Reducibility of Quasi-Periodic Skew-Products and the Spectrum of Schrödinger Operators

Joaquim Puig i Sadurní

Departament de Matemàtica Aplicada i Anàlisi
Universitat de Barcelona

Programa de doctorat de Matemàtiques Bienni 1999-2001

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Certifico que aquesta memòria ha estat
realitzada per Joaquim Puig i Sadurní
i dirigida per mi.

Barcelona, 20 d'abril de 2004

Carles Simó i Torres

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que em vas dur al món
i has fet que pogués arribar aquí.

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per tots els secrets
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Chapter I

Introduction

I have yet more to say, which I have thought upon,
and I am filled, like the moon at the full.
Book of Sirach (Ecclesiasticus) 39:12

The purpose of this thesis is to study the reducibility and other dynamical properties of linear quasi-periodic skew-products, with special emphasis to those arising from eigenvalue equations of quasi-periodic Schrödinger operators. This study has revealed to be fruitful since it combines dynamical and spectral methods to give a unified approach and new results, both from the dynamical and spectral point of view.

In this Introduction we want to give an overview of the contents and main results of the thesis. Chapters II and III are devoted to preliminaries, whereas chapters IV, V, VI, VII and the Appendix A contain novel results.

To present the results in this thesis and the motivation for studying this kind of problems, let us focus on Hill's equation with quasi-periodic forcing, which is the following second-order linear differential equation

$$x'' + (a - bq(t))\, x = 0, \tag{I.1}$$

where a, b are real parameters and q is a quasi-periodic function. This quasi-periodicity means that there exists a continuous map from $\mathbb{T}^d = (\mathbb{R}/2\pi\mathbb{Z})^d$ to $\mathbb{R}$, Q, and a rationally independent frequency vector $\omega \in \mathbb{R}^d$ such that

$$q(t) = Q(\omega t),$$

for all $t \in \mathbb{R}$. In most of the thesis we will assume that Q is real analytic and ω satisfies some standard Diophantine condition, which will be discussed in Section II.2.2.

Hill's equation with quasi-periodic forcing is a generalization of the classical Hill's equation, where the forcing q is a periodic function. This periodic equation was introduced by George Hill in the 19th century and it is a prototypical example of the linearization of a non-linear system with periodic forcing around a fixed point. The dynamical properties of this linearization can be used to study the problem of stability of the fixed point in the non-linear system. When the forcing is quasi-periodic and the same linearization process is followed, Hill's equation with quasi-periodic forcing also appears as a standard model.

To study (I.1) from a dynamical point of view it is convenient to consider the following flow on $\mathbb{R}^2 \times \mathbb{T}^d$,

$$\begin{pmatrix} x \\ x' \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ bQ(\theta) - a & 0 \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}, \qquad \theta' = \omega, \tag{I.2}$$

where $\theta \in \mathbb{T}^d$. This is a linear differential equation with quasi-periodic coefficients, which we will also call a quasi-periodic skew-product flow on $\mathbb{R}^2 \times \mathbb{T}^d$. Such skew-products occur as the linearization of a non-linear system around quasi-periodic orbits. Since many stable solutions of non-linear systems are quasi-periodic, the stability properties of these linear equations have implications for the dynamics of the non-linear system around these orbits. This is particularly interesting for the problem of lower dimensional tori in Hamiltonian systems [Eli88, JV97, Bou97].

A third motivation to study Hill's equation with quasi-periodic forcing comes from the spectral theory of Schrödinger operators because we can look at (I.1) as the eigenvalue equation of the following one-dimensional Schrödinger operator with quasi-periodic potential

$$\left(H^c_{bq}x\right)(t) = -x''(t) + bq(t)x(t), \tag{I.3}$$

which appears in many models of Quantum Physics, notably in the comprehension of the "Quantum Hall effect" [vKDP80, Frö94, OA01] and the electronic properties of quasi-crystals [Jan92] (see the beginning of Chapter III). From this point of view, the spectral parameter a in (I.1) is called the energy. Using the quasi-periodicity of the potential, one can consider the following family of quasi-periodic Schrödinger operators.

$$\left(H^c_{bQ,\omega,\phi}x\right)(t) = -x''(t) + bQ(\omega t + \phi)x(t), \tag{I.4}$$

which includes (I.3) if $\phi = 0$. The operators $H^c_{bQ,\omega,\phi}$ have a unique self-adjoint extension to $L^2(\mathbb{R})$, which we denote again by $H^c_{bQ,\omega,\phi}$.

These physical applications have motivated that, in the last twenty years, a lot of attention has been devoted to the spectral properties of Schrödinger operators with quasi-periodic potential. In this thesis we try to link the dynamical theory of the eigenvalue equation, Hill's equation (I.1), with the spectral properties of the corresponding Schrödinger operator (I.3).

A central topic of interest will be the study of the spectrum of Schrödinger operators as above (and some discrete variants to be described later) as a function of the parameter b. Thanks to the rational independence of the frequency vector ω, the spectrum of the operators $H^c_{bQ,\omega,\phi}$ does not depend on the specific $\phi \in \mathbb{T}^d$ chosen and will be denoted as

$$\sigma^c(bQ, \omega) = \mathrm{Spec}\left(H^c_{bQ,\omega,\phi}\right).$$

To present the main results of this thesis in a natural way it is convenient to introduce briefly the rotation number of a Hill's equation (I.1), $\mathrm{rot}^c(a - bQ, \omega)$, which is the following limit

$$\lim_{t\to\infty} \frac{\arg\left(x'(t) + \mathrm{i}x(t)\right)}{t},$$

where x is any non-trivial solution of (I.1). This rotation number does not depend on the chosen solution, nor on ϕ, and it has the property that the continuous map

$$a \in \mathbb{R} \mapsto \mathrm{rot}^c(a - bQ, \omega)$$

is non-decreasing and increases exactly at the spectrum $\sigma^c(bQ, \omega)$. Moreover, as it will be explained in Section III.2.2, the value of the rotation number in the open intervals of constancy (which do not belong to the spectrum, they are spectral gaps), must be of the form

$$\frac{1}{2}\langle \mathbf{k}, \omega \rangle,$$

where $\mathbf{k} \in \mathbb{Z}^d$ is such that $\langle \mathbf{k}, \omega \rangle \geq 0$. This is the Gap Labelling Theorem, by Johnson & Moser [JM82]. Sometimes the spectral gaps above will be called non-collapsed, to distinguish them from collapsed gaps, which are those $\{a_0\}$ for which there is a $\mathbf{k} \in \mathbb{Z}^d$ such that a_0 is the only a satisfying

$$\mathrm{rot}^c(a - bQ, \omega) = \frac{1}{2}\langle \mathbf{k}, \omega \rangle. \tag{I.5}$$

Therefore, and using this gap labelling, a convenient way to study the spectra $\sigma^c(bQ, \omega)$ as a function of b is by means of "resonance tongues". These are the connected components in the (a, b)-plane where the rotation number is constant and of the form (I.5) for some $\mathbf{k}$ fixed.

In Chapter IV these resonance tongues are studied in the perturbative regime, $|b|$ small. The reason to restrict ourselves to this perturbative analysis is that, for these values of the parameter b, the skew-product (I.2) is reducible to constant coefficients at the boundaries of resonance tongues, thanks to a theorem by Eliasson [Eli92]. This property of reducibility, which will be explained in chapters II (in a general setting) and III (in the Schrödinger case), describes the existence of a quasi-periodic change of variables conjugating (I.2) to a linear system with constant coefficients.

The combination of reducibility at tongue boundaries, normal form techniques and dynamical characterizations of the spectrum and of the resolvent set, allow us to show that the formal expressions for the Taylor expansion of the tongue boundaries given by normal form correspond to the actual expansion of these tongue boundaries, which are C^∞ functions (for $|b|$ small). Using this procedure to construct the expansion of resonance tongue boundaries several results analogous to the periodic case (where tongue boundaries are real analytic) are proved. These include a criterion for the transversality of tongue boundaries at the origin and the creation of instability pockets, which is the pinching of a tongue for two different values of b. The latter has direct implications to the collapse of spectral gaps in the corresponding Schrödinger operators.

Although many results from Chapter IV are an extension of the periodic case, there is a substantial difference between the periodic and the quasi-periodic case. For the sake of simplicity let us restrict to the perturbative regime, $|b|$ small, where there is reducibility at tongue boundaries both in the periodic and quasi-periodic case. We remind that this assumes real analyticity of Q and a Diophantine condition on ω. In the periodic case resonance tongues are separated one from another at a positive distance, whereas in the quasi-periodic case ($d \geq 2$) the set of possible rotation number of tongues

$$\mathcal{M}_+(\omega) = \left\{ \langle \mathbf{k}, \omega \rangle / 2; \mathbf{k} \in \mathbb{Z}^d \text{ and } \langle \mathbf{k}, \omega \rangle \geq 0 \right\},$$

is dense in $[0, +\infty)$. Due to the continuity and monotonicity of the rotation number, resonance tongues are dense in the parameter plane, which implies that classical methods to study the smoothness of tongue boundaries based on the separation of the eigenvalues fail [Rel69, Kat76].

In Chapter V the question of the analyticity of tongue boundaries in Hill's equation is put in a more general framework. There we study the existence of analytic families of linear differential equations with quasi-periodic coefficients which are reducible to constant coefficients with a fixed Floquet matrix. The techniques are from KAM theory adapted to a Lie algebra formalism, which is very convenient for this kind of problems.

Under the current hypothesis of real analyticity of Q and Diophantine character of ω, tongue boundaries are proved to be real analytic if $|b|$ is small. In combination with the techniques of Chapter IV one can prove the following result on Cantor spectrum for quasi-periodic Schrödinger operators.

Theorem. *Let ω satisfy a Diophantine condition and $C^a_\rho(\mathbb{T}^d,\mathbb{R})$, for some fixed $\rho > 0$, be the space of real analytic functions $Q : \mathbb{T}^d \to \mathbb{R}$ with analytic extension to $|\operatorname{Im}\theta| < \rho$ satisfying*

$$|Q|_\rho = \sup_{|\operatorname{Im}\theta|<\rho} |Q(\theta)| < \infty.$$

Then, there exists a constant $C = C(\omega,\rho)$ such that, for a generic potential in

$$\left\{Q \in C^a_r(\mathbb{T}^d,\mathbb{R}),\ |Q|_r < C\right\},$$

with respect to the $|\cdot|_\rho$-topology, the operator $H^c_{Q,\omega,\phi}$ has all spectral gaps open (and therefore, it is a Cantor set if $d \geq 2$) for almost all $|b| < 1$.

We would like to stress that, compared to previous results on Cantor spectrum, our methodology, based on the real analyticity of tongue boundaries, allows to prove Cantor spectrum (and opening of all spectral gaps) for a fixed Q (with some generic checkable conditions) and almost all values of b. An example is given by the following quasi-periodic generalization of the operator associated to Mathieu's equation, $H_{bQ,\omega,0}$, where

$$Q(\theta) = \sum_{j=1}^{d} c_j \cos(\omega_j t) \tag{I.6}$$

and the constants c_j are all different from zero.

Theorem. *Let $d \geq 2$. Then there is a set $\mathcal{A} \subset \mathbb{R}^d$, of zero measure, such that if $\omega = (\omega_1,\ldots,\omega_d) \notin \mathcal{A}$, there exists a constant $C = C(\omega)$ such that, for almost all values of b, with $|b| < C$, the spectrum of $H_{bQ,\omega,\phi}$, with Q as in (I.6), has all spectral gaps open.*

Apart from the significance of these results on Cantor spectrum from the point of view of spectral theory of Schödinger operators, they serve as valuable examples of quasi-periodic Hamiltonians whose Birkhoff Normal Form is divergent. In Appendix A these examples are applied, together with potential theory, to show that the Birkhoff Normal form of a quasi-periodic Hamiltonian with fixed frequencies and a quadratic part which is totally elliptic at the origin is generically divergent.

Up to now, the results that we have presented for quasi-periodic Schrödinger operators on $L^2(\mathbb{R})$ are perturbative in the sense that the bound for $|b|$ depends on the precise Diophantine conditions on ω. In the last years there has been significant progress on nonperturbative results for discrete quasi-periodic Schrödinger operators with $d = 1$. In chapters VI and VII we try to extend the perturbative analysis of chapters IV and V using these nonperturbative results.

Discretizing a Hill's equation with respect to x one obtains an equation of the form

$$x_{n+1} + x_{n-1} + bv(n)x_n = ax_n, \qquad n \in \mathbb{Z}, \tag{I.7}$$

where $(x_n)_{n\in\mathbb{Z}}$ is a sequence in $\mathbb{R}$, a and b are real parameters $(v(n))_{n\in\mathbb{Z}}$ is a real analytic quasi-periodic sequence. This means that there is a real analytic function $V : \mathbb{T}^d \to \mathbb{R}$ such that

$$v(n) = V\,(2\pi\omega n)$$

for all $n \in \mathbb{Z}$. The frequency ω in this discrete case version will satisfy the following nonresonance condition

$$\langle \mathbf{k}, \omega \rangle \notin \mathbb{Z}$$

for any $\mathbf{k} \in \mathbb{Z}^d$ different from zero. The relation between rational independence and nonresonance, together with the corresponding Diophantine conditions, is described in Section II.2.2. Equation (I.7) is the eigenvalue equation of the following discrete Schrödinger operator with quasi-periodic potential

$$(H^d_{bv}x)_n = x_{n+1} + x_{n-1} + bv(n)x_n, \tag{I.8}$$

which is a bounded and self-adjoint operator on $l^2(\mathbb{Z})$. As in the continuous case of operators H^c_{bq} and $H^c_{bQ,\omega,\phi}$, the quasi-periodicity of q defines a family of quasi-periodic operators

$$(H^d_{bV,\omega,\phi}x)_n = x_{n+1} + x_{n-1} + bV(2\pi\omega n + \phi)x_n,$$

for $\phi \in \mathbb{T}^d$, whose spectrum does not depend on ϕ. The spectral properties of these operators are related to the dynamics of the following discrete skew-product on $\mathbb{R}^2 \times \mathbb{T}^d$

$$\begin{pmatrix} x_{n+1} \\ x_n \end{pmatrix} = \begin{pmatrix} a - bV(\theta_n) & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_n \\ x_{n-1} \end{pmatrix}, \qquad \theta_{n+1} = \theta_n + 2\pi\omega. \tag{I.9}$$

As in the periodic case there exists a notion of reducibility of skew-product flows which describes the existence of a change of variables which transforms a discrete skew-product to the iteration of a constant matrix.

In Chapter VI a result on nonperturbative localization for the "Almost Mathieu" operator by Jitomirskaya [Jit99] is used to solve the so-called "Ten Martini Problem". In Chapter VII we extend some ideas from Chapter VI to prove a nonperturbative version of Eliasson's reducibility theorem. Let us now present the main results in these two chapters.

The best-studied discrete Schrödinger operator is probably the "Almost Mathieu" operator,

$$(H_{b,\omega,\phi}x)_n = x_{n+1} + x_{n-1} + b\cos\left(2\pi\omega n + \phi\right)x_n$$

whose eigenvalue equation,

$$x_{n+1} + x_{n-1} + b\cos\left(2\pi\omega n + \phi\right)x_n = ax_n,$$

is sometimes called Harper's equation. In 1981 Simon [Sim82], after an offer by Kac, proposed the Ten Martini Problem: "for all $b \neq 0$ and nonresonant ω the Almost Mathieu Operator has Cantor spectrum". In Chapter VI we solve this problem for $b \neq \pm 2$ and Diophantine ω.

Corollary. *If ω is Diophantine and $b \neq 0, \pm 2$, the spectrum of the Almost Mathieu operator is a Cantor set.*

Along Chapter VI we clarify why this result is a corollary as it illustrates the fruitful interaction between spectral and dynamical points of view. On one hand it is a corollary of a theorem by Jitomirskaya [Jit99] which, under the current hypothesis, states that for almost all ϕ and $|b| > 2$ the Almost Mathieu operator above has only pure-point spectrum with exponentially decaying eigenfunctions. On the other, it is a corollary of the dynamical theory of reducibility for quasi-periodic skew-products like (I.9) as above. Aubry duality, which is the invariance of Harper equations, with different parameters, by Fourier transform, is used to link these two points of view. Let us now briefly sketch the argument.

According to Jitomirskaya's theorem if $|b| > 2$, ω is Diophantine, and ϕ belongs to a total measure set which includes $\phi = 0$, Harper's equation has exponentially localized solutions for a set of eigenvalues which is dense in the spectrum. Let a be one of these eigenvalues and consider the Fourier transform of the associated eigenvector $\psi = (\psi_k)_{k\in\mathbb{Z}}$,

$$\tilde{\psi}(\theta) = \sum_{k\in\mathbb{Z}} \psi_k e^{\mathrm{i}k\theta},$$

which, due to the exponential decay of ψ, is an analytic function. A computation shows that the following analytic quasi-periodic Bloch wave

$$x_n = \tilde{\psi}\left(2\pi\omega n\right)$$

is a solution of

$$x_{n+1} + x_{n-1} + \frac{4}{b}\cos\left(2\pi\omega n\right) x_n = \frac{2a}{b} x_n,$$

which is Harper's equation for other values of the parameters a and b.

In Chapter VI we prove that the existence of this Bloch solution implies that a is the endpoint of a spectral gap of the operator $H_{4/b,\omega,\phi}$ and that this gap is collapsed if, and only if, the equation has another linearly independent Bloch wave as solution of the corresponding eigenvalue equation. Undoing the Aubry duality process we would obtain, if the gap was collapsed, that the original Harper's equation (I.7) would have two linearly independent solutions decaying exponentially. This is a contradiction with the limit-point character of quasi-periodic Schrödinger operators.

The argument above implicitly uses the notion reducibility of a quasi-periodic skew-product. Eliasson's Theorem can be extended to the discrete case for Harper-like equations (I.7) with real analytic potentials and Diophantine frequencies if $|b| < C$, where C is a constant which depends on the precise Diophantine conditions on ω. We can be use this to give a partial answer to the Strong (or Dry) Ten Martini Problem. Indeed, in Chapter VI we show that, for these small values of b and a Diophantine frequency, the Almost Mathieu operator has all spectral gaps open.

For general Schrödinger operators we cannot expect that the constant C in Eliasson's theorem does not depend on the specific Diophantine condition. Nevertheless, in Chapter VII we prove that, when $d = 1$, a nonperturbative version of Eliasson's theorem can be obtained.

Theorem. *Let $\rho > 0$ be a positive constant. Then there exists a $\varepsilon_0 = \varepsilon_0(\rho)$ such that, for any real analytic V and $b \in \mathbb{R}$ satisfying*

$$|bV|_\rho < \varepsilon_0,$$

the skew-product (I.9) is reducible to constant coefficients for almost every $a \in \mathbb{R}$ (with respect to Lebesgue measure) and for any Diophantine ω.

This is only a partial generalization of Eliasson's Theorem because the values of a for which the above theorem grants reducibility are not characterized in terms of its rotation number. This result has been recently obtained by Avila & Krikorian [AK03] under more restrictive hypothesis.

The main burden in the proof of Theorem I is to show that for almost every point in the spectrum there exist analytic quasi-periodic Bloch waves. Indeed, on one hand, if a does not belong to the spectrum, then the skew-product is reducible to constant coefficients thanks to the exponential dichotomy. On the other, if a Harper-like equation like (I.7) has such a Bloch wave as solution then the corresponding skew-product (I.9) is reducible to constant coefficients, as we prove in Chapter VII.

To show the existence of analytic quasi-periodic Bloch waves of (I.7) for almost all values of a in the spectrum of the associated operator we can try to use the trick of Aubry duality. In this case, however, we will not recover the same operator under Aubry duality, but looking for Bloch waves for (I.7) will lead us to study the existence of exponentially localized solutions of the difference equation

$$\sum_{k\in\mathbb{Z}} V_k x_{n-k} + 2\cos(2\pi\omega n + \varphi)\, x_n = a x_n \qquad n \in \mathbb{Z},$$

where $\varphi \in \mathbb{T}$ and $(V_k)_k$ are the Fourier coefficients of V. This is the eigenvalue equation of the long-range operator

$$(L_{V,\omega,\varphi} x)_n = \sum_{k\in\mathbb{Z}} V_k x_{n-k} + 2\cos(2\pi\omega n + \varphi)\, x_n$$

on $l^2(\mathbb{Z})$ which is also bounded and self-adjoint. The role of Jitomirskaya's theorem for the Almost Mathieu operator is now played by an extension, due to Bourgain & Jitomirskaya [BJ02a], which states the pure-point character of the spectrum with exponentially decaying eigenfunctions of such long-range operators and almost all $\varphi \in \mathbb{T}$.

As this introduction has tried to present, the combination of dynamical and spectral points of view has proved to be very fruitful and we believe that, exploiting even more this connection, more interesting results can be obtained. From the spectral point of view, we have seen that it is possible to accurately describe the behaviour of spectral gaps in terms of the dynamics of the associated skew-products. It is expected that this analysis of gaps can be extended to more general Schrödinger operators. From the dynamical point of view a valuable source of examples and methods has been studied. These include non-uniform hyperbolicity, nonperturbative techniques and accurate methods to describe the transition from regular to irregular behaviour in these examples. We expect to apply these methods to fully nonlinear systems in the future.

Chapter II

Linear quasi-periodic differential equations, skew-products and cocycles

In this chapter some of the basic notions and formalism to be used in this thesis are introduced. We begin with linear differential equations with quasi-periodic coefficients as a motivating example in Section II.1 before giving the formalism of cocycles and skew-product flows in Section II.2. In both sections, the concept of reducibility is emphasized. Finally in Section II.3, the concepts of exponential dichotomy, Sacker-Sell spectrum and invariant splittings, together with their properties, are discussed.

II.1 Linear quasi-periodic differential equations: a first approach

In this section we introduce the differential equations we shall study together with some basic properties. These differential equations are linear, homogeneous and their coefficients depend quasi-periodically on time. Let us try to give a meaning to all the terms used in this sentence.

II.1.1 Linear equations with time-depending coefficients

A (homogeneous) linear differential equation is a differential equation of the form

$$x'(t) = a(t)x(t), \tag{II.1}$$

where $'$ stands for the derivative with respect to the time t, $a(t)$ is a square matrix of dimension n and the function $x = x(t)$ is the unknown. We can look at a as a map from an interval to $gl(n, \mathbb{R})$, the set of square real matrices of dimension n.

Assuming continuity of the map $t \in \mathbb{R} \mapsto a(t) \in gl(n, \mathbb{R})$ (which is equivalent to the continuity of each of its entries) the Cauchy problem associated to (II.1),

$$x'(t) = a(t)x(t), \qquad x(t_0) = x_0 \tag{II.2}$$

has a unique solution for any $x_0 \in \mathbb{R}^n$ and $t_0 \in \mathbb{R}$ which we denote by $x(t; t_0, x_0)$.

Fixing $t_0 \in \mathbb{R}$ and due to the linear character of the equation (II.1) the set of its solutions,

$$\{x(t; t_0, x_0),\ x_0 \in \mathbb{R}^n\}$$

is a linear space of dimension n. Let $X = X(t)$ be a matrix whose columns are n linearly independent solutions of (II.1). Clearly, this matrix satisfies the equation

$$X'(t) = a(t)X(t). \tag{II.3}$$

and it is nonsingular for all t. Any nonsingular matrix $X(t)$ which satisfies this differential equation is called a *fundamental matrix* of the equation (II.1). All the information of a linear differential system can be obtained from a fundamental matrix because, if $x(t; t_0, x_0)$ is a solution of the Cauchy problem (II.2) and X is a fundamental matrix, then one can write

$$x(t; t_0, x_0) = X(t)X(t_0)^{-1}x_0.$$

If $X_1 = X_1(t)$ and $X_2 = X_2(t)$ are two matrices satisfying (II.3) then, for any $t_0 \in \mathbb{R}$,

$$X_2(t)X_2(t_0)^{-1} = X_1(t)X_1(t_0)^{-1}.$$

This means that the solution of the Cauchy problem

$$X'(t) = a(t)X(t), \qquad X(t_0) = X_0. \tag{II.4}$$

being X_0 any square matrix of dimension n and $t_0 \in \mathbb{R}$ is

$$X(t; t_0, X_0) = X(t)X(t_0)^{-1}X_0$$

where X is any fundamental matrix of (II.1).

The simplest example of linear systems are linear equations with constant coefficients

$$x'(t) = a\,x(t)$$

where a is a constant matrix of dimension n. The Cauchy problem (II.4) for these equations can be solved explicitly,

$$X(t; t_0, X_0) = \exp\left((t - t_0)a\right) X_0 = \left(I + \sum_{k=1}^{\infty} \frac{(t - t_0)^k}{k!} a^k\right) X_0.$$

Linear equations and matrix Lie algebras

The example above of linear equations with constant coefficients shows that if a lies in a certain matrix Lie algebra, then the fundamental matrices can be chosen in the corresponding Lie group (choosing an initial condition for the Cauchy problem (II.4) in the group, for example the identity). This fact can be generalized to linear nonautonomous differential equations.

A *matrix Lie group* $G \subset GL(n, \mathbb{R})$ is a smooth submanifold of $GL(n, \mathbb{R})$, which we will assume to be connected, such that the induced matrix product is a smooth operation. The corresponding *matrix Lie algebra* is defined by means of the exponential map: all one-parameter subgroups of G are of the form $(\exp(tA))_{t\in\mathbb{R}}$, where A belongs to some Lie subalgebra g of $gl(n,\mathbb{R})$. A Lie subalgebra of matrices is a vector space of $gl(n,\mathbb{R})$, the space of square n-dimensional real matrices, which is invariant by the *Lie bracket* or *commutator* of matrices

$$[A,B] = AB - BA.$$

In particular, for any $A \in g$ the *adjoint operator* of A,

$$B \in g \mapsto \mathrm{ad}\,_A(B) = [A,B] \in g$$

is a well-defined map. Regarded as a map from g to G, the exponential map

$$A \in g \mapsto \exp(A) = I + \sum_{k=1}^{\infty} \frac{1}{k!} A^k$$

defines a diffeomorphism between the Lie algebra g and a neighbourhood of the identity in G. More generally, for any $X \in G$ the maps

$$A \in g \mapsto X\exp(A) \quad \text{ and } A \in g \mapsto \exp(A)X$$

define diffeomorphisms between g and a neighbourhood of X in G. A computation shows that, for any $C \in G$ and $A \in g$,

$$\exp\left(C^{-1}AC\right) = C^{-1}\exp(A)C,$$

so that $C^{-1}AC$ is also in g. Classical examples of Lie groups and corresponding Lie algebras are

- $g = gl(n,\mathbb{R})$ the algebra of real matrices of dimension n and $G = GL(n,\mathbb{R})$ the group of real invertible matrices of dimension n.

- $g = sl(n,\mathbb{R})$ the subalgebra of $gl(n,\mathbb{R})$ of matrices with trace zero and $G = SL(n,\mathbb{R})$, the *special linear group*, which is the subgroup of$GL(n,\mathbb{R})$ of matrices with determinant one.

- $g = sp(n,\mathbb{R})$ the subalgebra of $gl(2n,\mathbb{R})$ of *infinitesimally symplectic* matrices, that is

  $$sp(n,\mathbb{R}) = \left\{A \in gl(2n,\mathbb{R}) \text{ such that } A^T J = -JA\right\}$$

  where J is the matrix
  $$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

  and I_n the n-dimensional identity matrix. $G = Sp(n,\mathbb{R})$ is the subgroup

  $$Sp(n,\mathbb{R}) = \left\{A \in gl(2n,\mathbb{R}) \text{ such that } A^T JA = J\right\}.$$

  of *symplectic matrices* of dimension n.

- $g = so(n,\mathbb{R})$ the subalgebra of $gl(n,\mathbb{R})$ of *skew-symmetric matrices* and $G = SO(n,\mathbb{R})$, the *special orthogonal group*, the subgroup of$GL(n,\mathbb{R})$ of orthogonal matrices with determinant one.

As it happens with linear equations with constant coefficients, fundamental matrices of linear equations taking values on some matrix Lie algebras can always be chosen to be in the Lie group. More specifically one has the following properties:

Proposition II.1. *Let g be a matrix Lie algebra and $Y : \mathbb{R} \to g$ any smooth function. Then,*

(i) For all $t \in \mathbb{R}$ the element $(\exp(Y(t)))' \exp(-Y(t))$ belongs to g.

(ii) If $a, b : \mathbb{R} \to gl(n,\mathbb{R})$ are continuous functions which satisfy the equation

$$(\exp(Y(t)))' = a(t)\exp(Y(t)) - \exp(Y(t))\, b(t), \qquad t \in \mathbb{R},$$

then $a(t) \in g$ for $t \in \mathbb{R}$ if, and only if, $b(t) \in g$ for $t \in \mathbb{R}$.

(iii) The solution of the Cauchy problem

$$Y' = a(t)Y, \qquad Y(0) = Y_0$$

with $a : \mathbb{R} \to g$ and $Y_0 \in G$ belongs to G.

Proof: First of all, note that item (ii) is a direct consequence of item (i). Indeed, if $X(t) = \exp(Y(t))$, then one has the identities

$$b = -X^{-1}X' + X^{-1}aX = (X^{-1})'X + X^{-1}aX,$$

and also

$$a = X'X^{-1} + XbX^{-1}.$$

Then item (ii) follows from (i), applying (i) to $-Y$ and Y respectively, and using the invariance of g by conjugations by matrices in G. The proof of (i) makes use of the following property of Lie algebras (see Postnikov [Pos86], for instance). Let $Y_0, Y_1 \in g$. Then

$$\exp(Y_0 + tY_1)\exp(-Y_0) = \exp\left(t\frac{\exp(\mathrm{ad}\,_{Y_0}) - I}{\mathrm{ad}\ Y_0}Y_1 + o(t)\right)$$

where by $\frac{\exp(\mathrm{ad}\ Y_0)-I}{\mathrm{ad}Y_0}$ is meant the sum of an operator series

$$I + \frac{\mathrm{ad}\,_{Y_0}}{2!} + \ldots + \frac{(\mathrm{ad}\,_{Y_0})^k}{(k+1)!} + \ldots$$

and $\mathrm{ad}\,_{Y_0} : g \to g$ is the adjoint operator of Y_0. Let $t_0 \in \mathbb{R}$ and write $Y_0 = Y(t_0)$ and $Y_1 = Y'(t_0)$, belonging both to g. Note that

$$\frac{d}{dt}\left(\exp\left(Y(t)\right)\right)_{t=t_0}\exp\left(-Y(t_0)\right) = \frac{d}{dt}\left(\exp\left(Y_0 + (t-t_0)Y_1\right)\right)_{t=t_0}\exp\left(-Y_0\right) =$$
$$\frac{d}{dt}\left(\exp\left(Y_0 + (t-t_0)Y_1\right)\exp\left(-Y_0\right)\right)_{t=t_0} = \frac{\exp(\mathrm{ad}\,_{Y_0}) - I}{\mathrm{ad}\,_{Y_0}}Y_1 \in g.$$

Finally, item (iii) follows from (ii). $\square$

II.1.2 Quasi-periodic functions

Before studying linear equations with quasi-periodic coefficients let us give some notions about quasi-periodic functions. Quasi-periodic functions are a generalization of periodic functions and appear naturally in modeling of phenomena involving several frequencies.

A function $f : \mathbb{R} \to \mathbb{R}$ is *quasi-periodic* if there exist real constants $\omega_1, \ldots, \omega_d$ and a continuous function $F : \mathbb{R}^d \to \mathbb{R}$, 2π-periodic in each variable such that we can express f as

$$f(t) = F(\omega_1 \cdot t, \ldots, \omega_d \cdot t)$$

for every $t \in \mathbb{R}$. We call F the *lift* of F and $\omega_1, \ldots, \omega_d$ its *basic frequencies* or simply *frequencies*. The vector

$$\omega = (\omega_1, \ldots, \omega_d)^d$$

is also called the *frequency vector*.

The fact that F is continuous and 2π-periodic in each variable means that one can consider it as a continuous function from $\mathbb{T}^d \to \mathbb{R}$. Here $\mathbb{T}$, the torus, is the quotient space

$$\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$$

whose elements will be called *angles*.

Without imposing extra conditions on the frequencies neither the frequency ω nor the lift of a quasi-periodic function are uniquely determined. Indeed, consider the function

$$f(t) = \cos(t) + \cos(2t),$$

which is quasi-periodic according to the definition above, since we can write

$$f(t) = F(t, 2t),$$

with

$$F(\theta_1, \theta_2) = \cos(\theta_1) + \cos(\theta_2), \qquad (\theta_1, \theta_2) \in \mathbb{T}^2.$$

However, one also has

$$f(t) = G(t),$$

where

$$G(\theta) = \cos(\theta) + 2\cos^2(\theta) - 1, \qquad \theta \in \mathbb{T}.$$

To overcome this lack of uniqueness, we will impose the condition that the basic frequencies of a quasi-periodic function, ω_1,...,ω_d, are *rationally independent*. This means that the only combination of integers $k_1, \ldots, k_d \in \mathbb{Z}$ which satisfies the relation

$$k_1\omega_1 + k_2\omega_2 + \ldots + k_d\omega_d = 0$$

is the trivial one $k_1 = \ldots = k_d = 0$. The above scalar product will be denoted in the sequel as

$$\langle \mathbf{k}, \omega \rangle = k_1\omega_1 + k_2\omega_2 + \ldots + k_d\omega_d,$$

where $\mathbf{k} = (k_1, \ldots, k_d)^T$. Assuming rationally independence of the basic frequencies, then the following result is easy to prove.

Lemma II.2. *Let $f : \mathbb{R} \to \mathbb{R}$ be quasi-periodic and let $\omega_1, \ldots, \omega_d$ be one set of basic frequencies (assumed rationally independent). Define the $\mathbb{Z}$-module*

$$\mathcal{R}(\omega) = \left\{ \langle \mathbf{k}, \omega \rangle ; \mathbf{k} \in \mathbb{Z}^d \right\}.$$

Then any other (rationally independent) set of basic frequencies $\nu_1, \ldots, \nu_m$ is a basis of $\mathcal{R}(\omega)$ (as $\mathbb{Z}$-module). In particular $m = d$, so that any two choices of basic frequencies have the same number of elements (which is the dimension of the module).

Remark II.3. *In the sequel, and unless otherwise stated, all frequency vectors will be assumed rationally independent.*

Remark II.4. *Periodic functions with period $T > 0$ are quasi-periodic functions with $2\pi/T$ as the only basic frequency.*

Given a quasi-periodic function f with lift F and frequency vector $\omega \in \mathbb{R}^d$, the function

$$f_\phi = F(\omega t + \phi)$$

for any $\phi \in \mathbb{T}^d$ is quasi-periodic with frequency ω and lift $F_\phi = F(\cdot + \phi)$. The set $\{f_\phi\}_{\phi \in \mathbb{T}^d}$ is a compact subspace of the continuous functions which is called the *Hull* of f. This space is homeomorphic to $\mathbb{T}^d$. This construction admits a generalization, which are the *almost periodic functions* An introduction can be found in the monographs by Bohr [Boh47], Fink [Fin74] and Levitan & Zikhov [LZ82]. Many properties of this chapter hold also for almost periodic functions and linear equations with almost periodic coefficients, although we will not consider them.

Regularity of quasi-periodic functions

As a general idea, we want that the properties of quasi-periodic functions hold for all functions in the hull. That is, any nice class of quasi-periodic functions must contain the hull of functions in it.

Concerning regularity properties, we have already seen that any quasi-periodic function is continuous because its lift is a continuous function. The continuity of the lift implies continuity of any function in the hull.

Imposing regularity conditions on a quasi-periodic function without taking into account the regularity of its lift may lead to surprising results. For instance, it is possible to produce examples of quasi-periodic functions f which are real analytic but whose lift F is only continuous (see Johnson & Moser [JM82]). This is why one defines regularity classes of quasi-periodic functions as the set of quasi-periodic functions whose lift belongs to the regularity class. For example, C^k quasi-periodic functions are those whose lift is of class C^k.

Analytic quasi-periodic functions, and especially *real analytic quasi-periodic functions*, deserve an special attention, since this will be the usual regularity condition in this thesis. Again, the definition of a real analytic quasi-periodic function f will be given in terms of the analyticity of its lift F. Consider F as a function from $\mathbb{R}^d$ to $\mathbb{R}$. It is real analytic if there exists U, an open neighbourhood of $\mathbb{R}^d$ considered as a subset of $\mathbb{C}^d$, and an analytic function $G : U \to \mathbb{C}$ such that

$$G_{|\mathbb{R}^d} = F.$$

By periodicity of F, the domain U can be made uniform. That is, there exists a positive ρ such that U contains the *complex strip*

$$|\mathrm{Im}\, z| < \rho \tag{II.5}$$

where $z = (z_1, \ldots, z_d)^T \in \mathbb{C}^d$ and

$$|z| = \max_{k=1,\ldots,d} |z_k|\,. \tag{II.6}$$

The number ρ in (II.5) will be called *width of the analyticity strip*. For the sake of simplicity in the notation we will denote by θ the extension to $\mathbb{C}^d$ of the angular variables $\theta \in \mathbb{T}^d$. On real or complex spaces we will always consider the norm defined by (II.6). However, for *multi-indices* or *multi-integers*, which are the elements of $\mathbb{Z}^d$, $\mathbf{k} = (k_1, \ldots, k_n) \in \mathbb{Z}^n$, the following norm will be used

$$|\mathbf{k}| = |k_1| + \ldots + |k_n|.$$

We denote both norms by the same symbol.

A convenient norm for real analytic functions $F : \mathbb{T}^d \to \mathbb{R}$ with analytic extension to $|\mathrm{Im}\,\theta| < \rho$ is

$$|F|_\rho := \sup_{|\mathrm{Im}\,\theta|<\rho} |F(\theta)|$$

which may be infinite. To obtain a useful function space we restrict to the set of those real analytic functions $F : \mathbb{T}^d \to \mathbb{R}$ with analytic extension to $|\mathrm{Im}\,\theta| < \rho$ and

$$|F|_\rho < \infty.$$

This set will be denoted by $C^a_\rho(\mathbb{T}^d, \mathbb{R})$ and

$$C^a(\mathbb{T}^d, \mathbb{R}) = \bigcup_{\rho>0} C^a_\rho(\mathbb{T}^d, \mathbb{R})$$

will stand for the set of all real analytic $F : \mathbb{T}^d \to \mathbb{R}$. A real analytic function quasi-periodic function f will be said to be real analytic if its lift belongs to $C^a_\rho(\mathbb{T}^d, \mathbb{R})$ for some $\rho > 0$. If there is no danger of confusion we will also denote $C^a_\rho(\mathbb{T}^d, \mathbb{R})$ by $C^a_\rho(\mathbb{T}^d)$.

Real analytic quasi-periodic functions are nice because we can apply the Cauchy estimates to its lift.

Theorem II.5 (Cauchy Estimates). *Let F be in $C^a_\rho(\mathbb{T}^d)$ for some $\rho > 0$. Then, for any $0 < \delta < \rho$ and any $\mathbf{j} \in (\mathbb{N} \cup \{0\})^d$,*

$$\partial^{\mathbf{j}} F = \frac{\partial^{|\mathbf{j}|} F}{\partial\theta_1^{j_1} \ldots \partial\theta_d^{j_d}}\,(\theta_1, \ldots, \theta_d)$$

belongs to $C^a_{\rho-\delta}(\mathbb{T}^d)$ and

$$\left|\partial^{\mathbf{j}} F\right|_{\rho-\delta} \le C_{\mathbf{j},d}\,\delta^{-|\mathbf{j}|}\,|F|_\rho$$

where $C_{\mathbf{j},d}$ is a constant depending only on $\mathbf{j}$ and d.

Fourier analysis of quasi-periodic functions

Since quasi-periodic functions are a generalization of periodic functions it is natural to look for an analog of the Fourier series for quasi-periodic functions. This is achieved resorting again to the lift $F : \mathbb{T}^d \to \mathbb{R}$ of the quasi-periodic function f with frequency ω. At a formal level, we can associate a Fourier series with d angular variables to F. This series is

$$\sum_{\mathbf{k}\in\mathbb{Z}^d} \hat{F}_{\mathbf{k}} \exp(\mathrm{i}\langle \mathbf{k}, \theta\rangle), \tag{II.7}$$

where $\mathbf{k} = (k_1, \ldots, k_d) \in \mathbb{Z}^d$ and the $\hat{F}_{\mathbf{k}}$ are the *Fourier coefficients* of F which are computed via the formula

$$\hat{F}_{\mathbf{k}} = \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} F(\theta)\exp(-\mathrm{i}\langle \mathbf{k}, \theta\rangle)d\theta.$$

Here the integration is taken with respect to the normalized Lebesgue measure on the torus. Since F is continuous these Fourier coefficients always exist.

Similarly to periodic functions, the regularity properties of F impose a certain rate of convergence of the above sum, so that it is not only formal. Focusing on real analytic functions, one can give the following convergence result for the Fourier series.

Theorem II.6. *Let $F \in \mathbb{C}^a_\rho(\mathbb{T}^d)$ be a real analytic function for some $\rho > 0$. Then its Fourier coefficients $\left(\hat{F}_{\mathbf{k}}\right)_{\mathbf{k}\in\mathbb{Z}^d}$ satisfy that*

$$\hat{F}_{\mathbf{k}} = \overline{\hat{F}_{-\mathbf{k}}}$$

and

$$\left|\hat{F}_{\mathbf{k}}\right| e^{-\rho|\mathbf{k}|} \leq |F|_\rho$$

for all $\mathbf{k} \in \mathbb{Z}^d$. Conversely, if $\left(\hat{F}_{\mathbf{k}}\right)_{\mathbf{k}\in\mathbb{Z}^d}$ is such that

$$\sup_{\mathbf{k}\in\mathbb{Z}^d} \left|\hat{F}_{\mathbf{k}}\right| e^{\rho|\mathbf{k}|} = C < \infty$$

for some $\rho > 0$, then the Fourier transform of the $(\hat{F}_{\mathbf{k}})_{\mathbf{k}}$, which we define as

$$F(\theta) = \sum_{\mathbf{k}\in\mathbb{Z}^d} \hat{F}_{\mathbf{k}} \exp(\mathrm{i}\langle \mathbf{k}, \theta\rangle),$$

belongs to $C^a_{\rho-\delta}(\mathbb{T}^d)$ for any $0 < \delta < \rho$ and

$$|F|_{\rho-\delta} \leq \frac{C}{\delta^d}K(d).$$

where $K(d)$ is a constant depending on d.

Let us now use this facts on Fourier analysis of functions $F : \mathbb{T}^d \to \mathbb{R}$ to derive a Fourier representation of quasi-periodic functions. The basic link between the Fourier coefficients of F and f, ω is the following theorem.

Theorem II.7 (Averages of quasi-periodic functions). *Let f be a quasi-periodic function. Then the limit*

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T f(t)dt$$

exists in $\mathbb{R}$ and it is called the average *of f, which is denoted by $[f]$ or $\bar{f}$. Moreover, if F is a lift of f,*

$$f(t) = F(\omega t),$$

with $\omega \in \mathbb{R}^d$ rationally independent, then the average of f is

$$[f] = \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} F(\theta)d\theta = \hat{F}_{\mathbf{0}},$$

the $\mathbf{0}$th Fourier coefficient of F. Sometimes we will also write $[F]$ for $[f]$.

This result, whose proof can be found in Arnol'd [Arn96], illustrates the property of *ergodicity* of a flow on the torus $\mathbb{T}^d$, in our case the flow given by the irrational rotation

$$(t,\phi)\in\mathbb{R}\times\mathbb{T}^d \mapsto \tau_t(\phi) = \omega t + \phi \in \mathbb{T}^d.$$

This flow is ergodic with respect to the normalized Lebesgue measure because it is invariant by the flow τ and any invariant measurable subset of $\mathbb{T}^d$ has zero or total measure. In the case of the irrational rotation we can even say that it is *uniquely ergodic* because the normalized Lebesgue measure on the torus is the only invariant Borel normalized measure (and therefore the flow is ergodic with respect to this measure, see Katok & Hasselblatt [KH95]). The fact that the irrational rotation is uniquely ergodic is called *Kronecker-Weyl Equidistribution Theorem* [KH95].

This result implies that one can obtain the Fourier coefficients of the lift of a quasi-periodic function only from the knowledge of a set of basic frequencies and the quasi-periodic function itself. Indeed, any other Fourier coefficient of F, say $\hat{F}_{\mathbf{k}}$ for $\mathbf{k}\in\mathbb{Z}^d$, can be obtained through the average

$$\hat{F}_{\mathbf{k}} = \lim_{T\to\infty}\frac{1}{T}\int_0^T f(t)\exp(-\mathrm{i}\langle\mathbf{k},\omega t\rangle)dt. \tag{II.8}$$

This yields a representation of a quasi-periodic function as series of exponentials of $-\mathrm{i}\langle\mathbf{k},\omega t\rangle$.

Theorem II.8. *Let f be a real analytic function with lift in $C^a_\rho(\mathbb{T}^d)$ and frequency $\omega\in\mathbb{R}^d$ for some $\rho>0$. Then the coefficients $\left(\hat{F}_{\mathbf{k}}\right)_{\mathbf{k}\in\mathbb{Z}^d}$ defined by (II.8) satisfy that*

$$f(t) = \sum_{\mathbf{k}\in\mathbb{Z}^d}\hat{F}_{\mathbf{k}}\exp\left(\mathrm{i}\langle\mathbf{k},\omega t\rangle\right)$$

converges uniformly in $|\mathrm{Im}\, t| < \rho/(d|\omega|)$. Moreover such a representation is unique once a frequency vector ω has been fixed.

Remark II.9. *Here we can recover the Fourier coefficients of F from f if we know the frequencies. The problem of determining the Fourier coefficients and the frequencies of a quasi-periodic function only from a knowledge of f is not so trivial, but very interesting for the applications (see Laskar, Froeschlé & Celletti* [LFC92]*, Laskar* [Las99]*, and Gómez, Mondelo & Simó* [GMC01] *and references therein).*

II.1.3 Linear equations with quasi-periodic coefficients

Linear differential equations with quasi-periodic coefficients are equations of the form

$$x' = a(t)x, \qquad x(t_0) = x_0 \tag{II.9}$$

where a is a square matrix of dimension n whose entries depend quasi-periodically on t. That is, there exist a rationally independent frequency vector $\omega \in \mathbb{R}^d$ and a lift A defined on $\mathbb{T}^d$ such that

$$a(t) = A(\omega t)$$

for all $t \in \mathbb{R}$. Thanks to the quasi-periodicity of a one can lift (II.9) to an autonomous system of linear equations defined on $\mathbb{R}^n \times \mathbb{T}^d$ writing

$$x' = A(\theta)x, \quad \theta' = \omega, \qquad (x,\theta) \in \mathbb{R}^n \times \mathbb{T}^d. \tag{II.10}$$

The time evolution of these new angular variables is trivial,

$$\theta(t) = \omega t + \phi$$

so that Equation (II.9) corresponds to (II.10) with initial condition $\theta(t_0) = \omega t_0$ (and hence $\phi = 0$). In fact, any equation

$$x' = b(t)x$$

with b in the hull of the quasi-periodic function a can be obtained with a suitable choice of initial condition for the variables θ by letting ϕ take arbitrary values on $\mathbb{T}^d$.

The matrix equation of (II.9),

$$X' = a(t)X, \tag{II.11}$$

can also be lifted to a system on $GL(n,\mathbb{R}) \times \mathbb{T}^d$ writing

$$X' = A(\theta)X, \quad \theta' = \omega, \qquad (X,\theta) \in GL(n,\mathbb{R}) \times \mathbb{T}^d. \tag{II.12}$$

This system contains all relevant information on the dynamics of (II.9).

Remark II.10. *Although we will mainly focus on the properties of linear equations with quasi-periodic coefficients in $\mathbb{R}$, it is clear that all the above construction can be performed to deal with the case of complex coefficients.*

Remark II.11. *If a belongs to some matrix Lie algebra g, then any lift $A(\theta)$ belongs to g (due to the rational independence of the frequencies) and (II.12) can be considered in $G \times \mathbb{T}^d$, where G is the corresponding Lie group.*

Linear periodic differential equations

As a motivating example, let us consider linear equations with periodic coefficients. These are linear quasi-periodic equations of the form

$$x' = a(t)x,$$

being a a periodic matrix function, which means that $a(t) = a(t+T)$ for some fixed $T > 0$ and all $t \in \mathbb{R}$. This can be seen as a quasi-periodic linear system with only one basic frequency $\omega = 2\pi/T$. It can be lifted to an autonomous system on $\mathbb{R}^n \times \mathbb{T}$ if

$$x' = A(\theta)x, \quad \theta' = \omega, \qquad (x,\theta) \in \mathbb{R}^n \times \mathbb{T}.$$

and we set $A(\theta) = a(\theta/\omega)$. Again, one can consider the associated matrix system

$$X' = A(\theta)X, \quad \theta' = \omega, \qquad (X,\theta) \in GL(n,\mathbb{R}) \times \mathbb{T}. \tag{II.13}$$

Let us denote by $X = X(t; X_0, \phi)$, for some $(X_0, \phi) \in GL(n,\mathbb{R}) \times \mathbb{T}$, the solution of (II.13) with initial conditions

$$X(0; X_0, \phi) = X_0, \qquad \theta(0) = \phi.$$

We are going to see that the behaviour of a linear system with periodic coefficients is basically the same than linear systems with constant coefficients: there exists a change of variables that takes the original system to constant coefficients. Let us consider the map

$$(X_0, \theta) \in GL(n,\mathbb{R}) \times \mathbb{T} \mapsto (X(2\pi; X_0, \phi), \phi + 2\pi) \in GL(n,\mathbb{R}) \times \mathbb{T},$$

which will be called the *Poincaré map* or *return map* of (II.13). Since

$$(X(t+2\pi; X_0, \phi), \phi + \omega t)$$

is a solution of (II.13) with initial conditions (X_0, ϕ) there exists a nonsingular matrix $P(\phi)$ such that

$$X(2\pi; X_0, \phi) = P(\phi)X_0$$

Fix one initial condition ϕ and let B be a square matrix such that

$$P(\phi) = \exp(2\pi B).$$

Such a matrix can be always found if we do not require it to be real (if $P(\phi)$ has negative eigenvalues, B cannot be real). B will be called a *Floquet matrix* (by construction it is not uniquely determined).

In terms of this Floquet matrix the relation

$$X(t+2\pi; X_0, \phi) = \exp(tB)X(t) \tag{II.14}$$

holds for all $t \in \mathbb{R}$ because it is true when $t = 0$ and both sides of (II.14) satisfy the same differential equation.

Let us now define

$$Z(t) = \exp(-tB)X(t).$$

This is a periodic transformation with period 2π. Indeed,

$$\begin{aligned} Z(t+2\pi) &= \exp(-(t+2\pi)B)X(t+2\pi) = \\ &= \exp(-2\pi B)\exp(-tB)\exp(2\pi B)X(t) = \exp(-tB)X(t) = Z(t). \end{aligned}$$

The change of variables

$$x = Z(t)y$$

satisfies that

$$y' = By.$$

We say that we have reduced our system to constant coefficients: we have obtained a nonsingular transformation Z which defines a change of variables such that in the new variables this system takes the form of constant coefficients. This procedure of reducing to constant coefficients is called *Floquet theory*.

Remark II.12. *If we want the reduced matrix (the Floquet matrix) to be real, then the reducing transformation cannot be always T-periodic but just $2T$-periodic, because*

$$X(2T) = X(T)^2$$

and for a matrix of this form (the square of a nonsingular matrix) there exists always a real logarithm. This is called period-doubling.

Remark II.13. *If G is a matrix Lie group and g is its Lie algebra then, by Proposition II.1, a is in g (equivalently A is in g) implies that B and Z can be chosen in $g_{\mathbb{C}}$ and G, where $g_{\mathbb{C}}$ is the complexification of g,*

$$g_{\mathbb{C}} = \left\{X_1 + \mathrm{i}X_2 \text{ such that } X_1, X_2 \in g\right\}.$$

II.1.4 Reducibility of linear quasi-periodic differential equations

The construction in the previous section of a reducing transformation for linear periodic equations leads naturally to the concept of *reducibility* of a linear system with quasi-periodic coefficients. Consider Equation (II.9) with lift (II.10). Following the same strategy than in the previous section, we may try to find a change of variables of the form

$$y = Z(\omega t)x, \qquad (\text{or } y = Z(\theta)x)$$

being $Z : \mathbb{T}^d \to GL(n, \mathbb{R})$, at least of class C^1, such that the equation satisfied by y is

$$y' = By \qquad (\text{or } y' = By, \;\; \theta' = \omega)$$

for some constant matrix B. If such Z and B exist we will say that (II.9) or (II.10), is *reducible to constant coefficients* or simply *reducible*. Any such B and Z are called *Floquet matrix* and *reducing transformation* respectively. The eigenvalues of B are called the *Floquet exponents* and their real parts are the *Lyapunov exponents*. From the previous conditions it is seen that Z and B need to satisfy the following relation

$$\frac{d}{dt} Z(\omega t) = A(\omega t)Z(\omega t) - Z(\omega t)B$$

for all $t \in \mathbb{R}$. Due to the rational independence of the frequency vector ω this is equivalent to the following *homological equation*

$$\partial_\omega Z(\theta) = A(\theta)Z(\theta) - Z(\theta)B, \qquad \theta \in \mathbb{T}^d$$

where we set

$$\partial_\omega Z(\theta) = (D_\theta Z)\, \omega.$$

In particular, reducibility of a quasi-periodic linear equation implies reducibility of any equation in its hull with the same Floquet matrix B. Any fundamental matrix of a reducible linear quasi-periodic equations can be expressed by means of the following *Floquet representation*

$$X\,(t; \phi, X_0) = Z\,(\omega t + \phi)\, e^{Bt} Z(\phi)^{-1} X_0.$$

As it already happens with periodic linear differential equations, we cannot expect the transformation Z to be defined on $\mathbb{T}^d$ but on some finite covering of it (unless we complexify the system). That is, we consider

$$x' = A(\omega t)x$$

as a linear equation with frequency ω/χ for some positive integer χ (usually $\chi = 2$) and consider A defined on $\mathbb{T}^d_\chi = (\mathbb{R}/(2\pi\chi\mathbb{Z}))^d$ instead of $\mathbb{T}^d$. We then look for reducing transformations on $\mathbb{T}^d_\chi$.

II.2 Cocycles and skew-products

In this section we investigate the properties of cocycles and skew-product flows, which are generalizations of concepts in the previous section. We will mainly deal with the discrete version, referring to the particularities of linear differential equations when necessary.

II.2.1 Cocycles, skew-products and basic properties

Let G a matrix Lie subgroup of $GL(n, \mathbb{R})$,

$$\omega = (\omega_1, \ldots, \omega_d) \in \mathbb{R}^d$$

be a frequency vector and $A : \mathbb{T}^d \to G$ a continuous map. We define the *quasi-periodic cocycle* or just *cocycle* (A, ω) as the map

$$\begin{array}{rccc} (A,\omega): & G \times \mathbb{T}^d & \longrightarrow & G \times \mathbb{T}^d \\ & (X,\theta) & \mapsto & (A(\theta)X, \theta + 2\pi\omega)\,. \end{array}$$

We say that it is a C^r-cocycle if A is a C^r-map from $\mathbb{T}^d$ to G. In the same way, the cocycle is real analytic if A is real analytic so that we can speak of C^a-cocycles or C^a_ρ-cocycles for $\rho > 0$.

The iteration of this map gives rise to the following dynamical system on $G \times \mathbb{T}^d$,

$$X_{k+1} = A(\theta_k)X_k \qquad \theta_{k+1} = \theta_k + 2\pi\omega, \tag{II.15}$$

where $(X_k, \theta_k)_{k\in\mathbb{Z}}$ is a sequence in $G \times \mathbb{T}^d$. This sequence is determined by an initial condition (X_0, θ_0) in $G \times \mathbb{T}^d$. The map

$$(k, X_0, \theta_0) \in \mathbb{Z} \times G \times \mathbb{T}^d \mapsto (X_k, \theta_k)$$

where

$$X_{j+1} = A\,(\theta_0 + 2\pi\omega j) \cdot \ldots \cdot A\,(\theta_0)\, X_0, \quad \theta_{j+1} = \theta_0 + 2\pi\omega(j+1)$$

is called a *discrete quasi-periodic skew-product flow* or simply *discrete quasi-periodic skew-product*. By an abuse of language, (II.15) will also be called a quasi-periodic skew-product.

A quasi-periodic cocycle (A, ω) can also be seen as a map from $\mathbb{R}^n \times \mathbb{T}^d$ to itself which we write in the same way as before

$$(A, \omega)(x, \theta) = (A(\theta)x, \theta + 2\pi\omega),$$

for any $(x, \theta) \in \mathbb{R}^n \times \mathbb{T}^d$, and the iteration of this last map also gives rise to a quasi-periodic skew-product, now on $\mathbb{R}^n \times \mathbb{T}^d$,

$$x_{k+1} = A(\theta_k)x_k \qquad \theta_{k+1} = \theta_k + 2\pi\omega, \tag{II.16}$$

being $(x_k, \theta_k)_{k\in\mathbb{Z}}$ a sequence in $\mathbb{R}^n \times \mathbb{T}^d$. This quasi-periodic skew-product shows up as the autonomization of the following linear difference equation on $\mathbb{R}^n$:

$$x_{k+1} = A(2\pi k\omega + \theta_0)x_k \qquad k \in \mathbb{Z}$$

which means picking an initial condition for the sequence $(\theta_k)_{k\in\mathbb{Z}}$ in (II.16). Any quasi-periodic cocycle on $G \times \mathbb{T}^d$ can also be seen as a cocycle on $\mathbb{R}^n \times \mathbb{T}^d$ and one can shift from one formulation to another when necessary.

A particularly important case is that of *constant cocycles* or cocycles with *constant coefficients*. These cocycles, (A, ω) satisfy that

$$A(\theta) = A(0) \equiv A_0 \quad \text{for all} \qquad \theta \in \mathbb{T}^d.$$

so that

$$(A, \omega)^k = \left(A_0^k, k\omega\right)$$

and the solutions of the skew-product are of the form

$$(X_k, \theta_k) = \left(A_0^k X_0, \theta_0 + 2\pi k\omega\right)$$

for any $k \in \mathbb{Z}$.

Remark II.14. *For the moment we have not yet considered arithmetic conditions on the frequency vector ω. This allows us to consider cocycles $(Z, 0)$ with frequency identically zero. These act on $G \times \mathbb{T}^d$ as*

$$(Z, 0)(X, \theta) = (Z(\theta)X, \theta).$$

so that they can be regarded as changes of variables.

Relation with quasi-periodic differential equations

Let us now see how one can derive a quasi-periodic cocycle and a skew-product from a linear quasi-periodic differential equation. Consider

$$x' = A(\theta)x, \qquad \theta' = \omega, \tag{II.17}$$

a linear quasi-periodic differential equation where $(x, \theta) \in \mathbb{R}^n \times \mathbb{T}^d$. Let

$$(x(t; x_0, \phi), \phi + \omega t)$$

be the solution of the associated Cauchy problem with initial conditions $(x_0, \phi) \in \mathbb{R}^n \times \mathbb{T}^d$. Then

$$x(t; x_0, \phi) = X(t; X_0, \phi)X_0^{-1}x_0$$

where $X(t; X_0, \phi)$ is a nonsingular solution of the matrix equation

$$X' = A(\theta)X, \quad \theta' = \omega \tag{II.18}$$

with $X(0) = X_0$ and $\theta(0) = \phi$. The maps

$$(t; x_0, \phi) \in \mathbb{R} \times \left(\mathbb{R}^n \times \mathbb{T}^d\right) \mapsto \left(X(t; X_0, \phi)X_0^{-1}x_0, \phi + \omega t\right) \in \mathbb{R}^n \times \mathbb{T}^d$$

and

$$(t; X_0, \phi) \in \mathbb{R} \times \left(G \times \mathbb{T}^d\right) \mapsto \left(X(t; X_0, \phi)X_0^{-1}x_0, \phi + \omega t\right) \in G \times \mathbb{T}^d$$

are called *continuous linear quasi-periodic skew-product flows* or *continuous quasi-periodic skew-product* on $\mathbb{R}^n \times \mathbb{T}^d$ and $G \times \mathbb{T}^d$ respectively. Also by an abuse of language, we will say that equations (II.17) and (II.18) are skew-product flows on $\mathbb{R}^n \times \mathbb{T}^d$ and $G \times \mathbb{T}^d$ respectively.

To produce a discrete flow from this continuous flow (and a quasi-periodic cocycle, also) let us take

$$\tilde{\theta} = \left(\theta^1, \ldots, \theta^{d-1}\right) \in \mathbb{T}^{d-1}, \quad \tilde{\phi} = \left(\theta_0^1, \ldots, \theta_0^{d-1}\right) \in \mathbb{T}^{d-1}, \quad \tilde{\omega} = \left(\frac{\omega_1}{\omega_d}, \ldots, \frac{\omega_{d-1}}{\omega_d}\right)$$

and, for some fixed $\theta_0^d \in \mathbb{T}$,

$$\tilde{A}(\tilde{\theta}) = X\left(2\pi/\omega_d; I, (\tilde{\theta}, \theta^d)\right).$$

Then the sequence

$$x_k = x\left(\frac{2\pi k}{\omega_d}; x_0, \left(\tilde{\theta}, \theta^d\right)\right), \qquad k \in \mathbb{Z}$$

satisfies the recursion

$$x_{k+1} = \tilde{A}\left(\tilde{\phi} + 2\pi k\tilde{\omega}\right)x_k, \qquad k \in \mathbb{Z}$$

so that, if $\tilde{\theta}_k = \tilde{\phi} + 2\pi\tilde{\omega}k$, then

$$x_{k+1} = \tilde{A}\left(\tilde{\theta}_k\right)x_k, \quad \tilde{\theta}_{k+1} = \tilde{\theta}_{k-1} + 2\pi\tilde{\omega}, \qquad k \in \mathbb{Z},$$

is a discrete quasi-periodic skew-product flow on $\mathbb{R}^n \times \mathbb{T}^{d-1}$ which comes from the iteration of the cocycle $\left(\tilde{A}, \tilde{\omega}\right)$ on $\mathbb{R}^n \times \mathbb{T}^{d-1}$. This is called a *Poincaré cocycle* of the skew-product flow (II.17). In the same way, one can consider Poincaré cocycles coming from the matrix equation (II.18).

