+, \rho_2^+), \qquad P_4 = \mathrm{diag}(\rho_1^-, \rho_2^-). \tag{3.26}$$

P is nonsingular and

$$P^{-1} B_p P = \frac{Q_p}{4} I + \bar{D}$$

where $\bar{D} = \dfrac{Q_p}{4}\mathrm{diag}(\beta_1, \beta_2, -\beta_1, -\beta_2)$. We introduce a new variable

$$w = \exp\left(-\frac{Q_p}{4}(\tau - \tau_1)\right) P^{-1} y$$

and we get the following system for w

$$w' = (\bar{D} + P^{-1} B_1(\tau) P) w. \tag{3.27}$$

Let $W(\tau)$ be the fundamental matrix of (3.27) such that $W(\tau_1) = I$. A simple computation shows that

$$\tilde{\Psi}(\tau, \tau_1) = \exp\left(\frac{Q_p}{4}(\tau - \tau_1)\right) P W(\tau) P^{-1}.$$

Then, from (3.21), we get

$$\Psi\left(\frac{\mathcal{T}}{2}\right) = \sigma \tilde{\Psi}\left(\frac{\mathcal{T}}{2}, \tau_2\right) P W(\tau_2) P^{-1} \tilde{\Psi}(\tau_1, 0), \tag{3.28}$$

where $\sigma = \exp\left(\dfrac{Q_p}{4}(\tau_2 - \tau_1)\right)$.

Next lemma give us an approximation of $W(\tau)$ by a simpler matrix.

Lemma 3.3.4. *Let $\epsilon > 0$ be small enough. If $\delta > 0$ is sufficiently small we have for all $\tau \in [\tau_1, \tau_2]$*

$$W(\tau) = (I_4 + \Delta(\tau))\mathcal{D}(\tau)(I_4 + R),$$

where $\mathcal{D}(\tau) = diag(e^{\nu_1(\tau-\tau_1)}, e^{\nu_2(\tau-\tau_1)}, e^{-\nu_1(\tau-\tau_1)}, e^{-\nu_2(\tau-\tau_1)})$, $\nu_i = \dfrac{Q_p}{4}\beta_i$, $i = 1, 2$, and $\Delta(\tau)$ is a matrix such that $\|\Delta(\tau)\| \le c_1\varepsilon$ for any $\tau \in [\tau_1, \tau_2]$ and R is a constant matrix such that $\|R\| \le c_2\varepsilon$, for some constants c_1, c_2, uniformly in δ.

The proof of this lemma is given in section 3.4.

We note that $W(\tau)$ can be approximated by a diagonal matrix for $\varepsilon > 0$ small enough.

Up to now, we have approximated the transition matrices $\tilde{\Psi}(\tau_1, 0)$, $\tilde{\Psi}\left(\frac{\mathcal{T}}{2}, \tau_1\right)$ by $Z_1(\tau_{L_1})$, $Z_2(0)$, respectively, and $W(\tau_2)$ by $\mathcal{D}(\tau_2)$. Then, using (3.28), we have that

$$\Psi\left(\frac{\mathcal{T}}{2}\right) = \sigma\left[Z_2(0)P\mathcal{D}(\tau_2)P^{-1}Z_1\tau_{L_1}\right](I_4 + \Delta_3),$$

where $\|\Delta_3\| = O(\varepsilon, q_0, \delta)$. We remark that we are assuming that

$$\|Z_2(0)P\mathcal{D}(\tau_2)P^{-1}Z_1(\tau_{L_1})\|$$

has the same order of magnitud as the product of norms. We shall see that this is the case if the coefficient $d_1 \neq 0$.

Now, using (3.15), we get

$$\Psi(\mathcal{T}) = \frac{q_0}{q_a}\sigma^2\mathcal{M}(I_4 + \mathcal{O}), \qquad \mathcal{M} = A_1\mathcal{D}A_2\mathcal{D}A_3, \tag{3.29}$$

where $A_1 = G_{00}A_3^T$, $A_2 = P^T Z_2(0)^T G_M Z_2(0)P$, $A_3 = P^{-1}Z_1(\tau_{L_1})$, $\mathcal{D} = \mathcal{D}(\tau_2)$, $G_{00} = \begin{pmatrix} 0 & C \\ C & 0 \end{pmatrix}$, $C = \mathrm{diag}(-1, 1)$, and $\mathcal{O}$ stands for a matrix with contains terms of order ε, q_0 and δ_0. The same remark concerning the product of norms holds for (3.29). We note also that matrices A_1, A_2 and A_3 are independent of δ.

Now we want to obtain the stability parameters of (3.5). We recall that $\Phi(T)$ has the same eigenvalues that $\Psi(\mathcal{T})$. Moreover, as $\Phi(T)$ is symplectic, the characteristic polynomial of $\Psi(\mathcal{T})$ is

$$p(x) = x^4 + a_1x^3 + a_2x^2 + a_1x + 1,$$

where

$$\begin{aligned} a_1 &= -(\mathrm{tr}_1 + \mathrm{tr}_2), \\ a_2 &= 2 + \mathrm{tr}_1\mathrm{tr}_2. \end{aligned} \tag{3.30}$$

Then, the stability parameters can be obtained from a_1 and a_2 as the zeroes of $x^2 + a_1x + a_2 - 2$. To estimate the dominant terms of these coefficients we shall use the matrix $\mathcal{M}$.

Let us denote by

$$q(x) = x^4 + b_3x^3 + b_2x^2 + b_1x + b_0,$$

the characteristic polynomial of $\mathcal{M}$. We denote $k = \dfrac{q_a}{q_0\sigma^2}$. From (3.29),

$$\det(\Psi(\mathcal{T}) - xI_4) = \frac{1}{k^4}\det(\mathcal{M} + \mathcal{O} - kxI_4).$$

Using the equality above we obtain the following relation between a_1, a_2 and $b_0, \ldots, b_3$,

$$a_1 = \frac{b_3}{k}, \quad a_2 = \frac{b_2}{k^2}, \quad a_1 = \frac{b_1}{k^3}, \quad \frac{b_0}{k^4} = 1. \tag{3.31}$$

From (3.30) and (3.31) we get that the stability parameters tr_1 and tr_2 satisfy, up to order 1 in ε, the quadratic equation

$$k^2\tilde{t}^2 + kb_3\tilde{t} + b_2 - 2k^2 \quad = \quad 0. \tag{3.32}$$

We are interested in the dominant terms of b_2 and b_3. The following lemma give us these terms.

Lemma 3.3.5. *Let $\epsilon > 0$ be small enough. Assume that λ_1 and λ_2 satisfy the hypothesis of teorem 3.3.1. Then*
(a) There exist some constants d_i, $i = 1, \ldots 5$ such that

$$-b_3 = d_1\sigma^{2\beta_1} + d_2\sigma^{2\beta_2} + d_3\sigma^{-2\beta_1} + d_4\sigma^{-2\beta_2} + d_5. \tag{3.33}$$

The coefficient d_1 is the product of two constants, $d_1 = d_n d_g$ with d_n depending on $\lambda_1, \lambda_2, \alpha$ and γ but not on the function V_1 defined in section 3.1. d_g depends also on V_1. If λ_1 and λ_2 are different from zero, then $d_n \neq 0$.
(b) The coefficient b_2 does not contain terms in $\sigma^{\pm 4\beta_1}$ nor $\sigma^{\pm 4\beta_2}$, that is the dominant terms are

$$b_2 = e_1\sigma^{2\beta_1} + e_2\sigma^{2\beta_2} + e_3\sigma^{2(\beta_1+\beta_2)} + e_4\sigma^{2(\beta_1-\beta_2)} + \ldots \tag{3.34}$$

for some constants $e_1, e_2, e_3, \ldots$ The coefficient e_3 is the product of two constants $e_3 = e_n e_g$ where e_n depends on $\lambda_1, \lambda_2, \alpha$ and γ but not on the function V_1 defined in section 3.1. e_g depends also on V_1. If λ_1, λ_2 are different from zero then $e_n \neq 0$.

Moreover, if $\lambda_2 < \gamma\frac{(2-\alpha)^2}{8}$, then $d_4 = \bar{d}_2$, and $e_4 = \bar{e}_3$ where the bar stands for the complex conjugate.

The proof of this lemma will be given in section 3.5.

Now, the stability parameters are obtained by solving the quadratic equation (3.32). The solutions of this equation are

$$\tilde{t} \quad = \quad \frac{-b_3 \pm \sqrt{d}}{2k},$$

where $d = k[b_3^2 - 4b_2 + 8k^2]$.

We assume that the constant d_1 in Lemma 3.3.5 is different from zero. If λ_1, λ_2 satisfy the hypothesis of Theorem 3.3.1 then either $\beta_1 > \beta_2 > 0$ or $\beta_1 > 0$ and

$\beta_2 = \gamma_2 \mathrm{i}$, $\gamma_2 \in \mathbb{R}$. In any case, the dominant term in $-b_3$ is $d_1\sigma^{2\beta_1}$. Therefore, we have that the dominant terms of d are

$$d \;=\; d_1^2\sigma^{4\beta_1}\tilde{d},$$

where $\tilde{d} = 1 + 2n_1\sigma^{-2\beta_1} + 2n_2\sigma^{-2(\beta_1-\beta_2)} + 2n_3\sigma^{-2(\beta_1+\beta_2)} + \ldots$, and

$$n_1 = \frac{d_1d_5 - 2e_1}{d_1^2}, \qquad n_2 = \frac{d_1d_2 - 2e_3}{d_1^2}, \qquad n_3 = \frac{d_1d_4 - 2e_4}{d_1^2} \quad .$$

Then,

$$\mathrm{tr}_1 = \frac{-b_3 + d_1\sigma^{2\beta_1}\sqrt{\tilde{d}}}{2k} = \frac{q_0}{q_a}\sigma^2(d_1\sigma^{2\beta_1} + \ldots),$$

and

$$\mathrm{tr}_2 \;=\; \frac{-b_3 - d_1\sigma^{2\beta_1}\sqrt{\tilde{d}}}{2k}.$$

As $\sqrt{\tilde{d}} = 1 + n_1\sigma^{-2\beta_1} + n_2\sigma^{-2(\beta_1-\beta_2)} + n_3\sigma^{-2(\beta_1+\beta_2)} + \ldots$, the dominant terms in tr_2 are

$$\mathrm{tr}_2 \;=\; \frac{1}{2k}[(d_2 - d_1n_2)\sigma^{2\beta_2} + (d_4 - d_1n_3)\sigma^{-2\beta_2} + d_5 - d_1n_1 + \ldots]. \tag{3.35}$$

If β_2 is real the dominant term is given by $(d_2 - d_1n_2)\sigma^{2\beta_2}$ and then

$$\mathrm{tr}_2 \;=\; \frac{q_0}{q_a}\sigma^2\left(\frac{e_3}{d_1}\sigma^{2\beta_2} + \ldots\right).$$

If β_2 is imaginary, all the terms written in (3.35) are of the same order. From lemma 3.3.5 we have that in this case $d_4 = \overline{d}_2$ and $e_4 = \overline{e}_3$. Then,

$$\mathrm{tr}_2 \;=\; \frac{q_0}{q_a}\sigma^2\left(\frac{e_1}{d_1} + 2\mathrm{Re}\left(\frac{e_3}{d_1}\sigma^{2\beta_2}\right) + \ldots\right).$$

Summarizing, we have that if $d_1 \neq 0$, then the stability parameters are obtained as

$$\mathrm{tr}_1 \;=\; \frac{q_0}{q_a}\sigma^2(d_1\sigma^{2\beta_1} + \ldots), \tag{3.36}$$

and

$$\mathrm{tr}_2 \;=\; \frac{q_0}{q_a}\sigma^2\left(\frac{e_3}{d_1}\sigma^{2\beta_2} + \ldots\right), \quad \text{if} \quad \beta_2 > 0$$

$$\mathrm{tr}_2 \;=\; \frac{q_0}{q_a}\sigma^2\left(\frac{e_1}{d_1} + 2\mathrm{Re}\left(\frac{e_3}{d_1}\sigma^{2\beta_2}\right) + \ldots\right), \quad \text{if} \quad \beta_2 = \gamma_2\mathrm{i}, \gamma_2 \in \mathbb{R}.$$

Now, using lemma 3.2.1 the result on the theorem 3.3.1 follows.

Remark 3.3.6. We recall that matrices A_1, A_2 and A_3 in (3.29) do not depend on δ, so their norms are finite. Therefore $\|A_1\|\|\mathcal{D}\|\|A_2\|\|\mathcal{D}\|\|A_3\|$ depends mainly on $\|\mathcal{D}\|^2$ for $\delta > 0$ small enough. Furthermore $\mathcal{D} = \mathcal{D}(\tau_2)$ is a diagonal matrix and so, $\|\mathcal{D}\|^2$ is of the order or $\sigma^{2\beta_1}$. However, if $d_1 \neq 0$ from (3.36) we have that tr_1 is of order $\sigma^{2\beta_1}$. This gives an estimation of the spectral radius of $\mathcal{M}$. Using that $\|\mathcal{M}\|$ is bounded from below by the spectral radius we conclude that it is of the same order of magnitude of the product of norms and then (3.29) holds.

3.4 Proof of lemma 3.3.4

In order to prove lemma 3.3.4 it will be useful the following lemma.

Lemma 3.4.1. *Let us consider the system*

$$\mathbf{x}' = D\mathbf{x} + C(t)\mathbf{x}, \tag{3.37}$$

where D is a diagonal matrix $n \times n$ and $C(t)$ is a continuous matrix in t. Assume that there exists some constant $\hat{\epsilon} < 1/4$ such that

$$\int_0^{\hat{t}} \|C(s)\| ds < \hat{\epsilon}. \tag{3.38}$$

Let λ be an eigenvalue of D and $\mathbf{v}$ an eigenvector corresponding to λ. Then, there exists a solution, $\varphi(t)$, of (3.37) such that

$$\|e^{-\lambda t}\varphi(t) - \mathbf{v}\| \leq \|\mathbf{v}\| \frac{3\hat{\epsilon}}{1 - 3\hat{\epsilon}},$$

for all $t \in [0, \hat{t}]$.

Proof

Let us consider $\lambda_1, \ldots, \lambda_n$ the eigenvalues of D. That is, $D = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$. Assume that $\mathrm{Re}(\lambda_j) < \mathrm{Re}(\lambda)$ for $0 < j \leq k < n$, $\mathrm{Re}(\lambda_j) > \mathrm{Re}(\lambda)$ for $k+1 \leq j \leq m < n$ and $\mathrm{Re}(\lambda_j) = \mathrm{Re}(\lambda)$ for $m+1 \leq j \leq n$.

We denote by $D_1 = \mathrm{diag}(\lambda_1, \ldots, \lambda_k)$ the $k \times k$ diagonal matrix obtained from D such that their eigenvalues have real part less than $\mathrm{Re}(\lambda)$. Analogously, we define $D_2 = \mathrm{diag}(\lambda_{k+1}, \ldots, \lambda_m)$ the $(m-k) \times (m-k)$ diagonal matrix with eigenvalues with real part bigger than $\mathrm{Re}(\lambda)$, and $D_3 = \mathrm{diag}(\lambda_{m+1}, \ldots, \lambda_n)$ a $(n-m) \times (n-m)$ diagonal matrix, which has eigenvalues with the same real part as λ.

It could be possible that D_1 or D_2 does not contain any term, but it is clear that λ is in D_3. We take $\lambda_n = \lambda$.

We have that $D = \mathrm{diag}(D_1, D_2, D_3)$. Let us denote by $X(t)$ the fundamental matrix of the system $\mathbf{x}' = D\mathbf{x}$ with $X(0) = I_n$. Then, $X(t) = e^{Dt} =$

$\mathrm{diag}(e^{D_1 t}, e^{D_2 t}, e^{D_3 t})$. We can write $X(t) = X_1(t) + X_2(t) + X_3(t)$ where $X_1(t) = \mathrm{diag}(e^{D_1 t}, 0, 0)$, $X_2(t) = \mathrm{diag}(0, e^{D_2 t}, 0)$, $X_3(t) = \mathrm{diag}(0, 0, e^{D_3 t})$.

$\tilde{X}_j(t) = e^{D_j t}$ is the fundamental matrix of the system

$$\mathbf{x}_j' \;=\; D_j \mathbf{x}_j,$$

satisfying $\tilde{X}_j(0)$ the identity matrix, $j = 1, 2, 3$. We note that $\mathbf{x}_1 \in \mathbb{R}^k$, $\mathbf{x}_2 \in \mathbb{R}^{m-k}$, $x_3 \in \mathbb{R}^{n-m}$.

Then $e^{-\lambda t}\tilde{X}_j(t)$ is a fundamental matrix of the system

$$\mathbf{x}_j' \;=\; \tilde{A}_j \mathbf{x}_j,$$

where $\tilde{A}_j = D_j - \lambda Id$, and Id denotes the identity matrix of suitable dimension. $\tilde{A}_1$ ($\tilde{A}_2$) has eigenvalues with negative (positive) real part. Then, there exists a positive constant $a > 0$ such that $\|e^{-\lambda t}\tilde{X}_1(t)\| \le e^{-at}$ for all $t \ge 0$, and $\|e^{-\lambda t}\tilde{X}_2(t)\| \le e^{at}$ for all $t \le 0$.

Now we consider $\tilde{A}_3$, that has eigenvalues with zero real part. Let us write $\lambda = \alpha + \mathrm{i}\beta$. Then, $\lambda_j = \alpha + i\beta_j$, $j = m+1, \ldots, n$, and

$$e^{-\lambda t}\tilde{X}_3(t) = \mathrm{diag}(e^{\mathrm{i}(\beta_{m+1}-\beta)t}, \ldots, e^{\mathrm{i}(\beta_{n-1}-\beta)t}, 1).$$

Therefore $\|e^{-\lambda t}\tilde{X}_3(t)\| = 1$.

As $\|e^{-\lambda t}X_j(t)\| = \|e^{-\lambda t}\tilde{X}_j(t)\|$ for $j = 1, 2, 3$ we get $\|e^{-\lambda t}X_1(t)\| \le e^{-at}$ for all $t \ge 0$, $\|e^{-\lambda t}X_2(t)\| \le e^{at}$ for all $t \le 0$, for a constant $a > 0$, and $\|e^{-\lambda t}X_3(t)\| = 1$.

It is easy to check that the solution $\varphi(t)$ of the integral equation

$$\begin{aligned} \varphi(t) \;&=\; e^{t\lambda}\mathbf{v} + \int_0^t X_1(t-s)C(s)\varphi(s)ds - \int_t^{\hat{t}} X_2(t-s)C(s)\varphi(s)ds \\ &\quad - \int_t^{\hat{t}} X_3(t-s)C(s)\varphi(s)ds \end{aligned}$$

is a solution of (3.37).

We want to obtain the solution of the integral equation above. To this end we use an iterative scheme with $\varphi_0(t) \equiv 0$. For $m \ge 1$ we define

$$\begin{aligned} \varphi_m(t) \;&=\; e^{t\lambda}\mathbf{v} + \int_0^t X_1(t-s)C(s)\varphi_{m-1}(s)ds - \int_t^{\hat{t}} X_2(t-s)C(s)\varphi_{m-1}(s)ds \\ &\quad - \int_t^{\hat{t}} X_3(t-s)C(s)\varphi_{m-1}(s)ds. \end{aligned}$$

Then, for all $t \in [0, \hat{t}]$ the following inequalities hold

$$\|\varphi_m(t) - \varphi_{m-1}(t)\| \;\le\; \|\mathbf{v}\|(3\hat{\epsilon})^{m-1}|e^{t\lambda}| \le \|\mathbf{v}\|(3\hat{\epsilon})^{m-1}|e^{\hat{t}\lambda}|, \qquad (3.39)$$

$$\|e^{-t\lambda}\varphi_m(t) - \mathbf{v}\| \;\le\; \|\mathbf{v}\| \sum_{k=1}^{m-1} (3\hat{\epsilon})^k. \qquad (3.40)$$

The inequalities above will be proved now by induction.

For (3.39) is clear that

$$\|\varphi_1(t) - \varphi_0(t)\| = \|\varphi_1(t)\| \leq \|\mathbf{v}\||e^{\lambda t}|.$$

Assume that (3.39) is true for $m-1$. We note that

$$\begin{aligned}
\varphi_m(t) - \varphi_{m-1}(t) \quad &= \quad e^{t\lambda}\left\{\int_0^t e^{-\lambda(t-s)}X_1(t-s)C(s)e^{-s\lambda}(\varphi_{m-1}(s) - \varphi_{m-2}(s))ds\right.\\
&\quad - \int_t^{\hat{t}} e^{-\lambda(t-s)}X_2(t-s)C(s)e^{-s\lambda}(\varphi_{m-1}(s) - \varphi_{m-2}(s))ds\\
&\quad \left.- \int_t^{\hat{t}} e^{-\lambda(t-s)}X_3(t-s)C(s)e^{-s\lambda}(\varphi_{m-1}(s) - \varphi_{m-2}(s))ds\right\}.
\end{aligned}$$

Then, using the inequalities for $\|e^{-\lambda t}X_j(t)\|$, $j = 1, 2, 3$, one obtains

$$\begin{aligned}
&\|\varphi_m(t) - \varphi_{m-1}(t)\| \leq |e^{t\lambda}|\left\{\int_0^t e^{-a(t-s)}\|C(s)\|\|\mathbf{v}\|(3\hat{\epsilon})^{m-2}ds + \right.\\
&\quad \left.+\int_t^{\hat{t}} e^{a(t-s)}\|C(s)\|\|\mathbf{v}\|(3\hat{\epsilon})^{m-2}ds + \int_t^{\hat{t}}\|C(s)\|\|\mathbf{v}\|(3\hat{\epsilon})^{m-2}ds\right\} \leq\\
&\leq |e^{t\lambda}|\|\mathbf{v}\|(3\hat{\epsilon})^{m-1}.
\end{aligned}$$

This proves (3.39).

The proof of (3.40) for $m = 1$ is trivial. For $m \geq 2$, first we note that

$$\begin{aligned}
&e^{-t\lambda}\varphi_2(t) - \mathbf{v} = \int_0^t e^{-\lambda(t-s)}X_1(t-s)C(s)\mathbf{v}ds -\\
&\int_t^{\hat{t}} e^{-\lambda(t-s)}X_2(t-s)C(s)\mathbf{v}ds \quad - \int_t^{\hat{t}} e^{-\lambda(t-s)}X_3(t-s)C(s)\mathbf{v}ds.
\end{aligned}$$

Then

$$\begin{aligned}
&\|e^{-t\lambda}\varphi_2(t) - \mathbf{v}\| \leq \int_0^t e^{-a(t-s)}\|C(s)\|\|\mathbf{v}\|ds + \int_t^{\hat{t}} e^{a(t-s)}\|C(s)\|\|\mathbf{v}\|ds\\
&\quad + \int_t^{\hat{t}}\|C(s)\|\|\mathbf{v}\|ds \leq 3\hat{\epsilon}\|\mathbf{v}\|.
\end{aligned}$$

For the general step we get

$$\begin{aligned}
&e^{-t\lambda}\varphi_m(t) - \mathbf{v} = \int_0^t e^{-\lambda(t-s)}X_1(t-s)C(s)(e^{-s\lambda}\varphi_{m-1}(s) - \mathbf{v})ds\\
&\quad - \int_t^{\hat{t}} e^{-\lambda(t-s)}X_2(t-s)C(s)(e^{-s\lambda}\varphi_{m-1}(s) - \mathbf{v})ds\\
&\quad - \int_t^{\hat{t}} e^{-\lambda(t-s)}X_3(t-s)C(s)(e^{-s\lambda}\varphi_{m-1}(s) - \mathbf{v})ds + (e^{-t\lambda}\varphi_2(t) - \mathbf{v}).
\end{aligned}$$

Therefore

$$\begin{aligned}
\|e^{-t\lambda}\varphi_m(t)-\mathbf{v}\| &\le \int_0^t e^{-a(t-s)}\|C(s)\|\|e^{-t\lambda}\varphi_{m-1}(t)-\mathbf{v}\|ds \\
&+\int_t^{\hat t} e^{a(t-s)}\|C(s)\|\|e^{-t\lambda}\varphi_{m-1}(t)-\mathbf{v}\|ds \\
&+\int_t^{\hat t} \|C(s)\|\|e^{-t\lambda}\varphi_{m-1}(t)-\mathbf{v}\|ds+\|e^{-t\lambda}\varphi_2(t)-\mathbf{v}\| \\
&\le \|\mathbf{v}\|\left(\sum_{k=1}^{m-2}(3\hat\epsilon)^k\left[3\int_0^{\hat t}\|C(s)\|ds\right]+3\hat\epsilon\right)\le \|\mathbf{v}\|\sum_{k=1}^{m-1}(3\hat\epsilon)^k
\end{aligned}$$

and (3.40) follows.

From (3.39) we have that φ_m tends to φ uniformly on compacts.

Then, using (3.40) we have that

$$\|e^{-\lambda t}\varphi(t)-\mathbf{v}\|\le\|\mathbf{v}\|\textstyle\sum_{k=1}^{\infty}(3\hat\varepsilon)^k=\|\mathbf{v}\|\dfrac{3\hat\varepsilon}{1-3\hat\varepsilon}.$$

This ends the proof of the lemma. □

To prove lemma 3.3.4 we shall apply the lemma 3.4.1 to the system (3.27).

For a fixed value of ε we consider $q_0>0$ small enough (we have that $q_0=\left(\dfrac{\delta}{|\gamma|}\right)^{\frac{2-\alpha}{2\alpha}}(1+O(\delta^{\frac{s}{\alpha}})))$ and $\tau\in[\tau_1,\tau_2]$ where we recall that τ_1,τ_2 depend on q_0. After a translation of time defined by $s=\tau-\tau_1$ we can restrict to the system

$$\frac{dw}{ds} = (\overline{D}+P^{-1}B_1(s+\tau_1)P)w, \tag{3.41}$$

for $s\in[0,\hat s]$, where $\hat s=\hat s(q_0)=\tau_2-\tau_1$. First we prove that the hypothesis (3.38) is satisfied for $C(s):=P^{-1}B(s+\tau_1)P$. We recall that $B_1(\tau)$ is defined in (3.22) and it depends on δ. We get

$$\|C(s)\| \le \|P\|\|P^{-1}\|\|B_{11}(s+\tau_1)\|,$$

For any δ small enough, using lemma 3.2.2 we get

$$\textstyle\int_0^{\hat s}\|C(s)\|ds\le\|P\|\|P^{-1}\|\left(\dfrac{4\varepsilon}{Q_p-\varepsilon}+\dfrac{c_0\varepsilon}{2}\right).$$

Then (3.38) is satisfied with $\hat\varepsilon=\varepsilon c_1$ being $c_1=\|P\|\|P^{-1}\|\left(\dfrac{4}{Q_p-\varepsilon}+\dfrac{c_0}{2}\right)$. If ε is small enough we have $\hat\varepsilon<\dfrac{1}{4}$. For technical reasons we shall assume that $\varepsilon<\dfrac{1}{12c_1}$.

We recall that $\bar{D} = \dfrac{Q_p}{4}\mathrm{diag}(\nu_1, \nu_2, \nu_3, \nu_4)$ where $\nu_j = \dfrac{Q_p}{4}\beta_j$, $j = 1, 2$, $\nu_3 = -\nu_1$, $\nu_4 = -\nu_2$. Using lemma 3.4.1, there exist $\hat{\varphi}_1(s), \hat{\varphi}_2(s), \hat{\varphi}_3(s), \hat{\varphi}_4(s)$ solutions of (3.41) such that for any $s \in [0, \hat{s}]$

$$\|e^{-\nu_i s}\hat{\varphi}_i(s) - \mathbf{e}_i\| \leq \|\mathbf{e}_i\|\tfrac{3\hat{\varepsilon}}{1-3\hat{\varepsilon}}, \quad i = 1, \ldots, 4,$$

where $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4$ is the canonical basis. Then, $\varphi_i(\tau) := \hat{\varphi}(\tau - \tau_1)$, $j = 1, \ldots, 4$, are solutions of (3.27) such that for any $\tau \in [\tau_1, \tau_2]$

$$\|e^{-\nu_i(\tau-\tau_1)}\varphi_i(\tau) - \mathbf{e}_i\| \leq \tfrac{3\hat{\varepsilon}}{1-3\hat{\varepsilon}}, \quad i = 1, \ldots, 4. \tag{3.42}$$

We denote as $Y(t)$ the matrix defined by $\varphi_1, \varphi_2, \varphi_3$ and φ_4 as column vectors. We write

$$\mathcal{D}^{-1}(\tau) = \mathrm{diag}(e^{-\nu_1(\tau-\tau_1)}, e^{-\nu_2(\tau-\tau_1)}, e^{-\nu_3(\tau-\tau_1)}, e^{-\nu_4(\tau-\tau_1)}),$$

and define $\Delta(\tau) = Y(\tau)\mathcal{D}^{-1}(\tau) - I_4$. Then, using (3.42)

$$\|\Delta(\tau)\| \leq \frac{3\hat{\varepsilon}}{1 - 3\hat{\varepsilon}} = \varepsilon\frac{3c_1}{1 - 3c_1\varepsilon},$$

for any $\tau \in [\tau_1, \tau_2]$. The last inequality above comes from the assumption that $\varepsilon < \dfrac{1}{12c_1}$. We remark that matrices $Y(\tau), \mathcal{D}(\tau)$, and then $\Delta(\tau)$, depend on δ. Moreover, we can say that $\|\Delta(\tau)\| < \dfrac{1}{3}$ for any $\tau \in [\tau_1, \tau_2]$. In particular, $\|\Delta(\tau_1)\| < 1$. Then, $Y(\tau_1) = I_4 + \Delta(\tau_1)$ is a non singular matrix and $Y(\tau) = (I_4 + \Delta(\tau))\mathcal{D}(\tau)$ is a fundamental matrix for the system (3.27). Furthermore, $W(\tau)$ is the fundamental matrix of (3.27) such that $W(\tau_1) = I_4$. Then

$$W(\tau) = Y(\tau)Y(\tau_1)^{-1} = (I_4 + \Delta(\tau))\mathcal{D}(\tau)(I_4 + \Delta(\tau_1))^{-1}.$$

Furthermore, using standard results for natural matrix norms (see [I.K.],[Ch.]) if we define

$$R \;:=\; I_4 - (I_4 + \Delta(\tau_1))^{-1},$$

we get $\|R\| \leq \dfrac{\hat{\hat{\varepsilon}}}{1 - \hat{\hat{\varepsilon}}}$ where $\hat{\hat{\varepsilon}} = \dfrac{3\hat{\varepsilon}}{1 - 3\hat{\varepsilon}}$. Therefore,

$$\|R\| \;\leq\; \varepsilon\frac{3c_1}{1 - 6c_1\varepsilon} < 6c_1\varepsilon.$$

3.5 Proof of lemma 3.3.5

In this section we prove lemma 3.3.5. To this end we need some information about the matrices involved in $\mathcal{M}$. We recall that

$$\mathcal{M} \quad = \quad A_1\mathcal{D}(\tau_2)A_2\mathcal{D}(\tau_2)A_3,$$

where A_1, A_2, A_3 do not depend on δ.

We begin with A_3. We recall that $A_3 = P^{-1}Z_1(\tau_{L_1})$ where P is the constant matrix defined in (3.26) and $Z_1(\tau_{L_1})$ the fundamental matrix of the system (3.23), that is,

$$\mathbf{u}' \quad = \quad B_{L_1}(\tau)\mathbf{u}, \tag{3.43}$$

with $Z_1(0) = I_4$, where

$$B_{L_1}(\tau) \quad = \quad \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \lambda_1 & 0 & -\frac{Q_{L_1}(\tau)}{2} & 0 \\ 0 & \lambda_2 & 0 & -\frac{Q_{L_1}(\tau)}{2} \end{pmatrix}, \tag{3.44}$$

and $P = \begin{pmatrix} I_2 & I_2 \\ P_3 & P_4 \end{pmatrix}$, $P_3 = \mathrm{diag}(\rho_1^+, \rho_2^+)$, $P_4 = \mathrm{diag}(\rho_1^-, \rho_2^-)$.

Then, the system (3.43) splits in two uncoupled systems, one for u_1, u_3 and the second for u_2, u_4. Let us denote by $\mathbf{e}_j$, $j = 1, \ldots, 4$, the canonical basis of $\mathbb{R}^4$. If we take as initial condition $\mathbf{e}_1, \mathbf{e}_3$ ($\mathbf{e}_2, \mathbf{e}_4$) then $u_2(\tau) \equiv u_4(\tau) \equiv 0$ ($u_1(\tau) \equiv u_3(\tau) \equiv 0$). Then, $Z_1(\tau_{L_1})$ is a 4×4 block diagonal matrix, that is,

$$Z_1(\tau_{L_1}) \quad = \quad \begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix}, \quad \text{with} \quad C_j = \mathrm{diag}(c_{j1}, c_{j2}), j = 1, \ldots, 4. \tag{3.45}$$

In order to compute A_3 we need to know P^{-1}. A simple check shows that $P^{-1} = \begin{pmatrix} Q_1 & Q_2 \\ Q_3 & Q_4 \end{pmatrix}$, where

$$Q_1 = \mathrm{diag}\left(\frac{\rho_1^-}{\rho_1^- - \rho_1^+}, \frac{\rho_2^-}{\rho_2^- - \rho_2^+}\right), \qquad Q_2 = \mathrm{diag}\left(\frac{1}{\rho_1^+ - \rho_1^-}, \frac{1}{\rho_2^+ - \rho_2^-}\right),$$
$$Q_3 = \mathrm{diag}\left(\frac{\rho_1^+}{\rho_1^+ - \rho_1^-}, \frac{\rho_2^+}{\rho_2^+ - \rho_2^-}\right), \qquad Q_4 = \mathrm{diag}\left(\frac{1}{\rho_1^- - \rho_1^+}, \frac{1}{\rho_2^- - \rho_2^+}\right),$$

and $\rho_1^\pm$, $\rho_2^\pm$ are given in (3.25) Then,

$$A_3 \quad = \quad \begin{pmatrix} E_1 & E_2 \\ E_3 & E_4 \end{pmatrix} \quad \text{with} \quad E_j = \mathrm{diag}(e_{j1}, e_{j2}), j = 1, \ldots, 4. \tag{3.46}$$

Moreover,

$$
\begin{aligned}
E_1 &= Q_1C_1 + Q_2C_3, \quad E_2 = Q_1C_2 + Q_2C_4, \\
E_3 &= Q_3C_1 + Q_4C_3, \quad E_4 = Q_3C_2 + Q_4C_4.
\end{aligned} \tag{3.47}
$$

As $A_1 = G_{00}A_3^T$ where $G_{00} = \begin{pmatrix} 0 & C \\ C & 0 \end{pmatrix}$, $C = \mathrm{diag}(-1, 1)$, we get that

$$
A_1 \;=\; \begin{pmatrix} H_1 & H_2 \\ H_3 & H_4 \end{pmatrix} \quad \text{with} \quad H_j = \mathrm{diag}(h_{j1}, h_{j2}), j = 1, \ldots, 4. \tag{3.48}
$$

Moreover, $H_1 = CE_2$, $H_2 = CE_4$, $H_3 = CE_1$, $H_4 = CE_3$. Therefore, we get the following relations between h_{ji} and e_{ji},

$$
\begin{aligned}
&h_{11} = -e_{21}, \quad h_{12} = e_{22}, \quad h_{21} = -e_{41}, \quad h_{22} = e_{42}, \\
&h_{31} = -e_{11}, \quad h_{32} = e_{12}, \quad h_{41} = -e_{31}, \quad h_{42} = e_{32}.
\end{aligned} \tag{3.49}
$$

From (3.29) we know that $A_2 = P^T Z_2(0)^T G_M Z_2(0)P$, being $Z_2(\tau)$ the fundamental matrix of the system (3.24), that is $\mathbf{u}' = B_{L_2}(\tau)\mathbf{u}$, such that $Z_2(-\tau_{L_2}) = I_4$. We denote

$$
A_2 \;=\; \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}, \tag{3.50}
$$

for some 2×2 matrices X_i, $i = 1, \ldots, 4$. We write also $A_2 = (x_{ij})$.

We note that using remark (3.3.3), the matrices A_1 and A_3 depend on λ_1, λ_2, α and γ but they do not depend on function V_1. However, A_2 depends on V_1.

We recall that the matrix $\mathcal{D}(\tau)$ is defined in lemma 3.3.4. We can write $\mathcal{D}(\tau_2) = \mathrm{diag}(D_1, D_1^{-1})$, where $D_1 = \mathrm{diag}(\sigma^{\beta_1}, \sigma^{\beta_2})$ and $\sigma = \exp\left(\dfrac{Q_p}{4}(\tau_2 - \tau_1)\right)$. It is not difficult to check that $\mathcal{M} = \tilde{D}_1^{-1}\tilde{\mathcal{M}}\tilde{D}_1$ where $\tilde{D}_1 = \mathrm{diag}(D_1, D_1)$ and

$$
\tilde{\mathcal{M}} \;=\; \tilde{D}_1^2\tilde{\mathcal{M}}_1 + \tilde{\mathcal{M}}_2 + \tilde{D}_1^2\tilde{\mathcal{M}}_3(\tilde{D}_1^{-1})^2 + \tilde{\mathcal{M}}_2(\tilde{D}_1^{-1})^2, \tag{3.51}
$$

$$
\begin{aligned}
\tilde{\mathcal{M}}_1 &= \begin{pmatrix} H_1X_1E_1 & H_1X_1E_2 \\ H_3X_1E_1 & H_3X_1E_2 \end{pmatrix}, \quad \tilde{\mathcal{M}}_2 = \begin{pmatrix} H_2X_3E_1 & H_2X_3E_2 \\ H_4X_3E_1 & H_4X_3E_2 \end{pmatrix}, \\
\tilde{\mathcal{M}}_3 &= \begin{pmatrix} H_1X_2E_3 & H_1X_2E_4 \\ H_3X_2E_3 & H_3X_2E_4 \end{pmatrix}, \quad \tilde{\mathcal{M}}_4 = \begin{pmatrix} H_2X_4E_3 & H_2X_4E_4 \\ H_4X_4E_3 & H_4X_4E_4 \end{pmatrix}.
\end{aligned} \tag{3.52}
$$

We note that $\tilde{\mathcal{M}}_j$, $j = 1, \ldots, 4$, only depend on A_1, A_2 and A_3 and so, they do not depend on δ.

So, we can reduce to consider the characteristic polynomial of $\tilde{\mathcal{M}}$. For the elements of these matrices we shall use the following notation $\tilde{\mathcal{M}}_1 = (u_{ij})$, $\tilde{\mathcal{M}}_2 = (v_{ij})$, $\tilde{\mathcal{M}}_3 = (p_{ij})$, $\tilde{\mathcal{M}}_4 = (w_{ij})$.

From (3.51) we get

$$\begin{gathered}\operatorname{trace}(\tilde{\mathcal{M}}) = \sigma^{2\beta_1}(u_{11}+u_{33}) + \sigma^{2\beta_2}(u_{22}+u_{44}) + \sigma^{-2\beta_1}(w_{11}+w_{33}) + \\ \sigma^{-2\beta_2}(w_{22}+w_{44}) + \operatorname{trace}(\tilde{\mathcal{M}}_2) + \operatorname{trace}(\tilde{\mathcal{M}}_3),\end{gathered}$$

where $\operatorname{trace}(\tilde{\mathcal{M}}_2)$ and $\operatorname{trace}(\tilde{\mathcal{M}}_3)$ do not depend on σ.

Taking into account that $b_3 = -\operatorname{trace}(\tilde{\mathcal{M}})$, we obtain (3.33) by taking $d_1 = u_{11}+u_{33}$, $d_2 = u_{22}+u_{44}$, $d_3 = w_{11}+w_{33}$, $d_4 = w_{22}+w_{44}$ and $d_5 = \operatorname{trace}(\tilde{\mathcal{M}}_2) + \operatorname{trace}(\tilde{\mathcal{M}}_3)$.

From (3.52) and (3.49) we get

$$u_{11} = h_{11}x_{11}e_{11}, \quad u_{33} = h_{31}x_{11}e_{21}, \quad u_{13} = h_{11}x_{11}e_{21}, \quad u_{31} = h_{31}x_{11}e_{11} \tag{3.53}$$

$d_1 = h_{11}x_{11}e_{11} + h_{31}x_{11}e_{21} = -2x_{11}e_{11}e_{21}$. Then, if we denote by

$$d_n = -2e_{11}e_{21} \quad \text{and} \quad d_g = x_{11},$$

d_n depends on $\lambda_1, \lambda_2, \alpha$ and γ but not on V_1 and d_g depends also on V_1.

Now we study the dominant terms in b_2.

Given a 4×4 matrix $A = (a_{ij})$, the coefficient of x^2 in the characteristic polynomial of A is

$$\begin{gathered}a_{11}(a_{22}+a_{33}+a_{44}) + a_{22}(a_{33}+a_{44}) + a_{33}a_{44} - a_{12}a_{21} - a_{13}a_{31} - a_{14}a_{41} - \\ a_{23}a_{32} - a_{34}a_{43} - a_{24}a_{42}.\end{gathered}$$

In our case, the terms on $\tilde{\mathcal{M}}$ are of the form $k_1\sigma^{2\beta_i} + k_2\sigma^{-2\beta_i} + k_3\sigma^{2(\beta_i-\beta_j)} + k_4$ for $i \neq j$, $i,j \in \{1,2\}$, and some constants k_m, $m = 1,\dots,4$. By taking into account this fact, it is easy to see that the dominant terms in b_2 is $(u_{11}u_{33} - u_{13}u_{31})\sigma^{4\beta_1}$. However using (3.53) one has $u_{11}u_{33} - u_{13}u_{31} = 0$.

An analogous computation shows that the coefficient of $\sigma^{4\beta_2}$ in b_2 is $u_{22}u_{44} - u_{24}u_{42}$, which is equal to zero. Then, b_2 does not contain terms in $\sigma^{4\beta_1}$ nor $\sigma^{4\beta_2}$.

Then, b_2 is written as (3.34) with

$$e_3 \;=\; (u_{11}+u_{33})(u_{22}+u_{44}) - u_{12}u_{21} - u_{14}u_{41} - u_{23}u_{32} - u_{34}u_{43}.$$

It is easy to check that $e_3 = -4e_{11}e_{12}e_{21}e_{22}\det(X_1)$. If we denote by

$$e_n = -4e_{11}e_{12}e_{21}e_{22} \quad \text{and by} \quad e_g = \det(X_1),$$

we have that e_n depends on $\lambda_1, \lambda_2, \alpha$ and γ, but not on V_1, and e_g depends also on V_1.

Now we assume that $\lambda_2 < \gamma \frac{(2-\alpha)^2}{8}$. In this case β_2 is pure imaginary. Due to the real character of $\mathcal{M}$ its characteristic polynomial is real and therefore

$$\begin{aligned} d_2\sigma^{2\beta_2} + d_4\sigma^{-2\beta_2} \in \mathbb{R} \Rightarrow d_4 = \overline{d}_2 \quad , \\ e_3\sigma^{2\beta_2} + e_4\sigma^{-2\beta_2} \in \mathbb{R} \Rightarrow e_4 = \overline{e}_3. \end{aligned}$$

To finish the proof of lemma 3.3.5 we only need to prove that $d_n, e_n \neq 0$. To do that, we shall see that $e_{11}, e_{12}, e_{21}, e_{22} \neq 0$.

From (3.47) and (3.45), a simple computation shows that

$$\begin{aligned} e_{11} &= \frac{1}{\rho_1^+ - \rho_1^-}(c_{31} - \rho_1^- c_{11}), \qquad & e_{21} &= \frac{1}{\rho_1^+ - \rho_1^-}(c_{41} - \rho_1^- c_{21}), \\ e_{12} &= \frac{1}{\rho_2^+ - \rho_2^-}(c_{32} - \rho_2^- c_{12}), \qquad & e_{22} &= \frac{1}{\rho_2^+ - \rho_2^-}(c_{42} - \rho_2^- c_{22}). \end{aligned} \tag{3.54}$$

System (3.43) can be written as two uncoupled systems of the following type

$$\begin{aligned} v_1' &= v_2, \\ v_2' &= \lambda v_1 - \frac{Q_{L_1}(\tau)}{2} v_2, \end{aligned} \tag{3.55}$$

where $\lambda = \lambda_1$ for the system corresponding to u_1, u_3 and $\lambda = \lambda_2$ for u_2, u_4.

We note that $\begin{pmatrix} c_{11} & c_{21} \\ c_{31} & c_{41} \end{pmatrix}$ and $\begin{pmatrix} c_{12} & c_{22} \\ c_{32} & c_{42} \end{pmatrix}$ are the fundamental matrices of (3.55) for $\lambda = \lambda_1$ and $\lambda = \lambda_2$, respectively.

Therefore any e_{ij} in (3.54) can be written in terms of $v_2(\tau_{L_1}) - \rho^- v_1(\tau_{L_1}))$ being $(v_1(\tau), v_2(\tau))$ a solution of (3.55) and

$$\rho^- = \frac{Q_p}{4}(1-\beta), \quad \text{with} \quad \beta = \sqrt{1 - \frac{8\lambda}{\gamma(2-\alpha)^2}}. \tag{3.56}$$

We note that if $\lambda = 0$, $\rho^- = 0$ and (3.55) has solutions with $v_2(\tau) \equiv 0$. Then, if $\lambda_1 = 0$, $c_{31} = 0$ and $e_{11} = 0$. In a similar way, if $\lambda_2 = 0$, $c_{32} = 0$ and $e_{12} = 0$.

Next lemma applied to our systems will be very useful in order to finish the proof of lemma 3.3.5.

Lemma 3.5.1. *Assume $\lambda \neq 0$. Let $\mathbf{v}(\tau) = (v_1(\tau), v_2(\tau))^T$ be one of the solutions of (3.55) with initial conditions $\mathbf{v}(0) = (1,0)^T$ or $\mathbf{v}(0) = (0,1)^T$. Let be ρ^- defined in (3.56). Then, for any $\tau > 0$ sufficiently large*

$$v_2(\tau) - \rho^- v_1(\tau) \neq 0. \tag{3.57}$$

Proof

Assume that $\lambda < \gamma(2-\alpha)^2/8 < 0$. In this case ρ^- is a complex number and as we consider real solutions of the real system (3.55) then the result of the lemma follows trivially.

We assume $\lambda > \dfrac{\gamma(2-\alpha)^2}{8}$. We introduce polar coordinates in (3.55) as $v_1 = r\cos\phi$, $v_2 = r\sin\phi$. Then

$$\begin{aligned} r' &= r\left[(1+\lambda)\cos\phi\sin\phi - \frac{Q_{L_1}(\tau)}{2}\sin^2\phi\right], \\ \phi' &= \lambda\cos^2\phi - \sin^2\phi - \frac{Q_{L_1}(\tau)}{2}\sin\phi\cos\phi. \end{aligned} \tag{3.58}$$

We are interested in the solutions of (3.55) with $\mathbf{v}(0) = (1,0)^T$ and $\mathbf{v}(0) = (0,1)^T$, that is, the solutions of (3.58) with $r(0) = 1$ and, $\phi(0) = 0$ and $\phi(0) = \dfrac{\pi}{2}$, respectively. Moreover we have $v_2(\tau) - \rho^- v_1(\tau) = r(\tau)[\sin\phi(\tau) - \rho^-\cos\phi(\tau)]$. We note that $r = 0$ is invariant for (3.58). Then, we have to prove that $\sin\phi(\tau) - \rho^-\cos\phi(\tau) \neq 0$, or equivalently, that

$$\tan\phi(\tau) \neq \rho^-, \tag{3.59}$$

for any $\tau > 0$ sufficiently large. To do that we only need to consider the non autonomous equation for ϕ in (3.58). If we consider time as a variable we are faced to a system defined on a cilinder. For convenience, we shall consider a 2–dimensional system on the plane taking into account the corresponding identifications.

So, we define new variables $u = \tanh\left(\dfrac{\alpha}{2(2-\alpha)}Q_p\tau\right)$ and $w = \tan(\phi(\tau))$. Condition (3.59) reduces to $w(\tau) \neq \rho^-$ for τ sufficiently large.

We get for w and u the following planar system

$$\begin{aligned} w' &= -w^2 + \frac{Q_p}{2}uw + \lambda, \\ u' &= \frac{\alpha}{2(2-\alpha)}Q_p(1-u^2), \end{aligned} \tag{3.60}$$

which is well defined for any u, w. However, for us, it only makes sense for $|u| \leq 1$. It is also clear that $u(\tau)$ is an increasing function for $|u| < 1$. Moreover, in order to recover the solutions of (3.58) from (3.60) we must identify the solutions of (3.60) with $w(\tau) \to -\infty$ for $\tau \to \tau_*^-$ with the corresponding ones with $w(\tau) \to +\infty$ for $\tau \to \tau_*^+$.

If $\lambda > \gamma(2-\alpha)^2/8$ the system (3.60) has two equilibrium points on the line $u = 1$ located at $(w,u) = (\rho^-, 1)$ and $(w,u) = (\rho^+, 1)$ respectively, where

$$\rho^\pm = \frac{Q_p}{4}(1 \pm \beta).$$

We study the stability of these equilibrium points. The Jacobian of the planar vector field that defines (3.60) is

$$\begin{pmatrix} -2w + \frac{Q_p}{2}u & \frac{Q_p}{2}w \\ 0 & -\frac{\alpha}{2-\alpha}Q_p u \end{pmatrix} \quad . \tag{3.61}$$

Then, for $(\rho^-, 1)$ matrix (3.61) has eigenvalues $-\dfrac{\alpha}{2-\alpha}Q_p < 0$ and $\dfrac{Q_p}{2}\beta > 0$, and then $(\rho^-, 1)$ is a saddle point. The eigenvalues of (3.61) on $(\rho^+, 1)$ are $-\dfrac{\alpha}{2-\alpha}Q_p < 0$ and $-\dfrac{Q_p}{2}\beta < 0$. Therefore ρ^+ is an attractor. Moreover, (3.60) has a vertical isocline defined by $-w^2 + \dfrac{Q_p}{2}uw + \lambda = 0$.

First we assume that $\lambda > 0$. In this case, $\rho^- < 0$ and $\rho^+ > 0$. The region $\mathcal{R}_1 = \{(w,u) \,|\, w \geq 0, 0 \leq u \leq 1\}$ is positively invariant for the flow defined by (3.60). In this region all the orbits tend to the attractor. Then, the orbits we are interested in are contained for all positive time in $\mathcal{R}_1$. As $\rho^- \notin \mathcal{R}_1$, $w(\tau) \neq \rho^-$ for $\tau > 0$. (see Figure 3.5).

Now we consider values of λ such that $\gamma(2-\alpha)^2/8 < \lambda < 0$. Then $0 < \rho^- < \rho^+$. Let W^s be the branch of the unstable invariant manifold of the point $(\rho^-, 1)$ contained in the band $\{(w,u) \,|\, |u| \leq 1\}$, and $\mathcal{R}_2 \subset \{(w,u) \,|\, w \geq 0, \quad 0 \leq u \leq 1\}$ the unbounded region with boundaries W^s and $\{(w,u) \,|\, w \geq \rho^-, u = 1\}$. Then $\mathcal{R}_2$ is positively invariant and all the orbits in $\mathcal{R}_2$ tend to $(\rho^+, 1)$ when $\tau \to \infty$. As the orbits we are interested in enter in $\mathcal{R}_2$ for some τ large enough, then $w(\tau) \neq \rho^-$ if τ is sufficiently large (see Figure 3.6).

□

In order to apply this lemma to our case, first we take $\mathbf{v}(\tau) = (c_{11}(\tau), c_{31}(\tau))^T$ and $\lambda = \lambda_1$. We know that $\mathbf{v}(0) = (1, 0)^T$. Then,

$$c_{31}(\tau) - \rho^- c_{11}(\tau) \neq 0,$$

for τ sufficiently large. We take δ small enough in order to obtain the relation above. Then, $e_{11} \neq 0$.

Analogously one can see that $e_{12}, e_{21}, e_{22} \neq 0$.

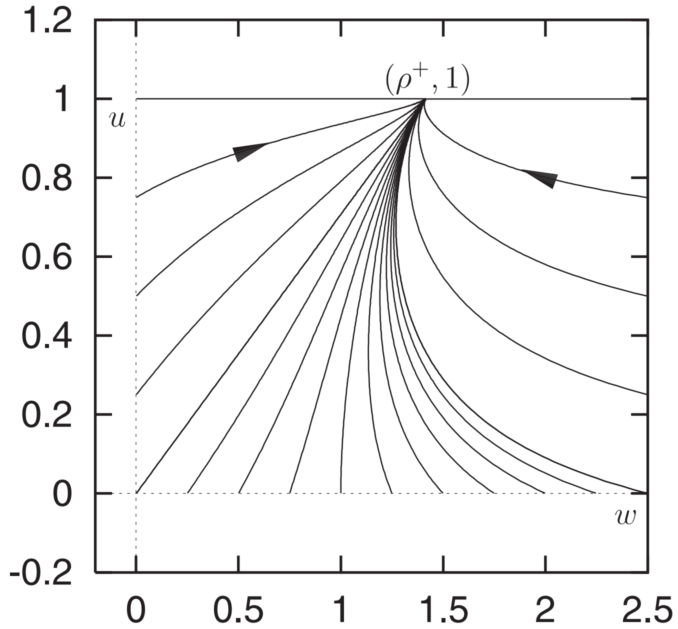


Figure 3.5: Phase portrait of system (3.60) for $\gamma = -1$ and $\lambda = 1$ on the region $\mathcal{R}_1$.

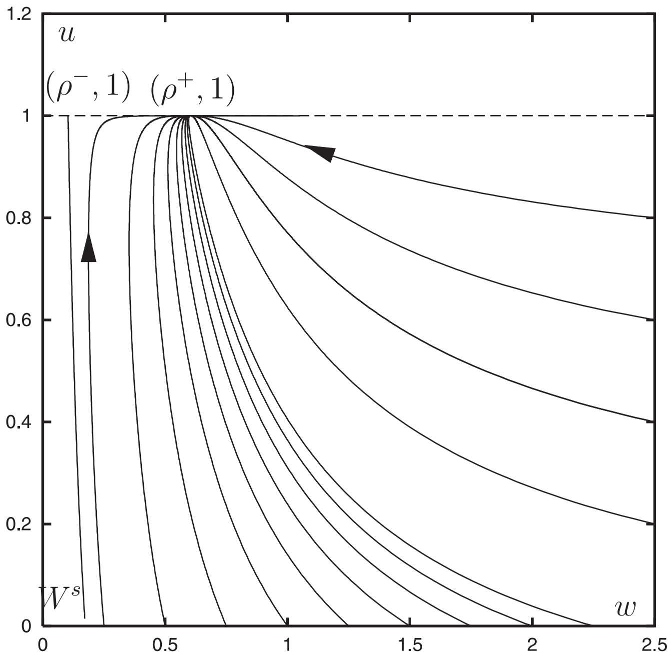


Figure 3.6: Phase portrait of system (3.60) for $\gamma = -1$ and $\lambda = -\frac{1}{16}$ on the region $\mathcal{R}_2$.

Chapter 4

Linear stability of homographic solutions

4.1 Introduction

In this chapter we study the stability parameters of the homographic solutions of the Planar Three Body Problem with homogeneous potential of order $-\alpha$, $0 < \alpha < 2$.

In chapter 1 we have seen that the system that gives us the non–trivial characteristic multipliers for the homographic solutions (see (1.56)) is

$$\dot{\mathbf{x}} = A(f, e)\mathbf{x},$$

where $\mathbf{x} \in \mathbb{R}^4$, $\dot{} = \dfrac{d}{df}$ and

$$A(f, e) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ g^{\alpha-2}\lambda_1 & 0 & 0 & -2 \\ 0 & g^{\alpha-2}\lambda_2 & 2 & 0 \end{pmatrix}. \tag{4.1}$$

The parameters λ_1, λ_2 are defined in table 1.1 for the triangular and collinear case, respectively, g is the periodic solution of the potential equation

$$\ddot{z} = -\frac{dU}{dz}(z) \quad \text{with} \quad U(z) = \frac{z^2}{2} - \frac{z^\alpha}{\alpha}, \tag{4.2}$$

on the energy level $E = -\dfrac{1}{2}\omega^{\frac{2\alpha}{2-\alpha}}$ being $0 < \omega \leq \omega_c$, $\omega_c = \left(\dfrac{2-\alpha}{\alpha}\right)^{\frac{2-\alpha}{2\alpha}}$ (see (1.34)) and f is defined in (1.11).

Notice that λ_1 and λ_2 depend on a unique mass parameter, β_c or β_t (see table 1.1). Moreover, (4.1) depends on ω. So the system (4.1) depends on two parameters, ω and β being $\beta = \beta_c$ in the collinear case and $\beta = \beta_t$ in the triangular case. In the following we shall denote by β the parameters β_c or β_t. When talking about collinear configurations, we will take $\beta = \beta_c$. In the triangular case, we shall take $\beta = \beta_t$. If we are talking about the both cases, we shall write β.

We also recall that once the configuration, triangular or collinear, is fixed we can characterize an homographic solution using $\omega \in (0, \omega_c]$ or the generalized eccentricity $e \in [0, 1)$ defined in (1.57), that is, $e = \sqrt{1 - \frac{\alpha}{2-\alpha}\omega^{\frac{2\alpha}{2-\alpha}}}$. Our purpose is to study the linear stability for $e \in [0, 1)$ and the range of β defined in the table 1.1.

If $e = 0$ or equivalently $\omega = \omega_c$, the homographic solution is a relative equilibrium. In that case, (4.1) is a constant linear system and some resonant points on the λ_1, λ_2 plane are obtained. Therefore, it is expected that some resonant 'tongues' will appear for $e \gtrsim 0$ in the plane of parameters β, e, giving rise to regions with a different stability character. These kind of bifurcations as well as the width of the respective tongues can be studied using the results of chapter 2.

When $e \lesssim 1$, that is $\omega \gtrsim 0$, (4.1) is near the singular case. Notice that (see (4.2)) $U(z) = z^\alpha \left(-\frac{1}{\alpha} + \frac{z^{2-\alpha}}{2}\right)$ satisfies the hypothesis (A1) and (A2) in chapter 3 with

$$\gamma = -\frac{1}{\alpha}, \qquad s = 2 - \alpha, \qquad V_1(z) = \frac{1}{2}. \tag{4.3}$$

Moreover we recall that in chapter 3, g is taken as a periodic solution of the conservative system (3.2) on the energy level $-\delta$. In the homographic case g is a periodic solution of (4.2) on the energy level $E = -\frac{1}{2}\omega^{\frac{2\alpha}{2-\alpha}}$. Therefore, the hypothesis (B) holds by taking $\delta = \frac{1}{2}\omega^{\frac{2\alpha}{2-\alpha}}$, or, using the generalized eccentricity,

$$\delta \;\; = \;\; \frac{2-\alpha}{2\alpha}(1 - e^2). \tag{4.4}$$

For intermediate values of the eccentricity $e \in (0, 1)$ the bifurcation diagram is computed numerically. In section 4.2 we consider small eccentricity and, section 4.3 is devoted to the near singular case, $e \lesssim 1$.

4.2 Stability parameters near the constant case

First (section 4.2.1) we determine resonances when the generalized eccentricity, e, equals zero. Then we study the stability parameters for small positive e. In section

4.2.2 we consider the Newtonian case. We shall apply the Normal Form technique developed in chapter 2 in order to obtain the boundaries of the resonant regions. Section 4.2.3 is devoted to the general case.

4.2.1 Eccentricity equal to zero

For the triangular configuration λ_1 and λ_2 are the zeroes of the polynomial

$$p(\lambda) = \lambda^2 - (\alpha+2)\lambda + \frac{\beta_t}{4},$$

(see table 1.1). Then, for $\beta_t \in (0, (\alpha+2)^2]$, (λ_1, λ_2) describes a segment on the plane with endpoints

$$(\alpha+2, 0), \qquad \left(\frac{\alpha+2}{2}, \frac{\alpha+2}{2}\right). \tag{4.5}$$

This segment goes from region $\mathcal{R}_2$ to $\mathcal{R}_3$ (see figure 4.1) using the notation introduced in chapter 2. The change from $\mathcal{R}_2$ to $\mathcal{R}_3$ takes place when $(\lambda_1+\lambda_2-4)^2 - 4\lambda_1\lambda_2 = 0$, that is,

$$(\alpha-2)^2 - \beta_t = 0.$$

For $0 < \beta_t \leq (\alpha-2)^2$ the characteristic exponents are $\pm i\omega_1$, $\pm i\omega_2$ and for $(\alpha-2)^2 < \beta_t \leq (\alpha+2)^2$ they are complex, $\pm a \pm ib$.

Assume $0 < \beta_t < (\alpha-2)^2$. In this case, $\omega_1 \neq \omega_2$. To look for resonant points we compute ω_1, ω_2 as

$$\omega_1^2 = \frac{2-\alpha+\sqrt{(2-\alpha)^2-\beta_t}}{2}, \qquad \omega_2^2 = \frac{2-\alpha-\sqrt{(2-\alpha)^2-\beta_t}}{2}.$$

Resonances are obtained when ω_1 or ω_2 satisfy $\omega T = n\pi$ for some $n \in \mathbb{N}$ where $T = \dfrac{2\pi}{\sqrt{2-\alpha}}$ or, equivalently when $\dfrac{4\omega^2}{2-\alpha} = n^2$. Moreover, if $0 < \beta_t < (\alpha-2)^2$ then

$$2 < \frac{4\omega_1^2}{2-\alpha} < 4, \qquad 0 < \frac{4\omega_2^2}{2-\alpha} < 2.$$

Therefore, we get a unique resonance when $\omega_2 T = \pi$ for $\beta_t = \dfrac{3}{4}(2-\alpha)^2$.

Let us assume $(\alpha-2)^2 < \beta_t \leq (\alpha+2)^2$, that is, (λ_1, λ_2) belongs to the region $\mathcal{R}_3$. The characteristic exponents are $\pm a \pm ib$ with $b^2 = \dfrac{1}{4}(2-\alpha+\sqrt{\beta_t})$.

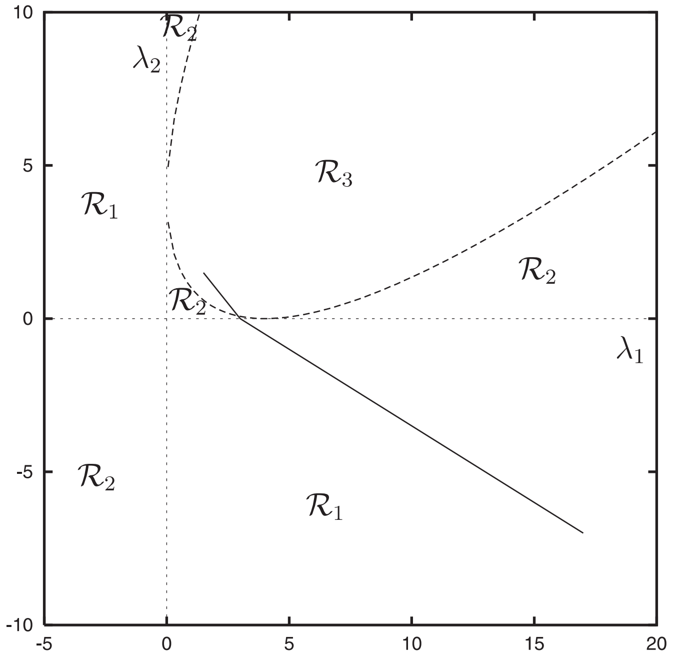


Figure 4.1: Segments corresponding to the collinear and triangular Newtonian case in the plane (λ_1, λ_2)

A resonance is attained if $Tb = n\pi$ for some $n \in \mathbb{N}$ or equivalently $\dfrac{4b^2}{2-\alpha} = n^2$. However for the allowed range of β_t we get

$$2 < \frac{4b^2}{2-\alpha} \leq \frac{4}{2-\alpha},$$

that is, in order to have a resonance with $Tb = n\pi$ we need

$$2 \leq n \leq \frac{2}{\sqrt{2-\alpha}}. \tag{4.6}$$

In that case, a simple computation shows that $\beta_t = (2-\alpha)^2(n^2-1)^2$. We note that (4.6) has no solution if $\alpha < 1$. Then, there is no resonance for $(\alpha-2)^2 < \beta_t < (\alpha+2)^2$ if $\alpha < 1$. Moreover, we have that $\dfrac{2}{\sqrt{2-\alpha}} \to +\infty$ when $\alpha \to 2^-$. Then, as $\alpha \geq 1$ increases we get more resonant points.

Table 4.1 summarizes the critical values of β_t, such that bifurcations are expected for $e > 0$ small enough.

Let us consider now the collinear case. From table 1.1 we get easily that as β_c

β_t^*	characteristic exponents	transition
$\frac{3}{4}(2-\alpha)^2$	$\pm\mathrm{i}\omega_j,\ j=1,2$ $\omega_1 T \neq n\pi,\ n\in\mathbb{N},$ $\omega_2 T=\pi$	EE↔EH
$(2-\alpha)^2$	$\pm\mathrm{i}\omega_j,\ j=1,2$ $\omega_1=\omega_2=\sqrt{\dfrac{2-\alpha}{2}}$	EE↔CS
$(2-\alpha)^2(n^2-1)^2$ for $n\in\mathbb{N}$ $2\leq n\leq \dfrac{2}{\sqrt{2-\alpha}}$	$\pm a\pm \mathrm{i}b$ $bT=n\pi$	CS↔HH

Table 4.1: Resonances for $e=0$ in the triangular case and expected transitions for small e

ranges from 0 to $2^{\alpha+2}-1$, the point (λ_1,λ_2) moves on a segment with endpoints

$$(\alpha+2,0) \qquad ((\alpha+1)2^{\alpha+2}+1, 1-2^{\alpha+2}). \tag{4.7}$$

For $\beta_c\neq 0$ this segment is contained in the region $\mathcal{R}_1$ (see figure 4.1). So, the characteristic exponents are $\pm\lambda$, $\pm\mathrm{i}\omega$. Only single resonances can be attained when $\omega T=n\pi$ for some $n\in\mathbb{N}$.

To get the resonant points we write ω^2 as

$$\omega^2 \;=\; \frac{2-\alpha(\beta_c+1)+\sqrt{\beta_c^2(\alpha+2)^2+2\beta_c(\alpha^2+4)+(\alpha-2)^2}}{2}.$$

It is easy to check that ω^2 is an increasing function of β_c. Then, $\omega\in(\sqrt{2-\alpha},\omega_M)$ being

$$\omega_M \;=\; \sqrt{1-2^{\alpha+1}\alpha+2^{\frac{\alpha}{2}}\sqrt{2^{\alpha+2}(\alpha+2)^2-8\alpha}}. \tag{4.8}$$

In terms of n, we have that

$$2<n<\frac{2\omega_M}{\sqrt{2-\alpha}}.$$

We note that $\omega_M^2 \to -15 + 8\sqrt{15} > 0$ and then $\dfrac{2\omega_M}{\sqrt{2-\alpha}} \to +\infty$ when $\alpha \to 2^-$. Then, as α approaches 2, the number of resonances increase.

4.2.2 The Newtonian case

We shall assume $\alpha = 1$, that is Newtonian potential. System (4.1) can be written as (2.1) by taking $G_1 = G_2 = g^{-1}$. The periodic solution of (4.2) is $g = 1 + e\cos f$, where e is the eccentricity of the orbit and f the true anomaly. Then, $g^{-1} = 1 - F(f,e)$ being $F(f,e)$ an even 2π–periodic function on f which satisfies the d'Alembert property. That is, the hypothesis assumed in chapter 2 holds for (4.1).

We begin with the triangular homographic solutions. From table 4.1 we get the following resonant points for $e = 0$.

β_t^*	characteristic exponents
$\frac{3}{4}$	$\pm i\omega_1,\ \pm i\omega_2,\ \omega_1 = \frac{\sqrt{3}}{2},\ \omega_2 = \frac{1}{2}$
1	$\pm i\omega_1,\ \pm i\omega_2,\ \omega_1 = \omega_2 = \frac{\sqrt{2}}{2}$
9	$\pm a \pm i$

Table 4.2: Resonances for $e = 0$ in the triangular case for Newtonian potential

We can apply the theory of chapter 2 to the case $\beta_t^* = \dfrac{3}{4}$, that is, near the point $(a_1, a_2) = \left(\dfrac{3}{2} + \dfrac{1}{4}\sqrt{33}, \dfrac{3}{2} - \dfrac{1}{4}\sqrt{33}\right)$ in the plane λ_1, λ_2. From (2.6), the resonant curve for $\omega_1 = \dfrac{1}{2}$ is given by

$$\gamma_t(\lambda_1, \lambda_2) := \left(\lambda_1 + \frac{1}{4}\right)\left(\lambda_2 + \frac{1}{4}\right) - 1 = 0. \tag{4.9}$$

Figure 4.2 shows the intersection of the resonant curve (4.9) with the segment with endpoints (4.5).

We take $\lambda_1 = a_1 + \delta_1$ and $\lambda_2 = a_2 + \delta_2$ with $|\delta_1|, |\delta_2|$ small enough. As $(a_1, a_2) \in \mathcal{R}_2$, using Proposition 2.3.4, we know that the Normal Form up to a

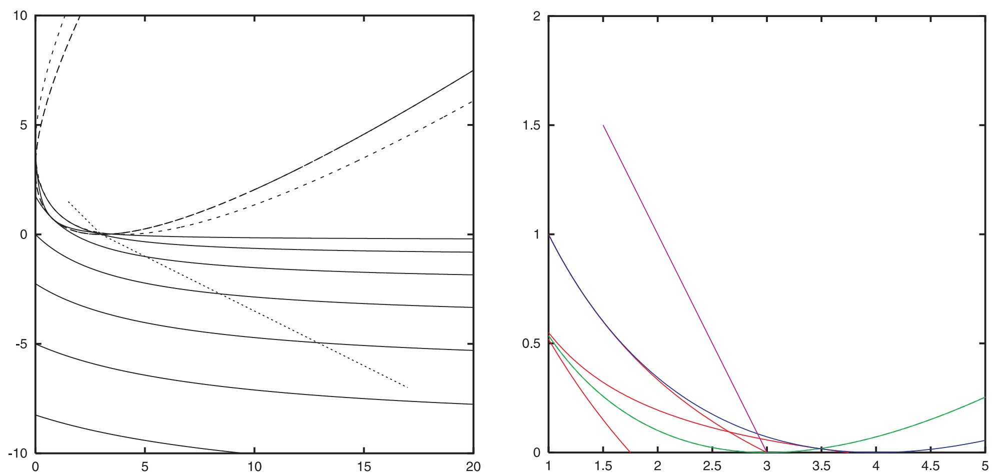


Figure 4.2: Left: Some resonant curves and the segments corresponding to the triangular and collinear case in the plane (λ_1, λ_2). Right: Magnification of the triangular case (color codes are the same as in figure 2.2)

given order in δ_1, δ_2 and e is

$$\begin{aligned} NF &= K + \mathrm{i}\omega_1 z_2 z_4 + \frac{1}{2}\mathrm{i} z_2 z_4 + \mathrm{i}\sigma_1 z_1 z_3 + i\sigma_2 z_2 z_4 + \\ &\quad + \sigma_4 z_2^2 e^{-it} - \sigma_4 z_4^2 e^{it}, \end{aligned}$$

where $\sigma_j \in \mathbb{R}$, $j = 1, 2, 4$, depend on δ_1, δ_2 and e. Here, $\omega_1 = \dfrac{\sqrt{3}}{2}$. Then, one of the traces satisfies $|\mathrm{tr}_2| < 2$ if δ_1, δ_2, e are small enough, giving an elliptic component. A region EH is created, and their boundaries are defined by the equation

$$\sigma_2^2 - 4\sigma_4^2 = 0. \tag{4.10}$$

As we are in a single resonance case and the function F satisfies d'Alembert property, we can use the theory in section 2.5.1. In particular, we can compute the dominant terms in the contribution of δ_1 and δ_2 to σ_1. Using lemma 2.5.1, a simple computation shows that

$$\sigma_1 = \left(\frac{7}{2} - \frac{1}{2}\sqrt{33}\right)\delta_1 + \left(\frac{7}{2} + \frac{1}{2}\sqrt{33}\right)\delta_2 + O_2.$$

Now we introduce the parameter δ as in (2.60), that is,

$$\begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix} = \delta \nabla \gamma_t(a_1, a_2),$$

where γ is given in (4.9) and (a_1, a_2) is the resonant point for $e = 0$. Then,

$$\delta_1 = \frac{\delta}{4}(7 - \sqrt{33}), \qquad \delta_2 = \frac{\delta}{4}(7 + \sqrt{33}).$$

We can write σ_1 in terms of δ as

$$\sigma_1 \quad = \quad \frac{41}{2}\delta + O_2,$$

We have implemented an algebraic manipulator that computes the Normal Form up to a given order in δ_1, δ_2 and e. In this case, we obtain that

$$\sigma_4 \quad = \quad -0.035903516541\ldots e + O_2.$$

Let us consider $\delta_+(e)$ and $\delta_-(e)$ the solutions of the two equations given by (4.10), that is, $\sigma_2 - 2\sigma_4 = 0$ and $\sigma_2 + 2\sigma_4 = 0$, respectively. We write $\delta_\pm(e) = d_\pm e + O(e^2)$. By proposition 2.5.3, the width of $\delta_+(e) - \delta_-(e)$ is of order 1 in e. Moreover, we can compute explicitly the values of $\delta_\pm$. We have that

$$\delta_\pm = \mp 0.0350278210155\ldots e + O(e^2).$$

In the plane (λ_1, λ_2) the boundaries of the region EH are given by

$$\begin{aligned} \lambda_1 = a_1 - d_1 e + O(e^2), &\qquad \lambda_2 = a_2 - d_2 e + O(e^2), \\ \lambda_1 = a_1 + d_1 e + O(e^2), &\qquad \lambda_2 = a_2 + d_2 e + O(e^2), \end{aligned}$$

where $d_1 = 0.0109938087283\ldots$ and $d_2 = 0.1116035648259\ldots$. Taking into account that $\beta_t = 4\lambda_1\lambda_2$ the equations above defines the following curves in the plane (β_t, e)

$$\beta_t^- = \frac{3}{4} - de + O(e^2), \qquad \beta_t^+ = \frac{3}{4} + de + O(e^2),$$

where $d = 0.4903894921666\ldots$.

We conclude that a resonant tongue $\mathcal{T}$ is born at the point $(\beta_t, e) = \left(\frac{3}{4}, 0\right)$ and the width of $\mathcal{T}$ is of order $O(e)$.

Remark 4.2.1. The existence of this tongue was proved by G.Roberts in [R.] using a different method.

Figure 4.3 shows the bifurcation diagram on the plane (β_t, e) computed numerically. On the range $\beta_t \in (0, 9)$ we distinguish the tongue $\mathcal{T}$ born at $\beta_t^* = \frac{3}{4}$. The behaviour for $e \lesssim 1$ will be described in section 4.3.

Now we study the collinear case. For $e = 0$, the characteristic exponents are $\pm\lambda$, $\pm\mathrm{i}\omega$, where $\omega \in (1, \omega_M)$, $\omega_M = 2.88335022\ldots$ (see (4.8)) as $\beta_c \in (0, 7)$.

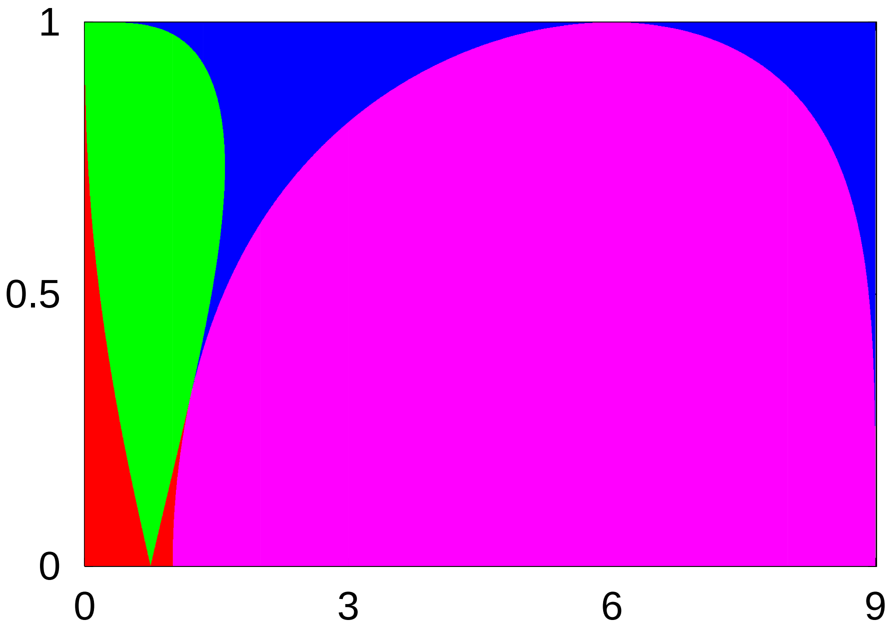


Figure 4.3: Bifurcation diagram of the triangular Newtonian homographic solutions in the plane (β_t, e). Color codes: Red for EE, Green for EH, Magenta for CS and Blue for HH

Resonances $\omega = \frac{3}{2}, 2, \frac{5}{2}$ are found on that range of β_c. The corresponding critical values of β_c are given in table 4.3. We expect resonant tongues $\mathcal{T}_{\frac{3}{2}}$, $\mathcal{T}_2$ and ,$\mathcal{T}_{\frac{5}{2}}$ associated to that resonances. Our purpose now is to compute the width of $\mathcal{T}_{\frac{3}{2}}$ and $\mathcal{T}_{\frac{5}{2}}$ using the Normal Form method.

Using the data in tables 4.3 and 1.1, we compute the following resonant points on the plane λ_1, λ_2,

$$
\begin{aligned}
(a_1, a_2) &= \left(\frac{3}{8}(\sqrt{41}+7), \frac{3}{16}(1-\sqrt{41})\right), \\
(a_1, a_2) &= \left(\frac{1}{8}(37+\sqrt{4369}), -\frac{1}{16}(13+\sqrt{4369})\right),
\end{aligned}
$$

for $\omega = \frac{3}{2}$ and $\omega = \frac{5}{2}$, respectively. The corresponding resonant curves are

$$
\begin{aligned}
\gamma_1(\lambda_1, \lambda_2) &= \left(\lambda_1 + \frac{9}{4}\right)\left(\lambda_2 + \frac{9}{4}\right) - 9 = 0, \\
\gamma_2(\lambda_1, \lambda_2) &= \left(\lambda_1 + \frac{25}{4}\right)\left(\lambda_2 + \frac{25}{4}\right) - 25 = 0,
\end{aligned}
$$

β_c^*	characteristic exponents $\pm \mathrm{i}\omega$	width of $\mathcal{T}_\omega$	transition
$\frac{3}{16}(\sqrt{41}-1)$	$\omega = \frac{3}{2}$	$O(e^3)$	HE$\longleftrightarrow$HH
$\frac{1}{4}(1+\sqrt{97})$	$\omega = 2$		no bifurcation
$\frac{1}{16}(13+\sqrt{4369})$	$\omega = \frac{5}{2}$	$O(e^5)$	HE$\longleftrightarrow$HH

Table 4.3: Resonances for $e = 0$ in the collinear case for Newtonian potential

respectively. Figure 4.2 shows the intersection of the resonant curves with the segment defined by the collinear homographic solutions.

We take $\lambda_1 = a_1 + \delta_1$ and $\lambda_2 = a_2 + \delta_2$ with $|\delta_1|, |\delta_2|$ small enough. As $(a_1, a_2) \in \mathcal{R}_1$ the Normal Form up to a given order of δ_1, δ_2, e is

$$\begin{aligned} NF &= K + \lambda z_1 z_3 + \mathrm{i}\omega z_2 z_4 + \sigma_1 z_1 z_3 + \mathrm{i}\sigma_2 z_2 z_4 + \\ &\quad \sigma_3 z_2^2 e^{it} - \sigma_3 z_4^2 e^{it}, \end{aligned}$$

where $\sigma_1, \sigma_2, \sigma_3 \in \mathbb{R}$ depend on δ_1, δ_2 and e, and $(\lambda, \omega) = \left(\frac{1}{4}\sqrt{3\sqrt{41}+17}, \frac{3}{2}\right)$ or $\left(\frac{1}{4}\sqrt{97+\sqrt{4369}}, \frac{5}{2}\right)$. We have that $|\mathrm{tr}_1| > 2$. A region HH is created. Its boundaries are defined by the equation

$$\sigma_2^2 - 4\sigma_3^2 = 0.$$

We take $\beta_c = \beta_c^* + \delta$. Then, $\delta_1 = 2\delta$ and $\delta_2 = -\delta$. We are in a single resonance case. Moreover, function F satisfies d'Alembert property. Then, using lemma 2.5.1 we obtain that

$$\begin{aligned} \sigma_2 &= \frac{53\sqrt{41}-123}{610}\delta + O_2 \quad \text{if} \quad \omega = \frac{3}{2}, \\ \sigma_2 &= \frac{197\sqrt{4369}-4369}{43050}\delta + O_2 \quad \text{if} \quad \omega = \frac{5}{2}. \end{aligned}$$

Using the algebraic manipulator we obtain that

$$\begin{aligned}\sigma_3 &= -5.9623466927\ldots\cdot 10^{-3}e^3 + O_4 \quad \text{if} \quad \omega = \frac{3}{2},\\ \sigma_3 &= -3.3038513137\ldots\cdot 10^{-5}e^5 + O_6 \quad \text{if} \quad \omega = \frac{5}{2}.\end{aligned}$$

Let us consider $\delta_+(e)$, $\delta_-(e)$ the solutions of $\sigma_2 - 2\sigma_3 = 0$ and $\sigma_2 + 2\sigma_3 = 0$, respectively. If we write $\delta_\pm(e) = d_1^\pm e + d_2^\pm e^2 + d_3^\pm e^3 + d_4^\pm e^4 + d_5^\pm e^5 + O(e^6)$ and taking into account the expression up to order 3 and 5 in the case $\omega = \frac{3}{2}$ and $\omega = \frac{5}{2}$, given by the algebraic manipulator, we obtain that the boundaries of the resonant tongues in the plane (β_c, e) are given by

$$\begin{aligned}\beta_c - \beta_c^* &= -0.4208699384\ldots e^2 \pm 0.03361931602\ldots e^3 + O(e^5) \quad \text{if} \quad \omega = \frac{3}{2},\\ \beta_c - \beta_c^* &= -1.9578203867\ldots e^2 - 0.5109418802\ldots e^4\\ &\quad \pm 0.00032876661\ldots e^5 + O(e^6) \quad \text{if} \quad \omega = \frac{5}{2},\end{aligned}$$

where we recall that $\beta_c - \beta_c^* = \delta$.

Therefore, two resonant tongues $\mathcal{T}_{\frac{3}{2}}$ and $\mathcal{T}_{\frac{5}{2}}$ are born at $e = 0$ being their width of order e^3, e^5, respectively (see table 4.3).

In the case $\omega = 2$ the computations up to a given order using the algebraic manipulator shows that the two boundaries coincide up to that order. We prove now that, in fact, if $\omega = 2$ there is no bifurcation. To this end, we consider the system (4.1) in the Newtonian case for arbitrary $(\lambda_1, \lambda_2) \in \mathcal{R}_1 \cup \mathcal{R}_2$.

Lemma 4.2.2. *Let us consider the system (4.1) in the Newtonian case and assume that for $e = 0$, $(\lambda_1, \lambda_2) = (a_1, a_2) \in \mathcal{R}_1 \cup \mathcal{R}_2$, we get a single resonance frequency $\omega = n$ with $n \in \mathbb{N}$. Then, the two boundaries of the resonant region coincide. There is no bifurcation in this case.*

Proof

For $\omega = n$, $n \in \mathbb{N}$, one stability parameter, tr_2, is equal to 2 for $e = 0$. Then the boundaries of the resonant region are defined by $\mathrm{tr}_2 = 2$. Furthermore, if $(\lambda_1, \lambda_2, e)$ belongs to the boundary, the linear system (4.1) has a 2π–periodic solution. To finish the proof we need the following lemma.

Lemma 4.2.3. *Assume that (4.1) has a 2π–periodic solution, φ, for a fixed value of $e \in (0,1)$ and $\lambda_j \neq 0$, $j = 1, 2$. Then, there exists a second periodic solution with the same period which is independent of φ.*

Now we prove Lemma 4.2.2.

Let us define $\Phi(2\pi)$ the monodromy matrix of (4.1). After lemma 4.2.3, if $(\lambda_1, \lambda_2, e)$ belongs to the boundary of the resonant region then $\Phi(2\pi)$ can be written (in a suitable basis) as

$$\Phi(2\pi) = \begin{pmatrix} Q & 0 \\ 0 & I_2 \end{pmatrix},$$

for some 2×2 matrix Q. Using the Normal Form we can compute $\Phi(2\pi)$ up to a given order in δ_1, δ_2, e. As we are in a single resonance case we know that the reduced system becomes uncoupled. Assume that $(a_1, a_2) \in \mathcal{R}_1$. Then the subsystem that defines tr_2 is (2.39), that is,

$$\begin{aligned} \dot{u} &= \mathrm{i}\sigma_2 u - 2\sigma_3 v, \\ \dot{v} &= -2\sigma_3 u - \mathrm{i}\sigma_2 v. \end{aligned}$$

(In the case $(a_1, a_2) \in \mathcal{R}_2$ a similar subsystem is obtained). We define for this system the symplectic change of coordinates

$$\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \frac{\sqrt{2}}{2} \begin{pmatrix} 1 & \mathrm{i} \\ \mathrm{i} & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}.$$

Then the new system is

$$\begin{pmatrix} \dot{\eta}_1 \\ \dot{\eta}_2 \end{pmatrix} = S_1 \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix},$$

where $S_1 = \begin{pmatrix} 0 & \sigma_2 - 2\sigma_3 \\ -(\sigma_2 + 2\sigma_3) & 0 \end{pmatrix}$. The corresponding monodromy matrix is $\exp(2\pi S_1)$.

Let us assume that $(\lambda_1, \lambda_2, e)$ belongs to the boundary such that $\sigma_2 - 2\sigma_3 = 0$. Then, $S_1 = \begin{pmatrix} 0 & 0 \\ -(\sigma_2 + 2\sigma_3) & 0 \end{pmatrix}$ and $\exp(2\pi S_1) = \begin{pmatrix} 1 & 0 \\ -2\pi(\sigma_2 + 2\sigma_3) & 1 \end{pmatrix}$.

If for these values of the parameters, $\sigma_2 + 2\sigma_3 \neq 0$, then system (4.1) would have a unique 2π–periodic solution. This gives a contradiction with lemma (4.2.3). In this way we have proved that the two boundaries coincide up to arbitrary order in e, once $\delta_1 = \delta_1(e)$ and $\delta_2 = \delta_2(e)$. Using the analycity they coincide for any value of the eccentricity. □

Proof of Lemma 4.2.3

System (4.1) can be written as the following system of second order equations

$$\begin{aligned} (1 + e\cos f)\ddot{x}_1 &= \lambda_1 x_1 - 2\dot{x}_2(1 + e\cos f), \\ (1 + e\cos f)\ddot{x}_2 &= \lambda_2 x_2 + 2\dot{x}_1(1 + e\cos f). \end{aligned} \tag{4.11}$$

A 2π–periodic solution of the system above can be written as

$$
\begin{aligned}
x_1(f) &= a_0 + \sum_{n\geq 1} a_n \cos(nf) + \sum_{n\geq 1} b_n \sin(nf),\\
x_2(f) &= c_0 + \sum_{n\geq 1} c_n \cos(nf) + \sum_{n\geq 1} d_n \sin(nf).
\end{aligned} \tag{4.12}
$$

Then, the coefficients must satisfy the following uncoupled sets of recurrences

$$
\begin{aligned}
&\lambda_1 a_0 = e\left(d_1 - \frac{a_1}{2}\right),\\
&eA_2\mathbf{u}_2 = B_1\mathbf{u}_1,\\
&eA_{n+1}\mathbf{u}_{n+1} = B_n\mathbf{u}_n - eA_{n-1}\mathbf{u}_{n-1}, \quad n\geq 2, \quad \mathbf{u} = (a_n, d_n)^T,
\end{aligned} \tag{4.13}
$$

$$
\begin{aligned}
&\lambda_2 c_0 = -e\left(b_1 + \frac{c_1}{2}\right),\\
&eA_2S\mathbf{v}_2 = B_1S\mathbf{v}_1,\\
&eA_{n+1}S\mathbf{v}_{n+1} = B_nS\mathbf{v}_n - eA_{n-1}S\mathbf{v}_{n-1}, \quad n\geq 2, \quad \mathbf{v} = (b_n, c_n)^T,
\end{aligned} \tag{4.14}
$$

where $A_n = -\dfrac{n}{2}\begin{pmatrix} n & -2\\ -2 & n\end{pmatrix}$, $B_n = \begin{pmatrix} \lambda_1 + n^2 & -2n\\ -2n & \lambda_2 + n^2\end{pmatrix}$ and $S = \mathrm{diag}(1,-1)$.

We note that if $\mathbf{u}_n$, $n\geq 1$ is a non trivial solution of the last two equations in (4.13) then $\mathbf{v}_n = S\mathbf{u}_n = (a_n, -d_n)^T$, $n\geq 1$, is a non trivial solution of the second and third equations in (4.14). Moreover, A_n is a non singular matrix for $n > 2$. However, $\det(A_2) = 0$. But if $\det(B_1) = (\lambda_1+1)(\lambda_2+1) - 4 \neq 0$, given $\mathbf{u}_2$ we can compute $\mathbf{u}_1$ from the second equality in (4.13), and from the last equation we obtain $\mathbf{u}_n$ for $n\geq 3$.

We assume that (4.12) is a non trivial 2π–periodic solution of (4.11). Then, both (4.13) and (4.14) have a solution. We assume that (4.13) admits a non trivial solution. Then, $\sum_{n\geq 1} a_n \cos(nf)$ and $\sum_{n\geq 1} d_n \sin(nf)$ are convergent. Therefore $\mathbf{v}_n = S\mathbf{u}_n$, that is, $b_n = a_n$ and $c_n = -d_n$, for $n\geq 1$, is a solution of (4.14). Then, we can built two independent periodic solutions of (4.11) as

$$
\begin{aligned}
&x_1^{(1)}(f) = a_0 + \textstyle\sum_{n\geq 1} a_n \cos(nf), && x_2^{(1)}(f) = \textstyle\sum_{n\geq 1} d_n \sin(nf),\\
&x_1^{(2)}(f) = \textstyle\sum_{n\geq 1} a_n \sin(nf), && x_2^{(2)}(f) = c_0 - \textstyle\sum_{n\geq 1} d_n \cos(nf),
\end{aligned} \tag{4.15}
$$

where $a_0 = \dfrac{e}{\lambda_1}\left(d_1 - \dfrac{a_1}{2}\right)$ and $c_0 = \dfrac{e}{\lambda_2}\left(\dfrac{d_1}{2} - a_1\right)$. $\square$

Figure 4.4 shows the bifurcation diagram on the plane (β_c, e) computed numerically for $\beta_c \in (0,7)$, $e\in[0,1)$. The first tongue borns at $\beta_c^* = \dfrac{3}{16}(\sqrt{41}-1) = 1.013\ldots$. We recall that the width of $\mathcal{T}_{\frac{3}{2}}$ is of order e^3. So, to distinguish the two boundaries we have to look at big values of the eccentricity. In figure 4.4 the line inside the resonant tongue corresponds to a minimum of the stability parameter. The second 'tongue', $\mathcal{T}_2$, is only a curve defined by points (β_c, e) for which the second stability parameter is equal to 2, as predicted by lemma 4.2.2. For the third tongue $\mathcal{T}_{\frac{5}{2}}$ the width is of order e^5. We can distinguish the two boundaries in figure 4.5 which is a magnification of 4.4 for big values of e. Other curves in figures 4.4 and 4.5 are resonant tongues $\mathcal{T}_\omega$ for $\omega = \dfrac{m}{2}$, $m \in \mathbb{N}$, $m > 5$. They are born at values $\beta_c^* > 7$. The behaviour of $\mathcal{T}_\omega$ as e goes to 1 will be described in section 4.3.

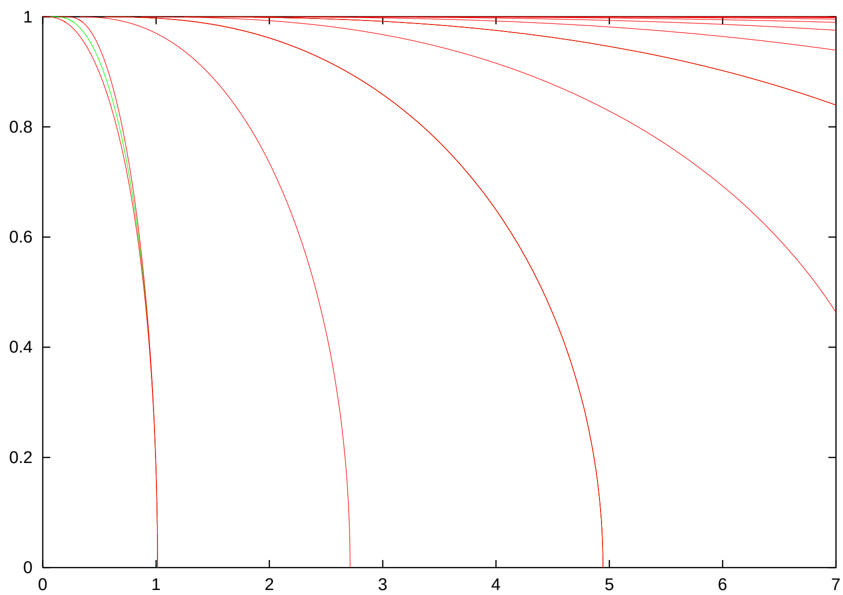


Figure 4.4: Bifurcation diagram of the collinear Newtonian homographic solutions in the plane (β_c, e)

4.2.3 The general case

For the general case we do not know explicitly the expression of $g^{\alpha-2}$. In this section we shall see that system (4.1) satisfies the properties of system (2.1). Then, the theory in this chapter can be applied for the homographic solutions in the general case. Moreover, we will see that $g^{\alpha-2}$ satisfies d'Alembert property, and then we can compute as in 2.5 the boundaries of the resonant regions.

Let $g(f)$ be the solution of (4.2) such that $\dot{g}(0) = 0$ and $g(0)$ is the minimum of $g(f)$. We introduce a new variable $v = g^{\alpha-2} - 1$. Then, the second order equation

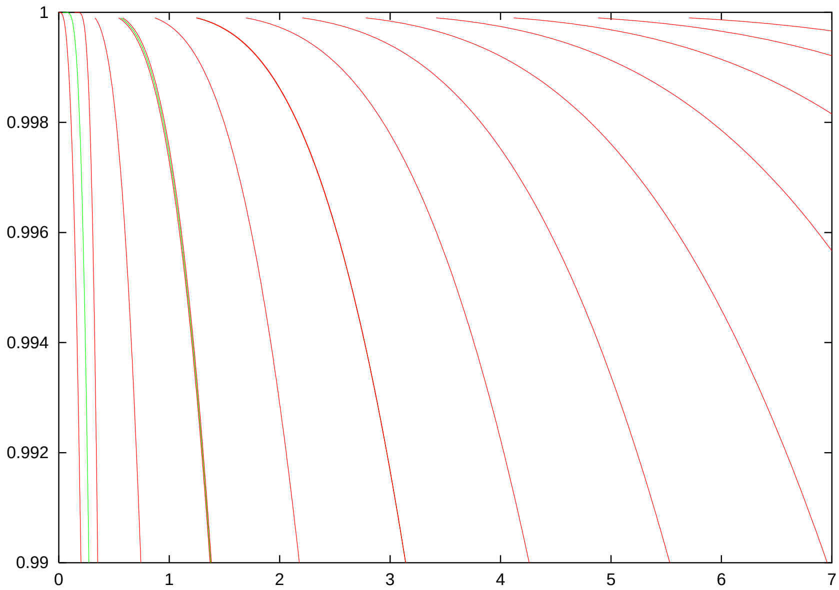


Figure 4.5: Magnification of the bifurcation diagram of the collinear Newtonian homographic solutions in the plane (β_c, e)

of v is

$$\ddot{v} = 2(\alpha-2)(\alpha-3)E(v+1)^{\frac{4-\alpha}{2-\alpha}} + (\alpha-2)^2(v+1)\left(\frac{3}{\alpha}(v+1)-1\right), \quad (4.16)$$

where E denotes the energy of (4.2), that is, $E = \frac{\dot{z}}{2} + U(z)$.