Remark II.15. *Instead of taking $\omega = (\tilde{\omega}, 1)/\omega_d$ one could, in principle, choose any other component of the frequency vector ω and consider the corresponding Poincaré cocycle.*

Remark II.16. *If g is the Lie algebra of some matrix Lie group G and A is in g then $\tilde{A}$ belongs to G and $\left(\tilde{A}, \tilde{\omega}\right)$ is a quasi-periodic cocycle on $G \times \mathbb{T}^{d-1}$.*

Remarks about frequency vectors

In the construction above of a linear cocycle from a quasi-periodic differential equation we have not yet considered which arithmetic conditions on $\tilde{\omega}$ are derived from the fact that ω is rationally independent. Recall that the condition of rationally independence on ω is that

$$\langle \mathbf{k}, \omega \rangle \neq 0 \tag{II.19}$$

for all $\mathbf{k} \in \mathbb{Z}^d$ not identically zero. Since

$$\omega = \frac{1}{\omega_d} (\tilde{\omega}, 1)$$

condition (II.19) is equivalent to

$$\langle \tilde{\mathbf{k}}, \tilde{\omega} \rangle \notin \mathbb{Z} \tag{II.20}$$

or

$$\sin \Big(\pi \langle \tilde{\mathbf{k}}, \tilde{\omega} \rangle \Big) \neq 0 \tag{II.21}$$

for every $\tilde{\mathbf{k}} \in \mathbb{Z}^{d-1}$ not identically zero. If $\tilde{\omega}$ satisfies (II.20) or (II.21) we say that it is *nonresonant*. According to this definition, $\tilde{\omega}$ is nonresonant if, and only if, ω is rationally independent.

II.2.2 Conjugation and reducibility of cocycles

In the previous section we have linked linear differential equations with quasi-periodic coefficients to quasi-periodic cocycles. Since for the former a concept of reducibility was given, we now want to define this in the more general framework of quasi-periodic cocycles.

Given two C^r-cocycles in $G \times \mathbb{T}^d$ with nonresonant frequency ω, (A, ω) and (B, ω), we say that they are *C^s-conjugated*, or simply *conjugated* if $s = r$, if there exists a C^s-map $Z : \mathbb{T}^d \to G$ such that the commutation

$$(A, \omega) \circ (Z, 0) = (Z, 0) \circ (B, \omega)$$

holds. In terms of the maps A, B and Z this is equivalent to the fulfillment of

$$A(\theta) Z(\theta) = Z(\theta + 2\pi\omega) B(\theta) \tag{II.22}$$

for all $\theta \in \mathbb{T}^d$. The conjugation between the cocycles induces a conjugation between the skew-products

$$X_{n+1} = A(\theta_n) X_n \qquad \theta_{n+1} = \theta_n + 2\pi\omega$$

and

$$Y_{n+1} = B(\theta_n) Y_n \qquad \theta_{n+1} = \theta_n + 2\pi\omega$$

by means of the change of variables

$$(X_n, \theta_n) = (Z(\theta_n) Y_n, \theta_n).$$

If the conjugation $(Z, 0)$ is seen as a map from $\mathbb{R}^n \times \mathbb{T}^d$ to itself, then the quasi-periodic skew products

$$x_{n+1} = A(\theta_n) x_n, \qquad \theta_{n+1} = \theta_n + 2\pi\omega$$

and

$$y_{n+1} = B(\theta_n) y_n, \qquad \theta_{n+1} = \theta_n + 2\pi\omega$$

are conjugated through the change of variables

$$(x_n, \theta_n) = (Z(\theta_n) y_n, \theta_n)\,.$$

Particularly interesting is the case of cocycles conjugated to constant cocycles: a cocycle is said to be *C^s-reducible to constant coefficients* or simply *reducible* whenever it is C^s-conjugate to a cocycle with constant coefficients. We also say that the corresponding skew-products, both on $G \times \mathbb{T}^d$ and $\mathbb{R}^n \times \mathbb{T}^d$, are reducible to constant coefficients, which means that by a change of variables it is transformed to constant coefficients.

Given a cocycle which is reducible to a constant cocycle (B, ω) by a conjugation Z we call B its *Floquet matrix* and Z its *reducing transformation* although from the conjugation equation (II.22) it is readily seen that they are not uniquely determined.

If (A, ω) is reducible with Floquet matrix B and reducing transformation Z, then one has the identity

$$(A, \omega)^n = (Z, 0) \circ (B, \omega)^n \circ (Z, 0)^{-1}\,,$$

which allows the following *Floquet representation* of the solutions (X_n, θ_n) of the corresponding skew-product flows

$$(X_n, \theta_n) = \big(Z(\theta_n) \cdot B^n \cdot Z(\theta_0)^{-1} X_0, \theta_n\big)$$

on $G \times \mathbb{T}^d$ and also for the corresponding skew-product on $\mathbb{R}^n \times \mathbb{T}^d$,

$$(x_n, \theta_n) = \big(Z(\theta_n) \cdot B^n \cdot Z(\theta_0)^{-1} x_0, \theta_n\big)$$

for all $n \in \mathbb{Z}$.

As we have said before, the Floquet matrix of a reducible system is not uniquely determined. However, it is seen from the previous Floquet representation that the eigenvalues of B have some dynamical meaning and they are called the *Floquet multipliers*. The logarithm of the modulus of the eigenvalues are the *Lyapunov exponents*.

Remark II.17. *Similarly to linear differential equations with quasi-periodic coefficients, it may be necessary to* half the frequency *to conjugate a cocycle to constant coefficients. This means conjugating $(A(2\cdot), \omega/2)$ to constant coefficients instead of (A, ω).*

To relate this concept of reducibility of cocycles with that given for linear quasi-periodic differential equations given in Section II.1.4, it is enough to consider the corresponding Floquet representations:

$$X\,(t; \phi, X_0) = Z\,(\omega t + \phi)\, e^{Bt} Z(\phi)^{-1} X_0.$$

for a fundamental matrix of a continuous skew-product flow and

$$(X_k, \theta_k) = \big(Z(\theta_0 + 2\pi\omega k) B^k Z(\theta_0)^{-1} X_0, \theta_0 + 2\pi\omega k\big)$$

for a discrete skew-product. With this in mind one sees the following

Proposition II.18. *A continuous quasi-periodic skew-product flow with irrational frequency vector is reducible to constant coefficients if, and only if, any Poincaré cocycle is reducible to constant coefficients. If*

$$x' = Bx, \qquad \theta' = \omega$$

is the reduced system of the skew-product flow (maybe complex-valued) and $(\tilde{B}, \tilde{\omega})$ *the reduced cocycle of the Poincaré cocycle (also possibly complex) with*

$$\omega = (\tilde{\omega}, 1)/\omega_d$$

then

$$\tilde{B} = \exp\left(\frac{2\pi}{\omega_d} B\right).$$

In particular the Floquet multipliers are of the form $\exp(2\pi\lambda/\omega_d)$ *with* λ *a Floquet exponent.*

Reducibility of scalar cocycles

As a motivating example let us discuss scalar quasi-periodic cocycles, that is, cocycles (A, ω) where $A : \mathbb{T}^d \to G \subset GL(1, \mathbb{R})$, being G a subgroup of $GL(1, \mathbb{R})$ which, for definiteness, we take $\mathbb{R}_+ = (0, +\infty)$. The iteration of this cocycle gives rise to the quasi-periodic skew-product

$$x_{k+1} = A(\theta_k)x_k, \qquad \theta_{k+1} = \theta_0 + 2\pi\omega k \tag{II.23}$$

where $(x_k, \theta_k) \in \mathbb{R}_+ \times \mathbb{T}^d$. To reduce (II.23) to constant coefficients means to look for a map $Z : \mathbb{T}^d \to \mathbb{R}_+$ and a constant $B \in \mathbb{R}_+$ such that the identity

$$A(\theta)Z(\theta) = Z(\theta + 2\pi\omega)B$$

holds for all $\theta \in \mathbb{T}^d$. Again we restrict our consideration to C^a_ρ-cocycles, real analytic cocycles for some fixed $\rho > 0$. Any C^a_ρ-map from $\mathbb{T}^d$ to $\mathbb{R}_+$ is the exponential of a C^a_r-map from $\mathbb{T}^d$ to $\mathbb{R}$ and vice-versa. Therefore, the previous conjugation is equivalent to

$$e^{a(\theta)}e^{z(\theta)} = e^{z(\theta+2\pi\omega)}e^b$$

being $a, z : \mathbb{T}^d \to \mathbb{R}$ real analytic maps and $b \in \mathbb{R}$. Rearranging this equation one obtains

$$z(\theta + 2\pi\omega) - z(\theta) = a(\theta) - b \tag{II.24}$$

where the unknowns are z and b. Such an equation is known as a *small divisors equation*. Integrating both sides of this equation on $\mathbb{T}^d$ one obtains

$$b = \int_{\mathbb{T}^d} a(\theta)d\theta,$$

which determines b. Using that a belongs to C^a_ρ the Fourier series

$$a(\theta) = \sum_{\mathbf{k}\in\mathbb{Z}^d} a_{\mathbf{k}} e^{\mathrm{i}\langle \mathbf{k},\theta\rangle}$$

converges for $|\mathrm{Im}\,\theta| < \rho$ and if z is a real analytic map then its Fourier representation

$$z(\theta) = \sum_{\mathbf{k}\in\mathbb{Z}^d} z_{\mathbf{k}} e^{\mathrm{i}\langle \mathbf{k},\theta\rangle} \tag{II.25}$$

must satisfy that

$$z_{\mathbf{k}} e^{2\pi\mathrm{i}\langle \mathbf{k},\omega\rangle} - z_{\mathbf{k}} = a_{\mathbf{k}}, \qquad \mathbf{k} \in \mathbb{Z}^d - \{0\}.$$

To be able to obtain the $z_{\mathbf{k}}$ from the above expression it is necessary that

$$e^{2\pi\mathrm{i}\langle \mathbf{k},\omega\rangle} \neq 1$$

for all $\mathbf{k} \neq \mathbf{0}$ which is equivalent to the nonresonance of ω:

$$\langle \mathbf{k},\omega\rangle \notin \mathbb{Z}$$

if $\mathbf{k} \neq \mathbf{0}$. In such a case the z_k are uniquely determined

$$z_{\mathbf{k}} = \frac{a_{\mathbf{k}}}{e^{2\pi\mathrm{i}\langle \mathbf{k},\omega\rangle} - 1}. \tag{II.26}$$

The problem of reducibility is not yet solved because it is not true that for every nonresonant frequency vector ω the Fourier transform of $(z_{\mathbf{k}})_{\mathbf{k}}$ given by (II.25) is real analytic. The real character comes from the fact that a is real analytic and formula (II.26). Using Theorem II.6, the function z defined by the series (II.25) will belong to $C^a_\rho(\mathbb{T}^d)$ if

$$\sup_{\mathbf{k}\in\mathbb{Z}^d} |z_{\mathbf{k}}|\, e^{\rho|\mathbf{k}|} < \infty.$$

By Formula (II.25) and the fact that $a \in C^a_\rho(\mathbb{T}^d)$, a convenient condition to grant the above bound is

$$\left| e^{2\pi\mathrm{i}\langle \mathbf{k},\omega\rangle} - 1 \right| \geq \frac{C}{2|\mathbf{k}|^\sigma}, \tag{II.27}$$

or, equivalently,

$$|\sin \pi\langle \mathbf{k},\omega\rangle| \geq \frac{C}{|\mathbf{k}|^\sigma}, \tag{II.28}$$

where $C > 0$ and $\sigma \geq d+1$ are some fixed positive constants. We say that ω is a *strongly nonresonant frequency vector* if such constants exists and denote it by $\omega \in DC^d(C,\tau,\mathbb{R}^d)$ (here d stands for discrete). A condition like (II.28) is called a *strong nonresonance condition* or, more generally, a *Diophantine condition*. This condition is not optimal for one-dimensional homological equations (see Rüssmann [Rüs75]), but suitable for KAM schemes, see Chapter V, where this kind of homological equations are solved an infinite number of times.

Remark II.19. *The set of strongly nonresonant vectors in $DC^d(C,\sigma,\mathbb{R}^d)$ has positive measure for any $C, \sigma > d+1$ and it is a dense and open subset of $\mathbb{R}^d$. The set of all strongly nonresonant vectors has full measure in $\mathbb{R}^d$ and its complementary are the so-called* Liouville-type numbers *which are a dense and uncountable subset of $\mathbb{R}^d$. Finally, let us mention that, if $\sigma = d+1$, then $DC^d(C,\sigma,\mathbb{R}^d)$ is dense with zero measure.*

Summing up, we have the following result.

Proposition II.20 (Reducibility of scalar cocycles). *Let (A, ω) a C^a_ρ-cocycle on $(0, \infty) \times \mathbb{T}^d$, with $\rho > 0$, and assume that ω belongs to $DC^d(C, \sigma)$,*

$$|\sin \pi \langle \mathbf{k}, \omega \rangle| \geq \frac{C}{|\mathbf{k}|^\sigma}, \qquad \mathbf{k} \in \mathbb{Z}^d - \{0\},$$

for some fixed $C, \sigma > 0$. Then the cocycle (A, ω) is C^a_ρ-reducible to the cocycle (B, ω) where

$$B = \exp\left(\int_{\mathbb{T}^d} \log A(\theta) d\theta\right)$$

and, therefore, reducible to constant coefficients.

Remark II.21. *In the same way one can prove by induction the reducibility of* triangular *quasi-periodic cocycles. These are cocycles (A, ω) of $GL(n, \mathbb{R}) \times \mathbb{T}^d$ where $A(\theta)$ is a triangular matrix. This will be used in Chapter VI.*

One can proceed analogously with linear equations with quasi-periodic coefficients,

$$X' = A(\theta)X, \qquad \theta' = \omega,$$

with $(X, \theta) \in \mathbb{R}_+ \times \mathbb{T}^d$. There are two ways of doing so. The first one is to adapt the above proof to the context of reducibility in Section II.1.4. In this case the Diophantine condition which is met, for some frequency vector $\omega \in \mathbb{R}^d$, is that the bound

$$|\langle \mathbf{k}, \omega \rangle| \geq \frac{C}{|\mathbf{k}|^\sigma} \tag{II.29}$$

must hold for all $\mathbf{k} \in \mathbb{Z}^d$ not identically zero. If such a condition is fulfilled, ω is said to be *strongly rationally independent* or *strongly irrational*. The class of strongly irrational vectors which satisfy condition (II.29) is denoted by $DC^c(C, \sigma, \mathbb{R}^d)$.

The second way of considering the reducibility of these scalar linear differential equations with quasi-periodic coefficients is to try to use the results for cocycles given above. In such a case, the frequency of the cocycle is given by

$$\tilde{\omega} = \left(\frac{\omega_1}{\omega_d}, \dots, \frac{\omega_{d-1}}{\omega_1}\right).$$

which must satisfy the Diophantine condition

$$|\sin \pi \langle \mathbf{k}, \tilde{\omega} \rangle| \geq \frac{C}{|\mathbf{k}|^\sigma}, \qquad \mathbf{k} \in \mathbb{Z}^d - \{0\}, \tag{II.30}$$

fore some $C, \sigma > 0$. These two conditions are equivalent: $\omega = (\tilde{\omega}, 1)$ is strongly irrational if, and only if, $\tilde{\omega}$ is strongly nonresonant.

It is customary to call strongly irrational vectors and strongly nonresonant vectors simply *Diophantine* vectors when the context, either continuous or discrete, is clear.

II.3 Exponential dichotomy, Sacker-Sell spectrum and invariant splittings

In the previous section it has been seen that a reducible cocycle (A, ω) has a Floquet representation

$$(A, \omega)^k = (Z, 0) \circ (B^k, k\omega) \circ (Z, 0)^{-1} \tag{II.31}$$

where $Z : \mathbb{T}^d \to G$ and B is a constant matrix. Assume that the Floquet matrix B has all eigenvalues with modulus different from one (we say then that it is *hyperbolic*). Then any solution of the quasi-periodic skew-product flow on $\mathbb{R}^n \times \mathbb{T}^d$,

$$x_{k+1} = A(\theta_k)x_k \qquad \theta_{k+1} = \theta_k + 2\pi\omega,$$

with $(x_k)_{k\in\mathbb{Z}}$ bounded in $\mathbb{R}^n$ satisfies that $x_k = 0$ for all $k \in \mathbb{Z}$. In this section, we will see the converse of this result and some related facts.

II.3.1 Exponential dichotomy of cocycles

Let (A, ω) a quasi-periodic cocycle on $G \times \mathbb{T}^d$, with $G \subset GL(n, \mathbb{R})$ a matrix Lie group and nonresonant frequency vector ω. This generates a quasi-periodic skew-product flow on $\mathbb{R}^n \times \mathbb{T}^d$,

$$x_{k+1} = A(\theta_k)x_k \qquad \theta_{k+1} = \theta_k + 2\pi\omega, \tag{II.32}$$

which, for any given initial condition $(x_0, \theta_0) \in \mathbb{R}^n \times \mathbb{T}^d$, generates a sequence $(x_k, \theta_k)_{k\in\mathbb{Z}}$.

We say that a quasi-periodic cocycle (A, ω), or rather its associated skew-product flow on $\mathbb{R}^n \times \mathbb{T}^d$ has an *exponential dichotomy* if the only bounded solutions are the trivial ones: that is, if $(x_k, \theta_k)_{k\in\mathbb{Z}}$ is a solution of (II.32) which is bounded in $k \in \mathbb{Z}$ then necessarily $x_k = 0$ for all $k \in \mathbb{Z}$. A continuous skew-product flow with irrational frequency vector ω,

$$x' = A(\theta)x, \qquad \theta' = \omega \tag{II.33}$$

on $\mathbb{R}^n \times \mathbb{T}^d$, has an *exponential dichotomy* if any Poincaré cocycle has an exponential dichotomy. Equivalently if any solution $(x(t), \theta(t))$ of (II.33) with $x(t)$ bounded satisfies $x \equiv 0$.

Remark II.22.

(i) If the group G is compact (resp. g is a compact Lie algebra), then all solutions of (II.32) (resp. of (II.33)) are bounded and the skew-product has no exponential dichotomy. In the following we will implicitly consider G noncompact and, for many things, $G = GL(n, \mathbb{R})$.

(ii) The notion of exponential dichotomy of cocycles is invariant by conjugation of cocycles.

(iii) If (A, ω) is a quasi-periodic cocycle on $G \times \mathbb{T}^d$ with an exponential dichotomy and $\phi \in \mathbb{T}^d$ then the cocyles $(A(\cdot + \phi), \omega)$ also has an exponential dichotomy (if ω is nonresonant).

(iv) The concept of exponential dichotomies was introduced by Massera & Schäffer [MS66]. *For nonautonomous equations see Coppel* [Cop78] *and the papers by Sacker, Sell and Johnson* [SS74, SS76b, SS76a, SS78, Joh80, JS81]. *This theory has been extended to*

more general equations, see Chicone & Latushkin [CL99] *and references therein. The interaction between exponential dichotomies and dynamics of cocyles or more general dynamical systems has been studied in a series of papers by Haro & de la Llave* [HdlL03c, HdlL03d, HdlL03e] *and* [HdlL03b, HdlL03a] *where the numerical implementation is discussed.*

If a cocycle (A, ω) is reducible to constant coefficients one has the Floquet representation (II.31). In such a case, the cocycle has an exponential dichotomy if, and only if, B is hyperbolic: all the Floquet multipliers (the eigenvalues of B) have modulus different from one. Then one can define the set of stable solutions of (A, ω) as

$$\mathcal{S}(A, \omega) = \left\{ (Z(\theta)v, \theta) \in \mathbb{R}^n \times \mathbb{T}^d ; v \in S(B) \right\} ,$$

where $S(B)$ is the spectral subspace of B associated to those eigenvalues with modulus less than one. The unstable set is defined as

$$\mathcal{U}(A, \omega) = \left\{ (Z(\theta)v, \theta) \in \mathbb{R}^n \times \mathbb{T}^d ; v \in U(B) \right\} ,$$

where $U(B)$ is now the spectral subspace of B associated to eigenvalues greater than one in modulus.

Remark II.23. *In the continuous case, the condition of hyperbolicity of a reducible skew-product flow is that all the Floquet exponents have nonzero real part.*

These subsets of $\mathbb{R}^n \times \mathbb{T}^d$ are invariant, have a natural structure of *vector subbundles* and they are called the *stable* and *unstable* subbundles of (A, ω). Due to the fact that the cocycle (A, ω) is C^s-reducible to constant coefficients, then these subbundles are also of class C^s. Moreover, due to the fact that B is hyperbolic

$$\mathbb{R}^n = S(B) \oplus U(B)$$

as vector spaces and, therefore, the stable and unstable subbundles are *complementary*, which means that, as subbundles, they generate $\mathbb{R}^n \times \mathbb{T}^d$ and that their intersection is the trivial subbundle $\{0\} \times \mathbb{T}^d$. This trivial subbundle can be seen as the kernel of the projection

$$\pi : (x, \theta) \in \mathbb{R}^n \times \mathbb{T}^d \mapsto \pi(x, \theta) = x \in \mathbb{R}^n .$$

The fact that they are complementary will be written as

$$\mathbb{R}^n \times \mathbb{T}^d = \mathcal{S}(A, \omega) \oplus \mathcal{U}(A, \omega) \qquad \text{(Whitney sum)}.$$

It turns out that these stable and unstable subbundles can be defined even if the cocycle is not reducible to constant coefficients. Moreover, we will see that the decomposition in terms of stable and unstable subbundles of (A, ω) can be defined for any cocycle having an exponential dichotomy, but not necessarily reducible to constant coefficients.

Exponential dichotomy and invariant subbundles

Let (A,ω) be a quasi-periodic cocycle. We define its *stable subbundle*

$$\mathcal{S}(A,\omega) = \left\{(x,\theta) \in \mathbb{R}^n \times \mathbb{T}^d ; \lim_{k\to+\infty} \pi\left((A,\omega)^n(x,\theta)\right) = 0\right\} = \left\{(x,\theta) \in \mathbb{R}^n \times \mathbb{T}^d ; \lim_{k\to+\infty} A(\theta+2\pi\omega k)\cdot \ldots \cdot A(\theta)x = 0\right\}$$

where the limit is taken with the usual Euclidean norm on $\mathbb{R}^n$, and the *unstable subbundle*

$$\mathcal{U}(A,\omega) = \left\{(x,\theta) \in \mathbb{R}^n \times \mathbb{T}^d ; \lim_{k\to-\infty} \pi\left((A,\omega)^n(x,\theta)\right) = 0\right\} = \left\{(x,\theta) \in \mathbb{R}^n \times \mathbb{T}^d ; \lim_{k\to-\infty} A(\theta+2\pi\omega k)^{-1}\cdot \ldots \cdot A(\theta)^{-1}x = 0\right\}$$

Sacker & Sell [SS78] proved that the exponential dichotomy of (A,ω) is equivalent to $\mathcal{S}(A,\omega)$ and $\mathcal{U}(A,\omega)$ being continuous to have the decomposition

$$\mathbb{R}^n \times \mathbb{T}^d = \mathcal{S}(A,\omega) \oplus \mathcal{U}(A,\omega) \qquad \text{(Whitney sum).}$$

Johnson & Sell [JS81] and Johnson [Joh80] proved that if (A,ω) is a C^r-cocycle (resp. C^a_ρ-cocycle for some positive ρ) with exponential dichotomy, then these two subbundles are not just continuous but of class C^r (resp. C^a_ρ).

Another way to express the decomposition in terms of the stable and unstable subbundles is the following. Assume that (A,ω) is a cocycle with an exponential dichotomy and let $\mathcal{S}(A,\omega)$ and $\mathcal{U}(A,\omega)$ be the stable and unstable subbundles, respectively. Since

$$\mathbb{R}^n \times \mathbb{T}^d = \mathcal{S}(A,\omega) \oplus \mathcal{U}(A,\omega)$$

one can define a projector $\Pi : \mathbb{R}^n \times \mathbb{T}^d \to \mathbb{R}^n \times \mathbb{T}^d$ of the form

$$\Pi(x,\theta) = (P(\theta)x,\theta)\,, \qquad (x,\theta)\in\mathbb{R}^n \times \mathbb{T}^d,$$

where $P : \mathbb{T}^d \to \mathbb{R}^n$ is a continuous map with $P^2 = P$, and such that

$$\mathcal{S}(A,\omega) = \left\{(x,\theta)\in\mathbb{R}^n \times \mathbb{T}^d ; \Pi(x,\theta) = (x,\theta)\right\}$$

and

$$\mathcal{U}(A,\omega) = \left\{(x,\theta)\in\mathbb{R}^n \times \mathbb{T}^d ; \Pi(x,\theta) = (0,\theta)\right\}.$$

Equivalently,

$$\Pi(\mathbb{R}^n \times \mathbb{T}^d) = \mathcal{S}(A,\omega)$$

and

$$(Id-\Pi)\left(\mathbb{R}^n \times \mathbb{T}^d\right) = \mathcal{U}(A,\omega).$$

The existence of such a continuous projector for the cocycle (A,ω) is equivalent to its exponential dichotomy. In this case, the regularity of the projector is the same than that of the cocycle [Joh80, JS81].

II.3.2 Sacker-Sell spectrum of quasi-periodic cocycles

The concept of exponential dichotomy induces a generalization of the Floquet multipliers for quasi-periodic cocycles. Let (A, ω) a quasi-periodic cocycle. The *Sacker-Sell spectrum of a quasi-periodic cocycle* (A, ω), $\Sigma^d(A, \omega)$ (or of the corresponding skew-product) is defined as the set of those $\lambda > 0$ for which the cocycle $(\lambda^{-1}A, \omega)$ does not have an exponential dichotomy.

Remark II.24.

(i) Since λA needs not to be in the group G, the group structure may be lost.

(ii) The Sacker-Sell spectrum of a cocycle with constant coefficients is the modulus of the spectrum of the constant matrix.

(iii) In principle, one could consider $\lambda \in \mathbb{C} - \{0\}$ in the definition above and obtain a different spectrum $\Sigma_{\mathbb{C}}$. In such a case, for any quasi-periodic cocycle, the identity

$$\Sigma^d_{\mathbb{C}}(A, \omega) = \Sigma^d_{\mathbb{C}}\left(e^{\mathrm{i}t}A, \omega\right)$$

holds for $t \in \mathbb{R}$. Therefore one has the relation

$$\Sigma^d_{\mathbb{C}}(A, \omega) = \bigcup_{0 \le t < 2\pi} e^{\mathrm{i}t}\Sigma^d(A, \omega)$$

and this is why we will only consider $\lambda \in (0, \infty)$.

The spectral Theorem of Sacker & Sell, [SS78] states that the Sacker-Sell spectrum is the union of, at most, n disjoint closed intervals in $(0, +\infty)$,

$$\Sigma^d(A, \omega) = [a_1, b_1] \cup \ldots \cup [a_m, b_m], \qquad a_j \le b_j < a_{j+1}$$

with $m \le n$, called the *spectral intervals*. Moreover, if $\lambda_1 < \lambda_2$ are such that

$$\Sigma(A, \omega) \cap (\lambda_1, \lambda_2) = [a_j, b_j]$$

for some $j = 1, \ldots, m$ then

$$\mathcal{S}(\lambda_1 A, \omega) \cap \mathcal{U}(\lambda_2 A, \omega)$$

is also an invariant subbundle of $\mathbb{R}^n \times \mathbb{T}^d$ which is of class C^r (as the original cocycle). Since it is an invariant subbundle we can consider the restriction of the skew-product generated by (A, ω) to this subbundle, which is again a quasi-periodic skew-product. It turns out that the Sacker-Sell spectrum of the restriction of the skew-product to this subbundle is precisely the spectral interval $[a_j, b_j]$.

Moreover, if $\lambda_1' < \lambda_2'$ are any other complex numbers such that

$$\Sigma^d(A, \omega) \cap (\lambda_1', \lambda_2') = [a_j, b_j]$$

then

$$\mathcal{S}(\lambda_1 A, \omega) \cap \mathcal{U}(\lambda_2 A, \omega) = \mathcal{S}(\lambda_1' A, \omega) \cap \mathcal{U}(\lambda_2' A, \omega).$$

and we call this invariant subbundle the *spectral subbundle associated to* $[a_j, b_j]$ which we write as $\mathcal{V}_j(A, \omega)$. In particular, if n_j, is the dimension of the jth spectral subbundle one has

$$n_1 + \ldots + n_m = n.$$

All these invariant subbundles generate the whole phase space. Given any quasi-periodic cocycle one has the following Whitney decomposition

$$\mathbb{R}^n \times \mathbb{T}^d = \mathcal{V}_1(A, \omega) \oplus \ldots \oplus \mathcal{V}_m(A, \omega)$$

into spectral invariant subbundles, which is called the *Sacker-Sell decomposition*.

Remarks for continuous skew-products

Let us briefly consider all the previous properties and definitions for continuous skew-product flows on $\mathbb{R}^n \times \mathbb{T}^d$,

$$x' = A(\theta)x, \qquad \theta' = \omega. \tag{II.34}$$

According to the definition of exponential dichotomy, (II.34) has such a property if, and only if, the only solution $(x(t), \theta(t))$ of (II.34) for which $|x(t)|$ is bounded for all t is the trivial one, $x \equiv 0$.

One can follow the same ideas than in the discrete case to introduce the Sacker-Sell spectrum for this situation. Indeed, the *Sacker-Sell spectrum of a continuous skew-product flow* (II.34) is the set of $\lambda \in \mathbb{R}$ such that

$$x' = (A(\theta) - \lambda I)\, x, \qquad \theta' = \omega. \tag{II.35}$$

has an exponential dichotomy. We will denote it by $\Sigma^c(A, \omega)$. For linear equations with constant coefficients, the Sacker-Sell spectrum is the real part of the spectrum of the constant matrix.

Let $(\tilde{A}, \tilde{\omega})$ be a Poincaré cocycle of (II.35) which has some Sacker-Sell spectrum $\Sigma^d(\tilde{A}, \tilde{\omega})$. We want to relate these two spectra, $\Sigma^c(A, \omega)$ and $\Sigma^d(\tilde{A}, \tilde{\omega})$.

Note that $(X(t), \theta(t)) \in GL(n, \mathbb{R}) \times \mathbb{T}^d$ is a solution of

$$X' = (A(\theta) - \lambda I)\, X, \qquad \theta' = \omega$$

if and only if $Y(t) = e^{\lambda t} X(t)$ satisfies

$$Y = A(\theta)Y, \qquad \theta' = \omega.$$

Therefore, if

$$\omega = \left(\frac{\omega_1}{\omega_d}, \ldots, \frac{\omega_{d-1}}{\omega_d} \right)$$

then

$$\left(e^{2\pi/\omega_d} \tilde{A}, \tilde{\omega} \right)$$

is a Poincaré cocycle of (II.35) so we obtain the relation:

$$\Sigma^d(\tilde{A}, \tilde{\omega}) = \exp\left(\frac{2\pi}{\omega_d} \Sigma^c(A, \omega) \right),$$

which “explains” why Σ^c is a subset of $\mathbb{R}$ while Σ^d is a subset of $(0, \infty)$.

II.3.3 Sacker-Sell spectrum and conjugation of cocycles

We would like to relate this Sacker-Sell decomposition of $\mathbb{R}^n \times \mathbb{T}^d$ given by a cocycle (A, ω) to the reduction of (A, ω) to another cocycle (N, ω) of block structure. More precisely, if

$$\Sigma^d(A, \omega) = [a_1, b_1] \cup \ldots \cup [a_m, b_m]$$

is the Sacker-Sell spectrum of (A, ω), then we would like to conjugate it to another cocycle (N, ω), being $N : \mathbb{T}^d \to GL(\mathbb{R}, n)$ of block-diagonal form:

$$N = \text{diag}\left(N^1, N^2, \ldots, N^m\right)$$

where N^j are square n_j-dimensional matrices such that the Sacker-Sell spectrum $\Sigma^d(N^j, \omega)$ is precisely the jth spectral interval of $\Sigma^d(A, \omega)$.

Pick $\mathcal{V}_j$ one of the spectral subbundles. Since it is a C^r-subbundle, around every $\theta \in \mathbb{T}^d$ there is a basis of this subbundle of the form $x_1^j, \ldots, x_{n_j}^j$ which are C^r functions defined in a neighbourhood of θ. This can be done for all $\theta \in \mathbb{T}^d$, but only if some geometric relations are satisfied (think of the Möbius strip) they form a global basis of C^r-functions $x_1^j, \ldots, x_{n_j}^j : \mathbb{T}^d \to \mathbb{R}^n$.

If all spectral subbundles have a global basis $x_1^j, \ldots, x_{n_j}^j : \mathbb{T}^d \to \mathbb{R}^n$, then it is possible to perform such reduction. Indeed,

$$A(\theta) x_k^j(\theta) = \sum_{l=1}^{n_j} N_{k,l}^j(\theta) x_l^j(\theta + 2\pi\omega),$$

for all $k = 1, \ldots, n_j$, where $N_{k,l}^j : \mathbb{T}^d \to \mathbb{R}$. These are C^r-maps because both A and the basis are of class C^r. Letting $N^j(\theta)$ be the matrix $\left(N_{k,l}^j(\theta)\right)_{1 \le k,l \le n_j}$ and

$$Z(\theta) = \left(x_1^1(\theta), \ldots, x_{n_1}^1(\theta), \ldots, x_1^m(\theta), \ldots, x_{n_m}^m\right)$$

then

$$A(\theta) Z(\theta) = Z(\theta + 2\pi\omega) N(\theta),$$

which yields the desired conjugation.

Remark II.25. *Here we have considered the reduction of (A, ω) which has a exactly one block per Sacker-Sell spectral interval. One can, of course, consider reductions with respect to bigger blocks which correspond to clusters of spectral intervals.*

In general, it is not true that the spectral subbundles are trivial (in fact, in the continuous case, it is possible to realize an arbitrary vector bundle over $\mathbb{R}^n \times \mathbb{T}^d$ as the stable or unstable subbundle of some skew-product flow, see Daltetskii & Krein [DK70]). In the next sections, we will consider some cases where such a reduction to block cocycles with lower dimension is possible. The first one is a perturbation of constant cocycles (where all the invariant subbundles are trivial) and the second one is the case of one-dimensional subbundle. Finally we will give Coppel's criterion for exponential dichotomy.

Perturbation of constant coefficients

Let A_0 be a constant matrix, ω rationally independent and $\Sigma^d(A_0, \omega)$ the Sacker-Sell spectrum of the cocycle (A_0, ω) (which is the union of the modulus of the eigenvalues of A_0). Let

$$\Sigma^d(A_0, \omega) = \Sigma_1^d(A_0, \omega) \cup \dots \Sigma_k^d(A_0, \omega)$$

be a disjoint decomposition of spectrum where each of the $\Sigma_j^d(A_0, \omega)$ is a union of Sacker-Sell intervals. We take $0 < \lambda_1^- < \lambda_1^+ < \lambda_2^- < \dots$ be such that

$$\Sigma^d(A_0, \omega) \cap (\lambda_j^-, \lambda_j^+) = \Sigma_j^d(A_0, \omega)$$

for $j = 1, \dots, k$. In particular, this holds if we take the decomposition given by Sacker-Sell spectral intervals. For each $j = 1, \dots, k$, let n_j be the dimension of the spectral subbundles associated to $\Sigma_j(A_0, \omega)$.

By the *roughness of exponential dichotomy* [SS78, JS81, Joh80], there is an constant ε depending on A_0, but not on ω (as long as it is rationally independent), such that for any cocycle (A, ω) of $\mathbb{R}^n \times \mathbb{T}^d$ with

$$||A_0 - A||_{C^0} < \varepsilon$$

the Sacker-Sell spectrum of (A, ω) has also a decomposition

$$\Sigma^d(A, \omega) = \Sigma_1^d(A, \omega) \cup \dots \Sigma_k^d(A, \omega)$$

with

$$\Sigma^d(A_0, \omega) \cap (\lambda_j^-, \lambda_j^+) = \Sigma_j^d(A_0, \omega)$$

for $j = 0, \dots, k$. Moreover, the invariant subbundles associated to this decomposition are trivial and of dimension n_j because these depend continuously on the cocycle and they are trivial for the constant cocycle.

This reduction can also be made smoothly with respect to external parameters. That is, if (A_μ, ω) is a family of quasi-periodic cocycles which depends smoothly (with some degree of regularity) of the parameter μ in some open neighbourhood of $\mu_0 \in \mathbb{R}^p$ and the cocycle (A_{μ_0}, ω) is in constant coefficients, then there is some constant $\varepsilon > 0$ such that for $|\mu - \mu_0| < \varepsilon$ the cocycle (A_μ, ω) is conjugated to one of the form (N_μ, ω), with the diagonal block structure determined by the spectral subspaces of A_{μ_0} as it was done above.

Cocycles with full spectrum

The Sacker-Sell spectrum of a quasi-periodic cocycle (A, ω) in $\mathbb{R}^n \times \mathbb{T}^d$ is the union of, at most, n disjoint intervals in $(0, \infty)$. Each of these annuli has an associated invariant subbundle and the whole product space $\mathbb{R}^n \times \mathbb{T}^d$ can be expressed as the Whitney sum of these subbundles. In particular, the sum of the dimensions of the subbundles must be n so that, if there are exactly n nonvoid spectral intervals, the corresponding subbundles must be one-dimensional. In such a case we say that the cocycle has *full spectrum*.

Geometrical properties of one-dimensional subbundles of $\mathbb{R}^n \times \mathbb{T}^d$ allowed to Johnson & Sell [JS81] to prove the following result in the continuous case.

Theorem II.26 ([JS81]). *Let (A,ω) be a C^r-quasi-periodic cocycle, $r \in \{0,\dots,\infty,a\}$, with nonresonant frequency ω. If (A,ω) has full spectrum then the cocycle is conjugated to (N,ω) on $\mathbb{R}^n \times (\mathbb{R}/(2\mathbb{Z}))^d$ (that is, halving the frequency), where*

$$N(\theta) = \mathit{diag}\left(N^1(\theta),\dots,N^n(\theta)\right)$$

is a diagonal C^r-cocycle.

Diagonal cocycles can be reduced to constant coefficients under appropriate hypothesis on the regularity of the cocycle and strong nonresonance of the frequency (see Section II.2.2). Putting together these reducibility of diagonal cocycles with the above theorem we obtain the following.

Theorem II.27 ([JS81]). *Let (A,ω) be a real analytic quasi-periodic cocycle on $\mathbb{R}^n \times \mathbb{T}^d$, with strongly nonresonant frequency ω. If (A,ω) has full spectrum and $a_1 < \dots < a_n$ are the elements of the Sacker-Sell spectrum, then it is C^a-reducible to (Λ,ω) on $\mathbb{R}^n \times (\mathbb{R}/(2\mathbb{Z}))^d$, where*

$$\Lambda = \mathit{diag}\left(a_1,\dots,a_n\right)$$

is a diagonal C^r-cocycle.

For future considerations it is interesting to have the following particular version of these two theorems for cocycles on $SL(2,\mathbb{R}) \times \mathbb{T}^d$.

Theorem II.28. *Let (A,ω) be a C^r-quasi-periodic cocycle on $SL(2,\mathbb{R}) \times \mathbb{T}^d$ with nonresonant frequency. If it has an exponential dichotomy then it is conjugated to the C^r-cocycle (N,ω) on $SL(2,\mathbb{R}) \times (\mathbb{R}/(2\mathbb{Z}))^d$ where*

$$N(\theta) = \left(n(\theta), n(\theta)^{-1}\right)$$

Moreover, if (A,ω) is real analytic (or smooth enough) and ω is strongly nonresonant, then the cocycle is reducible to (Λ,ω) on $SL(2,\mathbb{R}) \times (\mathbb{R}/(2\mathbb{Z}))^d$ where

$$\Lambda = \left(\lambda, \lambda^{-1}\right)$$

is a constant matrix.

Coppel's criterion for exponential dichotomy

Finally, we give an effective criterion for exponential dichotomy of quasi-periodic cocycles and skew-products. Since we will use it for the continuous and discrete case and the formulation for the continuous case is easier, we give first the version for continuous skew-product flows. Here we freely quote from Coppel [Cop78] in a suitable formulation.

Coppel's criterion can be seen as a generalization of Gerschgorin Theorem (see, for instance, Isaacson & Keller [IK94]): when a skew-product flow has a diagonal dominant part and this diagonal part is hyperbolic then the skew-product flow has an exponential dichotomy.

Theorem II.29 (Coppel's Criterion [Cop78]). *Let*

$$x' = A(\theta)x, \qquad \theta' = \omega \tag{II.36}$$

be a continuous quasi-periodic skew-product flow on $\mathbb{R}^n \times \mathbb{T}^d$ *whose frequency vector is rationally independent. If there is a constant* $\delta > 0$ *such that the matrix* $A(\theta) = (a_{ij}(\theta))$ *satisfies*

$$||a_{ii}||_{C^0} \geq \delta + \sum_{j=1, j\neq i}^{n} ||a_{ij}||_{C^0}\,, \quad i = 1, \ldots, n.$$

Then (II.36) has an exponential dichotomy. Moreover, if a_{ii} *is positive for* k *indices (and negative for* $n-k$*) then the dimension of the stable subbundle is* k *(and the dimension of the unstable* $n-k$*.*

In the case of discrete skew-products

$$x_{k+1} = A(\theta)x_k, \qquad \theta_{k+1} = \theta_k + 2\pi\omega \tag{II.37}$$

arising from quasi-periodic cocycles (A, ω) on $\mathbb{R}^n \times \mathbb{T}^d$ one can adapt the proof of the previous theorem to deal with cocycles close to the identity.

Theorem II.30. *Let* (A, ω) *be a quasi-periodic cocycle on* $\mathbb{R}^n \times \mathbb{T}^d$ *with nonresonant frequency* ω*. Assume that one has*

$$A(\theta) = \exp\left(\tilde{A}(\theta)\right)$$

where $\tilde{A} : \mathbb{T}^d \to gl(n, \mathbb{R})$ *is continuous. If there is a constant* $\delta > 0$ *such that the matrix* $\tilde{A}(\theta) = (\tilde{a}_{ij}(\theta))$ *satisfies*

$$||\tilde{a}_{ii}||_{C^0} \geq \delta + \sum_{j=1, j\neq i}^{n} ||\tilde{a}_{ij}||_{C^0}\,, \quad i = 1, \ldots, n.$$

Then the cocycle (A, ω) *(and therefore also the skew-product (II.37)) has an exponential dichotomy. Moreover, if* $\tilde{a}_{ii}$ *is positive for* k *indices (and negative for* $n-k$*) then the dimension of the stable subbundle is* k *(and the dimension of the unstable* $n-k$*).*

Chapter III

Quasi-periodic Schrödinger operators and cocycles

In the previous chapter we gave some basic definitions of quasi-periodic cocycles and linear differential equations on $\mathbb{R}^n \times \mathbb{T}^d$. In this chapter, and in most of the thesis also, we will focus on the important family of quasi-periodic Schrödinger skew-products on $\mathbb{R}^2 \times \mathbb{T}^d$.

Our starting point is *Hill's equation with quasi-periodic forcing*, which is the following second order differential equation

$$x'' + (a - q(t))\, x = 0, \tag{III.1}$$

where a is a real parameter, q is a quasi-periodic function and $x \in \mathbb{R}$. Since q is a quasi-periodic function, we can write

$$q(t) = Q(\omega t), \qquad t \in \mathbb{R}$$

for some suitable $Q : \mathbb{T}^d \to \mathbb{R}$ and $\omega \in \mathbb{R}^d$ rationally independent. In most of the thesis, the lift Q will be assumed real analytic, although many of the results in this chapter hold for a merely continuous Q. Using the lift Q, the hull of equation (III.1) is

$$x'' - Q(\theta)x = ax, \qquad \theta' = \omega \tag{III.2}$$

which gives rise to the following skew-product flow on $\mathbb{R}^2 \times \mathbb{T}^d$,

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ Q(\theta) - a & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \qquad \theta' = \omega. \tag{III.3}$$

Since the matrix of this system has trace zero, it belongs to $sl(2, \mathbb{R})$ and the associated matrix skew-product flow can be considered in $SL(2, \mathbb{R}) \times \mathbb{T}^d$.

Hill's equation with quasi-periodic forcing is a generalization of the classical periodic Hill's equation, where q is a periodic function, deviced by George Hill in the 19th century to study the motion of the Moon, see Barrow-Green [BG97] and references therein. Both the periodic and the quasi-periodic case occur as a first variation equation in the stability analysis of periodic solutions and lower dimensional tori in Hamiltonian with few degrees of freedom (see Eliasson [Eli88], Jorba & Villanueva [JV97] and Bourgain [Bou97]).

Remark III.1. *Sometimes it is useful to put a parameter b in front of the forcing q in Hill's equation (III.1) so that $b = 0$ corresponds to constant coefficients and can be directly integrated. When $|b|$ is small one can use perturbative methods to study Hill's equation with forcing bq and this is why b is called a* perturbation parameter. *For periodic Hill's equation the presence of such a parameter is customary in classical works on Hill's equation, e.g. Whittaker & Watson* [WW62], *Ince* [Inc44], *Magnus & Winkler* [MW79]. *In Chapter IV we will extend many of these perturbative techniques to the quasi-periodic case. In the rest of this chapter, however, we consider Hill's equation without this perturbing parameter.*

In the classical literature of Hill's equation, the formulation of (III.1) is somehow different because the forcing q is considered with a negative sign in front of it. The reason for the change of sign in this chapter is that equation (III.1) can be seen as the *eigenvalue equation* of the Schrödinger operator with potential q,

$$\left(H_q^c x\right)(t) = -x''(t) + q(t)x(t) \tag{III.4}$$

and a is a *spectral parameter*. Here the superscript c stands for continuous. An operator of this kind is called a *Schrödinger operator with quasi-periodic potential* and q is called the *potential*. Each of the equations in the hull (III.2) defines a Schrödinger operator with quasi-periodic potential,

$$\left(H_{Q,\omega,\phi}^c x\right)(t) = -x''(t) + Q(\omega t + \phi)x(t). \tag{III.5}$$

In Section III.1.2 it will be seen how this operator, which is in principle only defined for smooth enough functions, can be extended to a self-adjoint operator on $L^2(\mathbb{R})$, the space of square integrable functions on $\mathbb{R}$.

A main topic of interest in this thesis is to understand the spectrum of such operators when the potential is a real analytic and quasi-periodic with strongly irrational frequencies. We will be particularly interested in the dependence of this spectrum in terms of b when the potential is of the form bq, being b a perturbative parameter and q a quasi-periodic potential.

These spectral considerations are simpler in *discrete quasi-periodic Schrödinger operators*, which we now describe. A discretization of time in Hill's equation with quasi-periodic forcing leads to an equation of the form

$$x_{n+1} - 2x_n + x_{n-1} + v(n)x_n = ax_n,$$

where $(x_n)_{n\in\mathbb{Z}}$ is a sequence in $\mathbb{R}$, a is a parameter and $(v(n))_{n\in\mathbb{Z}}$ is a *quasi-periodic sequence*, which means that there is a continuous $V : \mathbb{T}^d \to \mathbb{R}$ and a nonresonant $\omega \in \mathbb{R}^d$ such that

$$v(n) = V\left(2\pi\omega n\right), \qquad n \in \mathbb{Z}.$$

As it is customary in this context, we will suppress the term $-2x_n$, which can be clearly included in v or a. An equation like

$$x_{n+1} + x_{n-1} + v(n)x_n = ax_n, \tag{III.6}$$

will sometimes be called of *Harper type* or *Harper-like*. It is the eigenvalue equation of the operator

$$(H_v^d x)_n = x_{n+1} + x_{n-1} + v(n)x_n \tag{III.7}$$

which is bounded and self-adjoint as a map from $l^2(\mathbb{Z})$ to itself. Here the superscript d stands for discrete. As in the continuous case, the hull of a Harper-like equation

$$x_{n+1} + x_{n-1} + V(\theta_n)x_n = ax_n, \qquad \theta_{n+1} = \theta_n + 2\pi\omega \tag{III.8}$$

generates a quasi-periodic skew-product flow on $\mathbb{R}^2 \times \mathbb{T}^d$,

$$\begin{pmatrix} x_{n+1} \\ x_n \end{pmatrix} = \begin{pmatrix} a - V(\theta_n) & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_n \\ x_{n-1} \end{pmatrix}, \qquad \theta_{n+1} = \theta_n + 2\pi\omega \tag{III.9}$$

which is the iteration of the quasi-periodic cocycle (A^d_{a-V}, ω) on $SL(2,\mathbb{R}) \times \mathbb{T}^d$, where

$$A^d_{a-V}(\theta) = \begin{pmatrix} a - V(\theta) & -1 \\ 1 & 0 \end{pmatrix}. \tag{III.10}$$

A cocycle of this form, will be called a *Schrödinger cocycle*. In the same way the skew-products (III.9) and (III.3) will be called *discrete* and *continuous Schrödinger skew-products* respectively.

In this Chapter we will see that the dynamics of Schrödinger skew-products and cocycles and the spectral properties of the corresponding Schrödinger operators are intimately related. In next chapters, this relation will be exploited to derive both dynamical and spectral consequences.

Remark III.2.

(i) Quasi-periodic Schrödinger operators have been studied intensively in the last twenty years. For reviews and references see Johnson [Joh83, JM03] *and Eliasson* [Eli98a, Eli99, Eli98b, Eli01, Eli02b]*, Chulaevsky* [Chu89]*, Chulaevsky & Sinai* [CS90, CS91] *and Dinaburg* [Din97] *from a dynamical point of view. From a spectral point of view see Simon* [Sim82, Sim00a]*, Jitomirskaya* [Jit02]*, Bourgain* [Bou04a, Bou04b] *and the books by Cycon, Froese, Kirsch & Simon* [CFKS87]*, Pastur & Figotin* [PF92] *and Carmona & Lacroix* [CL90]*. For physical applications see Sokoloff* [Sok85].

(ii) The study of quasi-periodic Schrödinger operators is relevant for the comprehension of the so-called Quantum Hall effect*, see Klitzing et al.* [vKDP80]*, Fröhlich* [Frö94] *and Osadchy & Avron* [OA01]*. In a two-dimensional metal or semiconductor at low temperatures a series of steps appear in the Hall resistance as a function of the magnetic field instead of the monotonic increase (which would correspond to the classical Hall effect). What is more, these steps occur at very precise values of resistance which are the same no matter what sample is investigate. This can be explained by means of the Gap Labelling Theorem which we will present in Section III.2.2. Another physical application comes from the electronic properties of quasi-crystals* [Jan92]*, which are not exact crystals but very nearly so which display symmetries forbidden by actual crystals. These have been found to be common structures in alloys of aluminum with such metals as cobalt, iron, and nickel. Unlike their constituent elements, quasi-crystals are poor conductors of electricity. From a mathematical point of view, see Bellissard et al.* [BIST89, BIT91, BHZ00] *and references therein.*

(iii) *The Schrödinger operators that we have presented are one-dimensional. In the continuous case, the higher-dimensional generalization involves partial derivatives and, in the discrete case, partial differences. Many of the spectral properties that we will present in Section III.1 extend to higher dimensions, but the connection with the dynamics is less clear. For an introduction to the spectral theory of Schrödinger operators in higher dimensions, see Reed & Simon* [RS75, RS78].

III.1 Spectral theory

In this section we present the basic spectral objects needed in our study of Schrödinger operators with quasi-periodic potential. Second order differential operators on functions

$$\left(H_q^c x\right)(t) = -\frac{d^2 x}{dt^2}(t) + q(t)x(t)$$

and second order difference operators on sequences

$$\left(H_v^d x\right) = x_{n+1} + x_{n-1} + v(n)x_n$$

will be considered at the same time. Along this section no quasi-periodicity will be assumed on the potentials q and v but only that they are bounded as functions of $t \in \mathbb{R}$ and $n \in \mathbb{Z}$ respectively.

III.1.1 Discrete Schrödinger operators. Operators in Hilbert spaces

Let us consider first a discrete Schrödinger operator H_v^d, being $v = (v(n))_{n\in\mathbb{Z}}$ a bounded sequence of $\mathbb{R}$. This is a *linear operator* or simply *operator* on $l^2(\mathbb{Z})$, the *Hilbert space* of sequences $x = (x_n)_{n\in\mathbb{Z}} \subset \mathbb{R}$ such that

$$\|x\|_{l^2} = \sum_{n\in\mathbb{Z}} |x_n|^2 < \infty.$$

The scalar product of this Hilbert space is

$$\langle x, y\rangle = \sum_{n\in\mathbb{Z}} x_n y_n$$

for any $x, y \in l^2(\mathbb{Z})$. Such discrete Schrödinger operators are *self-adjoint*, because $H_v^d = (H_v^d)^*$, and *bounded*:

$$\left\|H_v^d x\right\|_{l^2} \leq \left(2 + \|v\|_{l^\infty}\right)\|x\|_{l^2}.$$

The *spectrum* of a linear operator H (for example H_v^d, although H needs not to be bounded) on a Hilbert space $\mathcal{H}$ (e.g. $l^2(\mathbb{Z})$) is the set of those $\lambda \in \mathbb{C}$ for which the operator $H - \lambda I$, being I the identity on $\mathcal{H}$, does not have a bounded inverse. This set is denoted by $\sigma(H)$. If the operator is self-adjoint, then this is a subset of the real line which is bounded if the operator is bounded. The complementary on $\mathbb{R}$ of the spectrum

$$\rho(H) = \mathbb{R} - \sigma(H)$$

is an open set which is called the *resolvent set* of the operator. Therefore, discrete Schrödinger operators have a spectrum which is a compact subset of the real line:

$$\sigma\left(H_v^c\right) \subset \left[-\|v\|_{l^\infty}, \|v\|_{l^\infty}\right].$$

These discrete Schrödinger operators on $l^2(\mathbb{Z})$ can be seen as infinite matrices acting on the space of sequences,

$$H_v^d = \begin{pmatrix} \ddots & & & & 0 \\ & v(1) & 1 & & \\ & 1 & v(0) & 1 & \\ & & 1 & v(-1) & \\ 0 & & & & \ddots \end{pmatrix},$$

and finite dimensional real symmetric matrices can be regarded as self-adjoint operators on a finite set of integers. A difference between Hilbert spaces with infinitely many dimensions and finite-dimensional ones is that not all elements of the spectrum of an operator are eigenvalues. An *eigenvalue* of an operator H in a space $\mathcal{H}$ is a value $a \in \mathbb{C}$ such that there exists a $\psi \in \mathcal{H}$, called *eigenvector*, with

$$H\psi = a\psi.$$

Clearly, the eigenvalues of an operator belong to its spectrum (and therefore they are real for self-adjoint operators), but not all elements of the spectrum are eigenvalues. The set of eigenvalues of an operator (sometimes called the *point eigenvalues*) is called its *point spectrum*. The *multiplicity* of an eigenvalue is the dimension of the space of eigenvectors of this eigenvalue.

Adapting this to our case of interest, discrete Schrödinger operators H_v^d, $a \in \mathbb{R}$ is an eigenvalue if there exists a $\psi = \left(\psi_n\right)_{n\in\mathbb{Z}}$ such that

$$\psi_{n+1} + \psi_{n-1} + v(n)\psi_n = a\psi_n, n \in \mathbb{Z}.$$

We are going to show that the multiplicity of eigenvalues of discrete Schrödinger operators is always one: square integrable solutions of the eigenvalue equation cannot *coexist*. The tools will be completely dynamical:

Lemma III.3. *Let $v = (v_n)_{n\in\mathbb{Z}}$ a bounded sequence of $\mathbb{R}$ and $a \in \mathbb{R}$. Then the Harper-like equation*

$$x_{n+1} + x_{n-1} + v(n)x_n = ax_n, n \in \mathbb{Z} \tag{III.11}$$

cannot have more than one linearly independent solutions tending to 0 as $n \to +\infty$. In particular, the multiplicity of a as an eigenvalue of the corresponding discrete Schrödinger operator H_v^d is at most one. A similar result holds when $n \to \infty$.

Proof: The proof follows from a consideration of the iterations of the corresponding Schrödinger cocycle. Indeed, assume that $x = (x_n)_{n\in\mathbb{Z}}$ and $y = (y_n)_{n\in\mathbb{Z}}$ are two solutions of Equation (III.11). Let

$$X_{n+1} = \begin{pmatrix} x_{n+1} & y_{n+1} \\ x_n & y_n \end{pmatrix}, \quad \theta_{n+1} = 2\pi\omega(n+1).$$

Then the sequence $(X_n, \theta_n)_{n\in\mathbb{Z}}$ satisfies

$$\left(X_{n+1}, \theta_{n+1}\right) = (A, \omega)\left(X_n, \theta_n\right) = (A, \omega)^{n+1}\left(X_0, 0\right)$$

for every $n \in \mathbb{Z}$. Since (A, ω) is a cocycle on $SL(2,\mathbb{R}) \times \mathbb{T}^d$, the *Wronskian* of x and y

$$W_n(x,y) = x_n y_{n-1} - y_n x_{n-1} = \det X_n$$

does not depend on n and it is different from zero if x and y are linearly independent. In particular both solutions cannot tend to zero when n goes to $+\infty$ (or $-\infty$), hence the contradiction with both x and y linearly independent and in $l^2(\mathbb{Z})$. □

This property about the multiplicity of eigenvalues is called the *limit-point character* at $\pm\infty$ of the potential v.

Remark III.4. *The spectrum of the discrete Schrödinger operator on $l^2(\mathbb{Z})$ with potential identically zero, H_0^d, has no eigenvalues and*

$$\sigma(H_0^d) = [-2, 2],$$

see Isaacson & Keller [IK94] *for instance. In terms of the dynamical characterization which will be given in Section III.2.1 this will be clear.*

III.1.2 Continuous Schrödinger operators. Essential self-adjointness

For continuous operators H_q^d the situation is a bit more involved. The Hilbert space to be considered is $L^2(\mathbb{R})$, the space of functions $f : \mathbb{R} \to \mathbb{R}$ with

$$\|f\|_{L^2} = \left(\int_{\mathbb{R}} |f(t)|^2 \, dt \right)^{\frac{1}{2}} < \infty,$$

which comes from the scalar product on complex-valued functions

$$\langle f, g \rangle = \int_{\mathbb{R}} f(t)\overline{g(t)} dt.$$

Contrary to the case of discrete Schrödinger operators, these continuous operators are not bounded and the definition of the operators,

$$\left(H_q^c x\right)(t) = -x''(t) + q(t)x(t),$$

in terms of the derivatives does not make sense for all functions in the Hilbert space $L^2(\mathbb{R})$. Nevertheless, it can be defined on a dense subset of $L^2(\mathbb{R})$, namely the set of infinitely differentiable functions of compact support, which we denote by $C_0^\infty(\mathbb{R})$. We say that H_q^c is *essentially self-adjoint* in the *domain* $C_0^\infty(\mathbb{R})$ if there is a unique extension of H_q^d to the whole $L^2(\mathbb{R})$ which is a self-adjoint operator.

For a continuous Schrödinger operator, self-adjointness depends on its limit-point character at $\pm\infty$. More precisely one has the following.

Theorem III.5 (Weyl's limit-point criterion, [RS80]**).** *Let q be a continuous real valued function on $\mathbb{R}$. Then the corresponding Schrödinger operator*

$$\left(H_q^c x\right)(t) = -x''(t) + q(t)x(t),$$

is essentially self-adjoint on $C_0^\infty(\mathbb{R})$ if, and only if, q is in the limit-point case both at $+\infty$ and $-\infty$. That is, for any $a \in \mathbb{R}$, the Hill equation

$$-x''(t) + q(t)x(t) = ax(t) \tag{III.12}$$

has at most one linearly independent solution which is square integrable at $+\infty$. The same holds for $-\infty$.

The proof of the limit-point character is not as direct as in the discrete case, where it was a consequence of the preservation of the Wronskian and the decay at $\pm\infty$ of square-integrable sequences. In the continuous case, the preservation of the *Wronskian* of two solutions x, y of Hill's equation (III.12)

$$W(x,y)(t) = x'(t)y(t) - y'(t)x(t), \quad t \in \mathbb{R}$$

also holds, but this does not contradict with the fact that both x and y may be square integrable at $\pm\infty$. A convenient criterion for our purposes will be the following, which can be found in Coddington & Levinson [CL55].

Theorem III.6. *Let q be a continuous function and $C_1, C_2 \geq 0$ constants such that*

$$q(t) \geq -C_1(t^2+1), \qquad |t| \geq C_2.$$

Then q is in the limit-point case around $\pm\infty$ and the Schrödinger operator

$$\left(H_q^c x\right)(t) = -x''(t) + q(t)x(t)$$

is essentially self-adjoint on $C_0^\infty(\mathbb{R})$.

In particular, if q is bounded from below (which is true for any quasi-periodic function q), the corresponding Schrödinger operator H_q^c is essentially self-adjoint on $C_0^\infty(\mathbb{R})$ and we can consider its unique extension to $L^2(\mathbb{R})$. In what follows, H_q^d will denote also this extension to $L^2(\mathbb{R})$.

Given a potential q, the spectrum and the resolvent set of H_q^c can be defined as it was done in the previous section. This spectrum, $\sigma(H_q^c)$, is no longer a bounded set of $\mathbb{R}$, because the operator is unbounded. Nevertheless, if q is bounded then one has the inclusion

$$\sigma(H_q^c) \subset [\|q\|_\infty, +\infty),$$

which means that that the spectrum is bounded from below (see Section III.2.1).

Remark III.7. *The Schrödinger operator with potential identically zero, H_0^c, has no eigenvalues in $L^2(\mathbb{R})$ and*

$$\sigma(H_0^c) = [0, +\infty).$$

This will be proved in Section III.2.1.

III.1.3 The spectral theorem for bounded self-adjoint operators

In this section we give a formulation of the Spectral Theorem for bounded and self-adjoint operators. One can give a version for unbounded operators, for example continuous Schrödinger operators, but we prefer to give the version for bounded operators, because it is simpler and is the one that will be used in Chapter VII. The approach to the spectral theorem is taken from Reed & Simon [RS80] where the reader will find proofs and references to other formulations.

Let H be a bounded and self-adjoint operator in a Hilbert space $\mathcal{H}$. The spectral theorem tries to give a meaning to the expression $f(H)$ for any bounded Borel function defined on the spectrum of the operator. Let us first consider the finite-dimensional situation: $\mathcal{H}$ is a finite dimensional space (isomorphic to the Euclidean space $\mathbb{R}^n$ for a suitable dimension n) and H is a self-adjoint matrix in this space.

If $\sigma(H) = \{\lambda_1, \ldots, \lambda_n\}$ is the spectrum of H and P_k denotes the spectral projection onto the kth eigenspace, we may write

$$H = \sum_{k=1}^{n} \lambda_k P_k$$

and, given any complex function $f : \sigma(H) \to \mathbb{C}$, a convenient definition of $f(H)$ is

$$f(H) = \sum_{k=1}^{n} f(\lambda_k) P_k.$$

This construction satisfies the following properties

(i) $(\alpha f + \beta g)(H) = \alpha f(H) + \beta g(H)$.

(ii) $(fg)(H) = f(H)g(H)$.

(iii) $\overline{f}(H) = f(H)^*$.

(iv) If $f(x) = x$ then $f(H) = H$.

(v) If $H\psi = \lambda\psi$ then $f(H)\psi = f(\lambda)\psi$.

(vi) If $f \geq 0$ then $f(H) \geq 0$. This means that $\langle f(H)x, x\rangle \geq 0$ for all $x \in \mathcal{H}$.

(vii) $\|f(H)\| = \sup_{\lambda\in\sigma(H)} f(\lambda)$.

(viii) $\sigma(f(H)) = f(\sigma(H))$.

for $\alpha, \beta \in \mathbb{C}$ and complex-valued functions on $\sigma(H)$.

We now turn to the general situation where H is a bounded self-adjoint operator on a Hilbert space $\mathcal{H}$. We will construct $f(H)$ for any bounded Borel function on the spectrum of the operator H (this is a compact set) which will satisfy the properties (i-viii) above. The idea for this construction is to approximate it first by polynomials and then by continuous functions on the spectrum.

Given any complex-valued polynomial $p : \mathbb{C} \to \mathbb{C}$,

$$p(x) = a_0 + a_1 x + \ldots + a_n x^n,$$

the natural definition of $p(H)$,

$$p(H) = a_0 I + a_1 H + \ldots + a_n H^n$$

is a bounded operator on $\mathcal{H}$ (not necessarily self-adjoint) which satisfies the properties (i-viii) above. This is called the *polynomial functional calculus.*

The next step is to define $f(H)$ for any $f \in C(\sigma(H))$, the set of continuous functions on the spectrum of H, where we take the supremum norm

$$\|f\|_{C(\sigma(H))} = \sup_{\lambda \in \sigma(H)} |f(\lambda)|\,.$$

Using Weierstrass Approximation Theorem, polynomials are dense in $C(\sigma(H))$. Therefore, given any $f \in C(\sigma(H))$ there is a sequence of polynomials $(p_n)_{n\geq 0}$ converging to f on $C(\sigma(H))$. The sequence $\left(p_n(H)\right)_{n\geq 0}$ is a Cauchy sequence in the space of bounded linear operators on $\mathcal{H}$, $L(\mathcal{H})$. We call $f(H)$ the limit of this Cauchy sequence. One can show that this $f(H)$ does not depend on the sequence chosen and that the construction satisfies the properties of the functional calculus (i-viii). This is the *continuous functional calculus*.

The last step is to define $f(H)$ for any bounded Borel function on $C(\sigma(H))$. Given any $\psi \in \mathcal{H}$ the map

$$f \in C(\sigma(H)) \mapsto \langle \psi, f(H)\psi \rangle$$

defines a positive linear functional due to property (vi) of the continuous functional calculus and, by the Riesz Representation Theorem, there exists a unique positive Borel measure μ_ψ such that

$$\langle \psi, f(H)\psi \rangle = \int_{C(\sigma(H))} f(\lambda) d\mu_\psi(\lambda)$$

for every $f \in C(\sigma(H))$. This measure μ_ψ is called the *spectral measure associated to* ψ. These spectral measures are supported in the spectrum $\sigma(H)$ (this means that the measure of the resolvent set is zero). Using spectral measures one can construct $f(H)$ for Borel measurable functions: $f(H)$ is the only bounded operator on $\mathcal{H}$ such that

$$\langle \psi, f(H)\psi \rangle = \int_{\sigma(H)} f(\lambda) d\mu_\psi(\lambda)$$

for every $\psi \in \mathcal{H}$. This is the *spectral theorem* in the *functional calculus form*: for any bounded and self-adjoint operator H in a Hilbert space $\mathcal{H}$ there exists a unique map from the algebra of bounded Borel functions on $\mathbb{R}$ to the algebra of bounded linear operators from $\mathcal{H}$ to itself

$$f \in \mathcal{B}(\mathbb{R}) \mapsto \hat{\phi}(f) = f(H) \in \mathcal{L}(\mathcal{H})$$

such that

(a) $\hat{\phi}$ is an algebraic $*$-homomorphism.

(b) $\hat{\phi}$ is norm-continuous: $\|\hat{\phi}(f)\|_{\mathcal{L}(\mathcal{H})} \leq \|f\|_\infty$.

(c) If f is the identity, then $\hat{\phi}(f) = H$.

(d) If $f_n \to f$ pointwise and $\|f_n\|_\infty$ is bounded, then $\hat{\phi}(f_n) \to \hat{\phi}(f)$ in the strong topology.

(e) If $H\psi = \lambda\psi$, then $\hat{\phi}(f)\psi = f(\lambda)\psi$.

(f) If $f \geq 0$, then $\hat{\phi}(f) \geq 0$.

(g) If $BH = HB$ for some $B \in \mathcal{L}(\mathcal{H})$, then $\hat{\phi}(f)B = B\hat{\phi}(f)$.

Recall that any measure μ on $\mathbb{R}$ has a unique decomposition into

$$\mu = \mu_{pp} + \mu_{ac} + \mu_{sc}$$

where μ_{pp} is a pure point measure, μ_{ac} is absolutely continuous with respect to Lebesgue measure and μ_{sc} is singular continuous with respect to Lebesgue measure. It can be seen that for a bounded and self-adjoint operator on $\mathcal{H}$ the sets

$$\mathcal{H}_{pp} = \{\psi \in \mathcal{H};\ \mu_\psi \text{ is pure point}\},$$

$$\mathcal{H}_{ac} = \{\psi \in \mathcal{H};\ \mu_\psi \text{ is absolutely continuous}\}$$

and

$$\mathcal{H}_{sc} = \{\psi \in \mathcal{H};\ \mu_\psi \text{ is singular continuous}\}$$

are invariant under H and

$$\mathcal{H} = \mathcal{H}_{pp} \oplus \mathcal{H}_{ac} \oplus \mathcal{H}_{sc}.$$

Moreover $H_{|\mathcal{H}_{pp}}$ has a complete set of eigenvectors (a basis of eigenvectors), $H_{|\mathcal{H}_{ac}}$ has only absolutely continuous spectral measures and $H_{|\mathcal{H}_{sc}}$ has only singular continuous spectral measures.

According to this decomposition above we let $\sigma_{pp}(H)$ be the set of eigenvalues, which we call the *pure point spectrum*,

$$\sigma_{ac}(H) = \sigma\left(H_{|\mathcal{H}_{ac}}\right),$$

which is called the *absolutely continuous spectrum* and

$$\sigma_{sc}(H) = \sigma\left(H_{|\mathcal{H}_{sc}}\right)$$

the *singular continuous spectrum*. Even if the union of these spectra is not always $\sigma(H)$ the following holds:

$$\sigma(H) = \overline{\sigma_{pp}(H)} \cup \sigma_{ac}(H) \cup \sigma_{sc}(H).$$

III.2 Dynamical approaches to the spectral theory

In the previous section we considered some aspects of the spectral theory of Schrödinger operators, not necessarily quasi-periodic. In this section we will only consider the quasi-periodic case, both continuous and discrete, and our main interest will be to link the spectral properties of these operators to the dynamical properties of the skew-products that their eigenvalue equations define on $SL(2,\mathbb{R}) \times \mathbb{T}^d$.

Let us begin fixing the notation for the rest of the section. Given a quasi-periodic function q with continuous lift Q to $\mathbb{T}^d$ and irrational frequency vector ω, we define the family of operators $H^c_{Q,\omega,\phi}$, for $\phi \in \mathbb{T}^d$ as the self-adjoint extension to $L^2(\mathbb{R})$ of the following operator

$$\left(H^c_{Q,\omega,\phi}x\right)(t) = -x''(t) + Q(\omega t + \phi)x(t). \tag{III.13}$$

This is a quasi-periodic Schrödinger operator. Its eigenvalue equation is the following Hill's equation

$$x''\,(a - Q(\omega t + \phi))\,x = 0, \tag{III.14}$$

which defines the skew-product flow (III.3) on $\mathbb{R}^2 \times \mathbb{T}^d$.

For a quasi-periodic sequence $(v(n))_{n\in\mathbb{Z}}$ with continuous lift V to $\mathbb{T}^d$ and nonresonant frequency vector $\omega \in \mathbb{R}^d$, the discrete quasi-periodic Schrödinger operators $H^d_{V,\omega,\phi}$, with $\phi \in \mathbb{T}^d$, are defined as

$$(H^d_{V,\omega,\phi}x)_n = x_{n+1} + x_{n-1} + V\,(2\omega n + \phi)\,x_n \tag{III.15}$$

on $l^2(\mathbb{Z})$. The corresponding eigenvalue equations are

$$x_{n+1} + x_{n-1} + V\,(2\omega n + \phi)\,x_n = a x_n, \qquad n \in \mathbb{Z} \tag{III.16}$$

which gives rise to the quasi-periodic skew-product (III.9) on $\mathbb{R}^2 \times \mathbb{T}^d$. This is the iteration of the Schrödinger cocycle (A^d_{a-V}, ω) on $SL(2,\mathbb{R}) \times \mathbb{T}^d$ defined by (III.10).

III.2.1 Dynamical characterization of the resolvent set

Our first goal is to give a characterization of the spectrum of quasi-periodic Schrödinger operators in terms of the dynamics of the corresponding skew-products on $\mathbb{R}^2 \times \mathbb{T}^d$. Recall that the resolvent set of these operators (the complementary of the spectrum) is the set of those $a \in \mathbb{R}$ for which $H^c_{Q,\omega,\phi} - aI$ and $H^d_{V,\omega,\phi} - aI$ respectively have a bounded inverse on the corresponding Hilbert space. A dynamical characterization of these resolvent sets was given by Johnson [Joh82] using the notion of exponential dichotomy.

Theorem III.8. *Let $Q : \mathbb{T}^d \to \mathbb{R}$ (respectively V) be continuous and $\omega \in \mathbb{R}^d$ rationally independent (resp. nonresonant). Then the resolvent set of $H^c_{Q,\omega,\phi}$ (resp. $H^d_{V,\omega,\phi}$) is the set of $a \in \mathbb{R}$ for which the associated skew-product flow (III.3) (resp. (III.9)) has an exponential dichotomy. Equivalently $a \in \mathbb{R}$ belongs to $\sigma\left(H^c_{Q,\omega,\phi}\right)$ if, and only if, the Hill equation*

$$x'' - (a - Q(\omega t + \phi))\,x = 0$$

has a nontrivial bounded solution for some $\phi \in \mathbb{T}^d$. Also $a \in \sigma(H^d_{V,\omega,\phi})$ if, and only if, the Harper-like equation

$$x_{n+1} + x_{n-1} + V\,(2\omega n + \phi)\,x_n = a x_n, \qquad n \in \mathbb{Z}$$

has a nontrivial bounded solution for some $\phi \in \mathbb{T}^d$.

The proof in [Joh82] is for the continuous case and in a more general setting, but it extends to the difference case without problems (see Johnson [Joh83]). An immediate consequence is the independence of the spectra of $H^c_{Q,\omega,\phi}$ and $H^d_{V,\omega,\phi}$ on the phase $\phi \in \mathbb{T}^d$.

Corollary III.9. *Let $Q : \mathbb{T}^d \to \mathbb{R}$ (resp. V) be continuous and $\omega \in \mathbb{R}^d$ rationally independent (resp. nonresonant). Then the spectrum of $H^c_{Q,\omega,\phi}$ (resp. $H^d_{V,\omega,\phi}$) does not depend on ϕ and one can write*

$$\sigma^c(Q,\omega) = \sigma\left(H^c_{Q,\omega,\phi}\right)$$

and

$$\sigma^d(V,\omega) = \sigma\left(H^d_{V,\omega,\phi}\right).$$

for all $\phi \in \mathbb{T}$.

Even if the spectra of these quasi-periodic Schrödinger operators do not depend on ϕ, the spectral decomposition of these operators in terms of absolutely continuous, singular continuous and pure point spectra given in Section III.1.3 depend on the ϕ chosen. In fact, this decomposition may depend in a subtle way on the precise arithmetic conditions on ω and ϕ (see Chapters VI and VII).

At the end of the previous chapter we saw that a quasi-periodic skew-product flow, either continuous or discrete, with exponential dichotomy is conjugated to a skew-product with diagonal matrix by a transformation which is defined on $(\mathbb{R}/(4\pi\mathbb{Z}))^d$. In the continuous case such diagonal matrix is of the form

$$\begin{pmatrix} \lambda(\theta) & 0 \\ 0 & -\lambda(\theta) \end{pmatrix}$$

and in the discrete case can be written as

$$\begin{pmatrix} \kappa(\theta) & 0 \\ 0 & \kappa(\theta)^{-1} \end{pmatrix}.$$

Using the Floquet representation one has the following result on the type of solutions in the resolvent set of a Schrödinger operator

Corollary III.10. *Let $Q : \mathbb{T}^d \to \mathbb{R}$ (resp. V) be continuous and ω rationally independent (resp. nonresonant). Then $a \in \mathbb{R}$ is in the resolvent set of $H^c_{Q,\omega,\phi}$ (resp. $H^d_{V,\omega,\phi}$) if, and only if, the equation*

$$x'' + (a - Q(\omega t + \phi))\, x = 0$$

(resp.

$$x_{n+1} + x_{n-1} + V(2\pi\omega n + \phi)x_n = ax_n, \qquad n \in \mathbb{Z})$$

has two linearly independent solutions one of which is square integrable (resp. summable) at $+\infty$ and the other one at $-\infty$.

As a consequence of this result the identities

$$\sigma^c_{0,\omega} = [0, +\infty)$$

and

$$\sigma^d_{0,\omega} = [-2, 2]$$

follow from direct integration of the corresponding eigenvalue equations.

III.2.2 Rotation number and applications to the spectrum

The rotation number is a very convenient object for the description of the spectrum of quasi-periodic Schrödinger operators. Basically it is a continuous and increasing function on $\mathbb{R}$ whose support (the set of points of increase) is the spectrum of the Schrödinger operator. The rotation number can be obtained from several points of view which we now present. First of all we give the original formulation of Johnson & Moser [JM82] for the continuous case together with some equivalent definitions. The adaption to the discrete case is given in afterwards. Finally the so-called *Gap Labelling Theorem*, which describes the possible values of the rotation number in the complement of the spectrum, is stated.

The continuous case

Let us introduce the rotation number for quasi-periodic equations of Hill's type

$$x'' + (a - Q(\omega t + \phi))\, x = 0, \tag{III.17}$$

and its link with the spectrum of the operator $H^c_{Q,\omega}$ following the approach by Johnson & Moser [JM82]. Given any nontrivial solution of (III.17) the map

$$t \in \mathbb{R} \mapsto z(t) = x'(t) + ix(t) \in \mathbb{C} - \{0\}$$

is continuous and well-defined, so that one can take its argument

$$\varphi(t) = \arg z(t)$$

which is a continuous function $\varphi : \mathbb{R} \to \mathbb{R}$. Any two argument functions differ by a fixed integer. The differential equation satisfied by φ is

$$\varphi' = (a - Q(\omega t + \phi)) \cos^2 \varphi - \sin^2 \varphi.$$

The fact that this differential equation depends only on φ (and t) comes from the linear character of the skew-product flow (III.3). In fact, for any continuous skew-product flow, not necessarily coming from a differential equation of Hill's type, such an equation for the argument of z can be constructed.

The *rotation number* of Equation (III.17) is the following limit

$$\lim_{t \to +\infty} \frac{\varphi(t)}{t}$$

and therefore, measures the average winding of z around the origin in $\mathbb{C}$. The following result confirms the naturality of such object:

Theorem III.11 ([JM82]). *Let $Q : \mathbb{T}^d \to \mathbb{R}$ be a continuous function, $\omega \in \mathbb{R}^d$ a rationally independent frequency vector and $a \in \mathbb{R}$. If x is a nontrivial solution of*

$$x'' + (a - Q(\omega t + \phi))\, x = 0,$$

for some $\phi \in \mathbb{T}^d$, the limit

$$\mathrm{rot}^c(a - Q, \omega) = \lim_{t \to \infty} \frac{\arg (x'(t) + \mathrm{i}x(t))}{t}$$

exists in $\mathbb{R}$ and it does not depend on x nor on ϕ. Moreover the map

$$a \in \mathbb{R} \mapsto \operatorname{rot}^c(a - Q, \omega)$$

is increasing, continuous, equal to zero for $a \leq a^*$, for some $a^* \in \mathbb{R}$ and

$$\lim_{a \to \infty} \operatorname{rot}^c (a - Q) = +\infty.$$

Remark III.12. *If the average of Q is zero then $a^* \leq 0$.*

If ω has some rational dependencies, the rotation number also exists, but it is not independent on ϕ. We stress that the rotation number will be constant in the closure of the orbit $\omega t + \phi$ in $\mathbb{T}^d$. Apart from this, the rotation number has a good dependence on Q and ω:

Theorem III.13 ([JM82])**.** *Let $\operatorname{rot}^c(a - Q, \omega, \phi)$ denote the rotation number of Hill's equation (III.17). Then the map*

$$(Q, \omega) \in C(\mathbb{T}^d, \mathbb{R}) \times \mathbb{R}^d \mapsto \operatorname{rot}^c(a - Q, \omega, \phi)$$

is continuous at rationally independent frequency vectors ω and any $\phi \in \mathbb{T}^d$.

The spectrum of a quasi-periodic Schrödinger operator and the rotation number of the corresponding eigenvalue equations are linked in the following way

Theorem III.14 ([JM82])**.** *Let $\operatorname{rot}^c(a - Q, \omega)$ be the the rotation number of*

$$x'' + (a - Q(\omega t + \phi))\, x = 0$$

for a continuous Q and a rationally independent ω. Then the spectrum of the Schrödinger operator $H^c_{Q,\omega,\phi}$, is the set of points of increase of the map

$$a \in \mathbb{R} \mapsto \operatorname{rot}^c (a - Q, \omega)\,.$$

Figure III.1 displays a numerical computation of the rotation number for a Hill's equation. The intervals of constancy of the rotation number that there appear correspond to intervals in the resolvent set of the associated Schrödinger operator. The connected components of the resolvent set of a Schrödinger operator are called the *spectral gaps*. In next sections, we will see how the rotation number in these intervals can only take values in a countable set which depends on the frequencies of the potential.

Before ending this introduction let us give two other definitions of the rotation number of a Hill's equation. Take a nontrivial solution x and let $N([t_0, t_1], x)$ be the number of zeroes of x in the interval $[t_0, t_1]$. Then, the rotation number equals to

$$\lim_{|t_1 - t_0| \to \infty} \frac{\pi N([t_0, t_1], x)}{|t_1 - t_0|},$$

see [JM82]. Apart from this *Sturmian* definition, one can give a definition of the rotation number in terms of the following regular eigenvalue problem

$$x'' - Q(t)x = ax, \quad \text{with} \quad x(t_0) = x(t_1) = 0$$

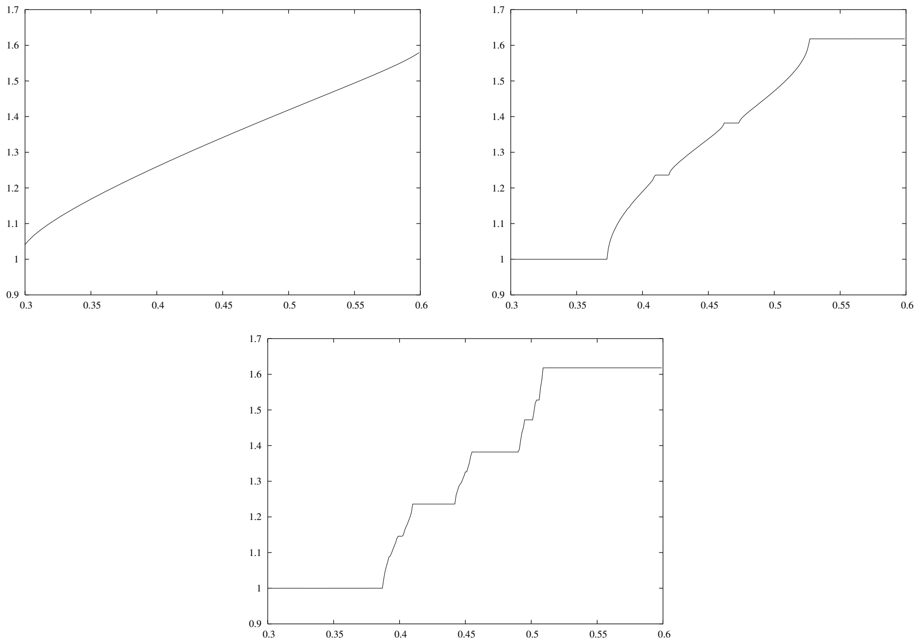

Figure III.1: Illustration of a numerical approximation of the rotation number of equation $x'' + (a + b(\cos t + \cos \gamma t))\, x = 0$ as a function of a for different values of b and $\gamma = (1+\sqrt{5})/2$. Top: at the left, $b = 0.1$, and at the right $b = 0.35$. Bottom: $b = 0.6$. For the methodology of the computation see Broer & Simó [BS98].

It follows from Sturm's comparison theorem (see, for instance, [CL55]) that the distribution function of the eigenvalues of the above problem $k(a, t_0, t_1)$ differs from $N([t_1, t_0], x)$ by ± 1, so that

$$\lim_{|t_1 - t_0| \to \infty} \frac{k(a, t_0, t_1)}{t_1 - t_0} = \frac{\mathrm{rot}^c(a - Q)}{\pi},$$

The left hand side of this expression is often called the *integrated density of states*. It is the distribution function of the density of states, a fundamental tool in quantum physics. It can be defined for more general operators, see Pastur & Figotin [PF92].

The discrete case. The integrated density of states

The definition by Johnson & Moser of the rotation number of a continuous quasi-periodic equation of Hill's type in terms of the average rate of growth of the argument of solutions is not the most direct in the discrete case. Nevertheless some of the previous equivalent definitions are easy to adapt to the discrete case. Let us start with the *Sturmian* approach to the rotation number. If $(x_n)_{n\in\mathbb{Z}}$ is a nontrivial solution of

$$x_{n+1} + x_{n-1} + V\left(2\pi\omega n + \phi\right) x_n = a x_n \tag{III.18}$$

for some $a \in \mathbb{R}$, $V : \mathbb{T}^d \to \mathbb{R}$ continuous, ω nonresonant and $\phi \in \mathbb{T}^d$, let $N(a, [n_1, n_2])$ be the number of changes of sign of such a solution for $n_1 \le n \le n_2$, adding one if $x(n_2) = 0$. Then the limit

$$\lim_{|n_2 - n_1| \to \infty} \frac{N(a, [n_1, n_2])}{2(n_2 - n_1) + 1}$$

exists, it does not depend on the chosen solution x, nor on ϕ and it is denoted by $\mathrm{rot}_s^d(a-V,\omega)$. Here the superscript d stands for discrete and the underscript s for Sturmian. This approach was followed by Delyon & Souillard [DS83] in a more general context.

The integrated density of states can also be introduced as in the continuous case. Indeed, let $k_{[n_1,n_2]}(a,V,\omega,\phi)$ be $(n_2-n_1)^{-1}$ times the number of eigenvalues less than or equal to a for the restriction of $H^d_{V,\omega,\phi}$ to the set

$$[n_1,n_2]=\{n_1,\dots,n_2\}$$

with zero boundary conditions at n_1-1 and n_2+1. Then, as $|n_2-n_1|\to\infty$, $k_{[n_1,n_2]}(a,V,\omega,\phi)$ converges to a continuous function $k(a,V,\omega)$, which is the called the *integrated density of states* of (III.18) or $H^d_{V,\omega,\phi}$, see Avron & Simon [AS83], and it is independent of ϕ. This integrated density of states and its relation with the spectral decomposition of the operators $H^d_{V,\omega,\phi}$ will be described in more detail in Chapter VII. The Sturmian rotation number and the integrated density of states of (III.18) are very close:

$$2\mathrm{rot}_s^d(a-V,\omega)=k(a,V,\omega),\qquad a\in\mathbb{R}.$$

This connection between both objects allows to adapt some of the properties of the rotation number from the continuous to the discrete case.

Theorem III.15. *Let $V:\mathbb{T}^d\to\mathbb{R}$ be continuous and ω nonresonant. The Sturmian rotation number of (III.18), $rot_s^d(a-V,\omega)$ satisfies that the map*

$$(a,V)\in\mathbb{R}\times C^0(\mathbb{T}^d,\mathbb{R})\mapsto \mathrm{rot}_s^d(a-V,\omega)\in[0,1/2]$$

is continuous and increasing for fixed V. Moreover the points of increase are exactly $\sigma^d(V,\omega)$, the spectrum of $H^d_{V,\omega,\phi}$.

Figure III.2 displays the numerical computation of the rotation number for a discrete equation. As in the continuous case, the intervals of constancy correspond to intervals in the resolvent set of the corresponding operator which are called *spectral gaps*.

To extend the definition by Johnson & Moser to the discrete case it is convenient to switch to the more general context of quasi-periodic cocycles on $SL(2,\mathbb{R})\times\mathbb{T}^d$ (with a suitable geometrical hypothesis to be described later) which will allow the definition for continuous time skew-product flows on $SL(2,\mathbb{R})\times\mathbb{T}^d$ using Poincaré cocycles. The definition of such rotation number was given by Herman [Her83]. Let us follow the presentation by Krikorian [Kri].

Let (A,ω) be a quasi-periodic cocycle on $SL(2,\mathbb{R})\times\mathbb{T}^d$, for example a Schrödinger cocycle where

$$A^d_{a-V}(\theta)=\begin{pmatrix} a-V(\theta) & -1\\ 1 & 0\end{pmatrix},$$

for some continuous $V:\mathbb{T}^d\to\mathbb{R}$ and ω nonresonant. We say that it is *homotopic to the identity* if $A:\mathbb{T}^d\to SL(2,\mathbb{R})$ is homotopic to the identity. Recall $SL(2,\mathbb{R})$ is not simply connected and its first homotopy group is isomorphic to $\mathbb{Z}$, with generator the rotation $R_1:\mathbb{T}\to SL(2,\mathbb{R})$ given by

$$R_1(\theta)=\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$$

for all $\theta\in\mathbb{T}$.

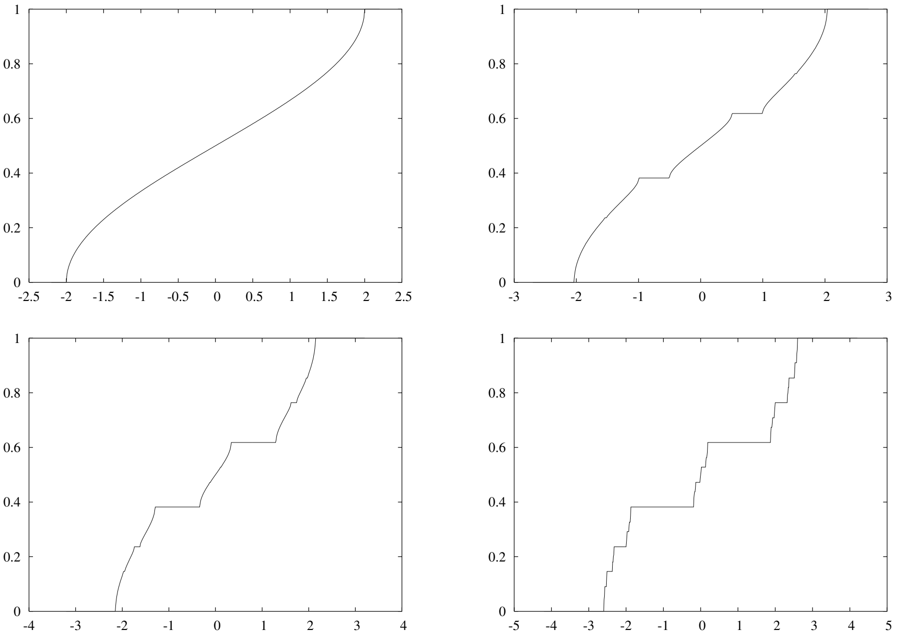

Figure III.2: Numerical approximation of the rotation number of the so-called Harper equation $x_{n+1} + x_{n-1} + b\cos(2\pi\omega n)x_n = ax_n$ as a function of a for different values of b and $\gamma = (\sqrt{5}-1)/2$. From top to bottom and from left to right $b = 0, 0.5, 1, 2$.

Remark III.16. *Any Schrödinger cocycle is homotopic to the identity. Also any Poincaré cocycle arising from a linear quasi-periodic differential equation on $SL(2,\mathbb{R}) \times \mathbb{T}^d$ is homotopic to the identity. In both cases it is enough to place a parameter in front of the potential, ranging from zero to one.*

Let $\mathbb{S}^1$ be the set of unit vectors of $\mathbb{R}^2$ and let us denote by $p : \mathbb{R} \to \mathbb{S}^1$ the projection given by the exponential $p(t) = e^{\mathrm{i}t}$, identifying $\mathbb{R}^2$ with $\mathbb{C}$. Because of the linear character of the cocycle and the fact that it is homotopic to the identity, the continuous map

$$\begin{array}{rccc} F: & \mathbb{S}^1 \times \mathbb{T} & \longrightarrow & \mathbb{S}^1 \times \mathbb{T} \\ & (v,\theta) & \mapsto & \left(\frac{A(\theta)v}{\|A(\theta)v\|}, \theta + 2\pi\omega\right) \end{array} \tag{III.19}$$

is also homotopic to the identity. Therefore, it admits a continuous lift $\tilde{F} : \mathbb{R} \times \mathbb{T}^d \to \mathbb{R} \times \mathbb{T}^d$ of the form:

$$\tilde{F}(t,\theta) = (t + f(\theta,t), \theta + 2\pi\omega)$$

such that

$$f(t + 2\pi, \theta + 2\pi\omega) = f(t,\theta) \text{ and } p\left(t + f(t,\theta)\right) = \frac{A(\theta)p(t)}{\|A(\theta)p(t)\|}$$

for all $t \in \mathbb{R}$ and $\theta \in \mathbb{T}$. The map f is independent of the choice of $\tilde{F}$ up to the addition of a constant $2\pi k$, with $k \in \mathbb{Z}$. Since the map $\theta \mapsto \theta + 2\pi\omega$ is uniquely ergodic on $\mathbb{T}$ for all $(t,\theta) \in \mathbb{R} \times \mathbb{T}^d$, one has that the limit

$$\lim_{N\to\infty} \frac{1}{2\pi N} \sum_{n=0}^{N-1} f\left(\tilde{F}^n(t,\theta)\right)$$

exists, it is independent of (t,θ) and the convergence is uniform in (t,θ), see Herman [Her83] and Johnson & Moser [JM82]. This object is called the *fibered rotation number* of (A,ω), which will be denoted by $\mathrm{rot}_f^d(A,\omega)$. By construction it is defined modulus $\mathbb{Z}$. The fibered rotation number of an equation of Harper type is defined as the fibered rotation number of the associated Schrödinger cocycle on $SL(2,\mathbb{R})\times\mathbb{T}^d$ and it is denoted as $\mathrm{rot}_f^d(a-V,\omega)$.

Example III.17. *If $A_0\in SL(2,\mathbb{R})$ is a constant matrix and $\omega\in\mathbb{R}^d$ is nonresonant, then the fibered rotation number of the cocycle (A_0,ω), is the absolute value of the argument of the eigenvalues divided by 2π.*

Since we have two definitions of rotation number for Harper equations it is sensible to try to link them. Using a suspension argument (see Johnson [Joh83]) it can be seen that, if $H^d_{V,\omega,\phi}$ is a quasi-periodic Schrödinger operator, $k(a-V,\omega)$ its integrated density of states and (A^d_{a-V},ω) the corresponding Schrödinger cocycle then

$$\mathrm{rot}_f^d(a-V,\omega)=\frac{1}{2}k(a,V,\omega)\qquad(\text{mod. }\mathbb{Z})$$

Therefore, one has the following relation between the Sturmian and the fibered rotation numbers of a Harper equation:

$$\mathrm{rot}_s^d(a-V,\omega)=\mathrm{rot}_f^d(a-V,\omega)\qquad(\text{mod. }\mathbb{Z}).$$

for all $a\in\mathbb{R}$.

Reducibility and rotation number

In this section we want to see to what extend the fibered rotation number is invariant through conjugation of cocycles. Recall that two quasi-periodic cocycles (A_1,ω) and (A_2,ω) on the space $SL(2,\mathbb{R})\times\mathbb{T}^d$ are conjugated if there is a continuous map $Z:\mathbb{T}^d\to SL(2,\mathbb{R})$ such that

$$(A_1,\omega)\circ(Z,0)=(Z,0)\circ(A_2,\omega)\tag{III.20}$$

which is equivalent to

$$A_1(\theta)Z(\theta)=Z(\theta+2\pi\omega)A_2(\theta)$$

for all $\theta\in\mathbb{T}^d$. A cocycle is reducible to constant coefficients if it is conjugated to constant coefficients. Let us first consider the case when the conjugation Z is defined on $\mathbb{T}^d$ (i.e. there is no frequency halving).

Recall that the fundamental group of $SL(2,\mathbb{R})$ is isomorphic to $\mathbb{S}^1$. The degree of a map $Z:\mathbb{T}^d\to SL(2,\mathbb{R})$ will be $\mathbf{k}=(k_1,\dots,k_d)\in\mathbb{Z}^d$ if, for each $1\le j\le d$, the degree of the map

$$\theta_j\in\mathbb{T}\mapsto Z(0,\dots,0,\theta_j,0,\dots,0)\in SL(2,\mathbb{R})$$

is k_j.

Proposition III.18 ([Kri]). *Let ω be nonresonant and (A_1,ω) and (A_2,ω) be two quasi-periodic cocycles on $SL(2,\mathbb{R})\times\mathbb{T}^d$ homotopic to the identity. If they are conjugated, see Equation III.20, for some continuous $Z:\mathbb{T}^d\to SL(2,\mathbb{R})$, then*

$$\mathrm{rot}_f^d(A_1,\omega)=\mathrm{rot}_f^d(A_2,\omega)+\langle\mathbf{k},\omega\rangle\ \textit{modulus}\ \mathbb{Z},$$

where $\mathbf{k}\in\mathbb{Z}^d$ is the degree of the map $Z:\mathbb{T}^d\to SL(2,\mathbb{R})$.

Remark III.19. *If the conjugation Z is not defined on $\mathbb{T}^d$ but on $(\mathbb{R}/(4\pi\mathbb{Z}))^d$ and it has degree $\mathbf{k} \in \mathbb{T}^d$, then*

$$\mathrm{rot}_f^d(A_1, \omega) = \mathrm{rot}_f^d(A_2, \omega) + \frac{1}{2}\langle \mathbf{k}, \omega \rangle.$$

This is so because

$$\mathrm{rot}_f^d(A, \omega) = \mathrm{rot}_f^d(A(2\cdot), \omega/2)$$

for any quasi-periodic cocycle on $SL(2, \mathbb{R}) \times \mathbb{T}^d$ homotopic to the identity.

We see, therefore, that the fibered rotation number is not invariant under conjugation of cocycles, although it can only change by half integer multiples of the fundamental frequencies. We will come to this again in the next section. Now let us discuss the rotation number for continuous quasi-periodic skew-products.

In the continuous case, one can define the rotation number of a continuous skew-product flow without making use of the Poincaré cocycle. Indeed, let

$$x' = A(\theta)x, \qquad \theta' = \omega, \tag{III.21}$$

be a quasi-periodic skew-product flow on $\mathbb{R}^2 \times \mathbb{T}^d$ with $A : \mathbb{T}^d \to sl(2, \mathbb{R})$ continuous and ω rationally independent. Then the rotation number of the skew-product flow (III.21) is the limit

$$\lim_{t\to\infty} \frac{-\arg x(t)}{t}$$

where (x, θ) is any nontrivial solution of (III.21). There is also a differential equation for this argument, $\varphi(t)$, which is given as a quadratic function in terms of $\cos\varphi(t)$ and $\sin\varphi(t)$. This will be used in the next chapters. The definition above does not depend on x, nor on the initial condition for θ so that one can write $\mathrm{rot}^c(A, \omega)$, see Eliasson [Eli92]. This definition is consistent with the definition of rotation number when the continuous skew-product comes from a Hill's equation. This rotation number is not invariant by conjugation, but if $Z : \mathbb{T}^d \to SL(2, \mathbb{R})$ and $B : \mathbb{T}^d \to sl(2, \mathbb{R})$ satisfy the conjugation

$$\partial_\omega Z(\theta) = A(\theta)Z(\theta) - Z(\theta)B(\theta), \qquad \theta \in \mathbb{T}^d,$$

then

$$\mathrm{rot}^c(A, \omega) = \mathrm{rot}^c(B, \omega) + \langle \mathbf{k}, \omega \rangle$$

where $\mathbf{k}$ is the degree of Z. If Z is only defined on $(\mathbb{R}/(4\pi\mathbb{Z}))^d$ and $\mathbf{k}$ is the degree of $Z(2\cdot)$ then

$$\mathrm{rot}^c(A, \omega) = \mathrm{rot}^c(B, \omega) + \frac{1}{2}\langle \mathbf{k}, \omega \rangle.$$

Similar to what was done with the Sacker-Sell spectrum in Section II.3.2 it is possible to link the above rotation number of a linear quasi-periodic differential equation on $SL(2, \mathbb{R}) \times \mathbb{T}^d$ to the fibered rotation number of any Poincaré cocycle.

Gap Labelling theorem and structure of the spectrum

The results in the previous section on the change of rotation number of quasi-periodic cocycles and linear quasi-periodic differential equations under their conjugation lead naturally to the definition of classes of rotation numbers which are invariant through conjugation. These classes depend only on the frequency vector $\omega \in \mathbb{R}^d$. If $\alpha \in \mathbb{R}$ the class of continuous rotation numbers preserved by conjugation is

$$\mathcal{M}_\alpha^c(\omega) = \left\{ \alpha + \frac{1}{2}\langle \mathbf{k}, \omega \rangle, \mathbf{k} \in \mathbb{Z}^d \right\}$$

and the class of discrete fibered rotation numbers preserved by conjugation is

$$\mathcal{M}_\alpha^d(\omega) = \left\{ \alpha + \frac{1}{2}\langle \mathbf{k}, \omega \rangle, \mathbf{k} \in \mathbb{Z}^d, (\mathrm{mod}\ \frac{1}{2}\mathbb{Z}) \right\}.$$

The following classes of conjugation are important. A continuous rotation number is said to be *rational with respect to* ω if it belongs to the class of conjugation of zero. The set of these numbers will be denoted as

$$\mathcal{M}_{rat}^c(\omega) = \mathcal{M}_0^c(\omega) = \left\{ \frac{1}{2}\langle \mathbf{k}, \omega \rangle, \mathbf{k} \in \mathbb{Z}^d \right\}.$$

In the discrete case the corresponding class of conjugation will include 0, $1/2$ and all fibered numbers which are conjugated to these two. The fibered rotation numbers of the set

$$\mathcal{M}_{ress}^d(\omega) = \left\{ \frac{1}{2}\langle \mathbf{k}, \omega \rangle, \mathbf{k} \in \mathbb{Z}^d, (\mathrm{mod}\ \frac{1}{2}\mathbb{Z}) \right\}.$$

are called *resonant with respect to* ω. Note that

$$\mathcal{M}_{ress}^d = \mathcal{M}_0^d \cup \mathcal{M}_{1/2}^d$$

and

$$\mathcal{M}_{ress}^d(\omega) = \mathcal{M}_{rat}^c(\omega) \quad (\mathrm{mod}\ \frac{1}{2}\mathbb{Z}).$$

We saw in Section III.2.1 that points in the resolvent set of quasi-periodic Schrödinger operators are characterized by the fact that the corresponding eigenvalue equations induce skew-products with exponential dichotomy to $SL(2,\mathbb{R}) \times \mathbb{T}^d$. The latter are conjugated to diagonal skew-product flows, maybe halving the frequency. The following result follows from direct integration.

Lemma III.20. *Let*

$$x' = \begin{pmatrix} \lambda(\theta) & 0 \\ 0 & -\lambda(\theta) \end{pmatrix} x, \qquad \theta' = \omega$$

(resp.

$$x_{n+1} = \begin{pmatrix} \kappa(\theta) & 0 \\ 0 & \kappa(\theta)^{-1} \end{pmatrix} x_n, \qquad \theta_{n+1} = \theta_n + 2\pi\omega)$$

be a continuous (resp. discrete) skew-product on $SL(2,\mathbb{R}) \times \mathbb{T}^d$. Then its rotation number is zero (resp. modulus $\frac{1}{2}\mathbb{Z}$).

Therefore one has the following description of the possible rotation numbers at spectral gaps (intervals in the resolvent set) of continuous quasi-periodic Schrödinger operators.

Theorem III.21 (Gap labelling, [JM82]**).** *Let $Q : \mathbb{T}^d \to \mathbb{R}$ be continuous and $\omega \in \mathbb{R}^d$ be a rationally independent frequency vector. If I is an open interval in the resolvent set of $\sigma^c(Q,\omega)$ then the continuous rotation number in this interval is rational with respect to ω:*

$$\mathrm{rot}^c(a - Q, \omega) = \frac{\langle \mathbf{k}, \omega \rangle}{2}, \quad a \in I$$

for a fixed $\mathbf{k} \in \mathbb{Z}^d$.

The result for discrete operators is as follows.

Theorem III.22. *Let $V : \mathbb{T}^d \to \mathbb{R}$ be continuous and $\omega \in \mathbb{R}^d$ be a nonresonant frequency vector. If I is an open interval in the resolvent set of $\sigma^d(V,\omega)$ then the discrete fibered rotation number is resonant with respect to ω:*

$$\mathrm{rot}_f^d(a - V, \omega) = \frac{\langle \mathbf{k}, \omega \rangle}{2} \quad \textit{(mod. } \frac{1}{2}\mathbb{Z})$$

in the fibered case and either

$$\mathrm{rot}_s^d(a - V, \omega) = \frac{1}{2}\left\{\langle \mathbf{k}, \omega \rangle\right\} \quad \textit{or} \quad \mathrm{rot}_s^d(a - V, \omega) = 1/2,$$

in the Sturmian case, for some fixed $\mathbf{k} \in \mathbb{Z}^d$. Here $\{\cdot\}$ denotes the fractional part of a real number.

Remark III.23. *Using the fact that the rotation number of a continuous quasi-periodic Schrödinger operator is always positive, the rotation number in the Gap Labelling Theorem must belong to the following* set of positive rationals with respect to ω

$$\mathcal{M}_+^c(\omega) = \left\{ \frac{1}{2}\langle \mathbf{k}, \omega \rangle \geq 0, \mathbf{k} \in \mathbb{Z}^d \right\}.$$

The Gap Labelling Theorem allows to assign a multi-integer (or equivalent a rotation number in $\mathcal{M}_+^c(\omega)$ or $\mathcal{M}_{ress}^d$) to each nonvoid interval in the resolvent set of a Schrödinger operator, see Figure III.3. If there is no spectral gap corresponding to a certain $\alpha \in \mathcal{M}_+^c(\omega)$ (or $\alpha \in \mathcal{M}_0^d(\omega)$ in the discrete fibered case) there is only one $a \in \mathbb{R}$ whose rotation number is precisely α. In this case the set $\{a\}$, which is a subset of the spectrum, is called a *collapsed spectral gap* of the operator. To distinguish between these collapsed gaps and the actual gaps of the spectrum of the operator (nonvoid open intervals in the resolvent set) the latter are called *noncollapsed spectral gaps*, even if this terminology is redundant. With this definition, there is a correspondence between the sets $\mathcal{M}_+^c(\omega)$ and $\mathcal{M}_{ress}^d(\omega)$ and spectral gaps, either collapsed and noncollapsed.

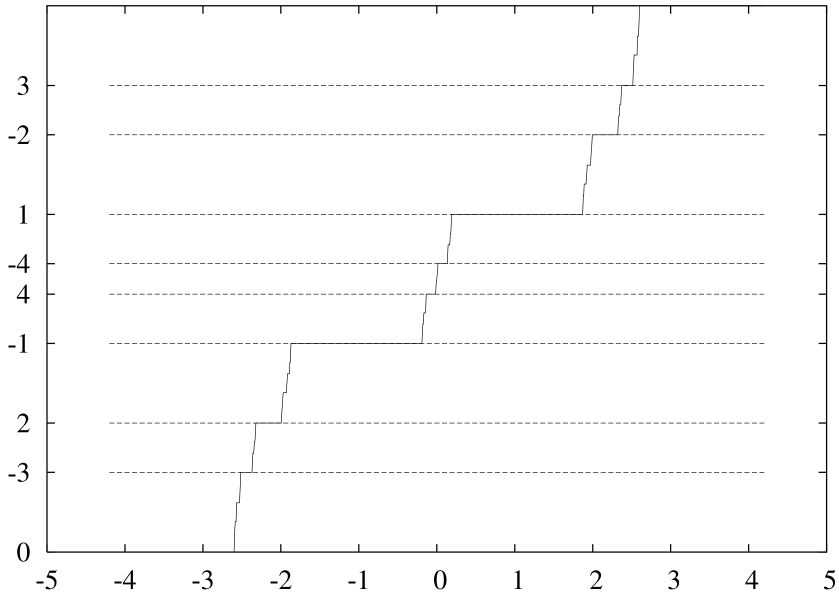

Figure III.3: Schematic view of the gap-labelling of a discrete quasi-periodic Schrödinger operator, the Almost Mathieu operator (see Chapter VI), $V(\theta) = 2\cos(\theta)$. The integer in the vertical direction corresponds to the integer label of the gap and the horizontal direction displays the spectral parameter.

Some results on Cantor spectrum

To end this section let us consider briefly the implications of this Gap Labelling for the structure of the spectrum of quasi-periodic Schrödinger operators. To fix ideas, let us consider the continuous case with $\omega \in \mathbb{R}^d$ rationally independent. The set

$$\mathcal{M}_+^c(\omega) = \left\{ \frac{1}{2}\langle \mathbf{k}, \omega \rangle \geq 0, \mathbf{k} \in \mathbb{Z}^d \right\}$$

is dense in $[0, \infty)$ if and only if $d > 1$, that is if the potential q is not periodic. In the periodic case all spectral gaps, either collapsed or not, are separated one from another by closed nonvoid intervals which are called the *spectral bands* or *energy bands*. This is because the rotation number is continuous, increasing and constant only at gaps (where it must take values in $\mathcal{M}_+^c$, which has no accumulation points).

In the quasi-periodic case, $d > 1$, this situation may change completely. Indeed, if $d > 1$ the set $\mathcal{M}_+^c(\omega)$ is dense in $[0, \infty)$ so, if all spectral gaps are open the spectrum must be a *Cantor set*, i.e. a nowhere dense set. This means that in every neighbourhood of a point in the spectrum there is a noncollapsed spectral gap. Of course, Cantor structure of the spectrum does not imply that all gaps are open, but only an infinite number of them, and that these are dense in $\mathbb{R}$. The Cantor structure of the spectrum is one of the main topics of interest in this thesis. Let us now finish this section with some results on this Cantor structure for continuous quasi-periodic Schrödinger operators. The results for the discrete case will be presented in Chapter VI.

Not all Schrödinger operators with quasi-periodic potential have Cantor spectrum. The easiest example is the identically zero potential but it is not the only one which does not have Cantor spectrum. Similarly to the periodic case, see McKean & van Moerbeke [MvM75] and references therein, De Concini & Johnson [DCJ87] showed that it is possible to produce examples of Schrödinger operators with algebraic-geometric quasi-periodic potentials having all spectral gaps, except a finite number, collapsed. In particular these operators do not display Cantor spectrum. Nevertheless, these examples are not very *generic* in a sense that will be explained in Chapter IV.

One way to prove the existence of Cantor spectrum is based on the approximation by periodic potentials where Floquet theory makes the discussion easier. Using this periodic case Moser [Mos81], constructed a limit-periodic potential with Cantor spectrum. Limit-periodic functions are uniform limit (in $\mathbb{R}$) of quasi-periodic functions. Therefore, these are not quasi-periodic but almost periodic functions.

Theorem III.24 ([Mos81])**.** *Given $\eta > 0$ and q_0 a continuous function of period π, there exists a limit-periodic analytic function q with basic frequencies 2^{-j} ($j = 0, 1, \dots$) with $\|q - q_0\|_\infty < \eta$ for which the Schrödinger equation with potential q has all spectral gaps open and, hence, the spectrum is a Cantor set.*

In Johnson [Joh91] (see also Fabbri, Johnson & Pavani [FJP02]), Moser techniques were heavily used to provide a more general statement using again the periodic approximation.

Theorem III.25 ([Joh91])**.** *There is a residual subset (i.e., countable intersection of open dense sets) $\mathcal{F} \subset \mathbb{R}^d$ such that, if $\omega \in \mathcal{F}$, then the following statement holds. There is a residual subset $\mathcal{V} = \mathcal{V}(\omega) \subset C^\delta(\mathbb{T}^d)$, with $0 \leq \delta < 1$, such that, if $Q \in \mathcal{V}$ and $\phi \in \mathbb{T}^d$, then the operator*

$$\left(H^c_{Q,\omega,\phi} x\right)(t) = -x''(t) + Q(\omega t + \phi)x(t) \tag{III.22}$$

has Cantor spectrum.

The core of the proof is the following theorem on the *genericity of exponential dichotomy*:

Theorem III.26 ([Joh91]**, cf.** [FJ00])**.** *Let $\mathcal{A} = \mathbb{R}^d \times C^\delta(\mathbb{T}^d)$, where $0 \leq \delta < 1$. There is an open dense subset $\mathcal{W} \subset \mathcal{A}$ such that, if $(\omega, Q) \in \mathcal{W}$, then the skew-product defined by*

$$-x''(t) + Q(\omega t + \phi)x(t) = 0$$

has an exponential dichotomy for all $\phi \in \mathbb{T}^d$.

Finally, Cantor spectrum can also be obtained in combination with reducibility results, as it is the case in the remarkable paper by Eliasson [Eli92] which will be discussed in the following section. This kind of results on Cantor spectrum based on reducibility at tongue boundaries will be given in Chapter V.

III.3 Eliasson's Theorem

In this section we present a result by Eliasson [Eli92] on the reducibility of Schrödinger skew-products, both continuous

$$x' = \begin{pmatrix} 0 & 1 \\ a - Q(\theta) & 0 \end{pmatrix} x \qquad \theta' = \omega, \tag{III.23}$$

where $(x,\theta) \in \mathbb{R}^2 \times \mathbb{T}^d$, and discrete

$$x_{n+1} = \begin{pmatrix} a - V(\theta) & -1 \\ 1 & 0 \end{pmatrix} x_n \qquad \theta_{n+1} = \theta_n + 2\pi\omega, \tag{III.24}$$

with $(x_n, \theta_n) \in \mathbb{R}^2 \times \mathbb{T}^d$.

In the previous sections we have shown reducibility of these Schrödinger skew-products when a lies in the resolvent set of the operator, the potential is real analytic and the frequencies satisfy a suitable Diophantine condition.

In the spectrum the situation is quite different and the reducibility can be obtained close to constant coefficients by means of KAM theory (from Kolmogorov-Arnol'd-Moser). A KAM technique will be used in Chapter V to prove some reducibility results.

Regarded as dynamical systems on $\mathbb{R}^2 \times \mathbb{T}^d$, the skew-products (III.23) and (III.24) have $\{0\} \times \mathbb{T}^d$ as an invariant torus. The flow restricted to this invariant torus is quasi-periodic and has frequency ω. The frequencies in the normal directions to the torus, the *normal frequencies*, are given by the rotation number of the skew-products (III.23) and (III.24).

A KAM method needs some Diophantine conditions on the frequency vector ω and the normal frequencies to the invariant torus, in our case the rotation number. Let us thus define these Diophantine conditions for continuous and discrete rotation numbers.

Let $\omega \in \mathbb{R}^d$ be a rationally independent frequency vector. We say that a number α is *strongly rationally independent with respect to* ω if there exist $K, \tau > 0$ such that

$$\left| \alpha - \frac{\langle \mathbf{k}, \omega \rangle}{2} \right| \geq \frac{K}{|\mathbf{k}|^\tau}, \qquad \mathbf{k} \in \mathbb{Z}^d - \{0\}.$$

The class of such α is denoted by $DC^c_\omega(K, \tau, \mathbb{R}^d)$.

In the discrete case, for a nonresonant ω, a number α is *strongly nonresonant with respect to* ω whenever the bound

$$|\sin(\pi(2\alpha - \langle \mathbf{k}, \omega \rangle))| \geq \frac{K}{|\mathbf{k}|^\tau},$$

holds for all $\mathbf{k} \in \mathbb{Z}^d - \{0\}$ and suitable fixed positive constants K and τ. The class is denoted by $DC^d_\omega(K, \tau, \mathbb{R}^d)$.

In [Eli92], Eliasson obtained the following remarkable result in the continuous case.

Theorem III.27 (Eliasson's Reducibility Theorem [Eli92]**).** *Let $\omega \in DC^c(c, \sigma, \mathbb{R}^d)$ be strongly rationally independent and $Q : \mathbb{T}^d \to \mathbb{R}$ be real analytic in a strip of width $\rho > 0$. Then there is a constant $C = C(c, \sigma, \rho) > 0$ such that if we define*

$$\lambda_0(s) = \begin{cases} \left(\frac{s}{C}\right)^2 & s \geq C \\ -\infty & s < C \end{cases}$$

then the following holds for $a > \lambda_0(|Q|_\rho)$.

(i) If the rotation number $\mathrm{rot}^c(a - Q, \omega)$ is strongly rationally independent or rational, with respect to ω, then the skew-product flow (III.23) is reducible to constant coefficients (with frequency $\omega/2$).

(ii) If $a > \lambda_0$ is at the endpoint of a noncollapsed spectral gap then the Floquet matrix $B \in sl(2, \mathbb{R})$ satisfies $B^2 = 0$, being $B = 0$ if, and only if, the gap is collapsed.

This theorem is semiperturbative since the rotation numbers for which the skew-product is reducible do not depend on the specific Diophantine condition of the rotation number. This achievement was obtained by the use of a technique by Moser & Pöschel [MP84] which introduced an adapted KAM scheme where a finite number of transformations, not close to the identity, were allowed in the iterative scheme. Previous perturbative KAM results proved reducibility in a large set in measure of rotation numbers, following the pioneering work of Dinauburg & Sinai [DS75] (see Rüssmann [Rüs80], Chierchia [Chi86, Chi87], Jorba & Simó [JS92, JS96], Nunes & Yuan [NY01]).

This theorem can be adapted to the discrete quasi-periodic case to obtain the following.

Theorem III.28. *Assume that $V : \mathbb{T}^d \to \mathbb{R}$ is real analytic in a complex strip of width $\rho > 0$ and ω is strongly nonresonant (i.e. $\omega \in DC^d(c, \sigma, \mathbb{R}^d)$ for some positive c and σ). Then there is a constant $C = C(c, \sigma, \rho)$ such that, if*

$$|V|_\rho < C$$

and the rotation number of

$$x_{n+1} + x_{n-1} + V(2\pi n\omega)x_n = ax_n$$

is either resonant or strongly nonresonant with respect to ω then the Schrödinger cocycle (A^d_{a-V}, ω), where

$$A^d_{a-V}(\theta) = \begin{pmatrix} a - V(\theta) & -1 \\ 1 & 0 \end{pmatrix},$$

is reducible to constant coefficients, with Floquet matrix B, by means of a quasi-periodic (with frequency $\omega/2$) and analytic transformation. Moreover, if a is at an endpoint of a spectral gap of the spectrum $\sigma^d(V, \omega)$, then the trace of B is ± 2, being $B = \pm I$ if, and only if, the gap collapses.

Remark III.29. *In Chapter VII we will prove a* nonperturbative *version of this theorem, namely that when $d = 1$ the constant C above does not depend on the precise Diophantine conditions on the frequency ω as long as it is strongly nonresonant.*

What happens with rotation numbers which are not rational with respect to ω? Eliasson [Eli92] showed that, generically in the continuous case, the eigenvalue equation has solutions which are unbounded but whose rotation number is not rational (hence they are are not at endpoints of spectral gaps). This is a contradiction with the reducibility Theorem III.27.

Theorem III.30 ([Eli92])**.** *Assume Q real analytic and ω strongly rationally independent as in Theorem III.27. Then, for a generic Q in the $|\cdot|_\rho$-topology, there exist values of $a > \lambda_0(|Q|_\rho)$ for which the fundamental matrix of (III.23) is unbounded and $\mathrm{rot}^c(a - Q, \omega)$ is neither Diophantine nor rational.*

Remark III.31. *In particular, in the quasi-periodic case, one cannot expect reducibility of Schrödinger cocycles for values of a in an open set. This is a substantial difference with the periodic case.*

Chapter IV

Resonance tongues and instability pockets in Hill's equation with quasi-periodic forcing

In this chapter we consider Hill's equation with quasi-periodic forcing,

$$x'' + (a + bq(t))x = 0, \tag{IV.1}$$

where a, b are real parameters and the real analytic function q is quasi-periodic in t, with a fixed frequency vector $\omega = (\omega_1, \ldots, \omega_d) \in \mathbb{R}^d$. Note the change of sign in the forcing q with respect to the notations in the previous chapter. A particular case appears when the forcing q is even, so that Hill's equation is then reversible, but for the main result of this chapter we shall consider general, including nonreversible, Hill's equations. The lift of the quasi-periodic function, $Q : \mathbb{T}^d \to \mathbb{R}$ such that $q(t) = Q(t\omega)$, will be assumed real analytic and the frequency vector ω is assumed to be strongly irrational with constants $c > 0$ and $\tau \geq d - 1$, i.e., that

$$|\langle \mathbf{k}, \omega \rangle| \geq c|\mathbf{k}|^{-\tau},$$

for all $\mathbf{k} \in \mathbb{Z}^d - 0$.

The objects of our interest are resonance tongues as they occur in the parameter space $\mathbb{R}^2 = \{a, b\}$. These are the connected components in the parameter plane where the rotation number is constant. Therefore, resonance tongues describe how spectral gaps of the associated Schrödinger operator vary as functions of b. The main result in this chapter states that in the present analytic case, for small $|b|$, the tongue boundaries are infinitely smooth curves and that using normal form techniques one can study the geometry of the resonance tongues. In this geometrical discussion we restrict to reversible near-Mathieu cases, which are a small perturbation of the exact Mathieu equation where $q(t) = \sum_{i=1}^{d} c_i \cos(\omega_i t)$, with $c_1, \ldots, c_d$ real constants. In Remark IV.1 a geometric reason is given for restricting to reversible systems when looking for instability pockets. An example of a near-Mathieu case with $d = 2$ and a deformation parameter ϵ is given by

$$q_\varepsilon(t) = \cos(\omega_1 t) + \cos(\omega_2 t) + \varepsilon \cos(\omega_1 + \omega_2)t. \tag{IV.2}$$

It is shown that the occurrence of instability pockets is generic in the reversible setting and a concise description of its complexity if given in terms of singularity theory, extending the results in the periodic case. We shall draw several consequences regarding the spectral behaviour of the corresponding Schrödinger operator, in particular regarding the effect of instability pockets on the collapsing of gaps. We develop examples where collapsed spectral gaps occur in a way that is persistent for perturbation of the b-parameterized, reversible family.

The approach to this problem in this chapter is similar to the classical theory of Hill's periodic equations. Unlike in the periodic case, smoothness of the tongue boundaries is not easy to obtain. This result uses a reducibility result by Eliasson [Eli92]. This makes an analysis possible as in the periodic case. However due to accumulation of tongues we need a delicate averaging technique.

Most of this chapter has been published in Broer, Puig & Simó [BPS03].

IV.1 Introduction. Main result

Our motivation on the one hand rests on the analogy with the periodic Hill equation, where several results in the same direction were known. On the other hand, the present results were motivated by the interest they have for certain spectral properties of the Schrödinger operator.

As explained in the previous chapter, for fixed $b \in \mathbb{R}$, Hill's equation shows up as the eigenvalue equation of the continuous quasi-periodic Schrödinger operator

$$(H^c_{-bq}x)(t) = -x''(t) - bq(t)x(t), \tag{IV.3}$$

which can be extended to a self-adjoint operator on $L^2(\mathbb{R})$. In this setting, the parameter a is called the energy- or the spectral-parameter.

In contrast with Chapter III, b is not considered as a constant, but as a parameter. This will give a better understanding of certain spectral phenomena as these were observed for fixed values of b. One example concerns the fact that generically no collapsed gaps occur, as shown by Moser & Pöschel [MP84]. Including b as a parameter, gives a deeper insight in the generic opening and closing behaviour of such gaps in dependence of b. Therefore our main interest is with the quasi-periodic analogue of the stability diagrams as these occur for the periodic Hill equation in the parameter plane $\mathbb{R}^2 = \{a, b\}$.

The periodic Hill equation revisited

We briefly reconsider Hill's equation with periodic forcing (the case $d = 1$), compare Broer & Levi [BL95], Broer & Simó [BS00], who study resonances in the near-Mathieu equation

$$x'' + (a + b\,q(t))\,x = 0,\ q(t + 2\pi) \equiv q(t), \tag{IV.4}$$

with q even and where a and b are real parameters. As is well-known, in the (a, b)-plane, for all $k \in \mathbb{N}$, resonance tongues emanate from the points $(a, b) = ((\frac{k}{2})^2, 0)$, see Figure IV.1. Inside these tongues, or instability domains, the trivial periodic solution $x = x' = 0$ is unstable. Compare Van der Pol & Strutt [VdPS28], Stoker [Sto92], Rellich [Rel69], Hochstadt [Hoc86], Keller & Levy [LK63], Magnus & Winkler [MW79], Arnol′d [Arn96, Arn83b, Arn83a] or Avron & Simon [AS81, AS78]. For related work on nonlinear parametric forcing, see Hale [Hal92] and Broer et al. [BHLV98, BHLV03, BHN97, BHvNV99, BLV97, BV92]. For nonlinear discrete versions see [OS89, OS98].

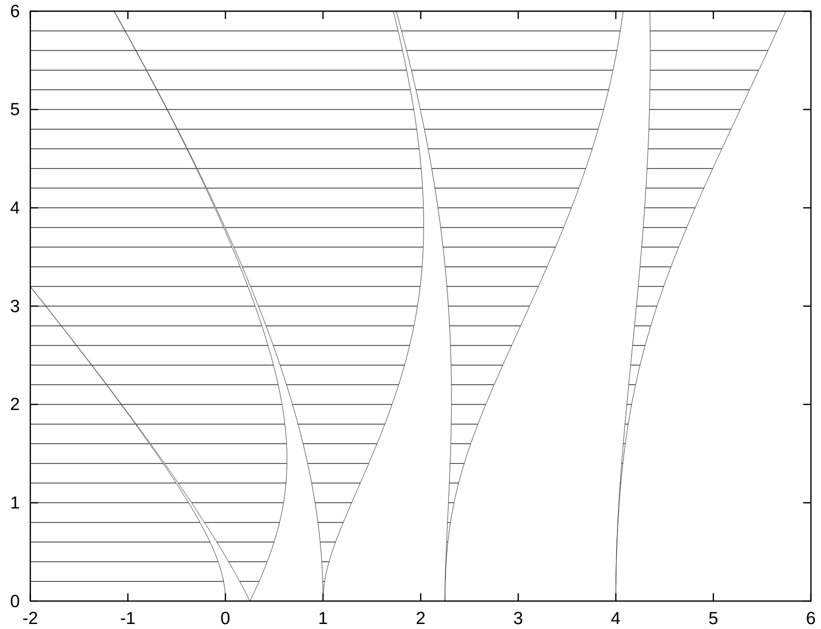

Figure IV.1: Resonance tongues in the classical periodic Mathieu equation $x'' + (a + b \cos t)x = 0$ in the (a, b)-plane, see Broer & Simó [BS00]. Shaded regions correspond to resonance tongues

Using Floquet theory (see Section II.1.3), the stability properties of the trivial solution of the periodic Hill equation are completely determined by the eigenvalues of the Poincaré map $P_{a,b}$ which belongs to $SL(2, \mathbb{R})$, the space of 2×2-matrices with determinant 1. In fact, elliptic eigenvalues correspond to stability and hyperbolic eigenvalues to instability.