Let $e > 0$ be small enough. We look for a solution of (4.16) which satisfies initial conditions $v(0) = e$ and $\dot{v}(0) = 0$. We shall write

$$v(f) \;=\; v_1(f)e + v_2(f)e^2 + v_3(f)e^3 + \ldots, \quad (4.17)$$

where $v_1(0) = 1$, $v_j(0) = 0$ for $j \geq 2$ and $\dot{v}_j(0) = 0$ for $j \geq 1$. We remark that writing the energy of (4.2) in terms of v we have that

$$E \;=\; \frac{1}{2}(e+1)^{\frac{2}{\alpha-2}} - \frac{1}{\alpha}(e+1)^{\frac{\alpha}{\alpha-2}} = E_1 + \Delta, \qquad E_1 = -\frac{2-\alpha}{2\alpha}, \quad (4.18)$$

and $\Delta = \alpha_2 e^2 + \alpha_3 e^3 + \alpha_4 e^2 + O(e^5)$ with

$$\alpha_2 = \frac{1}{2(2-\alpha)}, \qquad \alpha_3 = -\frac{4-\alpha}{3(2-\alpha)^2}, \qquad \alpha_4 = \frac{(4-\alpha)(3-\alpha)}{4(2-\alpha)^2}, \ldots$$

To get $v(f)$ we use a Lindstedt–Poincaré method. So, we introduce a new independent variable $\tau = \nu f$ with

$$\nu \;=\; \nu_0 + \nu_1 e + \nu_2 e^2 + \ldots.$$

The coefficients ν_j, $j \geq 0$ will be determined in order to eliminate resonant terms. Using (4.18) the equation (4.16) can be written as

$$\nu^2 \frac{d^2 v}{d\tau^2} = f(v) + g(v)\Delta, \tag{4.19}$$

where

$$\begin{aligned} f(v) &= E_1 g(v) + (\alpha-2)^2 (v+1)\left(\frac{3}{\alpha}(v+1) - 1\right), \\ g(v) &= 2(2-\alpha)(3-\alpha)(v+1)^{\frac{4-\alpha}{2-\alpha}}. \end{aligned}$$

By substituting (4.17) in (4.19) we get

$$\nu_0^2 \frac{d^2 v_1}{d\tau^2} = -(2-\alpha)v_1, \quad v_1(0) = 1, \quad \frac{dv_1}{d\tau}(0) = 0.$$

We choose $\nu_0^2 = (2-\alpha)$ and then trivially $v_1(\tau) = \cos\tau$. In a similar way we get

$$\begin{aligned} v_2(\tau) &= \frac{1}{2(2-\alpha)} + \frac{\alpha-4}{3(2-\alpha)}\cos\tau - \frac{2\alpha-5}{6(2-\alpha)}\cos(2\tau), \\ v_3(\tau) &= \frac{\alpha-4}{3(\alpha-2)^2} + \left(\frac{(\alpha-4)(7-\alpha)}{9(2-\alpha)^2} - \frac{9\alpha^2-47\alpha+62}{96(2-\alpha)^2}\right)\cos\tau \\ &\quad - \frac{(2\alpha-5)(\alpha-4)}{9(2-\alpha)^2}\cos(2\tau) + \frac{9\alpha^2-47\alpha+62}{96(2-\alpha)^2}\cos(3\tau), \end{aligned}$$

$\nu_1 = 0$ and

$$\nu_2 = -\frac{\sqrt{2-\alpha}}{2(2-\alpha)^2}\left(\frac{1}{6}(2\alpha-5)(11-2\alpha) - \frac{3}{4}(\alpha-3)(4-\alpha)\right).$$

In this way we can obtain $g^{2-\alpha} = 1 + v(\tau)$ up to a given order. Then, $g^{2-\alpha} = 1 + v(\nu f)$ is a periodic function of f with period $T = \dfrac{2\pi}{\nu}$.

Now we shall see that $g^{2-\alpha}$ is an even function of f and satisfies the d'Alembert property.

Lemma 4.2.4. *Let $v(\tau) = \sum_{m\geq 1} v_m(\tau)e^m$ be the solution of (4.19) such that $v_1(0) = 1$, $v_j(0) = 0$ for $j \geq 2$ and $\dot{v}_j(0) = 0$ for $j \geq 1$. Then, $v_m(\tau)$, $m \in \mathbb{N}$, is an even function on τ which satisfies the d'Alembert condition, that is, for $m \in \mathbb{N}$,*

$$v_m(\tau) = \sum_{l=0}^{m} a_{ml}\cos(l\tau). \tag{4.20}$$

Proof

We know that $g(f)$ is an even periodic function of f. So, $v(\tau)$ is also an even function. Moreover $v_1(\tau) = \cos\tau$. Assume that $v_m(\tau)$ for $m = 1, 2, \ldots, k-1$ are known and satisfy the (4.20). If we define $w = e^{i\tau}$ then $v_m(\tau)$ contains terms w^l with $l \leq m$.

The equation for $v_k(\tau)$ is obtained by equating in (4.19) terms of order k in e. It is clear that $v_1(\tau), \ldots, v_{k-1}(\tau)$ give terms with w^l, with $l \leq k-1$, in $\ddot{v}$.

Concerning the right part of (4.19) to get the terms of order k in e from $f(v)$ it is sufficient to consider

$$f(v) \quad = \quad f'(0)v_k(\tau) + \sum_{j=2}^{k} \frac{f^{(j)}(0)}{j!}(v^{(k)})^j,$$

where $v^{(k)}(\tau) = v_1(\tau)e + \ldots + v_k(\tau)e^k$.

The terms of order k in e which come from $(v^{(k)})^j$ can be written as

$$(v^{(k)})^j \quad = \quad \sum_{\substack{l_1 + \ldots + l_k = j, \\ l_1 + 2l_2 + \ldots + kl_k = k}} v_1^{l_1} v_1^{l_2} \cdots v_k^{l_k} e^k. \tag{4.21}$$

In (4.21) we consider $j \geq 2$. This implies $l_k = 0$ in the summatory (4.21). Using the hypothesis on $v_1(\tau), \ldots, v_{k-1}(\tau)$ we get that the highest term in w which appears in $v_1^{l_1} v_2^{l_2} \cdots v_k^{l_k}$ is $w^{l_1+2l_2+\ldots+(k-1)l_{k-1}} = w^k$. In a similar way it can be proved that $g(v)\Delta$ contributes to the equation of v_k with terms w^l, $l \leq k-2$. Therefore we can write the equation for $v_k(\tau)$ as a linear non homogeneous differential equation

$$\nu_0^2 \ddot{v}_k \quad = \quad f'(0)v_k + F(\tau),$$

where $F(\tau)$ depends on $v_1(\tau), \ldots, v_{k-1}(\tau)$. The terms of $F(\tau)$ contain w^l with $l \leq k$. This proves the lemma. □

4.3 Stability parameters near the singular case

Our purpose in this section is to apply the theorem 3.3.1 to the system (4.1). First we note that using (4.3) we obtain $\hat{\lambda} = -\dfrac{(2-\alpha)^2}{8\alpha}$. Moreover we recall that the parameter δ in theorem 3.3.1 is related to the generalized eccentricity through (4.4). So, we are interested now in small $\delta > 0$. We shall assume that the non degeneracy conditions of theorem 3.3.1 are satisfied.

We begin with the collinear case. Using table 1.1 we see that in the collinear case $\lambda_1 > 0$ and $\lambda_2 < 0$. Therefore, theorem 3.3.1 can be applied if $\lambda_2 \neq \hat{\lambda}$, that is, $\beta_c \neq \dfrac{(2-\alpha)^2}{2\alpha}$. The parameters β_1, β_2 in the theorem are easily computed (see table 4.4).

	$\alpha \in (0,2)$	$\alpha = 1$
β_1	$\dfrac{1}{2-\alpha}\sqrt{8\alpha(\alpha+1)\beta_c + (3\alpha+2)^2}$	$\sqrt{25+16\beta_c}$
β_2	$\sqrt{1-\dfrac{8\alpha\beta_c}{(2-\alpha)^2}}$	$\sqrt{1-8\beta_c}$

Table 4.4: The parameters β_1, β_2 in theorem 3.3.1 for the collinear case

We remark that in the case $\alpha = 1$ the values of β_j, $j = 1, 2$, are related to the eigenvalues of the equilibrium points on the triple collision manifold (see [Mo.2]).

We note that $\beta_1 > 0$. Then $\mathrm{tr}_1 > 2$ if $\delta > 0$ is small enough. Furthermore if $\beta_c < \dfrac{(2-\alpha)^2}{8\alpha}$, $\beta_2 \in \mathbb{R}$ and the second stability parameter is greater than 2. In this case, the system is hyperbolic–hyperbolic for $\delta > 0$ small enough. On the other hand, if $\beta_c > \dfrac{(2-\alpha)^2}{8\alpha}$ then β_2 is pure imaginary. From (3.14) we know that tr_2 oscillates as δ tends to 0.

In the Newtonian case the behaviour of tr_2 changes at $\beta_c = \dfrac{(2-\alpha)^2}{8\alpha} = \dfrac{1}{8} = 0.125$. We have computed numerically tr_2 as a function of the eccentricity for several values of β_c. Their plots are represented in figure 4.6 by taking $-\log_{10}(1-e)$ on the x axis. The computations shows that if $\beta_c < \dfrac{1}{8}$, tr_2 goes to $-\infty$. However, if $\beta_c > \dfrac{1}{8}$, tr_2 oscillates between 2 and a negative value $k < -2$. Moreover numerically we see that k decreases as $\beta_c \to \left(\dfrac{1}{8}\right)^+$. As tr_2 goes beyond -2, several intervals on e of hyperbolic–hyperbolic (HH) type are created. Therefore, for a fixed $\beta_c = b$ with $b > \dfrac{1}{8}$ we must have on the bifurcation diagram a sequence of infinite intervals of type HH which accumulate at $e = 1$. These HH intervals are in fact the intersections of the infinitely many resonant tongues $\mathcal{T}_\omega$ with the line $\beta_c = b$ (see figure 4.5). This implies that $\mathcal{T}_\omega$ with $\omega = \dfrac{m}{2}$, $m \geq 4$ tends to $\beta_c = \dfrac{1}{8}$ as e tends to 1.

Figure 4.7 shows the typical behaviour of the stability parameter tr_2 as a

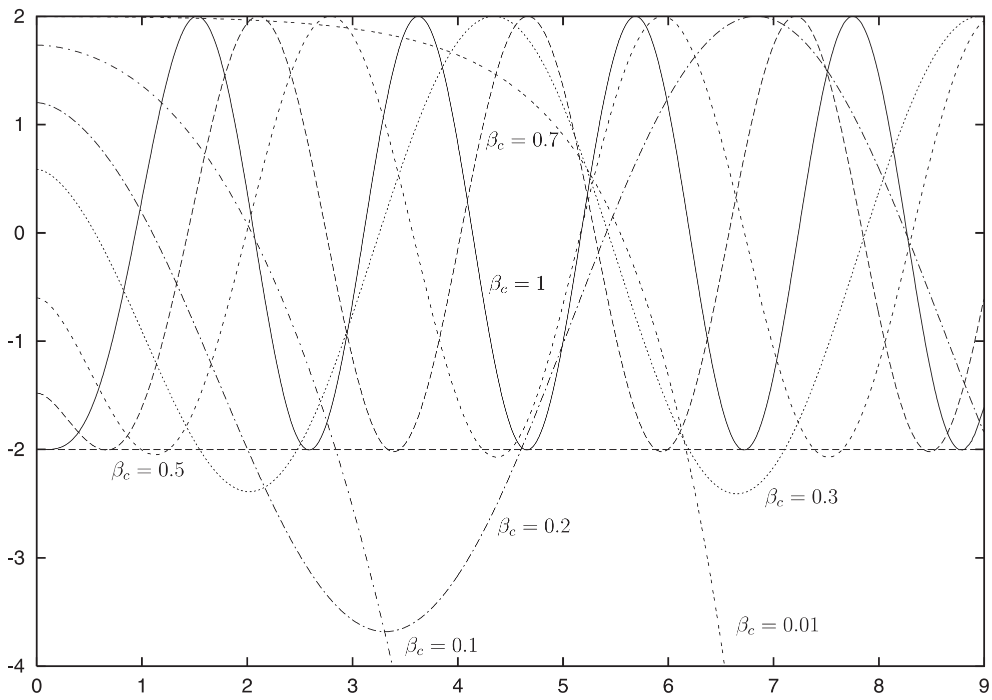


Figure 4.6: Stability parameter tr_2 in the collinear Newtonian case for several values of β_c. $-\log_{10}(1-e)$ is taken on the x axis

function of β_c when e is near 1. The plot corresponds to $e = 1 - 10^{-4}$. We distinguish clearly the first interval with $\mathrm{tr}_2 < -2$ when β_c is small. This interval corresponds to the first tongue $\mathcal{T}_{\frac{3}{2}}$. In the following oscillations the parameter goes under -2 by a small quantity defining the successive tongues. The numerical computations show that the first minimum goes to infinity as e goes to 1.

It is also interesting to point out that figures 4.6 and 4.7 show that tr_2 does not cross the horizontal line $\mathrm{tr}_2 = 2$, which corresponds to resonances $\omega = n$, $n \in \mathbb{N}$. This means that there is no bifurcation when $\omega = n$ (the two boundaries of $\mathcal{T}_n$ coincide) as it was predicted by lemma 4.2.2.

Now we consider the triangular case. From table 1.1 we get $\lambda_1 > \lambda_2 > 0$ and theorem 3.3.1 holds. The parameters β_1, β_2 in the theorem are given in the table 4.5.

Now, $\beta_1 \in \mathbb{R}$, $\beta_2 \in \mathbb{R}$. Then, if $\delta > 0$ is sufficiently small the system is HH provided that the coefficient d_1 in theorem 3.3.1 is different from 0. From lemma 3.3.5 we know that $d_1 = d_n d_g$ where $d_n \neq 0$ and d_g depends on the potential and on λ_1, λ_2.

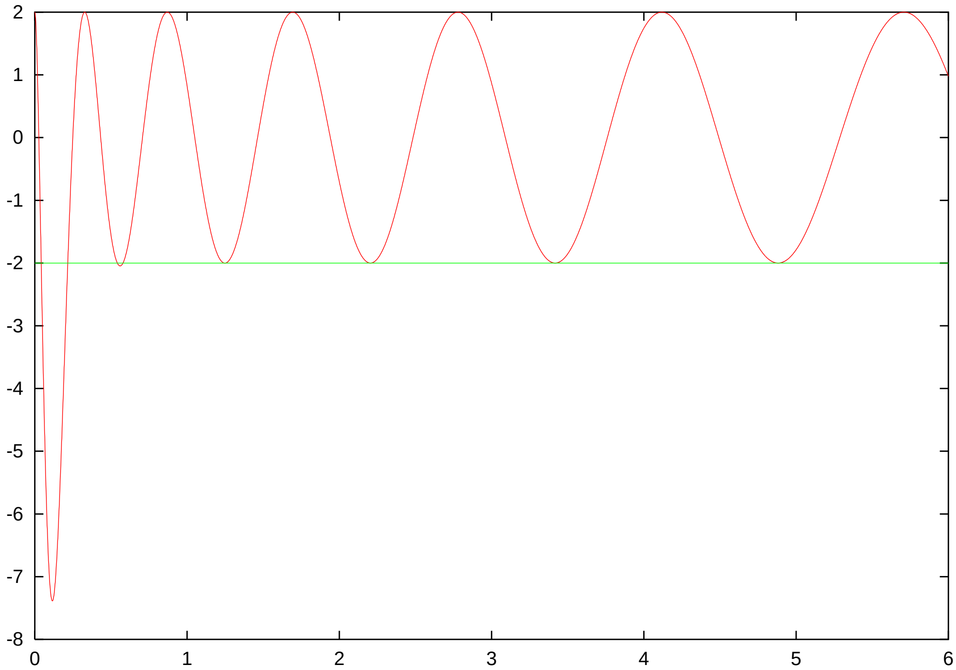


Figure 4.7: Behaviour of tr_2 for $e = 0.9999$ in the plane (β_c, tr_2)

Figure 4.3 shows the bifurcation diagram for the triangular Newtonian homographic solutions in the parameter space β_t, e. We see that for $e \lesssim 1$, the system is HH for any β_t except in a neighbourdhood of some critical value $\tilde{\beta}_t$ near 6. Numerical computations of d_g seems to indicate that it is equal to zero. However we do not have a proof of this fact.

Figure 4.8 shows the bifurcation diagram for the triangular homographic solutions for different values of α. Concerning the behaviour for e near 1 we see numerically that as α increases more critical values $\tilde{\beta}_t$ appear.

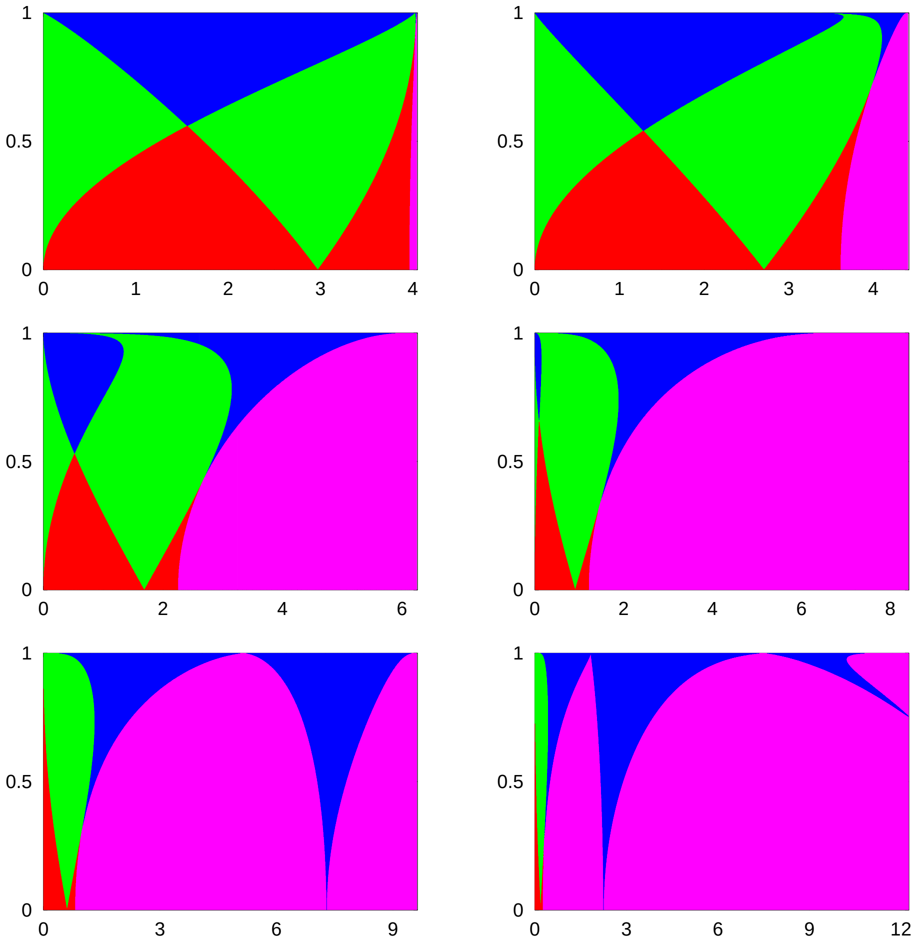


Figure 4.8: Bifurcation diagram for the triangular homographic solutions. The values of α are: top file $\alpha = 0.01$, $\alpha = 0.1$; center file $\alpha = 0.5$, $\alpha = 0.9$; bottom file $\alpha = 1.1$, $\alpha = 1.5$. The color codes are the same as in figure 4.3

	$\alpha \in (0,2)$	$\alpha = 1$
β_1	$\sqrt{1+\dfrac{4\alpha(2+\alpha)}{(2-\alpha)^2}(1+\tilde{\gamma})}$	$\sqrt{13+12\tilde{\gamma}}$
β_2	$\sqrt{1+\dfrac{4\alpha(2+\alpha)}{(2-\alpha)^2}(1-\tilde{\gamma})}$	$\sqrt{13-12\tilde{\gamma}}$

Table 4.5: The parameters β_1, β_2 in theorem 3.3.1 for the triangular case where $\tilde{\gamma} = \sqrt{1-3\kappa} \in \mathbb{R}$

Chapter 5

Some heteroclinic connections in the Spatial RTBP

In this chapter we study analytically the existence of homoclinic orbits to the centre manifold of the Spatial Restricted Three Body Problem (SRTBP).

The SRTBP has five relative equilibrium points, two triangular and three collinear. The collinear relative equilibrium points are of centre–centre–saddle type and then have 1–dimensional stable and unstable invariant manifolds and a 4–dimensional centre manifold.

In a neighbourhood of the collinear equilibrium points there are two families of Lyapunov periodic orbits, the planar and the vertical families. A Lyapunov periodic orbit has 2–dimensional stable and unstable invariant manifolds. There also exist 2–dimensional invariant tori with 3–dimensional stable and unstable invariant manifolds.

We shall study the existence of homoclinic orbits to the centre manifold of one of the relative equilibrium points. To this end, we consider the SRTBP as a perturbation of the three dimensional Hill's problem and also as a perturbation of the spatial synodic two body problem.

For the existence of homoclinic orbits on small perturbations of integrable system under generic assumptions see [L.], [K.L.1], [K.L.2].

5.1 The Spatial Restricted Three Body Problem

Let us consider two bodies, called primaries, describing circular orbits in the plane (x, y) around their center of masses that we assume located at the origin. If we consider a system of coordinates that rotates with the primaries and suitable units, the bodies can assumed to have masses $m_1 = 1-\mu$ and $m_2 = \mu$, with $\mu \in \left(0, \frac{1}{2}\right]$, and to be fixed located at coordinates $(\mu, 0, 0)$ and $(\mu - 1, 0, 0)$, respectively. It can also be assumed that they complete one inertial revolution in 2π time units.

It is well known that the equations of motion of a massless particle under the gravitational action of the primaries are

$$\begin{aligned} \ddot{x} - 2\dot{y} &= \Omega_x, \\ \ddot{y} + 2\dot{x} &= \Omega_y, \\ \ddot{z} &= \Omega_z, \end{aligned} \tag{5.1}$$

where

$$\Omega(x, y, z) = \frac{1}{2}(x^2 + y^2) + \frac{1-\mu}{r_1} + \frac{\mu}{r_2} + \frac{1}{2}\mu(1-\mu), \tag{5.2}$$

$$r_1^2 = (x-\mu)^2 + y^2 + z^2, \quad r_2^2 = (x - \mu + 1)^2 + y^2 + z^2. \tag{5.3}$$

Equations (5.1) are called the equations of the spatial restricted three body problem.

Equations (5.1) have a first integral , called the **Jacobi integral**, given by

$$F(x, y, z, \dot{x}, \dot{y}, \dot{z}) = -(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + 2\Omega(x, y, z). \tag{5.4}$$

They also have the following symmetry

$$\mathcal{S}(x, y, z, \dot{x}, \dot{y}, \dot{z}, t) = (x, -y, z, -\dot{x}, \dot{y}, -\dot{z}, -t), \tag{5.5}$$

It is well known that system (5.1) has five equilibrium points, three collinear points L_1, L_2, L_3, located on the x axis, and two triangular points, L_4, L_5 forming an equilateral triangle with the masses and located at the x, y plane. Figure 5.1 shows the equilibrium points of the SRTBP in the plane (x, y). If we denote by C_i the value of (5.4) on the L_i points, $i = 1, \ldots, 5$, we have that $3 = C_4 = C_5 \leq C_3 < C_1 < C_2 < 4.25$ for all $\mu \in \left(0, \frac{1}{2}\right)$.

Let us denote by $M(\mu, C)$ the hypersurface given by

$$M(\mu, C) = \{(x, y, z, \dot{x}, \dot{y}, \dot{z}) \in \mathbb{R}^6 | \quad F(x, y, z, \dot{x}, \dot{y}, \dot{z}) = C\}. \tag{5.6}$$

Due to the existence of the Jacobi integral, we can restrict to study the behaviour of the orbits in $M(\mu, C)$. The projection of $M(\mu, C)$ in the position space (x, y, z) is called Hill's region. We shall denote it by

$$R(\mu, C) = \{(x, y, z) \in \mathbb{R}^3 | \quad 2\Omega(x, y, z) \geq C\}. \tag{5.7}$$

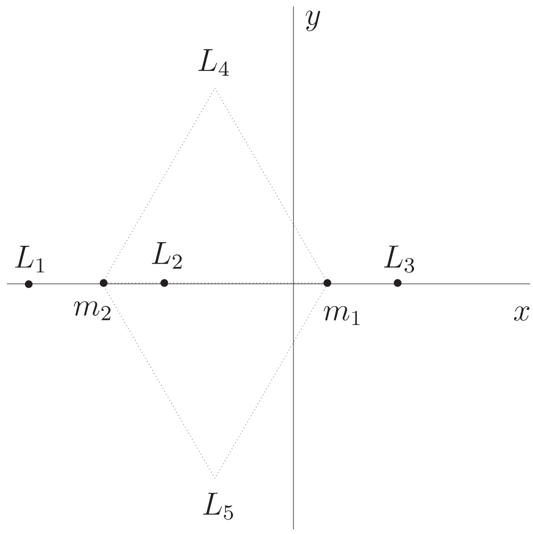


Figure 5.1: Equilibrium points of the SRTBP in the plane (x, y)

The boundary of $R(\mu, C)$ is the surface of zero velocity. For $C > C_2$ Hill's region consists in two ovoids enclosing the two primaris and a cilindrical surface outside the ovoids. We denote by $R_b(\mu, C)$ the bounded components of the Hill's region. In this case, $R_b(\mu, C)$ is formed by the two ovoids. As the value of C decreases the ovoids in $R_b(\mu, C)$ meet at L_2 (see figure 5.2). The three dimensional picture

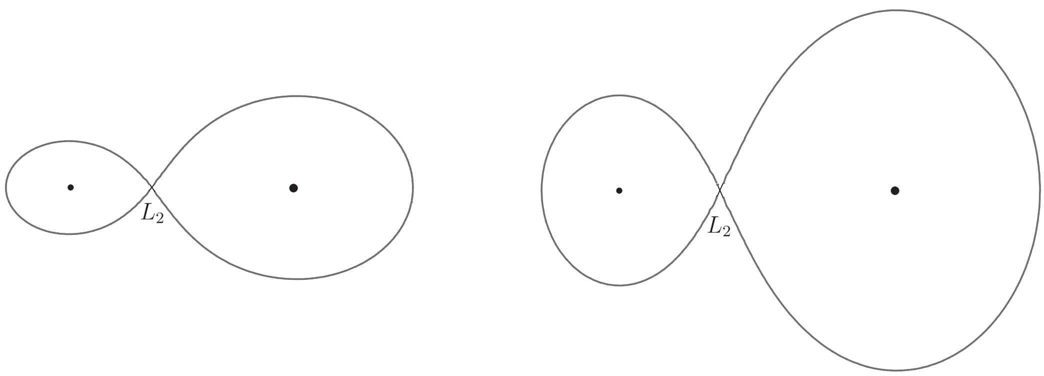


Figure 5.2: Intersections of $R_b(\mu, C)$ for $C = C_2$ in the planes (x, y) and (x, z), respectively, for $\mu = 0.2$

corresponding to this fact is formed by two ovoids that have a contact in L_2. For values of $C_1 < C < C_2$ the two ovoids converts in a surface homeomorphic to a sphere. In this case $R_b(\mu, C)$ has a unique connected component and so, it is possible that the massless particle travels from a neighbourhood of a primary to a neighbourhood of the other primary (see figure 5.3). We denote by $M_b(\mu, C)$ the

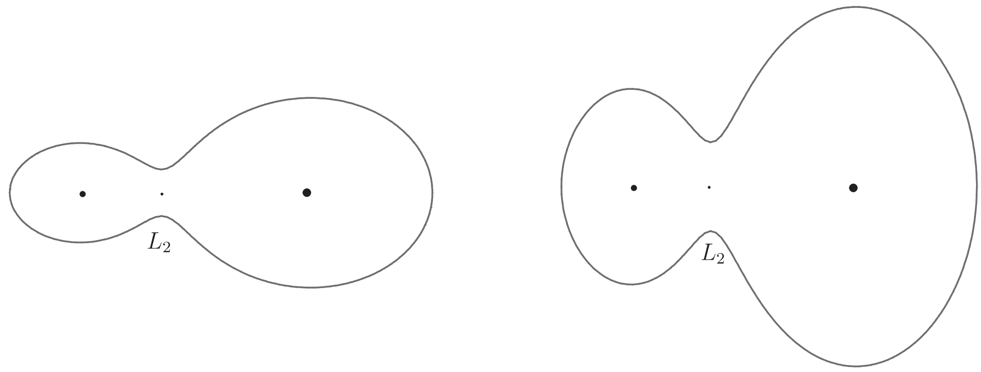


Figure 5.3: Intersections of $R_b(\mu, C)$ for $C \lesssim C_2$ in the planes (x, y) and (x, z), respectively, for $\mu = 0.2$

component of $M(\mu, C)$ that projects in $R_b(\mu, C)$. We shall study the behaviour of the orbits in $M_b(\mu, C)$ for $C \lesssim C_2$.

5.1.1 Qualitative description of a neighbourhood of L_2

As we are interested in the orbits near L_2, we shall give a qualitative description of a neighbourhood of this point. For details see [Sz.]. In fact, the same arguments hold for the other collinear equilibrium points.

By introducing momenta as $p_x = \dot{x} - y$, $p_y = \dot{y} + x$ and $p_z = \dot{z}$, the SRTBP can be written in Hamiltonian form, and the Hamiltonian function is

$$H(x, y, z, p_x, p_y, p_z) = \frac{1}{2}(p_x^2 + p_y^2 + p_z^2) - xp_y + yp_x - \frac{1-\mu}{r_1} - \frac{\mu}{r_2}, \quad (5.8)$$

where r_1 and r_2 are defined in (5.3). The relation between the energy h and the Jacobi constant of an orbit is given by

$$C = -2h - \mu(1 - \mu).$$

L_2 is located between the two primaries (see figure 5.1). We introduce ρ by $r_2 = \rho$. Then, $r_1 = 1 - \rho$ and $x = \mu - 1 + \rho$ for this equilibrium point. Figure 5.4 shows the situation.

ρ is the solution of Euler's quintic equation

$$\rho^5 - (3 - \mu)\rho^4 + (3 - 2\mu)\rho^3 - \mu\rho^2 + 2\mu\rho - \mu = 0, \quad (5.9)$$

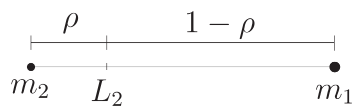


Figure 5.4: Coordinates of L_2

The linearized equations around the collinear equilibrium point are given by the second order terms of the Hamiltonian. These terms can be written as

$$H_2 = \frac{1}{2}(p_x^2+p_y^2) - xp_y + yp_x - c_2x^2 + \frac{c_2}{2}y^2 + \frac{1}{2}p_z^2 + \frac{c_2}{2}z^2, \qquad (5.10)$$

where $c_2 = \dfrac{1-\mu}{(1-\rho)^2} + \dfrac{\mu}{\rho^3}$. Figure 5.5 shows the values of c_2 depending on the parameter μ.

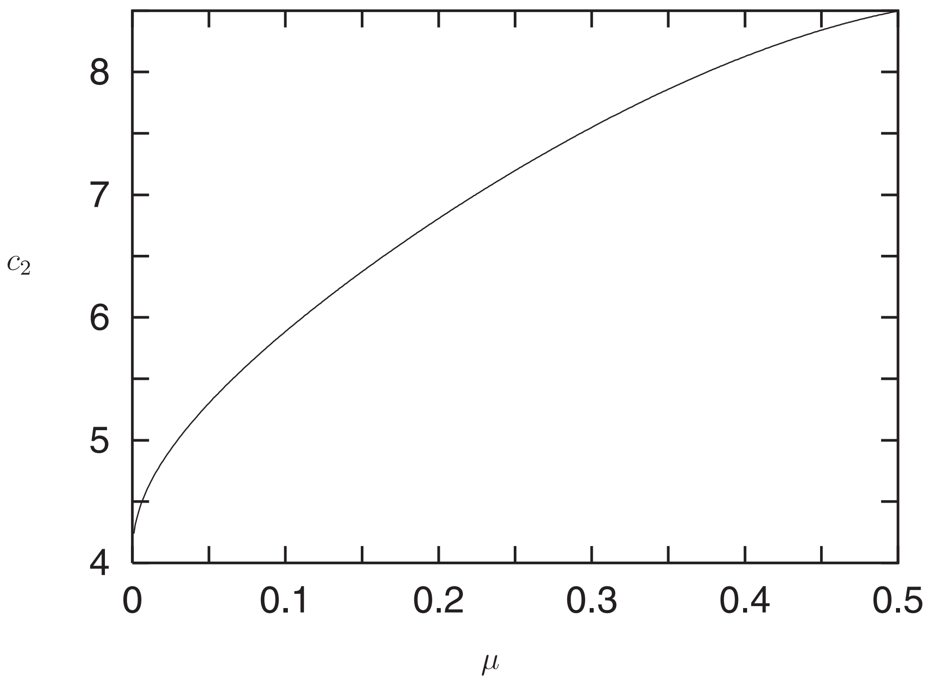


Figure 5.5: Values of c_2 depending on μ

From the expression of H_2 it is clear that, linearly, the direction z is uncoupled from the planar directions. The linearized system for $z, \dot{z}$ is an harmonic oscillator with frequency $\omega_v = \sqrt{c_2}$. It is well–known that $\omega_v \in (2,3)$ (see figure 5.6). For the planar directions, x, y, the characteristic polynomial of the linearized system is

$$p(\lambda) = \lambda^4 + (2 - c_2)\lambda^2 + (1 + c_2 - 2c_2^2).$$

Then, if we denote by $\eta = \lambda^2$, the zeroes of $p(\lambda)$ are given by

$$\eta_{1,2} = \frac{c_2 - 2 \pm \sqrt{9c_2^2 - 8c_2}}{2},$$

where, according to the values of c_2, $\eta_1 > 0$ and $\eta_2 < 0$. Then, L_2 is a centre–centre–saddle point. The frequency $\omega_p = \sqrt{-\eta_2}$ is known as planar frequency. It is easy to see that $\omega_p \in (2, 3)$. Figure 5.6 shows the graphic of ω_p in terms of μ.

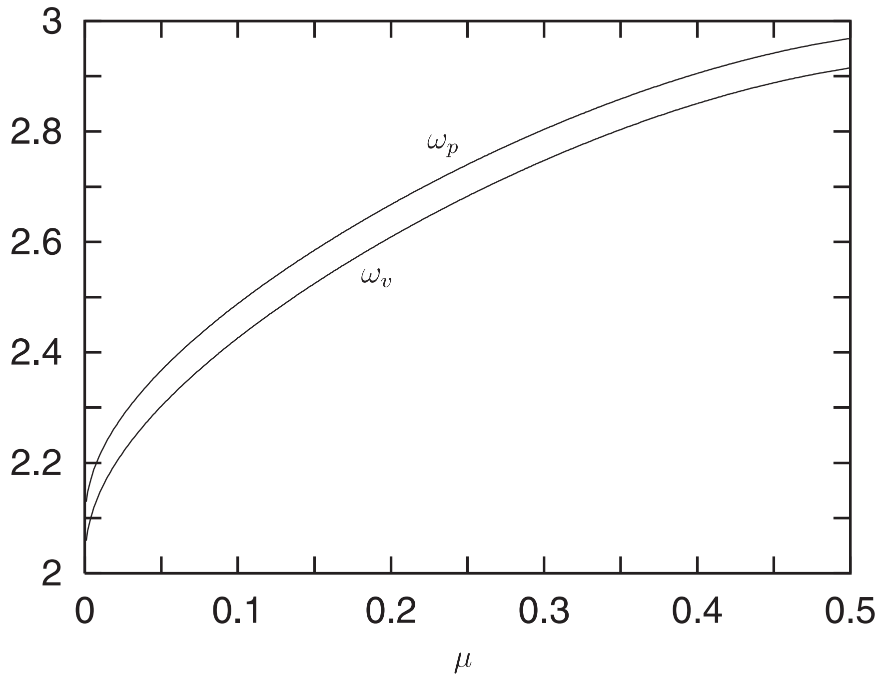


Figure 5.6: Frequencies ω_p and ω_v in terms of $\mu \in \left(0, \frac{1}{2}\right)$

As L_2 is of centre–centre–saddle type, it has 1–dimensional stable and unstable manifolds and a 4–dimensional centre manifold. Fixed an energy level C of the Jacobi constant, $W^c_{L_2} \cap M(\mu, C)$ is homeomorphic to $\mathbb{S}^3$, where $W^c_{L_2}$ denotes the centre manifold of L_2 (see appendix D).

It is well–known that there exists two families of periodic orbits in a neighbourhood of L_2, the planar and the vertical Lyapunov periodic orbits. The first family is associated to the frequency ω_p and the second one to the frequency ω_v.

These families are hyperbolic, and they have 2–dimensional stable and unstable invariant manifolds. We remark that the planar family of periodic orbits also exists in the Planar Restricted Three–Body Problem. Therefore, the stable and unstable invariant manifolds of the planar periodic orbits lie on $z = 0, \dot{z} = 0$.

For the linearized SRTBP and fixed C, the centre manifold, $W^c_{L_2} \cap M(\mu, C)$ is foliated by a one parameter family of 2–dimensional invariant tori. Generically, using KAM theory most of these tori subsist for the general SRTBP. Moreover, the tori have 3–dimensional stable and unstable invariant manifolds.

5.2 Homoclinic connections in the planar case

The Planar Restricted Three Body Problem is obtained from the equations of the SRTBP by taking $(z, \dot{z}) = (0, 0)$. In this case, on a neighbourhood of L_2 and for values of $C \leq C_2$ there also exist the planar family of Lyapunov periodic orbits. In this case, they have two–dimensional stable and unstable invariant manifolds, that we shall denote by $W^s_{p.o.}$ and $W^u_{p.o.}$, respectively. The existence of transversal homoclinic orbits in the planar problem for $\mu \gtrsim 0$ and $C \lesssim C_2$ has been studied in [L.M.S.]. In this section we shall summarize some of the results obtained in [L.M.S.].

In this case, we consider the hypersurface defined by

$$\tilde{M}(\mu, C) = M(\mu, C)_{|(x,\dot{x},y,\dot{y},0,0)}.$$

The Hill's region $\tilde{R}(\mu, C)$ is the projection of $\tilde{M}(\mu, C)$ in the position space. Let us consider $\tilde{R}_b(\mu, C)$ the bounded components of $R(\mu, C)$. We have that for $C = C_2$, $\tilde{R}(\mu, C)$ is formed by two connected components that have a contact point in L_2 (see the left figure in 5.2). We shall denote by S the connected component that contains the larger primary. For $C \leq C_2$ we can take two segments in $\tilde{R}_b(\mu, C)$ joining points in the zero velocity curve (see figure 5.7) that divides the region in three components. One of the components contains the projection of the periodic orbit near L_2, and the other components contain one of the primaries each one. Naming $\tilde{M}_b(\mu, C)$ the component of $\tilde{M}(\mu, C)$ that projects on $\tilde{R}_b(\mu, C)$, we shall denote again the component that contains the large primary as S.

The main result in [L.M.S.] is the following.

Theorem 5.2.1. *1. For values of $\mu \gtrsim 0$ of the form $\mu_k = \dfrac{1}{N^3_\infty k^3}(1 + o(1))$, where N_∞ is a suitable constant and $o(1)$ denotes terms that go to zero when μ does, there exists an homoclinic orbit to L_2.*

2. If μ and $\Delta C = C_2 - C > 0$ are small enough, the branch $W^{u,S}_{p.o.}$ of $W^u_{p.o.}$ contained initially in the region S intersects $y = 0$ and $x > 0$ in a curve

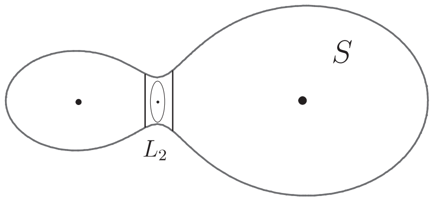


Figure 5.7: Hill's region in the Planar Restricted Three Body Problem for $C \lesssim C_2$ and region S

diffeomorphic to a circle. In particular, for the points in the μ, C plane such that there exists μ_k satisfying $\Delta C > L\mu_k^{\frac{4}{3}}(\mu-\mu_k)^2$, being L a constant, there exist transversal symmetric homoclinic orbits to the periodic orbit.

5.3 Statement of the results in the Spatial case

Fixed a value C of the Jacobi constant, from (5.4) we have that

$$C = F(x, y, z, \dot{x}, \dot{y}, \dot{z}). \tag{5.11}$$

This is equivalent to fix a level hypersurface of the form (5.6). We shall denote as C_p the constant that one obtains from (5.11) by taking $z = 0, \dot{z} = 0$, that is,

$$C_p = F(x, y, 0, \dot{x}, \dot{y}, 0). \tag{5.12}$$

Then, C_p is the Jacobi constant of the Planar problem. We shall refer to it as **planar component of the Jacobi constant**. We define C_v by

$$C_v = C - C_p, \tag{5.13}$$

and when talking about this constant we shall say **vertical component of the Jacobi constant**. Then, we have written the Jacobi constant as the sum of a planar component and a vertical one.

Let be $\Delta C = C_2 - C_p$ and define

$$\Delta C_p = C_2 - C_p, \qquad \Delta C_v = -C_v,$$

We introduce constants α, β by

$$\begin{aligned} \mu &= \mu_k + \alpha\mu_k^{\frac{4}{3}}, \\ \Delta C &= \beta\mu_k^{\frac{4}{3}}, \end{aligned} \tag{5.14}$$

where μ_k is defined in Theorem 5.2.1 and $\alpha, \beta = O(1)$ depend on μ_k. Once μ is fixed we can take μ_k as the value of the sequence given by Theorem 5.2.1 which is at minimum distance to μ. We define $\psi \in \left[0, \frac{\pi}{2}\right]$ by

$$\Delta C_p = \beta \cos^2(\psi) \cdot \mu_k^{\frac{4}{3}}, \qquad \Delta C_v = \beta \sin^2(\psi) \cdot \mu_k^{\frac{4}{3}}. \tag{5.15}$$

We note that once $\Delta C > 0$ is fixed, if $\psi = 0$ then $\Delta C = \Delta C_p$ which corresponds to the planar problem. For $\psi = \frac{\pi}{2}$ then $\Delta C = \Delta C_v$.

Given α and β, a torus in the centre manifold of L_2 is characterized by ψ.

The following theorem gives the existence of heteroclinic orbits between two tori provided some inequalities are satisfied. Some non degeneracy conditions will be required in the sense that $A \neq 1$ and $C_1 \neq 0$ for some coefficients to be introduced in section 5.4. The geometrical meaning is that some ellipse in the $(z, \dot{z})$ plane taken in the initial conditions of $W_T^{u,\mu}$ does not degenerate into a cercle and its axes do not coincide with the z, $\dot{z}$ axes. Here, $W_T^{u,\mu}$ denotes the unstable manifold of an invariant torus T in the centre manifold of L_2 once the Jacobi constant $C \lesssim C_2$ is fixed.

Theorem 5.3.1. *Let us consider α, β fixed and μ_k sufficiently small. Assume non degeneracy conditions. Let be $\psi, \psi' \in \left(0, \frac{\pi}{2}\right) \setminus \mathcal{E}$, being $\mathcal{E}$ a set of small measure, such that*

$$\cos\psi, \cos\psi' > \frac{\mathcal{K}}{\sqrt{\beta}} |\hat{\alpha} - k|, \tag{5.16}$$

for some integer k, where $\mathcal{K} > 0$ is a constant and $|\hat{\alpha}| = \left|\frac{\alpha}{3N_\infty}\right| \leq \frac{1}{2}$. Let be m the number of integers k such that (5.16) is satisfied and assume

$$\frac{\sin\psi}{\sin\psi'} \in (\kappa^{-1}, \kappa), \tag{5.17}$$

where $\kappa > 1$ is a constant. Then there exist $16m$ transversal heteroclinic orbits between the tori characterized by ψ and ψ'.

In particular, there exist at least $16m$ homoclinic orbits to the centre manifold.

In order to prove the theorem, we shall consider the intersection of invariant manifolds of the tori characterized by ψ and ψ' with $y = 0$. The hypothesis (5.16) is needed in order to get points in the invariant manifolds for which the components $x, \dot{x}$ coincide. The condition (5.17) is required in order that z and $\dot{z}$ also coincide.

Remark 5.3.2. The constants $\mathcal{K}$ and κ are effectively computed in section 5.4. They depend on the parameters of some ellipses in the planes $x, \dot{x}$ and $z, \dot{z}$, respectively.

Assume $\dfrac{\mathcal{K}|\hat{\alpha}|}{\sqrt{\beta}} < 1$. Let be $\psi_{max} = \arccos\left(\dfrac{\mathcal{K}}{\sqrt{\beta}}|\hat{\alpha}|\right)$. If $\psi, \psi' \in (0, \psi_{max})$ then (5.16) holds at least for $k = 0$. This situation is represented in figure 5.8.

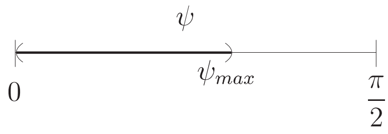


Figure 5.8: Admissible values of ψ

Furthermore, if ψ and ψ' satisfy (5.17), then there exists an heteroclinic orbit from the torus characterized by ψ to the one characterized by ψ'.

Notice that if $\hat{\alpha} = 0$ then $\mu = \mu_k$. In this case (5.16) is satisfied trivially for $k = 0$. Moreover as β increases, (5.16) holds for other values of k. This is in agreement with the previous results in the Planar RTBP ([L.M.S.]).

Notation 5.3.3. For $\psi, \psi' \in \left(0, \dfrac{\pi}{2}\right)$ we shall denote by

$$\psi \longrightarrow \psi'$$

the existence of an heteroclinic orbit between the tori characterized by ψ and ψ'.

Corollary 5.3.4. *We fix α, β such that $\dfrac{\mathcal{K}|\hat{\alpha}|}{\sqrt{\beta}} < 1$. If $\psi_1, \ldots, \psi_n, \ldots \in \left(0, \frac{\pi}{2}\right)$ is a sequence of values such that*

$$\psi_n \in (0, \psi_{max}),$$

and

$$\frac{\sin\psi_n}{\sin\psi_{n-1}} \in (\kappa^{-1}, \kappa),$$

then

$$\psi_1 \longrightarrow \psi_2 \longrightarrow \psi_3 \longrightarrow \ldots.$$

Corollary 5.3.5. *If $\frac{\mathcal{K}}{\beta}|\hat{\alpha}|$ is sufficiently small then there exist some orbits that go from a small neighbourhood of a planar periodic orbit to a small neighbourhood of a vertical periodic orbit. The orbit is close to a heteroclinic chain, whose length goes to infinity when κ approaches* 1.

5.4 Proof of Theorem 5.3.1

In order to obtain homoclinic connections to the centre manifold of L_2 we follow the same ideas given in [L.M.S.].

Let us denote by $W_T^{u,\mu}$ the unstable manifold of an invariant torus T on a level manifold $M(\mu, C)$ with $C \lesssim C_2(\mu)$ of the SRTBP. In order to prove theorem 5.3.1 we shall obtain an analytic expression for this invariant manifold when $\mu \gtrsim 0$ to obtain later heteroclinic and homoclinic orbits to the torus. Our purpose is to obtain the first intersection of $W_T^{u,\mu}$ with $\Sigma = \{y = 0,\ \ x > 0\}$. We shall assume that Σ is a Poincaré section for any orbit of $W_T^{u,\mu}$. Due to the transversality of $W_{L_2}^{u,\mu}$ with Σ, this assumption holds if ΔC is sufficiently small. In order to obtain this intersection, we shall approximate the SRTBP by the spatial Hill's problem in a neighbourhood of the equilibrium point. Then, as outside of a neighbourhood of L_2 the SRTBP can be seen as a perturbation of the Spatial Two Body Problem, we shall use it in order to obtain the expression of $W_T^{u,\mu} \cap \Sigma$. Once this intersection is obtained, we shall use the symmetries of the equations in order to obtain the stable manifold. Then, studying the intersections of the unstable manifold of one torus and the stable manifold of another torus, we shall obtain heteroclinic orbits. By taking the unstable and stable manifolds of the same torus we will obtain homoclinic orbits to that torus.

This section is structurated as follows. In section 5.4.1 we study the geometry of $W_T^{u,\mu} \cap \{y = -k\mu^{\frac{1}{3}}\}$ in a neighbourhood of L_2, that is, we shall take k and μ such that $k\mu^{\frac{1}{3}}$ is sufficiently small but k is large. To this end we shall approximate the SRTBP by the Spatial Hill's Problem. Using the geometry of $W_T^{u,\mu} \cap \{y = -k\mu^{\frac{1}{3}}\}$, in section 5.4.2 we obtain an analytic expression of the first cut of this manifold with $y = 0, x > 0$. To do that we shall approximate the SRTBP by the Spatial Synodic Two Body Problem (SSTBP) outside of a neighbourhood of L_2. In section 5.4.3 we shall compute the stable manifold of a torus from its unstable manifold using the symmetries given in (5.5). Then, once we have obtained both stable and unstable manifold of a torus, in section 5.4.4 we will compute the intersections of these manifolds in order to obtain homoclinic orbits to the torus. Taking the stable manifold of one torus and the unstable manifold of another torus, the intersections of these manifolds will give heteroclinic orbits from one torus to the other. All these homoclinic and heteroclinic orbits are homoclinic orbits to the centre manifold of L_2. The proofs of some lemmas are given in the section 5.5.

5.4.1 Geometry in a neighbourhood of the equilibrium point

In order to study the geometry of the unstable manifold in a small neighbourhood of the equilibrium point, we shall consider the intersection of this manifold with the section $y = -k\mu^{\frac{1}{3}}$, where k is large enough amb μ sufficiently small, in such a way that $k\mu^{\frac{1}{3}}$ is small enough. To this end we shall approximate the equations of the SRTBP by the Spatial Hill's problem in a neighbourhood of the equilibrium point. If (X, Y, Z) denotes the coordinates of the Spatial Hill's problem, then we need to study the geometry of the invariant manifold in the section $Y = -k$, with k large enough. The analysis of the Poincaré map between two sections $Y = -\tilde{k}$ and $Y = -\tilde{\tilde{k}}$, $0 < \tilde{k} < \tilde{\tilde{k}}$, with $\tilde{k}$ large enough, will give us the geometry of the manifold intersected with different hyperplanes.