The geometry of the tongue boundaries was studied in [BL95] and [BS00]. It turns out that generically the boundaries of a given tongue may exhibit several crossings and tangencies, thereby also creating instability pockets, see Figure IV.2. This term was coined by Broer-Levi prompted by the term 'instability interval' [LK63] as it occurs for fixed values of b. In [BL95, BS00] normal forms and averaging techniques provide a setting for singularity theory (see also [Afs86]). It turns out that in the near-Mathieu case close to the $k : 2$ resonance, one can have between 0 and $k - 1$ instability pockets, with all kinds of intermediate tangencies: the whole scenario has at least the complexity of the singularity $\mathbb{A}_{2k-1}$, compare [Arn94].

Remark IV.1. *For a description and analysis of more global phenomena in the periodic case, see* [BLS04]. *A singularity theory approach of resonances in a general dissipative context is given in* [BGV03].

Resonance tongues and spectral gaps

Preliminary to formulating our main result let us recall some concepts from the previous chapter. We start rewriting the quasi-periodic Hill equation (IV.1) as a quasi-periodic skew-product flow on $\mathbb{R}^2 \times \mathbb{T}^d$. This yields a vector field $\mathcal{X}$, given by

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ Q(\theta) - a & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \qquad \theta' = \omega. \tag{IV.5}$$

In this setting of skew-product flows, the evenness of Q leads to time-reversibility, which here is expressed as follows: if $R : \mathbb{R}^2 \times \mathbb{T}^d \to \mathbb{R}^2 \times \mathbb{T}^d$ is given by $R(x, y, \theta) = (x, -y, -\theta)$, then

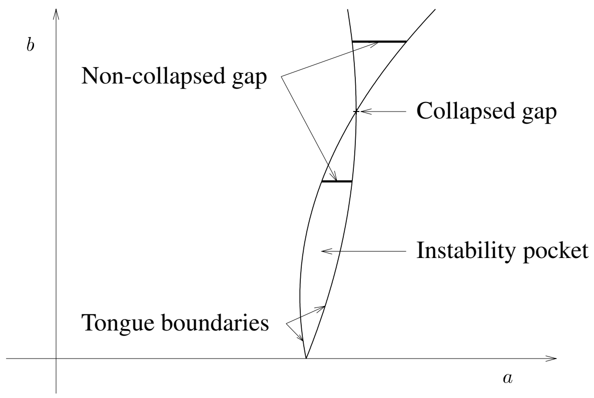

Figure IV.2: Resonance tongue with pocket in the (a, b)-plane giving rise to spectral gaps on each horizontal line with constant b. Note how collapses of gaps correspond to crossings of the tongue boundaries at the extremities of an instability pocket.

$R_*(\mathcal{X}) = -\mathcal{X}$. Reversibility will not be assumed for the main result.

Even if there is no Floquet theory for such skew-product flows, resonance tongues can be defined by means of the rotation number, $\mathrm{rot}(a, b) = \mathrm{rot}^c(a - Q, \omega)$, of Hill's equation IV.1, see Section III.2.2. We recall that the open intervals where the rotation number is constant are the spectral gaps of the operator H^c_{-bq} . In these gaps, the rotation number must be of the form

$$\alpha = \frac{\langle \mathbf{k}, \omega \rangle}{2},$$

where $\mathbf{k} \in \mathbb{Z}^d$ is a suitable multi-integer such that $\langle \mathbf{k}, \omega \rangle \geq 0$ by the Gap Labelling Theorem III.21.

$$\mathcal{M}^c_+(\omega) = \left\{ \tfrac{1}{2}\langle \mathbf{k}, \omega \rangle \in \mathbb{R} \;\middle|\; \mathbf{k} \in \mathbb{Z}^d \text{ and } \langle \mathbf{k}, \omega \rangle \geq 0 \right\}$$

is called the module of positive half-resonances of ω. By analogy with the periodic case we define resonance tongues as the following subsets in the parameter plane.

Definition IV.2. *Let $\mathbf{k} \in \mathbb{Z}^d$. The resonance tongue of the quasi-periodic Hill equation* (IV.1) *associated to the multi-index $\mathbf{k}$ is the set*

$$\mathcal{R}(\mathbf{k}) = \left\{ (a, b) \in \mathbb{R}^2 \;\middle|\; \mathrm{rot}(a, b) = \tfrac{1}{2}\langle \mathbf{k}, \omega \rangle \right\}.$$

This statement means that, for any fixed b_0 and any resonance $\frac{1}{2}\langle \mathbf{k}, \omega \rangle \in \mathcal{M}^c_+(\omega)$, the set of all a for which (a, b_0) belongs to the resonance tongue $\mathcal{R}(\mathbf{k})$ is precisely the closure of the spectral gap of $H_{-b_0 q}$ (either collapsed or noncollapsed) corresponding to this resonance by the Gap Labelling Theorem. See Figure IV.2 for illustration.

Formulation of the Main Theorem

As said before, the present chapter is concerned with the geometry and regularity of resonance tongue boundaries for the quasi-periodic Hill equation (IV.1) in the parameter plane $\mathbb{R}^2 = \{a, b\}$, where the function q is fixed.

For $\alpha_0 \in \mathcal{M}_+(\omega)$, if $\alpha_0 = \frac{1}{2}\langle \mathbf{k}, \omega \rangle$, let $\hat{\mathcal{R}}(\alpha_0) = \mathcal{R}(\mathbf{k})$. Each tongue $\hat{\mathcal{R}}(\alpha_0)$ has the form

$$\hat{\mathcal{R}}(\alpha_0) = \left\{ (a, b) \in \mathbb{R}^2 \mid a_-(b; \alpha_0) \leq a \leq a_+(b; \alpha_0) \right\} \tag{IV.6}$$

and $a_+(0;\alpha_0) = a_-(0;\alpha_0) = \alpha_0^2$. Indeed, if $b = 0$ and $a > 0$, the solutions of (IV.1), which now is autonomous, are linear combinations of $e^{\pm i\sqrt{a}t}$. By the above definition of the rotation number it follows that $\mathrm{rot}(a, 0) = \sqrt{a}$.

Mostly the value of α_0 is fixed, in which case we suppress its occurrence in the boundary functions $a_\pm$. Note that in (IV.6) one can ask, in general, for not more than continuity of the mappings $a_\pm : \mathbb{R} \to \mathbb{R}$, since we are imposing that $a_- \leq a_+$. Compare with the periodic case [BL95, BS00].

Nevertheless, recall that in the periodic case $d = 1$ there exist real analytic boundary curves $a_{1,2} = a_{1,2}(b)$ such that $a_- = \min\{a_1, a_2\}$ and $a_+ = \max\{a_1, a_2\}$. For the present case $d \geq 2$ we have the following result.

Theorem IV.3 (Smoothness of tongue boundaries). *Assume that in Hill's equation* (IV.1)

$$x'' + (a + bq(t))x = 0$$

with $a, b \in \mathbb{R}$, the function q is real analytic and quasi-periodic with strongly irrational frequency vector $\omega \in \mathbb{R}^d$, with $d \geq 2$. Then, for some constant $C = C(q, \omega)$ and for any $\alpha_0 \in \mathcal{M}_+^c(\omega)$, there exist C^∞-functions $a_1 = a_1(b)$ and $a_2 = a_2(b)$, defined for $|b| < C$, satisfying

$$a_- = \min\{a_1, a_2\}$$

and

$$a_+ = \max\{a_1, a_2\}.$$

Remark IV.4.

(i) In Theorem V.12 we will prove tongue boundaries are real analytic. The importance of the above result is that one can compute explicitly the Taylor expansion of such tongue boundaries to obtain C^r-approximations by means of Normal Form techniques, while Theorem V.12 is less constructive.

(ii) Using a C^∞-version of Eliasson's Theorem III.27, the result only needs the lift Q to be C^∞.

(iii) These results can be applied, a fortiori, to the periodic case $d = 1$.

Instability pockets, collapsed gaps and structure of the spectrum

We sketch the remaining results of this chapter, regarding instability pockets and the ensuing behaviour of spectral gaps.

In the quasi-periodic Hill equation, instability pockets can be defined as in the periodic case: a resonance tongue has an *instability pocket* when their boundaries cross at two different points. The fact that a tongue has a boundary crossing at (a_0, b_0) means that $\{a_0\}$ is a collapsed gap for the Schrödinger operator (IV.3) with $b = b_0$. An example of this occurs at the tongue tip $b = 0$.

Moser & Pöschel [MP84] showed that, for small analytic quasi-periodic potentials with strongly irrational frequencies, collapsed gaps can be opened by means of arbitrarily small perturbations having the suitable harmonics. This is done in a constructive way and implies that it is a generic property to have no collapsed gaps for fixed values of b (with $|b|$ small so that reducibility can be granted). In this chapter we go one step beyond, studying how gaps behave when the system is depending on the parameter b in a generic way.

By Theorem 1 we know that for analytic forcing (potential), for small $|b|$ and for a strongly irrational frequency vector ω, the tongue boundaries are infinitely smooth. Hence the computational techniques regarding normal forms and singularity theory, for studying the tongue boundaries, carry over from the periodic to the quasi-periodic setting. In particular this leads to a natural condition for the tongue boundaries to meet transversally at the tip $b = 0$, implying that there are no collapsed gaps for small $|b| \neq 0$. As a result we find, that for reversible Hill equations of near-Mathieu type, after excluding a subset of strongly irrational frequency vectors ω of measure zero, the situation is completely similar to the periodic case. Compare with the description given before in Section IV.1.

We shall present examples of families of reversible quasi-periodic Hill equations of near-Mathieu type with instability pockets. These examples are persistent in their (reversible) setting. To our knowledge, so far the existence of collapsed gaps in quasi-periodic Schrödinger operators has only been detected by De Concini and Johnson [DCJ87] in the case of algebraic-geometric potentials. These potentials only have a finite number of noncollapsed gaps, while all other gaps are collapsed. In view of the present chapter, this is a quite degenerate situation. See Figure IV.3 for an actual instability pocket for which normal form methods are needed up to second order, the results of which are compared with direct numerical computation (the agreement between numerical and analytical approximations can be improved using a higher order normal form). The techniques just described are useful when studying a fixed resonance. We note, however, that for investigating 'all' resonance tongues at once, even in a concrete example, we will use certain direct methods, which amount to refined averaging techniques. Compare with the periodic case [BS00].

Remark IV.5.

(i) In the nonreversible case generically no instability pockets can be expected (although they can be constructed following see Broer & Simó [BS00]*). To explain this, consider the classical periodic case $d = 1$, compare* [BL95]. *Recall that in this periodic case the stability diagram can be described in terms of Hill's map, which assigns to every parameter point (a, b) the Poincaré matrix $P_{a,b} \in SL(2, \mathbb{R})$, which is a 3-dimensional Lie group of 2×2-matrices. The tongue boundaries just are pull-backs under Hill's map of the unipotent cone, which has dimension 2 (except for singularities at $\pm\mathrm{Id}$). In the 3-dimensional matrix space the surfaces formed by the cone and the image under Hill's map of the (a, b)-plane generically meet in a transversal way. However, the intersection curves (which correspond to the tongue boundaries) generically do not meet away from the tip $b = 0$. Boundary crossings however do occur generically under the extra condition of reversibility, which reduces the dimension of the ambient matrix space to 2.*

(ii) At this moment we like to comment on global aspects of the geometry, as related to the spectrum of the corresponding Schrödinger operator. Unlike in the periodic case the union of resonance tongues is a dense subset of $\mathbb{R}^2 = \{a, b\}$ and quite usually it is a Cantor set with positive measure, see Section III.2.2. In a 2-dimensional strip where $|b|$ is sufficiently small, this gives a Cantor foliation of curves in between the dense collection of resonance tongues. This makes that classical methods to study the smoothness of tongue

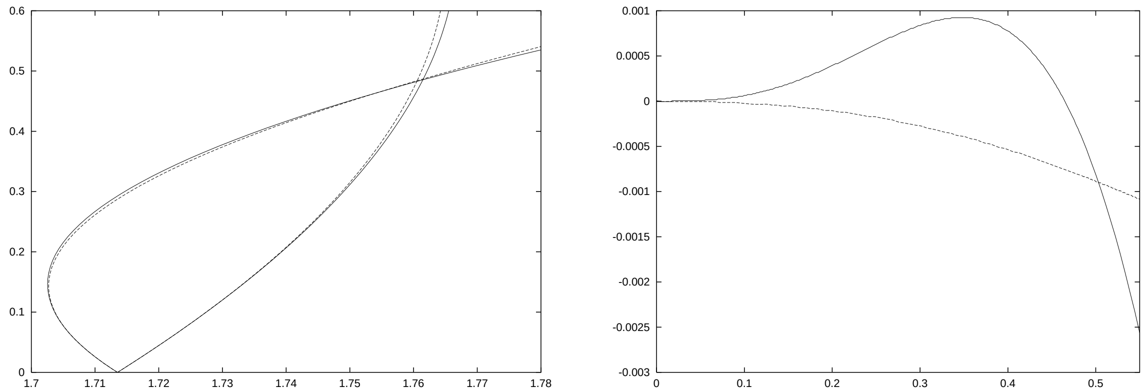

Figure IV.3: Left: Numerical computation of the instability pocket of the near-Mathieu equation with $q_\varepsilon(t) = \cos(\omega_1 t) + \cos(\omega_2 t) + \epsilon \cos(\omega_1 + \omega_2)t$, see (IV.2), in the (a, b)-plane with $\omega_1 = 1, \omega_2 = (1+\sqrt{5})/2$, and $\epsilon = 0.3$. Solid lines correspond to the approximation of the boundaries by second order averaging in (a, b); Dashed lines correspond to direct numerical computation. Right: Difference between the averaging and the direct numerical approximation as a function of b. Solid lines correspond to the tongue boundary that for small b turns to the left, dashed lines to the boundary as it turns to the right.

boundaries based on the separation of the eigenvalues fail, see Rellich [Rel69] Kato [Kat76].

(iii) Quasi-periodic Hill's equations can be written as a Hamiltonian with one degree of freedom. In a similar way one can consider linear Hamiltonian equations with quasi-periodic coefficients with more degrees of freedom. For the regularity of the boundaries where changes of stability occur in that case see Chapter V.

Outline of the chapter

Let us briefly outline the rest of this chapter. In Section IV.2 we present the ingredients for our proof of Theorem IV.3. Only a sketch of this proof is presented, a detailed proof is postponed to Section IV.4. In fact, most of the proofs are postponed to the latter section.

Section IV.3 contains applications of Theorem IV.3. For the criterion for transversality of the tongue boundaries at the tip see Section IV.3.1. A more thorough asymptotics at the tongue tip $b = 0$ and the ensuing creation of instability pockets is studied in a class of reversible near-Mathieu equations which is contained in Section IV.3.2. A proof is given in Section IV.6. The zero measure set of strongly irrational frequency vector ω to be excluded for this analysis, is considered in Section IV.7. A concrete example with instability pockets is studied in Section IV.3.3. Finally in Section IV.5 a Lipschitz property of the tongue boundaries is given under very general conditions.

IV.2 Towards a proof of the Main Theorem 1

We consider parameter values (a_0, b_0) at a tongue boundary, i.e., at an endpoint of a spectral gap, which may possibly be collapsed. At a boundary point (a_0, b_0) the rotation number $\mathrm{rot}(a_0, b_0) =$

$\frac{1}{2}\langle \mathbf{k}, \omega \rangle$, i.e. it is rational with respect to ω. Eliasson's Theorem III.27 will allow us to reduce the skew-product (IV.5) to constant coefficients at (a_0, b_0). In the reduced system the tongue boundary near (a_0, b_0) gets a simpler form, that can even be further simplified by repeated time-averaging, where the time-dependence is pushed to higher order in the localized parameters (a, b).

IV.2.1 Dynamical properties. Reducibility and rotation numbers

Recall from Section IV.1 the skew-product flow (IV.5) on $\mathbb{R}^2 \times \mathbb{T}^d$,

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ Q(\theta) - a & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \qquad \theta' = \omega$$

associated to the quasi-periodic Hill equation (IV.1). Also recall that evenness of Q leads to time-reversibility.

Since this is a linear equation with quasi-periodic coefficients, a main tool to study its dynamical behaviour is its possible reducibility to constant coefficients by a suitable transformation of variables. We always require that the transformation is quasi-periodic with the same basic frequencies as the original equation (or a rational multiple of these). The reduced matrix, which is not uniquely determined, is called the Floquet matrix. Note that for $d = 1$ reduction to Floquet form is always possible [Hal92, MW79, Pui03].

Eliasson's Theorem III.27, proves reducibility for small values of b and suitable conditions on the forcing. For the sake of completeness let us restate it adapted to the framework of this chapter.

Theorem IV.6 ([Eli92]). *Consider the quasi-periodic Hill equation* (IV.1)*, or, equivalently the skew-product flow* (IV.5)*. Assume that the following conditions hold*

- *The frequency vector ω is strongly irrational, $\omega \in DC^c(c, \tau, \mathbb{R}^d)$ for some constants $c > 0$ and $\tau \geq d - 1$.*

- *The function $Q : \mathbb{T}^d \to \mathbb{R}$ is real analytic, $Q \in C^a_\sigma(\mathbb{T}^d, \mathbb{R})$ for some $\sigma > 0$.*

Then, there exists a constant $C = C(c, \tau, \sigma) > 0$ such that if $|bQ|_\sigma < C$, while the rotation number of (IV.1) *is either rational or strongly irrational, with respect to ω, then the skew-product is reducible to constant coefficients (with frequency $\omega/2$) and $B = B(a, b)$ as Floquet matrix. Moreover,*

(i) $B(a, b)$ is nilpotent and nonzero if and only if a is an endpoint of a noncollapsed spectral gap;

(ii) $B(a, b)$ is zero if and only if $\{a\}$ is a collapsed gap.

Remark IV.7.

(i) In the present setting generically, for Liouville-type rotation numbers (i.e., which are neither rational nor strongly irrational) the normal behaviour of the invariant torus $\mathbb{T}^d \times \{(0, 0)\}$ is irreducible. In fact, there exist nearby solutions that are unbounded, where the growth is less than linear [Eli92]. *We recall that the Liouville-type rotation numbers form a residual subset (dense G_δ, second Baire category) of the positive half line. This is why we cannot expect to have analytic dependence on the parameters a, b.*

(ii) We shall use Theorem 2 only to arrive at a suitable perturbative setting around any point (a_0, b_0) at a tongue boundary. We shall construct a formal power series for the tongue boundary, which is shown to be the actual Taylor expansion at (a_0, b_0). Here we make a direct use of the definition of the derivative as a differential quotient. Limits are taken by constructing a series of shrinking wedge-like neighbourhoods of the curve, with increasing order of tangency at (a_0, b_0). The construction of the wedges uses dynamical properties of Hill's equation, e.g., concerning the variation of the rotation number outside the tongue and its constancy in the interior. Note that this will determine regions of exponential dichotomy in the interior of the tongue.

Using the reducing transformation provided by Theorem IV.6, we turn to co-rotating coordinates associated to parameter values at the tongue boundaries. Let $(a_0, b_0) \in \mathbb{R}^2$ be at a tongue boundary. Then Theorem 2 provides us with a reducing matrix of the form

$$Z\left((a_0, b_0)\right)(t) = \begin{pmatrix} z_{11}(t) & z_{12}(t) \\ z_{21}(t) & z_{22}(t) \end{pmatrix} \tag{IV.7}$$

where $z_{ij}(t) = Z_{ij}(\omega t/2)$ and a Floquet matrix of the form

$$B = \begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix}.$$

Observe that $c = 0$ if and only if (a_0, b_0) is an extreme point of an instability pocket. Also observe that $z_{ij}(t)$, $1 \le i, j \le 2$, is quasi-periodic with frequency vector $\frac{1}{2}\omega$.

To construct the co-rotating coordinates around (a_0, b_0), again consider Hill's equation (IV.1)

$$x'' + (a + bq(t))x = 0$$

and perform the linear, t-dependent change of variables given by Z.

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} z_{11}(t) & z_{12}(t) \\ z_{21}(t) & z_{22}(t) \end{pmatrix} \phi, \tag{IV.8}$$

where $\phi \in \mathbb{R}^2$. The differential equation for ϕ reads

$$\phi' = \left(\begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix} + \delta_\mu q \begin{pmatrix} z_{11}z_{12} & z_{12}^2 \\ -z_{11}^2 & -z_{11}z_{12} \end{pmatrix} \right) \phi, \tag{IV.9}$$

where $\mu = (a - a_0, b - b_0)$ is the new local multi-parameter and where $\delta_\mu q = (a - a_0) + (b - b_0)q$. Note that, if $\mu = 0$, this system is in constant coefficients. Also note that, since the eigenvalues of the Floquet matrix are 0 then the index of the map $Z(2\cdot) : \mathbb{T}^d \to SL(2, \mathbb{R})$ is $\mathbf{k}$ (see Section III.2.2).

Proposition IV.8. *In the above circumstances:*

(i) The functions $z_{11}z_{12}$, z_{11}^2 and z_{12}^2 are quasi-periodic in t with frequency vector ω.

(ii) In the case of even q we have

$$\begin{aligned} z_{12}(-t) &= -z_{12}(t) \\ z_{11}(-t) &= z_{11}(t). \end{aligned}$$

(iii) Let $\delta\mathrm{rot}(a,b)$ *be the rotation number of* (IV.9). *Then*

$$\delta\mathrm{rot}(a,b) = \mathrm{rot}(a,b) - \tfrac{1}{2}\langle \mathbf{k}, \omega\rangle.$$

Moreover, the tongue boundaries coincide with the boundaries of the set where $\delta\mathrm{rot}(a,b) = 0$.

Reversibility follows from a consideration of Floquet representations. If X is any fundamental matrix, then item *(ii)* above follows from the identity

$$X(-t) = X(t)^{-1}$$

in this reversible case.

IV.2.2 Proof of Theorem IV.3

By repeated averaging we recursively push the time dependence of the equation (IV.9) to higher order in the local parameter $\mu = (a - a_0, b - b_0)$. See [BS00, BV92, Sim94] and Appendix A for details. As before, by θ we denote the angular variables in $\mathbb{T}^d$.

At each step of this averaging or Birkhoff normalization process, for some $\sigma_1 < \sigma$, a linear change of variables

$$\psi = (I + R(\theta, \mu))\phi,$$

is found, which is complex analytic in θ on the strip $|\mathrm{Im}\ \theta| < \sigma_1$ and in the local parameters μ in a neighbourhood of 0. Furthermore, if the system is reversible, then the change of variables preserves this reversibility. To be more precise we have

Proposition IV.9. *In the above situation, after* r *steps of averaging system* (IV.9) *takes the form*

$$\phi' = \left(\begin{pmatrix} S_3^{(r)}(\mu) & c + S_2^{(r)}(\mu) \\ -S_1^{(r)}(\mu) & -\ S_3^{(r)}(\mu) \end{pmatrix} + M^{(r)}(\omega t, \mu) \right) \phi, \tag{IV.10}$$

where

$$S_i^{(r)}(\mu) = \sum_{1 \le s \le r} D_{s,i}^{(r)}(\mu)$$

for $i = 1, 2, 3$ *and where the functions* $D_{s,i}^{(r)}$, *for* $i = 1, 2, 3$ *and* $1 \le s \le r$, *have the following properties:*

(i) $D_{s,i}^{(r)}(\mu)$ *are homogeneous polynomials of degree* s *in* μ;

(ii) $D_{s,i}^{(t)} = D_{s,i}^{(s)}$ *for* $s \le t \le r$;

(iii) $D_{1,i}^{(1)} = (a - a_0)[z_{1i}^2] + (b - b_0)[q z_{1i}^2]$, *where* $[\,\cdot\,]$ *denotes the time average, for* $i = 1, 2$, *and* $D_{1,3}^{(1)} = (a - a_0)[z_{11} z_{12}] + (b - b_0)[q z_{11} z_{12}]$;

(iv) The remainder $M^{(r)}(\theta, \mu)$ is complex analytic in both θ and μ, (when $|\mathrm{Im}\ \theta|$ and $|\mu|$ are sufficiently small) while it is of order $r+1$, that is, the function

$$(\theta, \mu) \mapsto \frac{1}{|\mu|^{r+1}} \left| M_{ij}^{(r)}(\theta, \mu) \right|,$$

for $1 \leq i, j \leq 2$, is bounded on a neighbourhood of $\mu = 0$.

In the case of even q we have $S_3^{(r)} \equiv 0$.

In the application of this result, key idea is that the equation

$$S_1^{(r)}(\mu) \left(c + S_2^{(r)}(\mu) \right) - S_3^{(r)}(\mu)^2 = 0, \tag{IV.11}$$

which is the determinant of the averaged part of equation (IV.10), determines the derivatives of the tongue boundaries up to order r. In the analysis of (IV.11) we distinguish between the cases $c \neq 0$ (noncollapsed gap) and $c = 0$ (collapsed gap). This will be done next.

Non-collapsed gap ($c \neq 0$)

We first treat the case $c \neq 0$ of a noncollapsed gap. We will assume that $c > 0$, which means that (a_0, b_0) is at the right boundary of a resonance tongue, as it will be seen later on. The case of $c < 0$ can be treated similarly. Also, if the nonzero element c is in the lower off-diagonal element of the Floquet matrix, the procedure is analogous. Let

$$G^{(r)}(\mu) \equiv S_1^{(r)}(\mu) \left(c + S_2^{(r)}(\mu) \right) - S_3^{(r)}(\mu)^2.$$

We solve the equation $G^{(r)}(\mu) = 0$ by the Implicit Function Theorem, which provides a polynomial

$$a^{(r)}(b) = a_0 + \sum_{1 \leq k \leq r} \nu_k (b - b_0)^k.$$

The coefficients ν_k, $1 \leq k \leq r$, are uniquely determined by the functions $D_{s,i}^{(r)}$, $1 \leq s \leq r$, $i = 1, 2, 3$ and $G^{(r)}((a^{(r)}(b) - a_0, b - b_0)) = O_{r+1}(b - b_0)$. Here and in what follows, $g(\xi) = O_m(\xi)$, means that

$$\left| \frac{g(\xi)}{|\xi|^m} \right|$$

is bounded around $\xi = 0$.

In order to apply the Implicit Function Theorem, we compute

$$\frac{\partial}{\partial a} G^{(r)}(\mu)_{|\mu=0} = c[z_{11}^2] > 0.$$

This yields a unique polynomial $a^{(r)} = a^{(r)}(b)$ with the properties stated above.

Our next purpose is to show that, if $b \mapsto a(b)$ is a tongue boundary with $a(b_0) = a_0$, then

$$\lim_{b \to b_0} \frac{|a(b) - a^{(r)}(b)|}{|b - b_0|^r} = 0.$$

More precisely we have

Proposition IV.10. *Consider equation* (IV.10) *with $c > 0$. There exist positive constants N and Δ, such that if a_{N_+} and a_{N_-} are defined by*

$$a_{N_\pm}(b) = a^{(r)}(b) \pm N|b-b_0|^{r+1},$$

the following holds. For $0 < |b - b_0| < \Delta$

(i) the rotation number $\operatorname{rot}(a_{N_+}(b), b)$ is different from $\operatorname{rot}(a_0, b_0)$,

(ii) the system (IV.10) *(or equivalently* (IV.5)*) for $\mu = (a_{N_-}(b) - a_0, b - b_0)$ has zero rotation number.*

The proof is postponed to Section IV.4. As a direct consequence we have

Corollary IV.11. *Let (a_0, b_0) be at the tongue boundary as above and assume that $\{a_0\}$ is not a collapsed gap. Then, there exists a function $b \mapsto a(b)$ defined in a small neighbourhood of $b = b_0$, such that in this neighbourhood,*

(i) $(a(b), b)$ is at the tongue boundary of the same tongue as (a_0, b_0),

(ii) The map $b \mapsto a(b)$ at b_0 is r-times differentiable at b_0 and can be written as

$$a(b) = a_0 + \sum_{1\le k\le r} \nu_k (b-b_0)^k + O_{r+1}(b-b_0).$$

Proof: From now on, assume that $0 < |b - b_0| < \Delta$. Then, by Proposition IV.10, the set

$$\left\{(a_{N_-}(b), b) : 0 < |b-b_0| < \Delta\right\}$$

is a subset of the tongue's interior. Again by Proposition IV.10, for each $0 < |b - b_0| < \Delta$, the set $\left\{(a_{N_+}(b), b) : 0 < |b-b_0| < \Delta\right\}$ is a subset of the complement of the tongue. Now, for each fixed b, the map $a \mapsto \delta\operatorname{rot}(a, b)$ is monotonous, while, moreover, $\operatorname{rot}(a, b)$ is continuous in a and b. Therefore, for each $0 < |b - b_0| < \Delta$ there exists a unique $a(b)$ such that $(a(b), b)$ is at the tongue boundary.

Putting $a(b_0) = a_0$, the map $b \mapsto a(b)$ is continuously extended to $b = b_0$. The above argument also implies that for $0 < |b - b_0| < \Delta$,

$$a_{N_-}(b) \le a(b) \le a_{N_+}(b) \tag{IV.12}$$

and as $a_{N_-}(b_0) = a(b_0) = a_{N_+}(b_0) = a_0$, this inequality directly extends to $b = b_0$. Thus, due to the form of both a_{N_+} and a_{N_-}, we have that for $|b - b_0| < \Delta$,

$$\left|a(b) - a^{(r)}(b)\right| \le N|b-b_0|^{r+1},$$

from which the corollary follows. □

Collapsed gap ($c = 0$)

In the case $c = 0$ of a collapsed gap, system (IV.10) reads

$$\phi' = \left(\begin{pmatrix} S_3^{(r)}(\mu) & S_2^{(r)}(\mu) \\ -S_1^{(r)}(\mu) & -S_3^{(r)}(\mu) \end{pmatrix} + M^{(r)}(\omega t, \mu) \right) \phi. \tag{IV.13}$$

Thus, the analogue of (IV.11) now is

$$G^{(r)}(\mu) = S_1^{(r)}(\mu) S_2^{(r)}(\mu) - S_3^{(r)}(\mu)^2 = 0.$$

We will see in section IV.4 that there exist two polynomials of order r, $a_1^{(r)}(b)$ and $a_2^{(r)}(b)$ such that

$$G^{(r)}\left(a_i^{(r)}(b) - a_0, b - b_0\right) = O_{r+1}(b - b_0)$$

and, using the same tools as in the case of a noncollapsed gap the following result, whose proof is postponed to Section IV.4, is true.

Proposition IV.12. *Under the above assumptions, there exist positive constants N and Δ, such that if $|b - b_0| \leq \Delta$, then*

$$\begin{aligned} |a_+(b) - \max_{i=1,2}\{a_i^{(r)}(b)\}| &\leq N|b - b_0|^{r+1} \text{ and} \\ |a_-(b) - \min_{i=1,2}\{a_i^{(r)}(b)\}| &\leq N|b - b_0|^{r+1}. \end{aligned}$$

As a direct consequence we now have

$$\lim_{b \to b_0} \frac{|a_+(b) - \max_{i=1,2}\{a_i^{(r)}(b)\}|}{|b - b_0|^r} = 0 \tag{IV.14}$$

and

$$\lim_{b \to b_0} \frac{|a_-(b) - \min_{i=1,2}\{a_i^{(r)}(b)\}|}{|b - b_0|^r} = 0 \tag{IV.15}$$

and we can choose a_+ and a_- in such a way (skipping the restriction $a_- \leq a_+$) that both maps are continuous and r times differentiable at b_0. Moreover their Taylor expansions up to order r at b_0 are given by $a_1^{(r)}$ and $a_2^{(r)}$. Compare Corollary IV.11 of Proposition IV.10.

Conclusion of Proof of Theorem IV.3

Summarizing we conclude that Theorem IV.3 follows from the previous subsections, since we have shown that the tongue boundaries are infinitely smooth. Indeed, by Eliasson's Theorem IV.6 a positive constant C exists only depending on c, τ, σ and $|Q|_\sigma$ (see (IV.5)), such that for any $r \in \mathbb{N}$ the following holds. For any $|b_0| < C$, polynomials $a_1^{(r)}$ and $a_2^{(r)}$ exist of degree r in $(b - b_0)$, such that (IV.14) and (IV.15) hold.

IV.3 Applications and examples

In this section the methods and results of Section IV.2 are applied to study the geometric structure of resonance tongues in Hill's equations with quasi-periodic forcing (IV.1). In the previous section we saw that the tongue boundaries are smooth around $b = 0$. Also we found that their Taylor expansions around a certain point can be obtained by an averaging procedure for which one needs to know the reducing matrix at that point. In general this is not known unless $b = 0$, i.e., when the system has constant coefficients. In this section this fact will be used to obtain generalizations of results as these hold for Hill's equations with periodic coefficients, compare [BL95, BS00]. From a spectral point of view we will describe gap opening and closing as a function of b.

IV.3.1 A criterion for transversality at the tongue tip

The first application will be a criterion for the transversality of the tongue boundaries at the origin, i.e., at the tongue tip. In the periodic case it is known [Arn83b, BL95, BS00] that the two boundaries of a certain resonance tongue are transversal at $b = 0$ if, and only if, the corresponding harmonic (or Fourier coefficient) of q does not vanish. In the quasi-periodic case, the situation is the same.

Proposition IV.13. *In the quasi-periodic Hill equation*

$$x'' + (a + b\,q(t))\,x = 0,$$

where $q(t) = Q(\omega t)$ and $Q : \mathbb{T}^d \to \mathbb{R}$, assume that Q is real analytic and that the frequency vector $\omega \in \mathbb{R}^d$ is strongly irrational. Then the tongue boundaries of the $\mathbf{k}$th resonance, $\alpha_0 = \frac{1}{2}\langle \mathbf{k}, \omega\rangle \in \mathcal{M}_+^c(\omega)$, $\alpha_0 \neq 0$, meet transversally at $b = 0$ if and only if the $\mathbf{k}$th harmonic of Q does not vanish.

Proof: Let $a_0 = \alpha_0^2$ and $\alpha_0 = \frac{1}{2}\langle \mathbf{k}, \omega\rangle \in \mathcal{M}_+^c(\omega)$. Then a fundamental solution for $a = a_0$ and $b = 0$ is given by

$$X(t) = \begin{pmatrix} \cos(\alpha_0 t) & \alpha_0^{-1}\sin(\alpha_0 t) \\ -\alpha_0 \sin(\alpha_0 t) & \cos(\alpha_0 t) \end{pmatrix}. \tag{IV.16}$$

Following the notation of the previous section, let

$$z_{11}(t) = \cos(\alpha_0 t) \qquad \text{and} \qquad z_{12}(t) = \frac{1}{\alpha_0}\sin(\alpha_0 t).$$

Then

$$\begin{aligned} z_{11}^2(t) &= \cos^2(\alpha_0 t) = \frac{1}{2} + \frac{1}{2}\cos(2\alpha_0 t), \\ z_{12}^2(t) &= \frac{1}{a_0}\sin^2(\alpha_0 t) = \frac{1}{2a_0} - \frac{1}{2a_0}\cos(2\alpha_0 t), \\ z_{11}(t) z_{12}(t) &= \frac{1}{2\alpha_0}\sin(2\alpha_0 t) \end{aligned}$$

Denoting the tongue boundaries by $a_i = a_i(b)$, for $i = 1, 2$, their derivatives at $b = 0$ are obtained averaging once and considering the equation

$$S_1^{(1)}(a - a_0, b - b_0) S_2^{(1)}(a - a_0, b - b_0) - \left(S_3^{(1)}(a - a_0, b - b_0) \right)^2 = 0.$$

Using Proposition IV.9, it is seen by means of a computation that

$$\begin{aligned} a_1'(0) &= -Q_0 + |Q_{\mathbf{k}}|, \\ a_2'(0) &= -Q_0 - |Q_{\mathbf{k}}|, \end{aligned}$$

from which follows that $a_1'(0) \neq a_2'(0)$ if, and only if, $Q_{\mathbf{k}} \neq 0$. This concludes our proof. □

IV.3.2 Order of tangency at the tongue tip and creation of instability pockets

We now focus on a special class of quasi-periodic Hill equations of the reversible near-Mathieu type:

$$x'' + \left(a + b \left(\sum_{j=1}^{d} c_j \cos(\omega_j t) + \varepsilon \cos\left(\langle \mathbf{k}^*, \omega \rangle t\right) \right) \right) x = 0, \tag{IV.17}$$

compare with Section IV.1. Here ε is a small deformation parameter and $\omega = (\omega_1, \ldots, \omega_d)^T$ is a strongly irrational frequency vector. Also we take $c_j \neq 0$ for all $j = 1, \ldots, d$ and fix $\mathbf{k}^* = (k_1^*, \ldots, k_d^*)^T$ a nonzero vector in $\mathbb{Z}^d$. We often abbreviate $\langle \mathbf{k} \rangle = \langle \mathbf{k}, \omega \rangle$. Let us, for a moment, consider the periodic case, that is, the *Mathieu equation*, which is the following particular case of Hill's equation

$$x'' + (a + b \cos t)\, x = 0.$$

It was proved by Ince [Inc44] (see also Chapter VI for a related result) that this equation has no instability pockets. Also, the order of tangency of the kth resonance tongue at $b = 0$ is exactly $|k|$ (see Harrell [Har79] and references therein). However, when it is perturbed in the following way

$$x'' + (a + b\,(\cos t + \varepsilon \cos jt))\, x = 0.$$

where $\varepsilon \neq 0$ and $j \geq 2$ is a integer, an instability pocket is created for small $|\varepsilon|$ [BL95, BS00]. The following theorem is a generalization of this periodic case.

Theorem IV.14. *Consider the reversible near-Mathieu equation with quasi-periodic forcing* (IV.17) *as above. Then*

- *(i) If $\varepsilon = 0$, the order of tangency at $b = 0$ of the $\mathbf{k}^*$th resonance tongue is greater or equal than $|\mathbf{k}^*|$ and it is exactly $|\mathbf{k}^*|$ if, and only if, ω does not belong to $\mathcal{A}(\mathbf{k}^*)$, where $\mathcal{A}(\mathbf{k}^*)$ is a zero measure subset of the strongly irrational frequency vectors.*
- *(ii) If $\varepsilon \neq 0$, $\omega \notin \mathcal{A}(\mathbf{k}^*)$ is strongly irrational and $|\varepsilon|$ is small enough, there exists at least one pocket at the $\mathbf{k}^*$th resonance tongue with ends $b = 0$ and $b = b(\varepsilon) \neq 0$. Here ε needs to have a suitable sign if $|\mathbf{k}^*|$ is odd.*

Remark IV.15.

(i) Note that the above result only applies to quasi-periodic near-Mathieu equations of the type (IV.17)*. For more general quasi-periodic forcings, the problem of the order of tangency of the tongue boundaries at $b = 0$ is at least as complicated as in the periodic case, see* [BS00].

(ii) The sets $\mathcal{A}(\mathbf{k}^*)$ are not empty in general. For examples and some properties of these sets, see Section IV.7.

(iii) Instead of fixing ε one can also fix $|b_0|$ sufficiently small and show that, for a suitable value of $\varepsilon = \varepsilon(b_0)$, one can create an instability pocket in equation (IV.17) *with ends at $b = 0$ and $b = b_0$. A suitable choice of the components of c also allows several pockets (associated to different $\mathbf{k}^*$) with ends at $b = 0$ and $b = b_0$ and for the same value of ε. In general the same complexity holds here as in the periodic case, compare with the general discussion in Section* IV.1.

A proof of Theorem IV.14 is given in Section IV.6. One consequence of the Theorem is

Corollary IV.16. *Assume that in Hill's equation*

$$x'' + (a + bq(t))\, x = 0$$

the forcing q is a real analytic even quasi-periodic function whose frequency vector ω is strongly irrational. Suppose that, for some $\mathbf{k}^* \neq \mathbf{0}$, the $\mathbf{k}^*$th harmonic of q does not vanish and that $\omega \notin \mathcal{A}(\mathbf{k}^*)$. Then, the following equation

$$x'' + \left(a + b\left(\sum_{j=1}^{d} c_j \cos(\omega_j t) + q(t)\right)\right) x = 0 \tag{IV.18}$$

has a pocket at the $\mathbf{k}^*$th resonance tongue provided that the $|c_j|$ are sufficiently large.

Proof: Let $\varepsilon > 0$ be a small parameter and define $\tilde{c}_j = c_j\varepsilon$, for $j = 1, \ldots, d$. Writing $\tilde{b} = b/\varepsilon$, the equation (IV.18) reads

$$x'' + \left(a + \tilde{b}\left(\sum_{j=1}^{d} \tilde{c}_j \cos(\omega_j t) + \varepsilon q(t)\right)\right) x = 0.$$

Since the $\mathbf{k}^*$th harmonic $q_{\mathbf{k}^*}$ of q does not vanish, this even function can be split into

$$q(t) = q_{\mathbf{k}^*} \cos(\langle \mathbf{k}^*, \omega\rangle t) + \tilde{q}(t)$$

where $\tilde{q}$ is an even function whose $\mathbf{k}^*$th harmonic vanishes. Let $\tilde{\varepsilon} = \varepsilon q_{\mathbf{k}^*}$. In these new parameters the equation (IV.18) gets the form

$$x'' + \left(a + \tilde{b}\left(\sum_{j=1}^{d} \tilde{c}_j \cos(\omega_j t) + \tilde{\varepsilon} \cos(\langle \mathbf{k}^*, \omega\rangle t) + \frac{\tilde{\varepsilon}}{q_{\mathbf{k}^*}}\tilde{q}(t)\right)\right) x = 0.$$

The only difference of the latter equation with (IV.17) is the term $\tilde{q}$. But since its $\mathbf{k}^*$th harmonic vanishes, the conclusions of Theorem IV.14 concerning the existence of pockets hold here, provided $\omega \notin \mathcal{A}(\mathbf{k}^*)$ is strongly irrational, $\tilde{c}_j$ do not vanish and ε is sufficiently small. The latter condition is equivalent to the c_j being sufficiently large. $\square$

IV.3.3 A reversible near-Mathieu example with an instability pocket

In this section the following concrete example of a reversible near-Mathieu equation with quasi-periodic forcing is investigated:

$$x'' + (a + b(\cos t + \cos\gamma t + \epsilon\cos(1+\gamma)t))\, x = 0. \tag{IV.19}$$

Here γ is a strongly nonresonant number (which means that $(1, \gamma)$ is strongly rationally independent) and ε a deformation parameter. We consider the resonance $\alpha_0 = \frac{1}{2}(1+\gamma)$, which means that (a, b, ϵ) will be near

$$\left(\left(\tfrac{1}{2}(1+\gamma)\right)^2, 0, 0 \right).$$

Since, moreover, the forcing q is entire analytic, there exists a constant $C = C(\gamma)$ such that if $|b| < C$ and $|\epsilon| < 1$, then there is reducibility at the tongue boundaries [Eli92]. Compare Section 2. After a twofold averaging and other suitable linear transformations which do not affect the resonance domains, the system is transformed into

$$\phi' = \left(\begin{pmatrix} 0 & X(\nu) + Y(\nu) \\ -X(\nu) + Y(\nu) & 0 \end{pmatrix} + M^{(2)}(\nu) \right) \phi,$$

where $\nu = (a - a_0, b, \epsilon)$ and where X and Y are given by the following expansions in ν

$$X(\nu) = -\frac{1}{1+\gamma} \left(\frac{a-a_0}{1+\gamma} - \frac{(a-a_0)^2}{(1+\gamma)^3} - \frac{b^2}{4(1+\gamma)^2} C(\epsilon) \right)$$

and

$$Y(\nu) = -\frac{b}{1+\gamma} \left(\frac{-\epsilon}{2(1+\gamma)} + \frac{\epsilon(a-a_0)}{(1+\gamma)^3} + \frac{b}{2(1+\gamma)^2}(1 + \frac{1}{\gamma}) \right).$$

Here

$$C(\epsilon) = \frac{\epsilon^2}{2(1+\gamma)} + \frac{1}{2+\gamma} + \frac{1}{1+2\gamma} + 1 + \frac{1}{\gamma}.$$

Hence, in the notation of Section 2 we have $S_1^{(2)}(\nu) = X(\nu) - Y(\nu)$ and $S_2^{(2)}(\nu) = X(\nu) + Y(\nu)$. Thus $S_1^{(2)}(\nu) = 0$ (resp. $S_2^{(2)}(\nu) = 0$) if and only if $X(\nu) = Y(\nu)$ (resp. $X(\nu) = -Y(\nu)$). The Taylor expansions of the tongue boundaries up to second order in (b, ϵ) are given by

$$a_1^{(2)}(b,\epsilon) = \left(\frac{1+\gamma}{2}\right)^2 - \frac{b\epsilon}{2} + \frac{b^2}{4(1+\gamma)} \left(3 + \frac{3}{\gamma} + \frac{1}{2+\gamma} + \frac{1}{1+2\gamma} \right)$$

and

$$a_2^{(2)}(b,\epsilon) = \left(\frac{1+\gamma}{2}\right)^2 + \frac{b\epsilon}{2} + \frac{b^2}{4(1+\gamma)} \left(-1 - \frac{1}{\gamma} + \frac{1}{2+\gamma} + \frac{1}{1+2\gamma} \right).$$

Therefore the second order Taylor expansions of the tongue boundaries have a transversal crossing both at $(b, \epsilon) = (0, 0)$ and at the point $(b, \epsilon) = (\gamma\epsilon, \epsilon)$ if $\epsilon \neq 0$. By Theorem IV.3 we know that the boundary functions $a_1^{(2)}$ and $a_2^{(2)}$ are of class C^∞ in b. With little more effort, one also establishes this same degree of smoothness in the parameter ϵ. Following the argument of the previous subsection, one has

Corollary IV.17. *For the reversible near-Mathieu equation* (IV.19) *there exists a positive constant C such that, if $|b| < C$ and $|\epsilon| < 1$, then the tongue boundaries of the resonance corresponding to $\alpha_0 = (1+\gamma)/2$ are C^∞ functions of (b, ϵ), while*

(i) For $\epsilon \neq 0$ the tongue boundaries have two transversal crossings, one at $(a, b) = (a_0, 0)$ and the other at $(a_1^{(2)}(\gamma\epsilon, \epsilon) + O_3(\epsilon), \gamma\epsilon + O_2(\epsilon))$,

(ii) For $\epsilon = 0$ the tongue boundaries at $b = 0$ have a second order tangency.

Remark IV.18. *Note that Corollary* 3 *exactly describes the $\mathbb{A}_3$-scenario, compare* [Arn94]*. For the periodic analogue see* [BL95]*, where Hill's map has a Whitney cusp singularity. Compare with Section IV.1.*

IV.4 Proofs

Main aim of this section is to prove the propositions IV.10 and IV.12 of Section IV.2.2. We recall the setting there. Around a point $(a_0, b_0) \in \mathbb{R}^2$ with $|b_0|$ sufficiently small, at the boundary of a resonance zone, there exists a symplectic reducing matrix Z. Let $\mu = (a - a_0, b - b_0) =: (\alpha, \beta)$ be the new local parameters and hence $\delta_\mu q = \alpha + \beta q$, see Equation (IV.9). The change of variables (IV.8) reduces the equation for $\phi = (\phi_1, \phi_2)$ to the form (IV.9).

The corresponding Hamiltonian, written in autonomous form by introducing new momenta $J \in \mathbb{R}^d$, reads

$$K(\phi_1, \phi_2, \theta, J) = \langle J, \omega\rangle + \frac{1}{2}c\phi_2^2 + \delta_\mu Q(\theta)\left(\frac{1}{2}z_{11}^2\phi_1^2 + z_{11}z_{12}\phi_1\phi_2 + \frac{1}{2}z_{12}^2\phi_2^2\right).$$

The first two terms of the right hand side form the unperturbed Hamiltonian K_0, the last one is K_1.

The rotation number of this Hamiltonian (or rather of the associated skew-product flow on $\mathbb{R}^2 \times \mathbb{T}^d$, see Section III.2.2) is $\delta\mathrm{rot}(a, b) = \mathrm{rot}(a, b) - \frac{1}{2}\langle \mathbf{k}, \omega\rangle$. The tongue boundaries are the boundaries of the set $\delta\mathrm{rot}(a, b) = 0$. After r steps of averaging, the system takes the form of (IV.10),

$$\phi' = \left(\begin{pmatrix} S_3^{(r)}(\mu) & c + S_2^{(r)}(\mu) \\ -S_1^{(r)}(\mu) & -S_3^{(r)}(\mu)\end{pmatrix} + M^{(r)}(\omega t, \mu)\right)\phi,$$

see Proposition IV.9. In what follows, the expression of the previous equation in polar coordinates will be used, see again Section III.2.2. Writing $\varphi = \arg(\phi_2 + \mathrm{i}\phi_1)$, the differential equation for φ becomes

$$\varphi' = (S_1^{(r)} + M_1^{(r)})\sin^2\varphi + 2(S_3^{(r)} + M_3^{(r)})\sin\varphi\cos\varphi + (c + S_2^{(r)} + M_2^{(r)})\cos^2\varphi, \quad \text{(IV.20)}$$

which is a quadratic form with matrix

$$\begin{pmatrix} S_1^{(r)} & S_3^{(r)} \\ S_3^{(r)} & c + S_2^{(r)}\end{pmatrix} + \begin{pmatrix} M_1^{(r)} & M_3^{(r)} \\ M_3^{(r)} & M_2^{(r)}\end{pmatrix}. \quad \text{(IV.21)}$$

We recall that $M_j^{(r)} = O_{r+1}(|\mu|)$ uniformly in θ in a complex neighbourhood of $\mathbb{T}^d$. It is now important to distinguish between the cases of a noncollapsed gap ($c \neq 0$) and of a collapsed gap ($c = 0$).

IV.4.1 Non-collapsed gap

Suppose we are in the case of a noncollapsed gap, i.e., with $c \neq 0$. Present aim is to prove Proposition IV.10 which deals with the case $c > 0$. The case $c < 0$ is treated similarly. Recall that, in Section IV.2.2 for any $r \geq 1$ we obtained a polynomial of order r in $b - b_0$, $a^{(r)}(b)$ such that, if

$$G^{(r)}(\mu) = S_1^{(r)}(\mu)\left(c + S_2^{(r)}(\mu)\right) - S_3^{(r)}(\mu)^2,$$

then

$$G^{(r)}\left(a^{(r)}(b) - a_0, b - b_0\right) = O_{r+1}(b - b_0).$$

In order to prove Proposition IV.10 we will show that there exist constants $N > 0$, sufficiently large, and $\Delta > 0$, sufficiently small, such that if $0 < |b - b_0| < \Delta$

(i) Equation (IV.10) for $(a, b) = (a^{(r)}(b) + N|b - b_0|^{r+1}, b)$ has rotation number strictly different from zero.

(ii) Equation (IV.10) for $(a, b) = (a^{(r)}(b) - N|b - b_0|^{r+1}, b)$ has zero rotation number.

In what follows we write again $(\alpha, \beta) = (a - a_0, b - b_0)$ and $\alpha^{(r)}(\beta) = a^{(r)}(b)$. Let, for some $N > 0$,

$$R_j^{\pm}(\beta) = S_j^{(r)}(\alpha^{(r)}(\beta) \pm N|\beta|^{r+1}, \beta), \qquad j = 1, 3,$$
$$R_2^{\pm}(\beta) = c + S_2^{(r)}(\alpha^{(r)}(\beta) \pm N|\beta|^{r+1}, \beta)$$

and $M^{\pm}(\theta, \beta) = M^{(r)}\left(\theta, (\alpha^{(r)}(\beta) \pm N|\beta|^{r+1}, \beta)\right)$. With these definitions, matrix (IV.21) becomes

$$\begin{pmatrix} R_1^{\pm} & R_3^{\pm} \\ R_3^{\pm} & c + R_2^{\pm} \end{pmatrix} + \begin{pmatrix} M_1^{\pm} & M_3^{\pm} \\ M_3^{\pm} & M_2^{\pm} \end{pmatrix}. \tag{IV.22}$$

Let $R^{\pm}$ be the first term of the previous expression. First of all, note that, since

$$\frac{\partial G^{(r)}}{\partial \alpha}|_{\mu=0} = c[z_{11}^2],$$

then

$$\det R^{\pm}(\beta) = R_1^{\pm}(\beta)\left(c + R_2^{\pm}(\beta)\right) - \left(R_3^{\pm}(\beta)\right)^2 = \pm\left(cN[z_{11}^2] + A\right)|\beta|^{r+1} + O_{r+2}(\beta),$$

being the time-dependent term A uniformly bounded for all $\theta \in \mathbb{T}^d$. This means that N and β_0 can be chosen so that

$$|\det R^{\pm}(\beta)| \geq \frac{cN}{2}[z_{11}^2]|\beta|^{r+1},$$

provided $|\beta| < \beta_0$, and the sign of $\det R^{\pm}$ is $\pm$. The elements of the time-depending part, the $M_j^{\pm}(\theta, \beta)$, can be uniformly bounded by $\frac{N}{4}|\beta|^{r+1}$ if N and β_0 are suitably modified. The modulus of the eigenvalues of $R^{\pm}$ can be bounded from below by $\frac{N}{3}|\beta|^{r+1}$ and $2c/3$. Now we distinguish between the cases of R^+ and R^-.

In the case of R^+, the symmetric matrix (IV.22) is definite positive and for all $\theta \in \mathbb{T}^d$, φ' in (IV.20) is bounded from below by $\frac{N}{12}|\beta|^{r+1}$, since the minimum of φ', ignoring the contribution of the time-dependent part, is $\frac{N}{3}|\beta|^{r+1}$. This implies that the rotation number is different from zero, if $0 < |\beta| < \beta_0$.

In the case of R^-, the time independent part of (IV.22) has a positive eigenvalue bounded from below by $2c/3$ and a negative one bounded from above by $-\frac{N}{12}|\beta|^{r+1}$. In particular, if $0 < |\beta|$ is small enough, there exist φ_1 and φ_2, independent of θ, such that the right-hand side of (IV.20) is positive and negative, respectively, uniformly for all $\theta \in \mathbb{T}^d$. In particular, the rotation number must be zero. □

Remark IV.19. *The Normal Form for $c \neq 0$ can be obtained without terms in $\phi_1\phi_2$ and without changing the term in ϕ_2^2. Indeed, at each step of normalization the homological equation is of the form $[G, H_0] = M$, where M contains known terms of the form $\phi_1^{j_1}\phi_2^{j_2}\exp(\mathrm{i}\langle\mathbf{k},\theta\rangle)$ with $j_1 + j_2 = 2$. Let us see the system to solve for a fixed $\mathbf{k}$. Let $T_1\phi_1^2 + T_3\phi_1\phi_2 + T_2\phi_2^2$ the terms having $\exp(\mathrm{i}\langle\mathbf{k},\theta\rangle)$ as a factor in the expression of M and $A_1\phi_1^2 + A_3\phi_1\phi_2 + A_2\phi_2^2$ the corresponding terms to be found in G. In matrix form we have*

$$\begin{pmatrix} \mathrm{i}\langle\mathbf{k},\omega\rangle & 0 & 0 \\ 2c & \mathrm{i}\langle\mathbf{k},\omega\rangle & 0 \\ 0 & c & \mathrm{i}\langle\mathbf{k},\omega\rangle \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = \begin{pmatrix} T_1 \\ T_2 \\ T_3 \end{pmatrix}.$$

If $\mathbf{k} \neq 0$ the matrix is invertible. If $\mathbf{k} = 0$ one cannot cancel T_1, which must be kept in the Normal Form, but the terms T_2 i T_3 can be canceled by suitable choices of A_1, A_2. The value of A_3 is arbitrary.

IV.4.2 Collapsed gap

Present aim is to prove Proposition IV.12, i.e., assuming that $c = 0$. Here we follow ideas similar to the above case $c = 0$. We shall see that the tongue boundaries can be divided over sectors, determined by whether the modulus of the modified rotation number is greater than some constant or whether the rotation number is zero and there is exponential dichotomy, see Figure IV.4. From this we obtain the tangency of the required order.

Recall that the first step of averaging gives

$$\langle J, \omega\rangle + \left(\frac{1}{2}S_1^{(1)}\phi_1^2 + S_3^{(1)}\phi_1\phi_2 + \frac{1}{2}S_2^{(1)}\phi_2^2\right) + O_2(\mu, \phi, \theta),$$

where O_2 denotes terms which are $O(|\mu|^2)$ (and quadratic in ϕ and depending on time through θ) and

$$S_1^{(1)} = \alpha[z_{11}^2] + \beta[Qz_{11}^2], \quad S_2^{(1)} = \alpha[z_{12}^2] + \beta[Qz_{12}^2], \quad S_3^{(1)} = \alpha[z_{11}z_{12}] + \beta[Qz_{11}z_{12}],$$

see Proposition IV.9. Hence the coefficients of α in $S_1^{(1)}$ and $S_2^{(1)}$ are positive and $[z_{11}z_{12}]^2 < [z_{11}^2]\cdot[z_{12}^2]$, a key fact in what follows. To order r the coefficient $S_j^{(1)}$ is replaced by $S_j^{(r)}$ for $j = 1, 2, 3$, of the form described before, and O_2 by O_{r+1}.

After r steps of normalization the matrix of the system is

$$\begin{pmatrix} S_3^{(r)} & S_2^{(r)} \\ -S_1^{(r)} & -S_3^{(r)} \end{pmatrix} + \begin{pmatrix} M_3 & M_2 \\ -M_1 & -M_3 \end{pmatrix},$$

where the M_j terms depend of θ analytically on the same domain as Q and are of order $r+1$ in α, β.

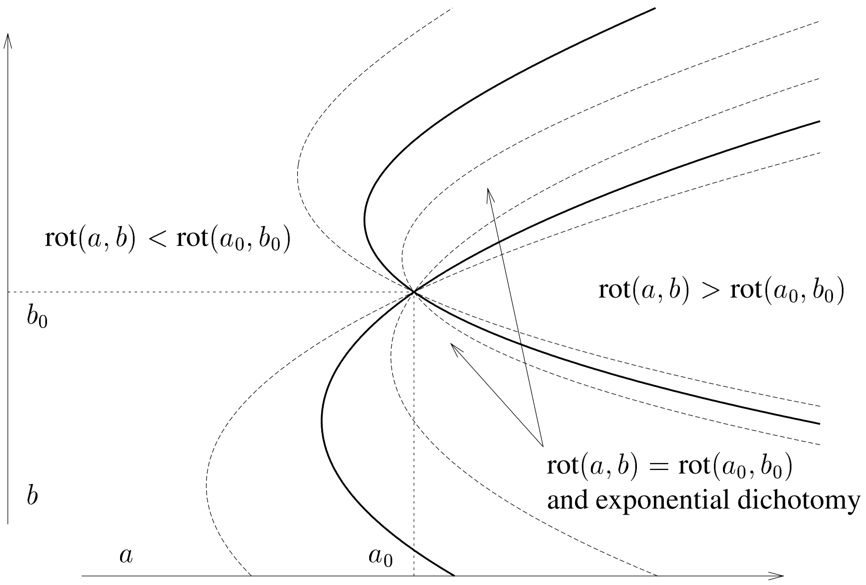

Figure IV.4: Areas of exponential dichotomy inside a resonance tongue as guaranteed by Lemma IV.21 and areas with rotation number different from $\mathrm{rot}(a_0, b_0)$ outside the tongue as guaranteed by Lemma IV.21. Solid lines denote tongue boundaries.

First we analyze the part coming from the Normal Form. As it is well-known, the boundaries of the resonance zone of this time-independent part correspond to μ-values such that the determinant of the system

$$G(\alpha, \beta) := S_1^{(r)} S_2^{(r)} - S_3^{(r)^2}$$

is equal to zero. As the terms of degree 1 in α in the S_j give rise to a positive definite part in the Hamiltonian, there exists a canonical change of variables (a rotation and scalings) such that $S_1^{(r)}$ and $S_2^{(r)}$ start as $n\alpha$ (for some $n > 0$) and $S_3^{(r)}$ contains no linear term in α. By scaling G we can assume $n = 1$ in the previous expressions. Hence, we are left with

$$\begin{array}{rcl} S_1^{(r)} & = & \alpha + \sigma_1(\beta) + \alpha\rho_1(\alpha, \beta), \\ S_2^{(r)} & = & \alpha + \sigma_2(\beta) + \alpha\rho_2(\alpha, \beta), \\ S_3^{(r)} & = & \sigma_3(\beta) + \alpha\rho_3(\alpha, \beta), \end{array}$$

where σ_j are polynomials in β of maximal degree r and starting, in principle, with linear terms, and ρ_j are polynomials in α, β of maximal degree $r - 1$. If $\sigma_j \not\equiv 0$ let k_j the minimal degree of β in σ_j, for $j = 1, 2, 3$. Otherwise we set $k_j = \infty$. Using Newton's polygon arguments (see, e.g., [Ful69]) to look to the relevant terms of the zero set of G, one can neglect the ρ_j terms.

Assume first $k = \min\{k_1, k_2, k_3\} \le r$. Introducing the change of variables $\alpha = \gamma\beta^k$ the function G can be written as

$$\beta^{2k}\left(\gamma^2 + (m_{1,k} + m_{2,k})\gamma + m_{1,k}m_{2,k} - m_{3,k}^2 + O(\beta)\right),$$

where $m_{j,k}$ denotes the coefficient of degree k in σ_j, $j = 1, 2, 3$ (some of them can be zero, but not all). Factoring out β^{2k} and neglecting the $O(\beta)$ term the zeros, γ_1 and γ_2, of the equation for

γ are simple, unless $m_{1,k} - m_{2,k}$ and $m_{3,k} = 0$. Hence, the Implicit Function Theorem implies that there are two different analytic functions

$$g_j(\beta) = \beta^k(\gamma_j + O(\beta)), \quad j = 1, 2$$

in the zero set of G, which differ at order $k \leq r$. An alternative way to prove this last fact is to use that the eigenvalues of a symmetric matrix depending analytically on one parameter depended analytically also on this parameter [Rel69] and to transform the matrix formed by the S to a symmetric matrix.

If $m_{1,k} - m_{2,k}$ and $m_{3,k} = 0$ let us introduce $\hat{\alpha} = \alpha + m_{1,k}\beta^k$ and rewrite G in terms of $\hat{\alpha}, \beta$. We rename $\hat{\alpha}$ again as α. Then the new equation for α, β is as before where k is at least replaced by $k+1$ and where the maximal degree of the σ_j and ρ_j polynomials also can increase. If the equation for the new γ has two different roots one obtains two curves $g_j(\beta)$ in the zero set of G, as before. Otherwise the procedure is iterated and ends when two different curves are obtained or when a value $k > r$ is reached.

If $k = \infty$ the procedure is stopped immediately. In this case, or when we reach $k > r$ in the iterative process, after a change of variables $\hat{\alpha} = \alpha - P(\beta)$, where P is a polynomial of degree r, the problem is equivalent to the initial one. Here the $S_j^{(r)}$ polynomials are replaced by S_j^*, where

$$\begin{aligned}
S_1^* &= \hat{\alpha} + \sigma_1^*(\beta) + \hat{\alpha}\rho_1^*(\hat{\alpha}, \beta), \\
S_2^* &= \hat{\alpha} + \sigma_2^*(\beta) + \hat{\alpha}\rho_2^*(\hat{\alpha}, \beta), \\
S_3^* &= \sigma_3^*(\beta) + \hat{\alpha}\rho_3^*(\hat{\alpha}, \beta),
\end{aligned}$$

and where the minimal degree of the σ_j^* is at least $k+1$. Hence, after a finite number of steps we obtain

Lemma IV.20. *Consider the Normal Form after r steps of normalization in the case $c = 0$. Let $G(\alpha, \beta) = 0$ be the defining equation of a boundary of the resonance zone. Then there exists $\beta_0 > 0$ such that, for $|\beta| < \beta_0$, one of the following statements holds:*

a) The zero set of G consists of two analytic curves $\alpha = g_j(\beta)$, $j = 1, 2$, with $g_2(\beta) - g_1(\beta) = d\beta^k(1 + O(\beta)), k \leq r,\ d > 0$. Furthermore

$$G(\alpha, \beta) = (\alpha - g_1(\beta))(\alpha - g_2(\beta))F(\alpha, \beta)$$

where F is an analytic function with $F(0, 0) > 0$.

b) There exists a curve $\alpha = P(\beta)$, with P a polynomial of degree r, and a constant $L > 0$ such that the zero set of G is contained in the domain bounded by $P(\beta) \pm L|\beta|^{r+1}$.

Proof: To complete the proof of the first item it is only necessary to remark that, from the previous discussion, only two branches of $G = 0$ can emerge from $(0, 0)$. Hence

$$\frac{G(\alpha, \beta)}{(\alpha - g_1(\beta))(\alpha - g_2(\beta))}$$

is an invertible function. The fact that $F(0, 0) > 0$ follows from the positive definite character of the linear terms in α.

Concerning the second item, using the variable $\hat{\alpha} = \alpha - P(\beta)$ one can work with the S_j^* functions. Let us denote as $G^*(\hat{\alpha}, \beta)$ the expression $S_1^* S_2^* - S_3^{*2}$, that is, the value of G in the new variables. Replacing $\hat{\alpha}$ by $\pm L|\beta|^{r+1}$ the function G^* becomes positive if L is large enough. It remains to show that the zero set is not empty, but this will be an immediate consequence of Lemma IV.21. $\square$

Next we consider the variations of the rotation number in different domains of the parameter plane. That is, we want to estimate $\delta\mathrm{rot}(a, b)$ which in the current parameters will be denoted simply by $\mathrm{rot}(\alpha, \beta)$. The differential equation for $\varphi = \arg(\phi_2 + \mathrm{i}\phi_1)$, i.e., equation (IV.20) for $c = 0$, reads

$$\varphi' = (S_1^{(r)} + M_1)\sin^2\varphi + 2(S_3^{(r)} + M_3)\sin\varphi\cos\varphi + (S_2^{(r)} + M_2)\cos^2\varphi. \tag{IV.23}$$

Lemma IV.21. *Consider the rotation number $\rho := \mathrm{rot}(\alpha, \beta)$ of the differential equation* (IV.20) *in co-rotating coordinates. Then there exist constants $N, \beta_0 > 0$ such that, for $0 < |\beta| < \beta_0$,*

a) In case a) of Lemma IV.20 let

$$g_-(\beta) = \min\{g_1(\beta), g_2(\beta)\}, \quad g_+(\beta) = \max\{g_1(\beta), g_2(\beta)\}.$$

Then one has

$$\rho < 0 \quad \textit{if} \quad \alpha < g_-(\beta) - N|\beta|^{r+1}; \qquad\qquad \rho > 0 \quad \textit{if} \quad \alpha > g_+(\beta) + N|\beta|^{r+1};$$

$$\rho = 0 \quad \textit{if} \quad g_-(\beta) + N|\beta|^{r+1} < \alpha < g_+(\beta) - N|\beta|^{r+1}.$$

b) In case b) of Lemma IV.20 one has

$$\rho < 0 \quad \textit{if} \quad \alpha < P(\beta) - N|\beta|^{r+1}; \qquad \rho > 0 \quad \textit{if} \quad \alpha < P(\beta) + N|\beta|^{r+1}.$$

Proof: Let us consider the quadratic form in the expression of φ', i.e., equation (IV.21) for $c = 0$, in case a) obtained by skipping the M_j terms and where α is taken equal $g_\pm(\beta)$. For definiteness let $\tilde{S}_j = S_j^{(r)}(g_\pm(\beta), \beta)$. This quadratic form is degenerate. As, in general, the discriminant of the quadratic form is $-G$, the form is indefinite for α in (g_-, g_+) and definite outside $[g_-, g_+]$. If $\alpha = g_-$ the form is negative definite everywhere except at one direction. Similarly, if $\alpha = g_+$ it is positive definite everywhere except at one direction.

We want to see the effect of adding the M_j terms and the change in the value of α. From the expression of the $S_j^{(r)}(\alpha, \beta)$ one has

$$\begin{aligned}
S_1^{(r)}(g_-(\beta) - N|\beta|^{r+1}, \beta) + M_1(g_-(\beta) - N|\beta|^{r+1}, \beta, \theta) &< \tilde{S}_1 - \frac{N}{2}|\beta|^{r+1}, \\
S_2^{(r)}(g_-(\beta) - N|\beta|^{r+1}, \beta) + M_2(g_-(\beta) - N|\beta|^{r+1}, \beta, \theta) &< \tilde{S}_2 - \frac{N}{2}|\beta|^{r+1}, \\
|S_3^{(r)}(g_-(\beta) - N|\beta|^{r+1}, \beta) + M_3(g_-(\beta) - N|\beta|^{r+1}, \beta, \theta) - \tilde{S}_3| &< \frac{N}{4}|\beta|^{r+1},
\end{aligned}$$

uniformly in θ, if N is large enough and $0 < |\beta| < \beta_0$ for some β_0. The current quadratic form is bounded from above by

$$\tilde{S}_1 \sin^2\varphi + 2\tilde{S}_3 \sin\varphi\cos\varphi + \tilde{S}_2\cos^2\varphi - \frac{N}{2}|\beta|^{r+1}(\sin^2\varphi + \sin\varphi\cos\varphi + \cos^2\varphi).$$

Hence $\varphi' < -\dfrac{N}{4}|\beta|^{r+1}$, proving the first of the assertions in a). The second assertion is proved in the same way.

To prove the third statement in a) it is better to shift α by $g_-(\beta)$. Let now $\hat{\alpha} = \alpha - g_-(\beta)$. Then the S_j functions are of the form

$$\begin{aligned}\hat{S}_1 &= \hat{\alpha} + \hat{\sigma}_1(\beta) + \hat{\alpha}\hat{\rho}_1(\hat{\alpha},\beta),\\ \hat{S}_2 &= \hat{\alpha} + \hat{\sigma}_2(\beta) + \hat{\alpha}\hat{\rho}_2(\hat{\alpha},\beta),\\ \hat{S}_3 &= \hat{\sigma}_3(\beta) + \hat{\alpha}\hat{\rho}_3(\hat{\alpha},\beta).\end{aligned}$$

It is clear that when $\hat{\alpha} = 0$ we have $G = 0$ by construction, and the other root is $g_+(\beta) - g_-(\beta) = d|\beta|^k(1 + O(\beta))$, $d > 0$. Therefore, $\hat{\sigma}_1(\beta) + \hat{\sigma}_2(\beta) = -d|\beta|^k(1 + O(\beta))$ and $\hat{\sigma}_1(\beta)\hat{\sigma}_2(\beta) = (\hat{\sigma}_3(\beta))^2$. Furthermore the $\hat{\sigma}_j$ functions have k as minimal degree for $j = 1, 2, 3$. For definiteness let $\hat{\sigma}_j(\beta) = h_j|\beta|^k\,(1 + O(\beta))$, with $h_j \neq 0$.

We set now $\hat{\alpha} = N|\beta|^{r+1}$ and add the M_j terms to the $\hat{S}_j$ functions. The new determinant is of the form

$$\left((N + A)|\beta|^{r+1} + \hat{\sigma}_1\right)\left((N + B)|\beta|^{r+1} + \hat{\sigma}_2\right) - \left(C|\beta|^{r+1} + \hat{\sigma}_3\right)^2$$

where $|A|, |B|, |C|$ are uniformly bounded for all θ by quantities which are $O_0(\beta)$. Therefore the determinant is uniformly bounded from above by $-dN|\beta|^{k+r+1}/2$ if N is large enough. This shows that the quadratic form is indefinite for all θ.

Furthermore, when $\hat{\alpha} = N|\beta|^{r+1}$ the $\hat{S}_j$ functions are $O(|\beta|^k)$. This, combined with the bound on the discriminant and the different terms contributing to the $\hat{S}_j$ shows that the slopes of the directions in the (ϕ_1, ϕ_2)-plane for which $\varphi' = 0$ are of the form

$$c_1(\beta) \pm |\beta|^{\frac{r+1-k}{2}}(c_2 + c_3(\beta,\theta)),$$

where c_1 and c_3 are analytic functions of their arguments and

$$c_1(0) = -\frac{h_3}{h_1} \neq 0,\ c_2 = \frac{\sqrt{dN}}{|h_1|}$$

and $|c_3(\beta,\theta)| < c_2/2$, uniformly in θ. The time dependence appears only in the c_3 term. One of the directions is attracting for the dynamics of φ in $\mathbb{S}^1$ and the other is repelling. We recall that these directions depend on t. However the slopes of both directions are bounded away from $c_1(\beta)$ uniformly in θ and therefore in t. Let $\varphi_r^*(t)$ the argument of a repelling direction. Any value of the form $\varphi_r^*(t) + m\pi$ is also repelling. Consider two consecutive repelling curves. For any fixed β with $|\beta| < \beta_0$ small enough, they are contained in a strip of the form $(\arg(c_1(\beta) - 2c_2), \arg(c_1(\beta) + 2c_2 + \pi)$. Any initial condition (ϕ_1, ϕ_2) between these repelling curves remains in the strip for all t. This shows that $\rho = 0$, as desired.

To prove the assertion for $\alpha = g_+(\beta) - N|\beta|^{r+1}$ one proceeds in a symmetric way. Then it follows for the full interval as in the statement, by monotonicity of ρ with respect to a.

Finally, we proceed to case b). By introducing now $\hat{\alpha} = \alpha - P(\beta)$ one obtains S functions like the S_j^* defined above, with $\sigma_j^*(\beta) = O_{r+1}(\beta)$. Then

$$\begin{aligned}
S_1^*(-N|\beta|^{r+1}, \beta) + M_1(\alpha, \beta, \theta) &< -\frac{N}{2}|\beta|^{r+1}, \\
S_2^*(-N|\beta|^{r+1}, \beta) + M_2(\alpha, \beta, \theta) &< -\frac{N}{2}|\beta|^{r+1}, \\
|S_3^*(-N|\beta|^{r+1}, \beta) + M_3(\alpha, \beta, \theta)| &< \frac{N}{4}|\beta|^{r+1},
\end{aligned}$$

uniformly in θ, if N is large enough and $|\beta| < \beta_0$ for some β_0. The current quadratic form is bounded from above as in the a) case by $-\frac{N}{4}|\beta|^{r+1}$. This proves the first assertion in b) and the second one is proved in a similar way. Furthermore, as was announced in Lemma IV.20, the zero set of G is contained between these two curves because the rotation number passes from < 0 to the left to > 0 to the right.

This finishes the proof of Lemma IV.21 and the last part of Lemma IV.20, case b). □

Remark IV.22. *In case a) we have constructed a domain with exponential dichotomy because it is an open set where the rotation number is constant. An alternative and equivalent way would be to perform a change of variables to render the time-independent part of the system to diagonal form and then apply Coppel's Criterion for exponential dichotomy II.29.*

Proposition IV.12 is now immediate. Indeed, let $a_1^{(r)}$ and $a_2^{(r)}$ the Taylor expansions up to order r in $b - b_0$ of $a_0 + g_1(b - b_0)$ and $a_0 + g_2(b - b_0)$ respectively. Then, letting $\Delta = \beta_0$, there is a constant N, given by the previous lemma, such that if $a_+(b)$ and $a_-(b)$ denote the right and left boundary of the tongue, then

$$\begin{aligned}
|a_+(b) - \max_{i=1,2}\{a_i^{(r)}(b)\}| &\leq N|b - b_0|^{r+1} \text{ and} \\
|a_-(b) - \min_{i=1,2}\{a_i^{(r)}(b)\}| &\leq N|b - b_0|^{r+1},
\end{aligned}$$

for $|b - b_0| < \Delta$, as we wanted to show.

IV.4.3 Differentiability of rotation number and Lyapunov exponent for a fixed potential

In this section we fix the parameter b_0 in a sufficiently small neighbourhood of the origin, to ensure reducibility according to Eliasson's Theorem IV.6. In this case we study rotation number $\rho = \rho(a)$ and (maximal) Lyapunov exponent $\lambda = \lambda(a)$ of the quasi-periodic Hill equation (IV.1), or equivalently (IV.5), in dependence of the parameter a. According to Eliasson [Eli92] for $|b| < C$ the Lyapunov exponent in the spectrum is zero.

The results in this setting are completely analogous to the periodic case, and proofs can be obtained from those of the previous section.

Corollary IV.23. *In the above situation, let a_0 be an endpoint of a spectral gap. Then*

(i) If a_0 is in the left (resp. right) endpoint of a noncollapsed spectral gap, then the functions

$$\alpha \in (-1,1) \mapsto \rho(a_0 - \alpha^2) \text{ and } \alpha \in (-1,1) \mapsto \lambda(a_0 + \alpha^2)$$

(resp $\alpha \in (-1,1) \mapsto \rho(a_0 + \alpha^2)$ and $\alpha \in (-1,1) \mapsto \lambda(a_0 - \alpha^2)$) are differentiable at zero.

(ii) If $\{a_0\}$ is a collapsed spectral gap, then the functions

$$a \mapsto \rho(a) \text{ and } a \mapsto \lambda(a)$$

are differentiable at a_0.

In particular, in any noncollapsed spectral gap $[a_-, a_+]$ the function $a \mapsto w(a) := \lambda(a)^2$ is analytic in (a_-, a_+) and has lateral derivatives at $a = a_-, a_+$.

The same result was obtained in [Nn95, OP92] in more general contexts (e.g., for the Schrödinger equation with almost periodic or ergodic potential).

Remark IV.24. *With a little more effort, one can recover the fact that for fixed, small potential the function $a \mapsto w(a)$ in a gap $[a_-, a_+]$ is of class $C^\omega((a_-, a_+)) \cap C^\infty([a_-, a_+])$, see Moser and Pöschel* [MP84].

Remark IV.25. *This shows that the best regularity that one can expect for the rotation number as a function of a is Hölder with exponent $1/2$. For references on this question of Hölder regularity, see Goldstein & Schlag* [GS01] *and Bourgain* [Bou00, Bou04a]. *Finally we would like to stress that our method of proof does not need the use of Thouless formula.*

IV.5 Lipschitz property of tongue boundaries in the large

In the chapter we approached the regularity of the tongue boundaries using reducibility. However, there exists numerical [BS98] and analytical evidence (see Frölich, Spencer & Wittver [FSW90] and Bjerklöv [Bje03]), that in cases far from constant coefficients this approach cannot be used. Presently we reconsider the quasi-periodic Hill equation (IV.1), or equivalently (IV.5), where we only assume the components of ω to be rationally independent (i.e., not necessarily with an additional Diophantine condition) and where the function $Q : \mathbb{T}^d \to \mathbb{R}$ is just continuous.

Proposition IV.26. *In the above situation, let*

$$C = \sup_{\theta \in \mathbb{T}^d} |Q(\theta)|.$$

and $b \in \mathbb{R} \mapsto a(b) \in \mathbb{R}$ be a (left or right) tongue boundary. Then for all $b, b' \in \mathbb{R}$ we have

$$|a(b) - a(b')| \leq C\,|b - b'|\,.$$

Our proof is based on Sturm-like arguments for the oscillation of the zeroes of a second order linear differential equation. Let us recall the Sturmian characterization of the rotation number, see Section III.2.2. For any nontrivial solution $x(t)$ of equation (IV.1) let $N(T;x)$ be the number of its zeroes in the interval $[0,T]$. Then, the limit

$$\lim_{T\to\infty} \frac{\pi N(T;x)}{T}$$

agrees with the rotation number of (IV.1), or equivalently, the system (IV.5).

Our idea to prove Proposition IV.26 is to use a suitable Sturm Oscillation Theorem to control the zeroes of a variation $Q+\delta Q$ of the original potential Q, with the property that $\delta Q(\theta)$ is either positive or negative for all $\theta \in \mathbb{T}^d$.

Lemma IV.27. *Assume that the maps $Q, \delta Q : \mathbb{T}^d \to \mathbb{R}$ are continuous and that $\delta Q(\theta) > 0$ for all $\theta \in \mathbb{T}^d$. Let ρ_1 be the rotation number of*

$$x'' + Q(\theta)x = 0, \;\; \theta' = \omega$$

and ρ_2 the rotation number of

$$y'' + (Q(\theta) + \delta Q(\theta))\, y = 0, \;\; \theta' = \omega.$$

Then $\rho_1 \leq \rho_2$.

Lemma IV.27 is a direct consequence of the Sturm Comparison Theorem, see, e.g., [CL55]. Indeed, by this result, the number of zeroes $N(T;x)$ in the interval $[0,T]$ is less than or equal to the number of zeroes $N(T,y)$ of y in the same interval, assuming that we have the same initial conditions $x(0) = y(0)$, $x'(0) = y'(0)$. Therefore, by the above considerations

$$\rho_1 = \lim_{T\to\infty} \frac{\pi N(T;x)}{T} \leq \lim_{T\to\infty} \frac{\pi N(T;y)}{T} = \rho_2,$$

as was to be shown.

We proceed showing how Lemma IV.27 can be used to check the Lipschitz condition stated in Proposition IV.26. First, note that, if the condition $\delta Q > 0$ is replaced by $\delta Q < 0$, then we have $\rho_1 \geq \rho_2$.

In the setting of Proposition IV.26, condition

$$\delta a - C|\delta b| > 0 \tag{IV.24}$$

implies that $\delta Q(\theta) = \delta a + \delta b Q(\theta) > 0$ for all $\theta \in \mathbb{T}^d$ and thus, by Lemma IV.27, that

$$\mathrm{rot}(a,b) \leq \mathrm{rot}(a+\delta a, b + C\delta b).$$

Now, if (a,b) is at the boundary of a certain tongue (for simplicity assume a is the right endpoint of the corresponding spectral gap), this means that for arbitrarily small perturbations in the a direction, the rotation number is strictly larger than that of the original equation. That is, for any $\delta' a > 0$,

$$\mathrm{rot}(a,b) < \mathrm{rot}(a+\delta' a, b).$$

The Lemma then yields that, if $(\delta a, \delta b)$ satisfies (IV.24), also

$$\mathrm{rot}(a,b) < \mathrm{rot}(a+\delta' a+\delta a, b+C\delta b).$$

As $\delta' a$ may be arbitrarily small the perturbations $(\delta a, \delta b)$ in the sector defined by condition (IV.24) do not contain any point in the boundary of the same tongue as (a,b). Therefore, in our proof of Proposition IV.26 we have

$$a(b_1) - a(b_2) \leq C\,|b_1 - b_2|\,.$$

In order to prove the remaining inequality, observe that perturbations in the sector

$$\delta a + C|\delta b| < 0 \tag{IV.25}$$

contain no points in the left boundary of the tongue of (a,b). By contradiction assume that such a point in the left boundary exists and let $(\delta a, \delta b)$ satisfying (IV.25) be the corresponding perturbation. Then, due to the openness of the above condition, there exists a positive constant $\delta' a$ such that $(\delta a + \delta' a, \delta b)$ still satisfies (IV.25). Moreover, as we are assuming $(a+\delta a, b+\delta b)$ to be in the endpoint of the left spectral gap and $\delta' a > 0$

$$\mathrm{rot}(a,b) = \mathrm{rot}(a+\delta a, b+\delta b) < \mathrm{rot}(a+\delta' a+\delta a, b+\delta b).$$

On the other hand, the perturbation $(-\delta a - \delta' a, -\delta b)$ satisfies condition (IV.24) and therefore

$$\mathrm{rot}(a+\delta' a+\delta a, \delta b) \leq \mathrm{rot}(a,b),$$

which implies $\mathrm{rot}(a,b) < \mathrm{rot}(a,b)$. This is the desired contradiction, whereby Proposition IV.26 is proved.

Remark IV.28. *The Lipschitz property in Proposition* IV.26 *regarding tongue boundaries also holds in the periodic case, where the proof runs exactly the same, and where this is referred to as the directional convexity of stability and instability domains, see Yakubovich & Starzhinskii* [YS75]. *The property also provides a bound on the derivatives of the tongue boundaries whenever they exist. This bound coincides with the one obtained in the averaging process of Section IV.2.2.*

IV.6 Proof of Theorem IV.14

Our proof follows from an analysis of the normal form to order $|\mathbf{k}^*|$. There are several normalization techniques and any such method for arbitrary $|\mathbf{k}^*|$ can be cumbersome. Therefore we only use the format of the normal form of order $|\mathbf{k}^*|$ to find out which terms are relevant. Subsequently, the coefficients of those terms are obtained by an alternative, recurrent and simpler method.

Let us set $\alpha_0 = \langle \mathbf{k}^* \rangle/2$ and $a = \alpha_0^2 + \alpha$. Next, a scaling

$$x = \frac{\xi}{\sqrt{\alpha_0}}, \quad y = \eta\sqrt{\alpha_0}$$

and passage to complex coordinates

$$\xi = \frac{q + \mathrm{i}p}{\sqrt{2}}, \quad \eta = \frac{\mathrm{i}q + p}{\sqrt{2}},$$

give the following form for the time-dependent Hamiltonian

$$H(q,p,t) = \alpha_0 iqp + \frac{q^2 - p^2 + 2iqp}{2}\left(\frac{\alpha}{2\alpha_0} + \frac{b}{4\alpha_0}\left(\sum_{j=1}^{d} c_j \cos(\omega_j t) + \varepsilon \cos\left(\langle \mathbf{k}^* \rangle t\right)\right)\right) \quad \text{(IV.26)}$$

where, again, we use the notation $\langle\cdot\rangle$ for $\langle\cdot,\omega,\rangle$. Now let J be canonically conjugate to the time t, and let $\zeta_j = \exp(i\ \mathrm{sign}(k_j^*)\omega_j t)$ for $j = 1,\ldots,d$. Then (IV.26) can be written as the sum of an integrable part H_0 and a perturbation H_1

$$H_0 = J + \alpha_0 iqp, \quad H_1 = \hat{b}\left(q^2 - p^2 + 2iqp\right)\left(\hat{\alpha} + \sum_{j=1}^{d} c_j\left(\zeta_j + \zeta_j^{-1}\right) + \varepsilon\left(\zeta^{\mathbf{k}^*} + \zeta^{-\mathbf{k}^*}\right)\right), \quad \text{(IV.27)}$$

where

$$\hat{\alpha} = 2\alpha/b, \qquad \hat{b} = b/(8\alpha_0)$$

act as perturbation parameters.

To carry out the normalization (averaging) one can use any Lie series method, for instance the Giorgilli-Galgani algorithm [GG78] as it is was done in [BS00]. Starting with $H_{0,0} = H_0$ and $H_{1,0} = H_1$, the terms

$$H_{j,k} = \sum_{l=1}^{k} \frac{l}{k}\left[G_l, H_{j,k-l}\right], \qquad j = 0,1, \quad k > 0,$$

where $[\cdot,\cdot]$ denotes the Poisson bracket, are computed recurrently. A term as $H_{j,k}$ contains $\hat{b}^{j+k}$ as a factor. The functions G_n are determined for canceling the time dependence as far as possible, i.e., if no resonances occur. To be precise, assume that $G_1,\ldots G_{n-1}$ are already computed. Then all terms in $H_{1,n-1} + H_{0,n}$ are known except the ones coming from $[G_n, H_{0,0}]$. Let K_n contain the known terms at order n. Then G_n is determined by requiring $K_n + [G_n, H_{0,0}]$ not to contain terms in the ζ_j variables. The transformed Hamiltonian then is $N = N_0 + N_1 + N_2 + \cdots$, where $N_0 = H_{0,0}$ and $N_n = H_{1,n-1} + H_{0,n}$. In particular, N_n is of order n with respect to $\hat{b}$.

It directly follows that

$$\left[H_{0,0}, q^r p^{2-r}\zeta^{\mathbf{k}}\right] = q^r p^{2-r}\zeta^{\mathbf{k}}\mathrm{i}\left(\alpha_0(2-2r) - \langle\mathbf{k}\rangle\right), \qquad r = 0,1,2,$$

and this shows that all terms with $\alpha_0(2-2r) - \langle\mathbf{k}\rangle$ different from zero can be canceled to any finite order. Proceeding by induction one observes that, if $j + k = m$ then $H_{j,k}$ has the form

$$H_{j,k} = q^2 d_1 - p^2 d_2 + iqp(d_3 + d_4),$$

with the corresponding G_m of the form

$$G_m = \mathrm{i}(q^2 d_1 + p^2 d_2) + qp(d_3 - d_4).$$

Here d_1 contains the terms with real coefficients of the form $\hat{\alpha}_0^{m-s}\zeta^{\mathbf{k}}$ with $|\mathbf{k}| = r$, where s and r have the same parity. The terms in d_2 can be obtained from d_1 by a replacement of $\zeta^{\mathbf{k}}$ by $\zeta^{-\mathbf{k}}$. Similarly, the expression of d_3 is identical to that of d_4 but replacing $\zeta^{\mathbf{k}}$ by $\zeta^{-\mathbf{k}}$.

Summing up, by a canonical change of variables, the Hamiltonian $H = H_0 + H_1$, up to a remainder of higher order in $\hat{b}$, can be reduced to the normal form

$$NF = J + a_0 iqp + \mathrm{coef}_1 \mathrm{i}qp + \mathrm{coef}_2 \left(q^2\zeta^{-\mathbf{k}^*} - p^2\zeta^{\mathbf{k}^*}\right), \tag{IV.28}$$

where

- $\mathrm{coef}_1 = \hat{\alpha} + r_1$, where r_1 a (real) function depending on $(\hat{b}, \hat{\alpha}, \varepsilon, c)$, and containing some power of $\hat{b}$ as a factor;
- $\mathrm{coef}_2 = \varepsilon \hat{b} f_2(\hat{b}, \hat{\alpha}, \varepsilon, c) + \hat{b}^{|\mathbf{k}^*|} \times r_2$, where r_2 a (real) function depending on $(\hat{b}, \hat{\alpha}, \varepsilon, c)$ and where $f_2(0, 0, 0, c) \neq 0$ does not depend on c;
- The order of the remainder in $\hat{b}$ is larger than $|\mathbf{k}^*|$.

Truncating away the remainder and passing to co-rotating coordinates (u, v) defined by

$$u = q\exp(-\mathrm{i}\langle \mathbf{k}^*\rangle t), \qquad v = p\exp(\mathrm{i}\langle \mathbf{k}^*\rangle t)$$

yields the system

$$u' = \mathrm{i}\,\mathrm{coef}_1\, u - 2\,\mathrm{coef}_2\, v, \qquad v' = -2\,\mathrm{coef}_2\, u - \mathrm{i}\,\mathrm{coef}_1\, v.$$

Therefore, up to the $|\mathbf{k}^*|$th order the tongue boundaries are given by the equation

$$\mathrm{coef}_1 = \pm 2\,\mathrm{coef}_2.$$

So if $r_2(0, 0, 0, c) \neq 0$ for $\varepsilon = 0$ there is a $|\mathbf{k}^*|$th order of tangency at $b = 0$, while for $\varepsilon \neq 0$ there is an instability pocket, see the end of this section. Hence, our proof of Theorem IV.14 is concluded by checking when $r_2(0, 0, 0, c)$ vanishes.

To find out whether $r_2(0, 0, 0, c)$ vanishes or not, it is only necessary to consider equation (IV.17) for $\varepsilon = 0$ at the exact resonance

$$x'' + \left(\frac{\langle \mathbf{k}^*\rangle}{4} + \frac{b}{2}\left(\sum_{j=1}^{d} c_j\left(\zeta_j + \zeta_j^{-1}\right)\right)\right) x = 0. \tag{IV.29}$$

Note that

$$r_2(0, 0, 0, c) = R(\omega, \mathbf{k}^*)c^{\mathbf{k}^*}$$

where now R does not depend on c. Therefore one may assume that $c_j = 1$ for $j = 1, \ldots, d$.

According to the normal form (IV.28), any nontrivial solution $x(t)$ of (IV.29) can be expanded in powers of b, the first $K - 1$ coefficients of which are quasi-periodic functions and where the Kth coefficient is also quasi-periodic if and only if $R(\omega, \mathbf{k}^*)$ vanishes. We are now going to compute this expansion directly from the differential equations, instead of using the Hamiltonian formulation.

Since we are interested in the $\mathbf{k}^*$th power in ζ, we first consider the equation (IV.29) only for positive powers of the ζ_j :

$$x'' + \left(\frac{\langle \mathbf{k}^* \rangle^2}{4} + \frac{b}{2}\sum_{j=1}^{d} \zeta_j\right) x = 0. \tag{IV.30}$$

Scaling time by $t = 2\tau$ turns (IV.30) into

$$\ddot{x} + \left(\langle \mathbf{k}^* \rangle^2 + \mu \sum_{j=1}^{d} z_j^2\right) x = 0. \tag{IV.31}$$

where the dot denotes derivation with respect to τ and where $\mu = 2b$ is the new perturbation parameter. Also note that after this change we have $\zeta_j = z_j^2$, where $z_j = \exp(i \text{ sign}(k_j^*)\omega_j \tau)$. Any solution of equation (IV.31) can be expanded in powers of μ as follows

$$x^{(1)} = x_0 + \mu x_1 + \mu^2 x_2 + \cdots + \mu_K x_K + O(\mu^{K+1})$$

where $K = |\mathbf{k}^*|$. Substitution of this expansion into (IV.31) leads to the following recursive relations:

$$\ddot{x}_r + \langle \mathbf{k}^* \rangle^2 x_r = -\left(\sum_{j=1}^{d} z_j^2\right) x_{r-1}$$

for $r = 1, \ldots, K$ and

$$\ddot{x}_0 + \langle \mathbf{k}^* \rangle^2 x_0 = 0$$

for $r = 0$. One of the two fundamental solutions of the latter equation is $x_0 = z^{-\mathbf{k}^*}$ so that the equation for x_1 becomes

$$\ddot{x}_1 + \langle \mathbf{k}^* \rangle^2 x_1 = -\sum_{j_1=1}^{d} z^{-\mathbf{k}^* + 2\mathbf{e}_{j_1}},$$

where $\mathbf{e}_j$ is the jth element of the canonical basis of $\mathbb{R}^d$. A solution of the latter equation is given by

$$x_1 = -\sum_{j_1=1}^{d} \frac{z^{-\mathbf{k}^* + 2\mathbf{e}_{j_1}}}{\langle \mathbf{k}^* \rangle^2 - \langle \mathbf{k}^* - 2\mathbf{e}_{j_1} \rangle^2}.$$

This recursive process can be continued up to any order. By induction it directly follows that at the rth step

$$x_r = (-1)^r \sum_{j_1,\ldots,j_r=1}^{d} \frac{z^{\left(-\mathbf{k}^* + 2\left(\mathbf{e}_{j_1} + \ldots + \mathbf{e}_{j_r}\right)\right)}}{\prod_{l=1}^{r} \left(\langle \mathbf{k}^* \rangle^2 - \langle \mathbf{k}^* - 2\mathbf{s}_l \rangle^2\right)}$$

where $\mathbf{s}_l = \mathbf{e}_{j_1} + \cdots + \mathbf{e}_{j_l}$ for $l = 1, \ldots, r$. Note that when $|r| < |\mathbf{k}^*|$, the denominator of the above expression never vanishes. At the order $K = |\mathbf{k}^*|$ the equation for x_K reads

$$\ddot{x}_K + \langle \mathbf{k}^* \rangle^2 x_K + (-1)^{K-1} \sum_{j_1,\ldots,j_K=1}^{d} \frac{z^{\left(-\mathbf{k}^* + 2\left(\mathbf{e}_{j_1} + \ldots + \mathbf{e}_{j_K}\right)\right)}}{\prod_{l=1}^{K-1} \left(\langle \mathbf{k}^* \rangle^2 - \langle \mathbf{k}^* - 2\mathbf{s}_l \rangle^2\right)} = 0.$$

In the summation we are only interested in terms with $z^{\mathbf{k}^*}$. Indeed, all other terms can be removed by a procedure similar to the one used for the previous $x_1, \dots, x_{K-1}$.

The remaining terms can be indexed by the paths of length K joining $\mathbf{0}$ and $\mathbf{k}^*$ in the lattice $\mathbb{Z}^d$. The set of these paths will be denoted by $\Gamma(\mathbf{k}^*)$, and for every path $\gamma \in \Gamma(\mathbf{k}^*)$ we consider the intermediate position vectors $\mathbf{s}_r(\gamma)$. In this way the equation for x_K becomes

$$\ddot{x}_K + \langle \mathbf{k}^* \rangle^2 x_K + z^{\mathbf{k}^*} F(\omega, \mathbf{k}^*) = 0$$

where

$$F(\omega, \mathbf{k}^*) = \sum_{\gamma \in \Gamma(\mathbf{k}^*)} \frac{(-1)^{K-1}}{\prod_{l=1}^{K-1} \left(\langle \mathbf{k}^* \rangle^2 - \langle \mathbf{k}^* - 2\mathbf{s}_l(\gamma) \rangle^2 \right)} = \frac{1}{4} \sum_{\gamma \in \Gamma(\mathbf{k}^*)} \frac{(-1)^{K-1}}{\prod_{l=1}^{K-1} \langle \mathbf{k}^* - \mathbf{s}_l(\gamma) \rangle \langle \mathbf{s}_l(\gamma) \rangle},$$

which has nonvanishing denominators for all irrational frequency vectors ω. The latter equation has as a nontrivial solution

$$x_K(\tau) = -2\mathrm{i}\tau \langle \mathbf{k}^* \rangle F(\omega, \mathbf{k}^*) z^{\mathbf{k}^*},$$

provided that $F(\omega, \mathbf{k}^*) \neq 0$, and this solution is clearly not quasi-periodic.

Next we proceed with the other fundamental solution $x^{(2)}$ of the equation (IV.31), starting with the zero order term $z^{\mathbf{k}^*}$. However it is better to study that solution via the conjugate equation

$$\ddot{x} + \left(\langle \mathbf{k}^* \rangle^2 + \mu \sum_{j=1}^{d} z_j^{-2} \right) x = 0. \tag{IV.32}$$

This leads to a recursive process as before for obtaining the coefficients of the expansion of $x^{(+)}$ in terms of μ. Now, taking $x^{(\pm)} = \frac{1}{2}\left(x^{(1)} \pm x^{(2)} \right)$ as fundamental solutions, we get the following equation

$$\ddot{x} + \left(\langle \mathbf{k} \rangle^2 + \mu \sum_{j=1}^{d} \left(z_j^2 + z_j^{-2} \right) \right) x = 0 \tag{IV.33}$$

which, undoing the changes in τ and μ, can be transformed into (IV.29).

In this way we have found two linearly independent solutions x^+ and x^- of this system, the expansion of which in powers of b have quasi-periodic coefficients in time up to order $K-1$ and where the Kth order coefficient is a function of the form $tz^{\mathbf{k}^*}$ times $F(\omega, \mathbf{k}^*)$. By comparison of coefficients it follows that $F(\omega, \mathbf{k}^*)$ and $R(\omega, \mathbf{k}^*)$ are identical except for a nonzero factor.

Note that $F(\omega, \mathbf{k}^*)$ is a rational function. We denote its numerator by $N(\omega, \mathbf{k}^*)$ and its denominator by $D(\omega, \mathbf{k}^*)$. Define $\mathcal{A}(\mathbf{k}^*)$ as the set of ω's for which $N(\omega, \mathbf{k}^*)$ is nonzero. We claim that $\mathcal{A}(\mathbf{k}^*)$ has measure zero, which follows from the fact that $N(\cdot, \mathbf{k}^*)$ is not identically zero. To check this first note that if $\omega = (1, \dots, 1)^T$, then $D(\omega, \mathbf{k}^*)$ does not vanish. Second we resort to the periodic case [BS00], noting that the equation now can be transformed to the classical Mathieu equation. It thereby follows that $N(\omega, \mathbf{k}^*)$ is nonzero for this value and, hence that $N(\cdot, \mathbf{k}^*)$ is not identically zero for any $\mathbf{k}^*$. Therefore the set $\mathcal{A}(\mathbf{k}^*)$, given by the zeroes of $N(\omega, \mathbf{k}^*)$, is a zero measure set and the theorem follows. □

Summarizing, the tongue boundaries at the $\mathbf{k}^*$th resonance, up to order $|\mathbf{k}^*|$, are given by the equation

$$\mathrm{coef}_1 = \pm 2\, \mathrm{coef}_2$$

which, in terms of $\hat{\alpha}$, $\hat{b}$, ε and c becomes

$$\hat{\alpha} + r_1(\hat{\alpha}, \hat{b}, \varepsilon, c) = \pm 2 \left(\varepsilon \hat{b} f_2(\hat{\alpha}, \hat{b}, \varepsilon, c) + \hat{b}^{|\mathbf{k}^*|} r_2(\hat{\alpha}, \hat{b}, \varepsilon, c) \right).$$

The tongue boundary crossings up to order $|\mathbf{k}^*|$ correspond to

$$\text{coef}_2 = 0$$

and a further analysis requires to distinguish between the cases of even and odd K.

When K is even, then for any $0 < |\varepsilon| \ll 1$ there is a pocket ending at $b = 0$ and at

$$b_{\text{tip}} = \left(\frac{-\varepsilon f_2(0,0,0,c)}{c^{\mathbf{k}^*} R_2(\omega, \mathbf{k}^*)} \right)^{\frac{1}{K-1}} + \cdots ,$$

where the dots denote higher order terms in ε. If K is odd then the sign of ε must be selected such that

$$\frac{-\varepsilon f_2(0,0,0,c)}{c^{\mathbf{k}^*} R_2(\omega, \mathbf{k}^*)}$$

is positive. If this is the case, then there are two instability pockets with ends at $b = 0$ and at

$$b_{\text{tip}}^{\pm} = \pm \left(\frac{-\varepsilon f_2(0,0,0,c)}{c^{\mathbf{k}^*} R_2(\omega, \mathbf{k}^*)} \right)^{\frac{1}{K-1}} + \cdots .$$

IV.7 Structure of the sets $\mathcal{A}(\mathbf{k})$

An interesting question related to Theorem IV.14 is whether the set of strongly irrational frequency vectors in $\mathcal{A}(\mathbf{k})$, for a fixed resonance $\mathbf{k}$, is empty or not.

When $d = 2$, we can always assume that $\omega = (1, \gamma)$, where γ is a real number. Note that any real irrational γ for which $N(\omega, \mathbf{k}) = 0$ for some $\mathbf{k}$, is strongly irrational, since it is algebraic. Direct computations, performed on $F(\omega, \mathbf{k})$, yield that if the order of the resonance is less than 5, all the roots of $N((1,\gamma), \mathbf{k})$ are either rational or complex (i.e., nonreal). However, for $\mathbf{k} = (3, 2)$,

$$N\left((1,\gamma),(3,2)\right) = 24 + 172\gamma + 454\gamma^2 + 505\gamma^3 + 232\gamma^4 + 49\gamma^5 + 4\gamma^6$$

which has real irrational zeroes. Direct computation also shows that the same happens for all resonances $6 \leq |\mathbf{k}| \leq 9$ with $k_1 \neq 1$, $k_2 \neq 1$. For $d \geq 3$ the situation is even simpler, since for $\mathbf{k}^* = (1,1,1)$ the polynomial $N(\omega, \mathbf{k}^*)$ has real strongly irrational zeroes.

There is one case when $|\mathbf{k}^*|$th order tangency at the $\mathbf{k}^*$th resonance always can be granted:

Proposition IV.29. *In the Mathieu equation with quasi-periodic forcing*

$$x'' + (a + b(\cos(t) + \cos(\gamma t)))\, x = 0, \tag{IV.34}$$

where $2\gamma \neq 0$ is not a negative integer, the order of tangency at $b = 0$ of the resonance tongue boundaries corresponding to $\mathbf{k}^* = (K, 1)$, for any K, exactly is $|\mathbf{k}^*| = K + 1$.

Proof: In this case the number of paths of minimal length in $\mathbb{Z}^2$ joining $(0,0)$ and $\mathbf{k}^*$ exactly is $K+1$ and any of these can be labelled by an integer between 0 and K. These paths will be denoted by $\sigma_0, \ldots, \sigma_K$. To show that the order of tangency is exactly $K+1$ we must compute $F((1,\gamma),(K,1)) =: f(\gamma,K)$, which amounts to

$$f(\gamma,K) = \frac{(-1)^K}{4} \sum_{j=0}^{K} \frac{1}{\prod_{l=1}^{K} \langle \mathbf{k}^* - \mathbf{s}_l(\sigma_j)\rangle \langle \mathbf{s}_l(\sigma_j)\rangle} \tag{IV.35}$$

and show that for $\gamma \notin \mathbb{Z}$, this does not vanish. For each of the paths σ_j, $j = 0, \ldots, K$, let α_j be the contribution to the sum in (IV.35). Then

$$\alpha_j = \frac{1}{(K-1+\gamma)1} \frac{1}{(K-2+\gamma)2} \cdots \frac{1}{(K-j+\gamma)j} \frac{1}{(K-j)(j+\gamma)} \cdots \frac{1}{1(K-1+\gamma)},$$

where the total number of factors is K. Using the Gamma-function it follows that

$$\sum_{j=0}^{K} \alpha_j = \sum_{j=0}^{K} \frac{\Gamma(K-j+\gamma)\Gamma(j+\gamma)}{j!(K-j)!\Gamma(K+\gamma)\Gamma(K+\gamma)}.$$

Since the denominator of $f(\gamma,K)$ is

$$d(\gamma,K) = D((1,\gamma),(K,1)) = (K-1+\gamma)1 \cdot \ldots \cdot (\gamma+1)(K-1) \cdot (\gamma)(K)$$

it is clear that

$$d(\gamma,K) = \frac{\Gamma(K+\gamma)K!}{\Gamma(\gamma)}$$

implying that

$$d(\gamma,K) \sum_{j=0}^{K} \alpha_j = \sum_{j=0}^{K} \binom{K}{j} \frac{\Gamma(K-j+\gamma)\Gamma(j+\gamma)}{\Gamma(K+\gamma)\Gamma(\gamma)} = \frac{1}{\Gamma(K+\gamma)\Gamma(\gamma)} \frac{\Gamma(K+2\gamma)\Gamma(\gamma)^2}{\Gamma(2\gamma)},$$

where the last identity is an application of Pochhammer's formula, see [WW62]. Therefore, the relevant coefficient is

$$f(\gamma,K) = \frac{(-1)^K}{4} \sum_{j=0}^{K} \alpha_j = \frac{1}{D} \frac{\Gamma(\gamma)^2}{\Gamma(K+\gamma)\Gamma(\gamma)\Gamma(2\gamma)} \Gamma(K+2\gamma)$$

which, if 2γ is not a negative integer, is different from zero. $\square$

Chapter V

Analytic families of reducible quasi-periodic equations

In this chapter we deal with linear equations in a certain matrix Lie algebra with quasi-periodic coefficients and depending on external parameters. This is motivated by the question raised in the previous chapter on the analyticity of tongue boundaries of Hill's equation with quasi-periodic forcing. We will consider systems of the form

$$x' = (A_0 + P(\theta, \mu))\, x, \qquad \theta' = \omega, \tag{V.1}$$

where $A_0 \in g$ is a constant matrix, $P = P(\theta, \mu)$ belongs to g, a matrix Lie algebra (see Chapter II) and it is real analytic.

Equation (V.1) is a perturbation of a system with constant coefficients if $P(\theta, \mu)$ is small. For this kind of systems and several different contexts, one has that (V.1) it is reducible to constant coefficients for almost all values of μ provided some generic conditions are met (see Eliasson [Eli92] for the case of $sl(2, \mathbb{R})$ and Krikorian [Kri99b] for compact Lie algebras). This seems to be also the situation for general analytic quasi-periodic perturbations of systems with constant coefficients (see Jorba & Simó [JS92, JS96] for results in positive measure). We would like to stress that, even in the cases where almost everywhere reducibility holds, there exist generic sets of μ for which reducibility does not hold (see Eliasson [Eli92, Eli02a]).

Even if a system like (V.1) is reducible to constant coefficients, the Floquet matrix will not be A_0 again. One can try, however, to modify (V.5) in a way such that the perturbed system is reducible with Floquet matrix A_0. This is an old idea going back to Moser [Mos67] (see Remark V.5). We will try to obtain a real analytic matrix function $\xi^* \in g$ such that

$$x' = (A_0 + P(\theta, \mu) - \xi^*(\mu))\, x, \qquad \theta' = \omega, \tag{V.2}$$

is reducible to the constant-coefficients system

$$y' = A_0 y, \qquad \theta' = \omega \tag{V.3}$$

by means of a transformation

$$x = \exp\left(X(\theta, \mu)\right) y, \tag{V.4}$$

where $X \in g$ is real analytic in both θ and μ. If we succeed in doing so then the equation $\xi^*(\mu) = 0$ will determine an analytic family of systems of (V.1) which are reducible to (V.3).

This allows us to study the problem of the persistence and analyticity of these families. To achieve our goal we will have to impose analyticity to the original system in both θ and μ, smallness of P and some arithmetic properties on the eigenvalues of A_0 and the frequencies ω.

One of our main motivations is to be able to detect bifurcations of the Floquet matrices of reducible systems. For example, think of Hill's equation with quasi-periodic forcing when b parameterizes some tongue boundary $b \mapsto a(b)$, with $|b|$ small. When $b = 0$ the Floquet matrix is identically zero but when $b \neq 0$ the reducible systems that we are looking for have a nilpotent, but not zero, Floquet matrix. Therefore, we will introduce a scaling function $\chi = \chi(\mu)$, also real analytic, a suitable order $k \in \mathbb{N}$ of scaling and consider the system

$$x' = \chi(\mu)^k \left(A_0 + \chi(\mu) P(\theta, \mu)\right) x, \qquad \theta' = \omega. \tag{V.5}$$

Under some additional hypothesis on A_0, we will show that a suitable modification of (V.5)

$$x' = \chi(\mu)^k \left(A_0 + \chi(\mu) P(\theta, \mu) - \chi(\mu)\xi^*(\mu)\right) x, \qquad \theta' = \omega$$

is reducible to constant coefficients with Floquet matrix $\chi(\mu)^k A_0$, where both ξ^* and the transformation depend analytically on μ. The treatment of this scaled case is postponed to Section V.1.1.

Remark V.1. *The requirement that the Floquet matrix of (V.2) is $\chi(\mu)^k A_0$ again is imposed not to destroy good Diophantine conditions.*

V.1 Formulation of the main result

Let us first formulate the main result without the scaling parameter χ. The reducibility of the modified system (V.2) to (V.3) by means of the transformation (V.4) requires that $Z = \exp(X)$ satisfies the homological equation

$$\partial_\omega Z(\theta, \mu) = \left(A_0 + P(\theta, \mu) - \xi^*(\mu)\right) Z, -Z\chi(\mu) A_0, \tag{V.6}$$

where the unknowns are X and ξ^*. If we try to solve the homological equation (V.6) by means of (modified) Newton's quadratic method we obtain the linear version (in X) of (V.6), namely

$$\partial_\omega X(\theta, \mu) = [A_0, X] + P(\theta, \mu) - \xi^*(\mu). \tag{V.7}$$

Without imposing extra conditions, this equation needs not to have a solution (even formally) and even if there is such a solution it may not be unique. Besides, considering different choices of ξ^* may be interesting in different contexts. As we want the convergence issues to be separated from the formal (algebraic) aspects, we will assume that equation (V.7) is solvable in the following way.

Definition V.2. *Given a matrix Lie algebra, g, a quartet (A_0, C, S, ω) is said to be* admissible *if $A_0 \in g$, $C, S : g \to g$ are linear operators with $C^2 = C$ and there exist positive constants c, ν such that, for all real analytic $P \in C^a_\rho(\mathbb{T}^d, g)$ the equations*

$$\partial_\omega X(\theta) = [A_0, X(\theta)] + P(\theta) - C(\bar{P}), \qquad \bar{X} = S(\bar{P}), \tag{V.8}$$

where the bar denotes the average of a quasi-periodic function, have a unique real analytic solution $X : \mathbb{T}^d \to g$ *which satisfies the estimates*

$$|X|_{\rho-\delta} \leq c\frac{|P|_\rho}{\delta^\nu} \tag{V.9}$$

for all $0 < \delta < \rho$*.*

The main result reads now as follows.

Theorem V.3. *Let* $g \subset gl(n, \mathbb{R})$ *be a matrix Lie algebra,* (A_0, C, S, ω) *an admissible quartet, with positive constants* c, ν*, and* ρ_0 *a positive number. Then there exists a constant* $\varepsilon = \varepsilon(\rho_0, c, \nu, |A_0|) > 0$ *such that for any real analytic matrix-function* $P : \mathbb{T}^d \to g$ *such that*

$$|P|_{\rho_0} \leq \varepsilon$$

there exists a $\xi^* \in g$*, with* $|\xi^*| \leq 2\varepsilon$ *and* $\xi^* = C(\xi^*)$*, such that the modified system*

$$x' = (A_0 + P(\theta) - \xi^*)\, x, \qquad \theta' = \omega \tag{V.10}$$

is reducible to the constant-coefficients system

$$y' = A_0 y, \qquad \theta' = \omega \tag{V.11}$$

by means of a transformation $x = Z(\theta)y$*, of the form* $Z = \exp(X)$*, where* $X : \mathbb{T}^d \to g$ *is real analytic and*

$$|X|_{\rho_0/2} \leq \sqrt{\varepsilon}.$$

Moreover, if P *depends real analytically on* $\mu \in \mathbb{R}^p$ *in a certain ball around the origin then both* X *and* ξ^* *depend real analytically on* μ *in a narrower ball.*

A more convenient version of the previous theorem for some applications will be stated in Section V.1.1. The proof of both theorems will be given in Section V.5.

Remark V.4. *The modifying term* $\xi^*(\mu)$ *will also be called* counterterm.

Remark V.5. *This result is a reformulation of Moser* [Mos67] *who introduced the counterterm. The adaption to the linear case was given by Bogoljubov, Mitropoliskii & Samoĭlenko* [BMS76] *and Katok* [Kat70]*, where the case of Lie algebras was also considered. For a similar result in the discrete and smooth context, see Krikorian* [Kri99a]*, who used techniques of nonlinear functional analysis.*

V.1.1 On admissible (A_0, C, S, ω)

Assume that $A_0 \in g$ and $\omega \in \mathbb{R}^d$, rationally independent, are fixed. One would like to have an effective method to determine operators $C, S : g \to g$ such that the quartet (A_0, C, S, ω) is admissible.

A criterion of this kind requires two conditions: one algebraic (which allows to compute a formal solution of this problem) and another Diophantine (so that the previous formal solution is an actual one).

Let us try to solve formally equation (V.8) in terms of the Fourier coefficients of $P : \mathbb{T}^d \to g$. Writing

$$X(\theta) = \sum_{\mathbf{k}\in\mathbb{Z}^d} X_{\mathbf{k}} e^{\mathrm{i}\langle \mathbf{k},\theta\rangle}, \qquad P(\theta) = \sum_{\mathbf{k}\in\mathbb{Z}^d} P_{\mathbf{k}} e^{\mathrm{i}\langle \mathbf{k},\theta\rangle}$$

and equating the Fourier coefficients in (V.8) one obtains

$$\mathrm{i}\langle \mathbf{k},\omega\rangle X_{\mathbf{k}} = [A_0, X_{\mathbf{k}}] + P_{\mathbf{k}} \tag{V.12}$$

for $\mathbf{k} \neq \mathbf{0}$ and

$$0 = [A_0, X_{\mathbf{0}}] + P_{\mathbf{0}} - C(P_{\mathbf{0}}) \tag{V.13}$$

for $\mathbf{k} = \mathbf{0}$. Thus the only condition on the operators S and C is that

$$0 = [A_0, S(P_{\mathbf{0}})] + P_{\mathbf{0}} - C(P_{\mathbf{0}}) \tag{V.14}$$

must hold for all $P_{\mathbf{0}} \in g$. This can bee understood at a more geometrical level making use of the adjoint operator (see Chapter II), which is the following linear operator on g:

$$\begin{array}{rccc} \mathrm{ad}_{A_0} : & g & \to & g \\ & X & \mapsto & [A_0, X], \end{array}$$

In terms of this operator, equation (V.14) holds for all $P_{\mathbf{0}} \in g$ if, and only if, the operator $C - \mathrm{ad}_{A_0} \circ S$ is the identity on g. To solve the equations for the remaining Fourier coefficients (V.12) one needs that $\mathrm{i}\langle \mathbf{k},\omega\rangle I - \mathrm{ad}_{A_0}$ is an invertible operator for all $\mathbf{k} \in \mathbb{Z}^d - \{\mathbf{0}\}$. Thus we meet the required condition on rational independence for the formal solution:

$$\lambda - \mathrm{i}\langle \mathbf{k},\omega\rangle \neq 0 \tag{V.15}$$

for all

$$\lambda \in \mathrm{Spec}\,(\mathrm{ad}_{A_0})$$

and $\mathbf{k} \neq \mathbf{0}$. Note that the eigenvalues of ad_{A_0} will be of the form $\lambda' - \lambda''$ for λ', λ'' in the spectrum of A_0.

If we want this formal solution to be an actual solution of the homological equation (V.8) one needs to strengthen this nonresonance condition to have a good control of the small divisors. This is very similar to the techniques in Section II.2.2 and it is summarized in the following lemma.

Lemma V.6. *Assume that $A_0 \in g$, $\omega \in \mathbb{R}^d$ and that there exist linear operators $C, S : g \to g$, with $C^2 = C$ such that $C - ad_{A_0} \circ S$ is the identity on g. Assume that there exist positive constants τ, K such that the following Diophantine condition*

$$\inf_{\lambda\in\mathrm{Spec}\left(\mathrm{ad}_{A_0}\right)} |\lambda - \mathrm{i}\langle \mathbf{k},\omega\rangle| \geq \frac{K}{|\mathbf{k}|^\tau}, \qquad \mathbf{k} \in \mathbb{Z}^d,\ \mathbf{k} \neq \mathbf{0} \tag{V.16}$$

is satisfied. Then the quartet (A_0, C, S, ω) is admissible.

To prove the lemma, it suffices to represent equations (V.12) and (V.13) in terms of a basis of the Lie algebra g and then use the standard Diophantine conditions.

Example V.7. *If $A_0 \in g$ has all eigenvalues equal and ω is strongly rationally independent, that is, there exist positive constants K and τ such that the frequency vector ω satisfies*

$$|\langle \mathbf{k}, \omega \rangle| \geq \frac{K}{|\mathbf{k}|^\tau}, \qquad \mathbf{k} \in \mathbb{Z}^d,\ \mathbf{k} \neq \mathbf{0},$$

then, choosing C to be the identity and S to be zero, the quartet (A_0, C, S, ω) is admissible.

Since the counter-term $\xi^*(\mu)$ in Theorem V.3 satisfies that $C(\xi^*) = \xi^*$ and the persistence of a family of reducible quasi-periodic systems of (V.5) with Floquet matrix A_0 requires the conditions $\xi^*(\mu) = 0$, it is important that the dimension of the image of C in g, which we denote as L_C, is as small as we can. Note that in Example V.7, the dimension of this space is not necessarily minimal, since the special properties of A_0 are not used. The following lemma gives a condition of this kind.

Lemma V.8. *Let $A_0 \in g$ and ω satisfy the Diophantine conditions (V.16). Then the minimal dimension of L_C in g is the dimension of g as a subspace of $gl(n, \mathbb{R})$ minus the dimension of the image of ad_{A_0} in g.*

Proof: Since

$$C - \mathrm{ad}_{A_0} \circ S = I,$$

it is clear that $\dim L_C \geq \dim \ker \mathrm{ad}_{A_0}$. If L_C is chosen to be exactly $\ker \mathrm{ad}_{A_0}$ and S so that $\mathrm{ad}_{A_0} \circ S = \mathrm{ad}_{A_0}$ the optimal bound is attained. □

Remark V.9. *Once L_C is chosen to be the kernel of ad_{A_0}, the operator S is any linear operator satisfying $ad_{A_0} \circ S = ad_{A_0}$.*

In particular, if $\mathrm{Spec}\,(\mathrm{ad}_{A_0}) = \{0\}$ and ω is strongly rationally independent, there exist choices of C and S such that the quartet $(\chi^k A_0, C, S, \omega)$ is admissible for all values of the parameter χ in $\mathbb{R}$. One can use this uniformity in χ to obtain the following theorem:

Theorem V.10. *Let $g \subset gl(n, \mathbb{R})$ be a matrix Lie algebra, let (A_0, C, S, ω) be admissible, with positive constants c, ν and such that Spec $\mathrm{ad}_{A_0} = \{0\}$. Let ρ_0 be a positive number. Then there exists a positive constant $\varepsilon = \varepsilon(\rho_0, c, \nu, |A_0|)$ such that for any real analytic matrix-function $P : \mathbb{T}^d \to g$ such that*

$$|P|_{\rho_0} \leq \varepsilon$$

and any $|\chi| \leq 1$, there exists a $\xi^* \in g$, with $\xi^* = C(\xi^*)$, such that the modified system

$$x' = \chi^k \left(A_0 + \chi P(\theta) - \chi \xi^*\right) x, \qquad \theta' = \omega \tag{V.17}$$

is reducible to the constant-coefficients system

$$y' = \chi^k A_0 y, \qquad \theta' = \omega \tag{V.18}$$

by means of a transformation $x = Z(\theta) y$, of the form $Z = \exp(\chi X)$, where $X : \mathbb{T}^d \to g$ is real analytic and

$$|X|_{\rho_0/2} \leq \sqrt{\varepsilon}.$$

Moreover, if P and χ depend real analytically on $\mu \in \mathbb{R}^p$ in a certain ball around the origin then both X and ξ^* depend real analytically on μ in a narrower ball.

An example, which will be used in the following section, is the following:

Example V.11. Let $g = sp(1,\mathbb{R}) = sl(2,\mathbb{R})$ and A_0 be the nilpotent matrix

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

In the algebra $sl(2,\mathbb{R})$ we can consider the basis formed by the elements

$$X_{11} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad X_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \text{and} \qquad X_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

and let $(x_{11}, x_{12}, x_{21})^T$ be the coordinates of an element $X \in g$ with respect to this basis. Since

$$[A_0, X] = \begin{pmatrix} x_{21} & -2x_{11} \\ 0 & -x_{21} \end{pmatrix},$$

the spectrum of $\mathrm{ad}_{A_0} : g \to g$ reduces to the zero eigenvalue with multiplicity three and its kernel is the linear subspace of g spanned by X_{12}. We can choose C, in the above coordinates, as

$$c_{11}(\xi) = 0, \qquad c_{12}(\xi) = 0, \qquad c_{21}(\xi) = \xi_{21}$$

and, for example,

$$s_{11}(\xi) = \frac{-1}{2}\xi_{12}, \qquad s_{12}(\xi) = 0, \qquad s_{21}(\xi) = \xi_{11}.$$

With these definitions (A_0, C, S, ω) is admissible.

V.1.2 Outline

Before ending the introduction, let us briefly outline the contents of the present chapter. In sections V.2, V.3 and V.4 we present some applications of theorems V.3 and V.10. More specifically, in Section V.2 we deal with Hill's equation with quasi-periodic forcing and we prove that resonance tongue boundaries are analytic functions of the perturbing parameter. This has applications to the genericity of "having all gaps open" (and in particular Cantor spectrum) for quasi-periodic Schrödinger operators.

In Section V.3 linear equations with quasi-periodic coefficients in $so(3,\mathbb{R})$ are considered. To end the applications we show how the results of Section V.2, together with some arguments of Sacker-Sell spectral theory can be used to study hyperbolicity boundaries of Hamiltonian systems in higher dimensions. This is done in Section V.4.

The proofs of theorems V.3 and V.10 are postponed to Section V.5, where a classical KAM scheme is presented. These proofs will be given in a unified way. Finally, we include two sections, V.6 and V.7, which deal with generalizations of Theorem V.3 to the context of multiple resonances and the presence of a time-reversing symmetry respectively.
Most of the results in this chapter will appear in Puig & Simó [PS03].

V.2 Analyticity of tongue boundaries in quasi-periodic Hill's equation and applications

In the proof of the C^∞ character of resonance tongues, Theorem IV.3, it was essential to have reducibility of the associated skew-product flow on $\mathbb{R}^2 \times \mathbb{T}^d$ for values of (a, b) at tongue boundaries which, by a theorem of Eliasson [Eli92], holds under the current hypothesis, that is real analyticity of Q and a strong rational independence on ω.

Theorem V.3 can be used to show that tongue boundaries are analytic for $|b|$ small enough. This is the contents of the following result.

Theorem V.12. *Consider Hill's equation with quasi-periodic forcing*

$$x'' + (a + bQ(\omega t))\, x = 0 \tag{V.19}$$

being the function $Q : \mathbb{T}^d \to \mathbb{R}$ real analytic and the frequency vector ω, strongly rationally independent.