Near the small mass μ, in suitable coordinates the SRTBP can be seen as a $\mu^{\frac{1}{3}}$ order perturbation of the Spatial Hill's problem. The three–dimensional Hill's problem studies the behaviour of the small mass for the SRTBP in the limit case when $\mu \to 0$. To obtain the limit equations we translate the small mass to the origin and we perform a scaling of the variables by the change of coordinates $(x, y, z) \longrightarrow (X, Y, Z)$ defined by

$$X = \mu^{-\frac{1}{3}}(x + 1 - \mu), \quad Y = \mu^{-\frac{1}{3}}y, \quad Z = \mu^{-\frac{1}{3}}z. \tag{5.18}$$

Then, equations (5.1) can be written as the second order system

$$\begin{aligned} \ddot{X} - 2\dot{Y} &= 3X - X(X^2 + Y^2 + Z^2)^{-\frac{3}{2}} + \mu^{\frac{1}{3}}\left(3X^2 - \frac{3}{2}Y^2 - \frac{3}{2}Z^2\right) + O(\mu^{\frac{2}{3}}), \\ \ddot{Y} + 2\dot{X} &= -Y(X^2 + Y^2 + Z^2)^{-\frac{3}{2}} - 3\mu^{\frac{1}{3}}XY + O(\mu^{\frac{2}{3}}), \\ \ddot{Z} &= -Z - Z(X^2 + Y^2 + Z^2)^{-\frac{3}{2}} - 3\mu^{\frac{1}{3}}XZ + O(\mu^{\frac{2}{3}}). \end{aligned}$$

If we take $\mu = 0$ we obtain the equations for the 3–dimensional Hill's problem

$$\begin{aligned} \ddot{X} - 2\dot{Y} &= 3X - X(X^2 + Y^2 + Z^2)^{-\frac{3}{2}}, \\ \ddot{Y} + 2\dot{X} &= -Y(X^2 + Y^2 + Z^2)^{-\frac{3}{2}}, \\ \ddot{Z} &= -Z - Z(X^2 + Y^2 + Z^2)^{-\frac{3}{2}}. \end{aligned} \tag{5.19}$$

We note that equations (5.19) can be written as

$$\begin{aligned} \ddot{X} - 2\dot{Y} &= \Omega^H_X, \\ \ddot{Y} + 2\dot{X} &= \Omega^H_Y, \\ \ddot{Z} &= \Omega^H_Z, \end{aligned} \tag{5.20}$$

with

$$\Omega^H(X,Y,Z) = \frac{1}{2}\left[3X^2 - Z^2 + 2(X^2+Y^2+Z^2)^{-\frac{1}{2}}\right].$$

Equations (5.19) have a first integral

$$F^H(X,Y,Z,\dot X,\dot Y,\dot Z) \;=\; 2\Omega^H(X,Y,Z) - (\dot X^2 + \dot Y^2 + \dot Z^2) = C^H. \quad (5.21)$$

As we have done with the Jacobi constant for the SRTBP, we can consider the value of the integral, C^H, as a sum of a planar and a vertical component. We shall write

$$C^H \;=\; C^H_p + C^H_v, \quad (5.22)$$

being $C^H_p = F^H(X,Y,0,\dot X,\dot Y,0)$ the planar component and C^H_v the vertical one.

The only two equilibrium points of the 3–dimensional Hill's problem are collinear. If we denote by L_1 and L_2 these equilibrium points, we have that $L_1 = (-3^{-\frac{1}{3}},0,0)$ and $L_2 = (3^{-\frac{1}{3}},0,0)$. The use of this notation is due to the fact that L_j for the Spatial Hill's problem corresponds to L_j for the SRTBP. Using (5.21) we obtain $C^H_{L_j} = 3^{\frac{4}{3}}$ for the collinear equilibrium points. We shall consider L_2. The eigenvalues for the linerized system at L_2 are $\pm\lambda$, $\pm\mathrm{i}\omega$, $\pm 2\mathrm{i}$ where $\lambda = \sqrt{1+2\sqrt{7}}$, $\omega = \sqrt{2\sqrt{7}-1}$. Then it is a centre–centre–saddle point.

We denote by $W^{u,H}_{L_2}$ the one–dimensional unstable manifold of L_2. It is known (see [L.M.S.], [McG.1]) that one of the branches of $W^{u,H}_{L_2}$ crosses the line $Y=-k$, for any value $k>0$ going down forwards, near the surface of velocity zero. Moreover, as in the SRTBP, there exist two families of periodic orbits in a neighbourhood of L^H_2, the planar and the vertical families. Furthermore using the KAM theorem, generically there exist invariant tori in the centre manifold of the collinear points.

For a fixed $C^H \le C^H_{L_2} = 3^{\frac{4}{3}}$ we take $\Delta C^H = C^H_{L_2} - C^H$. Let us consider a solution of the linearized Spatial Hill's Problem on the centre manifold of L_2. It can be written as

$$\begin{aligned} X(t) &= 2\omega a\cos(\omega t) + 3^{-\frac{1}{3}}, \\ Y(t) &= -(\omega^2+9)a\sin(\omega t), \\ Z(t) &= b\cos(2t) + c\sin(2t), \end{aligned} \quad (5.23)$$

where $\Delta C^H_p := C^H_{L_2} - C^H_p = 8a^2(5\omega^2+54)$, $\Delta C^H_v := -C^H_v = 4(b^2+c^2)$. In an equivalent way, instead of using b,c we can write $Z(t) = a_v\cos(2t+\varphi)$ for some phase φ. We note that $a = O((\Delta C^H_p)^{\frac{1}{2}})$ and $b,c,a_v = O((\Delta C^H_v)^{\frac{1}{2}})$. Once a is fixed we get a two dimensional torus, T, on the centre manifold for the linearized

Spatial Hill's Problem. Of course, a must be taken on a Cantor set of almost full measure. That torus is hyperbolic and has a three dimensional unstable invariant manifold to be denoted by $W_T^{u,H}$. The linear part is

$$\begin{aligned}\tilde{X}(t) &= X(t) + 2\lambda\tilde{c}e^{\lambda t},\\ \tilde{Y}(t) &= Y(t) + (\lambda^2 - 9)\tilde{c}e^{\lambda t},\\ \tilde{Z}(t) &= Z(t),\end{aligned} \tag{5.24}$$

for some $\tilde{c} > 0$ where $X(t)$, $Y(t)$ and $Z(t)$ are given in (5.23).

Let us consider a section $Y = -k_0$. If $k_0 > 0$ is small, orbits in $W_T^{u,H}$ has many intersections with $Y = -k_0$ unless ΔC is small enough. In fact, using the linear approximation (5.24) we get

$$\begin{aligned}\dot{\tilde{Y}} &= -(\omega^2 + 9)a\omega\cos(\omega t) + (\lambda^2 - 9)\lambda\tilde{c}e^{\lambda t} \leq\\ &\leq (\omega^2 + 9)a\omega + (\lambda^2 - 9)\lambda\tilde{c}e^{\lambda t} < 0,\end{aligned}$$

if a is small enough and $\tilde{c}e^{\lambda t}$ not too small. Therefore, if a is small enough all the orbits in $W_T^{u,H}$ cut transversally the section $Y = -k_0$. For the moment being we shall take the origin of time at $Y = -k_0$. Moreover, the intersection of $W_T^{u,H}$ with $Y = -k_0$ is a torus close to the product of two curves close to ellipses which live on the planes $(X, \dot{X})$ and $(Z, \dot{Z})$, approximately centered at $(3^{-\frac{1}{2}} + 2\lambda\tilde{c}, 2\lambda^2\tilde{c})$ and $(0, 0)$, respectively. The semiaxes are proportional to $(\Delta C_p^H)^{\frac{1}{2}}$ and $(\Delta C_v^H)^{\frac{1}{2}}$ respectively. The following lemma says that this structure is preserved for $W_T^{u,H} \cap \{Y = -\tilde{k}\}$ with $\tilde{k} > 0$ large if we take $\Delta C^H > 0$ small enough. The proof is postposed to section 5.5.

Lemma 5.4.1. *Let be $\tilde{k} \in \mathbb{R}^+$. If $\Delta C^H > 0$ is small enough, then $W_T^{u,H} \cap \{Y = -\tilde{k}\}$ is roughly a torus obtained as the product of two closed curves close to ellipses in the planes $(X, \dot{X})$, $(Z, \dot{Z})$ centered at $(X_{L_2}(\tilde{k}), \dot{X}_{L_2}(\tilde{k}))$ and $(0, 0)$ and semiaxes proportional to $(\Delta C_p^H)^{\frac{1}{2}}$ and $(\Delta C_v^H)^{\frac{1}{2}}$, respectively. Here, $X_{L_2}(\tilde{k})$ and $\dot{X}_{L_2}(\tilde{k})$ denote the coordinates $X, \dot{X}$ for $W_{L_2}^{u,H} \cap \{Y = -\tilde{k}\}$.*

The same is true for the SRTBP due to the fact that it is an arbitrarily small perturbation of the Spatial Hill's problem if μ is small enough. We study the geometry of $W_T^{u,\mu} \cap \{y = -\tilde{k}\mu^{\frac{1}{3}}\}$. The relations between the Jacobi constant for the SRTBP and the spatial Hill's problem is the following

$$C = 3 + \mu^{\frac{2}{3}} C^H + O(\mu).$$

Moreover,

$$C_p = 3 + \mu^{\frac{2}{3}} C_p^H + O(\mu), \qquad C_v = \mu^{\frac{2}{3}} C_v^H + O(\mu).$$

From lemma 5.4.1, if $\tilde{k} \in \mathbb{R}^+$, then $W_T^{u,\mu} \cap \{y = -\tilde{k}\mu^{\frac{1}{3}}\}$ can be written approximately as $E_1 \times E_2$, where E_1, E_2 are ellipses living in the planes $(x, \dot{x}), (z, \dot{z})$, with semiaxes proportional to $\mu^{\frac{1}{3}}\Delta C_p^H$ and $\mu^{\frac{1}{3}}\Delta C_v^H$ and centered at $(x_{L_2}(k), \dot{x}_{L_2}(k))$ and $(0, 0)$, respectively. Here, $x_{L_2}(k), \dot{x}_{L_2}(k)$ denote the coordinates $x, \dot{x}$ for $W_{L_2}^{u,\mu}$.

Now we study the geometry of $W_T^{u,H}$ from section $\{Y = -\tilde{k}\}$ to $\{Y = -\tilde{\tilde{k}}\}$ being $\tilde{\tilde{k}} > \tilde{k} > 0$.

If $-Y$ is large enough, equations (5.19) are well approximated by the linear equations

$$\begin{aligned} \ddot{X} - 2\dot{Y} &= 3X, \\ \ddot{Y} + 2\dot{X} &= 0, \\ \ddot{Z} &= -Z. \end{aligned} \tag{5.25}$$

The solution of this system is

$$\begin{aligned} X(t) &= \frac{2}{3}N + M\cos(t - t_0), \\ Y(t) &= B - Nt - 2M\sin(t - t_0), \\ Z(t) &= A\cos(t - t_0) + D\sin(t - t_0). \end{aligned}$$

We note that the constants M, N, B, t_0, A and D can be computed through

$$M^2 = \dot{X}^2 + \ddot{X}^2, \qquad N = 3(2X + \dot{Y}),$$
$$B = Y + 3(2X + \dot{Y})t - 2\dot{X}, \qquad t_0 = t - \arctan\left(-\frac{\ddot{Y}}{2\ddot{X}}\right),$$
$$A^2 + D^2 = Z^2 + \dot{Z}^2.$$

Now we want to estimate the effect of the neglected terms. We take the above constants as functions of t. Then, we have that

$$\begin{aligned} &(\dot{M^2}) = 2\ddot{X}[-2Yr^{-3} - \dot{X}r^{-3} + 3X(X\dot{X} + Y\dot{Y} + Z\dot{Z})r^{-5}], \\ &\dot{N} = -3Yr^{-3}, \\ &\dot{B} = (2X - 3Yt)r^{-3}, \\ &\dot{t}_0 = [-2r^{-3}(2\dot{Y}^2 + 2\dot{X}^2 + 3X\dot{Y} + 3\dot{X}Y) + 6r^{-5}(2Y\dot{Y} + 3XY + 2X\dot{X})(X\dot{X} + \\ &Y\dot{Y} + Z\dot{Z}) + r^{-6}(2X\dot{Y} - 3Y^2 - 2\dot{X}Y)][4M^2 + 4r^{-3}\dot{X}Y + r^{-6}Y^2]^{-1}, \\ &(\dot{A^2}) + (\dot{D^2}) = -2Z\dot{Z}r^{-3}, \end{aligned}$$

where $r^2 = X^2 + Y^2 + Z^2$.

Therefore, the contribution of the terms containing the factor $(X^2+Y^2+Z^2)^{-\frac{3}{2}}$ to the coefficients M, N, t_0, B and A^2+D^2 when Y goes from $-\tilde{k}$ to $-\tilde{\tilde{k}}$, $0 < \tilde{k} < \tilde{\tilde{k}}$, $\tilde{k}$ large, is $O(\tilde{k}^{-2})$, $O(\tilde{k}^{-1})$, $O\left(ln\left(\frac{\tilde{\tilde{k}}}{\tilde{k}}\right)\right)$ and $O(\tilde{k}^{-3})$, respectively.

If we choose a value of $\tilde{k}$ such that $W^{u,H}_{L_2}$ has a maximum of X when this manifold intersects $Y = -\tilde{k}$, we have that $W^{u,H}_{L_2}$ is expressed by

$$\begin{aligned} X(t) &= \frac{2}{3}N_\infty + M_\infty \cos t, \\ Y(t) &= -\tilde{k} - N_\infty t - 2M_\infty \sin t, \\ Z(t) &= 0, \end{aligned} \tag{5.26}$$

where N_∞, M_∞ stands for the values of N, M for this solution. These constants have been computed numerically, $N_\infty = 5.1604325\ldots$ and $M_\infty = 2.1320587\ldots$ (see [L.M.S.]).

It is easy to identify (5.26) with a Kepler orbit in synodical coordinates. Let $\omega = 1$ be the angular velocity of the rotating axes and consider a Kepler orbit

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -a(\cos E - e) \\ -a\sqrt{1-e^2}\sin E \end{pmatrix},$$

$M = nt = E - e\sin E$, $n = 1+\gamma$, $n^2a^3 = 1$ with γ, e small. Skipping all terms $O_2(e,\gamma)$ we obtain

$$\begin{aligned} &a = 1 - \frac{2}{3}\gamma, \quad E = M + e\sin M, \quad \cos E = \cos M - e\sin^2 M, \\ &\sin E = \sin M + e\sin M\cos M. \end{aligned}$$

We shall assume t large but bounded. Then, in rotating coordinates

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = a\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}\begin{pmatrix} -\cos M + e\sin^2 M + e \\ -\sin M - e\sin M\cos M \end{pmatrix} = \begin{pmatrix} -\gamma + \frac{2}{3}\gamma + e\cos t \\ -\gamma t - 2e\sin t \end{pmatrix}.$$

With respect to the point $\begin{pmatrix} -1 \\ 0 \end{pmatrix}$ we have

$$\begin{pmatrix} \tilde{x} - 1 \\ \tilde{y} \end{pmatrix} = \begin{pmatrix} \frac{2}{3}\gamma + e\cos t \\ -\gamma t - 2e\sin t \end{pmatrix}.$$

Hence $M_\infty \mu^{\frac{1}{3}}$ can be identified as e and $N_\infty \mu^{\frac{1}{3}}$ as γ (or $a = 1 - \frac{2}{3} N_\infty \mu^{\frac{1}{3}} + o(\mu^{\frac{1}{3}})$).

Let us consider rectangular coordinates $\xi_1, \dot{\xi}_1, \xi_2, \dot{\xi}_2$ in the hyperplane $Y = -\tilde{k}$, with $\tilde{k} \leq k \leq \tilde{\tilde{k}}$ and $\tilde{k}$ sufficiently large, where $\xi_i, \dot{\xi}_i$, $i = 1, 2$ defined by

$$\xi_1 = X - X_{L_2}(k), \quad \dot{\xi}_1 = \dot{X} - \dot{X}_{L_2}(k), \quad \xi_2 = Z, \quad \dot{\xi}_2 = \dot{Z}, \tag{5.27}$$

and $X_{L_2}(k), \dot{X}_{L_2}(k)$, denotes the $X, \dot{X}$ coordinates respectively, of $W^{u,H}_{L_2} \cap \{Y = -k\}$. We want to study the variation of this invariant object close to a torus defined by $W^{u,H}_T \cap \{Y = -k\}$ when we change the value of $k > 0$ large enough.

Lemma 5.4.2. *The Poincaré map for the approximated Hill's problem (5.25) which sends $(\xi_1, \dot{\xi}_1, \xi_2, \dot{\xi}_2)$ on the plane $Y = -\tilde{k}$ to $(\xi_1^*, \dot{\xi}_1^*, \xi_2^*, \dot{\xi}_2^*)$ on the plane $Y = -\tilde{\tilde{k}}$ is given by*

$$T_t^* = \begin{pmatrix} T_{p,t^*} & 0 \\ 0 & T_{v,t^*} \end{pmatrix},$$

where

$$T_{p,t^*} = \begin{pmatrix} \dfrac{4M_\infty N_\infty + (4M_\infty^2 + N_\infty^2)\cos t^* + 3M_\infty^2 t^* \sin t^*}{(N_\infty + 2M_\infty \cos t^*)(N_\infty + 2M_\infty)} & \dfrac{(2M_\infty + N_\infty)\sin t^*}{(N_\infty + 2M_\infty \cos t^*)} \\ \dfrac{-N_\infty^2 \sin t^* + 3M_\infty^2 t^* \cos t^*}{(N_\infty + 2M_\infty \cos t^*)(N_\infty + 2M_\infty)} & \dfrac{(2M_\infty + N_\infty)\cos t^*}{(N_\infty + 2M_\infty \cos t^*)} \end{pmatrix},$$

$$T_{v,t^*} = \begin{pmatrix} \cos t^* & \sin t^* \\ -\sin t^* & \cos t^* \end{pmatrix},$$

being t^* the time required for $W^{u,H}_{L_2}$ by going from $Y = -\tilde{k}$ to $Y = -\tilde{\tilde{k}}$.

The proof of this Lemma is given in section 5.5

Using the lemma above, if we write $\tilde{T} = \tilde{E}_1 \times \tilde{E}_2$, where $\tilde{E}_1$, $\tilde{E}_2$ are closed curves close to ellipses that live in the planes $(X, \dot{X})$ and $(Z, \dot{Z})$, respectively, when k increases $\tilde{E}_1$ rotates and one of the axes increases and $\tilde{E}_2$ only rotates. The standard symplectic form is preserved by T_{t^*} and the area enclosed by $\tilde{E}_1$ and $\tilde{E}_2$ is approximately preserved under T_{t^*}.

Now we study the behavior of $W^{u,H}_T$ for the SRTBP. To this end, we take $k > 0$ large and μ small such that $-k\mu^{\frac{1}{3}}$ is small enough. Then, on $y = -k\mu^{\frac{1}{3}}$, $W^{u,\mu}_T$ is roughly a torus in $(x, \dot{x}, z, \dot{z})$ variables, obtained as the product of two closed curves near ellipses. One of these curves lives in the plane $(x, \dot{x})$, it is centered approximately at $(x_{L_2}(k), \dot{x}_{L_2}(k))$, where $x_{L_2}(k), \dot{x}_{L_2}(k)$ denote the values of $x, \dot{x}$ of $W^{u,\mu}_{L_2} \cap \{y = -k\mu^{\frac{1}{3}}\}$, and with semiaxes proportional to $\mu^{\frac{1}{3}}(\Delta C^H_p)^{\frac{1}{2}} \approx (\Delta C_p)^{\frac{1}{2}}$. The other curve lies in the plane $(z, \dot{z})$, it is centered approximately at $(0, 0)$ and its semiaxes are proportional to $\mu^{\frac{1}{3}}(\Delta C^H_v)^{\frac{1}{2}} \approx (\Delta C_v)^{\frac{1}{2}}$. The major axes of the ellipse in $(x, \dot{x})$ rotates and increases when k does, while the ellipse close to $(z, \dot{z})$ only rotates.

5.4.2 Analytic expression of the unstable manifold

In this section we shall obtain an analytic expression of the first cut of $W^{u,\mu}_T$ with $y = 0, x > 0$ when μ is small enough. To this end, taking into account the geometry of $W^{u,\mu}_T$ on the sections $y = -k\mu^{\frac{1}{3}}$ for different values of k, we will take suitable initial conditions with $y \approx 0$ and, then we shall compute the first cut γ of $W^{u,\mu}_T$ with $y = 0, x > 0$. To this end we shall approximate the SRTBP by the SS2BP. Next lemma give us the initial conditions on $W^{u,\mu}_T \cap \{y = 0\}$. Its proof is given in section 5.5.

Lemma 5.4.3. *It is not restrictive to assume that the axes of the ellipse living in the plane $(x, \dot{x})$ are parallel to these axes. Then, we can take as initial condition*

$$
\begin{aligned}
x &= -1 + \mu + \left(\frac{2}{3}N_\infty + M_\infty \cos\tau\right)\mu^{\frac{1}{3}} + k_1 \Delta C_p^{\frac{1}{2}} \cos\sigma_1, \\
\dot{x} &= -M_\infty \sin\tau \cdot \mu^{\frac{1}{3}} + k_2 \Delta C_p^{\frac{1}{2}} \sin\sigma_1, \\
y &= 0, \\
\dot{y} &= -(N_\infty + 2M_\infty \cos\tau)\mu^{\frac{1}{3}} - (N_\infty + 2M_\infty \cos\tau)^{-1}[(2N_\infty + \\
&\quad 3M_\infty \cos\tau)k_1 \cos\sigma_1 + M_\infty k_2 \sin\tau \sin\sigma_1]\Delta C_p^{\frac{1}{2}}, \\
\begin{pmatrix} z \\ \dot{z} \end{pmatrix} &= \Omega(\tau) K \begin{pmatrix} \cos\sigma_2 \\ \sin\sigma_2 \end{pmatrix} \Delta C_v^{\frac{1}{2}},
\end{aligned}
$$

where $\dot{y}$ is obtained by the Jacobi relation, $\Omega(\tau) = \begin{pmatrix} \cos\tau & \sin\tau \\ -\sin\tau & \cos\tau \end{pmatrix}$, $K = \tilde{\rho}\begin{pmatrix} \cos\gamma & -A\sin\gamma \\ \sin\gamma & A\cos\gamma \end{pmatrix}$, $k_i, i = 1, 2$ and $\gamma, \tilde{\rho}$ and A (assumed to be $A \neq 1$) are finite quantities related to the axis of the ellipses in $(x, \dot{x})$ and $(z, \dot{z})$ in the torus, σ_1, σ_2 are the parameters for a point in the ellipse in $(x, \dot{x})$, $(z, \dot{z})$, respectively, and τ is related to the time for which $W^{u,H}_{L_2}$ reaches again $y = 0, x < 0$ when we start at $y = -k\mu^{\frac{1}{3}}$.

The assumption $A \neq 1$ means that the ellipse in z, $\dot{z}$ is not a perfect circle. Numerically it has been checked that this is the case.

Let us consider the constants α, β, ψ introduced in (5.14) and (5.15).

If we expand in power series in μ_k the initial condition given in Lemma 5.4.3 up to order $\mu_k^{\frac{2}{3}}$ we obtain

$$\begin{aligned}
x_0 &= -1+\left(\frac{2}{3}N_\infty + M_\infty\cos\tau\right)\mu_k^{\frac{1}{3}} + \left[\frac{\alpha}{3}\left(\frac{2}{3}N_\infty + M_\infty\cos\tau\right)+\right. \\
&\quad \left. k_1\tilde{\beta}_1\cos\sigma_1\right]\mu_k^{\frac{2}{3}} + O(\mu_k), \\
\dot{x}_0 &= -M_\infty\sin\tau\mu_k^{\frac{1}{3}} + \left[-\frac{\alpha}{3}M_\infty\sin\tau + k_2\tilde{\beta}_1\sin\sigma_1\right]\mu_k^{\frac{2}{3}} + O(\mu_k), \\
y_0 &= 0, \\
\dot{y}_0 &= -(N_\infty + 2M_\infty\cos\tau)\mu_k^{\frac{1}{3}} - \left\{\frac{\alpha}{3}(N_\infty + 2M_\infty\cos\tau)+\right. \\
&\quad + (N_\infty + 2M_\infty\cos\tau)^{-1}[(2N_\infty + 3M_\infty\cos\tau)k_1\cos\sigma_1 + \\
&\quad \left. M_\infty k_2\sin\tau\sin\sigma_1]\right\}\mu_k^{\frac{2}{3}}\tilde{\beta}_1 + \\
&\quad +O(\mu_k), \\
\begin{pmatrix} z_0 \\ \dot{z}_0 \end{pmatrix} &= \tilde{\beta}_2\Omega(\tau)K\begin{pmatrix} \cos\sigma_2 \\ \sin\sigma_2 \end{pmatrix}\mu_k^{\frac{2}{3}} + O(\mu_k),
\end{aligned} \tag{5.28}$$

where $\tilde{\beta}_1 = \tilde{\beta}\cos\psi, \tilde{\beta}_2 = \tilde{\beta}\sin\psi$, $\tilde{\beta} = \sqrt{\beta}$.

We note that if $\alpha = \beta = 0$ then we have an initial condition for a homoclinic orbit to L_2. We also note that if $\tilde{\beta}_2 = 0$ then we are on the unstable manifold of a planar periodic orbit and if $\tilde{\beta}_1 = 0$ then we obtain an initial condition for the unstable manifold of a vertical periodic orbit.

Now, our purpose is to compute the first cut with $y = 0, x > 0$ of the solution with initial condition (5.28). To this end, we compute the image under the Spatial Synodic two body problem (SSTBP) with initial condition (5.28). The solution of this problem is an ellipse in syderal system. We shall take into account the relations between the parameters of an orbit for the SSTBP. We shall use the mean anomaly M of an elliptic orbit in order to obtain these expression. It is known (see [L.M.S.]) that the first cut of $W_{L_2}^{u,\mu_k}$ with $y = 0, x > 0$ is given for $M_w = 0$ or $M_w = \pi$ where M_w denotes the mean anomaly for this intersection. The following lemma give us the first cut of this solution with $y = 0, x > 0$ assuming that $M_w = 0$. An analogous result is obtained assuming that $M_w = \pi$. The proof is given in 5.5.

Lemma 5.4.4. *The first cut of a solution with the initial conditions given in*

(5.28) with $y=0, x>0$ is

$$
\begin{aligned}
x(M_f)-x_w &= M_\infty(1-\cos M_f)\mu_k^{\frac{1}{3}}+\left\{\frac{\tilde\beta_1}{c_2}(2M_\infty+N_\infty\cos M_f)(c_3\sin\sigma_1-\right.\\
&\quad c_4\cos\sigma_1)-\frac{\alpha}{3}\left(\frac{2}{3}N_\infty+M_\infty\cos M_f\right)+M_\infty^2\sin^2 M_f+\\
&\quad \left. c_1(\cos M_f-1)\right\}\mu_k^{\frac{2}{3}}+O(\mu_k),\\
\dot x(M_f) &= M_\infty\sin M_f\mu_k^{\frac{1}{3}}+\sin M_f\left\{\frac{M_\infty}{3}\alpha-\tilde\beta_1\frac{N_\infty}{c_2}(c_3\sin\sigma_1-c_4\cos\sigma_1)+\right.\\
&\quad \left. 2M_\infty^2\cos M_f-c_1+N_\infty M_\infty\right\}\mu_k^{\frac{2}{3}}+O(\mu_k),\\
\begin{pmatrix} z(M_f)\\ \dot z(M_f)\end{pmatrix} &= \tilde\beta_2\Omega(M_f)K\begin{pmatrix}\cos\sigma_2\\ \sin\sigma_2\end{pmatrix}\mu_k^{\frac{2}{3}}+O(\mu_k),
\end{aligned}
$$

where $\Omega(M_f)=\begin{pmatrix}\cos M_f & \sin M_f\\ -\sin M_f & \cos M_f\end{pmatrix}$, *$x_w$ is the value of x for the first cut of $W_{L_2}^{u,\mu}$ with $y=0, x>0$,* $c_1=N_\infty M_\infty+\cos\tau[N_\infty(\alpha_2-\alpha_1)+M_\infty^2\cos^2\tau]$, $c_3=k_2\sin\tau$, $c_4=k_1\cos\tau$, $\alpha_1=\frac{2}{3}N_\infty+M_\infty\cos\tau$, $\alpha_2=N_\infty+2M_\infty\cos\tau$. *Moreover, M_f satisfies the following equation*

$$
\begin{aligned}
&\mu_k^{\frac{1}{3}}\left\{-\left(\frac{1}{3}N_\infty\alpha+3\frac{\tilde\beta_1}{\alpha_2}M_\infty(c_4\cos\sigma_1-c_3\sin\sigma_1)\right)\frac{\pi}{N_\infty}+N_\infty M_f+\right.\\
&\left. 2M_\infty\sin M_f\right\}+O(\mu_k^{\frac{2}{3}})=0. \qquad (5.29)
\end{aligned}
$$

The proof of this lemma is given in section 5.5.

5.4.3 The stable manifold via the symmetries of the problem

We consider an invariant torus T in the centre manifold of L_2. We want to study the conditions in order to obtain homoclinic or heteroclinic orbits to invariant tori. The following lemma give us a relation between the invariant manifolds of two tori.

Lemma 5.4.5. *Let us denote by $W_T^{u,\mu}$ the unstable manifold of an invariant torus T on the centre manifold of L_2. Then, $\mathcal{S}(W_T^{u,\mu},t)$ is the stable manifold of T.*

Proof

We have that $W_T^{u,\mu}=\{\boldsymbol\varphi$ solution of 5.1 such that $\boldsymbol\varphi(t)\to T$ when $t\to-\infty\}$, where $\boldsymbol\varphi(t)=(\varphi_1(t),\varphi_2(t),\varphi_3(t),\dot\varphi_1(t),\dot\varphi_2(t),\dot\varphi_3(t))^T$. Then,

$$\mathcal{S}(\varphi,t)\to\mathcal{S}(T,t) \text{ when } t\to-\infty \Leftrightarrow \tilde{\boldsymbol\varphi}(t)\to\mathcal{S}(T,-t) \text{ when } t\to+\infty,$$

where $\tilde{\boldsymbol{\varphi}}(t) = (\tilde{\varphi}_1(-t), -\tilde{\varphi}_2(-t), \tilde{\varphi}_3(-t), -\dot{\varphi}_1(-t), \dot{\varphi}_2(-t), \dot{\varphi}_3(-t))$. Therefore $\tilde{\boldsymbol{\varphi}} \in W_T^{s,\mu}$. Here, $W_T^{s,\mu}$ denotes the stable manifold of the torus on the same energy level that $W_T^{u,\mu}$.

Analogously one proves that if $\boldsymbol{\varphi} \in W_T^{s,\mu}$ then $\mathcal{S}(\boldsymbol{\varphi}, t) \in W_T^{u,\mu}$. □

Lemma 5.4.6. *Let us consider $\boldsymbol{\varphi} \in W_T^{u,\mu}$ such that $\boldsymbol{\varphi} \cap \{y = 0, z = 0, \dot{x} = 0\} \neq \emptyset$. Then there exists an homoclinic orbit to T.*

Proof

This lemma is easily proved by computing the fixed points of the symmetry $\mathcal{S}$ and using lemma 5.4.5. □

In order to obtain an analytic expression of the stable manifold we only need to take into account the symmetry $\mathcal{S}$. Then, the first cut of the stable manifold of a torus T, $W_T^{s,\mu}$, up to terms of order $\mu_k^{\frac{2}{3}}$ is given by the following expression

$$\begin{aligned} x^s(M_f) - x_w^s &= x(M_f) - x_w, \\ \dot{x}^s(M_f) &= -\dot{x}(M_f), \\ \begin{pmatrix} z(M_f) \\ \dot{z}(M_f) \end{pmatrix} &= \tilde{\beta}_2 J \Omega(M_f) K \begin{pmatrix} \cos\sigma_2 \\ \sin\sigma_2 \end{pmatrix} \mu_k^{\frac{2}{3}} + O(\mu_k), \end{aligned} \tag{5.30}$$

where $J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

5.4.4 Some homoclinic and heteroclinic orbits

In this section we study the intersections of the unstable manifold of one torus and the stable manifold of another torus in order to obtain heteroclinic orbits. The tori should live in the same level of energy. The parameter ψ describes one of the tori and ψ' the other one. These parameters belong to a Cantor set of relative large measure if β is small enough.

We fix μ and ΔC. This is equivalent to fix α and β. We take $\psi, \psi' \in \left(0, \frac{\pi}{2}\right)$. Let us denote by T_1 and T_2 the tori characterized by ψ and ψ', respectively.

Let us denote by

$$f(\alpha, \beta; \sigma_1, \sigma_2, M_f, \psi) = (f_1, f_2, f_3, f_4)^T,$$

and by

$$g(\alpha, \beta; \sigma_1', \sigma_2', M_f', \psi') = (g_1, g_2, g_3, g_4)^T,$$

the expression of the first cut of the invariant manifolds $W_{T_1}^{u,\mu}$ and $W_{T_2}^{s,\mu}$, respectively, with $y = 0, x > 0$, being

$$\begin{aligned} f_i &= f_i(\alpha, \beta; \sigma_1, M_f, \psi), \quad i = 1, 2, \\ f_i &= f_i(\alpha, \beta; \sigma_2, M_f, \psi), \quad i = 3, 4, \\ g_i &= g_i(\alpha, \beta; \sigma_1', M_f', \psi'), \quad i = 1, 2, \\ g_i &= g_i(\alpha, \beta; \sigma_2', M_f', \psi'), \quad i = 3, 4. \end{aligned}$$

Then, $f_1 = x(M_f)$, $f_2 = \dot{x}(M_f)$, $f_3 = z(M_f)$, $f_4 = \dot{z}(M_f)$ where $x(M_f)$, $\dot{x}(M_f)$, $z(M_f)$, $\dot{z}(M_f)$ are defined in Lemma 5.4.4. Moreover, as we have seen in the last section, $g_1 = x(M_f')$, $g_2 = -\dot{x}(M_f')$, $g_3 = z(M_f')$ and $g_4 = -\dot{z}(M_f')$.

We recall that M_f is the solution of the equation (5.29). Rearranging terms in this expression it can be written as

$$p(M_f) = \frac{\tilde{\beta}_1}{\alpha_2}(c_3 \sin\sigma_1 - c_4 \cos\sigma_1), \tag{5.31}$$

where

$$p(M_f) = \frac{N_\infty}{3M_\infty\pi}\left\{\frac{\pi\alpha}{3} - [N_\infty M_f + 2M_\infty \sin M_f]\right\} + O(\mu_k^{\frac{1}{3}}). \tag{5.32}$$

In order to obtain heteroclinic orbits we need to solve the system of equations

$$\begin{aligned} f_1(\sigma_1, \psi, M_f) &= g_1(\sigma_1', \psi', M_f'), \\ f_2(\sigma_1, \psi, M_f) &= g_2(\sigma_1', \psi', M_f'), \\ f_3(\sigma_2, \psi, M_f) &= g_3(\sigma_2', \psi', M_f'), \\ f_4(\sigma_3, \psi, M_f) &= g_4(\sigma_2', \psi', M_f'), \end{aligned} \tag{5.33}$$

under the restriction given by (5.31).

Using the relation (5.31) we have that f_1, f_2, g_1, g_2 can be written independent of σ_1, σ_1' and ψ, ψ'. In fact, we can write

$$\begin{aligned}
f_1(M_f) &= M_\infty(1-\cos M_f)\mu_k^{\frac{1}{3}} + \{(2M_\infty + N_\infty \cos M_f)p(M_f) - \\
&\quad \frac{\alpha}{3}\left(\frac{2}{3}N_\infty + M_\infty \cos M_f\right) + M_\infty^2 \sin^2 M_f + c_1(\cos M_f - 1)\Big\}\mu_k^{\frac{2}{3}} + \\
&\quad O(\mu_k), \\
f_2(M_f) &= M_\infty \mu_k^{\frac{1}{3}} \sin M_f + \sin M_f \left\{\frac{M_\infty}{3}\alpha - N_\infty p(M_f) + 2M_\infty^2 \cos M_f + \right. \\
&\quad N_\infty M_\infty - c_1\} \mu_k^{\frac{2}{3}} + O(\mu_k), \\
g_1(M_f') &= M_\infty(1-\cos M_f')\mu_k^{\frac{1}{3}} + \{(2M_\infty + N_\infty \cos M_f')p(M_f') - \\
&\quad \frac{\alpha}{3}\left(\frac{2}{3}N_\infty + M_\infty \cos M_f'\right) + M_\infty^2 \sin^2 M_f' + c_1(\cos M_f' - 1)\Big\}\mu_k^{\frac{2}{3}} + \\
&\quad O(\mu_k), \\
g_2(M_f') &= -M_\infty \mu_k^{\frac{1}{3}} \sin M_f' - \sin M_f' \left\{\frac{M_\infty}{3}\alpha - N_\infty p(M_f') + 2M_\infty^2 \cos M_f' + \right. \\
&\quad N_\infty M_\infty - c_1\} \mu_k^{\frac{2}{3}} + O(\mu_k),
\end{aligned}$$

where $f_j(M_f)$, $g_j(M_f')$ denotes f_j, g_j, $j=1,2$, respectively.

We look for the relation between M_f and M_f' in order to satisfy the first two equations in (5.33) up to order $\mu_k^{\frac{2}{3}}$. A computation shows that

$$\begin{aligned}
f_1(M_f) - g_1(M_f') &= M_\infty(\cos M_f' - \cos M_f)\mu_k^{\frac{1}{3}} + \{2M_\infty(p(M_f) - p(M_f')) + \\
&\quad N_\infty(p(M_f)\cos M_f - p(M_f')\cos M_f') + N_\infty(p(M_f)\cos M_f - \\
&\quad p(M_f')\cos M_f') + \frac{\alpha M_\infty}{3}(\cos M_f - \cos M_f') + \\
&\quad M_\infty^2(\sin^2(M_f) - \sin^2(M_f')) + c_1(\cos M_f - \cos M_f')\} \mu_k^{\frac{2}{3}} + \\
&\quad O(\mu_k), \\
f_2(M_f) - g_2(M_f') &= M_\infty(\sin M_f + \sin M_f')\mu_k^{\frac{1}{3}} + \left[(\sin M_f + \sin M_f')\left(\frac{M_\infty \alpha}{3} + \right.\right. \\
&\quad N_\infty M_\infty - c_1) - N_\infty(p(M_f)\sin M_f + p(M_f')\sin M_f') + \\
&\quad 2M_\infty^2(\sin M_f \cos M_f + \sin M_f' \cos M_f')] \mu_k^{\frac{2}{3}} + O(\mu_k).
\end{aligned}$$

We take

$$\begin{aligned}
M_f &= a_0 + a_1 \mu_k^{\frac{1}{3}} + O(\mu_k^{\frac{2}{3}}), \\
M_f' &= b_0 + b_1 \mu_k^{\frac{1}{3}} + O(\mu_k^{\frac{2}{3}}),
\end{aligned}$$

where a_0, b_0, a_1, b_1 are real constants that we need to determine. In order that the two first equations in (5.33) are satisfied up to terms of order $\mu_k^{\frac{2}{3}}$ it is necessary that the following conditions hold

$$
\begin{aligned}
M_f &= a_0 + a_1\mu_k^{\frac{1}{3}} + O(\mu_k^{\frac{2}{3}}),\\
M_f' &= 2k\pi - a_0 + b_1\mu_k^{\frac{1}{3}} + O(\mu_k^{\frac{2}{3}}),\\
M_\infty(a_1+b_1)\cos a_0 &+ \frac{2N_\infty^2}{3M_\infty\pi}\sin a_0[N_\infty(a_0-k\pi)+2M_\infty\sin a_0] = 0, \qquad (5.34)\\
M_\infty(a_1+b_1)\sin a_0 &- \frac{4N_\infty M_\infty}{3M_\infty\pi}[N_\infty(a_0-k\pi)+2M_\infty\sin a_0] -\\
\frac{2N_\infty^2}{3M_\infty\pi}&\cos a_0[N_\infty(a_0-k\pi)+2M_\infty\sin a_0] = 0. \qquad (5.35)
\end{aligned}
$$

Next lemma give us the solutions of the equations (5.34) and (5.35). Its proof is given in section 5.5.

Lemma 5.4.7. *The system formed for the equations (5.34) and (5.35) only has* $a_0 = m\pi$, $m \in \mathbb{Z}$, $b_1 = -a_1$ *as solution.*

Then,

$$
\begin{aligned}
M_f &= k\pi + a_1\mu_k^{\frac{1}{3}} + O(\mu_k^{\frac{2}{3}}),\\
M_f' &= k\pi - a_1\mu_k^{\frac{1}{3}} + O(\mu_k^{\frac{2}{3}}),
\end{aligned}
$$

where $k \in \mathbb{Z}$

Once we know the expression of M_f up to terms of order $\mu_k^{\frac{1}{3}}$, we want to know for which values of α and β equation (5.31) has solution σ_1.

We denote by $w = \cos\sigma_1$. Then, $\sin\sigma_1 = \tilde{s}\sqrt{1-w^2}$ where $\tilde{s} = \pm 1$. Therefore, (5.31) can be written as

$$
\tilde{\beta}_1 c_3\tilde{s}\sqrt{1-w^2} \quad = \quad \alpha_2 p(M_f) + \tilde{\beta}_1 c_4 w.
$$

If we consider the equation obtained from the above by raising up to square in both sides of the equality, the equation transforms in

$$
b_1 w^2 + b_2 w + b_3 \quad = \quad 0, \qquad (5.36)
$$

where $b_1 = \tilde{\beta}_1^2(c_3^2+c_4^2)$, $b_2 = 2\tilde{\beta}_1\alpha_2 p(M_f)c_4$, $b_3 = \alpha_2^2 p(M_f)^2 - \tilde{\beta}_1^2 c_3^2$. We assume that $b_1 \neq 0$. Due to the fact that $c_3^2 + c_4^2 = 0$ if and only if $k_1 = k_2 = 0$, and that in this case we are on an initial condition for the invariant manifold of a vertical periodic orbit, it is only necessary to suppose that $\tilde{\beta}_1 \neq 0$. We are interested in the solutions of (5.36) such that $w \in [-1,1]$. This is accomplished if and only if

$$
b_2^2 \leq 4b_1^2 \quad \text{and} \quad b_2^2 - 4b_1 b_3.
$$

It is easy to see that $b_2^2 - 4b_1b_3 \geq 0$ if and only if

$$\alpha_2^2 p(M_f)^2 \leq \tilde{\beta}_1^2(c_3^2 + c_4^2). \tag{5.37}$$

Under this condition, the inequality $b_2 \leq 4b_1^2$ holds.

Using lemma 5.4.7, if μ_k is sufficiently small we have that the condition (5.37) transforms in

$$\cos^2\psi \geq \frac{\mathcal{K}^2}{\beta}(\hat{\alpha} - k)^2, \tag{5.38}$$

where $\mathcal{K} = \dfrac{\alpha_2 N_\infty^2}{3M_\infty\sqrt{c_3^2 + c_4^2}}$ and $\hat{\alpha} = \dfrac{\alpha}{3N_\infty}$.

From Theorem 5.2.1 we get $\mu_{k+1} - \mu_k = 3N_\infty \mu_k^{\frac{4}{3}}(1 + o(1))$. We recall that for a fixed μ we consider the value μ_k which is at minimum distance from μ. Therefore we can assume $|\hat{\alpha}| \leq \dfrac{1}{2}$. Then if ψ satisfies the condition (5.38), the equation (5.31) has two solutions.

The same analysis can be done by changing M_f, σ_1 and β_1 by M'_f, σ'_1 and β'_1.

Now, for the values of M_f and M'_f given in Lemma (5.4.7) we look for the relations between $\sigma_2, \sigma'_2, \psi, \psi'$ in order to obtain intersections of the vertical components of the invariant manifolds up to terms of order $\mu_k^{\frac{2}{3}}$.

A simple check shows that last two equations in (5.33) can be written as

$$\tilde{\beta}_2 \begin{pmatrix} \cos\sigma_2 \\ \sin\sigma_2 \end{pmatrix} = \tilde{\beta}'_2 \begin{pmatrix} a_1 & -Aa_2 \\ -\dfrac{a_2}{A} & -a_1 \end{pmatrix} \begin{pmatrix} \cos\sigma'_2 \\ \sin\sigma'_2 \end{pmatrix}, \tag{5.39}$$

where $a_1 = \cos(2\gamma)$, $a_2 = \sin(2\gamma)$. We note that if $\tilde{\beta}'_2 = 0$ then $\tilde{\beta}_2 = 0$. In fact, we are on the invariant manifolds of some planar periodic orbits. We assume that $\tilde{\beta}'_2 \neq 0$ and we define $\tilde{\gamma} := \dfrac{\tilde{\beta}_2}{\tilde{\beta}'_2}$. Therefore, (5.39) is written as

$$\tilde{\gamma} \begin{pmatrix} \cos\sigma_2 \\ \sin\sigma_2 \end{pmatrix} = \begin{pmatrix} a_1 & -Aa_2 \\ -\dfrac{a_2}{A} & -a_1 \end{pmatrix} \begin{pmatrix} \cos\sigma'_2 \\ \sin\sigma'_2 \end{pmatrix}.$$

A solution of the equation above must satisfy

$$\begin{aligned} \tilde{\gamma}^2 &= [a_1\cos\sigma'_2 - Aa_2\sin\sigma'_2]^2 + \left[-\frac{a_2}{A}\cos\sigma'_2 - a_1\sin\sigma'_2\right]^2 = \\ & a_1^2 + \frac{a_2^2}{2}\left(\frac{1}{A^2} + A^2\right) + \frac{a_2^2}{2}\left(\frac{1}{A^2} - A^2\right)\cos(2\sigma'_2) + \\ & a_1a_2\left(\frac{1}{A} - A\right)\sin(2\sigma'_2). \end{aligned} \tag{5.40}$$

We want to write the last equation as

$$\tilde{c}_1 + \tilde{c}_2 w \;=\; \tilde{c}_3 \tilde{s}\sqrt{1-w^2},$$

where $w = \cos(2\sigma_2')$, $\sin(2\sigma_2') = \tilde{s}\sqrt{1-w^2}$, $\tilde{s} = \pm 1$, $\tilde{c}_1 = \tilde{\gamma}^2 - a_1 - \frac{a_2^2}{2}\left(\frac{1}{A^2} + A^2\right)$, $\tilde{c}_2 = -\frac{a_2^2}{2}\left(\frac{1}{A^2} - A^2\right)$, $\tilde{c}_3 = \tilde{s} a_1 a_2 \left(\frac{1}{A} - A\right)$. Then, we need to solve

$$C_1 w^2 + C_2 w + C_3 \;=\; 0, \tag{5.41}$$

where $C_1 = \tilde{c}_2^2 + \tilde{c}_3^2$, $C_2 = 2\tilde{c}_1\tilde{c}_2$, $C_3 = \tilde{c}_1^2 - \tilde{c}_3^2$. We assume that $C_1 \neq 0$. Note that $C_1 = 0$ if and only if $a_1 = 0$ or $a_2 = 0$. We are interested in the solutions of the above equation that satisfies $w \in [-1, 1]$. A simple computation shows that this fact occurs if and only if

$$\tilde{c}_2^2 + \tilde{c}_3^2 - \tilde{c}_1^2 \geq 0.$$

We have that

$$\tilde{c}_2^2 + \tilde{c}_3^2 - \tilde{c}_1^2 \;=\; -\tilde{\gamma}^4 + 2\tilde{\gamma}^2 d - 1,$$

where $d = 1 + \frac{a_2^2}{2A^2}(1-A^2)^2$. We note that $d \geq 1$. Therefore, $\tilde{\gamma}^2 \in [d - \sqrt{d^2-1}, d + \sqrt{d^2-1}]$. We define $\kappa = d - \sqrt{d^2-1}$.

If $\tilde{\gamma}^2 \in (d - \sqrt{d^2-1}, d + \sqrt{d^2-1})$ then we obtain two solutions of the equation (5.41). If $\tilde{\gamma}^2$ is in the boundary of this interval then we only have one solution.

Using the transversality of the solution obtained considering only the dominant terms in the equations, the Implicit Function Theorem assures the preservation of that solution if μ_k is sufficiently small.

This ends the proof of Theorem 5.3.1.

5.5 Proof of Lemmas

Proof of Lemma 5.4.1

We introduce coordinates $P_X = \dot{X}$, $P_Y = \dot{Y}$, $P_Z = \dot{Z}$ in the equations of the Spatial Hill's Problem. Then, (5.19) can be written as the following system of differential equations of order one

$$\begin{aligned}
\dot{X} &= P_X, \\
\dot{Y} &= P_Y, \\
\dot{Z} &= P_Z, \\
\dot{P}_X &= 2P_Y + \Omega^H_X, \\
\dot{P}_Y &= -2P_X + \Omega^H_Y, \\
\dot{P}_Z &= \Omega^H_Z,
\end{aligned} \tag{5.42}$$

where $\Omega^H_X, \Omega^H_Y, \Omega^H_Z$ denotes the partial derivatives respect to X, Y and Z, respectively, of the function $\Omega^H(X,Y,Z) = \frac{1}{2}[3X^2 - Z^2 + 2(X^2+Y^2+Z^2)^{-\frac{1}{2}}]$.