Assume that for $(a_0, b_0) \in \mathbb{R}^2$ the associated skew-product is reducible to constant coefficients and it is at a tongue boundary. Then

(i) If a_0 is at the end of a noncollapsed gap of $\sigma_{b_0 V,\omega}$, the tongue boundary $a = a(b)$ such that $a(b_0) = a_0$ is real analytic in a neighbourhood of b_0 and for $(a, b) = (a(b), b)$, the skew-product is reducible to constant coefficients.

(ii) If a_0 is a collapsed gap of $\sigma_{b_0 V,\omega}$, the two tongue boundaries $a_i = a_i(b)$ for $i = 1, 2$, with $a_i(b_0) = a_0$, are real analytic functions in a neighbourhood of b_0. Moreover, for $(a, b) = (a_i(b), b)$, $i = 1, 2$, the skew-product is reducible to constant coefficients.

In both cases the reducing transformations depend real analytically on both θ and b.

Since for $b = 0$ the skew-product associated to Hill's equation is always reducible to constant coefficients (it is already in this form) one has the following consequence.

Theorem V.13. *If the potential Q is analytic and ω strongly rationally independent, every tongue boundary is an analytic function of b in a neighbourhood of $b = 0$.*

V.2.1 Proof of Theorem V.12

To prove Theorem V.12 we will have to distinguish between collapsed and noncollapsed gaps at some point. However, both cases have the passage to a perturbative situation as a common starting point.

Fix (a_0, b_0) as in Theorem V.12. By hypothesis the skew-product is reducible to constant coefficients, whose Floquet matrix we denote by A_0. This matrix belongs to $sl(2, \mathbb{R})$ (since our setting is Hamiltonian) and satisfies that $A_0^2 = 0$, because a_0 is at the endpoint of a spectral gap. Moreover the gap is collapsed if, and only if, $A_0 = 0$.

Let $R : \mathbb{T}^d \to G$ be a real analytic reducing transformation for $(a, b) = (a_0, b_0)$ given by the hypothesis. After this transformation, the skew-product becomes

$$y' = \left(A_0 + (a - a_0 + (b - b_0)Q(\theta)) \begin{pmatrix} r_{11}r_{12} & r_{12}^2 \\ -r_{11}^2 & -r_{11}r_{12} \end{pmatrix} \right) y, \qquad \theta' = \omega, \tag{V.20}$$

where the r_{ij} are the components of R. We introduce now $\mu = (\alpha, \beta)$, where $\alpha = a - a_0$ and $\beta = b - b_0$, as the new local perturbation parameters. We denote as $P(\theta, \mu)$ the time dependent part of (V.20), that is,

$$P(\theta, \mu) = (\alpha + \beta Q(\theta)) \begin{pmatrix} r_{11}r_{12} & r_{12}^2 \\ -r_{11}^2 & -r_{11}r_{12} \end{pmatrix}. \tag{V.21}$$

Let us now distinguish between collapsed and noncollapsed gaps.

Non-collapsed gap

In this case, A_0 satisfies $A_0^2 = 0$ but $A_0 \neq 0$. After performing a change of basis if necessary, we may assume that

$$A_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Using Example V.11 the choices

$$C \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & -p_{11} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ p_{21} & 0 \end{pmatrix}.$$

and

$$S \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & -p_{11} \end{pmatrix} = \begin{pmatrix} -p_{12}/2 & 0 \\ p_{11} & p_{12}/2 \end{pmatrix}.$$

make (A_0, C, S, ω) admissible, since ω is strongly rationally independent. Thus, by Theorem V.3, there exist a real analytic function $\xi_{21}^*(\mu)$, defined in a neighbourhood of the origin, and a real analytic $X = X(\theta, \mu) \in g$ such that $Z(\theta, \mu) = \exp(X(\theta, \mu))$ satisfies

$$\partial_\omega Z = (A_0 + P(\theta, \mu) - \xi^*(\mu))\, Z - Z A_0, \tag{V.22}$$

where

$$\xi^*(\mu) = \begin{pmatrix} 0 & 0 \\ \xi_{21}^*(\mu) & 0 \end{pmatrix}.$$

Therefore, for the values of μ such that $\xi_{21}^*(\mu) = 0$, system (V.20) (and thus also the original skew-product for parameters $(a, b) = (a_0, b_0) + \mu$) is analytically reducible to a constant-coefficients system with Floquet matrix A_0.

Hence, to prove item (i), we only need to show that the equation

$$\xi_{21}^*(\alpha, \beta) = 0, \tag{V.23}$$

can be inverted to obtain an analytic function $\alpha = \alpha(\beta)$. Note that, passing to averages in equation (V.22) one sees that

$$\xi_{21}^*(\alpha, \beta) = -\alpha(\overline{r_{11}^2}) - \beta(\overline{Q r_{11}^2}) + O_2(\mu),$$

where $O_2(\mu)$ collects terms of order greater than one in μ. Since $\overline{r_{11}^2} \neq 0$ (because r_{11} is a nontrivial quasi-periodic solution of Hill's equation), the Implicit Function Theorem yields a real analytic function $\alpha = \alpha(\beta)$ for which (V.23) holds.

Collapsed gap

As we have said before, in this case the Floquet matrix is $A_0 = 0$. Using Example V.7 and since ω is strongly rationally independent, the choice $C = Id$ and $S = 0$ makes (A_0, C, S, ω) admissible so that we can apply Theorem V.3 to obtain real analytic functions $\xi^*_{11}(\mu)$, $\xi^*_{12}(\mu)$ and $\xi^*_{21}(\mu)$, defined in a neighbourhood of the origin, and a real analytic $X = X(\theta, \mu) \in g$ such that $Z(\theta, \mu) = \exp(X(\theta, \mu))$ satisfies

$$\partial_\omega Z = (P(\theta, \mu) - \xi^*(\mu))\, Z, \tag{V.24}$$

where P is defined as (V.21) and

$$\xi^*(\mu) = \begin{pmatrix} \xi^*_{11}(\mu) & \xi^*_{12}(\mu) \\ \xi^*_{21}(\mu) & -\xi^*_{11}(\mu) \end{pmatrix}.$$

As it will be seen in Section V.7, if q is even, $q(t) = q(-t)$ for all $t \in \mathbb{R}$, then $\xi_{11} \equiv 0$. Most of the considerations in the present section are simpler in this reversible setting.

We want to find two functions $\alpha_1(\beta)$ and $\alpha_2(\beta)$ such that system (V.20) is reducible to a Floquet matrix $B(\beta)$ satisfying $B^2 = 0$ if $\alpha = \alpha_{1,2}(\beta)$. In Chapter IV it was shown that these α_i were C^∞ functions. In principle, it could happen that these two boundaries have a C^∞-tangency but that they are not equal. First of all we shall rule out this possibility.

Note that the reducing transformation Z in (V.24) also defines a conjugation from the original unmodified system

$$\partial_\omega Z = P(\theta, \mu) Z - Z\left(Z^{-1}\xi^*(\mu) Z\right).$$

We will now study the analyticity of the boundaries of the resonance tongues of the system

$$x' = Z^{-1}\xi^*(\mu) Z x, \qquad \theta' = \omega. \tag{V.25}$$

This system has the property that for every fixed value of θ, the matrix S is similar to ξ^*, although the system is not necessarily conjugated to constant coefficients. In particular, the eigenvalues of S do not change with θ. For this system one has the following:

Lemma V.14. *If* $\det \xi^*(\mu) > 0$ *then the rotation number of the quasi-periodic system (V.25) is strictly different from zero.*

Proof: Going to polar coordinates, $\varphi = \arg(z_2 + \mathrm{i} z_1)$ the flow on $\mathbb{S}^1 \times \mathbb{T}^d$ is given by equations

$$\varphi' = -s_{21}(\theta)\sin^2\varphi + s_{12}(\theta)\cos^2\varphi + 2 s_{11}(\theta)\cos\varphi\sin\varphi, \qquad \theta' = \omega. \tag{V.26}$$

The right hand side is a quadratic form given by the matrix $-JS$. This last quadratic form is definite if, and only if, $\det S > 0$, which is equivalent to $\det \xi^*(\mu) > 0$. $\square$

Moreover, calculating the averages of (V.24) and keeping in mind the form of P, we can compute the first terms of $\xi^*(\mu)$:

$$\begin{aligned}
\xi^*_{12}(\alpha, \beta) &= \alpha(\overline{r^2_{12}}) + \beta(\overline{Q r^2_{12}}) + O_2(\mu), \\
\xi^*_{21}(\alpha, \beta) &= -\alpha(\overline{r^2_{11}}) - \beta(\overline{Q r^2_{12}}) + O_2(\mu), \\
\xi^*_{11}(\alpha, \beta) &= \alpha(\overline{r_{11} r_{12}}) + \beta(\overline{Q r_{11} r_{12}}) + O_2(\mu).
\end{aligned}$$

Since in the case of a collapsed gap the transformation R can always be chosen so that

$$\overline{r_{11}^2} = \overline{r_{12}^2} = 1, \qquad \text{and} \qquad \overline{r_{11} r_{12}} = 0,$$

the expression for the determinant of ξ^* becomes

$$\det \xi^*(\mu) = \alpha^2 + O(\alpha\beta, \beta^2, \alpha^3).$$

Using similar arguments to the proof of Lemma IV.20, when the polynomials appearing there are replaced by analytic functions with similar format, one has

$$\det \xi^*(\alpha, \beta) = F(\alpha, \beta) \left(\alpha - \alpha_1^*(\beta)\right) \left(\alpha - \alpha_2^*(\beta)\right),$$

where $\alpha_{1,2}^* = \alpha_{1,2}^*(\beta)$ are two real analytic functions and $F = F(\alpha, \beta)$, with $F(0,0) = 1$, is also real analytic. Since α_1^* and α_2^* are real analytic functions which both vanish at zero, either they coincide or they have a tangency of some order.

Let us assume first that they coincide, that is $\alpha_1^*(\beta) = \alpha_2^*(\beta)$ for all β. Using the continuity and the monotonicity in α of the rotation number, the rotation number of (V.25) is strictly positive if $\alpha > \alpha_1^*(\beta)$ and strictly negative if $\alpha < \alpha_1^*(\beta)$. Therefore, the two tongue boundaries coincide in a neighbourhood of zero and they are given by $\alpha_1^*(\beta)$.

If $\alpha_1^* \neq \alpha_2^*$, there exists an integer $p \geq 1$ and a constant $C \neq 0$ such that

$$\alpha_2^*(\beta) - \alpha_1^*(\beta) = C\beta^p + O_{k+1}(\beta).$$

We are going to see that this p is precisely the order of contact of the two tongue boundaries at $\beta = 0$ and that the latter are real analytic functions. Note that α_1^* and α_2^* need not to be the parameterization of the tongue boundaries.

After the changes in the parameters α and β from Section IV.4.2 we may assume that

$$\xi^*(\alpha, \beta) = \begin{pmatrix} S_3(\alpha, \beta) & S_2(\alpha, \beta) \\ -S_1(\alpha, \beta) & -S_3(\alpha, \beta) \end{pmatrix}$$

with

$$\begin{aligned} S_1(\alpha, \beta) &= \alpha + \sigma_1(\beta) + \alpha\rho_1(\alpha, \beta) \\ S_2(\alpha, \beta) &= \alpha + \sigma_2(\beta) + \alpha\rho_2(\alpha, \beta) \\ S_3(\alpha, \beta) &= \sigma_3(\beta) + \alpha\rho_3(\alpha, \beta) \end{aligned}$$

where

$$\sigma_j(\beta) = \sum_{k \geq p} m_{j,k}\beta^k, \quad j = 1, 2, 3, \qquad (m_{1,p} - m_{2,p})^2 + m_{3,p}^2 > 0$$

and the possible terms in S_1 and S_2 of degree less than p in β, which must be equal, are included inside α with a suitable redefinition of α.

The equation

$$\gamma^2 + (m_{1,p} + m_{2,p})\gamma + m_{1,p}m_{2,p} - m_{3,p}^2 = 0$$

has two different roots which we denote as γ_1 and γ_2. Taking one of these, for instance γ_1, we perform the change of variables $\alpha = \gamma_1\beta^p + \delta\beta^p$, which means that we restrict our study to a wedge around a boundary of the "unperturbed" problem of width $\delta\beta^p$, with δ small. In the new variables, δ and β, the matrix ξ^* becomes

$$\beta^p \left(\begin{pmatrix} m_{3,p} & (m_{1,p}+\gamma_1)+\delta \\ -(m_{2,p}+\gamma_1)-\delta & -m_{3,p} \end{pmatrix} + O(\beta) \right).$$

Therefore, the system (V.25) becomes

$$x' = \beta^p \left(\begin{pmatrix} m_{3,p} & (m_{1,p}+\gamma_1)+\delta \\ -(m_{2,p}+\gamma_1)-\delta & -m_{3,p} \end{pmatrix} + \beta P(\theta,\delta,\beta) \right) x, \qquad \theta' = \omega, \quad \text{(V.27)}$$

where we have used that X is of the order of β. The terms of order β^p in this expression will be written as

$$\begin{pmatrix} a & b+\delta \\ -c-\delta & -a \end{pmatrix}.$$

Due to the definition of δ and γ_1, $bc - a^2 = 0$. Also, b and c cannot be zero at the same time because in this case the order of contact of α_1^* and α_2^* would be greater than p. Let us assume that $b > 0$ (the other cases are treated similarly). This assumption means that the determinant of (V.27) is

$$\beta^{2p} \left(\delta(b+c+\delta) + O(\beta) \right).$$

If we fix a $\delta < 0$ then an application of Coppel's Criterion II.29 shows that (V.27) has an exponential dichotomy if $|\beta| > 0$ is small enough. As the same can be done for γ_2 this shows that the order of contact between the actual boundaries of the resonance tongue is exactly p. Therefore we are in the situation of Section IV.4.2 where, after some steps of normal form and suitable changes of variables in δ, we may assume that system (V.27) is of the form

$$x' = \beta^p \left(\begin{pmatrix} a & b+\delta \\ -c-\delta & -a \end{pmatrix} + \beta^2 P_1(\theta,\delta,\beta) \right) x, \qquad \theta' = \omega. \qquad \text{(V.28)}$$

The analytical (in α, β) conjugation given by

$$T = \begin{pmatrix} \sqrt{\frac{b+\delta}{b+c+\delta}} & 0 \\ \frac{-a}{\sqrt{(b+\delta)(b+c+\delta)}} & \sqrt{\frac{b+c+\delta}{b+\delta}} \end{pmatrix}$$

transforms (V.28) into

$$x' = \beta^p \left(\begin{pmatrix} 0 & b+c+\delta \\ -\delta & 0 \end{pmatrix} + \beta^2 Q_1(\theta,\delta,\beta) \right) x, \qquad \theta' = \omega,$$

where Q_1 is a new perturbation and the change given by

$$\begin{pmatrix} b+c+\delta & 0 \\ 0 & (b+c+\delta)^{-1} \end{pmatrix}$$

transforms it into

$$x' = \beta^p \left(\begin{pmatrix} 0 & 1 \\ -\delta(b+c+\delta) & 0 \end{pmatrix} + \beta^2 R_1(\theta,\delta,\beta) \right) x, \qquad \theta' = \omega, \qquad \text{(V.29)}$$

being R_1 a perturbation defined by the conjugation and Q_1.

Now we are in situation to apply Theorem V.10 to the system

$$x' = \beta^p \left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \beta^2 R_1(\theta, \delta, \beta) \right) x, \qquad \theta' = \omega. \tag{V.30}$$

This yields the existence of a real analytic function $\xi_{21} = \xi_{21}(\delta, \beta)$ such that

$$x' = \beta^p \left(\begin{pmatrix} 0 & 1 \\ -\beta\xi_{21}(\delta, \beta) & 0 \end{pmatrix} + \beta^2 R_1(\theta, \delta, \beta) \right) x, \qquad \theta' = \omega$$

is reducible to

$$x' = \beta^p \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x, \qquad \theta' = \omega, \tag{V.31}$$

for $|\delta|, |\beta|$ small enough. Clearly, one also has that the counterterm

$$\begin{pmatrix} 0 & 0 \\ -\delta(b + c + \delta) + \beta\xi_{21}(\delta, \beta) & 0 \end{pmatrix}$$

makes (V.29) reducible to (V.31). Therefore the equation

$$\delta(b + c + \delta) - \beta\xi_{21}(\delta, \beta) = 0$$

determines one of the components of the boundary of the resonance tongue. Note that, since $b + c > 0$, this can be written as $\delta_1 = \delta_1(\beta) = O(\beta)$, so that the expression for this part of the tongue boundary is

$$\alpha_1(\beta) = \gamma_1\beta^p + \delta_1(\beta)\beta^p = \gamma_1\beta^p + O(\beta^{p+1})$$

as we wanted to see (the case of γ_2 is treated similarly). This shows the analyticity of tongue boundaries around a collapsed gap and finishes the proof of Theorem V.12. □

V.2.2 Applications to the spectrum of quasi-periodic Schrödinger operators

Theorem V.12 on the analyticity of tongue boundaries can be strengthened in conjunction with Eliasson's reducibility theorem III.27, which states reducibility at tongue boundaries under the hypothesis of analyticity of the potential and strongly rationally independence of the frequency ω. According to this theorem, under the hypothesis made on ω and Q, Hill's equation is reducible at the tongue boundaries for small values of $|b|$ and all the values of a, or a large enough once b has been fixed. The analyticity of tongue boundaries holds in this domain as a consequence of V.12.

Corollary V.15. *Let $Q : \mathbb{T}^d \to \mathbb{R}$ be real analytic and $\omega \in DC^c(c, \tau, \mathbb{R}^d)$ be strongly rationally independent. Then there is a constant $C > 0$, such that the tongue boundaries are real analytic if $|b| < C$.*

The analyticity of tongue boundaries in an open domain for (a,b) can be used to study the genericity of "having all gaps open" for a certain value of b. That is, to study the opening of all spectral gaps for a certain value of b. Let us recall that our function space will be, for some $\rho > 0$, the space $C^a_\rho(\mathbb{T}^d)$ of real analytic functions $Q : \mathbb{T}^d \to \mathbb{R}$ with analytic extension to $|\mathrm{Im}\,\theta| < \rho$ with

$$|Q|_\rho < \infty.$$

Theorem V.16. *Let $\omega \in DC^c(c,\tau,\mathbb{R}^d)$ be strongly rationally independent. Then, there exists a constant $C = C(c,\tau,r)$ such that for a generic potential in*

$$\left\{Q \in C^a_\rho(\mathbb{T}^d,\mathbb{R}),\ |Q|_\rho < C\right\},$$

with respect to the $|\cdot|_\rho$-topology, the operator

$$\left(H_{Q,\omega,\phi}x\right)(t) = -x''(t) + Q(\omega t + \phi)x(t)$$

has all spectral gaps open and, thus, it is a Cantor set.

This result answers a problem raised by Moser & Pöschel [MP84] asking if having all spectral gaps open is generic or, at least, having all spectral gaps open for energies a big enough. Under the same hypothesis of the Theorem, Eliasson [Eli92] already proved the genericity of Cantor spectrum. For more results on Cantor spectrum, see Section III.2.2.

The proof uses the lemma IV.13 from last Chapter.

Lemma V.17 ([BPS03]). *Let $Q : \mathbb{T}^d \to \mathbb{R}$ be a real analytic potential and ω be strongly rationally independent. Let $a_1(b)$ and $a_2(b)$ be the two (analytic) tongue boundaries in a neighbourhood of zero for some resonance $\mathbf{k}$. Then these functions can be chosen so that*

$$a_1'(0) = Q_0 + |Q_{\mathbf{k}}| \text{ and } a_2'(0) = Q_0 - |Q_{\mathbf{k}}|,$$

being $Q_{\mathbf{k}}$ the Fourier coefficients of Q and $a_i' = da_i/db$.

The proof of Theorem V.16 is then a consequence of the analyticity of the tongue boundaries when the quasi-periodic system is reducible to constant coefficients. Indeed, if the two tongue boundaries of a certain resonance have a transversality at the origin, then the set of values of $|b| \le C$ for which the two tongue boundaries merge is finite. Since there is a countable set of resonance tongues the result follows.

Remark V.18. *As it will be shown by Theorem V.20, the condition that all the tongues are transversal at $b = 0$ is not necessary. The only requirement is that these tongues have some order of transversality at $b = 0$.*

Using Eliasson's result in the upper part of the spectrum, one can also conclude genericity of "having all gaps open" for quasi-periodic Schrödinger operators at large energies:

Corollary V.19. *Fix a frequency $\omega \in DC^c(c,\tau,\mathbb{R}^d)$. Then, the spectrum of the Schrödinger operator of a generic potential in $C^a_\rho(\mathbb{T}^d)$ has always a component in which all spectral gaps are open. That is, there is a constant $R > 0$, depending only on c, τ and the norm of Q, such that the spectrum of the operator restricted to the interval $[R,+\infty)$ has all gaps open.*

Let us now sketch the proof. Let Q have all harmonics different from zero. By Corollary V.15, the tongue boundaries of

$$x'' + (a - bQ(\omega t))\, x = 0$$

are analytic if $a \geq \lambda_0(|b||Q|_\rho)$ (see Eliasson's Theorem III.27). Fix $b_0 > 0$ and let $R_1 > 0$ be such that $R_1 \geq \lambda_0(|b||Q|_\rho)$ for all $|b| \leq b_0$. This means that in the domain $[R_1, +\infty) \times [0, b_0]$ of the parameter plane, tongue boundaries are analytic. Assume that a tongue boundary lies in this domain for $|b| \leq b_0$. Since it is analytic there, then their crossings form a finite set at most.

Since tongue boundaries are globally Lipschitz functions of b with uniform Lipschitz constant (see Proposition IV.26), there is a $R \geq R_1$ such that any tongue emanating from any $a_0 \geq R$ at $b = 0$ satisfies that $a \geq R_1$ for $0 \leq b \leq b_0$. In particular, the spectrum of a generic potential in $[R, +\infty)$ has all gaps open.

Finally, using Theorem IV.14, one can also study the question of the opening of all gaps for a particular potential.

Theorem V.20. *Let $d \geq 2$. Then, there is an exceptional set $\mathcal{A} \subset \mathbb{R}^d$, of zero measure, such that if $\omega = (\omega_1, \ldots, \omega_d) \notin \mathcal{A}$, then, there is a constant $C = C(c, \tau)$ such that for all values of b, except for a countable set, with $|b| \leq C$, the spectrum of the operator*

$$Hx = -x'' + b \sum_{j=1}^{d} c_j \cos(\omega_j t) x,$$

where the constants c_j are all different from zero and satisfy the normalization $c_1^2 + \ldots + c_d^2$, has all gaps open.

V.3 Analytic families of reducible linear quasi-periodic systems in $so(3, \mathbb{R})$

In this section we consider the existence of analytic families of reducible linear quasi-periodic systems in $so(3, \mathbb{R})$, the algebra of all real antisymmetric matrices (hence with zero trace). Quasi-periodic linear equations in $so(3, \mathbb{R})$ have been studied by Eliasson [Eli02a], Krikorian [Kri99a] and Moshchevitin [Mos98]. Let us start reviewing some basic facts on the geometry of $so(3, \mathbb{R})$.

If we denote by J_1, J_2, J_3 the following Pauli matrices

$$J_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad J_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \qquad J_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

then (J_1, J_2, J_3) is a basis of $so(3, \mathbb{R})$, and for the Lie bracket the following relation holds

$$[J_1, J_2] = J_3, \tag{V.32}$$

together with the other two circular permutations of indices $1, 2, 3$. Using this basis, we can express the Lie bracket $[X, Y]$ for any $X, Y \in so(3, \mathbb{R})$. Indeed, assume that $\mathbf{e}, \mathbf{f} \in \mathbb{R}^3$ are such that

$$X = e_1 J_1 + e_2 J_2 + e_3 J_3 \qquad \text{and} \qquad Y = f_1 J_1 + f_2 J_2 + f_3 J_3.$$

Then it follows that

$$[X,Y] = \begin{vmatrix} e_2 & f_2 \\ e_3 & f_3 \end{vmatrix} J_1 - \begin{vmatrix} e_1 & f_1 \\ e_3 & f_3 \end{vmatrix} J_2 + \begin{vmatrix} e_1 & f_1 \\ e_2 & f_2 \end{vmatrix} J_3.$$

This expression yields an identification

$$v : (so(3,\mathbb{R}), [\cdot,\cdot]) \to (\mathbb{R}^3, \wedge)$$

where $\wedge$ is the exterior product by sending

$$v([X,Y]) = v(X) \wedge v(Y).$$

If $A_0 \in so(3,\mathbb{R}^3)$, then its eigenvalues are 0 and $\pm \mathrm{i}|v(A_0)|$, where the norm on $(\mathbb{R}^3, \wedge)$ is assumed to be the Euclidean one.

Consider now a linear equation with quasi-periodic coefficients in $so(3,\mathbb{R})$. This means that there is a map $A : \mathbb{T}^d \to so(3,\mathbb{R})$ and a frequency vector such that

$$x' = A(\theta)x, \qquad \theta' = \omega, \tag{V.33}$$

where now $x \in \mathbb{R}^3$. Therefore, there exist $a_i : \mathbb{T}^d \to \mathbb{R}$ for $i = 1, 2, 3$, such that

$$A(\theta) = \begin{pmatrix} 0 & a_1(\theta) & a_3(\theta) \\ -a_1(\theta) & 0 & a_2(\theta) \\ -a_3(\theta) & -a_2(\theta) & 0 \end{pmatrix}.$$

Let us restrict our attention to systems which are perturbations of a constant matrix. That is, consider equations of the form

$$A(\theta,\mu) = A_0 + P(\theta,\mu) \tag{V.34}$$

with $A_0, P \in so(3,\mathbb{R})$ such that $P(\cdot;0) = 0$. To study analytic families of the above equations which have a constant Floquet matrix it is necessary to have an expression for the adjoint operator $\mathrm{ad}_{A_0} : g \to g$. In the basis (J_1, J_2, J_3) one can assume, after a change of basis,

$$A_0 = |v(A_0)| J_3$$

so that, if $X = x_1 J_1 + x_2 J_2 + x_3 J_3$, then

$$\mathrm{ad}_{A_0}(X) = \begin{pmatrix} 0 & |v(A_0)| & 0 \\ -|v(A_0)| & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

and the eigenvalues of ad_{A_0} are 0 and $\pm|v(A_0)|$. To check the nonresonance condition

$$|v(A_0)| - \langle \mathbf{k}, \omega \rangle \neq 0 \tag{V.35}$$

when $\mathbf{k} \in \mathbb{Z}^d$ we have to consider three possibilities: it is always different from zero (*irrational* case), it vanishes for $\mathbf{k} = \mathbf{0}$ (*degenerate* case) or for some $\mathbf{k} \neq \mathbf{0}$ (*rational* case). Let us treat these three cases separately.

Irrational case

Assume that the nonresonance condition (V.35) is satisfied for all $\mathbf{k} \in \mathbb{Z}^d$. To be under the hypothesis of Lemma V.6 one must impose the additional Diophantine hypothesis:

$$||v(A_0)| - \langle \mathbf{k}, \omega \rangle| \geq \frac{K}{|\mathbf{k}|^\tau}, \qquad \text{for} \qquad \mathbf{k} \neq \mathbf{0}. \tag{V.36}$$

where K, τ are some fixed positive constants. If, for

$$P_{\mathbf{0}} = p_1 J_1 + p_2 J_2 + p_3 J_3 \in so(3, \mathbb{R})$$

we define

$$S(P) = \left(-\frac{p_2}{|v(A_0)|}, \frac{p_1}{|v(A_0)|}, 0 \right)$$

and

$$C(P) = (0, 0, p_3)$$

in the (J_1, J_2, J_3)-basis of $so(3, \mathbb{R})$, the quartet (A_0, C, S, ω) is admissible by Lemma V.6. Therefore, there exists a real analytic function $p_3 = p_3(\mu)$ such that the system

$$x' = (A_0 + P(\theta, \mu) - p_3(\mu) J_3)\, x, \qquad \theta' = \omega \tag{V.37}$$

is reducible to

$$x' = A_0.$$

In particular, the condition $p_3(\mu) = 0$, which is real analytic, determines an analytic family of reducible systems with Floquet matrix A_0.

Degenerate case

This case corresponds to $|v(A_0)| = 0$ so that $A_0 = 0$. According to Example V.7, if ω is strongly rationally independent, we can choose the counter-term C to be the identity and the operator S to be identically zero.

Applying Theorem V.3, there exist real analytic functions ξ_1^*, ξ_2^* and ξ_3^* of μ an a real analytic matrix $X = X(\theta, \mu)$ in $so(3, \mathbb{R})$ such that the transformation $Z = \exp(X)$ satisfies

$$\partial_\omega Z(\theta, \mu) = \left(P(\theta, \mu) - \sum_{j=1}^{3} \xi_j^*(\mu) J_j \right).$$

In particular, the three conditions $\xi^*(\mu) = 0$ determine an analytic family of reducible subsystems of (V.33) with Floquet matrix A_0.

Rational case

This resonant case is characterized by the existence of some $\mathbf{k}_0 \neq \mathbf{0}$ such that

$$|v(A_0)| = \langle \mathbf{k}_0, \omega \rangle.$$

Note that, even if ω is strongly rationally independent, the Diophantine condition (V.16) does not hold, although this can be overcome, see Section V.6.

Nevertheless this situation of rational dependence can be reduced to the previous degenerate case. Indeed, denote $\alpha = |v(A_0)|$ as the positive eigenvalue of ad_{A_0} and assume, as before, A_0 of the form $A_0 = \alpha J_3$.

Let $y(t) = \exp(\alpha t J_3)$. Since $\alpha = \langle \mathbf{k}_0, \omega \rangle$, y is quasi-periodic with $y(t) = Y(\omega t) \in SO(3, \mathbb{R})$ being

$$Y(\theta) = \exp\left(\langle \mathbf{k}_0, \theta \rangle J_3\right),$$

which is real analytic. After the change of variables

$$x = Y(\theta) y$$

the new unperturbed matrix is 0 and we are in the degenerate case.

V.4 Hyperbolicity boundaries in higher dimensions

In this section we will consider the problem of the generalization of Theorem V.12 to higher dimensional Hamiltonian systems. A system of this kind is of the form

$$x' = H(\theta, \mu) x, \qquad \theta' = \omega \tag{V.38}$$

where $H \in sp(m, \mathbb{R})$ depends analytically on the angles $\theta \in \mathbb{T}^d$ and the external parameters $\mu \in \mathbb{R}^p$ in some neighbourhood of the origin. In what follows the frequency ω will also be assumed to be strongly rationally independent. The dimension of (V.38) is thus $n = 2m$.

We are interested in the regions in the parameter space $\mu \in \mathbb{R}^p$ such that system (V.38) has an exponential dichotomy. This property was discussed in Section II.3. Here we will be especially interested in the adaption of Sacker-Sell spectral theory to the Hamiltonian case. In Haro & de la Llave [HdlL03d], the structure of Sacker-Sell spectrum for discrete quasi-periodic skew-products was considered. These results apply to the case of continuous flows like (V.39) by taking Poincaré maps. They derive several additional properties when the flow is Hamiltonian like (V.38).

First of all, if λ belongs to the Sacker-Sell spectrum of a quasi-periodic Hamiltonian skew-product flow, then also $-\lambda$ belongs to it. In particular, if there is an spectral interval including the zero, it is symmetric with respect to zero. Also, it has as a consequence that the restriction of the flow to any invariant subbundle whose restricted flow has a nonsymmetric spectrum (like the stable or unstable subbundles) is not Hamiltonian.

There is one case when the restriction of a quasi-periodic Hamiltonian flow to an invariant subbundle is again Hamiltonian. Let $\mathcal{C}$ be an invariant subbundle of (V.38) whose spectrum (that is the spectrum of the restriction of the flow to it) is symmetric with respect the origin. Then, the restriction of the flow is Hamiltonian [HdlL03d]. This kind of subbundles, and their restricted flows, are particularly important, since the exponential dichotomy of (V.38) is equivalent to the exponential dichotomy of this reduced flow. Such a continuous subbundle will be called a *central subbundle*. Note that, trivially, the whole space $\mathbb{R}^{2m} \times \mathbb{T}^d$ is always a central subbundle.

Let us apply all this theory to the following quasi-periodic system of equations

$$\left\{\begin{array}{rcll} x_0'' &=& -ax_0 + b\sum_{j=0}^m Q_{0j}(\theta)x_j, & \\ x_k'' &=& \lambda_k^2 x_k + b\sum_{j=0}^m Q_{kj}(\theta)x_j, & k=1,\dots,m \\ \theta' &=& \omega, & \end{array}\right. \tag{V.39}$$

where $a, b, \lambda_1, \dots, \lambda_m$ are real parameters such that

$$\lambda_1, \dots, \lambda_m > \lambda_0 > 0, \tag{V.40}$$

for some λ_0 and the Q_{kj} are real analytic functions. This can be written as a Hamiltonian system like (V.38) if we set

$$H(\theta) = H_0 + bH_1(\theta) = \left(\begin{array}{cc|c} \multicolumn{2}{c|}{0} & I \\ \hline -a & 0 & \\ 0 & \Lambda & 0 \end{array}\right) + \left(\begin{array}{c|c} 0 & 0 \\ \hline Q & 0 \end{array}\right)$$

where Λ is the diagonal matrix $\Lambda = \operatorname{diag}(\lambda_1, \dots, \lambda_m)$ and $Q = (Q_{kj})_{k,j}$. Then, introducing $y = (x, x')^T$, it satisfies

$$y' = (H_0 + bH_1(\theta))\, y, \qquad \theta' = \omega. \tag{V.41}$$

When $b = 0$ this system is in constant coefficients and in this case the study of the Sacker-Sell spectrum and associated invariant subbundles is trivial. Assume that $a > -\lambda_0^2$. Then the Sacker-Sell spectrum is exactly the union

$$S_{a,0} \cup C_{a,0} \cup U_{a,0}$$

where

$$S_{a,0} = \{-\lambda_1\} \cup \dots \{-\lambda_k\}, \qquad U_{a,0} = \{\lambda_k\} \cup \dots \{\lambda_1\},$$

and $C_{a,0}$ is $\{0\}$ if $a \geq 0$ or $\{-\sqrt{-a}\} \cup \{\sqrt{-a}\}$ if $a < 0$. Therefore, in terms of system (V.41) has an exponential dichotomy if, and only if, $a < 0$.

Let $\mathcal{C}_{a,0}$ be the invariant subbundle corresponding to $C_{a,0}$. If b is small enough, then the Sacker-Sell spectrum of (V.41) has three components $S_{a,b}$, $C_{a,b}$ and $U_{a,b}$ separated by λ_0 and $-\lambda_0$. The corresponding spectral subbundles $\mathcal{S}_{a,b}$, $\mathcal{C}_{a,b}$ and $\mathcal{U}_{a,b}$, which are real analytic, depend real analytically on a, b for $a > -\lambda_0$ and b small enough, see Section II.3.3.

The flow on the central subbundle is again Hamiltonian and two-dimensional, since for $b = 0$ the central subbundle $\mathcal{C}_{a,0}$ is given by $x_1 = \dots = x_m = x_1' = \dots = x_m' = 0$ and the reduced flow (in the coordinates (x_0, x_0', θ)) is given by precisely

$$x_0'' = -ax_0', \qquad \theta' = \omega. \tag{V.42}$$

By the analytic dependence of the subbundles, the reduced flow on the central subbundle $\mathcal{C}_{a,b}$ is given, in some new coordinates $(\xi_1, \xi_2)^T$, by

$$\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}' = \left(\begin{pmatrix} 0 & 1 \\ -a & 0 \end{pmatrix} + bP(\theta, a, b)\right)\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}, \qquad \theta' = \omega, \tag{V.43}$$

where P is a Hamiltonian matrix-function depending real analytically on θ and (a,b) for $a > -\lambda_0$ and small values of $|b|$. Note that, after this reduction to the central subbundle, (V.41) has an exponential dichotomy if, and only if, (V.43) has an exponential dichotomy.

Since for $b = 0$ (V.43) reduces to (V.42), the resonances of our problem occur for the values of a of the form

$$\left(\frac{\langle \mathbf{k}, \omega\rangle}{2}\right)^2 \tag{V.44}$$

for some $\mathbf{k} \in \mathbb{Z}^d$ not identically zero, because for these values all the solutions of (V.42) are quasi-periodic with frequency $\omega/2$. These rational values of a lie precisely at the boundaries of hyperbolic regions:

Theorem V.21. *Assume that a_0 is of the form (V.44) for some nonzero $\mathbf{k} \in \mathbb{Z}^d$ and that $\lambda_1, \ldots, \lambda_m$ satisfy (V.40) for some $\lambda_0 > 0$. Then there exists a $\beta_0 > 0$ and two real analytic functions a_1 and a_2, such that for $|b| < \beta_0$*

(i) If (a,b) is such that $(a-a_1(b))(a-a_2(b)) < 0$ system (V.41) has an exponential dichotomy.

(ii) If $(a-a_1(b))(a-a_2(b)) = 0$ system (V.41) does not have an exponential dichotomy.

(ii) For all $\varepsilon > 0$ there exists an a, with $\varepsilon > (a-a_1(b))(a-a_2(b)) > 0$ such that (V.41) does not have an exponential dichotomy.

Proof: Theorem V.12 also holds for systems of the form (V.43) so that, for any rational value of a_0 there exist two real analytic functions a_1 and a_2 with $a_1(0) = a_2(0) = a_0$ which parameterize the boundaries of the region in the (a,b)-plane with zero rotation number for $|b| < \beta_0$. Since the regions of constancy of the rotation number of (V.43) correspond to the regions of exponential dichotomy, the result follows. □

V.5 Proof of Theorems V.3 and V.10

In this section we will prove theorems V.3 and V.10. The proof of both results follows the same guidelines and we will prove them at the same time. We give the proof of Theorem V.10 because there is an additional element, the scaling factor, which has to be taken into account. The reader interested in Theorem V.3 only, can replace χ by one whenever it appears. Recall that Theorem V.10 requires some additional properties on the eigenvalues of the adjoint of A_0.

We will first prove Theorem V.10 disregarding the dependence on the external parameters μ and then we will explain what needs to be done in order to prove the analyticity with respect to these parameters.

To prove Theorem V.3 we must show the existence of a constant element $\xi^* \in g$, with $C(\xi^*) = \xi^*$, and $Z : \mathbb{T}^d \to G$, of the form $Z = \exp(\chi X)$ with $X \in g$ small (therefore Z is close to the identity), such that

$$\partial_\omega Z(\theta) = \chi^k \left(A_0 + \chi P(\theta) - \chi\xi^*\right) Z - Z\chi^k A_0, \qquad \theta \in \mathbb{T}^d. \tag{V.45}$$

This is a nonlinear homological equation which we will try to solve by Newton's quadratic method (following [Mos67] and [BMS76]). It is an iterative process in which the final transformation Z will be given as the infinite composition of the transformations that will be defined at each step. Note that ξ^* is not yet known and it will have to be determined along the iterative process. To make this more evident we write this equation as

$$\partial_\omega Z(\theta) = \chi^k \left(A_0 + \chi P(\theta) - \chi\eta\right) Z - Z\chi^k A_0, \qquad \theta \in \mathbb{T}^d, \tag{V.46}$$

where now $\eta \in g$ is a variable. At each step of the iterative process we will define new transformations Z^r in G and $\xi^r : g \to g$ which will reduce the system to constant coefficients up to a certain perturbation which will become smaller and smaller. The domains in θ of the composition of the transformations $Z^0, \dots, Z^r$ will shrink to a narrower, but nonvoid, complex strip of $\mathbb{T}^d$. The domains for which the composition $\xi^0 \circ \dots \circ \xi^r$ is defined will quickly shrink to zero and the image of zero under this composition will define the sought ξ^*. To see this clearer, we proceed a bit further in this iterative process before giving the inductive lemma. Writing $Z = \exp(\chi X)$, the linear version of (V.46), with respect to the size of the perturbation, becomes

$$\partial_\omega X(\theta) = \chi^k \left([A_0, X] + P^0(\theta) - \eta^0\right), \qquad \theta \in \mathbb{T}^d, \tag{V.47}$$

where we have written $P^0 = P$ and $\eta^0 = \eta$ to stress that this is the first step of an iterative process. The admissibility of $(\chi^k A_0, C, S, \omega)$ implies that equation (V.47) can be uniquely solved in any strip of $\mathbb{T}^d$ narrower than ρ provided η^0 is taken equal to $\hat{\eta}^0 = C(\overline{P})$ and we set $\overline{X} = S(\overline{P})$. Let $X^0(\theta)$ be the solution for this choice of η^0. Then $Z^0 = \exp(\chi X^0)$ satisfies

$$\begin{aligned}\partial_\omega Z^0(\theta) = \chi^k \left(A_0 + \chi P^0(\theta) - \chi\eta^0\right) Z^0 - \\ Z^0\chi^k \left(A_0 + \chi P_1(\theta, \eta^0) - \chi\eta^0 + \chi C(\overline{P})\right), \qquad \theta \in \mathbb{T}^d.\end{aligned} \tag{V.48}$$

where $P_1(\theta, \eta^0)$ is the new perturbation defined by the above equation. Up to now we have defined the transformation Z^0, but we have not yet defined the transformation for η^0 to render it closer to zero. In order to put the right hand side of (V.48) in the form of the left hand side we introduce a new variable η^1 satisfying

$$\eta^1 = \eta^0 - C(\overline{P}). \tag{V.49}$$

This trivially defines a diffeomorphism $\xi^0 : g \to g$

$$\eta^1 \mapsto \xi^0(\eta^1) = \eta^0 = \eta^1 + C(\overline{P})$$

which allows to express equation (V.48) in the new variable η^1 as follows.

$$\begin{aligned}\partial_\omega Z^0(\theta) = \chi^k \left(A_0 + \chi P^0(\theta) - \chi\xi^0(\eta^1)\right) Z^0(\theta) - \\ - Z^0(\theta)\chi^k \left(A_0 + \chi P^1(\theta, \eta^1) - \chi\eta^1\right), \qquad \theta \in \mathbb{T}^d\end{aligned} \tag{V.50}$$

if we set $P^1(\theta, \eta^1) = P_1(\theta, \xi^0(\eta^1))$.

The point in choosing these transformations Z^0 and ξ^0 is that the perturbations on the right hand side are much smaller than those on the left. This will be shown in the following section. We would like to stress that each step of the transformation involves two changes of variables. First, using the admissibility of $(\chi^k A_0, C, S, \omega)$, we perform the change Z, which implies considering narrower strips around the torus $\mathbb{T}^d$. Secondly, inverting equation (V.49) we perform a change in the variable η so that the system in this new variable is closer to A_0. Of course, in this first step, the transformation ξ^0 defined by (V.49) is globally a diffeomorphism, but in the next steps the domains of definition of the transformation of η will shrink to zero in a very fast way.

V.5.1 The inductive lemma

Now we can state the inductive lemma:

Lemma V.22 (The inductive lemma). *Assume that (A_0, C, S, ω) is admissible with constants c and ν. Fix a complex domain*

$$\mathcal{D}^r : \qquad |\operatorname{Im}\theta| < \rho_r, \qquad |\eta^r| < \sigma_r$$

and a constant $0 < \delta_r < \rho_r$. Then there exists a constant $K = K(g)$ such that if P^r is analytic on $\mathcal{D}^r$, belongs to g for real values of (θ, η^r), and

$$|P^r|_{\mathcal{D}^r} = \sup_{(\theta,\eta^r)\in\mathcal{D}^r} |P^r(\theta,\eta^r)| \leq \varepsilon_r < K\sigma_r \tag{V.51}$$

then, in the domain

$$\mathcal{D}_{r+1} : \qquad |\operatorname{Im}\theta| < \rho_r - \delta_r, \qquad |\eta^r| < \sigma_r/2$$

the transformation

$$Z^r(\theta,\eta^r) = \exp\left(\chi X^r(\theta,\eta^r)\right), \tag{V.52}$$

where $X^r(\theta,\eta^r)$ satisfies

$$\partial_\omega X^r = \chi^k\left([A_0, X^r] + P^r - C(\overline{P^r}(\eta^r))\right), \qquad \overline{X^r} = S(\overline{P^r}(\eta^r)), \tag{V.53}$$

is real analytic and the equation

$$\eta^{r+1} = \eta^r - C(\overline{P^r}(\eta^r)) \tag{V.54}$$

defines an analytic diffeomorphism ξ^r

$$\eta^{r+1} \in D(0,\varepsilon_r) \mapsto \xi^r(\eta^{r+1}) \in D(0, 2\varepsilon_r)$$

such that the equation

$$\begin{aligned} \partial_\omega Z^r(\theta,\xi^r(\eta^{r+1})) = \chi^k\left(A_0 + \chi P^r(\theta,\xi^r(\eta^{r+1})) - \chi\xi^r(\eta^{r+1})\right) Z^r \\ - Z^r\chi^k\left(A_0 + \chi P^{r+1}(\theta,\eta^{r+1}) - \chi\eta^{r+1}\right) \end{aligned} \tag{V.55}$$

holds in the domain

$$\mathcal{D}^{r+1} : \qquad |\operatorname{Im}\theta| < \rho_r - \delta_r, \qquad |\eta^{r+1}| < \varepsilon_r$$

with the estimates

$$|X^r|_{\mathcal{D}_{r+1}} \leq M := c\frac{\varepsilon_r}{\delta_r^\nu}, \tag{V.56}$$

$$\left|P^{r+1}\right|_{\mathcal{D}^{r+1}} \leq \left(e^{|\chi|M} - 1\right)\left(\varepsilon_r + 5\varepsilon_r e^{|\chi|M} + 2|A_0|e^{|\chi|M}M\right) + |A_0|\,|\chi|M^2 e^{2|\chi|M} \tag{V.57}$$

and

$$\left|D_{\eta^{r+1}}\xi^r\right|_{\varepsilon_r} \leq 1 + c_1\frac{\varepsilon_r}{\sigma_r}, \tag{V.58}$$

where the constant c_1 depends only on g.

Remark V.23. *The estimate (V.56) comes from the admissibility of* $(\chi^k A_0, C, S, \omega)$ *and it is included in the statement of the lemma only for the sake of completeness.*

Remark V.24. *The difference between the domains* $\mathcal{D}_{r+1}$ *and* $\mathcal{D}^{r+1}$ *is due to the* η *component. The restriction for the* η *component in* $\mathcal{D}^{r+1}$ *allows us to define the map*

$$(\theta, \eta^{r+1}) \in \mathcal{D}^{r+1} \mapsto (\theta, \xi^r(\eta^{r+1})) \in \mathcal{D}_{r+1}$$

which inverts Equation (V.54). Similarly to what we did for the first step, a perturbation $P_{r+1} : \mathcal{D}_{r+1} \to g$ *is defined by*

$$\begin{aligned}\partial_\omega Z^r(\theta,\eta^r) = \chi^k \left(A_0 + \chi P^r(\theta,\eta^r) - \chi\eta^r\right) Z^r - \\ Z^r \chi^k \left(A_0 + \chi P_{r+1}(\theta,\eta^r) - \chi\eta^r + \chi C(\overline{P^r}(\eta^r))\right),\end{aligned} \quad \text{(V.59)}$$

and later on we will define the perturbation $P^{r+1} : \mathcal{D}^{r+1} \to g$ *as*

$$P^{r+1}(\theta,\eta^{r+1}) = P_{r+1}(\theta, \xi^r(\eta^{r+1}))$$

so that (V.55) holds.

Proof: First of all we compute P_{r+1} in terms of Z^r, X^r, A_0, P^r and η^r. The identities (V.52) and (V.53) determine P_{r+1} when $\chi \neq 0$:

$$\begin{aligned}P_{r+1}(\theta,\eta^r) = \left(I - (Z^r)^{-1}\right)\left(\eta^r - C(\overline{P^r}(\eta^r))\right) - \\ (Z^r)^{-1}\left(\frac{1}{\chi}[A_0, \chi X^r - Z^r] + P^r(I - Z^r) + \eta^r(Z^r - I) + \frac{1}{\chi^{k+1}}\partial_\omega(Z^r - \chi X^r)\right)\end{aligned} \quad \text{(V.60)}$$

on $\mathcal{D}_{r+1}$. For the proof of Theorem V.10, one also has to define the value for $\chi = 0$. This can be done taking the limit of the above expression when $\chi \to 0$ and obtain

$$P_{r+1}(\theta,\eta^r) = 0. \quad \text{(V.61)}$$

Since, A_0, X^r, and P^r belong to g for real values of θ and $\eta^r \in g$, then necessarily $P_{r+1} \in g$ for these real values, because of Proposition II.1. In order to be able to define

$$P^{r+1}(\theta,\eta^{r+1}) = P_{r+1}(\theta,\xi^r(\eta^{r+1})), \qquad (\theta,\eta^{r+1}) \in \mathcal{D}^{r+1}$$

we first need to know that (V.54) can be inverted so that the map ξ^r can be defined. This is what we do now.

Inversion of (V.54)

Let $F^r(\eta^r) = C(\overline{P^r}(\eta^r))$. Then F^r is analytic on the ball $D(0,\sigma_r)$. By Cauchy estimates we have

$$|D_{\eta^r} F^r|_{\sigma_r/2} \leq c' \frac{|F^r|_{\sigma_r}}{\sigma_r - \sigma_r/2} \leq 2c' \frac{\varepsilon_r}{\sigma_r},$$

where c' is a constant depending only on g. Assume

$$K < \min\left(\frac{1}{4}, \frac{1}{4c'}\right). \quad \text{(V.62)}$$

In this case, Equation (V.54) is invertible when $|\eta^r| < \sigma_r/2$ and, since

$$|\xi^r(\eta^{r+1})| \leq |\eta^{r+1}| + |F^r(\xi^r(\eta^r))|,$$

then for $|\eta^{r+1}| < \varepsilon_r$ one has

$$|\eta^r| < 2\varepsilon_r.$$

As $\varepsilon_r > K\sigma_r$,

$$2\varepsilon_r \leq \frac{\sigma_r}{2},$$

and the map ξ^r,

$$\eta^{r+1} \in D(0,\varepsilon_r) \mapsto \xi^r(\eta^{r+1}) \in D(0, \frac{\sigma_r}{2}),$$

is well-defined. Moreover,

$$|D_{\eta^{r+1}}\xi^r|_{\varepsilon_r} \leq \frac{1}{1-|D_{\eta^r}F^r|_{\sigma_r/2}} \leq \frac{1}{1-2c'\frac{\varepsilon_r}{\sigma_r}} \leq 1 + c_1\frac{\varepsilon_r}{\sigma_r},$$

writing $c_1 = 4c'$, as we wanted to show.

Bounds for P^{r+1}

Once we have inverted (V.54) we can now estimate $P^{r+1}(\theta,\eta^{r+1}) = P_{r+1}(\theta,\xi^r(\eta^{r+1}))$ on $\mathcal{D}^{r+1}$ which, in virtue of (V.60) can be expressed as follows

$$\begin{aligned} P^{r+1}(\theta,\eta^{r+1}) = &\left(I-Z^{-1}\right)\left(\xi^r(\eta^{r+1}) - C(\overline{P}(\xi^r(\eta^{r+1})))\right) \\ &- Z^{-1}\Bigg(\frac{1}{\chi}[A_0,\chi X - Z] + P^r\left(\theta,\xi^r(\eta^{r+1})\right)(I-Z) + \\ &\qquad \xi^r(\eta^{r+1})(Z-I) + \frac{1}{\chi^{k+1}}\partial_\omega(Z-\chi X)\Bigg) \end{aligned} \tag{V.63}$$

where we write $Z = Z^r(\theta,\xi^r(\eta^{r+1}))$ and $X = X^r(\theta,\xi^r(\eta^{r+1}))$ only for simplicity. To bound this remainder, we will estimate all the terms in the above expression. First of all note that, since $(\theta,\xi^r(\eta^{r+1})) \in \mathcal{D}_{r+1}$ for $(\theta,\eta^{r+1}) \in \mathcal{D}^{r+1}$, then

$$|X|_{\mathcal{D}^{r+1}} \leq |X^r|_{\mathcal{D}_{r+1}} \leq c\frac{\varepsilon_r}{{\delta_r}^{\nu}} =: M.$$

Now we are ready to bound the terms of (V.63):

$$\begin{aligned} |I - Z^{-1}|_{\mathcal{D}^{r+1}} = |\sum_{j=1}^{\infty}\frac{(-1)^j}{j!}(\chi X)^j|_{\mathcal{D}^{r+1}} &\leq e^{|\chi|M} - 1. \\ |\xi^r - C(\overline{P^r}(\xi^r))|_{\mathcal{D}^{r+1}} = |\eta^{r+1}|_{\mathcal{D}^{r+1}} &\leq \varepsilon_r. \\ |Z^{-1}|_{\mathcal{D}^{r+1}} &\leq e^{|\chi|M}. \end{aligned}$$

$$\begin{aligned} \left|\frac{1}{\chi}[A_0,\chi X - Z]\right|_{\mathcal{D}^{r+1}} &= \frac{1}{|\chi|}\,|[A_0, I + \chi X - Z]|_{\mathcal{D}^{r+1}} \leq \frac{2}{|\chi|}|A_0||I+\chi X - Z|_{\mathcal{D}^{r+1}} \leq \\ &\frac{2}{|\chi|}|A_0|\left(e^{|\chi|M} - 1 - |\chi|M\right) \leq |\chi|M^2|A_0|\exp\left(|\chi|M\right). \end{aligned}$$

$$\begin{aligned} |P^r|_{\mathcal{D}^{r+1}} &\leq \varepsilon_r. \\ |I-Z|_{\mathcal{D}^{r+1}} &\leq e^{|\chi|M}-1. \\ |\xi^r|_{\mathcal{D}^{r+1}} &\leq 2\varepsilon_r. \end{aligned}$$

$$\frac{1}{|\chi^{k+1}|}|\partial_\omega(Z-\chi X)|_{\mathcal{D}^{r+1}} = \frac{1}{|\chi^{k+1}|}|\sum_{j=2}^{\infty}\frac{1}{j!}\partial_\omega((\chi X)^j)|_{\mathcal{D}^{r+1}} \leq$$
$$\frac{1}{|\chi^{k+1}|}\left(\exp\left(|\chi||X|_{\mathcal{D}^{r+1}}\right)-1\right)|\chi||\partial_\omega X|_{\mathcal{D}^{r+1}} \leq$$
$$\left(e^{|\chi|M}-1\right)\left(2|A_0|M+2\varepsilon_r\right).$$

Collecting all these estimates we have

$$\begin{aligned} |P^{r+1}|_{\mathcal{D}^{r+1}} \leq \left(e^{|\chi|M}-1\right)\varepsilon_r + e^{|\chi|M}\Big\{&|\chi|M^2|A_0|e^{|\chi|M}+\varepsilon_r\left(e^{|\chi|M}-1\right)+ \\ &2\varepsilon_r\left(e^{|\chi|M}-1\right)+\left(e^{|\chi|M}-1\right)\left(2|A_0|M+2\varepsilon_r\right)\Big\} \leq \\ &\left(e^{|\chi|M}-1\right)\left(\varepsilon_r+5\varepsilon_r e^{|\chi|M}+2|A_0|e^{|\chi|M}M\right)+|A_0|\,|\chi|M^2e^{2|\chi|M} \end{aligned} \tag{V.64}$$

which holds for all χ, even for $\chi=0$, due to the choice of (V.61). This proves the last estimate (V.57). □

V.5.2 The iterative construction. End of proof

To finish the proof we must show that the iterative process that was started at the beginning of the section can be continued up to any order (by suitably choosing the right domains) and that this process is convergent. As the first step of an iterative process define:

$$P^0(\theta,\eta^0)=P(\theta),$$

which is analytic in the complex strip $|\operatorname{Im}\theta|<\rho_0$. Having fixed this constant, we will define sequences $(\rho_r)_r$, $(\delta_r)_r$, $(\varepsilon_r)_r$ and $(\sigma_r)_r$ such that the inductive lemma can be applied up to any finite order and which guarantee the existence of the constant ξ^* and the reducing transformation which we will call Z^*.

Take

$$\rho_r=\rho_0\left(\frac{1}{2}+\frac{1}{2^{r+1}}\right),\qquad \rho_{r+1}=\rho_r-\delta_r,\qquad \delta_r=\frac{\rho_0}{2^{r+2}},\qquad r\geq 0$$

as the sequences which will define the successive domains for the angles θ. In order to overcome the problems caused by the presence of small divisors, we will define the sequences $(\varepsilon_r)_r$ and $(\sigma_r)_r$ as

$$\varepsilon_{r+1}=\varepsilon_r^{3/2},\qquad \varepsilon_r=\varepsilon_0^{\left(\frac{3}{2}\right)^r},\qquad \sigma_{r+1}=\varepsilon_r,\qquad r\geq 0$$

and $\sigma_0=\varepsilon_0^{2/3}$, which will be completely determined once we fix the initial ε_0. In order to do so, we first state which inequalities we want the sequences $(\rho_r)_r$, $(\delta_r)_r$, $(\varepsilon_r)_r$ and $(\sigma_r)_r$ to satisfy. In the notations of the inductive lemma, the conditions we impose are

$$\varepsilon_r<K\sigma_r,\qquad r\geq 0, \tag{V.65}$$
$$M=c\frac{\varepsilon_r}{\delta_r^{\nu}}<\varepsilon_r^{1/2},\qquad r\geq 0 \tag{V.66}$$

and

$$\left(e^{|\chi|M}-1\right)\left(\varepsilon_r+5\varepsilon_r e^{|\chi|M}+2|A_0|e^{|\chi|M}M\right)+|A_0|\,|\chi|M^2e^{2|\chi|M}<\varepsilon_r^{3/2},\qquad r\geq 0. \quad \text{(V.67)}$$

Now we must choose ε_0 so that these conditions are satisfied.

Choice of ε_0

The choice of ε_0 will be very conservative. First of all, condition (V.65) is equivalent to

$$\frac{\varepsilon_r}{\varepsilon_{r-1}}=\varepsilon_0^{\frac{1}{2}\left(\frac{3}{2}\right)^{r-1}}<K,\qquad r\geq 0,$$

provided that we set $\varepsilon_{-1}=\varepsilon_0^{2/3}$ for consistency. The conditions for $r\geq 0$ hold if

$$\varepsilon_0<K^3, \quad \text{(V.68)}$$

because $K<1/4$. Writing (V.66) in terms of ε_0 and ρ_0 we obtain

$$c\left(\frac{4}{\rho_0}\right)^{\nu}\cdot 2^{r\nu}\varepsilon_0^{\frac{1}{2}\left(\frac{3}{2}\right)^{r}}<1.$$

which holds choosing

$$\varepsilon_0<\min\left(\exp\left(\frac{\nu\log(1/4)}{\log(3/2)}\right),\frac{1}{c^2}\left(\frac{\rho_0}{4}\right)^{2\nu}\right). \quad \text{(V.69)}$$

Finally using that, by the above assumptions, $M<\varepsilon_r^{1/2}$ (which is less than one), we can estimate the left hand side of (V.67) as follows

$$\left(e^{|\chi|M}-1\right)\left(\varepsilon_r+5\varepsilon_r e^{|\chi|M}+2|A_0|e^{|\chi|M}M\right)+|A_0|\,|\chi|M^2e^{2|\chi|M}$$
$$=C_1M\varepsilon_r+C_2M^2, \quad \text{(V.70)}$$

being

$$C_1=e^{|\chi|}|\chi|\left(1+5e^{|\chi|}\right)$$

and

$$C_2=3|A_0|\,|\chi|e^{2|\chi|}$$

where we have used that $M<1$. If we want the right hand side of (V.70) to be smaller than $\varepsilon_r^{3/2}$, as required, we need to impose extra conditions, apart from (V.68) and (V.69). Indeed, writing the definition of M, condition (V.67) holds if

$$C_1c\frac{\varepsilon_r^2}{\delta_r^{\nu}}+C_2c^2\frac{\varepsilon_r^2}{\delta_r^{2\nu}}<\varepsilon_r^{3/2},$$

which is equivalent to

$$C_1c\frac{\varepsilon_r^{1/2}}{\delta_r^{\nu}}+C_2c^2\frac{\varepsilon_r^{1/2}}{\delta_r^{2\nu}}<1.$$

The left hand side of this expression is bounded by

$$C_1 c \frac{\varepsilon_r^{1/2}}{\delta_r^\nu} + C_2 c^2 \frac{\varepsilon_r^{1/2}}{\delta_r^{2\nu}} < c\,\varepsilon_r^{1/2}\, 4^{r\nu} \left(C_1 \left(\frac{4}{\rho_0}\right)^\nu + C_2 c \left(\frac{4}{\rho_0}\right)^{2\nu} \right).$$

This expression is less than one (which implies condition (V.67)) if take

$$\varepsilon_0 < \min\left(\exp\left(\frac{2\nu \log(1/4)}{\log(3/2)}\right), \frac{1}{C_3^2} \right), \tag{V.71}$$

where

$$C_3 = c\left(C_1 \left(\frac{4}{\rho_0}\right)^\nu + C_2 c \left(\frac{4}{\rho_0}\right)^{2\nu} \right).$$

Therefore, taking ε_0 satisfying the bounds (V.68), (V.69) and (V.71) the estimates (V.65), (V.66) and (V.67) follow for all $r \geq 0$.

The iterative process

Once we have chosen ε_0, the sequences $(\varepsilon_r)_r$ and $(\sigma_r)_r$ are defined and the inductive lemma can be applied up to any finite order to obtain analytic maps

$$X^r : \mathcal{D}_{r+1} \to g, \qquad \text{and} \qquad Z^r = \exp(\chi X^r),$$
$$P^{r+1} : \mathcal{D}^{r+1} \to g$$

and

$$\xi^r : D(0, \sigma_{r+1}) \to D(0, 2\sigma_{r+1}) \subset D(0, \frac{\sigma_r}{2}),$$

which satisfy the homological equation (V.55) with the estimates

$$\begin{aligned} |P^{r+1}|_{\mathcal{D}^{r+1}} < \varepsilon_r^{3/2} &= \varepsilon_{r+1}, \\ |D_{\eta^{r+1}} \xi^r|_{\varepsilon_r} < 1 + c_1 \varepsilon_{r-1}^{1/2} &= 1 + c_1 \sigma_r^{1/2}, \\ |X^r|_{\mathcal{D}_{r+1}} &< \varepsilon_r^{1/2}, \end{aligned} \tag{V.72}$$

for $r \geq 0$. Writing

$$\xi_r = \xi_{r-1} \circ \xi^r = \xi^0 \circ \xi^1 \circ \ldots \circ \xi^r,$$

which is a real analytic map on $B(0, \sigma_{r+1})$, and

$$Z_r(\theta, \eta^{r+1}) = Z_{r-1}(\theta, \xi^r(\eta^{r+1})) \cdot Z^r(\theta, \xi^r(\eta^{r+1})), \qquad (\theta, \eta^{r+1}) \in \mathcal{D}^{r+1},$$

which is also G-real analytic, we obtain, for all $r \geq 0$, the equation

$$\begin{aligned} &\partial_\omega Z_r(\theta, \eta^{r+1}) = \\ &\qquad \chi^k \left(A_0 + \chi P^0(\theta) - \chi \xi_r(\eta^{r+1})\right) Z_r - Z_r \chi^k \left(A_0 + \chi P^{r+1}(\theta, \eta^{r+1}) - \chi \eta^{r+1}\right) \end{aligned}$$

for $(\theta, \eta^{r+1}) \in \mathcal{D}^{r+1}$. To prove the conjugation

$$\partial_\omega Z^*(\theta, 0) = \chi^k \left(A_0 + \chi P^0(\theta) - \chi \xi^*(0)\right) Z^* - Z^* \chi^k A_0,$$

for $|\mathrm{Im}\,\theta| < \rho_0/2$, we only need to show that the sequences $(Z_r)_r$ and $(\xi_r)_r$ converge uniformly on $\mathcal{D}^* = \lim_{r\to\infty} \mathcal{D}^r$ to Z^* and ξ^* respectively, because from the estimates (V.72) the perturbations $(P^r)_r$ converge to zero on $\mathcal{D}^*$.