Let us denote by $\boldsymbol{\varphi}(t, \mathbf{W}_0)$ the solution of (5.42) such that $\boldsymbol{\varphi}(t_0, \mathbf{W}_0) = \mathbf{W}_0$. Let us consider $\mathbf{W}_0^*$ an initial condition for $W^{u,H}_{L_2}$. Then, $W^{u,H}_{L_2}$ is given by $\boldsymbol{\varphi}(t, \mathbf{W}_0^*)$. We denote $\boldsymbol{\varphi}(t, \mathbf{W}_0^*)$ by $(X^*, Y^*, Z^*, P_X^*, P_Y^*, P_Z^*)^T$. We take $\mathbf{W}_0 = \mathbf{W}_0^* + \Delta \mathbf{W}_0$ an initial condition for a solution on $W^{u,H}_T$. We assume that $\mathbf{W}_0, \mathbf{W}_0^*$ are on $Y = -k_0$ with $k_0 > 0$ small, and we take $t_0 = 0$. Then,

$$\begin{gathered} X_0 = X_0^* + \Delta X_0, \quad Y_0 = Y_0^* = -k_0, \quad Z_0 = Z_0^* + \Delta Z_0, \\ P_{X,0} = P^*_{X,0} + \Delta P_{X,0}, \quad P_{Y,0} = P^*_{Y,0} + \Delta P_{Y,0}, \quad P_{Z,0} = P^*_{Z,0} + \Delta P_{Z,0}, \end{gathered}$$

with

$$\Delta X_0, \Delta P_{X,0} = O((\Delta C^H_p)^{\frac{1}{2}}), \qquad \Delta Z_0, \Delta P_{Z,0} = O((\Delta C^H_v)^{\frac{1}{2}}),$$

where $\mathbf{W}_0^* = (X_0^*, Y_0^*, Z_0^*, P^*_{X,0}, P^*_{Z,0}, P^*_{Z,0})^T$ and $\mathbf{W}_0 = (X_0, Y_0, Z_0, P_{X,0}, P_{Z,0}, P_{Z,0})^T$.

We fix $\tilde{k} \in \mathbb{R}^+$. Let us define $t^*(\tilde{k})$ the time for which $\boldsymbol{\varphi}(t, \mathbf{W}_0^*)$ cuts $Y = -\tilde{k}$. Let us denote by $t(\tilde{k})$ the time used for $\boldsymbol{\varphi}(t, \mathbf{W}_0)$ to attain $Y = -\tilde{k}$. Then, $t(\tilde{k}) = t^*(\tilde{k}) + \Delta t$ with Δt small.

Therefore, we have that

$$\begin{aligned} \boldsymbol{\varphi}(t(\tilde{k}), \mathbf{W}_0) &= \boldsymbol{\varphi}(t^*(\tilde{k}), \mathbf{W}_0^*) + \Delta t \dot{\boldsymbol{\varphi}}(t^*(\tilde{k}), \mathbf{W}_0^*) + \\ &\quad \frac{\partial \boldsymbol{\varphi}}{\partial \mathbf{W}_0}(t^*(\tilde{k}), \mathbf{W}_0^*) \Delta \mathbf{W}_0 + O_2, \end{aligned} \tag{5.43}$$

where O_2 denotes terms of order two in Δt and $\Delta \mathbf{W}_0$.

As $\boldsymbol{\varphi}(t, \mathbf{W}_0^*)$ stands for $W^{u,H}_{L_2}$, we know that $Z^* = P_Z^* = 0$. Then, $\dot{\boldsymbol{\varphi}}(t^*(k), \mathbf{W}_0^*))$ is uncoupled in variables $(X, \dot{X}, Y, \dot{Y})$ and $(Z, \dot{Z})$.

Now we compute $\frac{\partial \boldsymbol{\varphi}}{\partial \mathbf{W}_0}(t^*(k), \mathbf{W}_0^*)$. We know that it is the solution of

$$\begin{aligned} \dot{\mathbf{W}} &= A\mathbf{W}, \\ \mathbf{W}(t_0) &= I_6, \end{aligned} \tag{5.44}$$

where $A = D_2G(t, \boldsymbol{\varphi}(t^*(k), \mathbf{W}_0^*))$ being $G = (P_X, P_Y, P_Z, 2P_Y + \Omega^H_X, -2P_X + \Omega^H_Y, \Omega^H_Z)^T$ the vector field defined by (5.42). Then,

$$D_2G = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \Omega^H_{XX} & \Omega^H_{XY} & \Omega^H_{XZ} & 0 & 2 & 0 \\ \Omega^H_{XY} & \Omega^H_{YY} & \Omega^H_{YZ} & -2 & 0 & 0 \\ \Omega^H_{XZ} & \Omega^H_{XY} & \Omega^H_{ZZ} & 0 & 0 & 0 \end{pmatrix}.$$

A simple computation shows that

$$\Omega_{XZ}(X^*,Y^*,Z^*)=\Omega_{YZ}(X^*,Y^*,Z^*)=\Omega_{ZZ}(X^*,Y^*,Z^*)=0.$$

Then, $\dfrac{\partial\boldsymbol{\varphi}}{\partial\mathbf{W}_0}(t^*(k),\mathbf{W}_0^*)$ uncouples the variables $(X,\dot X,Y,\dot Y)$ and $(Z,\dot Z)$.

If we approximate the coordinates X,Y,P_X,P_Y of $\boldsymbol{\varphi}(t^*(k),\mathbf{W}_0^*)$ by its linear part we have that

$$\begin{array}{rcl} X^*(t^*(\tilde k)) &=& 3^{-\frac{1}{3}}+2\lambda c_1 e^{\lambda t^*(\tilde k)},\\ Y^*(t^*(\tilde k)) &=& (\lambda^2-9)c_1 e^{\lambda t^*(\tilde k)},\\ P_X^*(t^*(\tilde k)) &=& 2\lambda^2 c_1 e^{\lambda t^*(\tilde k)},\\ P_Y^*(t^*(\tilde k)) &=& \lambda(\lambda^2-9)c_1 e^{\lambda t^*(\tilde k)}, \end{array}$$

for some $c_1>0$. Then the linear part of $\dfrac{\partial\boldsymbol{\varphi}}{\partial\mathbf{W}_0}(t^*(\tilde k),\mathbf{W}_0^*)$ has the form

$$\begin{pmatrix} * & * & 0 & * & * & 0\\ * & * & 0 & * & * & 0\\ 0 & 0 & * & 0 & 0 & *\\ * & * & 0 & * & * & 0\\ * & * & 0 & * & * & 0\\ 0 & 0 & * & 0 & 0 & * \end{pmatrix},$$

where $*$ denotes some expressions that depend on $t^*(\tilde k)$. We recall that $t^*(\tilde k)$ is the time for which $W_{L_2}^{u,H}$ cuts $Y=-\tilde k$.

As from (5.43) we have that

$$\boldsymbol{\varphi}(t(\tilde k),\mathbf{W}_0)-\boldsymbol{\varphi}(t^*(\tilde k),\mathbf{W}_0^*)-\Delta t\dot{\boldsymbol{\varphi}}(t^*(\tilde k),\mathbf{W}_0^*) \;=\; \frac{\partial\boldsymbol{\varphi}}{\partial\mathbf{W}_0}(t^*(\tilde k),\mathbf{W}_0^*)\Delta\mathbf{W}_0+O_2,$$

this finishes the proof of the Lemma. □

Proof of Lemma 5.4.2

In order to compute the Poincaré map we will find the images $T_{t^*}\mathbf{P}_i$ where $\mathbf{P}_i\in\tilde T$, $i=1,\ldots,4$, being $\tilde T=W_T^{u,H}\cap\{Y=-\tilde k\}$ and $\mathbf{P}_1$ is such that $\xi_1>0$, $\dot\xi_1=0$, $\xi_2=0$, $\dot\xi_2=0$, $\mathbf{P}_2$ is such that $\xi_1=0$, $\dot\xi_1>0$, $\xi_2=0$, $\dot\xi_2=0$, $\mathbf{P}_3$ is such that $\xi_1=0$, $\dot\xi_1=0$, $\xi_2>0$, $\dot\xi_2=0$ and $\mathbf{P}_4$ is such that $\xi_1=0$, $\dot\xi_1=0$, $\xi_2=0$, $\dot\xi_2>0$. We remark that from the geometry of $\tilde T$ we have that $\xi_1,\dot\xi_1=O((\Delta C_p^H)^{\frac{1}{2}})$ and $\xi_2,\dot\xi_2=O((\Delta C_v^H)^{\frac{1}{2}})$.

As $\tilde k>0$ is large enough, the solutions can be approximated by the linear system (5.25). By taking $t=0$ the time for which a solution is on $Y=-\tilde k$, the general solution of this system can be written as

$$\begin{aligned} X(t) &= \frac{2}{3}N + M\cos(t - t_0), \\ Y(t) &= B - Nt - 2M\sin(t - t_0), \\ Z(t) &= A\cos(t - t_0) + D\sin(t - t_0), \end{aligned} \tag{5.45}$$

and their derivatives with respect to t.

We begin with $\mathbf{P}_1$. We denote by $\boldsymbol{\varphi}_1 = (X_1, Y_1, Z_1, \dot{X}_1, \dot{Y}_1, \dot{Z}_1)$ its coordinates in the space $(X, Y, Z, \dot{X}, \dot{Y}, \dot{Z})$. Then, we have that

$$\begin{aligned} X_1 = X_{L_2}(0) + \Delta X, \quad Y_1 = -\tilde{k}, \quad Z_1 = 0, \\ \dot{X}_1 = 0, \quad \dot{Y}_1 = \dot{Y}_{L_2}(0) + \Delta\dot{Y}, \quad \dot{Z}_1 = 0. \end{aligned} \tag{5.46}$$

First we compute the expression of $\Delta\dot{Y}$ in terms of ΔX. To this end we shall use the Jacobi integral (5.21). We note that this integral can be approximate for $Y = -k$, with k large enough, by

$$C^H \approx -(\dot{X}^2 + \dot{Y}^2 + \dot{Z}^2) + 3X^2 - Z^2.$$

From (5.22) we can uncouple this integral in two as

$$C_p^H \approx 3X^2 - \dot{X}^2 - \dot{Y}^2, \tag{5.47}$$

$$C_v^H \approx -Z^2 - \dot{Z}^2. \tag{5.48}$$

Using (5.47) we obtain that

$$\dot{Y}^2 \approx \Delta C_p^H - 3^{\frac{4}{3}} - \dot{X}^2 + 3X^2, \tag{5.49}$$

where $\Delta C_p^H = C_{L_2}^H - C_p^H$. Then, using (5.46), on $\mathbf{P}_1$ we have that

$$\dot{Y}_{L_2}(0)^2 + 2\dot{Y}_{L_2}(0)\Delta\dot{Y} + (\Delta\dot{Y})^2 \approx \Delta C_p^H - 3^{\frac{4}{3}} + 3[X_{L_2}(0)^2 + 2X_{L_2}(0)\Delta X + (\Delta X)^2].$$

On $W_{L_2}^{u,H}$ the value of the Jacobi constant is the same as on the equilibrium point. Moreover, $\Delta X, \Delta\dot{X} = O((\Delta C_p^H)^{\frac{1}{2}})$. Then, we obtain that

$$\Delta\dot{Y} \approx \frac{3X_{L_2}(0)\Delta X}{\dot{Y}_{L_2}(0)} + O(\Delta C_p^H). \tag{5.50}$$

Therefore, (5.46) can be written as

$$\begin{aligned} X_1 = X_{L_2}(0) + \Delta X, \quad Y_1 = -\tilde{k}, \quad Z_1 = 0, \\ \dot{X}_1 = 0, \quad \dot{Y}_1 = \dot{Y}_{L_2}(0) + \frac{3X_{L_2}(0)}{\dot{Y}_{L_2}(0)}\Delta X + O(\Delta C_p^H), \quad \dot{Z}_1 = 0. \end{aligned}$$

Now we determine the constants t_0, B, M, N, A, D in (5.45) for this point. It is clear that $A = D = 0$. We take $M = M_\infty + \Delta M$ and $N = N_\infty + \Delta N$. As $\dot{X}_1 = 0$ we can take $t_0 = 0$. Then, from the equation for Y, we have that $B = -\tilde{k}$. We compute ΔM and ΔN. The equations for X and $\dot{Y}$ give us the following system of linear algebraic equations for ΔM and ΔN.

$$\begin{aligned} \frac{2}{3}\Delta N + \Delta M &= \Delta X, \\ \Delta N + 2\Delta M &= -\Delta \dot{Y}. \end{aligned}$$

Solving the system above and using (5.50) we obtain

$$\begin{aligned} \Delta M &= \frac{N_\infty}{N_\infty + 2M_\infty}\Delta X, \\ \Delta N &= \frac{3M_\infty}{N_\infty + 2M_\infty}\Delta X. \end{aligned} \tag{5.51}$$

We denote by $\mathbf{P}_1^*$ the end point, that is, $\mathbf{P}_1^*$ is the intersection of $W_{L_2}^{u,H}$ with the hyperplane $Y = -\tilde{\tilde{k}}$ in coordinates $\xi_1, \dot{\xi}_1, \xi_2, \dot{\xi}_2$. We shall denote by $\boldsymbol{\varphi}_1^*$ the coordinates of $\mathbf{P}_1^*$ in the variables $(X, Y, Z, \dot{X}, \dot{Y}, \dot{Z})$. We can write $\boldsymbol{\varphi}_1^* = \varphi_{L_2} + \Delta\varphi_1$, where $\boldsymbol{\varphi}_{L_2}$ is the final point for $W_{L_2}^{u,H}$. From (5.26) we have that the coordinates of $\boldsymbol{\varphi}_{L_2}$ are given by

$$\begin{aligned} X_{L_2}^* = \frac{2}{3}N_\infty + M_\infty \cos t^*, \qquad Y_{L_2}^* = -\tilde{\tilde{k}}, \qquad Z_{L_2}^* = 0, \\ \dot{X}_{L_2}^* = -M_\infty \sin t^*, \qquad \dot{Y}_{L_2}^* = -N_\infty - 2M_\infty \cos t^*, \qquad \dot{Z}_{L_2}^* = 0. \end{aligned}$$

From the definition of t^* and using (5.26) we have that

$$-\tilde{k} - N_\infty t^* - 2M_\infty \sin t^* \quad = \quad -\tilde{\tilde{k}}. \tag{5.52}$$

As

$$-\tilde{\tilde{k}} = Y(t^* + \Delta t) = -\tilde{k} - Nt^* + N\Delta t - 2M(\sin(t^* + \Delta t \cos t^*)) + O((\Delta t)^2),$$

from (5.52) we have that

$$\Delta t \quad = \quad -\frac{t^*\Delta N + 2\Delta M \sin t^*}{N_\infty + 2M_\infty \cos t^*} + O(\Delta C_p^H). \tag{5.53}$$

A simple computation shows that

$$X_1^* = X_1(t^* + \Delta t) = X_{L_2} + \Delta X^*,$$

where $\Delta X^* = \dfrac{2}{3}\Delta N + \Delta M \cos t^* - M_\infty \Delta t \sin t^* + O((\Delta t)^2)$. Using (5.51) and (5.53) we obtain that

$$\Delta X^* = \frac{\Delta X[4N_\infty M_\infty + (4M_\infty^2 + N_\infty^2)\cos t^* + 3M_\infty^2 t^* \sin t^*]}{(N_\infty + 2M_\infty)(N_\infty + 2M_\infty \cos t^*)}.$$

Similar computations can be done in order to obtain $\dot{X}_1^*$. We obtain that

$$\dot{X}_1^* = \dot{X}_{L_2} + \Delta\dot{X},$$

where

$$\Delta\dot{X} = \frac{\Delta X}{(N_\infty + 2M_\infty)(N_\infty + 2M_\infty \cos t^*)}(-N_\infty^2 \sin t^* + 3M_\infty^2 t^* \cos t^*).$$

The computation of Z_1^* and $\dot{Z}_1^*$ is easiest due to the fact that $A = D = 0$. We obtain that $Z_1^* = \dot{Z}_1^* = 0$. With this, we have obtained the first column in T_{t^*}.

Analogously, by computing the images of $\mathbf{P}_2$, $\mathbf{P}_3$ and $\mathbf{P}_4$ on $W_T^{u,H} \cap \{Y = -\tilde{\tilde{k}}\}$ one can obtain the complete first order terms of the Poincaré map T_{t^*}. $\square$

Proof of Lemma 5.4.3

We begin by taking an initial condition on $W_T^{u,\mu} \cap \{y = -k\mu^{\frac{1}{3}}\}$ with $k > 0$ large enough. From the geometry of $W_T^{u,\mu} \cap \{y = -k\mu^{\frac{1}{3}}\}$ we can take initial conditions of the form

$$\begin{aligned} x_0 &= x_{L_2} + \Delta x, \quad &, y_0 = -k\mu^{\frac{1}{3}} \quad & z_0 = z_{L_2} + \Delta z, \\ \dot{x}_0 &= \dot{x}_{L_2} + \Delta\dot{x}, \quad &\dot{y}_0 = \dot{y}_{L_2} + \Delta\dot{y}, \quad & \dot{z}_0 = \dot{z}_{L_2} + \Delta\dot{z}, \end{aligned} \tag{5.54}$$

being $(x_{L_2}, y_{L_2}, z_{L_2}, \dot{x}_{L_2}, \dot{y}_{L_2}, \dot{z}_{L_2})$ the coordinates of $W_{L_2}^{u,\mu}$ and

$$\Delta x, \Delta y, \Delta\dot{x} = O((\Delta C_p)^{\frac{1}{2}}), \quad \Delta z, \Delta\dot{z} = O((\Delta C_v)^{\frac{1}{2}}).$$

In terms of the Hill's coordinates (5.18) we have that

$$X_0 = X_{L_2} + \Delta X, \quad Y = -k, \quad Z = Z_{L_2} + \Delta Z,$$

where $X_{L_2} = \mu^{-\frac{1}{3}}(x_{L_2} + 1 - \mu)$, $Z_{L_2} = \mu^{-\frac{1}{3}} z_{L_2}$, $\Delta X = \mu^{-\frac{1}{3}}\Delta x$ and $\Delta Z = \mu^{-\frac{1}{3}}$. Moreover, $\Delta X = O((\Delta C_p^H)^{\frac{1}{2}})$ and $\Delta Z = O((\Delta C_v^H)^{\frac{1}{2}})$. As k is large enough, we can approximate C_p^H and C_v^H by (5.47) and (5.48), respectively. Then, $\dot{Y}_{L_2} \approx 3X^2 - \dot{X}^2 + \Delta C_p^H - C_{L_2^H}$ where $C_{L_2^H} = 3X_{L_2}^2 - \dot{X}_{L_2}^2 - \dot{Y}_{L_2}^2$. As $\dot{Y} = \dot{Y}_{L_2} + \Delta\dot{Y}$, we obtain that

$$\Delta\dot{Y} \approx \frac{1}{\dot{Y}_{L_2}}(3X_{L_2}\Delta X - \dot{X}_{L_2}\Delta\dot{X}) + O(\Delta C_p^H).$$

From the expression of $W^H_{L_2}$ given in (5.26) we have that

$$\Delta \dot{Y} \approx -\frac{1}{N_\infty + 2M_\infty \cos t}[(2N_\infty + 3M_\infty \cos t)\Delta X + M_\infty \sin t \Delta \dot{X}],$$

and so,

$$\Delta \dot{y} \approx -\frac{1}{N_\infty + 2M_\infty \cos t}[(2N_\infty + 3M_\infty \cos t)\Delta x + M_\infty \sin t \Delta \dot{x}].$$

A simple computation shows that

$$\begin{aligned}
x_{L_2} &= -1+\mu+\left(\frac{2}{3}N_\infty + M_\infty \cos t\right)\mu^{\frac{1}{3}},\\
\dot{x}_{L_2} &= -\mu^{\frac{1}{3}} M_\infty \sin t,\\
\dot{y}_{L_2} &= -(N_\infty + 2M_\infty \cos t)\mu^{\frac{1}{3}},\\
z_{L_2} &= \dot{z}_{L_2} = 0.
\end{aligned}$$

Then, the initial condition (5.54) can be written as

$$\begin{aligned}
x_0 &= -1+\mu+\left(\frac{2}{3}N_\infty + M_\infty \cos t\right)\mu^{\frac{1}{3}} + \Delta x,\\
\dot{x} &= -M_\infty \sin t \mu^{\frac{1}{3}} + \Delta \dot{x},\\
y &= -k\mu^{\frac{1}{3}},\\
\dot{y} &= -(N_\infty + 2M_\infty \cos t)\mu^{\frac{1}{3}} - (N_\infty + 2M_\infty \cos t)^{-1}[(2N_\infty + \\
&\quad 3M_\infty \cos t)\Delta x + M_\infty \sin t \Delta \dot{x}],\\
z &= \Delta z,\\
\dot{z} &= \Delta \dot{z}.
\end{aligned}$$

We assume that $W^{u,\mu}_{L_2}$ needs a time τ in order to attain $y=0, x<0$ when we start at $y=-k$ and go back, and that for this value of τ the ellipse near $(x,\dot{x})$ has the axes parallel to x and $\dot{x}$, respectively. In fact, y is not exactly zero because we select τ with the smallest y such that the ellipse in $(x,\dot{x})$ is in suitable position.

As the ellipse near $(x,\dot{x})$ is centered approximately at $(x_{L_2}, \dot{x}_{L_2})$ and has semi-axes proportional to $(\Delta C_p)^{\frac{1}{2}}$, we can write

$$\begin{aligned}
\Delta x &= k_1(\Delta C_p)^{\frac{1}{2}} \cos \sigma_1,\\
\Delta \dot{x} &= k_2(\Delta C_v)^{\frac{1}{2}} \sin \sigma_1,
\end{aligned}$$

where k_1, k_2 are suitable constants and σ_1 is a parameter on the ellipse. For the ellipse near the $(z,\dot{z})$ plane we can write

$$\begin{aligned}
\Delta z &= \tilde{\rho}(\cos\gamma \cos\sigma_2 - A\sin\gamma \sin\sigma_2)(\Delta C_v)^{\frac{1}{2}},\\
\Delta \dot{z} &= \tilde{\rho}(\cos\gamma \sin\sigma_2 + A\sin\gamma \cos\sigma_2)(\Delta C_v)^{\frac{1}{2}},
\end{aligned}$$

being $,\gamma, \tilde{\rho}, A$ suitable constants and σ_2 a parameter on the ellipse.

We shall denote by $(\tilde{\xi}_1, \dot{\tilde{\xi}}_1, \tilde{\xi}_2, \dot{\tilde{\xi}}_2)$ a point with minimum $|y|$ and the ellipse near $(x, \dot{x})$ in suitable position, and $(\xi_1, \dot{\xi}_1, \xi_2, \dot{\xi}_2)$ its image on $Y = -k$. We recall that the coordinates ξ_j are defined in (5.27) and they are expressed in Hill's coordinates. From Lemma 5.4.2 we have that

$$\begin{pmatrix} \tilde{\xi}_2 \\ \dot{\tilde{\xi}}_2 \end{pmatrix} = T^{-1}_{v,-\tau} \begin{pmatrix} \xi_2 \\ \dot{\xi}_2 \end{pmatrix} = \begin{pmatrix} \cos\tau & \sin\tau \\ -\sin\tau & \cos\tau \end{pmatrix} \begin{pmatrix} \Delta Z \\ \Delta \dot{Z} \end{pmatrix}.$$

Then, we can take as initial condition the one given in the statement of the Lemma. □

Proof of Lemma 5.4.4

Let us consider (x, y, z) synodic coordinates for the spatial two body problem. In these variables, the equations of motion are given by

$$\begin{aligned} \ddot{x} - 2\dot{y} &= x\left(1 - \frac{1}{r^3}\right), \\ \ddot{y} + 2\dot{x} &= y\left(1 - \frac{1}{r^3}\right), \\ \ddot{z} &= -\frac{z}{r^3}, \end{aligned} \tag{5.55}$$

where $r^2 = x^2 + y^2 + z^2$.

We want to perform a change of variables in order to reduce the system above to the planar Kepler problem. To this end, we begin performing the change of variables $(x, y, z) \longrightarrow (q_1, q_2, q_3)$ given by

$$\begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}. \tag{5.56}$$

Then, system (5.55) is written in sideral coordinates by

$$\ddot{q}_i = -\frac{q_i}{r^3}, \quad i = 1, 2, 3,$$

with $r^2 = q_1^2 + q_2^2 + q_3^2$. Then, we obtain the Spatial Kepler's problem with $\mu = 1$,

$$\ddot{\mathbf{q}} = -\frac{\mathbf{q}}{r^3}, \quad \mathbf{q} = (q_1, q_2, q_3), \quad r = \|\mathbf{q}\|. \tag{5.57}$$

This problem has $\boldsymbol{\omega} = \mathbf{q} \wedge \dot{\mathbf{q}}$ as first integral, and if $\boldsymbol{\omega} \neq 0$ then the movement takes place in the ortogonal plane to $\boldsymbol{\omega}$. That is, if $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$ with $\boldsymbol{\omega} \neq 0$, then, given an initial condition $\mathbf{q}_0 = (q_1^0, q_2^0, q_3^0)$, $\dot{\mathbf{q}}_0 = (\dot{q}_1^0, \dot{q}_2^0, \dot{q}_3^0)$, the movement

takes place in the plane $\omega_1 q_1 + \omega_2 q_2 + \omega_3 q_3 = 0$, where $\boldsymbol{\omega} = \mathbf{q}_0 \wedge \dot{\mathbf{q}}_0$.
To reduce (5.57) to the Planar Kepler's problem, we introduce new variables $\mathbf{r} = (r_1, r_2, r_3)$ defined by

$$\mathbf{q} \;=\; A\mathbf{r}, \quad A = \begin{pmatrix} \frac{\omega_3}{\tilde{\omega}} & -\frac{\omega_1\omega_2}{\omega\tilde{\omega}} & \frac{\omega_1}{\omega} \\ 0 & \frac{\tilde{\omega}}{\omega} & \frac{\omega_2}{\omega} \\ -\frac{\omega_1}{\tilde{\omega}} & -\frac{\omega_2\omega_3}{\omega\tilde{\omega}} & \frac{\omega_3}{\omega} \end{pmatrix}, \tag{5.58}$$

where $\omega = \|\boldsymbol{\omega}\|$, $\tilde{\omega} = \sqrt{\omega_1^2 + \omega_3^2}$. We note that A is an orthogonal matrix. Then, equations (5.57) are reduced to the equations of the Planar Kepler's problem

$$\ddot{\mathbf{r}} \;=\; -\frac{\mathbf{r}}{r^3}, \;\text{ with }\; \mathbf{r} = (r_1, r_2), \;\; r = \|\mathbf{r}\|. \tag{5.59}$$

We note that the motion takes place in the plane $r_3 = 0$ and then, we do not consider this coordinate.

We take initial conditions (5.28). We can write these as

$$\begin{aligned} x_0 &= -1 + \alpha_1\mu_k^{\frac{1}{3}} + \left(\frac{\alpha}{3}\alpha_1 + \alpha_3\right)\mu_k^{\frac{2}{3}} + O(\mu_k), \\ \dot{x}_0 &= -M_\infty \sin\tau\mu_k^{\frac{1}{3}} + \left(-\frac{\alpha}{3}M_\infty + \alpha_4\right)\mu_k^{\frac{2}{3}} + O(\mu_k), \\ y_0 &= 0, \\ \dot{y}_0 &= -\alpha_2\mu_k^{\frac{1}{3}} - \left(\frac{\alpha}{3}\alpha_2 + \frac{3\alpha_1\alpha_3 + M_\infty\alpha_4\sin\tau}{\alpha_2}\right)\mu_k^{\frac{2}{3}} + O(\mu_k), \\ z_0 &= \tilde{z}_0\mu_k^{\frac{2}{3}} + O(\mu_k), \\ \dot{z}_0 &= \dot{\tilde{z}}_0\mu_k^{\frac{2}{3}} + O(\mu_k), \end{aligned} \tag{5.60}$$

where $\alpha_1 = \dfrac{2}{3}N_\infty + M_\infty\cos\tau$, $\alpha_2 = N_\infty + 2M_\infty\cos\tau$, $\alpha_3 = k_1\tilde{\beta}_1\cos\sigma_1$, $\alpha_4 = k_2\tilde{\beta}_1\sin\sigma_2$, $\tilde{z}_0 = \tilde{\beta}_2(k_3\cos\sigma_2 + k_4\sin\sigma_2)$, $\dot{\tilde{z}}_0 = \tilde{\beta}_2(k_5\cos\sigma_2 + k_6\sin\sigma_2)$, $k_6 = Ak_3 = A\tilde{\rho}\cos\gamma$, $k_4 = -Ak_5 = -A\tilde{\rho}\sin\gamma$.

We want to obtain the first cut of the solution of SSTBP with initial conditions (5.60) with $y = 0, x > 0$. Let us denote by t_f the time in which this fact occurs. Let us assume that we know the expressions of r_1 and r_2. Then, we also know r and $\dot{r}$. Let us see how to obtain the expressions of $x(t_f), z(t_f), \dot{x}(t_f), \dot{z}(t_f)$ from these terms.

First, we recall that

$$r^2 \;=\; x^2 + y^2 + z^2, \tag{5.61}$$

where $r^2 = r_1^2 + r_2^2$ and r_1, r_2 is the solution of the Planar Kepler problem. Then,

$$r^2(t_f) \;=\; x^2(t_f) + z^2(t_f). \tag{5.62}$$

Moreover, from the changes of coordinates (5.56) and (5.58), we can write

$$z = -\frac{\omega_1}{\tilde{\omega}} r_1 - \frac{\omega_2\omega_3}{\omega\tilde{\omega}} r_2. \tag{5.63}$$

From these two relations we obtain $x(t_f)$ and $z(t_f)$.

Now we want to see how to obtain $\dot{x}(t_f)$ and $\dot{z}(t_f)$. From (5.61) we have that

$$x\dot{x} + y\dot{y} + z\dot{z} = r\dot{r}.$$

If we evaluate on t_f we obtain

$$x(t_f)\dot{x}(t_f) + z(t_f)\dot{z}(t_f) = r(t_f)\dot{r}(t_f). \tag{5.64}$$

$x(t_f), z(t_f), r(t_f)$ and $\dot{r}(t_f)$ are known. Moreover, from (5.56) and (5.58) we have that

$$\dot{z} = -\frac{\omega_1}{\tilde{\omega}} \dot{r}_1 - \frac{\omega_2\omega_3}{\omega\tilde{\omega}} \dot{r}_2.$$

Then, we can compute $\dot{z}(t_f)$ and after $\dot{x}(t_f)$ from (5.64).

Now we will explain how to compute r_1, r_2 and so, r and $\dot{r}$. Let us consider (r_1, r_2) an elliptic solution of (5.59). It is well–known that

$$\begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = \Omega(\delta) \begin{pmatrix} \tilde{r}_1 \\ \tilde{r}_2 \end{pmatrix}, \tag{5.65}$$

where $\tilde{r}_1 = a(\cos E - e)$, $\tilde{r}_2 = a\sqrt{1-e^2}\sin E$, $\Omega(\delta) = \begin{pmatrix} \cos\delta & -\sin\delta \\ \sin\delta & \cos\delta \end{pmatrix}$, and a is the major axis of the ellipse, e the eccentricity, E the eccentric anomaly and δ denotes the argument of the pericenter.

These orbital parameters satisfy the following relations (see [S.S.])

$$\begin{aligned} \frac{1}{a} &= \frac{2}{r_0} - v_0^2, \\ e\cos E_0 &= r_0 v_0^2 - 1, \\ e\sin E_0 &= r_0\dot{r}_0 a^{-\frac{1}{2}}, \\ V_0 + \delta &= \text{Arg } \mathbf{r}_0, \end{aligned} \tag{5.66}$$

where E_0, V_0 denotes the initial eccentric and true anomalies, $r_0 = \|\mathbf{r}_0\|$, $v_0 = \|\dot{\mathbf{r}}_0\|$ and $r_0\dot{r}_0 =< \mathbf{r}_0, \dot{\mathbf{r}}_0 >$.

Now we want to express these orbital parameters in synodic coordinates. To this end, we only need to take into account that $\omega_1 = y_0\dot{z}_0 - z_0(x_0 + \dot{y}_0)$, $\omega_2 = z_0(\dot{x}_0 - y_0) - x_0\dot{z}_0$, $\omega_3 = x_0(x_0 + \dot{y}_0) - y_0(\dot{x}_0 - y_0)$. We note that $\omega_1, \omega_2 = O(\mu_k^{\frac{2}{3}})$, $\omega_3, \omega, \tilde{\omega} = O_1$.

Then,

$$\mathbf{r}_0 = \begin{pmatrix} \frac{\omega_3}{\tilde{\omega}} x_0 - \frac{\omega_1}{\tilde{\omega}} z_0 \\ -\frac{\omega_1\omega_2}{\omega\tilde{\omega}} x_0 + \frac{\tilde{\omega}}{\omega} y_0 - \frac{\omega_2\omega_3}{\omega\tilde{\omega}} z_0 \end{pmatrix}.$$

Using this expression for $\mathbf{r}_0$ we can compute all the orbital parameters in terms of the initial condition for a solution in synodic coordinates.

A straight forward computation shows that

$$\begin{aligned}
r_0 &= 1 - \alpha_1 \mu_k^{\frac{1}{3}} - \left(\frac{\alpha}{3}\alpha_1 + \alpha_3\right) \mu_k^{\frac{2}{3}} + O(\mu_k), \\
v_0^2 &= 1 + 2(\alpha_2 - \alpha_1)\mu_k^{\frac{1}{3}} + \left[\frac{N_\infty}{3}(3\alpha_2 - 4\alpha_1) + \frac{2\alpha}{3}(\alpha_2 - \alpha_1) + \right. \\
&\quad \left. 2\frac{M_\infty \alpha_4 \sin\tau - \alpha_3(\alpha_2 - 3\alpha_1)}{\alpha_2} + M_\infty^2\right] \mu_k^{\frac{2}{3}} + O(\mu_k), \\
r_0 \dot{r}_0 &= M_\infty \sin\tau \mu_k^{\frac{1}{3}} + \left[-\alpha_4 + M_\infty \sin\tau \left(\frac{\alpha}{3} - \alpha_1\right)\right] \mu_k^{\frac{2}{3}} + O(\mu_k), \\
a &= 1 - \frac{2N_\infty}{3}\mu_k^{\frac{1}{3}} + \left[\frac{4}{9}N_\infty^2 - 2\alpha_1^2 + \frac{N_\infty}{3}(3\alpha_2 - 4\alpha_1) + \frac{2\alpha}{3}(\alpha_2 - 2\alpha_1) + \right. \\
&\quad \left. \frac{2M_\infty}{\alpha_2}(\alpha_4 \sin\tau - \alpha_3 \cos\tau) + M_\infty^2\right] \mu_k^{\frac{2}{3}} + O(\mu_k), \\
e &= M_\infty \mu_k^{\frac{1}{3}} + \left\{\frac{M_\infty}{3}(\alpha - N_\infty) + \frac{N_\infty}{\alpha_2}(\alpha_3 \cos\tau - \alpha_4 \sin\tau) + \right. \\
&\quad \left. \cos\tau[N_\infty(\alpha_1 - \alpha_2) - M_\infty^2 \cos^2\tau]\right\} \mu_k^{\frac{2}{3}} + O(\mu_k), \\
\mathrm{Arg}(r_0) &= \pi + O(\mu_k^{\frac{4}{3}}).
\end{aligned}$$

Now, in order to compute E_0 it is only necessary to take into account that from (5.66) we have that $\tan E_0 = \dfrac{r_0 \dot{r}_0 a^{-\frac{1}{2}}}{r_0 v_0^2 - 1}$. Then, we obtain

$$\begin{aligned}
E_0 &= \tau + 2k\pi + \frac{1}{M_\infty}\left\{\sin\tau\left[N_\infty(\alpha_2 - \alpha_1) - M_\infty^2 \sin^2\tau - \right.\right. \\
&\quad \left.\left. \frac{2M_\infty \alpha_4 \sin\tau + N_\infty \alpha_3}{\alpha_2}\right] - \alpha_4 \cos\tau\right\} \mu_k^{\frac{1}{3}} + O(\mu_k^{\frac{2}{3}}).
\end{aligned}$$

We want to know the expression of V_0 and δ. In order to obtain the expression of V_0, we need the relation between the eccentric anomaly E and the true anomaly V. We have that

$$V = E + e \sin E + \frac{e^2}{2} \sin E \cos E + O(e^3). \qquad (5.67)$$

Using (5.67) we obtain that

$$\begin{aligned} V_0 &= \tau + 2k\pi + \frac{1}{M_\infty}\left\{\sin\tau\left[N_\infty(\alpha_2-\alpha_1) + M_\infty^2\cos^2\tau - \right.\right. \\ &\left.\left. \frac{2M_\infty\alpha_4\sin\tau + N_\infty\alpha_3}{\alpha_2}\right] - \alpha_4\cos\tau\right\}\mu_k^{\frac{1}{3}} + O(\mu_k^{\frac{2}{3}}), \end{aligned}$$

and then, from the equality $V_0 + \delta = \mathrm{Arg}(r_0)$ given in (5.66) we have that

$$\begin{aligned} \delta &= \pi - \tau - 2k\pi - \frac{1}{M_\infty}\left\{\sin\tau\left[N_\infty(\alpha_2-\alpha_1) + M_\infty^2\cos^2\tau - \right.\right. \\ &\left.\left. \frac{2M_\infty\alpha_4\sin\tau + N_\infty\alpha_3}{\alpha_2}\right] - \alpha_4\cos\tau\right\}\mu_k^{\frac{1}{3}} + O(\mu_k^{\frac{2}{3}}). \end{aligned}$$

We are interested in to express the solution in terms of the mean anomaly M. To this end, it will be useful the relation between the eccentric anomaly E and M, that it is given by

$$E = M + e\sin M + e^2\sin M\cos M + O(e^3). \tag{5.68}$$

Now we compute r_1 and r_2. From the relation (5.65) first we need to compute $\tilde{r}_1$ and $\tilde{r}_2$. We have that

$$\tilde{r}_1 = \cos M + O(\mu_k^{\frac{1}{3}}) \quad \text{and} \quad \tilde{r}_2 = \sin M + O(\mu_k^{\frac{1}{3}}).$$

Then,

$$r_1 = \cos(M+\pi-\tau) + O(\mu_k^{\frac{1}{3}}) \quad \text{and} \quad r_2 = \sin(M+\pi-\tau) + O(\mu_k^{\frac{1}{3}}).$$

From (5.63) we have that

$$z = [\tilde{z}_0\cos(M-\tau) + \dot{\tilde{z}}_0\sin(M-\tau)]\mu_k^{\frac{2}{3}} + O(\mu_k).$$

We denote by M_f the mean anomaly at physic time t_f. From (5.62) we have that $x(M_f) = r(M_f) + O(\mu_k)$. Therefore, in order to obtain $x(M_f)$ we need to compute $r = a(1 - e\cos E)$. We have that

$$\begin{aligned} r &= 1 - \left(\frac{2}{3}N_\infty + M_\infty\cos M\right)\mu_k^{\frac{1}{3}} + \left\{(2M_\infty + N_\infty\cos M)(\alpha_4\sin\tau - \alpha_3\cos\tau)\alpha_2^{-1} - \right. \\ &\frac{\alpha}{3}\left(\frac{2}{3}N_\infty + M_\infty\cos M\right) + M_\infty^2\sin^2 M + (\cos M\cos\tau - 2)\left(\frac{1}{3}N_\infty^2 + \right. \\ &\left.\left. N_\infty M_\infty\cos\tau + M_\infty^2\cos^2\tau\right) + \frac{1}{3}N_\infty^2 + N_\infty M_\infty\cos M + M_\infty^2\right\}\mu_k^{\frac{2}{3}} + O(\mu_k). \end{aligned}$$

For $W^{u,\mu_k}_{L_2}$ is known (see [L.M.S.]) that the mean anomaly of the first cut of this manifold with $y = 0, x > 0$ is $M_w = 0$ or π. If x_w denotes the value of x for $W^{u,\mu_k}_{L_2}$ with $M_w = 0$ we have that

$$\begin{aligned} x_w &= 1 - \left(\frac{2}{3}N_\infty + M_\infty\right)\mu_k^{\frac{1}{3}} + \left\{\frac{1}{3}N_\infty^2\cos\tau + N_\infty M_\infty\cos^2\tau + M_\infty^2\cos^3\tau - \right. \\ & \left. \frac{1}{3}N_\infty^2 - 2N_\infty M_\infty\cos\tau - 2M_\infty^2\cos^2\tau + N_\infty M_\infty + M_\infty^2\right\}\mu_k^{\frac{2}{3}} + O(\mu_k). \end{aligned}$$

Then, we obtain the value of $x(M_f)$ and $z(M_f)$ in the lemma.

Now we will obtain $\dot{x}(M_f)$ and $\dot{z}(M_f)$. We have that

$$\dot{\tilde{r}}_1 = -\frac{a^{\frac{1}{2}}}{r}\sin E \quad \text{and} \quad \dot{\tilde{r}}_2 = \frac{a^{\frac{1}{2}}}{r}\sqrt{1-e^2}\cos E.$$

Then, $\dot{\tilde{r}}_1 = -\sin M + O(\mu_k^{\frac{1}{3}})$ and $\dot{\tilde{r}}_2 = \cos M + O(\mu_k^{\frac{1}{3}})$. A simple computation shows that

$$\dot{z} = [-\tilde{z}_0\sin(M-\tau) + \dot{\tilde{z}}_0\cos(M-\tau)]\mu_k^{\frac{2}{3}} + O(\mu_k).$$

Then, using that $\dot{x} = (a^{\frac{1}{2}}e\sin E - z\dot{z})x^{-1}$, we obtain the value of $\dot{x}(M_f)$ given in the lemma.

Last step is to obtain the equation for M_f. We take $M_f = M_0 + n(V_f + \delta)$ where n is obtained from the relation $n^2a^3 = 1$. After some computations, one obtains the equation given in the statement of the Lemma.

□

Proof of Lemma 5.4.7

We assume that $\cos a_0 = 0$. Then $a_0 = \frac{\pi}{2} + n\pi$ with $n \in \mathbb{N}$ and equation (5.34) transforms in

$$\frac{N_\infty}{2}\pi(1 + 2n - 2k) \pm 2M_\infty = 0,$$

and then $1 + 2n - 2k = \mp\frac{4M_\infty}{N_\infty\pi}$, where we recall that $N_\infty = 5.1604325\ldots$ and $M_\infty = 2.1320587\ldots$ and then $\left|\frac{4M_\infty}{N\infty\pi}\right| < 1$. Therefore, it is no possible this value of a_0.

Now we assume that $\sin a_0 = 0$. So, $a_0 = n\pi$ with $n \in \mathbb{N}$. Equation (5.35) can be written as

$$\frac{2sN_\infty^2}{3\pi}\pi(n-k)\left(2 + \frac{N_\infty}{M_\infty}\right) = 0.$$

Then, $k = n$. Moreover, equation (5.34) transforms in $M_\infty(a_1 + b_1) = 0$ and therefore $b_1 = -a_1$.

Now we consider the general case in which $\cos a_0, \sin a_0 \neq 0$. If we multiply (5.34) by $\sin a_0$ and (5.35) by $\cos a_0$ and we substract the resultant equations, rearranging terms yields

$$\frac{2sN_\infty}{3\pi}\left(\frac{N_\infty}{M_\infty} + 2\cos a_0\right)[N_\infty(a_0 - k\pi) + 2M_\infty \sin a_0].$$

We note that the equation $\frac{N_\infty}{M_\infty} + 2\cos a_0 = 0$ gives no solution due to the fact that $\frac{N_\infty}{M_\infty} > 2$. From the other hand, $N_\infty(a_0 - k\pi) + 2M_\infty \sin a_0$ has a unique solution, located at $a_0 = k\pi$. But this give a contradiction because we have assumed that $\sin a_0 \neq 0$. This ends the proof of the lemma. $\square$

Chapter 6

Invariant tori in the centre manifold of collinear points in the Planar TBP

In this chapter we study the orbits in a neighbourhood of the collinear points of the Planar Three Body Problem with Newtonian potential for all positive values of the masses. Given admissible masses m_1, m_2, m_3, the collinear equilibrium points are centre–centre–saddle points. As in chapter 1 we shall consider the collinear equilibrium with the body of mass m_2 in the middle. Denote by $\pm\lambda, \pm i, \pm i\omega$ the eigenvalues of the linear part. So, in a neighbourhood of the equilibrium point the quadratic part of the Hamiltonian can be written as

$$H_2(\boldsymbol{\xi}, \boldsymbol{\eta}) = \lambda \xi_1 \eta_1 + \frac{1}{2}(\xi_2^2 + \eta_2^2) + \frac{\omega}{2}(\xi_3^2 + \eta_3^2).$$

The equilibrium point has an one–dimensional unstable manifold, an one–dimensional stable one and a four–dimensional centre manifold. Two families of periodic orbits which are born at the equilibrium point live on the centre manifold: the Lyapunov family with a period approaching $\dfrac{2\pi}{\omega}$ when the periodic orbits tend to the point, and the homographic family of periodic orbits with a period tending to 2π. However, for the quadratic part $H_2(\boldsymbol{\xi}, \boldsymbol{\eta})$, these periodic orbits are surrounded by two–dimensional invariant tori. The preservation of these invariant tori for the complete Hamiltonian is guaranteed by KAM theorem under some nondegeneracy conditions.

The purpose of this chapter is to study the applicability of KAM theorem in a neighbourhood of the collinear points for any values of the masses. To this end we do the following steps. First, we perform some canonical transformations to write the Hamiltonian in Normal Form. Then we reduce the Hamiltonian to the centre manifold. After that, we check, by numerical evaluation of the coefficients of the Normal Form, the nondegeneracy conditions of KAM theorem. The results presented in section 6.4 show that both conditions (either isoenergetic or not) are satisfied for any values of the masses in the triangle of masses.

However, we remark that for some values of the masses the eigenvalues at the collinear points are $\pm\lambda, \pm i, \pm 2i$ and a resonance $2:1$ takes place. In fact, $1 < \omega < \sqrt{8\sqrt{2}-3} \approx 2.88335022...$ So, up to order 4, we only need to take into account the resonance $2:1$. The corresponding resonant masses describe a curve in the triangle of masses. Therefore, for resonant masses it is expected to get resonant monomials of order three in the Normal Form of the Hamiltonian. We prove in section 6.3 that this is not the case. In fact, we prove that the coefficients of these monomials are different from zero for general masses but they become zero for resonant masses, and also in the symmetrical case $m_1 = m_3$. The existence of the homographic solutions allows us to compute analytically, in an easy way, the coefficients of the resonant monomials of order three. These coefficients have $(\omega - 2)$ as a factor.

6.1 Reduction of the order

We consider the equations of the Planar Three Body Problem with Newtonian potential (PTBPN). We know that the equations of this problem can be written as a Hamiltonian system with six degrees of freedom with Hamiltonian function

$$H(\mathbf{q},\mathbf{p}) = \frac{1}{2}\mathbf{p}^T M^{-1}\mathbf{p} - U(\mathbf{q}), \qquad (6.1)$$

where $\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3)^T$, $\mathbf{p} = (\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3)^T$, $\mathbf{q}_i, \mathbf{p}_i \in \mathbb{R}^2$,

$$M = \mathrm{diag}(m_1, m_1, m_2, m_2, m_3, m_3)$$

and

$$U(\mathbf{q}) = \sum_{1\le i<j\le 3} \frac{m_i m_j}{\|\mathbf{q}_i - \mathbf{q}_j\|}.$$

We recall that we have assumed that $m_1 + m_2 + m_3 = 1$.

We are interested in the solutions near the collinear equilibrium points.

In section 1.3 we have seen that the homographic solutions are equilibrium points of the system once a suitable rotating and pulsating coordinate system is introduced. Moreover, using the integrals of the centre of masses we have reduced the Hamiltonian to one with four degrees of freedom. As we are interested in the equilibrium points we have that the equations of motion (1.16) can be written as

$$\begin{aligned}
\dot{\mathbf{u}}_i &= \frac{1}{m_i}\mathbf{v}_i + \frac{1}{m_3}(\mathbf{v}_1+\mathbf{v}_2) + J_2\mathbf{u}_i, \qquad i=1,2,\\
\dot{\mathbf{v}}_i &= \frac{\partial U}{\partial \mathbf{u}_i} + J_2\mathbf{v}_i, \quad i=1,2,
\end{aligned}$$

where, as we did before, $J_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and the Hamiltonian function is

$$H(\mathbf{u},\mathbf{v}) = T(\mathbf{v}) - U(\mathbf{u}) - Q(\mathbf{u},\mathbf{v}),$$

where

$$\begin{aligned}
T(\mathbf{v}) &= \frac{1}{2m_3}\|\mathbf{v}_1+\mathbf{v}_2\|^2 + \sum_{i=1}^{2}\frac{1}{2m_i}\|\mathbf{v}_i\|^2,\\
U(\mathbf{u}) &= \frac{m_1m_2}{\|\mathbf{u}_1-\mathbf{u}_2\|} + \frac{m_1m_3}{\|\mathbf{u}_1\|} + \frac{m_2m_3}{\|\mathbf{u}_2\|}
\end{aligned}$$

and

$$Q(\mathbf{u},\mathbf{v}) = \sum_{i=1}^{2}\mathbf{u}_i^T J_2\mathbf{v}_i,$$

is the angular momentum.