Existence of ξ^*

The desired ξ^* will be the limit of the sequence $(\xi_r(0))_r$. First of all note that if the limit exists, then it will belong to g and satisfy the identity $C(\xi^*) = \xi^*$ and the bound $|\xi^*| \leq 2\varepsilon_0$, since this holds for all $r \geq 0$. Now let us prove the convergence of the sequence.

Using the estimates (V.72)

$$|\xi_{r+1}(0) - \xi_r(0)| = |\xi_r(\xi^{r+1}(0)) - \xi_r(0)| \leq |D\xi_r|_{\sigma_{r+1}} |\xi^{r+1}(0)| < |D\xi_r|_{\sigma_{r+1}} \sigma_{r+1}.$$

Since $\xi_r = \xi_{r-1} \circ \xi^r$, then

$$D\xi_r(\eta^{r+1}) = (D\xi_{r-1})\left(\xi^r(\eta^{r+1})\right)(D\xi^r)\left(\eta^{r+1}\right),$$

so, if $|\eta^{r+1}| < \sigma_{r+1}$,

$$\begin{aligned}|D\xi_r|_{\sigma_{r+1}} \leq |D\xi^r|_{\sigma_{r+1}} \cdot |D\xi^{r-1}|_{\sigma_r} \cdot \ldots \cdot |D\xi^1|_{\sigma_2} \cdot |D\xi^0|_{\sigma_1} \leq \\ (1 + c_1\sigma_r^{1/2})(1 + c_1\sigma_{r-1}^{1/2}) \cdot \ldots \cdot (1 + c_1\sigma_1^{1/2})(1 + c_1\sigma_0^{1/2}) \\ \leq \prod_{j=0}^{\infty}\left(1 + c_1\sigma_j^{1/2}\right) \leq \exp\left(\sum_{j=0}^{\infty}\left(c_1\sigma_j^{1/2}\right)\right) < \exp(2c_1\sigma_0^{1/2}) < \infty\end{aligned}$$

because, by (V.65), $\sigma_{r+1}/\sigma_r < K < 1/4$, so that

$$\sum_{j=0}^{\infty}\sigma_j^{1/2} < \sum_{j=0}^{\infty}\frac{\sigma_0^{1/2}}{2^j} = 2\sigma_0^{1/2}.$$

Therefore, $(\xi_r(0))_r$ is a Cauchy sequence and it converges to $\xi^* \in g$, with $C(\xi^*) = \xi^*$.

Existence of Z^*

We follow the same idea that for the existence of ξ^*. Since

$$Z_r(\theta, \eta^{r+1}) = Z_{r-1}(\theta, \xi^r(\eta^{r+1}))Z^r(\theta, \xi^r(\eta^{r+1})), \qquad (\theta, \eta^{r+1}) \in \mathcal{D}^{r+1}$$

for $r \geq 1$ and

$$Z_0(\theta, \eta^1) = Z^0(\theta, \xi^0(\eta^1)),$$

then, for $|\mathrm{Im}\,\theta| < \rho_0/2$,

$$\begin{aligned}|Z_{r+1}(\theta, 0) - Z_r(\theta, 0)| = \left|Z_r(\theta, \xi^{r+1}(0))Z^{r+1}(\theta, \xi^{r+1}(0)) - Z_r(\theta, 0)\right| = \\ = \left|Z_r(\theta, \xi^{r+1}(0)) - Z_r(\theta, 0) + Z_r(\theta, \xi^{r+1}(0))\left(Z^{r+1}(\theta, \xi^{r+1}(0)) - I\right)\right| \leq \\ \left|Z_r(\theta, \xi^{r+1}(0)) - Z_r(\theta, 0)\right| + \left|Z_r(\theta, \xi^{r+1}(0))\right| \cdot \left|Z^{r+1}(\theta, \xi^{r+1}(0)) - I\right|.\end{aligned}$$

We now estimate all these terms:

$$\begin{aligned}\left|Z^{r+1}(\theta, \xi^{r+1}(0)) - I\right| \leq \exp(|\chi||X^{r+1}|_{\mathcal{D}_{r+2}}) - 1 \leq 2|\chi|\varepsilon_{r+1}^{1/2} = 2|\chi|\sigma_r^{1/2}, \\ \left|Z_r(\theta, \xi^{r+1}(0))\right| \leq \prod_{j=0}^{\infty}\exp\left(|\chi|\varepsilon_j^{1/2}\right) = \exp\left(\sum_{j=0}^{\infty}|\chi|\varepsilon_j^{1/2}\right) < \exp(2|\chi|\varepsilon_0^{1/2}) < \infty, \\ \left|Z_r(\theta, \xi^{r+1}(0)) - Z_r(\theta, 0)\right| \leq |D_{\eta^{r+1}}Z_r|_{\mathcal{D}^{r+1}}\left|\xi^{r+1}(0)\right|.\end{aligned}$$

We now need a bound of $|D_{\eta^{r+1}} Z_r|_{\mathcal{D}^{r+1}}$. To do so we will apply Cauchy estimates:

$$\begin{aligned}|D_{\eta^{r+1}} Z_r|_{\mathcal{D}^{r+1}} &= |D_{\eta^{r+1}} \left(Z_{r-1}(\cdot, \xi^r(\cdot)) Z^r(\cdot, \xi^r(\cdot))\right)|_{\mathcal{D}^{r+1}} \leq \\ &|D_{\eta^r} \left(Z_{r-1} Z^r\right)|_{\{|\operatorname{Im}\theta|<\rho_{r+1}\}\times\{|\eta^r|<2\sigma_{r+1}\}} \, |D\xi^r|_{\sigma_{r+1}} \leq \\ &\frac{c'}{\sigma_r/2 - 2\sigma_{r+1}} \, |Z_{r-1} Z^r|_{\mathcal{D}_{r+1}} \, |D\xi^r|_{\sigma_{r+1}} \leq \\ &\frac{2c'}{\sigma_r - 4\sigma_{r+1}} \, |Z_{r-1}|_{\mathcal{D}_r} \, |\exp\left(\chi X^r\right)|_{\mathcal{D}_{r+1}} \, |D\xi^r|_{\sigma_{r+1}} \leq \\ &\frac{2c'}{\sigma_r - 4\sigma_{r+1}} \exp(2|\chi|\varepsilon_0^{1/2}) \exp(|\chi|\sigma_{r+1}^{1/2}) \left(1 + c_1 {\sigma_r}^{1/2}\right) < \frac{c_2}{\sigma_r - 4\sigma_{r+1}},\end{aligned}$$

where c_2 is a new constant. Therefore,

$$\left|Z_r(\theta, \xi^{r+1}(0)) - Z_r(\theta, 0)\right| < \frac{2c_2\,\sigma_{r+1}}{\sigma_r - 4\sigma_{r+1}} = \frac{2c_2\,\sigma_r^{3/2}}{\sigma_r - 4\sigma_r^{3/2}} = \frac{2c_2\,\sigma_r^{1/2}}{1 - 4\sigma_r^{1/2}} \leq c_3\,\sigma_r^{1/2},$$

being c_3 another constant. Collecting all these bounds,

$$|Z_{r+1}(\theta, 0) - Z_r(\theta, 0)| < \left(c_3 + 2|\chi| \exp(2|\chi|\varepsilon_0^{1/2})\right) \sigma_r^{1/2},$$

which implies that $(Z_r(\cdot, 0))_r$ is a Cauchy sequence on $|\operatorname{Im}\theta| < \rho_0/2$ and therefore it converges to some Z^* on $\mathcal{D}^*$. Moreover, since

$$|Z^*|_{\mathcal{D}^*} < \exp(2|\chi|\varepsilon_0^{1/2})$$

and $\varepsilon_0 < 1$, there exists a real analytic map $X^* : \mathcal{D}^* \to g$, with $|X^*|_{\mathcal{D}^*} < 2\varepsilon_0^{1/2}$, such that $Z^* = \exp(\chi X^*)$.

This ends the proof of theorems V.3 and V.10 disregarding the dependence on external parameters.

Analytic dependence on μ

Up to now we have proved theorems V.3 and V.10 disregarding the dependence with respect to the external parameters μ. First of all note that these proofs (not considering yet the dependence on μ) can be extended to apply to analytic $P : \mathbb{T}^d \to g_{\mathbb{C}}$ such that

$$|P|_{\rho_0} < \varepsilon_0, \tag{V.73}$$

and to complex χ with $|\chi| \leq 1$ (in case of Theorem V.10). Here $g_{\mathbb{C}}$ stands for the complexification of the Lie algebra g. Elements of $g_{\mathbb{C}}$ are of the form $P_1 + \mathrm{i}P_2$, where P_1 and P_2 belong to g. The bound of (V.73) holds because the admissibility of (A_0, C, S, ω) (respectively $(\chi^k A_0, C, S, \omega)$) implies that the equations

$$\partial_\omega X(\theta) = \chi^k \left([A_0, X(\theta)] + P(\theta) - C(\overline{P})\right), \qquad \overline{X} = S(\overline{P}),$$

for $P : \mathbb{T}^d \to g_{\mathbb{C}}$ with analytic extension to $|\operatorname{Im}\theta| < \rho_0$, have a unique analytic solution $X : \mathbb{T}^d \to g_{\mathbb{C}}$ which satisfies the estimates

$$|X|_{\rho_0 - \delta} \leq c\frac{|P|_{\rho_0}}{\delta^\nu}$$

for all $0 < \delta \leq \rho$. With this in mind it can be checked that all the other parts of the proof hold.

Let us now consider the dependence with respect to μ. That is, assume that both P and χ depend real analytically on μ in a certain ball around the origin. Again, we deal with theorems V.3 and V.10 at the same time. The reader interested only in Theorem V.3 can replace χ by one.

Let $\nu > 0$ such that, if $|\mu| < \nu$, then $|P(\cdot, \mu)|_\rho < \varepsilon$ and $|\chi(\mu)| < 1$. For these complex values of μ there exist $\xi^*(\mu) \in g_{\mathbb{C}}$ (with $C(\xi^*(\mu)) = \xi^*(\mu)$ and $\xi^*(\mu) \in g$ for real values of μ) and $X(\cdot, \mu) : \mathbb{T}^d \to g_{\mathbb{C}}$, with analytic extension to $|\mathrm{Im}\,\theta| < \rho/2$ such that $Z(\theta, \mu) = \exp(\chi(\mu)X(\theta, \mu))$ satisfies

$$\partial_\omega Z(\theta, \mu) = \chi(\mu)^k \left(A_0 + \chi(\mu)P(\theta, \mu) - \chi(\mu)\xi^*(\mu)\right) Z(\theta, \mu) - Z(\theta, \mu)\chi(\mu)^k A_0,$$

for $|\mathrm{Im}\,\theta| < \rho/2$ and $|\mu| < \nu$. Moreover, if μ is real then P is real analytic in θ and belongs to g, so $\xi^*(\mu) \in g$ and $X(\theta, \mu) \in g$ for real θ. Therefore, we need to show that the dependence of these objects on μ is analytic on $|\mu| < \nu$.

To do so, note that the transformations constructed in the inductive lemma can be made analytic on $\mathcal{D}^r \times \{|\mu| < \nu\}$. For this, it is essential to define P^{r+1} when $\chi = 0$ as (V.61) to avoid a discontinuity. Since the final solution is obtained as the uniform limit (in the complex domain $\mathcal{D}^* \times \{|\mu| < \nu\}$) of the approximations, the limits are analytic there. □

V.6 Multiple internal-external resonances

In this chapter we have considered the existence of analytic families of reducible linear quasi-periodic equations with frequency ω and Floquet matrix A_0 satisfying the Diophantine condition (V.16),

$$\inf_{\lambda \in \text{ Spec ad }_{A_0}} |\lambda - \mathrm{i}\langle \mathbf{k}, \omega\rangle| \geq \frac{K}{|\mathbf{k}|^\tau}, \qquad \mathbf{k} \in \mathbb{Z}^d, \ \mathbf{k} \neq \mathbf{0},$$

for some positive constants K, τ. This assumption does not cover the case of *multiple* resonances, which happens when the previous condition holds for all values of $\mathbf{k} \in \mathbb{Z}^d$ except for a finite set of multi-integers. Theorem V.3 (and also Theorem V.10) can be adapted to the case of multiple resonances, provided suitable conditions are imposed.

We first of all we impose a less restrictive Diophantine condition on the eigenvalues of ad_{A_0} and ω. Assume that there exist positive constants $c, \nu > 0$ and a finite set $\mathcal{R} \subset \mathbb{Z}^d$ such that the estimate

$$\inf_{\lambda \in \mathrm{Spec}\left(\mathrm{ad}_{A_0}\right)} |\lambda - \mathrm{i}\langle \mathbf{k}, \omega\rangle| \geq \frac{K}{|\mathbf{k}|^\tau}, \tag{V.74}$$

holds for all $\mathbf{k} \in \mathbb{Z}^d - \mathcal{R}$. In particular, $\mathbf{0}$ must belong to this resonant set $\mathcal{R}$.

Secondly, let us introduce the generalization of the operators S and C. We assume that for all $\mathbf{k} \in \mathcal{R}$ there exist linear operators $S_\mathbf{k}, C_\mathbf{k}$ of $g_{\mathbb{C}}$ such that $C_\mathbf{k}^2 = C_\mathbf{k}$ and, for all $P \in g_{\mathbb{C}}$, the identity

$$\mathrm{i}\langle \mathbf{k}, \omega\rangle S_\mathbf{k}(P_\mathbf{k}) = [A_0, S_\mathbf{k}(P_\mathbf{k})] + P_\mathbf{k} - C_\mathbf{k}(P_\mathbf{k}) \tag{V.75}$$

holds. Under these two hypothesis, Theorem V.3 has to be modified only in the following way. For $\mathbf{k} \in \mathcal{R}$ there exist $\xi_\mathbf{k}^* \in g_{\mathbb{C}}$, with $C_\mathbf{k}(\xi_\mathbf{k}^*) = \xi_\mathbf{k}^*$, such that the modified system is

$$x' = \chi^k \left(A_0 + P(\theta) - \sum_{\mathbf{k} \in \mathcal{R}} \xi_\mathbf{k}^* \exp\left(i\langle \mathbf{k}, \theta\rangle\right) \right) x, \qquad \theta' = \omega \tag{V.76}$$

instead of (V.10).

Nevertheless, in several practical situations it turns out that it is not needed to make use of this extended version of the theorem, because some preliminary transformations can be performed so that resonances for values of $\mathbf{k}$ different from zero are *removed* and the resonance of $\mathbf{0}$ has a higher multiplicity (see also Moser & Pöschel [MP84], Eliasson [Eli02a, Eli01] and Krikorian [Kri99a]).

To illustrate this procedure consider a perturbed system

$$x' = (A_0 + P(\theta, \mu))\, x, \qquad \theta' = \omega \tag{V.77}$$

for which the adjoint operator $\mathrm{ad}_{A_0} : g \to g$ has rational eigenvalues with respect to ω. Assume that we can find matrices $A_0^d, A_0^r \in g$ such that

(i) $A_0 = A_0^d + A_0^r$.

(ii) A_0^d and ω satisfy the Diophantine condition (V.16).

(iii) The map $t \mapsto \exp(tA_0^r)$ is quasi-periodic with frequency $\omega/2$. Denote by Z its lift to $\mathbb{T}^d$.

If these conditions are fulfilled (an example of this appears in Section V.3) then, the transformation

$$x = \exp(tA_0^r)y$$

sends system (V.77) to

$$y' = \left(A_0^d + Q(\theta, \mu)\right) y, \qquad \theta' = \omega,$$

where

$$Q(\theta, \mu) = Z(\theta)^{-1} P(\theta, \mu) Z(\theta),$$

which is quasi-periodic with frequency ω and we are under the conditions of Theorem V.3.

V.7 The case of reversible systems

In practical situations, given a linear differential equation on some Lie algebra, there be can additional symmetries to be taken into account. In this case it is interesting to know if we can use these symmetries to deduce more properties of the counter-term C, essentially reducing the dimension of the space $C(\xi) = \xi$ in the algebra g. In this section we focus on the reversible case (see Broer, Huitema & Sevryuk [BHS96] and references therein).

Definition V.25. *Given an element $R \in GL(n, \mathbb{R})$, with $R^2 = I$, we will say that a map $Q : \mathbb{R} \to g$ is R-reversible, whenever*

$$Q(-t)R = -RQ(t)$$

for all $t \in \mathbb{R}$.

In presence of such a symmetry, the solutions of a linear differential equation have the following properties

Proposition V.26. *Consider a reversibility with respect to the involution R. Let $g \subset gl(n,\mathbb{R})$ a Lie sub-algebra. Then the following is true:*

(i) Let $A_0 \in g$, $Q : \mathbb{R} \to g$, both R-reversible, and let $X : \mathbb{R} \to g$, smooth, such that

$$X'(t) = [A_0, X(t)] + Q(t)$$

for all $t \in \mathbb{R}$. Then

$$X(-t)R = RX(t)$$

for all $t \in \mathbb{R}$.

(ii) If $X : \mathbb{R} \to g$ satisfies $X(-t)R = RX(t)$ for all $t \in \mathbb{R}$, then $Z(t) = \exp(X(t))$ also does:

$$Z(-t)R = RZ(t).$$

(iii) If $A, X : \mathbb{R} \to g$ satisfy that $X(-t)R = RX(t)$, $A(-t)R = -RA(t)$ and the conjugacy

$$Z'(t) = A(t)Z(t) - Z(t)B(t),$$

with $Z(t) = \exp(X(t))$, $B(t) \in g$ holds for all $t \in \mathbb{R}$, then B is R-reversible:

$$B(-t)R = -RB(t)$$

for all $t \in \mathbb{R}$.

Proof: The first item follows from the identities

$$(RX(t)R)' = -\left([A_0, RX(t)R] + Q(t)\right), \qquad (X(-t))' = -\left([A_0, X(-t)] + Q(t)\right).$$

Since $RX(t)R$ and $X(-t)$ satisfy the same differential equation and they coincide for $t = 0$, then

$$RX(t)R = X(-t)$$

for all $t \in \mathbb{R}$ and the first statement follows. Item (ii) is a direct consequence of the definition of the exponential of a matrix. To prove (iii) we first note that $Z'(t)$ is R-reversible and, since

$$B(t) = Z^{-1}(t)A(t)Z(t) - Z(t)^{-1}Z'(t),$$

then $B(t)$ must be R-reversible because Z^{-1} satisfies

$$Z^{-1}(-t)R = RZ^{-1}(t)$$

for all $t \in \mathbb{R}$. □

With this proposition in mind one can modify theorems V.3 and V.10 to obtain additional symmetries of the counter-term C. Here we give only the adaption of Theorem V.3 to the reversible case.

Theorem V.27. *Assume that, in addition to the hypothesis of Theorem V.3, there is an involution* $R \in GL(n, \mathbb{R})$ *such that*

$$A_0 R = -RA_0$$

and

$$C(\xi)R = -RC(\xi) \qquad S(\xi)R = RS(\xi) \tag{V.78}$$

hold for all R*-reversible* $\xi \in g$*. Then, if* P *is* R*-reversible, the element* $\xi^* \in g$ *is also* R*-reversible,*

$$\xi^* R = -R\xi^*,$$

and the conjugation X *satisfies*

$$X(-\theta)R = RX(\theta)$$

for all $\theta \in \mathbb{T}^d$*.*

As an application, in Hill's equation with quasi-periodic forcing, assume that the quasi-periodic forcing q is even in t, i.e. it satisfies that $q(t) = q(-t)$ for all $t \in \mathbb{R}$. Then the matrix function

$$t \mapsto \begin{pmatrix} 0 & 1 \\ -(a + bq(t)) & 0 \end{pmatrix}$$

is reversible with respect to the involution

$$R = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Following the construction in section V.2, the operators C and S clearly satisfy the identities (V.78). Therefore, the counter-term $\xi^*(\mu)$ is also R-reversible and, thus $\xi^*_{11}(\mu) = 0$, so that the persistence of a collapsed gap is given by the two equations

$$\xi^*_{12}(\mu) = \xi^*_{21}(\mu) = 0,$$

compare with Remark IV.5.

Chapter VI

Cantor Spectrum for the Almost Mathieu Operator

In this chapter we study the spectrum of the Almost Mathieu operator, which is the following discrete quasi-periodic Schrödinger operator (see Chapter III)

$$(H_{b,\phi}x)_n = x_{n+1} + x_{n-1} + b\cos(2\pi\omega n + \phi)x_n, \qquad n \in \mathbb{Z}, \tag{VI.1}$$

on $l^2(\mathbb{Z})$, where b is a real parameter, ω is a nonresonant frequency and $\phi \in \mathbb{T}$. For each b this is a bounded self-adjoint operator whose spectrum, a compact subset of the real line which does not depend on ϕ, will be denoted by σ_b. In the notations from Chapter III we have, thus,

$$H_{b,\phi} = H^d_{b\cos,\omega,\phi}$$

and

$$\sigma_b = \sigma^d(b\cos,\omega).$$

Remark VI.1. *In the notations of Chapter III, the Almost Mathieu operator would be denoted by $H^d_{b\cos,\omega,\phi}$ and its spectrum by $\sigma^d(b\cos,\omega)$. In this chapter we will use $H_{b,\phi}$ and σ_b instead.*

The reason for the name of this operator comes from the similarity of its eigenvalue equation, the *Harper equation*,

$$x_{n+1} + x_{n-1} + b\cos(2\pi\omega n + \phi)x_n = ax_n, \qquad n \in \mathbb{Z} \tag{VI.2}$$

for $a \in \mathbb{R}$, with the *Mathieu equation*,

$$x'' + (a + b\cos t)\,x = 0,$$

see Section IV.3.2. In the rest of the chapter we will assume that the frequency ω is strongly nonresonant. That is, there exist c and $\tau > 0$ for which the bound

$$|\sin 2\pi k\omega| > \frac{c}{|k|^\tau}$$

holds for all $k \in \mathbb{Z}$ different from zero.

The nature of the spectrum of this operator has been studied intensively in the last twenty years (for a review, see Last [Las95]) and an open problem has been to know whether the spectrum is a Cantor set or not, which is usually referred as the "Ten Martini Problem" (see the end of this section for some historical background on it). In this chapter (which is mostly based on Puig [Pui04]) we derive two results on this problem. The first one is *nonperturbative* (it does not depend on the precise arithmetical conditions imposed on the frequency).

Corollary VI.2. *If ω is Diophantine, then the spectrum of the Almost Mathieu operator is a Cantor set if $b \neq 0, \pm 2$.*

Here, we prefer to call this result a corollary, rather than a theorem, because the proof requires just a combination of reducibility, point spectrum and duality developed quite recently for the Almost Mathieu operator and the related eigenvalue equation. In the critical case $|b| = 2$, Y. Last proved in [Las94] that the spectrum of the Almost Mathieu operator is a subset of the real line with zero Lebesgue measure and that it is a Cantor set for the values of ω which have an unbounded continued fraction expansion, which is a set of full measure. This last result has been obtained recently for the remaining nonresonant frequencies by Avila & Krikorian [AK03] using dynamical methods.

Before presenting the second result in this chapter, let us now recall the context of the Gap Labelling Theorem for this operator (see also Section III.2.2). If $(x_n)_{n\in\mathbb{Z}}$ is any nontrivial solution of (VI.2), for some fixed a, b, ϕ and $S(N)$ is the number of changes of sign of such solution for $1 \leq n \leq N$, then the limit

$$\lim_{N\to\infty} \frac{S(N)}{2N},$$

exists, it does not depend on the chosen solution x, nor on ϕ and it is called the Sturmian rotation number of (VI.2) which, in this chapter, will be denoted by $\operatorname{rot}(a, b)$. The Gap Labelling Theorem states that the rotation number, which is constant exactly at the gaps of the Almost Mathieu operator, must take the value

$$\operatorname{rot}(a, b) = \frac{1}{2}\left\{k\omega\right\},$$

for a suitable integer k, or

$$\operatorname{rot}(a, b) = \frac{1}{2}$$

in these spectral gaps. This integer k is the label of the spectral gap. If the closure of a spectral gap degenerates to a point we will say that it is a *collapsed gap* and otherwise that it is a *noncollapsed gap*. See Figure VI.1 for a numerical computation of the biggest gaps in the spectrum of the Almost Mathieu operator for several values b.

In view of this gap labelling and Corollary VI.2, it is natural to ask for the following: are all spectral gaps of the Almost Mathieu operator open if $b \neq 0$? This was called the "Strong (or Dry) Ten Martini Problem´´ by Simon [Sim82]. Our second result gives a *perturbative* answer to this problem.

Corollary VI.3. *Assume that $\omega \in \mathbb{R}$ is strongly nonresonant, $\omega \in DC^d(c, \tau, \mathbb{R})$. Then, there is a constant $C = C(c, \tau) > 0$ such that if $0 < |b| < C$ or $4/C < |b| < \infty$ all the spectral gaps of the spectrum of the Almost Mathieu operator are open.*

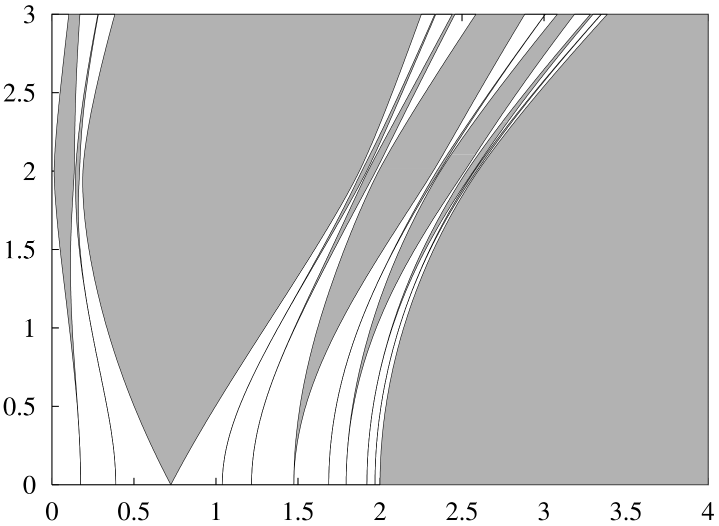

Figure VI.1: Numerical computation of the ten biggest spectral gaps for the Almost Mathieu operator with different values of b and $\omega = (\sqrt{5}-1)/2$. They correspond to the first $|k|$ such that $\{k\omega\}/2$ belongs to $[1/4, 1/2]$. The coupling parameter b is in the vertical direction whereas the spectral one, a, is in the horizontal one. Note that for $b = 0$, all gaps except the upper one are collapsed.

Before ending this introduction we give a short account of the existing results (to our knowledge) on the "Ten Martini Problem" for $|b| \neq 0, 2$. For results on Cantor spectrum for continuous quasi-periodic Schrödinger operators see Sections III.2.2 and V.2.2.

The Cantor structure of the spectrum for the Almost Mathieu operator was first conjectured by Azbel [Azb64] (see also Harper [Har55] and Sokoloff [Sok85] for physical approaches to this operator). The problem of the Cantor structure of the spectrum was named the "Ten Martini Problem" by Simon [Sim82] after an offer by Kac who conjectured that "all spectral gaps are open". Sinai [Sin87], proved that for Diophantine ω's and sufficiently large (or small $|b|$), depending on ω, the spectrum σ_b is a Cantor set. Choi, Elliott & Yui [CEY90] proved that the spectrum σ_b is a Cantor set for all $b \neq 0$ when ω is a Liouville number obeying the condition

$$\left|\omega - \frac{p}{q}\right| < D^{-q},$$

for a certain constant $D > 1$ and infinitely many rationals p/q. In particular, this means that for a G_δ-dense subset of pairs (b, ω) the spectrum is a Cantor set, which is the Bellissard-Simon result [BS82]. There is a great number of works devoted to the spectral properties of the Almost Mathieu operator. A list of them can be found in the papers by Last [Las95], Jitomirskaya [Jit95, Jit02], Bourgain [Bou04b, Bou02a, Bou02b, Bou02c], Simon [Sim82, Sim00b] and the book by Boca [Boc01]. Finally, let us mention that, if we consider the case of rational ω, all spectral gaps, apart from the middle one, are open if $b \neq 0$. This result was proved by van Mouche [vM89] and Choi, Elliott & Yui [CEY90].

Let us now outline the contents of the this chapter. In Section VI.1 we introduce Aubry duality for the Almost Mathieu operator and a reformulation of Ince's argument for the lack of coexisting quasi-periodic solutions adapted to the context of the Almost Mathieu operator. In

VI.1.1 w apply the reducibility results by Eliasson to prove Corollary VI.3. Finally, in Section VI.2, the proof of of Corollary VI.2 is given, which is based on a result of nonperturbative localization by Jitomirskaya.

VI.1 Aubry Duality, lack of coexistence and the Dry Ten Martini Problem

Aubry duality is specific feature of the Almost Mathieu operator which is basically the invariance through Fourier transform. It will be considered, for more general potentials, in next chapter. Let us now introduce the basic idea behind it.

Assume that we have $a \in \sigma_b$ and $\psi \in l^2(\mathbb{Z})$ which satisfy the Harper equation

$$\psi_{n+1} + \psi_{n-1} + b\cos(2\pi\omega n)\psi_n = a\psi_n, \qquad n \in \mathbb{Z},$$

for some $b > 0$. Assume, in addition, that this solution *decays exponentially*, which means that there exist positive constants $A, \beta > 0$ such that

$$|\psi_n| \leq A\exp(-\beta|n|), \qquad n \in \mathbb{Z}.$$

We will sometimes say that this solution is *exponentially localized*. The Fourier transform of ψ, defined as

$$\tilde{\psi}(\theta) = \sum_{n\in\mathbb{Z}} \psi_n e^{in\theta}, \qquad \theta \in \mathbb{T},$$

is real analytic in $|\mathrm{Im}\,\theta| < \beta$. Moreover, and here comes the specificity of the Almost Mathieu operator, the quasi-periodic sequence

$$x_n = \tilde{\psi}\left(2\pi\omega n + \theta\right), \qquad n \in \mathbb{Z},$$

for any $\theta \in \mathbb{T}$, satisfies the equation

$$(x_{n+1} + x_{n-1}) + \frac{4}{b}\cos(2\pi\omega n + \theta)x_n = \frac{2a}{b}x_n, \qquad n \in \mathbb{Z}.$$

This is again a Harper equation but with different parameters

$$(a, b) \mapsto \left(\frac{2a}{b}, \frac{4}{b}\right). \tag{VI.3}$$

Using the characterization of the spectrum in terms of the existence of a nontrivial bounded solution, Theorem III.8, we have that both $a \in \sigma_b$ and $2a/b \in \sigma_{4/b}$. This argument depends on the existence of an exponentially localized solution $(\psi_n)_{n\in\mathbb{Z}}$. Nevertheless, the relation between the spectra σ_b and $\sigma_{4/b}$ always holds, as it was shown by Avron & Simon [AS83]. This is known as *Aubry duality* or simply *duality*.

Theorem VI.4. *For every nonresonant frequency ω, the rotation number of (VI.2) satisfies the relation*

$$\mathrm{rot}(a, b) = \mathrm{rot}(2a/b, 4/b) \tag{VI.4}$$

for all $b \neq 0$ and $a \in \mathbb{R}$.

This means that the spectrum $\sigma_{4/b}$, for $b \neq 0$ is just a dilatation of the spectrum σ_b. In particular, σ_b is a Cantor set (resp. none of the spectral gaps of σ_b is collapsed) if, and only if $\sigma_{4/b}$ is a Cantor set (resp. none of the spectral gaps of $\sigma_{4/b}$ is collapsed).

Let us now see how the argument of Aubry duality given above implies the absence of coexisting quasi-periodic solutions. This will be used in the proof of VI.2 and VI.3 and it is very similar to Ince's argument for the classical Mathieu periodic differential equation (see [Inc44] §7.41).

In principle, the eigenvalue equation of a general quasi-periodic Schrödinger operator may have two linearly independent quasi-periodic solutions with frequency ω (or $\omega/2$). One may call this phenomenon *coexistence* of quasi-periodic solutions, in analogy with the classical Floquet theory for second-order periodic differential equations. A trivial example of this occurs in the Almost Mathieu case for $b = 0$ and suitable values of a.

Let us now show that in the Almost Mathieu case this does not happen if $b \neq 0$, i.e. two quasi-periodic solutions with frequency ω of the eigenvalue equation cannot coexist. Let $(x_n)_{n\in\mathbb{Z}}$ satisfy the equation

$$x_{n+1} + x_{n-1} + b\cos(2\pi\omega n + \phi)x_n = ax_n, \qquad n \in \mathbb{Z} \tag{VI.5}$$

for some a, $b \neq 0$ and $\phi \in \mathbb{T}$. If x is quasi-periodic with frequency ω, there exists a continuous function $\psi : \mathbb{T} \to \mathbb{R}$ such that $x_n = \psi(2\pi\omega n + \phi)$ for all $n \in \mathbb{Z}$. The Fourier coefficients of ψ, $(\psi_m)_{m\in\mathbb{Z}}$ satisfy the following Harper equation:

$$\psi_{m+1} + \psi_{m-1} + \frac{4}{b}\cos(2\pi\omega m)\psi_m = \frac{2a}{b}\psi_m, \qquad m \in \mathbb{Z}. \tag{VI.6}$$

Since ψ is at least continuous, then $(\psi_m)_{m\in\mathbb{Z}}$ belongs to $l^2(\mathbb{Z})$. Now the reason for the absence of coexisting quasi-periodic solutions is clear. Indeed, if $(y_n)_{n\in\mathbb{Z}}$ is another linearly independent quasi-periodic solution of (VI.5) with frequency ω, say $y_n = \chi(2\pi\omega + \phi)$, for some continuous χ, then the sequence of the Fourier coefficients of χ, $(\chi_m)_{m\in\mathbb{Z}}$, would be a solution of (VI.6) belonging to $l^2(\mathbb{Z})$. The sequences $(\psi_m)_{m\in\mathbb{Z}}$ and $(\chi_m)_{m\in\mathbb{Z}}$ would be two linearly independent solutions of (VI.6) which belong both to $l^2(\mathbb{Z})$. This is a contradiction with the limit point character of the cosine, see Lemma III.3.

Therefore, two quasi-periodic solutions with frequency ω cannot coexist if $b \neq 0$. A similar argument shows that quasi-periodic solutions of the form

$$(-1)^n\psi(2\pi\omega n + \phi), \tag{VI.7}$$

for a continuous $\psi : \mathbb{T} \to \mathbb{R}$ cannot coexist. Such a solution will be called *anti-quasi-periodic* in analogy with the periodic case [Inc44]. Also, if the frequency is $\omega/2$ or it is anti-quasi-periodic with frequency $\omega/2 + \pi$,

$$(-1)^n\psi(\pi\omega n + \phi),$$

two quasi-periodic solutions cannot coexist. Note that all these four cases correspond to the choices $\phi = 0, \pi, \omega/2, \omega/2 + \pi$ in Equation (VI.5).

Finally, note that the coexistence of two quasi-periodic solutions with frequency ω (resp. $\omega/2$) is equivalent to the reducibility with frequency ω (resp. $\omega/2$) of the corresponding Schrödinger cocycle, with the identity as Floquet matrix. Similarly the coexistence of two anti-quasi-periodic solutions with frequency ω (resp. $\omega/2$) (VI.7) is equivalent to the reducibility with frequency ω (resp. $\omega/2$) of the cocycle with minus the identity as Floquet matrix.

VI.1.1 The Strong Ten Martini Problem for small (and large) $|b|$

As an application of the argument above we will now show that for $0 < |b| < C$, where $C > 0$ is a suitable constant, and for $|b| > 4/C$ all spectral gaps are open. This will be a consequence of Eliasson's Theorem III.28 which, adapted to this context reads as follows.

Theorem VI.5. *Assume that ω is Diophantine with constants c and τ. Then there is a constant $C(c, \tau)$ such that, if $|b| < C(c, \tau)$ and $\mathrm{rot}(a, b)$ is either resonant or strongly nonresonant with respect to ω, then the corresponding Schrödinger cocycle (A, ω) with*

$$A(\theta) = \begin{pmatrix} a - b\cos\theta & -1 \\ 1 & 0 \end{pmatrix}, \tag{VI.8}$$

is reducible to constant coefficients, with Floquet matrix B, by means of a quasi-periodic (with frequency $\omega/2$) and analytic transformation. Moreover, if a is at an endpoint of a spectral gap of σ_b, then the trace of B is ± 2, being $B = \pm I$ if, and only if, the gap collapses.

Taking into account the arguments from the previous section, Corollary VI.3 is immediate. Indeed, let $|b| < C(c, \tau)$. Then the cocycle (VI.8) is reducible to constant coefficients and the Floquet matrix has trace ± 2 if a is an endpoint of a spectral gap. Moreover the gap is collapsed if, and only if, the Floquet matrix B is $\pm I$. Since we have seen in the previous section that (VI.8) for $b \neq 0$ cannot be reducible to these Floquet matrices, Corollary VI.3 follows. □

VI.2 Non-perturbative localization and Cantor spectrum for $b \neq 0$

In this section we will see how Corollary VI.2 is a consequence of the following theorem on nonperturbative localization, due to Jitomirskaya:

Theorem VI.6 ([Jit99]). *Let ω be strongly nonresonant, $\omega \in DC^d(c, \tau, \mathbb{R})$. Define Φ as the set of those $\phi \in \mathbb{T}$ such that the relation*

$$|\sin(\phi + \pi k\omega)| < \exp\left(-|k|^{\frac{1}{2\tau}}\right) \tag{VI.9}$$

holds for infinitely many values of k. Then, if $\phi \notin \Phi$ and $|b| > 2$ the operator $H_{b,\phi}$ has only pure point spectrum with exponentially decaying eigenfunctions. Moreover, any of these eigenfunctions $(\psi_n)_{n\in\mathbb{Z}}$ satisfies that

$$\beta(b) := -\lim_{|n|\to\infty} \frac{\log\left(\psi_n^2 + \psi_{n+1}^2\right)}{2|n|} = \log\left(\frac{|b|}{2}\right). \tag{VI.10}$$

An operator has only pure-point spectrum if the spectral measure is purely pure-point, see Section III.1.3. For the proof we will only need that if $\phi \notin \Phi$ then there is a set of eigenvalues of the operator which is dense in the spectrum and whose eigenfunctions decay exponentially.

Now we prove Corollary VI.2. Let $|b| > 2$. Then, according to Theorem VI.6, the operator $H_{b,0}$ (although one can also prove the result for $\phi = \pi, \omega/2, \omega/2 + \pi$) has only pure point spectrum with exponentially decaying eigenfunctions. The eigenvalue equation associated to this operator have the following property

Lemma VI.7. *Let $(x_n)_{n\in\mathbb{Z}}$ be a solution of the difference equation*

$$x_{n+1} + x_{n-1} + b\cos(2\pi n\omega)x_n = ax_n, \qquad n \in \mathbb{Z},$$

for some $a, b \in \mathbb{R}$. Then, $(x_{-n})_{n\in\mathbb{Z}}$ is also a solution of this equation.

According to Theorem VI.6, there exists a sequence of eigenvalues $(a^k(b))_{k\in\mathbb{Z}}$ with eigenvectors $(\psi^k(b))_{k\in\mathbb{Z}}$, exponentially localized and which form a complete orthonormal basis of $l^2(\mathbb{Z})$. Moreover the set of eigenvalues $(a^k(b))_{k\in\mathbb{Z}}$ must be dense in the spectrum σ_b. Again, we do not write the dependence on b for simplicity in what follows. None of these eigenvalues can be repeated, since we are in the limit point case. Writing each of the ψ^k as

$$\psi^k = (\psi^k_n)_{n\in\mathbb{Z}},$$

we define

$$\tilde{\psi}^k(\theta) = \sum_{k\in\mathbb{Z}} \psi^k_n e^{\mathrm{i}k\theta},$$

for $\theta \in \mathbb{T}$. All these functions belong to are real analytic functions on $\mathbb{T}$ with analytic extension to $|\mathrm{Im}\,\theta| < \beta(b)$, due to Equation (VI.10), and they are even functions of θ, because of Lemma VI.7 (here we have applied again that we are in the limit point case). Passing to the dual equation, we obtain that, for each $k \in \mathbb{Z}$, the sequence $(\tilde{\psi}^k(2\pi\omega n))_{n\in\mathbb{Z}}$ is a quasi-periodic solution of

$$x_{n+1} + x_{n-1} + \frac{4}{b}\cos\theta_n x_n = \frac{2a}{b}x_n, \qquad \theta_{n+1} = \theta_n + 2\pi\omega \qquad n \in \mathbb{Z}, \tag{VI.11}$$

provided a is now replaced by a^k. We are now going to see that $2a^k/b$ is at an endpoint of a spectral gap and this gap is noncollapsed. To do so we will use reducibility as in the proof of Theorem VI.3. For a direct proof that $2a^k/b$ is at an endpoint of a gap (it has rational rotation number), see Herman [Her83].

The fact that $(\tilde{\psi}^k(2\pi\omega n))_{n\in\mathbb{Z}}$ is a quasi-periodic solution with frequency ω of (VI.11) means that, for all $\theta \in \mathbb{T}$, the following equation is satisfied

$$\begin{pmatrix} \tilde{\psi}^k(4\pi\omega+\theta) \\ \tilde{\psi}^k(2\pi\omega+\theta) \end{pmatrix} = \begin{pmatrix} \frac{2a}{b} - \frac{4}{b}\cos\theta & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \tilde{\psi}^k(2\pi\omega+\theta) \\ \tilde{\psi}^k(\theta) \end{pmatrix}.$$

The following lemma shows that, if this is the case, then the quasi-periodic cocycle

$$\left(\begin{pmatrix} \frac{2a^k}{b} - \frac{4}{b}\cos\theta & -1 \\ 1 & 0 \end{pmatrix}, \omega \right) \tag{VI.12}$$

is reducible to constant coefficients.

Lemma VI.8. *Let $A : \mathbb{T} \to SL(2,\mathbb{R})$ be a real analytic map, with analytic extension to $|\mathrm{Im}\,\theta| < \delta$ for some $\delta > 0$. Assume that there is a nonzero real analytic map $v : \mathbb{T} \to \mathbb{R}^2$, with analytic extension to $|\mathrm{Im}\,\theta| < \delta$ such that*

$$v(\theta + 2\pi\omega) = A(\theta)v(\theta)$$

holds for all $\theta \in \mathbb{T}$. Then the cocycle (A, ω) on $SL(2, \mathbb{R}) \times \mathbb{T}$ is reducible to constant coefficients by means of a quasi-periodic transformation which is analytic in $|\mathrm{Im}\, \theta| < \delta$ and has frequency ω. Moreover the Floquet matrix can be chosen to be of the form

$$B = \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \qquad \text{(VI.13)}$$

for some $c \in \mathbb{R}$.

Proof: Since $v = (v_1, v_2)^T$ does not vanish, $d = v_1^2 + v_2^2$ is always different from zero and the transformation

$$Z(\theta) = \begin{pmatrix} v_1(\theta) & -v_2(\theta)/d(\theta) \\ v_2(\theta) & v_1(\theta)/d(\theta) \end{pmatrix},$$

is an analytic map $Z : \mathbb{T} \to SL(2, \mathbb{R})$. The transformation Z defines a conjugation of A with B^1, being

$$A(\theta)Z(\theta) = Z(2\pi\omega + \theta)B^1(\theta),$$

which means that B^1 is

$$B^1(\theta) = \begin{pmatrix} 1 & b_{12}^1(\theta) \\ 0 & 1 \end{pmatrix},$$

for some analytic $b_{12}^1 : \mathbb{T} \to \mathbb{R}$. The conjugated cocycle, (B^1, ω), is reducible to constant coefficients because it is in triangular form, the frequency ω is Diophantine and b_{12}^1 is analytic, see Section II.2.2. Indeed, if $y_{12} : \mathbb{T} \to \mathbb{R}$ is an analytic solution of the small divisors equation

$$y_{12}(2\pi\omega + \theta) - y_{12}(\theta) = b_{12}^1(\theta) - [b_{12}^1], \qquad \theta \in \mathbb{T},$$

where $[b_{12}^1]$ is the average of b_{12}^1 then the transformation

$$Y(\theta) = \begin{pmatrix} 1 & y_{12} \\ 0 & 1 \end{pmatrix}$$

conjugates (B^1, ω) with its averaged part (B, ω), where

$$B = [B^1] = \begin{pmatrix} 1 & [b_{12}^1] \\ 0 & 1 \end{pmatrix}$$

which is in the form of (VI.13). □

Thus, applying this lemma, the cocycle (VI.12) is reducible to constant coefficients with Floquet matrix B, of the form (VI.13). That is, there exists a real analytic map $Z : \mathbb{T} \to SL(2, \mathbb{R})$ such that

$$A(\theta)Z(\theta) = Z(\theta + 2\pi\omega)B \qquad \text{(VI.14)}$$

for all $\theta \in \mathbb{T}$. Moreover, since the trace of B is 2, the rotation number of (VI.11) is rational, so that we are at the endpoint of a gap, which we want to show that is noncollapsed.

By the arguments of Section VI.1, we rule out the possibility of B being the identity. Indeed, this would imply the coexistence of two quasi-periodic analytic solutions with frequency ω, which does happen in the Almost Mathieu case. Therefore $B \neq I$ and, thus, $c \neq 0$ in the definition above.

If $B \neq I$, it is a well-known fact of Floquet theory that $2a^k/b$ lies at the endpoint of a noncollapsed gap. In fact, adapting the techniques of the Chapter V, especially Section IV.4.3, the result follows. For the sake of self-completeness, let us sketch this adaption.

We will see that there exists a $\alpha_0 > 0$ such that if $0 < |\alpha| < \alpha_0$ and α is either positive or negative (depending on the sign of c) then $2a^k/b + \alpha$ lies in the resolvent set of $\sigma_{4/b}$. To do so, we will show that, for these values of α, the skew-product

$$\begin{pmatrix} x_{n+1} \\ x_n \end{pmatrix} = \begin{pmatrix} \frac{2a^k}{b} + \alpha - \frac{4}{b}\cos\theta_n & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_n \\ x_{n-1} \end{pmatrix}, \qquad \theta_{n+1} = \theta_n + 2\pi\omega \tag{VI.15}$$

has an exponential dichotomy which, by Theorem III.8, implies that $2a^k/b + \alpha \notin \sigma_{4/b}$. The reduction given by Z transforms this skew-product into

$$\begin{aligned} y_{n+1} &= \begin{pmatrix} 1 + \alpha\left(z_{11}z_{12} - cz_{11}^2\right) & c + \alpha\left(-cz_{11}z_{12} + z_{12}^2\right) \\ -\alpha z_{11}^2 & 1 - \alpha z_{11}z_{12} \end{pmatrix} y_n, \\ \theta_{n+1} &= \theta_n + 2\pi\omega, \end{aligned} \tag{VI.16}$$

where $y_n \in \mathbb{R}^2$ are the new variables. The z_{ij} are the elements of the matrix Z and we have used the relations given by (VI.14) and the special form of A and B. In the same calculation, we also see that $(z_{11}(2\pi n\omega))_{n\in\mathbb{Z}}$ is a quasi-periodic solution of equation (VI.11) and that it is not identically zero.

Next, we use averaging to conjugate the previous skew-product (VI.16) to the following one

$$\begin{aligned} y_{n+1} &= \left(\begin{pmatrix} 1 + \alpha\left([z_{11}z_{12}] - c[z_{11}^2]\right) & c + \alpha\left(-c[z_{11}z_{12}] + [z_{12}^2]\right) \\ -\alpha[z_{11}^2] & 1 - \alpha[z_{11}z_{12}] \end{pmatrix} + M\right) y_n \\ \theta_{n+1} &= \theta_n + 2\pi\omega \end{aligned} \tag{VI.17}$$

by means of a conjugation in $SL(2,\mathbb{R})$, with M analytic in both θ and α (in some narrower domains) and of order α^2. This is achieved imposing that the conjugation transformation, which is close to the identity, cancels the elements of (VI.16) which depend on θ and are of order α.

The trace of the skew-product (VI.17) is $2 - c\alpha[z_{11}^2] + O_2(\alpha)$ where $O_2(\alpha)$ stands for terms of order greater or equal to two in α (which also depend on θ). Thus, if we could forget about these higher order terms, the skew-product would have an exponential dichotomy for $c\alpha < 0$. We now want to apply Coppel's Criterion for exponential dichotomy II.30. To do so, note that

$$\begin{pmatrix} 1 + \alpha\left([z_{11}z_{12}] - c[z_{11}^2]\right) & c + \alpha\left(-c[z_{11}z_{12}] + [z_{12}^2]\right) \\ -\alpha[z_{11}^2] & 1 - \alpha[z_{11}z_{12}] \end{pmatrix} + M = e^{A(\theta,\alpha)},$$

where, by means of a computation, it is seen that

$$A(\theta,\alpha) = \begin{pmatrix} \alpha\left([z_{11}z_{12}] - \frac{c}{2}[z_{11}^2]\right) & c + \alpha\left(-c[z_{11}z_{12}] + [z_{12}^2]\right) \\ -\alpha[z_{11}^2] & -\alpha\left([z_{11}z_{12}] - \frac{c}{2}[z_{11}^2]\right) \end{pmatrix} + \tilde{M}(\theta,\alpha),$$

where $\tilde{M}$ is of order α^2 at $\alpha = 0$. After a change of variables for $c\alpha < 0$, the exponential dichotomy follows from Coppel's Criterion. Hence $2a^k/b + \alpha$ does not belong to $\sigma_{4/b}$. Since this works for all a^k, (which are dense in the spectrum), $\sigma_{4/b}$ is a Cantor set. By duality the result is also true for σ_b. This ends the proof of Corollary VI.2. □

Remark VI.9. *The same can be done for the operators $H_{b,\phi}$, for $\phi = \pi, \omega/2, \omega/2 + \pi$ instead of $H_{b,0}$. The corresponding point eigenvalues correspond to ends of noncollapsed gaps and are dense in the spectrum.*

Chapter VII

A Nonperturbative Eliasson's Theorem

In this chapter we exploit the techniques of the previous chapter to prove a nonperturbative version of Eliasson's Theorem III.28 on the reducibility of Schrödinger cocycles. We saw in the proof of the "Ten Martini Problem" how the combination of Aubry duality and Jitomirskaya nonperturbative localization Theorem VI.6 produced reducibility results at endpoints of spectral gaps. Here we will try to reproduce this idea for general real analytic potentials. The role of Aubry duality and Jitomirskaya's Theorem will be played by a convenient version of duality and a result by Bourgain & Jitomirskaya [BJ02b] on nonperturbative localization for a class of long-range quasi-periodic Schrödinger operators to be considered in a moment.

Before stating the main result in this chapter let us recall the context of discrete quasi-periodic Schrödinger operators on $l^2(\mathbb{Z})$ (see Chapter III). We will consider Schrödinger operators $H_{V,\omega,\phi}$ as follows

$$(H_{V,\omega,\phi}x)_n = x_{n+1} + x_{n-1} + V(2\pi\omega n + \phi)x_n, \qquad n \in \mathbb{Z}, \tag{VII.1}$$

where $V : \mathbb{T} \to \mathbb{R}$ is a real analytic potential, $\phi \in \mathbb{T}$ and ω a strongly nonresonant frequency. This means that there exist c and $\tau > 1$ such that the bounds

$$|\sin 2\pi k\omega| > \frac{c}{|k|^\tau} \tag{VII.2}$$

hold for any integer $k \neq 0$, $\omega \in DC^d(c, \tau, \mathbb{R})$ for short. The operator $H_{V,\omega,\phi}$ is a bounded and self-adjoint operator from $l^2(\mathbb{Z})$ to itself whose spectrum is a compact subset of the real line which does not depend on ϕ. Therefore, there is no confusion writing

$$\sigma(V, \omega) = \mathrm{Spec}\,(H_{V,\omega,\phi})\,.$$

The Almost Mathieu operator, studied in the previous chapter, occurs as a particular case if $V(\theta) = b\cos(\theta)$, with $b \in \mathbb{R}$ a real parameter.

As it has been seen along this thesis, many spectral properties of a quasi-periodic Schrödinger operator can be derived from a dynamical analysis of its eigenvalue equation. In the case of the operators $H_{V,\omega,\phi}$ this is the following Harper-like equation

$$x_{n+1} + x_{n-1} + V(2\pi\omega n + \phi)x_n = ax_n, \qquad n \in \mathbb{Z}, \tag{VII.3}$$

where $a \in \mathbb{R}$ is the spectral parameter. Such an equation defines a quasi-periodic skew-product on $\mathbb{R}^2 \times \mathbb{T}$,

$$\begin{pmatrix} x_{n+1} \\ x_n \end{pmatrix} = \begin{pmatrix} a - V(\theta_n) & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_n \\ x_{n-1} \end{pmatrix}, \qquad \theta_{n+1} = \theta_n + 2\pi\omega, \tag{VII.4}$$

which can be seen as an iteration of the associated Schrödinger cocycle, (A^d_{a-V}, ω) on $SL(2,\mathbb{R}) \times \mathbb{T}$, where

$$A^d_{a-V}(\theta) = \begin{pmatrix} a - V & -1 \\ 1 & 0 \end{pmatrix}, \qquad \theta \in \mathbb{T}. \tag{VII.5}$$

Eliasson's Theorem III.28 states that, if V is real analytic in $C^a_\rho(\mathbb{T},\mathbb{R})$ and the frequency is strongly nonresonant, $\omega \in DC^d(c,\tau,\mathbb{R})$ for some constants c, τ, then the Schrödinger cocycle is reducible to constant coefficients if its rotation number is either resonant or strongly nonresonant with respect to ω and $|V|_\rho < C(c,\tau,\rho)$, for a positive constant $C = C(c,\tau,\rho)$. This result is *semiperturbative* because the constant C depends on the arithmetic conditions on the frequency ω but not on the arithmetic conditions on the allowed rotation numbers (as long as they are strongly nonresonant or resonant with respect to the frequency).

The main result in this chapter states that a nonperturbative version of this result is true.

Theorem VII.1. *Let $\rho > 0$ be a positive number. Then, there is a constant $\varepsilon_0 = \varepsilon_0(\rho)$ such that, for any real analytic $V \in C^a_\rho(\mathbb{T},\mathbb{R})$ with*

$$|V|_\rho < \varepsilon_0,$$

the Schrödinger cocycle (A^d_{a-V}, ω) is reducible to constant coefficients for all strongly nonresonant frequencies and almost all $a \in \mathbb{R}$ (with respect to Lebesgue measure).

The proof of this Theorem will be given in Section VII.2.

Remark VII.2.

(i) Very recently Avila & Krikorian [AK03] *proved Theorem VII.1 but with more restrictive hypothesis on ω. In fact, we will see that both results follow from a nonperturbative theorem on localization by Bourgain & Jitomirskaya* [BJ02a].

(ii) Since Eliasson theorem works for potentials V defined on the d-dimensional torus one may wonder if the nonperturbative version above is true for this higher-dimensional situation. It turns out that it is not, as Bourgain showed in [Bou02a, Bou02b]. *He proved that, if $V : \mathbb{T}^2 \to \mathbb{R}$ is a trigonometric polynomial with a nondegenerate maximum, there is a set of $\omega \in \mathbb{R}^2$, with positive measure, for which the operators $H_{V,\omega,\phi}$ have some point spectrum. This point spectrum is incompatible with the reducible behaviour of the above theorem. See the review by Bourgain* [Bou04b] *for the differences between the cases of one and several frequencies.*

(iii) The main burden of the proof of Theorem VII.1 is to show that for almost all $a \in \sigma(V,\omega)$ the corresponding Schrödinger cocycle is reducible to constant coefficients. If a lies in the resolvent set, then the cocycle has an exponential dichotomy and, taking into account the hypothesis on V and ω, it is reducible to constant coefficients, see Theorem II.28.

(iv) *We would like to stress that Theorem VII.1 is not a full nonperturbative version of Eliasson's theorem because the set of spectral values a whose corresponding Schrödinger cocycle is reducible to constant coefficients is not characterized in terms of the rotation number.*

An immediate application of Theorem VII.1 is the existence of quasi-periodic *Bloch waves* for almost all a in the spectrum. An analytic quasi-periodic Bloch wave for a Harper-like equation (VII.3) is a solution of the form

$$x_n(\phi) = e^{\mathrm{i}\varphi n} f\left(2\pi\omega n + \phi\right), \qquad n \in \mathbb{Z}, \tag{VII.6}$$

where $\varphi \in [0, 2\pi)$ is called the *Floquet exponent* and $f : \mathbb{T} \to \mathbb{R}$ is an analytic function. In Section VII.2 we will also prove the following about the existence of quasi-periodic Bloch waves.

Corollary VII.3. *Let V, ω and ε_0 be as in Theorem VII.1. Then for (Lebesgue) almost all values of a in the spectrum $\sigma(V, \omega)$, the equation*

$$x_{n+1} + x_{n-1} + V(2\pi\omega n + \phi)x_n = ax_n,$$

has analytic quasi-periodic Bloch waves.

Using Theorem VII.1 one can adapt many of the results in the previous chapter to this nonperturbative and discrete setting. Here we mention only the existence of gaps. As in the Almost Mathieu case, if a cocycle (A^d_{a-V}, ω) is reducible to a Floquet matrix with trace ± 2, then a lies at the endpoint of a spectral gap of $\sigma(V, \omega)$. Moreover, this gap is collapsed if, and only if, the Floquet matrix is $\pm I$ (see Section VI.2). An adaption of Moser & Pöschel [MP84] to this discrete case shows that reducible collapsed gaps can be opened by means of arbitrarily small and suitable perturbations. Using this it is possible to produce examples of quasi-periodic Schrödinger operators which display open spectral gaps *nonperturbatively*. It can be shown that the values of a at endpoint of collapsed gaps whose corresponding Schrödinger cocycle is reducible to a Floquet matrix with trace ± 2 are dense in the spectrum if V and ω are as in Theorem VII.1. Using the previous genericity of gap opening it could be shown that Cantor spectrum is generic nonperturbatively. This has direct applications to the Hölder character of the integrated density of states, see Section IV.4.3. These applications will be given elsewhere.

Let us finally outline the contents of this chapter. In Section VII.1 we introduce the extension of Aubry duality for non Almost Mathieu operators, together with its link with the integrated density of states. In Section VII.2 this is used to prove VII.1 and VII.3 using a similar technique to the one used for the Almost Mathieu operator.

VII.1 Aubry duality and the Integrated Density of States

In this section we present some of the preliminaries that will be needed in the proof of Theorem VII.1. As said in the introduction, we plan to extend some of the ideas in the proof of the "Ten Martini Problem" in last chapter. More precisely we need a convenient version of Aubry duality, which will lead us to consider long-range operators, which are not of Schrödinger type. In last chapter the rotation number played an important role in the study of the spectral properties of the Almost Mathieu operator. For these long-range operators it will be more convenient to use the integrated density of states.

VII.1.1 Aubry Duality

Aubry Duality [AA80] was introduced for the study of the Almost Mathieu operator, where $V(\theta) = \cos(\theta)$, see Section VI.1, but the idea works for other potentials. Let us give first the heuristic approach and then a more functional one.

Assume that

$$x_{n+1} + x_{n-1} + V(2\pi\omega n + \phi)x_n = ax_n,$$

has an analytic quasi-periodic Bloch wave,

$$x_n = e^{\mathrm{i}\varphi n}\tilde{\psi}\left(2\pi\omega n + \phi\right), \tag{VII.7}$$

being $\tilde{\psi} : \mathbb{T} \to \mathbb{C}$ analytic and $\varphi \in [0, 2\pi)$. Letting $(\psi_n)_{n\in\mathbb{Z}}$ be the Fourier coefficients of $\tilde{\psi}$, a computation shows that these must satisfy the following difference equation

$$\sum_{k\in\mathbb{Z}} V_k\psi_{n-k} + 2\cos\left(2\pi\omega n + \varphi\right)\psi_n = a\psi_n \qquad n \in \mathbb{Z},$$

where V_k are the Fourier coefficients of V,

$$V(\theta) = \sum_{k\in\mathbb{Z}} V_k e^{\mathrm{i}k\theta}.$$

This difference equation is the eigenvalue equation of the operator

$$\left(L_{V,\omega,\varphi}\psi\right)_n = \sum_{k\in\mathbb{Z}} V_k\psi_{n-k} + 2\cos\left(2\pi\omega n + \varphi\right)\psi_n$$

which we call a *dual operator* of $H_{V,\omega,\phi}$. This is a self-adjoint and bounded operator on $l^2(\mathbb{Z})$, because V is real analytic, and it is not a Schrödinger operator unless V is exactly the cosine. Such an operator will be called a *long-range (quasi-periodic) operator* even if it may be a finite-difference operator (if V is a trigonometric polynomial).

If ω is nonresonant, the spectrum of the long-range operators $L_{V,\omega,\varphi}$ does not depend on the chosen φ, so that one can write

$$\sigma^L(V,\omega) = \mathrm{Spec}\left(L_{V,\omega,\varphi}\right).$$

To avoid confusion, in what follows we will write

$$\sigma^H(V,\omega) = \mathrm{Spec}\left(H_{V,\omega,\phi}\right),$$

which was previously denoted by $\sigma(V,\omega)$.

This naive approach to Aubry duality shows that if a is a value in the spectrum $\sigma^H(V,\omega)$ such that $(x_n)_{n\in\mathbb{Z}}$ is an analytic quasi-periodic Bloch wave of the form (VII.7) with Floquet exponent φ of the eigenvalue equation, then a is a point eigenvalue of the dual operator $L_{V,\omega,\varphi}$ whose eigenvector decays exponentially and therefore $a \in \sigma^L(V,\omega)$. The converse is also true: one can pass from exponentially decaying eigenvalues of $L_{V,\omega,\varphi}$ to quasi-periodic Bloch waves of $H_{V,\omega,\phi}$ with Floquet exponent φ.