Now we want to perform a change of coordinates in order to reduce by two the order of the system. To this end, we shall use the integral of the angular momentum. We consider the canonical transformation $(\mathbf{u},\mathbf{v}) \longrightarrow (\boldsymbol{\xi},\boldsymbol{\eta})$ defined by

$$\begin{aligned}
\mathbf{u}_1 &= \xi_1\mathbf{w},\\
\mathbf{u}_2 &= L\begin{pmatrix}\xi_2\\ \xi_3\end{pmatrix},\\
\eta_1 &= \mathbf{v}_1^T\mathbf{w},\\
\mathbf{v}_2 &= L\begin{pmatrix}\eta_2\\ \eta_3\end{pmatrix},\\
\eta_4 &= -Q,
\end{aligned}$$

where $L = \begin{pmatrix}\cos\xi_4 & \sin\xi_4\\ -\sin\xi_4 & \cos\xi_4\end{pmatrix}$, $\mathbf{w} = \begin{pmatrix}\cos\xi_4\\ -\sin\xi_4\end{pmatrix}$ and $\boldsymbol{\xi} = (\xi_1,\xi_2,\xi_3,\xi_4)^T$, $\boldsymbol{\eta} = (\eta_1,\eta_2,\eta_3,\eta_4)^T$ (see [S.M.]).

Figure 6.1 shows geometrically the changes of coordinates performed to (6.1).

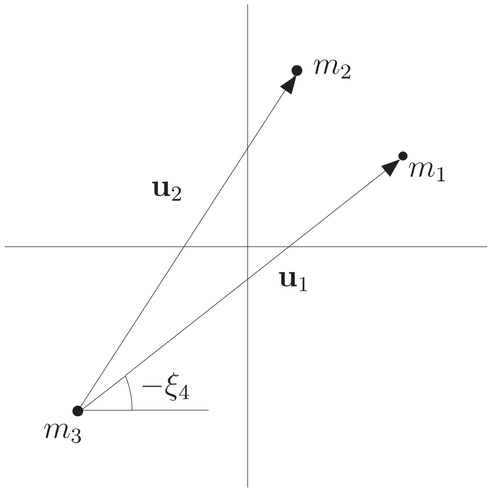


Figure 6.1: The change of variables of the centre of masses and the angular momentum

The new Hamiltonian can be written as

$$H(\boldsymbol{\xi},\boldsymbol{\eta}) \quad = \quad T(\boldsymbol{\xi},\boldsymbol{\eta}) - U(\boldsymbol{\xi}) + \eta_4, \tag{6.2}$$

where

$$\begin{aligned} T(\boldsymbol{\xi},\boldsymbol{\eta}) &= \frac{1}{2m_3}\left[(\eta_1+\eta_2)^2 + \left(\eta_3 + \frac{A}{\xi_1}\right)^2\right] + \frac{1}{2m_1\xi_1^2}(\xi_1^2\eta_1^2 + A^2) + \\ &\quad \frac{1}{2m_2}(\eta_2^2+\eta_3^2), \\ U(\boldsymbol{\xi}) &= \frac{m_1m_2}{r_{12}} + \frac{m_1m_3}{r_{13}} + \frac{m_2m_3}{r_{23}}, \end{aligned} \tag{6.3}$$

$$A = -\eta_4 + \xi_3\eta_2 - \xi_2\eta_3,$$
$$r_{12} = \left[(\xi_1-\xi_2)^2 + \xi_3^2\right]^{\frac{1}{2}}, \quad r_{13} = |\xi_1|, \quad r_{23} = (\xi_2^2+\xi_3^2)^{\frac{1}{2}},$$

and the equations of motion are

$$\begin{aligned} \dot\xi_1 &= \alpha_1\eta_1 + \frac{1}{m_3}\eta_2 \\ \dot\xi_2 &= \frac{1}{m_3}\eta_1 + \alpha_2\eta_2 + \xi_3 Z \\ \dot\xi_3 &= \alpha_2\eta_3 + \frac{1}{m_3\xi_1}A - \xi_2 Z \\ \dot\xi_4 &= 1 - Z \\ \dot\eta_1 &= -\frac{m_1m_2}{r_{12}^3}(\xi_1-\xi_2) - \frac{m_1m_3}{|\xi_1|^3}\xi_1 + \frac{A}{\xi_1}Z \end{aligned}$$

$$
\begin{aligned}
\dot\eta_2 &= \frac{m_1 m_2}{r_{12}^3}(\xi_1-\xi_2) - \frac{m_2 m_3}{r_{23}^3}\xi_2 + \eta_3 Z \\
\dot\eta_3 &= -\left(\frac{m_1 m_2}{r_{12}^3} + \frac{m_2 m_3}{r_{23}^3}\right)\xi_3 - \eta_2 Z \\
\dot\eta_4 &= 0
\end{aligned}
$$

where $Z = \dfrac{1}{\xi_1^2}\left(\alpha_1 A + \dfrac{\xi_1\eta_3}{m_3}\right)$, $\alpha_1 = \dfrac{m_1+m_3}{m_1 m_3}$, $\alpha_2 = \dfrac{m_2+m_3}{m_2 m_3}$. We note that the Hamiltonian (6.2) does not depend on ξ_4. So, this is an ignorable variable and fixing the value of η_4, that is, the angular momentum, we can restrict ourselves to consider a Hamiltonian system with three degrees of freedom

$$
\begin{aligned}
\dot\xi_i &= H_{\eta_i}, \\
\dot\eta_i &= -H_{\xi_i}.
\end{aligned}
\tag{6.4}
$$

From now on we will take $\boldsymbol{\xi} = (\xi_1,\xi_2,\xi_3)^T$ and $\boldsymbol{\eta} = (\eta_1,\eta_2,\eta_3)^T$.

Hamiltonian (6.2) admits some symmetries.

Lemma 6.1.1. *Let $H(\boldsymbol{\xi},\boldsymbol{\eta})$ the Hamiltonian defined in (6.2). Then*

(i) For any value of the masses $H(S_1(\boldsymbol{\xi},\boldsymbol{\eta})) = H(\boldsymbol{\xi},\boldsymbol{\eta})$ where

$$S_1 = diag(1,1,-1,-1,-1,1).$$

(ii) For values of the masses such that $m_1 = m_3$, $H(S_2(\boldsymbol{\xi},\boldsymbol{\eta})) = H(\boldsymbol{\xi},\boldsymbol{\eta})$ holds, where

$$S_2 = \begin{pmatrix} S_{11} & 0 \\ 0 & S_{22} \end{pmatrix},$$

$$\textit{with} \quad S_{11} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \textit{and} \quad S_{22} = diag(1,1,-1).$$

6.2 Expansion of the Hamiltonian in power series in a neighbourhood of L_2

In this section we shall expand the Hamiltonian of the PTBPN in a neighbourhood of L_2. Moreover, some properties of the eigenvalues and eigenvectors of the linear part of system (6.4) on the equilibrium point are proved. The eigenvalues can not be obtained, in general, explicitly due to the fact that the equilibrium points depend on the solution of an equation of degree five. However, we shall prove the necessary properties for our purposes. These properties will be useful in section 6.3 in order to obtain the Normal Form of the Hamiltonian up to order 4.

We consider the point L_2, that is, we take the collinear equilibrium point with the masses ordered from left to right as m_3, m_2, m_1. We recall that the other collinear points are obtained from L_2 by changing the values of the masses. In section 1.4 we have seen that for L_2, $u_1 = a(\rho+1)$, $u_2 = a$, being $\mathbf{u}_1 = (u_1, 0)$ and $\mathbf{u}_2 = (u_2, 0)$ the coordinates of these point, where ρ is the solution of Euler's quintic equation

$$\begin{aligned} \rho^5(m_2+m_3) + \rho^4(2m_2+3m_3) + \rho^3(m_2+3m_3) - \rho^2(3m_1+m_2) - \\ -\rho(3m_1+2m_2) - (m_1+m_2) = 0, \end{aligned} \qquad (6.5)$$

and $a^3 = -\dfrac{m_1}{\rho^2} + \dfrac{m_1}{(\rho+1)^2} + m_2 + m_3$. An easy computation shows that if $(\boldsymbol{\xi}^p, \boldsymbol{\eta}^p) = (\xi_1^p, \xi_2^p, \xi_3^p, \eta_1^p, \eta_2^p, \eta_3^p)$ denotes the coordinates of the point L_2 in variables $(\boldsymbol{\xi}, \boldsymbol{\eta})$ then

$$\begin{array}{lll} \xi_1^p = a(\rho+1), & \xi_2^p = a, & \xi_3^p = 0, \\ \eta_1^p = 0, & \eta_2^p = 0, & \eta_3^p = m_2 a(m_3 - m_1\rho). \end{array}$$

We also get

$$\eta_4^p \;=\; -a^2\left[m_1 m_3(\rho+1)^2 + m_2(m_1\rho^2 + m_3)\right]. \qquad (6.6)$$

We note that if $m_1 = m_3$ then $\rho = 1$. In this case, the equilibrium point is $(\boldsymbol{\xi}^p, \boldsymbol{\eta}^p) = (2a, a, 0, 0, 0, 0)$ where $a^3 = \dfrac{1}{4}(4m_2 + m_1)$ and the angular momentum is given by $\eta_4^p = -2m_1 a^2$.

From now on we fix the angular momentum equal to η_4^p, that is, the angular momentum at the equilibrium point, and we shall consider different values of the energy.

In order to expand the Hamiltonian in a neighbourhood of L_2 we translate the equilibrium point to the origin introducing new variables as

$$\begin{aligned} \mathbf{x} &= \frac{1}{a\rho}(\boldsymbol{\xi} - \boldsymbol{\xi}^p), \\ \mathbf{y} &= \boldsymbol{\eta} - \boldsymbol{\eta}^p. \end{aligned}$$

This change of variables defines a canonical transformation with multiplier $\dfrac{1}{a\rho}$. Then, the new Hamiltonian is

$$\mathcal{H}(\mathbf{x}, \mathbf{y}) \;=\; \frac{1}{a\rho} H(\boldsymbol{\xi}, \boldsymbol{\eta}). \qquad (6.7)$$

The following step consists in to expand the Hamiltonian (6.7) in power series in the variables $\mathbf{x}, \mathbf{y}$ in a neighbourhood of the origin. Then, it is only necessary to expand the Hamiltonian (6.2) in variables $\mathbf{x}, \mathbf{y}$ around the origin and to multiply the result by the factor $\dfrac{1}{a\rho}$.

Using (6.3) we have that

$$T(\mathbf{x}, \mathbf{y}) = \frac{1}{2}\alpha_1 y_1^2 + \frac{1}{m_3} y_1 y_2 + \frac{1}{2}\alpha_2 y_2^2 + \frac{1}{2}\alpha_2 y_3^2 + \alpha_2 \eta_3^p y_3 + \frac{1}{2}\alpha_2 (\eta_3^p) +$$

$$+ \frac{(y_3 + \eta_3^p) A(x, y)}{m_3 a(\rho + 1)} \sum_{n \geq 0} \left(\frac{-\rho}{\rho + 1} x_1 \right)^n + \frac{\alpha_1 (A(x, y))^2}{2a^2(\rho + 1)^2} \sum_{n \geq 0} (n + 1) \left(\frac{-\rho}{\rho + 1} x_1 \right)^n,$$

$$U(\mathbf{x}) = \frac{m_1 m_2}{a\rho} \sum_{n \geq 0} c_1^n P_n \left(\frac{x_2 - x_1}{c_1} \right) + \frac{m_1 m_3}{a(\rho + 1)} \sum_{n \geq 0} \left(\frac{-\rho}{\rho + 1} x_1 \right)^n +$$

$$+ \frac{m_2 m_3}{a} \sum_{n \geq 0} c_2^n P_n \left(\frac{-\rho x_2}{c_2} \right)$$

where $c_1^2 = (x_1 - x_2)^2 + x_3^2$, $c_2^2 = \rho^2 (x_2^2 + x_3^2)$, $A(\mathbf{x}, \mathbf{y}) = -\eta_4^p - a\eta_3^p - a(y_3 + \rho \eta_3^p x_2) + a\rho(x_3 y_2 - x_2 y_3)$ and P_n is the nth Legendre polynomial. We recall that the Legendre polynomials $P_n(x)$ are defined by

$$P_0(x) = 1, \quad P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} \left[(x^2 - 1)^n \right], \quad n \geq 1.$$

These polynomials satisfy $P_n(-x) = (-1)^n P_n(x)$ and the recurrence

$$\begin{aligned} P_0(x) &= 1, \quad P_1(x) = x, \\ P_{n+1}(x) &= \frac{2n + 1}{n + 1} x P_n(x) - \frac{n}{n + 1} P_{n-1}(x), \quad n \geq 1. \end{aligned}$$

Moreover, the generating function of the Legendre polynomials is $\dfrac{1}{\sqrt{1 - 2tx + t^2}}$, that is,

$$\frac{1}{\sqrt{1 - 2tx + t^2}} = \sum_{n=0}^{\infty} P_n(x) t^n.$$

Therefore, we can write the Hamiltonian (6.7) as

$$\mathcal{H}(\mathbf{x}, \mathbf{y}) = \mathcal{H}_0 + \sum_{k=2}^{\infty} \mathcal{H}_k(\mathbf{x}, \mathbf{y}),$$

where $\mathcal{H}_k(\mathbf{x}, \mathbf{y})$ is an homogeneous polynomial of degree k. In appendix E we give explicitly $\mathcal{H}_0$, $\mathcal{H}_2$, $\mathcal{H}_3$ and $\mathcal{H}_4$. Once the energy h is fixed we can consider $\mathcal{H}_0$ added to the energy. Then, we have the Hamiltonian

$$\mathcal{H}(\mathbf{x}, \mathbf{y}) = \sum_{k=2}^{\infty} \mathcal{H}_k(\mathbf{x}, \mathbf{y}). \tag{6.8}$$

Now we diagonalize the linear part of the system associated to this Hamiltonian. In order to write the quadratic part of the Hamiltonian in a canonical form we perform the following transformation

$$\begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} = \mathcal{D} \begin{pmatrix} \tilde{\mathbf{x}} \\ \tilde{\mathbf{y}} \end{pmatrix}, \tag{6.9}$$

where $\mathcal{D}$ is a 6×6 matrix defined by column vectors as

$$\mathcal{D} = (k_0\mathbf{z}_1, k_1\mathbf{e}_1, k_2\mathbf{e}_2, k_0\mathbf{z}_4, k_1\mathbf{f}_1, k_2\mathbf{f}_2). \tag{6.10}$$

In $\mathcal{D}$, $\mathbf{z}$ denotes the eigenvector of $D\tilde{F}(0,0)$ where $\tilde{F}$ is the vector field defined by $\mathcal{H}$ according to the following convention: $\mathbf{z}_1$ and $\mathbf{z}_4$ are eigenvectors corresponding to the real eigenvalues λ and $-\lambda$, respectively, and $\mathbf{z}_2 = \mathbf{e}_1 + \mathrm{i}\mathbf{f}_1$ and $\mathbf{z}_3 = \mathbf{e}_2 + \mathrm{i}\mathbf{f}_2$ with $\mathbf{e}_j, \mathbf{f}_j \in \mathbb{R}^6$, $j = 1, 2$, are eigenvectors for i and iω, respectively. In (6.10),

$$k_0^2 = \frac{1}{\mathbf{z}^T J_6 \mathbf{z}_4}, \qquad k_1^2 = \frac{1}{\mathbf{e}_1^T J_6 \mathbf{f}_1}, \qquad k_2^2 = \frac{1}{\mathbf{e}_2^T J_6 \mathbf{f}_2}.$$

Due to the Hamiltonian character of the matrix $D\tilde{F}(0,0)$, if $\mathbf{z}_a$ and $\mathbf{z}_b$ are eigenvectors corresponding to eigenvalues λ_a and λ_b such that $\lambda_a + \lambda_b \neq 0$, then $\mathbf{z}_a^T J_6 \mathbf{z}_b = 0$ (see [M.H.]). From this orthogonality property we have that the transformation (6.9) is canonical.

If we denote by $\mathcal{H}(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})$ the Hamiltonian in the new variables, the quadratic part is

$$\mathcal{H}_2(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}) = \lambda \tilde{x}_1 \tilde{y}_1 + \frac{1}{2}(\tilde{x}_2^2 + \tilde{y}_2^2) + \frac{1}{2}\omega(\tilde{x}_3^2 + \tilde{y}_3^2). \tag{6.11}$$

In order to simplify the computations needed to get the Normal Form it is convenient to introduce new complex variables $\mathbf{Q}, \mathbf{P}$ by

$$\begin{aligned} Q_1 &= k_0 \tilde{x}_1, \qquad & \tilde{x}_j &= \frac{1}{k_{j-1}}\left(Q_j + \mathrm{i}\frac{k_{j-1}}{2}P_j\right), \\ P_1 &= \frac{1}{k_0}\tilde{y}_1 \qquad & \tilde{y}_j &= \mathrm{i}\frac{1}{k_{j-1}}\left(Q_j - \mathrm{i}\frac{k_{j-1}}{2}P_j\right), \quad j = 2, 3, \end{aligned} \tag{6.12}$$

where $\mathbf{Q} = (Q_1, Q_2, Q_3)$, $\mathbf{P} = (P_1, P_2, P_3)$. This transformation is canonical and the Hamiltonian in $\mathbf{Q}, \mathbf{P}$ is written as

$$\mathcal{H}(\mathbf{Q}, \mathbf{P}) = \lambda Q_1 P_1 + \mathrm{i} Q_2 P_2 + \mathrm{i}\omega Q_3 P_3 + \sum_{k \geq 3} \mathcal{H}_k(\mathbf{Q}, \mathbf{P}). \tag{6.13}$$

We note that one can define a canonical transformation from $(\mathbf{x}, \mathbf{y})$ to $(\mathbf{Q}, \mathbf{P})$. Anyhow we shall use the intermediate Hamiltonian $\mathcal{H}(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})$ in order to get some properties of the homographic solutions that will be useful to compute the Normal Form.

Now we study qualitatively a neighbourhood of L_2. We consider the Hamiltonian $H(\tilde{\mathbf{x}},\tilde{\mathbf{y}}) = \sum_{k\geq 2} H_k(\tilde{\mathbf{x}},\tilde{\mathbf{y}})$ with $H_2(\tilde{\mathbf{x}},\tilde{\mathbf{y}})$ given in (6.11). The solutions of the linear system of equations given by $H_2(\tilde{\mathbf{x}},\tilde{\mathbf{y}})$ can be written as

$$\begin{array}{rclcl} \tilde{x}_1(t) & = & \tilde{x}_1^0 e^{\lambda t}, & & z_1(t) = \tilde{x}_2(t) + i\tilde{y}_2(t) = z_1^0 e^{-it}, \\ \tilde{y}_1(t) & = & \tilde{y}_1^0 e^{-\lambda t}, & & z_2(t) = \tilde{x}_3(t) + i\tilde{y}_3(t) = z_2^0 e^{-i\omega t}, \end{array} \tag{6.14}$$

where $\tilde{x}_1^0, \tilde{y}_1^0$ are real constants and z_1^0, z_2^0 complex ones. In (6.14) we distinguish two families of periodic orbits $\mathcal{F}_1$ and $\mathcal{F}_2$, with periods 2π and $\dfrac{2\pi}{\omega}$, respectively. These orbits can be parametrized by the energy h. Moreover, there exist a family of 2–dimensional invariant tori.

Now we consider the full Hamiltonian $H(\tilde{\mathbf{x}},\tilde{\mathbf{y}})$. We fix a value of the energy $h > 0$. Then, the intersection of the centre manifold of the equilibrium point with the corresponding energy level is homeomorphic to a three–dimensional sphere $\mathbb{S}^3$ (see appendix D). The preservation of the family of periodic orbits $\mathcal{F}_2$ associated to the eigenvalue $i\omega$ is obtained by using the Lyapunov theorem ([S.M.]). We know that $\omega > 1$ for all masses. Therefore $1/\omega$ is not an integer and we get a family of periodic solutions of $H(\mathbf{Q},\mathbf{P})$ with limit period $2\pi/\omega$. We shall denote again by $\mathcal{F}_2$ the family of periodic orbits of the full Hamiltonian $H(\mathbf{Q},\mathbf{P})$. However, from section 4.2 we have $1 < \omega < 3$, then if $\omega \neq 2$, the Lyapunov theorem gives the existence of a second family, $\mathcal{F}_1$, of periodic solutions with limit period 2π. In spite that Lyapunov theorem does not apply in the case $\omega = 2$, these periodic solutions exist for arbitrary masses because it is the family of homographic solutions. The preservation of the 2–dimensional invariant tori will be studied in section 6.4.

The rest of the section is devoted to study some properties of the eigenvalues and eigenvectors of the transformations described above. These properties will be used in section 6.3.

The characteristic polynomial of the linearized system of (6.4) at the point L_2 is

$$p(x) \quad = \quad (x^2+1)(x^4 + (1-\beta_c)x^2 - \beta_c(2\beta_c + 3)), \tag{6.15}$$

β_c is given in (1.53).

We have that the polynomial (6.15) has two real zeroes $\pm\lambda$ and two pairs of imaginary ones, $\pm\mathrm{i}$, $\pm\mathrm{i}\omega$. Therefore, L_2 is a centre–centre–saddle point. We note that the characteristic exponents $\pm\lambda$, $\pm\mathrm{i}\omega$ have been in chapter 4 from system (4.1). Moreover, in chapter 4 we have seen that $\omega \in (1, \omega_M)$ where $\omega_M = \sqrt{8\sqrt{2}-3} \approx$

2.88335022.... Figure 6.2 shows the level sets of ω and λ in the triangle of masses. They look similar because these curves are, in particular, level curves of β_c.

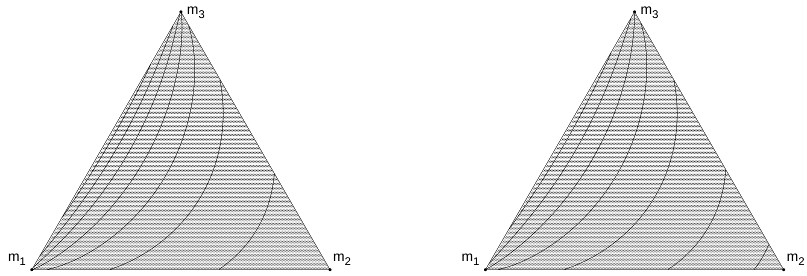


Figure 6.2: Level sets of ω and λ. In the first figure from left to right the values of ω are: $\frac{j}{4}$, $j = 11, \ldots, 5$. In the second figure from left to right the values of λ are: $\frac{j}{2}$, $j = 7, \ldots, 1$.

Now, we are interested in the eigenvectors of $D\tilde{F}(0,0)$. These vectors will be easily obtained from $DF(\boldsymbol{\xi}^p, \boldsymbol{\eta}^p)$, where F denotes the vector field of the system (6.4). After some computations we get

$$DF(\boldsymbol{\xi}^p, \boldsymbol{\eta}^p) = \begin{pmatrix} A_1 & A_2 \\ A_3 & -A_1^T \end{pmatrix}, \tag{6.16}$$

where

$$A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ k_1 & k_2 & 0 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} \alpha_1 & \frac{1}{m_3} & 0 \\ \frac{1}{m_3} & \alpha_2 & 0 \\ 0 & 0 & k_3 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} k_4 & k_5 & 0 \\ k_5 & k_6 & 0 \\ 0 & 0 & k_7 \end{pmatrix}, \tag{6.17}$$

with $\alpha_1 = \dfrac{m_1 + m_3}{m_1 m_3}$, $\alpha_2 = \dfrac{m_2 + m_3}{m_2 m_3}$ and k_i, $i = 1, \ldots, 7$, are constants that depend on the masses and ρ. They are given in appendix F. A simple computation shows that $k_3 \neq 0$ and so, A_2 is nonsingular for positive masses.

We denote by $\boldsymbol{\nu} = (\xi_1, \xi_2, \xi_3, \eta_1, \eta_2, \eta_3)^T$ an eigenvector of $DF(\boldsymbol{\xi}^p, \boldsymbol{\eta}^p)$ associated to the eigenvalue μ. It is easy to check that $\boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3)^T$ is a non–trivial solution of the system

$$(\mu^2 I_3 + \mu N - L)\boldsymbol{\xi} = 0, \tag{6.18}$$

where $N = A_2 A_1^T A_2^{-1} - A_1$, $L = A_2(A_3 + A_1^T A_2^{-1} A_1)$, and then $\boldsymbol{\eta} = (\eta_1, \eta_2, \eta_3)^T$ is obtained by

$$\boldsymbol{\eta} \quad = \quad -A_2^{-1}(A_1 - \mu I_3)\boldsymbol{\eta}. \tag{6.19}$$

In appendix F we give the expressions of

$$D(\mu) = \mu^2 I_3 + \mu N - L \quad \text{and} \quad E(\mu) = -A_2^{-1}(A_1 - \mu I_3). \tag{6.20}$$

We can obtain some trivial properties for these eigenvectors. We have that $A_1 DF(\boldsymbol{\xi}^p, \boldsymbol{\eta}^p) S_1^{-1} = -DF(\boldsymbol{\xi}^p, \boldsymbol{\eta}^p)$ where S_1 is the symmetry given in lemma 6.1.1. Therefore, if $\boldsymbol{\nu}$ is an eigenvector of $DF(\boldsymbol{\xi}^p, \boldsymbol{\eta}^p)$ corresponding to an eigenvalue μ, $S_1\boldsymbol{\nu}$ is an eigenvector for the eigenvalue $-\mu$. It is easy to see that if $\mu \in \mathbb{C} \setminus \mathbb{R}$ then we can take an eigenvector corresponding to eigenvalue μ such that ξ_1, ξ_2, η_3 are real and ξ_3, η_1, η_2 imaginary.

For arbitrary masses we get the following eigenvector for the eigenvalue i

$$\boldsymbol{\nu}_2 \quad = \quad (1, \rho_2, 0, \sigma_1 \mathrm{i}, \sigma_2 \mathrm{i}, -\sigma_2)^T,$$

where $\sigma_1 = m_1(m_2\rho\rho_2 + m_3)$, $\sigma_2 = m_2(m_3 - m_1\rho)\rho_2$ and $\rho_2 = \dfrac{1}{\rho+1}$ (see (1.54)). The eigenvectors corresponding to eigenvalues λ and iω do not have an easy expression. However, they satisfy a relation that will be used in section 6.3.

Lemma 6.2.1. *Let* $\boldsymbol{\nu} = (\xi_1, \xi_2, \xi_3, \eta_1, \eta_2, \eta_3)^T$ *be an eigenvector of* $DF(\boldsymbol{\xi}^p, \boldsymbol{\eta}^p)$ *corresponding to one of the eigenvalues* λ *or* $i\omega$. *Then*

$$\xi_1 m_1[m_3 + \rho(1 - m_1)] + \xi_2 m_2(m_3 - m_1\rho) \quad = \quad 0. \tag{6.21}$$

Proof

We consider the eigenvector corresponding to the eigenvalue i. We can write it as $\boldsymbol{\nu}_2 = \mathbf{e}_1 + \mathrm{i}\mathbf{f}_1$ with $\mathbf{e}_1 = (1, \rho_2, 0, 0, 0, -\sigma_2)^T$ and $\mathbf{f}_1 = (0, 0, 0, \sigma_1, \sigma_2, 0)^T$. We complete $\mathbf{e}_1$, $\mathbf{f}_1$ to a base of $\mathbb{R}^6$ taking vectors $\mathbf{r} \in \mathbb{R}^6$ skew–orthogonal to $\mathbf{e}_1$ and $\mathbf{f}_1$, that is, $\mathbf{e}_1^T J_3 \mathbf{r} = 0$ and $\mathbf{f}_1^T J_3 \mathbf{r} = 0$, being $J_3 = \begin{pmatrix} 0 & I_3 \\ -I_3 & 0 \end{pmatrix}$. From all these vectors we choose

$$\begin{array}{ll} \mathbf{r}_1 = (\sigma_2, -\sigma_1, 0, 0, 0, 0)^T, & \mathbf{r}_3 = (0, 0, 0, -\sigma_2, 1, 0)^T \\ \mathbf{r}_2 = (0, 0, 1, -\sigma_2, 0, 0)^T, & \mathbf{r}_4 = (0, 0, 0, 0, 0, 1)^T \end{array}.$$

We define the matrix

$$M_v \quad = \quad \begin{pmatrix} 1 & 0 & \sigma_2 & 0 & 0 & 0 \\ \rho_2 & 0 & -\sigma_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & \sigma_1 & 0 & -\sigma_2 & -\rho_2 & 0 \\ 0 & \sigma_2 & 0 & 0 & 1 & 0 \\ -\sigma_2 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

It is easy to check that

$$M_v^{-1} = \frac{1}{\sigma_1+\rho_2\sigma_2}\begin{pmatrix} \sigma_1 & \sigma_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma_2 & 1 & \rho_2 & 0 \\ \rho_2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma_1+\rho_2\sigma_2 & 0 & 0 & 0 \\ 0 & 0 & -\sigma_2^2 & -\sigma_2 & \sigma_1 & 0 \\ \sigma_1\sigma_2 & \sigma_2^2 & 0 & 0 & 0 & \sigma_1+\rho_2\sigma_2 \end{pmatrix},$$

and then

$$M_v^{-1}DF(\boldsymbol{\xi}^p,\boldsymbol{\eta}^p)M_v = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & \tilde{L} \end{pmatrix}, \quad \text{with} \quad \tilde{L} = \begin{pmatrix} 0 & a_1 & a_2 & 0 \\ a_3 & 0 & 0 & a_4 \\ a_5 & 0 & 0 & a_6 \\ 0 & a_7 & a_8 & 0 \end{pmatrix},$$

where $a_1,\dots,a_8$ are some constants depending on the masses and ρ. It is clear that $\tilde{L}$ has eigenvalues $\pm\lambda$ and $\pm\mathrm{i}\omega$. Moreover, if $(l_1,l_2,l_3,l_4)^T$ is an eigenvector or $\tilde{L}$ then $\mathbf{l}=(0,0,l_1,l_2,l_3,l_4)^T$ is an eigenvector of the matrix $M_v^{-1}DF(\boldsymbol{\xi}^p,\boldsymbol{\eta}^p)M_v$. Then, $M_v\mathbf{l}$ is an eigenvector of $DF(\boldsymbol{\xi}^p,\boldsymbol{\eta}^p)$. A simple check shows that these eigenvectors have the following expression

$$M_v\mathbf{l} = (\sigma_2 l_1, -\sigma_1 l_1, l_2, -\sigma_2 l_2 - \rho_2 l_3, l_3, l_4)^T.$$

Therefore, $\xi_1=\sigma_2 l_1$ and $\xi_2 = -\sigma_1 l_1$. After an easy computation (6.21) follows. □

In the symmetric case in which $m_1 = m_3$ the eigenvectors have a simplest expression due to the fact that the solution of the quintic equation (6.5) is $\rho=1$. We have that

$$N = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -2 \\ -1 & 2 & 0 \end{pmatrix} \quad \text{and} \quad L = \begin{pmatrix} -1 & 0 & 0 \\ l_{21} & l_{22} & 0 \\ 0 & 0 & l_{33} \end{pmatrix},$$

where

$$l_{21} = -\frac{1}{4a^3}(8m_2+9m_1), \qquad l_{22} = \frac{1}{4a^3}(12m_2+17m_1), \qquad l_{33} = -\frac{7m_1}{4a^3}.$$

Solving (6.18) for these matrices N and L and computing $\boldsymbol{\eta}$ from (6.19), we obtain the following lemma.

Lemma 6.2.2. *For positive masses such that $m_1 = m_3$ it is satisfied that*

$$\begin{aligned}
\boldsymbol{\nu}_1 &= (\boldsymbol{\xi}^T, \boldsymbol{\eta}^T)^T \quad \textit{with} \quad \boldsymbol{\xi}^T = (0, 1, v_\lambda), \\
&\quad \boldsymbol{\eta}^T = m_1 m_2(-\lambda + v_\lambda, 2(\lambda - v_\lambda), 2(1 + \lambda v_\lambda)), \quad v_\lambda = \frac{\lambda^2 - l_{22}}{2\lambda}, \\
\boldsymbol{\nu}_2 &= (\boldsymbol{\xi}^T, \boldsymbol{\eta}^T)^T \quad \textit{with} \quad \boldsymbol{\xi}^T = \left(1, \frac{1}{2}, 0\right), \\
&\quad \boldsymbol{\eta}^T = \left(\frac{m_1}{2} i, 0, 0\right), \\
\boldsymbol{\nu}_3 &= (\boldsymbol{\xi}^T, \boldsymbol{\eta}^T)^T \quad \textit{with} \quad \boldsymbol{\xi}^T = (0, 1, v_\omega i), \\
&\quad \boldsymbol{\eta}^T = a m_1 m_2((v_\omega - \omega)i, 2(\omega - v_\omega)i, 2(1 - \omega v_\omega)), \quad v_\omega = \frac{\omega^2 + l_{22}}{2\omega},
\end{aligned}$$

Now we study the eigenvectors of $D\tilde{F}(0,0)$. We have that

$$D\tilde{F}(0,0) = \begin{pmatrix} A_1 & \frac{1}{a\rho} A_2 \\ a\rho A_3 & -A_1^T \end{pmatrix},$$

where A_1, A_2 and A_3 are the matrices defined in (6.17). It is easy to check that if $\boldsymbol{\nu} = (\boldsymbol{\xi}^T, \boldsymbol{\eta}^T)$ is an eigenvector of $DF(\boldsymbol{\xi}^p, \boldsymbol{\eta}^p)$ for an eigenvalue μ then $(\boldsymbol{\xi}^T, a\rho\boldsymbol{\eta}^T)$ is the corresponding eigenvector of $D\tilde{F}(0,0)$. So, the matrix $\mathcal{D}$ in (6.10) can be written as

$$\mathcal{D} = \begin{pmatrix}
v_{11} & e_{11} & e_{21} & v_{11} & 0 & 0 \\
v_{12} & e_{12} & e_{22} & v_{12} & 0 & 0 \\
v_{13} & 0 & 0 & -v_{13} & 0 & f_{23} \\
v_{14} & 0 & 0 & -v_{14} & f_{14} & f_{24} \\
v_{15} & 0 & 0 & -v_{15} & f_{15} & f_{25} \\
v_{16} & e_{16} & e_{16} & v_{16} & 0 & 0
\end{pmatrix}.$$

We are interested in the monomials that appear in $\mathcal{H}(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})$. The non existence of certain monomials will be useful in section 6.3 in order to obtain the Normal Form of the Hamiltonian.

In $\mathcal{H}(\mathbf{x}, \mathbf{y})$ the monomials of degree 3 are the following:

$$x_1^3,\ x_1^2x_2,\ x_1^2y_3,\ x_1x_2^2,\ x_1x_2y_3,\ x_1x_3^2,\ x_1x_3y_2,\ x_1y_3^2,\ x_2^3,\ x_2^2y_2,\ x_2x_3^2,\ x_2x_3y_2,\ x_2y_3^2,\ x_3y_2y_3.$$

Using the particular form of $\mathcal{D}$ and the variables $\tilde{\mathbf{x}}, \tilde{\mathbf{y}}$ defined in (6.9) the following lemma follows easily.

Lemma 6.2.3. *$\mathcal{H}_3(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})$ does not contain the monomials $\tilde{x}_2^2\tilde{y}_3$, $\tilde{x}_3\tilde{y}_2^2$, $\tilde{y}_2^2\tilde{y}_3$ and $\tilde{x}_2\tilde{y}_2\tilde{y}_3$.*

6.3 The Normal Form

In this section we compute the Normal Form of the Hamiltonian reduced to the centre manifold of L_2. First of all we explain the method used in order to reduce the Hamiltonian to the centre manifold of the equilibrium point. Then, in section 6.3.2 we give some properties of the Normal Form that will be useful in section 6.4 in order to apply the KAM theorem to the Hamiltonian.

6.3.1 Reduction to the centre manifold

In this section we give a brief description of the method used in order to obtain the Hamiltonian reduced to the centre manifold of L_2.

In order to simplify we shall denote by $\mathbf{z} = (Q_1, P_1), \mathbf{w} = (Q_2, P_2, Q_3, P_3)$. We can write the Hamiltonian system associated to (6.13) as

$$\begin{aligned} \dot{\mathbf{z}} &= \Lambda_1 \mathbf{z} + f_1(\mathbf{z}, \mathbf{w}), \\ \dot{\mathbf{w}} &= \Lambda_2 \mathbf{w} + f_2(\mathbf{z}, \mathbf{w}), \end{aligned} \tag{6.22}$$

where $\Lambda_1 = \mathrm{diag}(\lambda, -\lambda), \Lambda_2 = \mathrm{diag}(\mathrm{i}, -\mathrm{i}, \mathrm{i}\omega, -\mathrm{i}\omega)$. From the Centre Manifold Theorem we know that in a neighbourhood $U_1 \times U_2$ of the origin small enough there exists a function $h(\mathbf{w})$ with $h(0) = 0$, $D_{\mathbf{w}} h(0) = 0$ such that $M_c = \{(h(\mathbf{w}), \mathbf{w}) \mid \mathbf{w} \in U_2\}$ is a centre manifold for the system (6.22). M_c is a local invariant manifold. Moreover, in a neighbourhood of the origin all bounded solution for all t of (6.22) is completely contained in M_c. The reduced equation to the centre manifold is given by

$$\dot{\mathbf{w}} = \Lambda_2 \mathbf{w} + f_2(h(\mathbf{w}), \mathbf{w}).$$

It is well known ([Mi.]) that, for Hamiltonian systems, if the centre manifold of an equilibrium point is $\mathcal{C}^2$, then the reduced equation is also Hamiltonian.

In order to obtain the reduced equation on a neighbourhood of a collinear equilibrium point of the PTBPN we shall use the flattening method of this manifold ([Mi.],[Si.],[J.M.]) that allow us to obtain an approximation of the reduced Hamiltonian up to a given order.

We note that for the linearized system of (6.22) the centre manifold is trivially obtained as $\mathbf{z} = 0$. The idea of the flattening method consists in to perform successives canonical transformations to (6.22) in such a way that to the obtained Hamiltonian, $\tilde{\mathcal{H}}(\mathbf{z}, \mathbf{w})$, the centre manifold is determined by $\mathbf{z} = h(\mathbf{w}) = O(|\mathbf{w}|^n)$ for a given n. Then, the reduced Hamiltonian is obtained as $\tilde{\mathcal{H}}(0, \mathbf{w}) + O(|\mathbf{w}|^{n+1})$.

We assume that performing several canonical transformations we can write the Hamiltonian (6.13) as

$$\mathcal{H}(\mathbf{z}, \mathbf{w}) = \mathcal{H}_n(Q_1 P_1, \mathbf{w}) + R_{n+1}(\mathbf{z}, \mathbf{w}) \tag{6.23}$$

for $n \geq 4$, where $H_n = \sum_{k=2}^{n} \mathcal{H}_k(Q_1P_1, \mathbf{w})$ in such a way that $\mathcal{H}_2 = \lambda Q_1 P_1 + iQ_2P_2 + i\omega Q_3 P_3$ and $\mathcal{H}_k$ is an homogeneous polynomial of order k in $Q_1P_1, \mathbf{w}$. R_{n+1} contains terms of order greater that n. We note that in (6.23), $\mathcal{H}_n$ only depends on the product Q_1P_1, but not on Q_1 nor P_1. We remark that for the Hamiltonian $\mathcal{H}_n(Q_1P_1, \mathbf{w})$ the centre manifold is trivially obtained as $Q_1 = 0$, $P_1 = 0$.

We assume that the centre manifold of the origin for the Hamiltonian system associated to (6.23) is locally defined by two functions $h_1(\mathbf{w}), h_2(\mathbf{w})$ in such a way that $Q_1 = h_1(\mathbf{w}), P_1 = h_2(\mathbf{w})$. As $h_1(\mathbf{w}), h_2(\mathbf{w})$ need to define an invariant manifold we have that

$$\sum_{j=2}^{3} \left(\frac{\partial h_1}{\partial Q_j}(\lambda_j Q_j + O_2) + \frac{\partial h_1}{\partial P_j}(-\lambda_j P_j + O_2) \right) = h_1(\mathbf{w})(\lambda + O_1) + O_n. \quad (6.24)$$

where $\lambda_2 = i, \lambda_3 = i\omega$.

For h_2, a similar equation is obtained. We assume that the terms of h_1 of minimal order are of order k for some $1 \leq k < n$. We consider a term in h_1 of order k of the form $\mathbf{Q}^{\mathbf{l}}\mathbf{P}^{\mathbf{s}} = Q_2^{l_2} Q_3^{l_3} P_2^{s_2} P_3^{s_3}$ with $l_2 + l_3 + s_2 + s_3 = k$. On the left hand of the equality (6.24), this term will give rise to one of the form $(l_j - s_j)\lambda_j \mathbf{Q}^{\mathbf{l}}\mathbf{P}^{\mathbf{s}}$. On the other hand, on the right hand of (6.24) we have $\lambda \mathbf{Q}^{\mathbf{l}}\mathbf{P}^{\mathbf{s}}$ with $\lambda \in \mathbb{R}$. Then, $h_1(\mathbf{w})$ can not contain terms of order $1 \leq k < n$. The same occurs for $h_2(\mathbf{w})$. Therefore, if $\mathcal{H}(\mathbf{z}, \mathbf{w})$ can be expressed in the form (6.23) then $h_1(\mathbf{w})$ and $h_2(\mathbf{w})$ are of order n. Now, the reduced system is obtained by taking $Q_1 = 0$, $P_1 = 0$ on (6.23).

The form (6.23) for the initial Hamiltonian is obtained by performing successives canonical transformations that eliminate the terms of the form $Q_1^{l_1} P_1^{s_1}$ with $l_1 \neq s_1$ at the different orders. This process can be done to any order due to the fact that does not have small divisors. As we will see in section 6.3.2 for a term of the form $Q_1^{l_1} Q_2^{l_2} Q_3^{l_3} P_1^{s_1} P_2^{s_2} P_3^{s_3}$ the divisor is $\lambda(s_1 - l_1) + i(s_2 - l_2) + i\omega(s_3 - l_3)$ that is different from zero if $l_1 \neq s_1$.

6.3.2 The Normal Form in a neighbourhood of L_2

To get the form (6.23) for the Hamiltonian we perform successive canonical transformations which eliminate the terms $Q_1^{l_1} P_1^{s_1}$ with $l_1 \neq s_1$ to different orders. This process can be formally done up to any order because it does not have small divisors even if the two central frequencies are resonant. But this does not mean that the complete process is convergent.

As we are interested in the applicability of KAM theorem we shall write the reduced Hamiltonian in action–angle variables. The canonical transformations will be chosen in order to cancel all the possible terms in the Hamiltonian. In particular the terms $Q_1^{l_1}P_1^{s_1}$ with $l_1 \neq s_1$ will be eliminated. We will see that for our purpose it is only necessary to simplify the Hamiltonian up to order 4. To this end we shall use the Giorgilli–Galgani algorithm (see appendix C) up to order 4.

We write the generating function as $G = G_3+G_4$ with $G_3 \in \Pi_3$, $G_4 \in \Pi_4$, where Π_k stands for the space of homogeneous polynomials in the variables $(\mathbf{Q},\mathbf{P}) \in \mathbb{C}^6$. We consider the canonical transformation T_G defined by the time one flow for the Hamiltonian G (see appendix C). The new Hamiltonian is $\tilde{\mathcal{H}} = \tilde{\mathcal{H}}_2+\tilde{\mathcal{H}}_3+\tilde{\mathcal{H}}_4+\dots$ where

$$\begin{aligned} \tilde{\mathcal{H}}_2 &= \mathcal{H}_2, \qquad \tilde{\mathcal{H}}_3 = \mathcal{H}_3 + L_{\mathcal{H}_2}G_3, \\ \tilde{\mathcal{H}}_4 &= \mathcal{H}_4 + L_{\mathcal{H}_2}G_4 + L_{\mathcal{H}_3}G_3 - \frac{1}{2}L_{G_3}L_{\mathcal{H}_2}G_3, \end{aligned} \tag{6.25}$$

where $L_f g$ stands for the Poisson bracket. We take G_3 and G_4 such that

$$\begin{aligned} L_{\mathcal{H}_2}G_3 &= N(F_3), \qquad \text{where} \quad N(F_3) = -\mathcal{H}_3, \\ L_{\mathcal{H}_2}G_4 &= N(F_4), \qquad \text{where} \quad N(F_4) = -\mathcal{H}_4 - L_{\mathcal{H}_3}G_3 + \frac{1}{2}L_{G_3}L_{\mathcal{H}_2}G_3. \end{aligned}$$

In order to solve the homological equation $L_{\mathcal{H}_2}G_k = N(F_k)$ for $k = 3, 4$, we write $G_k = \sum_{\|\mathbf{l}+\mathbf{s}\|=k} g_{\mathbf{l},\mathbf{s}}\mathbf{Q}^{\mathbf{l}}\mathbf{P}^{\mathbf{s}}$ and $N(F_k) = \sum_{\|\mathbf{l}+\mathbf{s}\|=k} n_{\mathbf{l},\mathbf{s}}\mathbf{Q}^{\mathbf{l}}\mathbf{P}^{\mathbf{s}}$, where $\mathbf{l} = (l_1, l_2, l_3)$, $\mathbf{s} = (s_1, s_2, s_3)$ and $\mathbf{Q}^{\mathbf{l}}\mathbf{P}^{\mathbf{s}} = Q_1^{l_1}Q_2^{l_2}Q_3^{l_3}P_1^{s_1}P_2^{s_2}P_3^{s_3}$. Then

$$L_{\mathcal{H}_2}G_k = \sum_{\|\mathbf{l}+\mathbf{s}\|=k} \boldsymbol{\lambda}\cdot(\mathbf{s}-\mathbf{l})g_{\mathbf{l},\mathbf{s}}\mathbf{Q}^{\mathbf{l}}\mathbf{P}^{\mathbf{s}},$$

where $\boldsymbol{\lambda}\cdot(\mathbf{s}-\mathbf{l}) := \lambda(s_1-l_1) + \mathrm{i}(s_2-l_2) + \mathrm{i}\omega(s_3-l_3)$. If $\boldsymbol{\lambda}\cdot(\mathbf{s}-\mathbf{l}) \neq 0$ we define

$$g_{\mathbf{l},\mathbf{s}} = \frac{n_{\mathbf{l},\mathbf{s}}}{\boldsymbol{\lambda}\cdot(\mathbf{s}-\mathbf{l})}.$$

If there exist $\mathbf{l},\mathbf{s} \in \mathbb{Z}^3$, $\mathbf{l},\mathbf{s} \neq 0$ satisfying $\boldsymbol{\lambda}\cdot(\mathbf{s}-\mathbf{l}) = 0$, then the Normal Form will contain a resonant monomial $\mathbf{Q}^{\mathbf{l}}\mathbf{P}^{\mathbf{s}}$. Next lemma give us these monomials.

Lemma 6.3.1. *(a) For arbitrary values of the masses,*

$$(Q_1P_1)^{s_1}(Q_2P_2)^{s_2}(Q_3P_3)^{s_3}$$

with $s_1+s_2+s_3 = 2k$, $k \geq 2$, are resonant monomial of order $2k$. Moreover, if $k = 2$, these are the unique resonant monomials of order 4.

(b) $Q_2^{2k}P_3^k$ and $Q_3^kP_2^{2k}$, $k \in \mathbb{N}$, are resonant monomials of order $3k$ associated to the resonant frequency vector $(\hat{\lambda}, i, 2i)$, $\hat{\lambda} = \frac{1}{2}\sqrt{13+\sqrt{97}}$. Moreover, if $k = 1$, these are the unique resonant monomials of order 3.

Proof

First we consider the resonant monomials of order 3 or 4.

We take $Q_1^{l_1}Q_2^{l_2}Q_3^{l_3}P_1^{s_1}P_2^{s_2}P_3^{s_3}$, $\quad l_i, s_i \in \mathbb{N}\cup\{0\}$, $i = 1, 2, 3$, a resonant monomial or order 3 or 4 associated to the frequency vector $(\lambda, i, i\omega)$ for some $\omega \in (1, \omega_M)$. We have that $|\mathbf{l}+\mathbf{s}| = 3$ or $|\mathbf{l}+\mathbf{s}| = 4$, respectively, and $(\mathbf{l}, \mathbf{s})$ is solution of the equation

$$\lambda(s_1 - l_1) + i(s_2 - l_2) + i\omega(s_3 - l_3) = 0. \tag{6.26}$$

Then, $s_1 = l_1$ i $s_2 - l_2 + \omega(s_3 - l_3) = 0$.

If $s_2 - l_2 \neq 0$, and then $s_3 - l_3 \neq 0$, we have that

$$\omega = \frac{l_2 - s_2}{s_3 - l_3} \Rightarrow \omega \in \mathbb{Q} \cap (1, 3).$$

In this case, if $|s_3 - l_3| > 1$ then $|\mathbf{l}+\mathbf{s}| > 4$. As we are not interested in this case, if $s_2 - l_2 \neq 0$, we only need to take into account the case in which $|s_3 - l_3| = 1 (\Rightarrow \omega = 2)$.

(a) We assume that $|\mathbf{l}+\mathbf{s}| = 4$. We distinguish two cases.

(i) If $s_2 = l_2$ then $s_3 = l_3$ and $(\mathbf{s}, \mathbf{s})$ is solution of the equation (6.26). Therefore, for all $\omega \in (1, \omega_M)$, $(Q_1P_1)^{s_1}(Q_2P_2)^{s_2}(Q_3P_3)^{s_3}$, with $s_1 + s_2 + s_3 = 2$, is a resonant monomial of order 4 associated to the frequency vector $(\lambda, i, i\omega)$. In fact, if $s_1 + s_2 + s_3 = k$, with $k \in \mathbb{N}$, then one obtains a resonant monomial of order $2k$.

(ii) If $s_2 \neq l_2$, we have seen that $\omega = 2$ and $|s_3 - l_3| = 1$. Then, $|s_2 - l_2| = 2$. From this, s_3 and l_3 have different parity. We note that s_1 and l_1, s_2 and l_2, have the same parity. So, in this case $|\mathbf{l}+\mathbf{s}|$ is odd, that give us a contradiction with the fact that $|\mathbf{l}+\mathbf{s}| = 4$.

(b) We assume that $|\mathbf{l}+\mathbf{s}| = 3$. We distinguish two cases.

(i) If $s_2 = l_2$, then $s_3 = l_3$ and $|\mathbf{l}+\mathbf{s}|$ is even, giving a contradiction with the fact that $|\mathbf{l}+\mathbf{s}| = 3$.

(ii) If $s_2 \neq l_2$ then $\omega = 2$ and $|s_3 - l_3| = 1$.

First we consider the case in which $s_3 - l_3 = 1$. In this case, $l_2 - s_2 = 2$ and, as $|\mathbf{l}+\mathbf{s}| = 3$, we have that $l_1 = 0, l_2 = 2, l_3 = 0, s_1 = 0, s_2 = 0, s_3 = 1$. Then, $Q_2^2P_3$ is a resonant monomial of order 3 associated to the frequency vector $(\hat{\lambda}, i, 2i)$.

Now, if $s_3 - l_3 = -1$ then $s_2 - l_2 = 2$ and, as $|\mathbf{l}+\mathbf{s}| = 3$, we obtain $l_1 = 0, l_2 = 0, l_3 = 1, s_1 = 0, s_2 = 2, s_3 = 0$, which corresponds to the resonant monomial of order 3 $Q_3P_2^2$.

We note that $Q_2^{2k}P_3^k$ and $Q_3^kP_2^{2k}$ with $k \in \mathbb{N}$ are resonant monomials of order $3k$ associated to the frequency vector $(\hat{\lambda}, \mathrm{i}, 2\mathrm{i})$.

□

Corollary 6.3.2. *The minimum $\mathbb{Z}$–modulus that contains the resonant terms associated to $\mathcal{H}_2(\mathbf{Q},\mathbf{P}) = \lambda Q_1P_1 + iQ_2P_2 + i\omega Q_3P_3$ up to order 4 is the one generated by $(s_1, s_2, s_3, s_1, s_2, s_3) \in \mathbb{Z}^6$ with $s_1+s_2+s_3 = 1$ in the case $\omega \neq 2$. If $\omega = 2$ there are also $(0, 2, 0, 0, 0, 1)$ and $(0, 0, 1, 0, 2, 0)$ as generators.*

The masses m_1, m_2, m_3 for which the frequency vector is $(\hat{\lambda}, \mathrm{i}, 2\mathrm{i})$ determine a curve in the triangle of masses. The following lemma gives this curve in an implicit form.

Lemma 6.3.3. *The resonant curve is defined implicitly by*

$$f(\nu_1, \nu_2) = 0, \qquad \textit{where} \quad \nu_1 = \frac{m_1 - m_3}{2}, \quad \nu_2 = \frac{m_1 + m_3}{2}, \tag{6.27}$$

where f is a polynomial in ν_1, ν_9 of degree 9, even in ν_1 and such that $f(\nu_1, 0) = 0$. Moreover, f can be written as $f(\nu_1, \nu_2) = \sum_{(j,k)\in J}(a_{j,k} + \sqrt{97}b_{j,k})\nu_1^j\nu_2^k$, $J \subset \mathbb{Z}^2$, where the coefficients $a_{j,k}, b_{j,k}$ are given in Table 6.1.

Figure 6.3 shows the resonant curve in the triangle of masses.

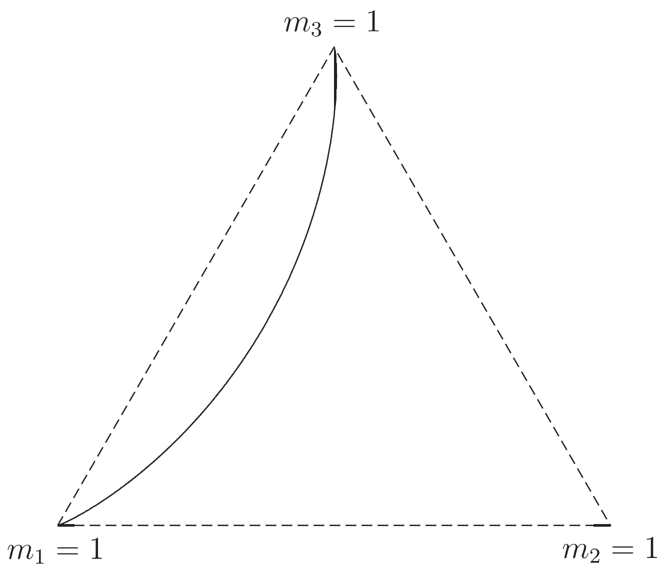


Figure 6.3: Gràfic de la corba ressonant sobre el triangle de masses.

Proof

j	k	$a_{j,k}$	$b_{j,k}$	$32a_{j,k}$	$32b_{j,k}$	$b_{j_k}\sqrt{97}$
0	2	146.00000	50.00000	4672	1600	0.638442890090E+3
2	1	1384.87500	142.87500	44316	4572	0.279203055843E+4
0	3	−463.50000	−679.50000	−14832	−21744	−0.715579887632E+4
2	2	−8871.87500	−765.87500	−283900	−24508	−0.164148689690E+5
0	4	−2843.62500	3618.37500	−90996	115788	0.327932358486E+5
4	1	7124.34375	711.84375	227979	22779	0.141351916208E+5
2	3	4145.56250	734.56250	132658	23506	0.113801641090E+5
0	5	13019.59375	−8780.90625	416627	−280989	−0.734623032772E+5
4	2	−32015.50000	−3559.50000	−1024496	−113904	−0.670725093455E+5
2	4	67611.00000	2763.00000	2163552	88416	0.948233941064E+5
0	6	−28851.50000	9972.50000	−923248	319120	0.693662344284E+5
6	1	2973.50000	317.50000	95152	10160	0.610051235207E+4
4	3	45344.50000	4496.50000	1451024	143888	0.896298891058E+5
2	5	−143849.50000	−8489.50000	−4603184	−271664	−0.227461378308E+6
0	7	45755.50000	−3860.50000	1464176	−123536	0.773398445617E+4
6	2	−6916.00000	−532.00000	−221312	−17024	−0.121555923506E+5
4	4	−21140.00000	28.00000	−676480	896	−0.208642319815E+5
2	6	96628.00000	8260.00000	3092096	264320	0.177979565443E+6
0	8	−32732.00000	−588.00000	−1047424	−18816	−0.385231283875E+5
8	1	−70.00000	−14.00000	−2240	−448	−0.207884009225E+3
6	3	1400.00000	280.00000	44800	8960	0.415768018450E+4
4	5	−6020.00000	−1204.00000	−192640	−38528	−0.178780247934E+5
2	7	−9800.00000	−1960.00000	−313600	−62720	−0.291037612915E+5
0	9	−3430.00000	−686.00000	−109760	−21952	−0.101863164520E+5

Table 6.1: Coefficients of the resonant curve

Using (1.53) we can write β_c in terms of ν_1 and ν_2 as

$$\beta_c = \frac{(\nu_1+\nu_2)(3\rho^2+3\rho+1)+(\nu_2-\nu_1)(\rho^4+3\rho^3+3\rho^2)}{2\nu_2\rho^2+(1-2\nu_2)(\rho^4+2\rho^3+2\rho^2+2\rho+1)}, \qquad (6.28)$$

where ρ is the solution of the Euler's quintic equation. We impose 2i to be a zero of the characteristic polynomial. Then, $\beta_c = \dfrac{1+\sqrt{97}}{2}$. We fix this value of β_c. Then, (6.28) can be written as $P_1(\rho,\nu_1,\nu_2)=0$ where P_1 is a polynomial in ρ,ν_1,ν_2.

On the other hand, we can write Euler's quintic equation in terms of ν_1 and ν_2. We obtain

$$\begin{aligned} P_2(\rho,\nu_1,\nu_2) &= \rho^5(1-\nu_1-\nu_2)+\rho^4(1-\nu_2-3\nu_1)+\rho^3(1-3\nu_1+\nu_2)\\ &\quad -\rho^2(1+3\nu_1+\nu_2)-\rho(2+3\nu_1-\nu_2)-(1-\nu_2+\nu_1)=0. \end{aligned}$$

The function $f(\nu_1,\nu_2)$ is the resultant of these polynomials. f has been computed using a specific algebraic manipulator built for this purpose. We note that the coefficients $a_{j,k}, b_{j,k}$ given in the table are rational numbers. The properties of f are obtained from the table. □

Remark 6.3.4. From the expression of the resonant curve given in (6.27) it can be seen that this curve is symmetrical with respect to the line $m_1=m_3$. In fact, this is a consequence of the symmetry of the function $\beta_c(m_1,m_3)$ given in (1.53) (see chapter 1).

If $m_1=0$ then $\nu_1=-\nu_2=-\dfrac{m_3}{2}$ and from Lemma 6.3.3

$$f(-\nu_2,\nu_2)=\sum_{(j,k)\in J}(a_{j,k}+\sqrt{97}b_{j,k})\nu_2^{j+k}.$$

Analogously, if $m_3=0$, $\nu_1=\nu_2=\dfrac{m_1}{2}$ and

$$f(\nu_1,\nu_1)=\sum_{(j,k)\in J}(a_{j,k}+\sqrt{97}b_{j,k})\nu_1^{j+k}.$$

Solving the equation $f(\nu_1,\nu_1)=0$ with $\nu_1=\dfrac{m_1}{2}$ we get two solutions $m_1=1$ and $m_1^*=0.9995998\ldots$ (zeroes of a polynomial of degree 6 with coefficients in $\mathbb{Q}[\sqrt{97}]$). These points correspond to intersections of the resonant curve with the side $m_3=0$ of the triangle of masses. In fact, the arc of the resonant curve between $m_1=1$ and m_1^* is outside the triangle of masses and it will not be considered.

From the symmetry of the resonant curve with respect to the line $m_1=m_3$, this curve intersects the side $m_1=0$ at $m_3^*=0.9995998\ldots$ and for $m_3=1$.

We are only interested in the arc of the resonant curve located inside the triangle of masses. We have that this arc does not cross the vertex of the triangle.

Assume we take non resonant masses, that is, $\omega \neq 2$. Then all the terms of degree 3 in $\mathcal{H}_3(\mathbf{Q},\mathbf{P})$ can be eliminated by the transformation T_G. The Normal Form for the Hamiltonian is the following

$$
\begin{aligned}
\mathcal{H}(\mathbf{X},\mathbf{Y}) \;=\; & \lambda X_1Y_1 + iX_2Y_2 + i\omega X_3Y_3 + \sum_{\substack{j_1, j_2, j_3 \in \mathbb{N}\cup\{0\} \\ j_1+j_2+j_3=2}} a_{j_1j_2j_3}(\mathbf{XY})^j + \\
& \sum_{k\geq 5}\mathcal{H}_k(\mathbf{X},\mathbf{Y}). \qquad (6.29)
\end{aligned}
$$

For resonant masses, that is, if $\omega = 2$, one could expect that the Hamiltonian $\mathcal{H}(\mathbf{Q},\mathbf{P})$ contains resonant monomials of order 3, $Q_2^2P_3$ and $Q_3P_2^2$, which give rise to the corresponding terms of order 3 in the transformed Hamiltonian $\mathcal{H}(\mathbf{X},\mathbf{Y})$. We will prove that this is not the case.

First we denote the resonant monomials of order 3 in $\mathcal{H}(\mathbf{Q},\mathbf{P})$ by $\mathcal{H}_R(\mathbf{Q},\mathbf{P}) = cQ_2^2P_3 + gQ_3P_2^2$ for some complex constants c and g. Using (6.12) we see that $\mathcal{H}_R(\mathbf{Q},\mathbf{P})$ is obtained from the following terms of $\mathcal{H}_3(\tilde{\mathbf{x}},\tilde{\mathbf{y}})$

$$a_1\tilde{x}_2^2\tilde{x}_3 + a_2\tilde{x}_2^2\tilde{y}_3 + a_3\tilde{y}_2^2\tilde{x}_3 + a_4\tilde{y}_2^2\tilde{y}_3 + b_1\tilde{x}_2\tilde{y}_2\tilde{x}_3 + b_2\tilde{x}_2\tilde{y}_2\tilde{y}_3.$$

The coefficients are related through

$$
\begin{aligned}
c &= \frac{\tilde{k}_2}{2\tilde{k}_1^2}\mathcal{C}, \quad \mathcal{C} = a_2 - a_4 - b_1 + i(a_1 - a_3 + b_2),\\
g &= \frac{\tilde{k}_1^2}{4\tilde{k}_2}\mathcal{G}, \quad \mathcal{G} = -a_1 + a_3 - b_2 + i(-a_2 + a_4 + b_1).
\end{aligned}
$$

We recall that $\tilde{k}_1$ and $\tilde{k}_2$ are constants defined in section 6.2. Note that $\mathcal{G} = -i\bar{\mathcal{C}}$. Moreover, from lemma 6.2.3 we have $a_2 = a_3 = a_4 = b_1 = 0$ and then $\mathcal{C} = i(a_1+b_2)$.

Proposition 6.3.5. *If m_1, m_2, m_3 are resonant masses, then the Hamiltonian $\mathcal{H}(\mathbf{Q},\mathbf{P})$ does not contain resonant monomials of order* 3.

Proof

To prove that $\mathcal{C} = 0$ we will obtain some expressions for the coefficients a_1 and b_2. To compute these expressions directly is a hard work. But using the existence of the homographic solutions we can get a_1 and b_2 in a rather simple way.

First we consider arbitrary values of the masses m_1, m_2 and m_3 such that $m_1 + m_2 + m_3 = 1$. We denote by $\gamma(t) = (\mathbf{Q}(t),\mathbf{P}(t))$ an homographic solution with eccentricity e. We recall that, identifying $\mathbb{R}^2$ with the complex plane, the homographic solutions can be written as $\mathbf{q}(t) = r(t)e^{if(t)}$ where $z(t)$ is a solution of the Kepler problem

$$\ddot{z} = -\frac{z}{|z|^3}, \qquad (6.30)$$

(see appendix A). Let $E_K = \frac{1}{2}|\dot{z}|^2 - \frac{1}{|z|}$ the energy of the Kepler problem (6.30).

It turns out that on the homographic solutions, we get for the total angular momentum $\|\mathbf{c}\| = \omega\eta_4^p$ where η_4^p is given in (6.6). As we consider fix angular momentum equal to η_4^p we have $\omega = 1$ and so we characterize the homographic solutions by their eccentricity e. Then $E = \frac{1}{2}(\omega^2 - 1)$ and the period is $P = 2\pi(1-e^2)^{-3/2}$. It is not difficult to write the homographic solutions in the variables $\tilde{\mathbf{x}}, \tilde{\mathbf{y}}$ defined in section 6.2. We obtain

$$\begin{aligned}
\tilde{x}_1(t) &= \tilde{k}_0(A - B), & \tilde{y}_1(t) &= \tilde{k}_0(A + B), \\
\tilde{x}_2(t) &= -\frac{\tilde{k}_1\eta_4^p}{a^2\rho(\rho+1)}(r(t) - 1), & \tilde{y}_2(t) &= -\frac{\tilde{k}_1\eta_4^p}{a^2\rho(\rho+1)}\dot{r}(t), \\
\tilde{x}_3(t) &= \tilde{k}_2\left\{\frac{f_{24}(\rho+1) + f_{25}}{\rho}(r(t) - 1) - f_{23}m_2a(m_3 - m_1\rho)(r(t)\dot{f}(t) - 1)\right\}, \\
\tilde{y}_3(t) &= \tilde{k}_2a\dot{r}(t)\left\{e_{21}m_1(m_3 + \rho(1 - m_1)) + e_{22}m_2(m_3 - m_1\rho)\right\},
\end{aligned} \tag{6.31}$$

where

$$\begin{aligned}
A &= -\frac{v_{14}(\rho+1) + v_{15}}{\rho}(r(t) - 1) + v_{13}m_2a(m_3 - m_1\rho)(r(t)\dot{f}(t) - 1), \\
B &= [v_{11}m_1a(m_3 + \rho(1 - m_1)) + v_{12}m_2a(m_3 - m_1\rho)]\dot{r}(t).
\end{aligned}$$

In (6.31) $\mathbf{z}_1 = (v_{11}, v_{12}, v_{13}, v_{14}, v_{15}, v_{16})^T$ and $\mathbf{z}_3 = \mathbf{e}_2 + i\mathbf{f}_2$ with

$$\mathbf{e}_2 = (e_{21}, e_{22}, 0, 0, 0, e_{26})^T, \mathbf{f}_2 = (0, 0, f_{23}, f_{24}, f_{25}, 0)^T$$

are the eigenvectors of $D\tilde{F}(0,0)$, introduced in section 6.2, corresponding to eigenvalues λ and iω respectively. We recall that $a, \rho, \tilde{k}_0, \tilde{k}_1$ and $\tilde{k}_2$ have been introduced in section 6.2. In (6.31) $\dot{f}$ means the derivative of the true anomaly, f, with respect t.

We write

$$\tilde{x}_j(t) = \sum_{n\geq 1} a_{jn}(t)e_k^n, \qquad \tilde{y}_j(t) = \sum_{n\geq 1} b_{jn}(t)e_k^n, \quad j = 1, 2, 3.$$

In order to finish the prove of the lemma we need the following result.

Lemma 6.3.6. *For any masses m_1, m_2, m_3, if $(\tilde{x}_1(t), \tilde{x}_2(t), \tilde{x}_3(t), \tilde{y}_1(t), \tilde{y}_2(t), \tilde{y}_3(t))$ is a collinear homographic solution then $\tilde{x}_1(t) = \tilde{y}_1(t)$, $\tilde{y}_2(t) = \dot{\tilde{x}}_2(t)$ and $\tilde{y}_3(t) = 0$. Moreover, if $m_1 = m_3$, then $\tilde{x}_1(t) = 0$, $\tilde{y}_1(t) = 0$ and $\tilde{x}_3(t) = 0$.*

Now we prove the proposition 6.3.5.

From lemma 6.3.6 we know that $\tilde{x}_1(t) = \tilde{y}_1(t)$ and so $\dot{\tilde{x}}_1(t) = \dot{\tilde{y}}_1(t)$. As $\gamma(t)$ is a solution of the differential equations associated to the Hamiltonian $\mathcal{H}(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})$, we get

$$\lambda \tilde{x}_1(t) + \frac{\partial \hat{\mathcal{H}}}{\partial \tilde{y}_1}(\gamma(t)) = -\lambda \tilde{y}_1(t) - \frac{\partial \hat{\mathcal{H}}}{\partial \tilde{x}_1}(\gamma(t)),$$

where $\hat{\mathcal{H}}$ stands for the terms of $\mathcal{H}(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})$ of order greater than 2. Therefore

$$2\lambda \tilde{x}_1(t) = -\frac{\partial \hat{\mathcal{H}}}{\partial \tilde{y}_1}(\gamma(t)) - \frac{\partial \hat{\mathcal{H}}}{\partial \tilde{x}_1}(\gamma(t)).$$

The right hand part of the equality above contain terms of order greater than or equal to 2 in e. Then $\tilde{x}_1(t) = O(e^2)$ and $\tilde{y}_1(t) = O(e^2)$. From (6.31) we get

$$\begin{aligned}
\tilde{x}_2(t) &= \frac{\tilde{k}_1 \eta_4^p}{a^2 \rho(\rho+1)} e \cos f + O(e^2), \\
\tilde{y}_2(t) &= -\frac{\tilde{k}_1 \eta_4^p}{a^2 \rho(\rho+1)} e \sin f, \\
\tilde{x}_3(t) &= \tilde{k}_2[-f_{23} m_2 a(m_3 - m_1 \rho) e^2 \cos^2 f] + O(e^3),
\end{aligned}$$

By derivating $\tilde{x}_3(t)$ and using $\dot{f} = (1 + e\cos f)^2$

$$\dot{\tilde{x}}_3(t) = \tilde{k}_2[2 f_{23} m_2 a(m_3 - m_1 \rho) e^2 \cos f \sin f] + O(e^3).$$

Now consider the differential equations for $\tilde{x}_3$ and $\tilde{y}_3$. Using that $\tilde{x}_3(t) = 0$ we get

$$\dot{\tilde{x}}_3 = \frac{\partial \hat{\mathcal{H}}}{\partial \tilde{y}_3}(\gamma(t)), \tag{6.32}$$

$$0 = -\omega \tilde{x}_3(t) - \frac{\partial \hat{\mathcal{H}}}{\partial \tilde{x}_3}(\gamma(t)). \tag{6.33}$$

By inspection of $\tilde{\mathcal{H}}$ one can see that in the right hand part of the equations (6.32) and (6.33) the only terms of order 2 in e come from the monomials $\tilde{x}_2^2, \tilde{y}_2^2$ and $\tilde{x}_2\tilde{y}_2$. In fact, in (6.32) there is only the contribution due to the term $b_2\tilde{x}_2\tilde{y}_2$, and in (6.33), only $a_1\tilde{x}_2^2$ contributes to terms of order e^2 in $\frac{\partial \hat{\mathcal{H}}}{\partial \tilde{x}_3}(\gamma(t))$. So, we get

$$\begin{aligned}
2\tilde{k}_2 f_{23} m_2 a(m_3 - m_1\rho)\cos f \sin f &= -b_2 \left(\frac{\tilde{k}_1 \eta_4^p}{a^2\rho(\rho+1)}\right)^2 \cos f \sin f, \\
-\tilde{k}_2 \omega f_{23} m_2 a(m_3 - m_1\rho)\cos^2 f &= -a_1 \left(\frac{\bar{k}_1 \eta_4^p}{a^2\rho(\rho+1)}\right)^2 \cos^2 f.
\end{aligned}$$

Then

$$\begin{aligned} b_2 &= -\frac{2\tilde{k}_2 f_{23} m_2 a^5 (m_3 - m_1\rho)\rho^2(\rho+1)^2}{\tilde{k}_1^2(\eta_4^p)^2}, \\ a_1 &= \frac{\omega \tilde{k}_2 f_{23} m_2 a^5 (m_3 - m_1\rho)\rho^2(\rho+1)^2}{\tilde{k}_1^2(\eta_4^p)^2}. \end{aligned}$$

Now,

$$\mathcal{C} = \mathrm{i}(a_1 + b_2) = \mathrm{i}(\omega - 2)(m_3 - m_1\rho)\frac{\tilde{k}_2 f_{23} m_2 a^5 \rho^2(\rho+1)^2}{\tilde{k}_1^2(\eta_4^p)^2}. \tag{6.34}$$

Therefore, in the resonant case, $\omega = 2$ and $c = 0$.

□

Proof of Lemma 6.3.6

The equality $\tilde{y}_2(t) = \dot{\tilde{x}}_2(t)$ follows directly from (6.31). Lemma 6.2.1 implies $\tilde{x}_1(t) = \tilde{y}_1(t)$ and $\tilde{x}_3(t) \equiv 0$. The corresponding relations in the case $m_1 = m_3$ are obtained using the exact expressions of the eigenvectors given by lemma 6.2.2.

□

Remark 6.3.7. In the symmetrical case $m_1 = m_3$ we have $\rho = 1$ and then using (6.34) $\mathcal{C} = 0$ independently of the value of ω. Moreover for masses m_1, m_2, m_3 such that $m_1 \neq m_3$ and $\omega \neq 2$ we have $\mathcal{C} \neq 0$.

We have implemented an algebraic manipulator to compute the Normal Form up to order 4 for given values of the masses. In Table 6.2 we give the coefficients of the monomials $Q_3P_2^2$ and $Q_2^2P_3$ for some values of the masses. We note that these monomials are 'numerically zero' in the symmetrical case, $m_1 = m_3$, in agreement with Remark 6.3.7. The same agreement between numerical and analytical results has also been checked for resonant masses.

Consider the term $cQ_2^2P_3$ in $\mathcal{H}(\mathbf{Q}, \mathbf{P})$. Using (6.34) and (6.6) we get

$$c = \frac{\tilde{k}_2}{2\tilde{k}_1^2}\mathcal{C} = \frac{\tilde{k}_2^2}{2\tilde{k}_1^4}\mathrm{i}(\omega - 2)(m_3 - m_1\rho)\frac{f_{23} m_2 a \rho^2(\rho+1)^2}{[m_1 m_3(\rho+1)^2 + m_2(m_1\rho^2 + m_3)]^2}.$$

If we consider non resonant masses we can define a generating function $G = G_3 + G_4$ where G_3 contains the monomial $g_1 Q_2^2 P_3$ with $g_1 = \dfrac{c}{i(-2+\omega)}$. Therefore for positive values of m_1, m_2, m_3, g_1 is a continuous function of the masses. In a similar way and due to the monomial $P_3Q_2^2$, G_3 contains the term $g_2 Q_3 P_2^2$ where g_2 is a continuous function of the masses. So, we can define a function G_3 on the triangle of masses taking the limit of g_1 and g_2 in the case of resonant masses. We remark that for $\omega = 2$, the terms $Q_2^2 P_3$ and $Q_3 P_2^2$ belong to the kernel of the linear operator L_{H_2}. This means that they do not contribute to $\tilde{H}_3$ (see (6.25)) . Therefore we get the following proposition.

m_1	m_3	coefficient of $Q_3P_2^2$	coefficient of $Q_2^2P_3$i
0.1	0.8	–1.179419348112622E-002	6.875177379696579E-003
0.1	0.7	–9.319371384346949E-002	7.758543543388584E-002
0.1	0.6	–1.413142569551888E-001	1.548904537008628E-001
0.1	0.5	–1.749179227510806E-001	2.286007158007460E-001
0.1	0.4	–2.011008770853154E-001	2.858515079866209E-001
0.1	0.3	–2.177563331662531E-001	3.070251869558777E-001
0.1	0.2	–2.007104469104191E-001	2.536095880359161E-001
0.1	0.1	9.235014541892594E-015	–1.183014251097389E-014
0.2	0.7	1.029366603579695E-002	–1.491208746569150E-002
0.2	0.6	–1.966944788903897E-002	2.895185423930213E-002
0.2	0.5	–4.152197150460450E-002	6.647693015509877E-002
0.2	0.4	–5.221842494622244E-002	8.869746541104991E-002
0.2	0.3	–4.586674997804073E-002	7.933390003814744E-002
0.2	0.2	6.940183773499024E-016	–8.210080641681364E-016
0.2	0.1	1.765861018277782E-001	–3.073164843015664E-001
0.3	0.6	6.238037823992740E-003	–1.768228493842869E-002
0.3	0.5	–4.438682094330806E-003	1.083995060532376E-002
0.3	0.4	–9.191599656495861E-003	2.207101058566123E-002
0.3	0.3	2.022770770177911E-016	3.132560016105415E-016
0.3	0.2	4.000070097083308E-002	–9.741018176772076E-002
0.3	0.1	1.668996116082525E-001	–4.575591110691087E-001
0.4	0.5	1.798577883741621E-003	–8.485012379500253E-003
0.4	0.4	–1.827228383071727E-017	7.311853698924998E-016
0.4	0.3	7.935569740810064E-003	–2.751323770250442E-002
0.4	0.2	3.926071923919142E-002	–1.360534267829678E-001
0.4	0.1	1.325417655797161E-001	–5.342349636043282E-001
0.5	0.4	–1.534726145643662E-003	1.076465252598653E-002
0.5	0.3	3.263655003503015E-003	–1.719301330985832E-002
0.5	0.2	2.649617500229235E-002	–1.304106710601642E-001
0.5	0.1	9.738989522637230E-002	–5.502476606321277E-001
0.6	0.3	–3.831843227559372E-003	3.672812785621461E-002
0.6	0.2	1.041828802723820E-002	–7.510521579183546E-002
0.6	0.1	6.478871988992192E-002	–4.989471222532697E-001
0.7	0.2	–4.374349135738403E-003	5.382987717642781E-002
0.7	0.1	3.381097753148919E-002	–3.550369383481710E-001
0.8	0.1	3.156516816489582E-003	–4.965627280827153E-002

Table 6.2: Coefficients of the monomials $Q_3P_2^2$ for several values of the masses

Proposition 6.3.8. *Consider $m_1, m_2, m_3 \in (0,1)$ such that $m_1 + m_2 + m_3 = 1$. Then there exists a generating function $G = G(m_1, m_2, m_3)$ which depends analytically on the masses such that after the transformation T_G, the new Hamiltonian is of the form (6.29).*

6.4 Existence of invariant tori

The study of the behaviour of the orbits of a non integrable Hamiltonian system is quite difficult. To this end, the idea it to associate to such a system an integrable Hamiltonian system. For this kind of system we know the dynamic of the orbits. KAM theorem give us information about the non integrable system depending on the associated integrable system (see [A.A.]). In this section we study the applicability of KAM theorem to the Hamiltonian reduced to the centre manifold of the equilibrium point L_2.

We write the Normal Form (6.29) as

$$\begin{aligned} \mathcal{H}(\mathbf{X},\mathbf{Y}) &= \lambda X_1Y_1 + iX_2Y_2 + i\omega X_3Y_3 - a_{020}(X_2Y_2)^2 - a_{002}(X_3Y_3)^2 - \\ & \quad a_{011}X_2Y_2X_3Y_3 - a_{200}(X_1Y_1)^2 - a_{110}iX_1Y_1X_2Y_2 - \\ & \quad a_{101}iX_1Y_1X_3Y_3 + \sum_{k\geq 5}\mathcal{H}_k(\mathbf{X},\mathbf{Y}). \end{aligned} \tag{6.35}$$

To get action–angle variables we introduce a canonical transformation $(\mathbf{X},\mathbf{Y}) \mapsto (X_1, \boldsymbol{\varphi}, Y_1, \mathbf{I})$ where

$$X_k = \sqrt{I_k}e^{i\varphi_k}, \qquad Y_k = -i\sqrt{I_k}e^{i\varphi_k}, \quad k = 2,3.$$

The new Hamiltonian is

$$\mathcal{H}(X_1,\boldsymbol{\varphi},Y_1,\mathbf{I}) = H_2(X_1Y_1,\mathbf{I}) + \mathcal{H}_4(X_1Y_1,\mathbf{I}) + R_5(X_1,\boldsymbol{\varphi},Y_1,\mathbf{I}),$$

where

$$\begin{aligned} \mathcal{H}_2(X_1Y_1,\mathbf{I}) &= \lambda X_1Y_1 + I_2 + \omega I_3, \\ \mathcal{H}_4(X_1Y_1,\mathbf{I}) &= -a_{020}I_2^2 - a_{011}I_2I_3 - a_{002}I_3^2 - a_{200}(X_1Y_1)^2 - a_{110}iX_1Y_1I_2 - \\ & \quad a_{101}iX_1Y_1I_3. \end{aligned}$$

To obtain the Hamiltonian reduced to the centre manifold, $\mathcal{H}_c$, we take $X_1 = 0$ and $Y_1 = 0$. Then we get $\mathcal{H}_c(\mathbf{I},\boldsymbol{\varphi}) = \mathcal{H}_{c0}(\mathbf{I}) + \mathcal{H}_{c1}(\mathbf{I},\boldsymbol{\varphi})$ where

$$\mathcal{H}_{c0}(\mathbf{I}) = I_2 + \omega I_3 - a_{020}I_2^2 - a_{011}I_2I_3 - a_{002}I_3^2. \tag{6.36}$$

The Hamiltonian system associated to $\mathcal{H}_{c0}$ is

$$\begin{aligned} \dot{I}_2 &= 0, \qquad \dot{\varphi}_2 = -(1 - 2a_{020}I_2 - a_{011}I_3), \\ \dot{I}_3 &= 0, \qquad \dot{\varphi}_3 = -(\omega - a_{011}I_2 - 2a_{002}I_3). \end{aligned} \tag{6.37}$$

Clearly (6.37) is an integrable system. So, we look at $H_c(\mathbf{I}, \boldsymbol{\varphi})$ as a perturbation of $H_{c0}(\mathbf{I})$.

The nondegeneracy conditions of KAM theorem are the following

$$\text{(a)} \qquad D_1 = \det\left(\frac{\partial^2 H_{c0}}{\partial \mathbf{I}^2}\right) \neq 0,$$

$$\text{(b)} \qquad D_2 = \det\begin{pmatrix} \dfrac{\partial^2 H_{c0}}{\partial \mathbf{I}^2} & \dfrac{\partial H_{c0}}{\partial \mathbf{I}} \\ \left(\dfrac{\partial H_{c0}}{\partial \mathbf{I}}\right)^T & 0 \end{pmatrix} \neq 0.$$

If the condition (a) is satisfied the theorem says that for almost every frequency vector there exists a 2–dimensional invariant torus near the unperturbed one. The condition (b) or isoenergetic condition gives the existence of invariant tori on every energy level.

From (6.36) we have that $D_1 = 4a_{020}a_{002} - a_{011}^2$ is constant. D_2 depends on I_2 and I_3. Anyhow it is sufficient to check the condition (b) at the origin, that is, $D_2 = 2a_{020}\omega^2 - 2a_{011}\omega + 2a_{002} \neq 0$.

We have evaluated D_1 and D_2 numerically for different values of the masses in the triangle of masses. Using a specific manipulator we compute the coefficients of the Normal Form up to order 4 for given values of the masses m_1, m_2, m_3. Once the Normal Form is known, D_1 and D_2 are obtained immediately. The values of D_1 and D_2 for a set of masses are given in table 6.3. Figures 6.4 and 6.5 show, respectively, D_1 and D_2 as functions of m_2. In figure 6.4 every continuous line corresponds to $D_1(m_2)$ for a fixed value of m_1. Note that once m_1 is fixed, $m_2 \in (0, 1 - m_1)$ and $m_3 = 1 - m_1 - m_2$. The same holds for figure 6.5. These results show that $D_1 > 0$ and $D_2 > 0$ for positive masses. Then KAM theorem ensures the existence of 2–dimensional invariant tori in a neighbourhood of L_2 for any energy level.

m_1	m_2	D_1	D_2
0.1	0.1	123.192444126190900	17.788785832091770
0.1	0.2	57.981216051192120	9.276140574712066
0.1	0.3	40.728298686281160	6.589952363214971
0.1	0.4	34.682311867438230	5.341187287868248
0.1	0.5	33.684503156474210	4.677695909919546
0.1	0.6	36.745287504522810	4.332526838718575
0.1	0.7	45.813213441577230	4.225259225545290
0.1	0.8	71.089285311101490	4.457041208199403
0.2	0.1	103.393517316096200	18.218294888530650
0.2	0.2	50.824193245420020	9.300570286444902
0.2	0.3	36.743296393792050	6.516670411071603
0.2	0.4	32.138454012679390	5.240962280420760
0.2	0.5	32.257632210686840	4.589675857720991
0.2	0.6	37.081352918128770	4.316558741864279
0.2	0.7	52.071994619538740	4.504639989181360
0.3	0.1	102.073743778695200	19.298538564906000
0.3	0.2	51.744613621402370	9.781560417457513
0.3	0.3	38.491722199751520	6.832744606754595
0.3	0.4	34.883295985464800	5.515028693081639
0.3	0.5	36.988170596588670	4.914702092384127
0.3	0.6	47.942107180607540	4.948789669577259
0.4	0.1	110.485328727285100	20.671363273493780
0.4	0.2	57.759778937352510	10.473316682437390
0.4	0.3	44.584133478133890	7.353628431238850
0.4	0.4	42.745509539056030	6.044269112305969
0.4	0.5	51.108291030540560	5.761858546700239
0.5	0.1	129.480083003787500	22.377846871814720
0.5	0.2	70.374438999527100	11.407304656778590
0.5	0.3	57.580164145066870	8.157801766505628
0.5	0.4	62.291451633215250	7.158104149041286
0.6	0.1	166.170648618560100	24.623203692516640
0.6	0.2	95.954711362321600	12.779312315293600
0.6	0.3	88.834742771266920	9.732525836537608
0.7	0.1	243.304387577001600	27.947054572142850
0.7	0.2	159.814511483404300	15.400391716639330
0.8	0.1	460.303431252741100	34.385581785583930

Table 6.3: Determinants D_1 and D_2 for several values of the masses.

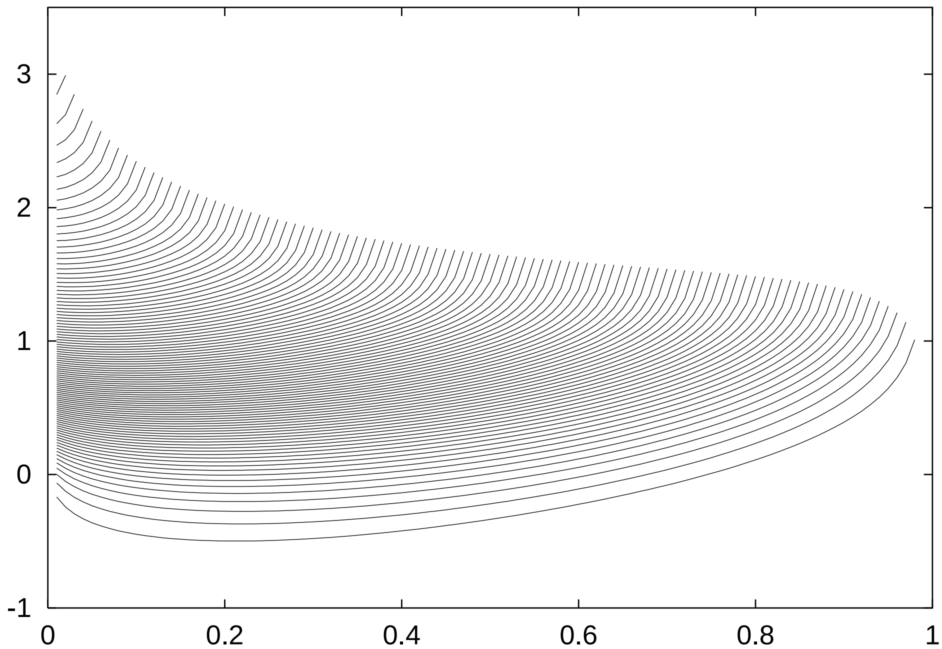


Figure 6.4: Representation of D_1. Every continuous line is the plot of $\log_{10}(m_1 m_2 D_1)$ as a function of m_2 for a fixed value of m_1.

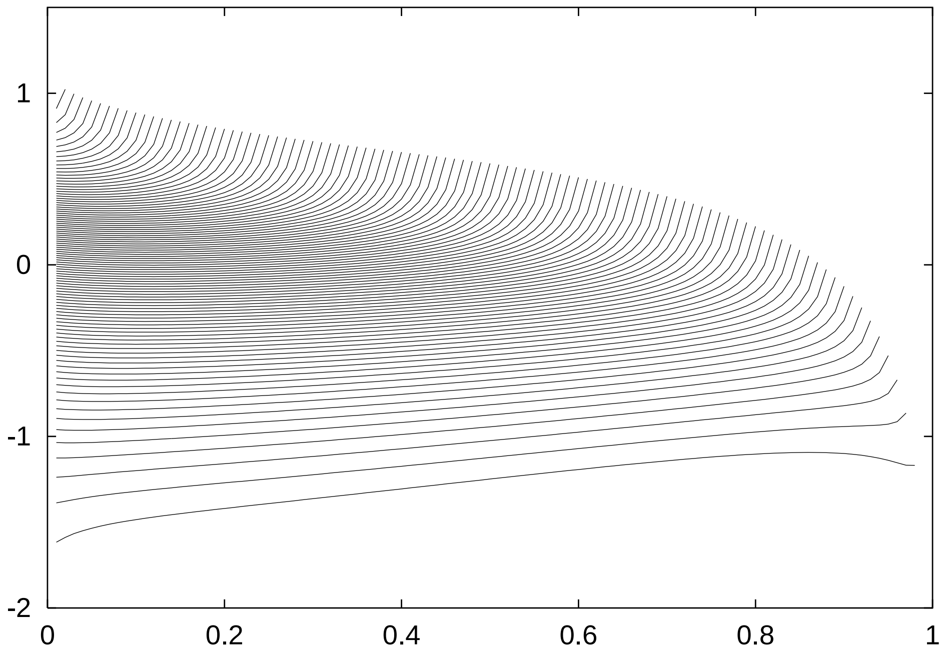


Figure 6.5: Representation of D_2. Every continuous line is the plot of $\log_{10}(m_1 m_2 D_2)$ as a function of m_2 for a fixed value of m_1.

Appendix A

The planar Kepler problem with homogeneous potential

In this appendix we discuss the solutions of the planar Kepler problem with some homogeneous potential. For details, see [A.].

We consider the equations of the planar Kepler problem with homogeneous potential of order $-\alpha$, $0 < \alpha < 2$

$$\mathbf{z}'' \quad = \quad -\frac{d\tilde{U}}{d\mathbf{z}}, \quad \text{where} \quad \tilde{U}(\mathbf{z}) = -\frac{\lambda}{\alpha\|\mathbf{z}\|^{\alpha}}. \tag{A.1}$$

with $\mathbf{z} = (z_1, z_2)^T \in \mathbb{R}^2$ and $\lambda > 0$. (A.1) corresponds to the motion of an unitary mass moving under a homogeneous potential due to a mass λ located in the origin.

This system has first integrals the energy and the angular momentum. Then, once the energy h is and the angular momentum ω are fixed, we have

$$\begin{aligned} h &= \frac{1}{2}\|\dot{\mathbf{z}}\|^2 - \frac{\lambda}{\alpha\|\mathbf{z}\|^{\alpha}}, \\ \boldsymbol{\omega} &= \mathbf{z} \wedge \dot{\mathbf{z}}. \end{aligned}$$

We introduce polar coordinates

$$z_1 = r\cos f, \qquad z_2 = r\sin f.$$

Using the integral of the angular momentum one can see that

$$f'r^2 = \omega, \tag{A.2}$$

where $\omega = \|\boldsymbol{\omega}\|$. Then, $f'r^2$ is constant over the solutions of (A.1).

From (A.1) we obtain

$$\begin{cases} r'' - (f')^2 r = -\dfrac{\lambda}{r^{\alpha+1}}, \\ 2r'\theta' + f'r = 0 \end{cases}$$

Second equation always holds due to the fact that $f'r^2$ is constant. First equation can be written as

$$r'' = -\frac{\partial V}{\partial r}(r), \tag{A.3}$$

where

$$V(r) = -\frac{\lambda}{\alpha r^\alpha} + \frac{\omega^2}{2r^2}. \tag{A.4}$$

From (A.2) we obtain

$$f(t) = \int_0^t \frac{\omega}{r(s)^2} ds. \tag{A.5}$$

In order to study the solutions of system (A.1) we begin studying the potential equation (A.3).

Function V has a zero in $\left(\dfrac{\alpha\omega^2}{2\lambda}\right)^{\frac{1}{2-\alpha}}$ and a minimum in $r^* = \left(\dfrac{\omega^2}{\lambda}\right)^{\frac{1}{2-\alpha}}$. We have that $V(r^*) = \dfrac{\omega^2(\alpha-2)}{2\alpha}\left(\dfrac{\omega^2}{\lambda}\right)^{\frac{-2}{2-\alpha}}$. Figure A.1 shows the plot of V for $0 < \alpha < 2$.

Let us define

$$E_K = \frac{(r')^2}{2} + V(r), \tag{A.6}$$

the energy of (A.3). If $E_K = V(r^*)$ we get a constant solution of (A.3). If $0 > E_K > V(r^*)$ then we obtain a periodic solution and if $E_K > 0$ the solutions are not periodic and unbounded. For the case $0 > E_K > V(r^*)$ let us define r_{min} and r_{max} the minimum and the maximum value of r, respectively, of the periodic solution obtained. The points such that $r = r_{min}$ are known as *pericenters* and if $r = r_{max}$ then are called *apocenters*.

If $r_{min} = r_{max}$, that is, $E_K = V(r^*)$, then the associated solution for (A.1) is circular and $\dfrac{2\pi}{\sqrt{2-\alpha}}$–periodic. If we consider a periodic orbit of (A.3), then the orbit for (A.1) is not necessary closed. An orbit is closed if the angle of successives pericenter and apocenter is commensurable with 2π. It is well known ([A.]) that all the bounded solutions of (A.1) are periodic if and only if $\alpha = 1$ or $\alpha = -2$. Other values of α give, in general, quasiperiodic solutions.

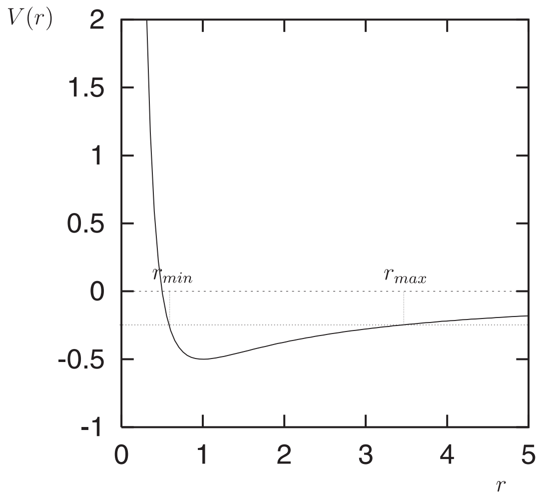


Figure A.1: Graphic of $V(r)$ for $\alpha = \omega = 1$.

Appendix B

Constants on the computation of the linearized system on a triangular solution of the Planar Three Body Problem with homogeneous potential

In this appendix we give the expression of the constants a_j, $j = 1, \ldots, 8$, that appears in (1.45). These constants depend on the masses and on ϱ as

$$
\begin{aligned}
a_1 &= \frac{\alpha m_1}{\varrho^{\alpha+2}}\left[m_2(\alpha+1) + \frac{1}{4}m_3(\alpha-2)\right],\\
a_2 &= \frac{\sqrt{3}\alpha(\alpha+2)m_1m_3}{4\varrho^{\alpha+2}},\\
a_3 &= -\frac{\alpha(\alpha+1)m_1m_3}{\rho^{\alpha+2}},\\
a_4 &= \frac{\alpha m_1}{\varrho^{\alpha+2}}\left[\frac{1}{4}m_3(3\alpha+2) - m_2\right],\\
a_5 &= \frac{\alpha m_1 m_2}{\varrho^{\alpha+2}},\\
a_6 &= \frac{\alpha m_2}{\varrho^{\alpha+2}}\left[m_1(\alpha+1) + \frac{1}{4}m_3(\alpha-2)\right],\\
a_7 &= -\frac{\sqrt{3}\alpha(\alpha+2)m_2m_3}{4\varrho^{\alpha+2}},
\end{aligned}
$$

$$a_8 = \frac{\alpha m_2}{\varrho^{\alpha+2}}\left[-m_1 + \frac{1}{4}m_3(3\alpha+2)\right].$$

Appendix C

The Giorgilli–Galgani algorithm for the Normal Form

The global idea of the Normal Form consists in to transform the Hamiltonian in a neighbourhood of the origin in a simpler one performing changes of coordinates. In this appendix we shall describe briefly the algorithm of A.Giorgilli and L.Galgani (see [G.G.]) that we have used in chapters 2 and 6.

We consider a Hamiltonian system with n degrees of freedom defined by the Hamiltonian function

$$H(\boldsymbol{\xi}, \boldsymbol{\eta}) \quad = \quad \sum_{k \geq 2} H_k(\boldsymbol{\xi}, \boldsymbol{\eta}), \tag{C.1}$$

where $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_n)$, $\boldsymbol{\eta} = (\eta_1, \ldots, \eta_n)$ and H_k is an homogeneous polynomial of order k, $k \geq 2$, in the variables $\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_n$. We note that the origin is an equilibrium point for the system. We assume that the second order terms are given by $H_2(\boldsymbol{\xi}, \boldsymbol{\eta}) = \sum_{i=1}^{n} \lambda_j \xi_j \eta_j$, being $\pm\lambda_j$, $j = 1, \ldots, n$ the eigenvalues of the linearized system on the equilibrium point.

We begin with some definitions.

If $\mathbf{l} = (l_1, \ldots, l_n) \in \mathbb{Z}^n$ and $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$ we shall denote by $\mathbf{l} \cdot \boldsymbol{\lambda} = l_1\lambda_1 + \ldots + l_n\lambda_n$.

Definition C.1. We say that the vector of eigenvalues $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$ is **non resonant** if the equation

$$\mathbf{l} \cdot \boldsymbol{\lambda} = l_1\lambda_1 + l_2\lambda_2 + \ldots + l_n\lambda_n = 0,$$

with $l_1, l_2, \ldots, l_n$ integers, only has the trivial solution, that is, if $l_1 = \ldots = l_n = 0$ is the unique solution.

In other case, we say that $\boldsymbol{\lambda}$ is **resonant**.

Definition C.2. We define the **resonant $\mathbb{Z}$-modulus** associated to H_2 as

$$M_{\boldsymbol{\lambda}} = \{\boldsymbol{\nu} \in \mathbb{Z}^n : \boldsymbol{\nu} \cdot \boldsymbol{\lambda} = 0\}.$$

We note that if $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$ is non resonant, then $M_{\boldsymbol{\lambda}} = \{0\}$.

Notation C.3. We shall denote by Π_k the space of all the homogeneous polynomials in the variables $(\boldsymbol{\xi}, \boldsymbol{\eta}) \in \mathbb{C}^{2n}$ and by $\Pi = \oplus_{k\in\mathbb{N}}\Pi_k$ the vector space of the formal series in $\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_n$. Given $f \in \Pi$, we write $f = \sum_{k\geq 1} f_k$ with $f_k \in \Pi_k$. If $f \in \Pi_k$, we shall write $f = \sum_{|\mathbf{l}+\mathbf{s}|=k} f_{\mathbf{l},\mathbf{s}}\boldsymbol{\xi}^{\mathbf{l}}\boldsymbol{\eta}^{\mathbf{s}}$, where $\mathbf{l} = (l_1, \ldots, l_n), \mathbf{s} = (s_1, \ldots, s_n) \in (\mathbb{N} \cup \{0\})^n$, $|\mathbf{l}+\mathbf{s}| = l_1+s_1+l_2+s_2+\ldots+l_n+s_n$ i $\boldsymbol{\xi}^{\mathbf{l}} = \xi_1^{l_1} \cdots \xi^{l_n}, \boldsymbol{\eta}^{\mathbf{s}} = \eta_1^{s_1} \cdots \eta_n^{s_n}$.

Definition C.4. Given a vector of resonant eigenvalues $\boldsymbol{\lambda}$, $\boldsymbol{\xi}^l\boldsymbol{\eta}^s$ is a **resonant monomial** associated to $\boldsymbol{\lambda}$ if $(s-l)\cdot\boldsymbol{\lambda} = 0$.

In this case, the order of the monomial is $|\mathbf{l}+\mathbf{s}| = |\mathbf{l}| + |\mathbf{s}|$.

Definition C.5. Given $G \in \Pi$, $G = G_3 + G_4 + \ldots$, $G_k \in \Pi_k$ we define the map $T_G : \Pi \longrightarrow \Pi$ by

$$T_G g = g_0 + g_1 + \ldots,$$

where $g \in \Pi$ and

$$g_0 \;=\; g, \quad g_k = \sum_{m=1}^{k} \frac{m}{k} L_{G_{2+m}} g_{k-m}.$$

Here, $L_g f = \{g, f\} = \sum_{j=1}^n \left(\frac{\partial g}{\partial \xi_j}\frac{\partial f}{\partial \eta_j} - \frac{\partial g}{\partial \eta_j}\frac{\partial f}{\partial \xi_j}\right)$ és el parèntesi de Poison.

In general, g is not of a given order s, and the same for g_k. However, if g has order s, then g_k is of order $k+s$. By this result, we can express the map T_G in a practice way for our purpose.

Remark C.6. Let us consider $f \in \Pi$. We write $f = \sum_{k\geq 1} f_k$ where $f_k \in \Pi_k$, and $T_G f = \sum_{k\geq 1} F_k$ with $F_k \in \Pi_k$. Then, by an application of the above definition to $T_G f$, we obtain $F_k = \sum_{l=1}^k f_{l,k-l}$ on

$$f_{l,0} \;=\; f_l, \quad f_{l,k} = \sum_{m=1}^{k} \frac{m}{k} L_{G_{2+m}} f_{l,k-m},$$

and $f_{l,k-l} \in \Pi_k$ for all $l = 1, \ldots, k$.

Lemma C.7. The map T_G is linear and invertible; moreover, if $f, g \in \Pi$, then

$$T_G\{f, g\} = \{T_G f, T_G g\}.$$

In particular, T_G is a canonical map.

We define a tranformation from $(\mathbf{X},\mathbf{Y}) \in \mathbb{C}^{2n}$ to $(\boldsymbol{\xi},\boldsymbol{\eta}) \in \mathbb{C}^{2n}$ given by $\xi_j = T_G X_j$, $\eta_j = T_G Y_j$, $j = 1,\dots,n$. From the above lemma, this change of coordinates is canonic. Moreover, if $f \in \Pi$, $f = f(\boldsymbol{\xi},\boldsymbol{\eta})$ and we take $T_G f$ as in the remark C.6, we have that

$$T_G f(\mathbf{X},\mathbf{Y}) = F(\mathbf{X},\mathbf{Y}) = f(T_G\boldsymbol{\xi}, T_G\boldsymbol{\eta}).$$

We are interested in to compute the Normal Form of the Hamiltonian (C.1). To this end we give a definition of Normal Form of a Hamiltonian function.

Definition C.8. Given a $\mathbb{Z}$-modulus $M \supset M_{\boldsymbol{\lambda}}$, $H^{(r)}$ is in Normal Form up to order r, respect to the modulus M, if $H^{(r)}$ is of the form $H^{(r)} = Z^{(r)} + R^{(r)}$, where

$$Z^{(r)} = \sum_{|\mathbf{l}+\mathbf{s}|\le r,\,\mathbf{l}-\mathbf{s}\in M} a_{\mathbf{l},\mathbf{s}}\boldsymbol{\xi}^{\mathbf{l}}\boldsymbol{\eta}^{\mathbf{s}}, \qquad R^{(r)} = \sum_{|\mathbf{l}+\mathbf{s}|>r} a_{\mathbf{l},\mathbf{s}}\boldsymbol{\xi}^{\mathbf{l}}\boldsymbol{\eta}^{\mathbf{s}}.$$

The idea is that $Z^{(r)}$ contains the terms of the Hamiltonian (C.1) that can not be eliminated by the change of variables. In particular, if $\mathbf{l} = \mathbf{s}$ with $2|\mathbf{l}| = r$, the corresponding monomial $\boldsymbol{\xi}^{\mathbf{l}}\boldsymbol{\eta}^{\mathbf{l}}$ is in the expression of $Z^{(r)}$.

We want to apply a transformation T_G, for some G, in such a way that $T_G H$ is in Normal Form up to order r. Therefore, G is the function that transforms the Hamiltonian (C.1) to Normal Form $H^{(r)}$.

We take $G^{(r)} = \sum_{k=3}^{r} G_k$ with $G_k \in \Pi_k$. As we want that $T_{G^{(r)}}H = H^{(r)}$ holds, then $T_{G^{(r)}}H = Z^{(r)} + R^{(r)}$. The idea is to determine G_k, Z_k recurrently. In order to do that we shall use the Giorgilli–Galgani algorithm (see [G.G.]).

Proposition C.9. Let be $H = \sum_{k\ge 2} H_k$, $H_2(\boldsymbol{\xi},\boldsymbol{\eta}) = \sum_{j=1}^{n} \lambda_j \xi_j \eta_j$ and $M \supset M_{\boldsymbol{\lambda}}$. Then, there exists a generating function $G^{(r)} = \sum_{k=3}^{r} G_k$ such that $T_{G^{(r)}}H$ is in Normal Form up to order r respect to M. If we write the Normal Form as $T_{G^{(r)}}H = Z^{(r)} + R^{(r)}$ where $Z^{(r)} = \sum_{k=2}^{r} Z_k$ then

$$\begin{aligned} Z_2 &= H_2, \quad \text{and} \quad L_{H_2}G_k + Z_k = F_k \ \text{ for } \ k \ge 3, \ \text{ where} \\ F_3 &= H_3 \quad \text{and} \\ F_k &= \sum_{m=1}^{k-3} \frac{m}{k-2} L_{G_{2+m}} Z_{k-m} + \sum_{m=1}^{k-2} \frac{m}{k-2} H_{2+m,k-m-2} \quad \text{for } \ k \ge 4 \end{aligned}$$

where $H_{l,0} = H_l, H_{l,k} = \sum_{m=1}^{k} \frac{m}{k} L_{G_{2+m}} H_{l,k-m}$.

Moreover, $T_{G^{(r)}}H = \sum_{m=1}^{r} \bar{H}_m + R^{(r)}$ where $\bar{H}_k = \sum_{l=1}^{k} H_{l,k-l}$.

For a proof see [Si.].

Appendix D

Qualitative description of a centre–centre–saddle point

In this appendix we describe the behaviour of the solutions in a neighbourhood of a centre–centre–saddle point. We give this description following the ideas introduced by Conley in [Co.2] for the restricted three body problem.

We consider the Hamiltonian

$$H(\mathbf{Q},\mathbf{P}) \;=\; \sum_{k\geq 2} H_k(\mathbf{Q},\mathbf{P}), \tag{D.1}$$

where $\mathbf{Q} = (Q_1, Q_2, Q_3) \in \mathbb{R}^3$, $\mathbf{P} = (P_1, P_2, P_3) \in \mathbb{R}^3$, H_k is an homogeneous polynomial of order k in the variables Q_j, P_j, $j = 1, \ldots, 3$, and H_2 has the following form

$$H_2(\mathbf{Q},\mathbf{P}) \;=\; \lambda Q_1 P_1 + \frac{1}{2}\omega_1(Q_2^2 + P_2^2) + \frac{1}{2}\omega_2(Q_3^2 + P_3^2), \tag{D.2}$$

being $\lambda, \omega_1, \omega_2$ positive real constants.

We note that the origin is an equilibrium point of centre–centre–saddle type. From $H_2(\mathbf{Q},\mathbf{P})$ one can describe qualitatively the solutions in the neighbourhood of the equilibrium point.

We consider the solutions of the linear system of equations given by $H_2(\mathbf{Q},\mathbf{P})$. They can be written as

$$\begin{array}{rclc} Q_1(t) & = & Q_1^0 e^{\lambda t}, & z_1(t) = Q_2(t) + \mathrm{i}P_2(t) = z_1^0 e^{-i\omega_1 t}, \\ P_1(t) & = & P_1^0 e^{-\lambda t}, & z_2(t) = Q_3(t) + \mathrm{i}P_3(t) = z_2^0 e^{-i\omega_2 t}, \end{array} \tag{D.3}$$

where Q_1^0, P_1^0 are real constants and z_1^0, z_2^0 complex ones. Clearly for the linearized system, the stable and unstable manifolds of the origin are obtained as $Q_1^0 = 0, z_1^0 =$

$z_2^0 = 0$ and $P_1^0 = 0, z_1^0 = z_2^0 = 0$ respectively. The centre manifold corresponds to $Q_1^0 = P_1^0 = 0$.

In (D.3) we distinguish two families of periodic orbits $\mathcal{F}_1$ and $\mathcal{F}_2$ which are obtained by taking $Q_1^0 = P_1^0 = z_2^0 = 0$ and $Q_1^0 = P_1^0 = z_1^0 = 0$ respectively. The period is $\dfrac{2\pi}{\omega_1}$ for the periodic orbits in the family $\mathcal{F}_1$ and $\dfrac{2\pi}{\omega_2}$ in family $\mathcal{F}_2$. These families can be parametrised by the energy h.

We fix some value of the energy $h \in \mathbb{R}$ and some constant $c > 0$ and we consider the set

$$L(h, c) = \{(\mathbf{Q}, \mathbf{P}) \in \mathbb{R}^6 \,|\, H_2(\mathbf{Q}, \mathbf{P}) = h, |Q_1 - P_1| \leq c\}.$$

From $H_2(\mathbf{Q}, \mathbf{P}) = h$ we get

$$\tau := \frac{1}{2}\omega_1(Q_2^2 + P_2^2) + \frac{1}{2}\omega_2(Q_3^2 + P_3^2) = h - \lambda Q_1 P_1 \tag{D.4}$$

and, hence, the motion is restricted to the region of the phase space such that $h - \lambda Q_1 P_1 \geq 0$.

We define in $L(h, c)$ the following sets

$$S_e \;=\; \{(\mathbf{Q}, \mathbf{P}) \in L(h, c) | Q_1 - P_1 = c\}$$

and

$$S_m \;=\; \{(\mathbf{Q}, \mathbf{P}) \in L(h, c) | Q_1 - P_1 = -c\}.$$

If $(\mathbf{Q}, \mathbf{P}) \in L(h, c)$ using $Q_1 P_1 = \dfrac{1}{4}[(Q_1 + P_1)^2 - (Q_1 - P_1)^2]$ we get

$$\frac{\lambda}{4}(Q_1 + P_1)^2 + \frac{1}{2}\omega_1(Q_2^2 + P_2^2) + \frac{1}{2}\omega_2(Q_3^2 + P_3^2) \leq h + \frac{\lambda c^2}{4}.$$

Therefore, if $h + \dfrac{\lambda c^2}{4} > 0$, S_e is homeomorphic to a sphere $\mathbb{S}^4$. In a similar way, S_m is homeomorphic to $\mathbb{S}^4$. This is the case if $h \geq 0$. Note that for a fixed value of h, these spheres separate the constant energy submanifold which is 5–dimensional. Figure D.1 shows the projection of the linear flow on the plane Q_1, P_1 in a neighbourhood of the origin. We note that the orbits enter in $L(h, c)$ through one of the spheres S_e or S_m. Only in the case $h > 0$ there are orbits which go out of $L(h, c)$ through the other sphere.

We note also that for $h > 0$ if we take $Q_1 = P_1 = 0$, (D.4) defines a three–dimensional sphere $\mathbb{S}^3$. In fact, this sphere is the intersection of the centre manifold of the equilibrium point with the corresponding energy level. The linear flow restricted to this sphere is given by the product of two harmonic oscillators with frequencies ω_1 and ω_2. So, for any $h > 0$, the sphere contains two periodic orbits,

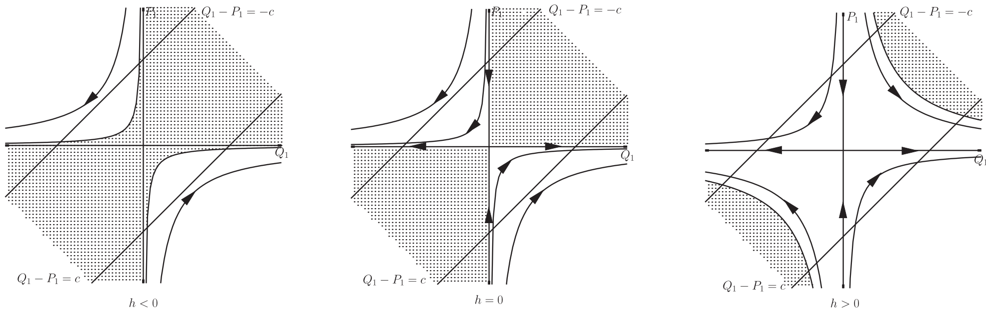


Figure D.1: Projection of the linear flow on the plane (Q_1, P_1) in a neighbourhood of the origin. The dashed area corresponds to the forbidden region.

corresponding to families $\mathcal{F}_1$ and $\mathcal{F}_2$ respectively, each one surrounded by two–dimensional invariant tori ([M.H.]). The invariant unstable and stable manifolds of this centre manifold reduced to the level h, are the objects which decide which one of the transitions $S_e \to S_e, S_e \to S_m$, $S_m \to S_e$, $S_m \to S_m$ occurs. To see the relevant role of these codimension one manifolds see [G.J.M.S.], [Si.2].

Now we consider the full Hamiltonian $H(\mathbf{Q}, \mathbf{P})$. In order to study the preservation of the families of periodic orbits we shall use the Lyapunov theorem ([S.M.]).

Theorem D.1 *(Lyapunov).* Let us consider the Hamiltonian system

$$\begin{aligned} \dot{\mathbf{Q}}_k &= H_{\mathbf{P}_k}, \\ \dot{\mathbf{P}}_k &= -H_{\mathbf{Q}_k}, \quad k = 1, \ldots, n. \end{aligned}$$

Let us assume that the origin is an equilibrium point.

Let us denote by $\lambda_1, \ldots \lambda_n, -\lambda_1, \ldots, -\lambda_n$ the eigenvalues of the linearized system on $(\mathbf{0}, \mathbf{0})$.

We assume that $\lambda_1 = is_1$, where $s_1 \in \mathbb{R}^+$, and $\frac{\lambda_2}{\lambda_1}, \ldots, \frac{\lambda_n}{\lambda_1}$ are not integer. Then, there exists a family of real periodic solutions of the Hamiltonian system that depends analytically on a real parameter ϵ, in such a way that $\epsilon = 0$ corresponds to the equilibrium point. Moreover, the period $\tau(\epsilon)$ is an analytic function on ϵ, and $\tau(\epsilon)$ tends to $\frac{2\pi}{|\lambda_1|}$ as $\epsilon \to 0$.

We consider H in (D.1). The quadratic part is given by H_2 in (D.2). If $\dfrac{\omega_2}{\omega_1} \notin \mathbb{N}$, the preservation of the family of periodic orbits $\mathcal{F}_1$ associated to the eigenvalue $i\omega_1$ is obtained by using the Lyapunov theorem. If $\dfrac{\omega_1}{\omega_2} \notin \mathbb{Z}$ we get a family of periodic solutions of $H(\mathbf{Q}, \mathbf{P})$ with limit period $\dfrac{2\pi}{\omega_2}$.

Appendix E

Expansion of the Hamiltonian of the Planar Three Body Problem with Newtonian potential

In order to perform the change of variables to compute the Normal Form of the Hamiltonian (6.13) it is necessary to know the expression of the terms of order 3 and 4 for these Hamiltonian. This terms can be obtained from the expression of the terms of order 3 and 4 of the Hamiltonian (6.7) applying the change of variables (6.9) and (6.12).

In this appendix we give the expression of the terms of order $0, 2, 3, 4$ of the Hamiltonian (6.7) in the variables $(\mathbf{x}, \mathbf{y})$. Then, if $\mathcal{H}_i$ denote the terms of order $i, i = 0, 2, 3, 4$ of (6.7), we have that $\mathcal{H}_i = \dfrac{1}{a\rho} H_i, i = 0, 2, 3, 4$ where

$$
\begin{aligned}
H_0(\mathbf{x}, \mathbf{y}) &= \frac{\alpha_2 \left(\eta_3^p\right)^2}{2} + \frac{A\eta_3^p}{m_3 a \left(\rho + 1\right)} + \frac{\alpha_1 \left(A\right)^2}{2a^2 \left(\rho + 1\right)^2} - \frac{m_1 m_2}{a\rho} - \frac{m_1 m_3}{a \left(\rho + 1\right)} - \\
&\quad \frac{m_2 m_3}{a} + \eta_4^p
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{H}_2(\mathbf{x}, \mathbf{y}) &= \frac{1}{a} \left[-\frac{m_1 m_2}{\rho} - \frac{m_1 m_3 \rho^2}{\left(\rho + 1\right)^3} + \frac{A\eta_3^p \rho^2}{m_3 \left(\rho + 1\right)^3} + \frac{3\alpha_1 \left(A\right)^2 \rho^2}{2a \left(\rho + 1\right)^4} \right] x_1^2 + \\
&\quad + \left(\frac{\rho^2 \left(\eta_3^p\right)^2}{m_3 \left(\rho + 1\right)^2} + \frac{2\alpha_1 \rho^2 \eta_3^p A}{a \left(\rho + 1\right)^3} + \frac{2m_1 m_2}{a\rho} \right) x_1 x_2 + \\
&\quad + \frac{\rho}{\left(\rho + 1\right)^2} \left(-\frac{A}{m_3 a} + \frac{\eta_3^p}{m_3} + \frac{2\alpha_1 A}{a \left(\rho + 1\right)} \right) x_1 y_3 +
\end{aligned}
$$

$$
\begin{aligned}
&+\left(\frac{\alpha_1\rho^2\left(\eta_3^p\right)^2}{2\left(\rho+1\right)^2}-\frac{m_1m_2}{a\rho}-\frac{m_2m_3\rho^2}{a}\right)x_2^2+\\
&+\frac{\rho}{\rho+1}\left(\frac{-2\eta_3^p}{m_3}-\frac{\alpha_1 A}{a\left(\rho+1\right)}+\frac{\alpha_1\eta_3^p}{\rho+1}\right)x_2y_3+\frac{m_2}{2a}\left(\frac{m_1}{\rho}+m_3\rho^2\right)x_3^2+\\
&+\frac{\rho}{\rho+1}\left(\frac{\alpha_1 A}{a\left(\rho+1\right)}+\frac{\eta_3^p}{m_3}\right)x_3y_2+\frac{\alpha_1}{2}y_1^2+\frac{\alpha_2}{2}y_2^2+\frac{1}{m_3}y_1y_2+\\
&+\left(\frac{\alpha_2}{2}-\frac{1}{m_3\left(\rho+1\right)}+\frac{\alpha_1}{2\left(\rho+1\right)^2}\right)y_3^2
\end{aligned}
$$

$$
\begin{aligned}
H_3(\mathbf{x},\mathbf{y}) \;=\;& \frac{1}{a}\left(\frac{-A\eta_3^p\rho^3}{m_3\left(\rho+1\right)^4}-\frac{2\alpha_1\rho^3\left(A\right)^2}{a\left(\rho+1\right)^5}+\frac{m_1m_2}{\rho}+\frac{m_1m_3\rho^3}{\left(\rho+1\right)^4}\right)x_1^3+\\
&+\left(\frac{-\rho^3\left(\eta_3^p\right)^2}{m_3\left(\rho+1\right)^3}-\frac{3\alpha_1\rho^3\eta_3^pA}{a\left(\rho+1\right)^4}-\frac{3m_1m_2}{a\rho}\right)x_1^2x_2+\\
&+\frac{\rho^2}{\left(\rho+1\right)^3}\left(\frac{A}{m_3a}-\frac{\eta_3^p}{m_3}-\frac{3\alpha_1 A}{a\left(\rho+1\right)}\right)x_1^2y_3+\\
&+\frac{\rho^2}{\left(\rho+1\right)^2}\left(\frac{2\eta_3^p}{m_3}+\frac{2\alpha_1 A}{a\left(\rho+1\right)}-\frac{2\alpha_1\eta_3^p}{\rho+1}\right)x_1x_2y_3+\\
&\frac{\rho}{\left(\rho+1\right)^2}\left(\frac{1}{m_3}-\frac{\alpha_1}{\rho+1}\right)x_1y_3^2+\\
&+\frac{\rho^2}{\left(\rho+1\right)^2}\left(\frac{-\eta_3^p}{m_3}-\frac{2\alpha_1 A}{a\left(\rho+1\right)}\right)x_1x_3y_2+\\
&\left(\frac{-\alpha_1\rho^3\left(\eta_3^p\right)^2}{\left(\rho+1\right)^3}+\frac{3m_1m_2}{a\rho}\right)x_1x_2^2+\\
&+\frac{\alpha_1\rho^2\eta_3^p}{\left(\rho+1\right)^2}x_2^2y_3-\frac{3m_1m_2}{2a\rho}x_1x_3^2+\frac{m_2}{a}\left(\rho^3m_3-\frac{m_1}{\rho}\right)x_2^3+\\
&\frac{\rho}{\rho+1}\left(\frac{1}{m_3}-\frac{\alpha_1}{\rho+1}\right)x_3y_2y_3+\\
&+\frac{\rho}{\rho+1}\left(\frac{\alpha_1}{\rho+1}-\frac{1}{m_3}\right)x_2y_3^2-\frac{\alpha_1\rho^2\eta_3^p}{\left(\rho+1\right)^2}x_2x_3y_2+\\
&\frac{3m_2}{2a}\left(\frac{m_1}{\rho}-m_3\rho^3\right)x_2x_3^2
\end{aligned}
$$

$$
\begin{aligned}
H_4(\mathbf{x},\mathbf{y}) \;=\;& \frac{1}{a}\left(\frac{A\eta_3^p\rho^4}{m_3\left(\rho+1\right)^5}+\frac{5\alpha_1\left(A\right)^2\rho^4}{2a\left(\rho+1\right)^6}-\frac{m_1m_2}{\rho}-\frac{m_1m_3\rho^4}{\left(\rho+1\right)^5}\right)x_1^4+\\
&+\frac{\rho^3}{\left(\rho+1\right)^4}\left(\frac{-A}{m_3a}+\frac{\eta_3^p}{m_3}+\frac{4\alpha_1 A}{a\left(\rho+1\right)}\right)x_1^3y_3+
\end{aligned}
$$

$$+\frac{\rho^3}{(\rho+1)^3}\left(\frac{-2\eta_3^p}{m_3}-\frac{3\alpha_1 A}{a(\rho+1)}+\frac{3\alpha_1\eta_3^p}{\rho+1}\right)x_1^2x_2y_3+$$
$$+\frac{\rho^2}{(\rho+1)^3}\left(\frac{-1}{m_3}+\frac{3\alpha_1}{2(\rho+1)}\right)x_1^2y_3^2+\frac{\rho^2}{(\rho+1)^2}\left(\frac{-1}{m_3}+\frac{2\alpha_1}{\rho+1}\right)x_1x_3y_2y_3+$$
$$+\frac{\rho^2}{(\rho+1)^2}\left(\frac{1}{m_3}-\frac{2\alpha_1}{\rho+1}\right)x_1x_2y_3^2+$$
$$\left(\frac{(\eta_3^p)^2\rho^4}{m_3(\rho+1)^4}+\frac{4\alpha_1 A\rho^4\eta_3^p}{a(\rho+1)^5}+\frac{4m_1m_2}{a\rho}\right)x_1^3x_2+$$
$$+\frac{\rho^3}{(\rho+1)^3}\left(\frac{\eta_3^p}{m_3}+\frac{3\alpha_1 A}{a(\rho+1)}\right)x_1^2x_3y_2+3\left(\frac{\alpha_1\rho^4(\eta_3^p)^2}{2(\rho+1)^4}-\frac{2m_1m_2}{a\rho}\right)x_1^2x_2^2+$$
$$+\frac{2\alpha_1\rho^3\eta_3^p}{(\rho+1)^3}x_1x_2x_3y_2-\frac{2\alpha_1\rho^3\eta_3^p}{(\rho+1)^3}x_1x_2^2y_3+\frac{\alpha_1\rho^2}{2(\rho+1)^2}x_3^2y_2^2-\frac{\alpha_1\rho^2}{(\rho+1)^2}x_2x_3y_2y_3+$$
$$+\frac{\alpha_1\rho^2}{2(\rho+1)^2}x_2^2y_3^2-\frac{m_2}{a}\left(\frac{m_1}{\rho}+\rho^4m_3\right)x_2^4-\frac{3m_2}{8a}\left(\frac{m_1}{\rho}+\rho^4m_3\right)x_3^4+$$
$$\frac{4m_1m_2}{a\rho}x_1x_2^3++\frac{3m_2}{a}\left(\frac{m_1}{\rho}+\rho^4m_3\right)x_2^2x_3^2-\frac{6m_1m_2}{a\rho}x_1x_2x_3^2+\frac{3m_1m_2}{a\rho}x_1^2x_3^2,$$

where the constants $a, \rho, \eta_3^p, \eta_4^p, A, \alpha_1, \alpha_2$ have been introduced in chapter 6.

In the symmetric case in which the masses satisfy $m_1 = m_3$ the expressions for H_2 and H_3 are reduced to

$$\begin{aligned}
H_2(\mathbf{x},\mathbf{y}) &= \frac{1}{m_1}y_1^2+\frac{\alpha_2}{2}y_2^2+\frac{1}{m_1}y_1y_2+\frac{1}{4m_1m_2}y_3^2+\frac{a}{2}x_1y_3+ax_3y_2- \\
&\quad -ax_2y_3+\frac{m_1}{16a}(m_1-4m_2)x_1^2-\frac{2m_1m_2}{a}x_2^2+\frac{2m_1m_2}{a}x_1x_2+ \\
&\quad \frac{m_1m_2}{a}x_3^2, \\
H_3(\mathbf{x},\mathbf{y}) &= \frac{m_1}{16a}(8m_2-m_1)x_1^3+\frac{3m_1m_2}{a}x_1x_2^2-\frac{3m_1m_2}{a}x_1^2x_2-\frac{3m_1m_2}{2a}x_1x_3^2- \\
&\quad -\frac{a}{2}x_1^2y_3-ax_1x_3y_2+ax_1x_2y_3.
\end{aligned}$$

Appendix F

Constants of the linear part of the system on L_2 of the Planar Three Body Problem with Newtonian potential

In this appendix we give the expression of the constants $k_i, i = 1, \ldots, 7$, that define the matrix $DF(\boldsymbol{\xi}^p, \boldsymbol{\eta}^p)$ in (6.16) in both general and symmetric case $m_1 = m_3$.

For any positive masses these constants are

$$
\begin{aligned}
k_1 &= \frac{\eta_4^p}{m_3 a^2(\rho+1)^2} + \frac{2}{\rho+1}, \\
k_2 &= \frac{\eta_3^p}{a(\rho+1)}\left(\frac{\alpha_1}{\rho+1} - \frac{1}{m_3}\right) - 1, \\
k_3 &= \alpha_2 - \frac{2}{m_3(\rho+1)} + \frac{\alpha_1}{(\rho+1)^2}, \\
k_4 &= \frac{2m_1m_2}{a^3\rho^3} + \frac{2m_1m_3}{a^3(\rho+1)^3} + \frac{A_p\eta_3^p}{m_3a^3(\rho+1)^3} - \frac{3A_p}{a^2(\rho+1)^2}, \\
k_5 &= -\frac{2m_1m_2}{a^3\rho^3} + \frac{(\eta_3^p)^2}{m_3a^2(\rho+1)^2} - \frac{2\eta_3^p}{a(\rho+1)}, \\
k_6 &= \frac{2m_1m_2}{a^3\rho^3} + \frac{2m_2m_3}{a^3} - \frac{(\eta_3^p)^2\alpha_1}{a^2(\rho+1)^2}, \\
k_7 &= -\frac{m_1m_2}{a^3\rho^3} - \frac{m_2m_3}{a^3},
\end{aligned}
$$

where $A_p = -\eta_4^p - a\eta_3^p, \eta_3^p = m_2a(-m_1\rho+m_3), \eta_4^p = -a^2[m_1m_3(\rho+1)^2+m_2(m_1\rho^2+m_3)], a^3 = -\dfrac{m_1}{\rho^2} + \dfrac{m_1}{(\rho+1)^2} + m_2 + m_3$, ρ is the solution of the Euler quintic equation (1.52) and $\alpha_i = \dfrac{m_i + m_3}{m_i m_3}, i = 1, 2.$

The matrices $D(\mu)$ and $E(\mu)$ defined in (6.20) can be expressed as

$$D(\mu) = \begin{pmatrix} \mu^2 - \alpha_1\tilde{c}_1 - \frac{1}{m_3}\tilde{c}_2 & -\alpha_1\tilde{c}_2 - \frac{1}{m_3}\tilde{c}_3 & \frac{\mu(\alpha_1 k_1 + \frac{1}{m_3}k_2)}{k_3} \\ -\frac{1}{m_3}\tilde{c}_1 - \alpha_2\tilde{c}_2 & \tilde{\mu}^2 - \frac{1}{m_3}\tilde{c}_2 - \alpha_2\tilde{c}_3 & \mu\left(\frac{\frac{1}{m_3}k_1 + \alpha_2 k_2}{k_3} - 1\right) \\ \mu(-m_1 m_2 k_3 - k_1) & \mu(m_1 m_2 m_3 \alpha_1 k_3 - k_2) & \mu^2 - k_3(m_1 m_2 m_3 \alpha_1 + k_7) \end{pmatrix}$$

and

$$E(\mu) = \begin{pmatrix} m_1 m_2 m_3 \alpha_2 \mu & -m_1 m_2 \mu & m_1 m_2 \\ -m_1 m_2 \mu & m_1 m_2 m_3 \alpha_1 \mu & -m_1 m_2 m_3 \alpha_1 \\ -\frac{k_1}{k_3} & -\frac{k_2}{k_3} & \frac{\mu}{k_3} \end{pmatrix},$$

where $\tilde{c}_1 = \dfrac{k_1^2}{k_3} + k_4$, $\tilde{c}_2 = \dfrac{k_1 k_2}{k_3} + k_5$ and $\tilde{c}_3 = \dfrac{k_2^2}{k_3} + k_6$.

In the symmetric case $m_1 = m_3$ we have

$$\begin{aligned} k_1 &= \frac{1}{2}, & k_5 &= -\frac{2m_1 m_2}{a^3}, & & \\ k_2 &= -1, & k_6 &= \frac{4m_1 m_2}{a^3}, & & \\ k_3 &= \frac{1}{2m_1 m_2}, & k_7 &= -\frac{2m_1 m_2}{a^3}, & k_4 &= \frac{m_1}{8a^3}(4m_2 - m_1), \end{aligned}$$

where $a^3 = \frac{1}{4}(4m_2 + m_1)$.

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Introducció

La Mecànica Celeste s'ocupa de l'estudi del moviment dels astres. El seu punt d'inici pot situar-se al segle XVII, quan Johannes Kepler va formular les lleis del moviment dels planetes a *Astronomia Nova* (1609). L'any 1687 Newton va donar a la seva obra *Philosophiae Naturalis Principia Mathematica* la formulació del principal objecte d'estudi de la Mecànica Celeste: el problema de n–cossos. Aquest problema estudia el moviment d'un sistema de n partícules que s'atrauen mútuament d'acord amb la Llei de Gravitació Universal de Newton. Tot i que la formulació de les equacions que descriuen el problema de n–cossos és senzilla, no ho és la seva resolució. De fet, l'únic cas completament resolt és el problema de 2–cossos. Tots els esforços per resoldre explícitament les equacions per $n \geq 3$ han estat fallits. Poincaré va demostrar que la principal dificultat prové de l'existència de petits divisors. En el seu famós treball *Méthodes Nouvelles de la Mécanique Céleste* (1899), Poincaré inicia l'estudi del problema des d'un punt de vista qualitatiu. En realitat, els mètodes qualitatius juguen un paper molt important en l'estudi de les equacions diferencials.

Tot i això, es coneixen algunes solucions especials del problema de n–cossos: les solucions homogràfiques. Per a aquestes solucions la configuració de les partícules es preserva en el temps. Això només s'aconsegueix en les anomenades configuracions centrals. És ben sabut que per al Problema Pla de Tres Cossos existeixen tres configuracions centrals col·lineals, on les masses estan situades sobre una recta, i dos de triangulars, en les que les masses es troben sobre els vèrtexs d'un triangle equilàter. Pel que fa a la quantitat i tipus de les configuracions centrals per $n \geq 4$ només es coneixen resultats parcials.

D'altra banda, per moltes aplicacions es poden fer diverses suposicions que simplifiquen el problema matemàtic. El Problema Restringit de Tres Cossos (RTBP) és un dels models més utilitzats com a una primera aproximació a moltes aplicacions. En aquest problema la principal suposició és que un dels cossos té massa infinitesimal, de forma que no influeix en el moviment dels altres dos cossos, anomenats primaris. Així es pot suposar que els primaris es mouen sobre una solució del problema de Kepler. El Problema Restringit de Tres Cossos prova d'explicar el moviment de la massa infinitesimal influïda per les forces gravitatòries exercides pels primaris. Els casos més interessants per les aplicacions es corresponen amb òrbites el·líptiques dels primaris. Si l'excentricitat, e, d'aquestes òrbites és zero obtenim el Problema Restringit de Tres Cossos Circular, i per $e \in (0, 1)$ el Problema Restringit de Tres Cossos El·líptic.

En un sistema de coordenades giratori el Problema Restringit de Tres Cossos Circular queda descrit per un sistema Hamiltonià amb dos graus de llibertat ([Sz.]). És ben sabut que en aquest problema hi ha tres punts d'equilibri col·lineals $L_{1,2,3}$ i dos de triangulars $L_{4,5}$. Els punts d'equilibri col·lineals són de tipus centre–sella. Sigui C_{L_i} el valor de la constant de Jacobi en l'equilibri L_i. El teorema de Lyapunov ([S.M.],[Ms.]) assegura l'existència d'una família d'òrbites periòdiques que neixen de l'equilibri. Aquesta família es pot parametritzar per la constant de Jacobi de tal manera que per a un nivell d'energia fixat de la constant de Jacobi tal que $C_{L_i} - C$ és prou petit, l'òrbita periòdica és l'única òrbita acotada que per a tot temps es manté en un petit entorn del punt d'equilibri. A més, aquestes òrbites són hiperbòliques. Tenen varietats invariants 2–dimensionals estables i inestables de codimensió 1 un cop fixada la constant de Jacobi. Fent servir aquestes varietats invariants és possible donar una classificació de les òrbites que passen per un petit entorn dels punts d'equilibri col·lineals ([Co.2],[McG.1]). L'existència d'òrbites homoclíniques transversals a l'òrbita periòdica de Lyapunov s'ha estudiat a [L.M.S.] per a diversos valors del paràmetre de masses i la constant de Jacobi. Això permet presentar una dinàmica simbòlica ([L.M.S.],[Ms.2]) que dóna l'existència d'òrbites que passen per diferents regions de l'espai de fase. L'aplicabilitat d'aquestes òrbites a les missions espacials s'ha estudiat a [K.L.M.R.].

En aquest treball distingim tres parts principals. A la primera estudiem algunes qüestions relacionades amb l'estabilitat de les solucions homogràfiques. La segona part es dedica al RTBP Espacial. Per a aquest problema estudiem l'existència de connexions heteroclíniques/homoclíniques als tors invariants continguts en la varietat central del RTBP Espacial. Finalment, estudiem l'aplicabilitat del teorema KAM a la varietat central dels punts d'equilibri col·lineals en el Problema Pla de Tres Cossos. A continuació presentem els tres temes.

Solucions homogràfiques

Considerem el Problema Pla de Tres Cossos amb potencial homogeni de grau $-\alpha$, $0 < \alpha < 2$, del següent tipus

$$U(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3) = \frac{m_1 m_2}{\|\mathbf{q}_1 - \mathbf{q}_2\|^\alpha} + \frac{m_1 m_3}{\|\mathbf{q}_1 - \mathbf{q}_3\|^\alpha} + \frac{m_2 m_3}{\|\mathbf{q}_2 - \mathbf{q}_3\|^\alpha}.$$

Notem que si $\alpha = 1$ obtenim el potencial Newtonià. Es poden generalitzar les solucions homogràfiques introduïdes per al potencial Newtonià al cas general ($0 < \alpha < 2$). Aquestes solucions es poden escriure com punts d'equilibri d'un sistema Hamiltonià periòdic amb 6 graus de llibertat. Per a aconseguir-ho s'ha d'introduir un canvi de variables que depèn de manera quasiperiòdica (periòdica amb $\alpha = 1$) del temps.

Com ens interessa l'estabilitat d'aquestes solucions serà necessari calcular els valors propis de la matriu de monodromia. Per reduir 2 graus de llibertat primer fem servir les integrals del centre de masses. En aquest punt el sistema linealitzat per les solucions homogràfiques té ordre 8. Aleshores demostrem que es pot escriure el sistema com dos sistemes de dimensió 4 desacoblats. La matriu de monodromia d'un dels sistemes té 1 com a valor propi amb multiplicitat 4. Per tant, per a obtenir els multiplicadors característics no trivials ens cal estudiar l'altre sistema de dimensió 4,

$$\dot{\mathbf{x}} = A(t)\mathbf{x} \tag{1}$$

on t és l'anomalia veritable en el cas Newtonià. A més del grau d'homogeneïtat $-\alpha$, el sistema depèn de dos paràmetres: β, que depèn de les masses, i e, una excentricitat generalitzada. Notem que el paràmetre β és diferent en els casos col·lineal i triangular.

Quan e val zero, el sistema (1) té coeficients constants i els exponents característics, o equivalentment els paràmetres d'estabilitat, es calculen trivialment. A mesura que e creix poden aparèixer algunes bifurcacions. A més, a mesura que e s'acosta a 1, en el cas límit tenim una matriu $A(t)$ en (1) amb una singularitat a $t = 0$.

El nostre objectiu és estudiar l'estabilitat de sistemes que generalitzen en algun sentit el comportament del cas homogràfic linealitzat per a e proper a 0 i e proper a 1. Així, considerem sistemes lineals del següent tipus

$$\dot{\mathbf{x}} = A(t,e)\mathbf{x}, \qquad A(t,e) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \lambda_1 G_1(t,e) & 0 & 0 & -2 \\ 0 & \lambda_2 G_2(t,e) & 2 & 0 \end{pmatrix}, \tag{2}$$

on $\mathbf{x} \in \mathbb{R}^4$, λ_1, λ_2 són paràmetres reals, $e \in [0,1)$, i G_1, G_2 són funcions periòdiques en t, depenent de e. Estudiarem l'estabilitat per a $e \gtrsim 0$ i $e \lesssim 1$. En tot cas, formularem diverses hipòtesis sobre G_1 and G_2 que es satisfaran en particular en el cas homogràfic.

Un sistema com (2) té diverses aplicacions. Una d'elles es l'estudi de l'estabilitat per als equilibris d'alguns sistemes Mecànics. A més, el sistema (2) pot obtenir-se com a primer sistema de variació d'una solució periòdica d'un sistema autònom. Els capítols 2 i 3 es dediquen a l'estudi dels paràmetres d'estabilitat de (2) per a $e > 0$ prou petit i per $e \lesssim 1$, respectivament.

Al capítol 2 estudiem l'estabilitat del sistema (2) per $e \gtrsim 0$ en el cas que G_1, G_2 siguin funcions periòdiques parelles de t i analítiques en e. En primer lloc, a la secció 2.2 estudiem el cas trivial $e = 0$ on es troben alguns punts ressonants. A mesura que e creix poden aparèixer bifurcacions donant lloc a regions en els espais de paràmetres amb diferent caràcter d'estabilitat. Per estudiar les superfícies frontera d'aquestes regions fem servir la tècnica de la Forma Normal. A [B.S.1] es va usar aquest mètode per estudiar les llengües ressonants per a l'equació de Hill quasiperiòdica, depenent de dos paràmetres, i a [B.S.2] per a l'unfolding d'equacions de tipus Mathieu en el cas periòdic.

En aquest treball, per a estudiar les superfícies frontera ens concentrem principalment en el cas de d'Alembert, és a dir, suposem que per a G_1 i G_2 el $k$èssim harmònic té una amplitud al menys d'ordre k en e. De fet, és una situació molt comú en els sistemes mecànics. Per exemple, es dóna en l'estudi de l'estabilitat de famílies d'òrbites periòdiques amb origen un punt equilibri amb valors propis imaginaris purs. Suposant la propietat de d'Alembert distingim entre ressonàncies simples i dobles. El cas més interessant és el segon. Sota condicions de no degeneració en un entorn d'una ressonància doble, canviant els paràmetres es poden obtenir regions de qualsevol tipus per $e \gtrsim 0$.

Al capítol 3 estudiem l'estabilitat del sistema (2) per a $e \lesssim 1$. Suposem $G_1 = G_2$ amb alguna singularitat per a $e = 1$ en $t = 0$. El principal resultat del capítol és una fórmula asimptòtica per als paràmetres d'estabilitat. Fem servir una espècie de tècnica de blow up per veure el cas límit com una connexió heteroclínica.

Al capítol 4 fem servir els resultats dels capítols 2 i 3 per a estudiar l'estabilitat de les solucions homogràfiques del Problema Pla de Tres Cossos. En aquest cas, els paràmetres λ_1, λ_2 depenen d'un únic paràmetre de massa β. Per tant, el diagrama de bifurcació es representa en el pla (β, e) per a α fixat. Calculem els paràmetres ressonants en $e = 0$ per a qualsevol α. Ara bé, ens concentrem principalment en el cas Newtonià. Fent servir el Mètode de la Forma Normal desenvolupat al capítol 2, obtenim les llengües ressonants que neixen a $e = 0$ fins a un cert ordre.

En el cas col·lineal, $\beta \in (0, 7)$ per al problema físic. Tot i això, matemàticament es pot considerar $\beta > 0$. S'obtenen ressonàncies en $e = 0$ per a les freqüències $\dfrac{k}{2}$, $k \geq 3$. Si $k = 2n$ no hi ha bifurcació per a $e > 0$. Si $k = 2n + 1$ les llengües ressonants $\mathcal{T}_{\frac{2n+1}{2}}$ tenen origen en $e = 0$. Malgrat que $\mathcal{T}_{\frac{3}{2}}$, $\mathcal{T}_{\frac{5}{2}}$ són les úniques llengües que emanen de $e = 0$ per a $\beta \in (0, 7)$, totes les altres llengües $\mathcal{T}_{\frac{2n+1}{2}}$ entren en aquest rang de β per a valors de e en $(0, 1)$. L'amplada de $\mathcal{T}_{\frac{3}{2}}$, $\mathcal{T}_{\frac{5}{2}}$ és d'ordre 3 i 5 en e, respectivament. A més, la fórmula asimptòtica per a e proper a

1 prediu que totes aquestes llengües s'acumulen en $\beta = \dfrac{1}{8}$ a mesura que e s'acosta a 1. Aquest comportament concorda amb els càlculs numèrics fets per a qualsevol $e \in (0, 1)$.

Pel que fa al cas triangular, per a $0 < \beta < 1$ i $e = 0$ el sistema és el·líptic–el·líptic i només es troba una llengua ressonant $\mathcal{T}$ que neix per $\beta = \dfrac{3}{4}$. Aquesta defineix una regió el·líptica–hiperbòlica en el pla (β, e). L'amplada és d'ordre 1 en e. El comportament per a $\beta = \dfrac{3}{4}$ i $e \gtrsim 0$ fou estudiat per G. Roberts (veure [R.]). En aquest treball, desenvolupant la matriu de monodromia en sèrie de potències en e, demostra l'existència d'una regió el·líptica–hiperbòlica per a aquest valor de β i per a e prou petit. El mètode utilitzat a [R.] no és útil en el cas col·lineal perquè els càlculs són durs. Això es deu al fet que en el cas col·lineal l'amplada de les llengües és de major ordre en e i, per tant, fa falta calcular els termes com a mínim d'ordre 3 en e de la matriu de monodromia del sistema linealitzat sobre la solució col·lineal.

El RTBP Espacial

En el capítol 5 ens dediquem a l'estudi de les òrbites homoclíniques a la varietat central de L_2 en el Problema Restringit de Tres Cossos Espacial. És ben sabut que L_2 és un punt d'equilibri de tipus centre–centre–sella. Per tant, té varietats invariants estable i inestable unidimensionals, i una varietat central de dimensió 4. En un entorn de L_2 existeixen les ben conegudes famílies d'òrbites periòdiques de Lyapunov plana i vertical. Aquestes famílies d'òrbites periòdiques tenen varietats estable i inestable bidimensionals. A més, a la varietat central existeixen tors invariants, amb varietats estable i inestable tridimensionals. Sobre la dinàmica en la varietat central consulteu [J.M.], [G.M.]. La intersecció de la varietat inestable d'un tor en la varietat central i la varietat estable d'un altre tor dóna òrbites heteroclíniques del primer tor al segon. Si considerem les varietat estable i inestable del mateix tor, obtenim òrbites homoclíniques al tor. Totes aquestes òrbites homoclíniques i heteroclíniques son òrbites homoclíniques a la varietat central de L_2. Per a obtenir òrbites heteroclíniques (o homoclíniques) seguim les idees principals desenvolupades a [L.M.S.] per al RTBP Pla. Calculem fins a un cert ordre la intersecció de la varietat invariant inestable d'un tor donat amb la secció $y = 0$ a l'altre costat del primari més gran. Per fer-ho considerem el RTBP Espacial com una perturbació de problema de Hill tridimensional en un entorn del punt d'equilibri i després com una perturbació del Problema Sinòdic de Dos Cossos Espacial. La varietat estable s'obté de la inestable fent servir les simetries del problema.

També donem algunes estimacions de la diferència en l'espai d'acció per dos tors per tal de tenir una connexió heteroclínica. Això ens permet construir cadenes heteroclíniques. En particular, des de tors invariants propers a l'òrbita periòdica plana a tors invariants propers a la vertical en entorn del punt L_2.

El Problema Pla de Tres Cossos

Finalment, al capítol 6 estudiem l'existència de tors invariants a la varietat central dels punts d'equilibri col·lineals en el Problema Pla de Tres Cossos amb potencial Newtonià. Per a fer-ho seguim els següents passos. Primer, fem algunes transformacions canòniques per escriure el Hamiltonià en forma normal. Aleshores reduïm el Hamiltonià a la varietat central. Després comprovem, per avaluació numèrica dels coeficients de la forma normal fins a ordre 4, les condicions de no degeneració del teorema KAM. Els resultats presentats a la secció 6.4 mostren que les dues condicions (tant isoenergètica com no) es satisfan per a valors qualssevol de les masses en el triangle de masses.

El sistema linealitzat en un punt d'equilibri col·lineal te valors propis $\pm\lambda$, $\pm\mathrm{i}$, $\pm\mathrm{i}\omega$, $\lambda, \omega \in \mathbb{R}^+$. Per tant els punts d'equilibri col·lineals són de tipus centre–centre–sella. Està demostrat que fins a ordre 4 només cal tenir en compte la ressonància $2:1$. Les corresponents masses ressonants descriuen una corba en el triangle de masses. Per tant, per a masses ressonants seria d'esperar obtenir monomis ressonants d'ordre tres en la forma normal del Hamiltonià. Demostrem a la secció 6.3 que aquest no és el cas. De fet, demostrem que els coeficients dels monomis són diferents de zero per a masses generals, però esdevenen zero per a masses ressonants, i també en el cas simètric $m_1 = m_3$. L'existència de les solucions homogràfiques ens permet calcular analíticament, de manera senzilla, els coeficients dels monomis ressonants d'ordre tres. Aquests coeficients tenen $(\omega - 2)$ com a factor. Els resultats donats al capítol 6 estan publicats a [M.S.].