The argument given above heavily relies on the existence of quasi-periodic Bloch waves or, equivalently, exponentially localized eigenvectors. It turns, however, that both operators can be related. This was done by see Avron & Simon [AS83]. Here we will follow the idea by Gordon, Jitomirskaya, Last & Simon [GJLS97] who studied duality for the Almost Mathieu operator although it can be extended to the general case, see Bourgain & Jitomirskaya [BJ02b]. The idea is to shift to more general spaces where the extensions of the operators H and their duals L are unitarily equivalent. Note that it is not true that the operators $H_{V,\omega,\phi}$ and $L_{V,\omega,\varphi}$ are unitarily equivalent, since their spectral measures may be very different.

Let us consider the following Hilbert space,

$$\mathcal{H} = L^2\left(\mathbb{T} \times \mathbb{Z}\right),$$

which consists of functions $\Psi = \Psi(\theta, n)$ such that

$$\sum_{n\in\mathbb{Z}} \int_{\mathbb{T}} \left|\Psi(\theta, n)\right|^2 d\theta < \infty.$$

This space "includes" at the same time $L^2(\mathbb{T})$ and $l^2(\mathbb{Z})$, considering the projections

$$\Psi \in \mathcal{H} \mapsto \pi_\theta(\Psi) \in l^2(\mathbb{Z}), \qquad \pi_\theta(\Psi)(n) = \Psi(\theta, n)$$

and

$$\Psi \in \mathcal{H} \mapsto \pi_n(\Psi) \in L^2(\mathbb{T}), \qquad \pi_n(\Psi)(\theta) = \Psi(\theta, n)$$

for any fixed $\theta \in \mathbb{T}$ and $n \in \mathbb{Z}$.

The extensions of the Schrödinger operators H and their long-range duals L are given in terms of their *direct integrals*, which we now define. The *direct integral* of the Schrödinger operator $H_{V,\omega,\phi}$, is the operator $\tilde{H}_{V,\omega}$, defined as

$$\left(\tilde{H}_{V,\omega}\Psi\right)(\theta, n) = \Psi(\theta, n+1) + \Psi(\theta, n-1) + V(2\pi\omega n + \theta)\Psi(\theta, n),$$

and the direct integral of $L_{V,\omega,\varphi}$, $\tilde{L}_{V,\omega}$, is

$$\left(\tilde{L}_{V,\omega}\Psi\right)(\theta, n) = \sum_{k\in\mathbb{Z}} V_k \Psi(\theta, n-k) + 2\cos\left(2\pi\omega n + \theta\right)\Psi(\theta, n).$$

These two operators are bounded and, for any fixed $\theta \in \mathbb{T}$, they satisfy

$$\pi_\theta \circ \tilde{H}_{V,\omega} = H_{V,\omega,\theta} \circ \pi_\theta$$

and

$$\pi_\theta \circ \tilde{L}_{V,\omega} = L_{V,\omega,\theta} \circ \pi_\theta.$$

We now want to see that, for any fixed real analytic V and nonresonant frequency ω, the direct integrals $\tilde{H}_{V,\omega}$ and $\tilde{L}_{V,\omega}$ are unitarily equivalent, which means that there exists a unitary operator U on $\mathcal{H}$ such that the conjugation

$$\tilde{H}_{V,\omega} U = U \tilde{L}_{V,\omega}$$

holds. By analogy with the heuristic approach to Aubry duality in the beginning of this section, let U be the following operator on $\mathcal{H}$,

$$\left(U\Psi\right)(\theta, n) = \hat{\Psi}\left(n, \theta + 2\pi\omega n\right),$$

where $\hat{\Psi}$ is the Fourier transform. Here the Fourier transform is the following operator

$$\Psi \in L^2(\mathbb{T} \times \mathbb{Z}) \mapsto \hat{\Psi} \in L^2(\mathbb{Z} \times \mathbb{T})$$

where

$$\hat{\Psi}(k, \phi) = \sum_{n \in \mathbb{Z}} \int_{\mathbb{T}} \Psi(\theta, n) e^{\mathrm{i}\phi n} e^{-\mathrm{i}\theta k} d\theta.$$

The map U is unitary and satisfies

$$\tilde{H}_{V,\omega} U = U \tilde{L}_{V,\omega}$$

by construction of the dual long-range operators in terms of the Schrödinger operators. Therefore, the direct integrals $\tilde{H}_{V,\omega}$ and $\tilde{L}_{V,\omega}$ are unitarily equivalent. In particular, due to the nonresonant character of ω one has

$$\sigma^H(V, \omega) = \bigcup_{\phi \in \mathbb{T}} \operatorname{Spec}(H_{V,\omega,\phi}) = \operatorname{Spec}\left(\tilde{H}_{V,\omega}\right) = \operatorname{Spec}\left(\tilde{L}_{V,\omega}\right) = \bigcup_{\varphi \in \mathbb{T}} \operatorname{Spec}(H_{V,\omega,\varphi}) = \sigma^L(V, \omega)$$

so that the spectrum of a quasi-periodic Schrödinger operator and its dual are the same. In the next section we will introduce the integrated density of states for long-range operators and we will see that this function is preserved by Aubry duality.

VII.1.2 The integrated density of states and duality

The integrated density of states of quasi-periodic Schrödinger operators, the IDS for short, has been introduced in Section III.2.2 in connection with the rotation number of the corresponding eigenvalue equations. Now we want to extend this definition to the dual long-range operators. In order to introduce this IDS in a unified way let us put the operators H and L in a more general framework. If $V, W : \mathbb{T} \to \mathbb{R}$ are real analytic functions, ω is a nonresonant frequency and $\phi \in \mathbb{T}$, let $K_{W,V,\omega,\phi}$ be the following operator

$$(K_{W,V,\omega,\phi} x)_n = \sum_{k \in \mathbb{Z}} W_k x_{n-k} + V(2\pi\omega n + \phi) x_n$$

acting on $l^2(\mathbb{Z})$, which is bounded and self-adjoint. The operators in the previous section occur as particular cases,

$$H_{V,\omega,\phi} = K_{2\cos,V,\omega,\phi} \quad \text{and} \quad L_{V,\omega,\phi} = K_{V,2\cos,\omega,\phi}.$$

Let us now define the IDS for the operators $K_{W,V,\omega,\phi}$. Take some integer $N > 0$ and consider $K^N_{W,V,\omega,\phi}$, the restriction of the operator $K_{W,V,\omega,\phi}$ to the interval $[-N, N]$ with zero boundary conditions. Let

$$k_N(a, W, V, \omega, \phi) = \frac{1}{2N+1} \# \left\{ \text{eigenvalues} \leq a \text{ of } K^N_{W,V,\omega,\phi} \right\}.$$

Then, due to nonresonant character of ω the limit

$$\lim_{N\to\infty} \frac{1}{2N+1} k_N(a, W, V, \omega, \phi)$$

exists, it is independent of ϕ and of the boundary conditions imposed above. It is called the *integrated density of states*, IDS , of the operator $K_{W,V,\omega,\phi}$ We will write this as $k(a, W, V, \omega)$. The map

$$a \in \mathbb{R} \mapsto k(a, W, V, \omega) \tag{VII.8}$$

is increasing and it is constant exactly at the spectral gaps of $K_{W,V,\omega,\phi}$. It is the distribution function of a Borel measure,

$$k(a, W, V, \omega) = \int_{-\infty}^{a} dn_{W,V,\omega}(\lambda)$$

which is the *density of states*, $n_{W,V,\omega}$. Its support is precisely the spectrum of $K_{W,V,\omega,\phi}$ (this comes from the characterization of the spectrum in terms of the growth of the IDS (VII.8)). In the Schrödinger case we will use the notations

$$k^H(a, V, \omega) = k(a, 2\cos, V, \omega), \quad n^H_{V,\omega} = n_{2\cos,V,\omega}$$

and

$$k^L(a, V, \omega) = k(a, V, 2\cos, \omega), \quad n^L_{V,\omega} = n_{V,2\cos,\omega}.$$

for their duals.

Remark VII.4. *In the Schrödinger case the Sturmian rotation number and the* IDS *satisfy the following relation*

$$k^H(a, V, \omega) = 2\mathrm{rot}^d_s(a - V, \omega),$$

for all $a \in \mathbb{R}$, see Section III.2.2.

Let us now relate the IDS of the operators $K_{W,V,\omega,\phi}$ to the spectral measures in Section III.1.3 (see Avron & Simon [AS83] and Cycon, Froese, Kirsch & Simon [CFKS87]).

Let $\delta_0 = (\delta_{0n})_{n\in\mathbb{Z}}$ be the Kronecker's delta function. In Section III.1.3 we saw that, for any bounded measurable function f, one can define $f(K_{W,V,\omega,\phi})$ using the spectral measure of $K_{W,V,\omega,\phi}$ associated to δ_0,

$$\langle \delta_0, f\left(K_{W,V,\omega,\phi}\right)\delta_0\rangle_{l^2(\mathbb{Z})} = \int f(\lambda) d\mu_{\delta_0}(\lambda).$$

Let μ_ϕ be the measure defined on Borel sets by

$$\mu_\phi(A) = \langle \delta_0, \chi_A\left(K_{W,V,\omega,\phi}\right)\delta_0\rangle_{l^2(\mathbb{Z})} = \int_A d\mu_{\delta_0}(\lambda),$$

where A is a Borel set and χ_A is its characteristic function. This is a *spectral measure* in the sense that $\chi_A(K_{W,V,\omega,\phi})$ is zero (as an operator) if, and only if, $\mu_\phi\left(A\right) = 0$.

Avron & Simon [AS83] proved that the IDS is the integral over $\phi \in \mathbb{T}$ of all these spectral measures μ_ϕ. That is, if f is any continuous function on the spectrum, then

$$\int f(\lambda) dn(\lambda) = \int_{\mathbb{T}} d\phi \int f(\lambda) d\mu_\phi(\lambda)$$

where, for simplicity in the notation, we write n for the density of states measure of $K_{W,V,\omega,\phi}$. Approximating characteristic functions by positive continuous functions, one readily sees that, for any Borel measurable set $A \subset \sigma(K_{W,V,\omega,\phi})$,

$$n(A) = \int_A dn(\lambda) = \int_{\mathbb{T}} d\phi \int_A d\mu_\phi(\lambda) = \int_{\mathbb{T}} \mu_\phi(A) d\phi. \tag{VII.9}$$

Using the fact that the scalar product in the space $L^2(\mathbb{T} \times \mathbb{Z})$ is the integral over $\mathbb{T}$ of the scalar product in $l^2(\mathbb{Z})$ one can prove the following,

Theorem VII.5 ([GJLS97]). *Let $k^L_{V,\omega}$ and $k^H_{V,\omega}$ be the integrated density of states of $H_{V,\omega,\phi}$ and $L_{V,\omega,\varphi}$ respectively, for some real analytic $V : \mathbb{T} \to \mathbb{R}$ and nonresonant frequency ω. Then*

$$k^L_{V,\omega}(a) = k^H_{V,\omega}(a)$$

for all $a \in \mathbb{R}$.

As a consequence of Equation (VII.9) we obtain the following result which will be crucial in the proof of Theorem VII.1. We state it for Schrödinger operators and their dual rather than in the full generality.

Proposition VII.6. *Let V be real analytic, ω non resonant and μ_ϕ a spectral measure of $L_{V,\omega,\phi}$. Assume that there is a measurable set A such that*

$$\mu_\phi(A) = 0$$

for almost every $\phi \in \mathbb{T}$. Then $n^L(A) = 0$ so that the Lebesgue measure of $A \cap \sigma^L(V, \omega)$ is zero. Also $n^H(A) = 0$ and, thus, the Lebesgue measure of $A \cap \sigma^H(V, \omega)$ is zero.

VII.2 Proof of Theorem VII.1

We are now ready to show that Theorem VII.1 is a direct consequence of the following theorem by Bourgain & Jitomirskaya [BJ02b], which we restate in a convenient way:

Theorem VII.7 ([BJ02b]). *Let $\rho > 0$ be a positive number. Then there is a constant $\varepsilon_0 = \varepsilon_0(\rho)$ such that, for any real analytic $V \in C^a_\rho(\mathbb{T}, \mathbb{R})$ with*

$$|V|_\rho < \varepsilon_0,$$

and strongly nonresonant ω there is a set $\Phi \subset \mathbb{T}$, of zero (Lebesgue) measure such that, if $\phi \notin \Phi$, the operator $L_{V,\omega,\phi}$ has pure point spectrum with exponentially decaying eigenfunctions.

Remark VII.8.

(i) In [BJ02a]*, the bound ε_0 depends on $\|V\|_1$, $\|V\|_2$, $\|V\|_\infty$ and ρ. If V belongs to $C^a_\rho(\mathbb{T},\mathbb{R})$, then all these previous norms can be controlled by $|V|_\rho$.*

(ii) The set Φ consists of those angles ϕ for which the relation

$$|\sin(\phi+\pi k\omega)| < \exp\left(-|k|^{\frac{1}{2\tau}}\right) \tag{VII.10}$$

holds for infinitely many values of k, where τ comes from the strong nonresonance of ω, $\omega \in DC^d(c,\tau,\mathbb{R})$. Note that the sets $\Phi = \Phi(\tau)$ have measure zero for any $\tau > 1$.

Let us now prove Theorem VII.1. Assume V, ω and Φ as in the Theorem VII.7. First of all we are going to prove Corollary VII.3 and from this we will prove the main result. The first step is to show that almost every $a \in \sigma^L(V,\omega)$ is a point eigenvalue of $L_{V,\omega,\phi}$, for some $\phi \in \mathbb{T}$, whose eigenvector in $l^2(\mathbb{Z})$ is exponentially localized. More precisely, let

$$\sigma^L_{pp}(V,\omega,\phi) \subset \sigma^L(V,\omega)$$

be the set of point eigenvalues of $L_{V,\omega,\phi}$. We will see that, if

$$A = \sigma^L(V,\omega) - \bigcup_{\phi\notin\Phi} \sigma^L_{pp}(V,\omega,\phi),$$

then

$$n^L(A) = |A| = 0,$$

where $|\cdot|$ stands for the Lebesgue measure on $\mathbb{R}$.

According to Proposition VII.6 we only need to show that $\mu_\phi(A) = 0$ for all $\phi \notin \Phi$, where μ_ϕ are the spectral measures defined in the previous section. This is a consequence from the fact that the spectral measures μ_ϕ, for $\phi \notin \Phi$ are supported on the set of point eigenvalues of the corresponding operator.

Using Proposition VII.6 we conclude that $n^L(A) = 0$ (and also $n^H(A) = 0$). In the beginning of Section VII.1.1 we saw that if a is a point eigenvalue of the operator $L_{V,\omega,\varphi}$ whose eigenfunction decays exponentially then the Harper-like equation

$$x_{n+1} + x_{n-1} + V(2\pi\omega n)x_n = ax_n, n \in \mathbb{Z} \tag{VII.11}$$

has analytic quasi-periodic Bloch wave with Floquet exponent φ. Putting everything together, the set of values of a for which Equation (VII.11) has a quasi-periodic Bloch wave is of total measure in the spectrum and Corollary VII.3 is proved.

VII.2.1 From Bloch waves to reducibility

Summing up the results from the previous section we have that, if V, ω and Φ are as in Theorem VII.7 then, for almost all $a \in \sigma^H(V,\omega)$, the equation (VII.11) has an analytic quasi-periodic Bloch wave with Floquet exponent $\varphi \notin \Phi$. Since we only want to prove a result for almost every a (and looking at the proof on full measure above), it is sufficient to show that if $\varphi \notin \Phi$ is such that

$$\varphi - \pi k\omega - \pi j \neq 0 \tag{VII.12}$$

for all $k, j \in \mathbb{Z}$ and (VII.11) has an analytic quasi-periodic Bloch wave with this Floquet exponent φ, then the corresponding Schrödinger cocycle (A^d_{a-V}, ω), where

$$A^d_{a-V}(\theta) = \begin{pmatrix} a - V(\theta) & -1 \\ 1 & 0 \end{pmatrix},$$

is reducible to constant coefficients.

Remark VII.9. *If $\varphi/2\pi$ is resonant with respect to ω,*

$$\varphi = \pi k + \pi j\omega,$$

for some integers k, j, then one can also prove reducibility using the same techniques of the previous chapter.

The existence of the Bloch wave implies that the Schrödinger cocycle has the following quasi-periodic solution

$$\begin{pmatrix} \tilde{\psi}(4\pi\omega + \theta) \\ e^{-\mathrm{i}\varphi}\tilde{\psi}(2\pi\omega + \theta) \end{pmatrix} = e^{-\mathrm{i}\varphi} \begin{pmatrix} a - V(\theta) & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \tilde{\psi}(2\pi\omega + \theta) \\ e^{-\mathrm{i}\varphi}\tilde{\psi}(\theta) \end{pmatrix} \tag{VII.13}$$

for all $\theta \in \mathbb{T}$. Writing

$$v(\theta) = \left(\tilde{\psi}(\theta + 2\pi\omega), e^{-\mathrm{i}\varphi}\tilde{\psi}(\theta)\right)^T \tag{VII.14}$$

and

$$Y(\theta) = \begin{pmatrix} v_1(\theta) & \bar{v}_1(\theta) \\ v_2(\theta) & \bar{v}_2(\theta) \end{pmatrix},$$

where the bar denotes complex conjugation, one always has the relation

$$A^d_{a-V}(\theta)Y(\theta) = Y(\theta + 2\pi\omega)\Lambda(\varphi), \tag{VII.15}$$

where

$$\Lambda(\varphi) = \begin{pmatrix} e^{\mathrm{i}\varphi} & 0 \\ 0 & e^{-\mathrm{i}\varphi} \end{pmatrix}.$$

Of course, Y will only define a conjugation if it is nonsingular. Note that, because of (VII.15), the determinant of Y is constant as a function of θ and it is purely imaginary. In particular, $v(\theta)$ and $\bar{v}(\theta)$ are linearly independent for all θ if, and only if, they are independent for some θ. In the case that these two v and $\bar{v}$ are linearly independent, it is not difficult to prove reducibility to constant coefficients of the cocycle.

Lemma VII.10. *Let $A : \mathbb{T} \to SL(2, \mathbb{R})$ be a real analytic map, with analytic extension to $|\mathrm{Im}\,\theta| < \delta$ and ω be Diophantine. Assume that there is a nonzero analytic map $v : \mathbb{T} \to \mathbb{R}^2$, with analytic extension to $|\mathrm{Im}\,\theta| < \delta$, with v and $\bar{v}$ linearly independent, such that*

$$v(\theta + 2\pi\omega) = e^{-\mathrm{i}\varphi}A(\theta)v(\theta)$$

holds for all $\theta \in \mathbb{T}$, where $\varphi \in [0, 2\pi)$. Then the cocycle (A, ω) is reducible to constant coefficients by means of a real analytic transformation in $SL(2, \mathbb{R})$ which has analytic extension to $|\mathrm{Im}\,\theta| < \delta$. Moreover, the Floquet matrix can be chosen to be of the form

$$B = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}. \tag{VII.16}$$

Proof: Let

$$Z^1(\theta) = \begin{pmatrix} v_1(\theta) & \bar{v}_1(\theta) \\ v_2(\theta) & \bar{v}_2(\theta) \end{pmatrix}$$

where

$$d(\theta) = v_1(\theta)\bar{v}_2(\theta) - \bar{v}_1(\theta)v_2(\theta).$$

is the determinant. Since v and $\bar{v}$ are linearly independent, Z^1 is real analytic and, for every $\theta \in \mathbb{T}$, $Z^1(\theta)$ is nonsingular. A computation shows that

$$A(\theta)Z^1(\theta) = Z^1(\theta + 2\pi\omega)B^1,$$

where

$$B^1 = \begin{pmatrix} e^{\mathrm{i}\varphi} & 0 \\ 0 & e^{-\mathrm{i}\varphi} \end{pmatrix}.$$

In particular, since ω is rationally independent and Z^1 is continuous, this shows that $d(\theta)$ is constant as a function of θ. By the linearity of our system, we choose this constant value to be $-\mathrm{i}/2$.

To obtain the real rotation consider the composition

$$Z(\theta) = Z^1(\theta)Z^2$$

where

$$Z^2 = \begin{pmatrix} 1 & -\mathrm{i} \\ 1 & \mathrm{i} \end{pmatrix},$$

which satisfies the desired conjugation

$$A(\theta)Z(\theta) = Z(\theta + 2\pi\omega)B$$

being B the rotation (VII.16). Thanks to the construction, it is readily checked that Z is real and with determinant one. □

Finally, it only remains to rule out the possibility that φ is as above (VII.12) and that v and $\bar{v}$ are linearly dependent at the same time. Note that both $v(\theta)$ and $\bar{v}(\theta)$ are different from zero for all $\theta \in \mathbb{T}$ by construction. Assume that $v(\theta)$ and $\bar{v}(\theta)$ were linearly dependent for all θ. Since these vectors depend analytically on θ, there would exist an analytic $h : \mathbb{T} \to \mathbb{R}$ and an integer $k \in \mathbb{Z}$ such that

$$\bar{v}(t) = e^{\mathrm{i}(h(t)+kt)}v(t)$$

for all $t \in \mathbb{R}$. Using that v and $\bar{v}$ are quasi-periodic solutions of (A, ω), this would imply that

$$e^{\mathrm{i}(h(t)+kt)}e^{\mathrm{i}\varphi} = e^{\mathrm{i}(h(t+2\pi\omega)+kt+2\pi k\omega)}e^{-\mathrm{i}\varphi}$$

for all $t \in \mathbb{R}$. Therefore, h must satisfy the following small divisors equation

$$h(\theta + 2\pi\omega) - h(\theta) = 2\varphi - 2\pi k\omega - 2\pi j$$

for all $\theta \in \mathbb{T}$, where $j \in \mathbb{Z}$. From the considerations of Section II.2.2 such analytic h cannot exist unless

$$\varphi = \pi\left(k\omega + j\right),$$

which is a contradiction with the nonresonance condition (VII.12). This ends the proof of Theorem VII.1. □

Appendix A

Quasi-Periodic Birkhoff Normal Forms

In this appendix we study the normalization of analytic Hamiltonians on $\mathbb{R}^n$ (and $\mathbb{C}^n$) which depend quasi-periodically on time. These are Hamiltonians of the form $h(x, y, t)$, being $t \mapsto h(x, y, t)$ a quasi-periodic function. The corresponding flow is given by the Hamilton equations

$$x'(t) = \frac{\partial h}{\partial y}(x, y, t), \qquad y'(t) = -\frac{\partial h}{\partial x}(x, y, t), \tag{A.1}$$

where $\cdot'$ stands for the derivation with respect to time.

Quasi-periodicity means that there exist a rationally independent frequency vector $\omega \in \mathbb{R}^d$ and a continuous function $\tilde{h} = \tilde{h}(x, y, \theta)$ such that

$$h(x, y, t) = \tilde{h}(x, y, \omega t)$$

for all $t \in \mathbb{R}$. If $d = 1$ the Hamiltonian is periodic in time and, therefore, periodic Hamiltonians occur as particular cases. In most of what follows, however, we will assume that $d \geq 2$.

Similarly to what we did with linear equations with quasi-periodic coefficients (see Chapter II), the flow (A.1) can be rendered autonomous

$$x' = \frac{\partial \tilde{h}}{\partial y}(x, y, \theta), \qquad y' = -\frac{\partial \tilde{h}}{\partial x}(x, y, \theta), \qquad \theta' = \omega. \tag{A.2}$$

where $\theta \in \mathbb{T}^d$. To make this flow Hamiltonian one can introduce new variables $I \in \mathbb{R}^d$, canonically conjugated to θ, and define

$$H(x, y, \theta, I) = \langle \omega, I \rangle + \tilde{h}(x, y, \theta).$$

This Hamiltonian defines a flow on $\mathbb{R}^{2n} \times \mathbb{T}^d \times \mathbb{R}^d$,

$$x' = \frac{\partial \tilde{h}}{\partial y}(x, y, \theta), \quad y' = -\frac{\partial \tilde{h}}{\partial x}(x, y, \theta), \quad \theta' = \omega, I' = 0.$$

We will focus on Hamiltonians which are perturbations of quadratic systems with constant coefficients. That is, we fix a rationally independent $\omega \in \mathbb{R}^d$ such that the Hamiltonian H can be written as

$$H(x, y, \theta, I) = H_0 + \varepsilon H_1 = \langle \omega, I \rangle + \sum_{i=1}^{n} \frac{\alpha_j}{2} \left(x_j^2 + y_j^2 \right) + \varepsilon H_1(x, y, \theta), \tag{A.3}$$

where ε is a real parameter, H_1 is of order greater or equal than two in x and y, and $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$ are such that the vector (ω, α) is strongly rationally independent. That is, there exist positive constants K and τ such that

$$|\langle \omega, \mathbf{k} \rangle + \langle \alpha, \mathbf{l} \rangle| \geq \frac{K}{(|\mathbf{k}| + |\mathbf{l}|)^\tau}, \tag{A.4}$$

holds for all $\mathbf{k} \in \mathbb{Z}^d$ and $\mathbf{l} \in \mathbb{Z}^n$ with $|\mathbf{k}| + |\mathbf{l}| \neq 0$.

Since we want to discuss the convergence of Birkhoff's normalization process and the resulting normal form for quasi-periodic Hamiltonians, it is important to fix the space of analytic Hamiltonians that we are considering and the norm there. Functions like

$$K = K(x, y, \theta, I, \varepsilon)$$

will be considered on open domains of the form

$$\mathcal{D}_\rho = \left\{(x, y, \theta, I, \varepsilon) \in \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^d \times \mathbb{C}^d \times \mathbb{C}; \max\left(|x|, |y|, |\mathrm{Im}\, \theta|, |I|, |\varepsilon|\right) < \rho\right\}$$

for a positive $\rho > 0$ which we now fix. A Hamiltonian belongs to the class $\mathcal{H}$ if

$$|K|_\rho := \sup_{(x,y,\theta,I,\varepsilon) \in \mathcal{D}_\rho} |K(x, y, \theta, I, \varepsilon)| < \infty.$$

The space of Hamiltonians in $\mathcal{H}$ of the form (A.3), for a fixed (ω, α), will be denoted by $\mathcal{H}_{(\omega,\alpha)}$. Real analytic Hamiltonians in $\mathcal{H}$ satisfy, in addition, that they are real for real values in $\mathcal{D}_\rho$. Particular cases are quadratic Hamiltonians, which satisfy that H_1 is a quadratic function of (x, y). In this case, the corresponding Hamilton equations are linear quasi-periodic differential equations.

A normal form reduction tries to transform a Hamiltonian like (A.3) into the simplest possible form suitable for the study to be done. This form depends strongly on the rational relations between ω and α and this is why the format of the normal form that will be presented in Theorem A.1 differs from the normal forms considered in Chapter IV. In Section A.1 the following theorem will be proved (compare with Arnol'd [Arn83a]).

Theorem A.1 (Quasi-Periodic Birkhoff Normal Form). *Consider a real (resp. complex) analytic quasi-periodic Hamiltonian of the form*

$$H_0 + \varepsilon H_1 = \langle \omega, I \rangle + \sum_{j=1}^{n} \frac{\alpha_j}{2} \left(x_j^2 + y_j^2\right) + \varepsilon H_1(x, y, \theta),$$

where the pair (ω, α) is strongly rationally independent and H_1 contains terms of order at least two in (x, y). Then, for any $N \geq 1$, there exists a real (resp. complex) analytic canonical transformation which takes the Hamiltonian to

$$H^N(x, y, \theta, I) = \langle \omega, I \rangle + \sum_{j=1}^{n} \frac{\alpha_j}{2} \left(x_j^2 + y_j^2\right) + \sum_{k=1}^{N} \varepsilon^k Z_k(x, y) + \varepsilon^{N+1} R_{N+1}(x, y, \theta),$$

where the Z_k are real (resp. complex) analytic functions of

$$\frac{1}{2}\left(x_j^2+y_j^2\right)$$

for $j \in \{1,\dots,n\}$ and are independent of θ. The remainder R_{N+1} is a real (resp. complex) analytic function in (x,y,θ) which is of order greater or equal than two in (x,y). Moreover, if H_1 is a quadratic function of (x,y) then H^N is also quadratic in (x,y).

Applying formally this result to a quasi-periodic Hamiltonian, one can transform the Hamiltonian to the normal form

$$H^\infty(x,y,\theta,I)=\langle\omega,I\rangle+\sum_{j=1}^{n}\frac{\alpha_j}{2}\left(x_j^2+y_j^2\right)+\sum_{k=1}^{\infty}\varepsilon^k Z_k(x,y),$$

which is integrable introducing the so-called Poincaré variables

$$x_j=\sqrt{2\tau_j}\cos\varphi_j,\qquad y_j=\sqrt{2\tau_j}\sin\varphi_j.$$

Also, if H is quadratic then the normalized Hamiltonian is also quadratic. Since H^∞ has no dependence on θ the corresponding skew-product flow has constant coefficients depending on ε. Therefore, for Hamiltonian linear differential equations with quasi-periodic coefficients, Birkhoff Normal Form allows to reduce to constant coefficients at a formal level. We will see in a moment, however, that both the normalization transformation and the resulting Hamiltonian H^∞ are generically divergent.

As it was already pointed out by Poincaré [Poi92], the transformation to Birkhoff Normal Form is generically divergent. We will prove that, generically, the Normal Form itself is divergent. We will follow Pérez-Marco [PM03] where the reader can find references on this problem. Our theorem is modelled after [PM03] and establishes the following dichotomy.

Theorem A.2. *Let (ω,α) be a strongly rationally independent pair and $\mathcal{H}_{(\omega,\alpha)}\subset\mathcal{H}$ as before. Then*

(i) If there exists a Hamiltonian $H_{div}\in\mathcal{H}_{(\omega,\alpha)}$ with divergent Birkhoff Normal Form, then for a generic Hamiltonian in $\mathcal{H}_{(\omega,\alpha)}$ its Birkhoff Normal Form is also divergent.

(ii) More precisely, all Hamiltonians in any real (resp. complex) affine one-dimensional subspace V of $\mathcal{H}_{(\omega,\alpha)}$ have a convergent Birkhoff Normal Form or only an exceptional zero Lebesgue measure (resp. polar) subset of Hamiltonians in V have this property.

The second theorem in this appendix provides us with an example of a quasi-periodic Hamiltonian with divergent Birkhoff Normal Form for each strongly rationally independent pair (ω,α).

Theorem A.3. *Let (ω,α) be strongly rationally independent with $d\geq 2$. Then, there exists a quasi-periodic Hamiltonian in $\mathcal{H}_{(\omega,\alpha)}$ with divergent normal form.*

This will be proved in Section A.2.1 by constructing examples of quadratic Hamiltonians with diverging Birkhoff Normal Form. As a consequence one has the following.

Corollary A.4. *Let (ω,α) be a strongly rationally independent pair. Then a generic Hamiltonian in $\mathcal{H}_{(\omega,\alpha)}$ has divergent Birkhoff Normal Form.*

A.1 Quasi-periodic Birkhoff Normal Form

In this section we will discuss quasi-periodic Hamiltonians and their normalization by means of Birkhoff method. Here we will follow the method by Giorgilli & Galgani [GG78] (see also [GG85], Fassò [Fas89] for the normal form theory and Gómez, Jorba, Masdemont & Simó [GJSM01], Jorba & Villanueva [JV97] and Gabern [Gab03] for the adaption to the nonautonomous case). For other approaches to quasi-periodic normal forms see Bryuno [Bry89], Chow *et al.* [CLS92], Braaksma & Broer [BB87].

Consider a quasi-periodic Hamiltonian of the form

$$H(x,y,\theta,I)=H_0+\varepsilon H_1=\langle\omega,I\rangle+\sum_{j=1}^{n}\frac{\alpha_j}{2}\left(x_j^2+y_j^2\right)+\varepsilon H_1(x,y,\theta), \tag{A.5}$$

where the terms of $H_1(x,y,\theta)$ are of, at least, order two in (x,y). Recall that $(x,y)\in\mathbb{R}^{2n}$ and $I\in\mathbb{R}^d$ is an auxiliary variable to make the system autonomous, which is canonically conjugated to $\theta\in\mathbb{T}^d$. Our aim is to define a canonical transformation to render (A.5) into the simplest form. That is, for each $N\geq 1$, we will transform our system to a Hamiltonian of the form

$$H^N(x,y,\theta,I)=Z_0+\sum_{k=1}^{N}\varepsilon^k Z_k(x,y)+\varepsilon^{N+1}R_{N+1}(x,y,\theta), \tag{A.6}$$

where $Z_0=H_0$ and $R_1=H_1$ for consistency. If this can be done for all $N\geq 1$ we will have, disregarding the convergence of the sequence of transformations and corresponding reduced Hamiltonians, that the original Hamiltonian becomes

$$H^\infty(x,y,\theta,I)=Z_0+\sum_{k=1}^{\infty}\varepsilon^k Z_k(x,y). \tag{A.7}$$

Note that this can be written as a formal power series in ε,x,y,

$$H^\infty(x,y,\theta,I)=Z_0+\sum_{k=1}^{\infty}\sum_{|\mathbf{l}_1|+|\mathbf{l}_2|\geq 2}Z_k^{\mathbf{l}}\varepsilon^k x^{\mathbf{l}_1}y^{\mathbf{l}_2},$$

where $\mathbf{l}=(\mathbf{l}_1,\mathbf{l}_2)\in\mathbb{Z}^n\times\mathbb{Z}^n$ have nonnegative components. As already noted by Poincaré [Poi92], this transformation is generically divergent, although, for any finite order of normalization N, it is analytic on some domain (which shrinks to the void set as N increases).

Before outlining the method of normalization, let us briefly discuss the role of the parameter ε. Assume that for a certain real value of $\varepsilon_0>0$, the series (A.7) is absolutely convergent as a function of (x,y) in a suitable neighborhood of the origin. Then for $|\varepsilon|<\varepsilon_0$, the normal form is also convergent on the variables (ε,x,y). Thus, the addition of a parameter ε is to some extent artificial if we only regard convergence of the normal form. However, it is customary to work with such a parameter in series expansions of classical mechanics, where it is usually a perturbing parameter. Moreover it is very convenient in the labelling of the terms of the normal form.

Remark A.5. *Instead of a single perturbation parameter one can use any number of them. The adaption to this multi-parameter case is straightforward although the notation may be a bit cumbersome.*

Let us proceed in the Birkhoff normalization process. For the moment, we disregard the question of the convergence of the normal form and the normalizing transformation.

Let $\chi = \chi(x, y, \theta, \varepsilon)$ a function of the form

$$\chi = \chi(x, y, \theta, \varepsilon) = \sum_{j \geq 1} \varepsilon^j \chi_j(x, y, \theta).$$

Then, for any $g = g(x, y, \theta, I)$, one can define the (canonical) transformation $g \mapsto T_\chi g$, with

$$T_\chi g = \sum_{j \geq 0} \varepsilon^j g_j,$$

where the coefficients g_j are defined by the following recursive relations

$$g_0 = g, \qquad g_r = \sum_{j=1}^{r} \frac{j}{r} L_{\chi_j} g_{r-j}, \tag{A.8}$$

being $L_{\chi_j} = \{\chi_j, \cdot\}$ defined in terms of the *Poisson bracket*,

$$L_{\chi_j} g = \{\chi_j, g\} = \frac{\partial \chi_j}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial \chi_j}{\partial y}\frac{\partial g}{\partial x} + \frac{\partial \chi_j}{\partial \theta}\frac{\partial g}{\partial I} - \frac{\partial \chi_j}{\partial I}\frac{\partial g}{\partial \theta}.$$

For a quasi-periodic Hamiltonian H like (A.5) and some $N \geq 2$ fixed, we will look for a $\chi = \chi(x, y, \theta, \varepsilon)$ such that the transformed Hamiltonian $T_\chi H$ is in normal form (A.6). Since $H = Z_0 + \varepsilon R_1$ and due to the linearity of the adjoint operator of χ one has

$$T_\chi H = Z_0 + \sum_{j \geq 1} \varepsilon^j \left(h_j + f_{j-1}\right),$$

where the coefficients h_j and f_j are defined by the recursion relations (A.8) with $h_0 = Z_0$ and $f_0 = R_1$. By comparison with (A.6) one gets

$$h_j + f_{j-1} = Z_j.$$

Using also

$$h_s = L_{\chi_s} h_0 + \sum_{j=1}^{s-1} \frac{j}{s} L_{\chi_j} h_{s-j}$$

and $L_{\chi_s} h_0 = -L_{h_0} \chi_s$ one has that for $s = 1, \ldots, r$,

$$L_h \chi_s + Z_s = \Psi_s, \tag{A.9}$$

where Ψ_s is given by

$$\Psi_1 = f_0$$

and

$$\Psi_s = f_{s-1} + \sum_{j=1}^{s-1} \frac{j}{s} L_{\chi_j} h_{s-j} \tag{A.10}$$

for $s \geq 2$, writing $h = h_0 = Z_0$ for simplicity.

The key point now is to solve the *homological equation* (A.9). Indeed, each Ψ_s depends only on f_{s-1}, $\chi_1, \ldots, \chi_{s-1}$ and $Z_1, \ldots, Z_{s-1}$, which are terms of order less than s in ε. If we are able to solve (A.9) then all the terms $Z_1, \ldots, Z_N$ and the remainder R_{N+1} can be computed recursively, as well as $\chi_1, \ldots, \chi_N$.

Let us now focus on (A.9) at a formal level. Since it is a linear equation on the space of formal power series in $x, ye^{\mathrm{i}\theta}$, it would be very convenient that the operator L_h was diagonal in the basis formed by monomials. This is achieved by means of a complexification of the variables (x, y). These new complex canonical variables are defined by means of the following canonical transformation:

$$x = \frac{1}{\sqrt{2}}\left(q + \mathrm{i}p\right), \qquad y = \frac{1}{\sqrt{2}}\left(p + \mathrm{i}q\right).$$

In these new variables, the quadratic part $h = h_0 = Z_0$ takes the form

$$h = \langle \omega, I\rangle + \sum_{j=1}^{n} \mathrm{i}\alpha_j q_j p_j,$$

so that

$$L_h\chi_s = \frac{\partial h}{\partial q}\frac{\partial \chi_s}{\partial p} - \frac{\partial h}{\partial p}\frac{\partial \chi_s}{\partial q} + \frac{\partial h}{\partial \theta}\frac{\partial \chi_s}{\partial I} - \frac{\partial h}{\partial I}\frac{\partial \chi_s}{\partial \theta} = \sum_{j=1}^{n} \mathrm{i}\alpha_j\left(p_j\frac{\partial \chi_s}{\partial p_j} - q_j\frac{\partial \chi_s}{\partial q_j}\right) - \langle \omega, \frac{\partial \chi_s}{\partial \theta}\rangle.$$

Consider a monomial of the form

$$\mu(q,p,\theta) = q^{\mathbf{l}_1}p^{\mathbf{l}_2}e^{\mathrm{i}\langle \mathbf{k},\theta\rangle}, \tag{A.11}$$

for some $(\mathbf{l}_1, \mathbf{l}_2) = (l_1^1, \ldots, l_1^n, l_2^1, \ldots, l_2^n) \in \mathbb{Z}^n \times \mathbb{Z}^n$, with nonnegative components satisfying $|\mathbf{l}_1| + |\mathbf{l}_2| \geq 2$ and $\mathbf{k} \in \mathbb{Z}^d$. Let us compute the action of L_h on this monomial. On one hand we have

$$\begin{aligned}\sum_{j=1}^{n} \mathrm{i}\alpha_j\left(p_j\frac{\partial \mu}{\partial p_j} - q_j\frac{\partial \mu}{\partial q_j}\right) &= \sum_{j=1}^{n} \mathrm{i}\alpha_j\left(l_2^j p_j q^{\mathbf{l}_1}p^{\mathbf{l}_2-\beta_j} - l_1^j q_j q^{\mathbf{l}_1-\beta_j}p^{\mathbf{l}_2}\right)e^{\mathrm{i}\langle \mathbf{k},\theta\rangle} = \\ &= \sum_{j=1}^{n} \mathrm{i}\alpha_j\left(l_2^j - l_1^j\right)\mu(q,p,\theta) = \mathrm{i}\langle \alpha, \mathbf{l}_2 - \mathbf{l}_1\rangle\mu(q,p,\theta),\end{aligned} \tag{A.12}$$

where β_j is the jth element of the canonical basis of $\mathbb{Z}^n$. On the other hand,

$$\langle \omega, \frac{\partial \mu}{\partial \theta}\rangle = \mathrm{i}\langle \omega, \mathbf{k}\rangle q^{\mathbf{l}_1}p^{\mathbf{l}_2}e^{\mathrm{i}\langle \mathbf{k},\theta\rangle}. \tag{A.13}$$

Hence, putting everything together,

$$L_h\mu = \mathrm{i}\left(\langle \alpha, \mathbf{l}_2 - \mathbf{l}_1\rangle - \langle \omega, \mathbf{k}\rangle\right)\mu(q,p,\theta),$$

so that every monomial like μ is an eigenvector of the operator L_h with $\mathrm{i}\left(\langle \alpha, \mathbf{l}_2 - \mathbf{l}_1\rangle - \langle \omega, \mathbf{k}\rangle\right)$ as eigenvalue. In particular the kernel of the adjoint operator L_h is generated by those monomials for which the relation

$$\langle \alpha, \mathbf{l}_2 - \mathbf{l}_1\rangle - \langle \omega, \mathbf{k}\rangle = 0 \tag{A.14}$$

holds. Since we are assuming that the pair (α, ω) is strongly rationally independent (and in particular rationally independent) the only $(\mathbf{l}_1, \mathbf{l}_2, \mathbf{k})$ that satisfy (A.14) are $\mathbf{l}_1 = \mathbf{l}_2$ and $\mathbf{k} = \mathbf{0}$. The corresponding monomials will be called *resonant monomials*.

To solve formally equation (A.9) in the unknowns χ_s and Z_s we can do the following. Split Ψ_s (which is already known) into the resonant and nonresonant parts:

$$\Psi_s = \Psi_s^{res} + \Psi_s^{nores}$$

which contain, respectively, resonant and nonresonant monomials and solve

$$L_h \chi_s = \Psi_s^{nores}$$

letting

$$Z_s = \Psi_s^{res},$$

where χ_s contains only nonresonant monomials and Z_s only resonant monomials. The arithmetic assumption on (ω, α) makes this formal solution χ_s and Ψ_s analytic if Ψ_s is also analytic, compare with the methods in Section II.2.2.

Undoing the changes in (q, p) and taking into account the identity

$$\mathrm{i} q_j p_j = \frac{1}{2}\left(x_j^2 + y_j^2\right)$$

for $j = 1, \ldots, n$ we obtain the desired normal form. Together with some other easy adaptions to the real analytic or quadratic cases, this proves Theorem A.1. □

A.2 Cantor spectrum and divergent normal forms

In this section we will construct examples of quasi-periodic Hamiltonians with divergent Birkhoff Normal Form. These examples are cooked after a reelaboration of the methods in Chapter IV which we now briefly recall. Later, in Section A.2, such examples are constructed.

A.2.1 Hill's equation, Quasi-periodic Schrödinger operators and Cantor spectrum

Hill's equation with quasi-periodic forcing,

$$x'' + (a + q(t))\, x = 0, \tag{A.15}$$

considered in chapters III and IV can be thought as one of the simplest examples of a quasi-periodic Hamiltonian system. Let us assume that the frequency ω of q is strongly rationally independent. Hill's equation defines a linear equation with quasi-periodic coefficients introducing $y = x'$ and $\theta \in \mathbb{T}^d$,

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -a - Q(\theta) & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \qquad \theta' = \omega \tag{A.16}$$

The fact that the matrix above belongs to $sl(2,\mathbb{R})$ indicates that it comes from the following quasi-periodic quadratic Hamiltonian,

$$H(x,y,\theta,I) = \langle\omega, I\rangle + \frac{1}{2}\left(y^2 - (a+Q(\theta))\,x^2\right), \tag{A.17}$$

with $I \in \mathbb{R}^d$, which is is real analytic if q is a real analytic quasi-periodic function.

In Section III.2.2 the rotation number for such equations, $\mathrm{rot}^c(a-Q,\omega)$, was introduced. An important link between the spectral properties of the corresponding Schrödinger operator,

$$(H_q x)(t) = -x''(t) - q(t)x(t), \tag{A.18}$$

and Hill's equation (A.15) is that the spectrum of H_q on $L^2(\mathbb{R})$ is the set of points of increase of the map

$$a \mapsto \mathrm{rot}^c(a-Q,\omega). \tag{A.19}$$

In Section III.2.2 the Cantor structure of the spectrum was discussed. In terms of the rotation number, the spectrum of H_q is a Cantor set if, and only if, the above map has a dense set of intervals of constancy. We will use a combination of Eliasson's results III.27 and III.30 to assume the following.

Let $\omega \in \mathbb{R}^d$ and $\alpha > 0$ such that (ω,α) is strongly rationally independent. Assume that $Q : \mathbb{T}^d \to \mathbb{R}$ is a real analytic function and $a_0 \in \mathbb{R}$ are such that

H1 The spectrum $\sigma^c(Q,\omega)$ is a Cantor set. In particular, the map (A.19) is not analytic around any point in the spectrum.

H2 The rotation number of a_0, $\mathrm{rot}^c(a_0-Q,\omega)$ is α.

H3 The skew-product flow (A.16) for $a = a_0$ is reducible to constant coefficients with frequency ω and Floquet matrix

$$B = \begin{pmatrix} 0 & \alpha \\ -\alpha & 0 \end{pmatrix}. \tag{A.20}$$

Moreover, the lift of the reducing transformation to $\mathbb{T}^d$ belongs to $C^a_\rho(\mathbb{T}^d,\mathbb{R})$.

A.2.2 Reducibility and diverging normal forms

Let us now show that, if a_0 and Q satisfy H1-H3 above, we can construct a quasi-periodic Hamiltonian in $\mathcal{H}_{(\omega,\alpha)}$ with diverging Birkhoff Normal Form. The reducibility of (A.16), for $a = a_0$, to constant coefficients means that there exists a symplectic quasi-periodic transformation

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} z_{11}(\omega t) & z_{12}(\omega t) \\ z_{21}(\omega t) & z_{22}(\omega t) \end{pmatrix} \xi, \tag{A.21}$$

where $\xi \in \mathbb{R}^2$, which reduces system (A.16) to

$$\xi' = \alpha J \xi, \qquad \theta' = \omega \tag{A.22}$$

where J is the usual 2×2 symplectic matrix. In particular, applying this change of variables to system (A.16) with $a = a_0 + \varepsilon$, being ε a real parameter, we obtain

$$\xi' = \left(\left(\begin{array}{cc} 0 & \alpha \\ -\alpha & 0 \end{array} \right) + \varepsilon \left(\begin{array}{cc} r_{11} & r_{12} \\ -r_{21} & -r_{11} \end{array} \right) \right) \xi, \qquad \theta' = \omega, \tag{A.23}$$

for some r_{ij} which are real analytic functions whose lifts R_{ij} have analytic extension to $|\mathrm{Im}\,\theta| < \rho$ by H3. System (A.23) can be written in terms of its Hamilton function:

$$H(\xi_1, \xi_2, \theta, J) = \langle I, \omega \rangle + \frac{1}{2}\alpha\xi_1^2 + \frac{1}{2}\alpha\xi_2^2 + \varepsilon \left(\frac{1}{2} r_{21}\xi_1^2 + r_{11}\xi_1\xi_2 + \frac{1}{2} r_{12}\xi_2^2 \right), \tag{A.24}$$

This is a Hamiltonian in $\mathcal{H}_{(\omega,\alpha)}$. Let us now apply Birkhoff normalization process, with

$$Z_0 = \langle \omega, I \rangle + \frac{1}{2}\alpha\xi_1^2 + \frac{1}{2}\alpha\xi_2^2.$$

Theorem A.1 implies that, for each $N \geq 2$, one can transform Hamiltonian (A.24) to

$$H^N(q, p, \theta, J) = Z_0 + \sum_{k=1}^{N} \varepsilon^k Z_k(q, p) + \varepsilon^{N+1} R_{N+1}(q, p, \theta),$$

where the $Z_1, \ldots, Z_N, R_{N+1}$ are quadratic and the Z_j do not depend on time,

$$Z_k(q, p) = \frac{\beta_k}{2}\left(q^2 + p^2\right)$$

for some constants β_k independent of ε. The functions H^N, R_{N+1} and the normalizing transformation are analytic in suitable neighborhoods of the origin. At a formal level, this can be continued up to infinite order and obtain a θ-free Hamiltonian:

$$H^\infty(q, p, I) = Z_0 + \sum_{k=1}^{\infty} \varepsilon^k Z_k(q, p) = \langle \omega, I \rangle + \frac{1}{2}\left(\alpha + \sum_{k=1}^{\infty} \beta_k \varepsilon^k \right)\left(q^2 + p^2\right).$$

whose corresponding flow is a linear equation with constant coefficients. Stopping at a finite order, the differential equation defined by H^N is

$$z' = \left(\left(\begin{array}{cc} 0 & \alpha^N(\varepsilon) \\ -\alpha^N(\varepsilon) & 0 \end{array} \right) + \varepsilon^{N+1} P_{N+1}(\theta) \right) z, \qquad \theta' = \omega \tag{A.25}$$

where $z = (q, p)^T$,

$$P_{N+1} = P_{N+1}(\theta, \varepsilon) = \left(\begin{array}{cc} P_{3,N+1} & P_{2,N+1} \\ -P_{1,N+1} & -P_{3,N+1} \end{array} \right)$$

is also symplectic and real analytic and

$$\alpha^N(\varepsilon) = \alpha + \sum_{k=1}^{N} \beta_k \varepsilon^k.$$

In polar coordinates, $\varphi = \arg(p + \mathrm{i}q)$, Equation (A.25) becomes

$$\begin{aligned} \varphi' &= \alpha^N(\varepsilon) + \varepsilon^{N+1}\left(P_{1,N+1}(\theta)\sin^2\varphi + 2P_{3,N+1}(\theta)\sin\varphi\cos\varphi + P_{2,N+1}(\theta)\cos^2\varphi\right), \\ \theta' &= \omega. \end{aligned}$$

which we rewrite as

$$\varphi' = \alpha^N(\varepsilon) + \varepsilon^{N+1}\Phi_{N+1}(\varphi,\theta,\varepsilon), \qquad \theta' = \omega,$$

where Φ_{N+1} is a real analytic function, quadratic in $(\cos\varphi, \sin\varphi)$. Let Φ_+^{N+1} and Φ_-^{N+1} be, respectively, the maximum and minimum of Φ^{N+1} in $\theta \in \mathbb{T}^d$, $\varphi \in \mathbb{S}^1$, $|\varepsilon| \leq \varepsilon_0$, for some $\varepsilon_0 > 0$. Then the rotation number of (A.25), which we denote by $\mathrm{rot}(\varepsilon)$ (and was $\mathrm{rot}^c(a_0 + \varepsilon - Q, \omega)$ in the notations of Chapter III) satisfies the bounds

$$\varepsilon^{N+1}\Phi_-^{N+1} \leq \mathrm{rot}(\varepsilon) - \alpha^N(\varepsilon) \leq \varepsilon^{N+1}\Phi_+^{N+1}.$$

In particular, since $\alpha^N(\varepsilon)$ is an Nth order polynomial, $\varepsilon \mapsto \mathrm{rot}(\varepsilon)$ is N-times differentiable at $\varepsilon = 0$ and its Taylor expansion up to order N is given by $\alpha^N(\varepsilon)$.

More importantly, if the normal form was convergent, this would imply that α^∞ would be convergent in a neighborhood of $\varepsilon = 0$. In particular, the rotation number would be analytic in a neighborhood of $\varepsilon = 0$. This is a contradiction with H1 and the Normal Form cannot be convergent.

To get the formulation of Theorem A.3 we only need to consider the product of n uncoupled Hill's equations. In this case the normal form will also be the product of the normal forms so that the divergence of the latter will imply the divergence of the first. □

A.3 Polar sets and proof of Theorem A.2

In this section we will prove Theorem A.2. This will follow the guidelines of Pérez-Marco [PM03] adapted to our setting. In particular, note that we work with polar sets instead of pluri-polar sets. We start giving a brief account of the theory that we will need and then give the proof of Theorem A.2.

A.3.1 Some potential theory

Here we give some ideas and definitions from potential theory on the complex plane. For a proper exposition and proofs see the monograph by Ransford [Ran95], which we now freely quote.

Let $U \subset \mathbb{C}$ be an open subset of the complex plane. A complex function defined on U is *subharmonic* if it is upper semi-continuous and it satisfies the *local submean inequality*. That is, for any $z \in U$, there exists a positive $\rho > 0$ such that for any radius $0 \leq r < \rho$ the following inequality holds

$$u(z) \leq \frac{1}{2\pi}\int_0^1 u\left(z + re^{2\pi\mathrm{i}t}\right) dt.$$

A basic example of a subharmonic function on $U \subset \mathbb{C}$ is given by $\log|f|$, being f any analytic function on U. Another example is the *potential* of a finite Borel measure μ with compact support, which is the function $p_\mu : \mathbb{C} \to [-\infty, \infty)$ given by

$$p_\mu(z) = \int \log|z - w| d\mu(w)$$

for all $z \in \mathbb{C}$, which is a subharmonic function on $\mathbb{C}$. For such a measure, one can define its *energy* as

$$I(\mu) = \int p_\mu(z) d\mu(z) = \int \int \log|z - w| d\mu(z) d\mu(w).$$

Different Borel measures supported on compact subsets of the same $E \subset \mathbb{C}$ can have different energies. Therefore, it is convenient to introduce the *logarithmic capacity* (or *capacity*) as

$$c(E) = \sup_\mu e^{I(\mu)}$$

where the supremum is taken over all Borel probability measures μ on $\mathbb{C}$ whose support is a compact subset of E and the convention $e^{-\infty} = 0$ is taken. The sets with zero capacity are called *polar sets*.

For any polar set $E \subset \mathbb{C}$, there exists a subharmonic function on $\mathbb{C}$ for which E is the preimage of $-\infty$. Borel and Polar subsets of $\mathbb{C}$ have Lebesgue area zero. Also, the intersection of any polar subset of $\mathbb{C}$ with $\mathbb{R}$ has zero measure in $\mathbb{R}$. Finally, let us mention that the countable union of polar sets is polar. Our main tool in the proof of Theorem A.2 will be the following result.

Theorem A.6 (Berstein's Lemma). *Let C be a nonpolar compact subset of $\mathbb{C}$. If P is a polynomial of degree r then, for all $z \in \mathbb{C}$,*

$$|P(z)| \leq \|P\|_{C^0(C)} e^{rV_C(z)},$$

where

$$V_C(z) = \sup\left\{u(z);\ u \text{ is a subharmonic function of minimal growth such that } u_{|C} \leq 0\right\}$$

is the Green's function for C.

Remark A.7. *A subharmonic function u like in the theorem is said to be of minimal growth if $u(z) - \log|z|$ is bounded from above when $|z| \to \infty$.*

As a consequence of A.6, if C_1 is any compact set which contains C, then

$$M = \sup_{z \in C_1} |V_C(z)| < \infty$$

and

$$\|P\|_{C^0(C_1)} \leq \|P\|_{C^0(C)} e^{rM}.$$

A.3.2 Proof of Theorem A.2

Let us first show that item (ii) implies (i) in Theorem A.2. That is, if there is a quasi-periodic Hamiltonian H_{div} in $\mathcal{H}_{(\omega,\alpha)}$ with diverging normal form then the normal form of a generic Hamiltonian in $\mathcal{H}_{(\omega,\alpha)}$ is also diverging.

Consider the set F_k of Hamiltonians $H_0 + \varepsilon H_1$ in $\mathcal{H}_{(\omega,\alpha)}$ whose Birkhoff Normal Form K_H converges in $\mathcal{D}_{\rho/k}$ and such that

$$|K_H|_{1/k} = \sup_{|x|,|y|,|\varepsilon|<\rho/k} |K_H(x,y,\varepsilon)| \leq 1.$$

To prove the first statement we will show that if F is the set of quasi-periodic Hamiltonians with converging normal form,

$$F = \bigcup_{k\geq 1} F_k,$$

then $\mathcal{H}_{(\omega,\alpha)} - F$ is the countable intersection of open and dense subsets of $\mathcal{H}_{(\omega,\alpha)}$. Since

$$\mathcal{H}_{(\omega,\alpha)} - F = \bigcap_{k\geq 1} \left(\mathcal{H}_{(\omega,\alpha)} - F_k\right)$$

we only need to prove that for k large enough the sets F_k are closed with void interior.

These F_k are closed. Indeed, consider a sequence of quasi-periodic Hamiltonians $(H_j)_j \subset F_k$ converging uniformly on compact sets to some $H \in \mathcal{H}_{(\omega,\alpha)}$. We want to see that the Birkhoff Normal Form of H, K_H, belongs to F_k. Let $(K_{H_j})_j$ be the sequence of normal forms of the $(H_j)_j$. Since the H_j belong to F_k,

$$\left|K_{H_j}\right|_{1/k} \leq 1$$

and, therefore, they form a normal family. This implies the existence of a subsequence which is convergent. Let us see that the whole sequence is convergent. At each step of the normal form the conjugation is analytic and previous terms are not changed by transformations of higher order, so that any limit point of the sequence $(K_{H_j})_j$ must be K_H because of the coefficient convergence. Therefore, $|K_H|_{\rho/k} \leq 1$ and, thus, it belongs to F_k.

Now we will show that the F_k have void interior. Let us assume the contrary, that is, there exists a quasi-periodic Hamiltonian H_1 in the interior of F_k. Let us consider the real (or complex) line

$$V = \{(1-s)H_{div} + sH_1;\ s \in \mathbb{R}\} \subset \mathcal{H}_{(\omega,\alpha)},$$

where H_{div} is the Hamiltonian in $\mathcal{H}_{(\omega,\alpha)}$ with diverging normal form.

Since we are assuming that (ii) holds, the set of $s \in \mathbb{R}$ (resp. $s \in \mathbb{C}$) such that the corresponding Hamiltonian in V has a diverging normal form must have zero Lebesgue measure (resp. it must be polar). However, since H_1 belongs to the interior of F_k, in a neighborhood of $s = 1$ the Hamiltonians H_s have converging normal form. This is a contradiction with the zero measure (resp. polarity).

Let us now prove item (i) in Theorem A.2. Note that the real case will follow from the complex one through the observation that the intersection of a polar set in $\mathbb{C}$ with $\mathbb{R}$ has zero Lebesgue measure on $\mathbb{R}$.

Let $V \subset \mathcal{H}_{(\omega,\alpha)}$ be a complex line, which we parameterize by $s \in \mathbb{C}$. This means that the coefficients of the Hamiltonians H_s in this line are linear functions of this parameter s.

Assume that the Birkhoff Normal Form of the Hamiltonians H_s corresponding to a set of values $s \in F \subset \mathbb{C}$, with F not polar, are converging. We will show that in this case all other Hamiltonians in V must have a converging normal form. Let

$$F = \bigcup_{k \geq 1} F_k,$$

being F_k the set of parameters $s \in \mathbb{C}$ such that the corresponding Hamiltonian H_s has a Birkhoff normal form K_s which satisfies that

$$|K_s|_{\rho/k} \leq 1.$$

If F is not a polar set, then there exist arbitrarily large values of k whose corresponding F_k are not polar. Also, these sets F_k are closed (the proof is the same than for item (i)). Write this normal form as

$$K_s(x, y, \varepsilon) = Z_0 + \sum_{r \geq 1}^{\infty} \varepsilon^r Z_r(x, y, s) = Z_0 + \sum_{r \geq 1} \sum_{|j_1|+|j_2| \geq 2} K_r^{\mathbf{j}}(s) \varepsilon^r x^{j_1} y^{j_2}$$

being $K_r^{\mathbf{j}}(s)$ a rth order polynomial in s due to the format of the Birkhoff Normal Form. If s belongs to F_k the above K_s is convergent in ρ/k, so there exists a $\rho_0 > 0$ such that

$$\varphi(s) = \sup_{r \geq 1, |\mathbf{j}| \geq 2} \left| K_r^{\mathbf{j}}(s) \right| \rho_0^{-r-|\mathbf{j}|} < \infty.$$

The function φ is lower semicontinuous. If

$$L_m = \{s \in F_k;\ \varphi(s) \leq m\}$$

then

$$F_k = \bigcup_m L_m$$

and the L_m are closed sets. For some m, L_m must have positive capacity, because $c(F_k) > 0$. Therefore, there exists a compact set $C \subset L_m$, of positive capacity and a $\rho_1 > 0$ with the property that for any $s \in C$ and all $r \geq 1$, $|\mathbf{j}| \geq 2$,

$$\left| K_r^{\mathbf{j}}(s) \right| \leq \rho_1^{r+|\mathbf{j}|}.$$

Using Berstein's Lemma A.6, we conclude that, for any compact set $C_0 \subset \mathbb{C}$,

$$\|K_r^{\mathbf{j}}\|_{C^0(C_0)} \leq \exp\left(r \max_{s \in C_0} V_C(s) \right) \rho_1^{r+|\mathbf{j}|} \leq \rho_2^{r+|\mathbf{j}|}$$

for some constant ρ_2 depending only on C_0 and C. Thus K_s is converging for any $s \in \mathbb{C}$. This proves Theorem A.2. $\square$

Appendix B

Resum

Després d'haver reflexionat,
estic com la lluna quan fa el ple,
curull d'idees que us vull explicar.
Siràcida (Eclesiàstic), 39:12

L'objectiu d'aquesta tesi és estudiar la reductibilitat i altres propietats dinàmiques de sistemes lineals quasiperiòdics, especialment els que provenen de les equacions de valors propis associades a operadors de Schrödinger quasiperiòdics. Aquest estudi és particularment fructífer, ja que permet combinar mètodes dinàmics i espectrals i obtenir aplicacions en cadascun dels dos camps.

El nostre punt de partença és l'anomenada equació de Hill amb forçament quasiperiòdic, que és la següent equació diferencial de segon ordre

$$x'' + (a - q(t))\, x = 0, \tag{B.1}$$

essent a un paràmetre real i q una funció quasiperiòdica. Això darrer vol dir que existeix un aixecament de q a $\mathbb{T}^d = (\mathbb{R}/2\pi\mathbb{Z})^d$, que és una funció $Q : \mathbb{T}^d \to \mathbb{R}$ contínua, i un vector de freqüències $\omega \in \mathbb{R}^d$ de manera que

$$q(t) = Q(\omega t),$$

per qualsevol $t \in \mathbb{R}$. En la majoria de la tesi suposarem que, a més, la funció Q és analítica real. Suposarem també que el vector de freqüències ω és racionalment independent, és a dir, que compleix

$$\langle \mathbf{k}, \omega \rangle = k_1\omega_1 + \ldots + k_d\omega_d \neq 0$$

per qualsevol multi-índex $\mathbf{k} = (k_1, \ldots, k_d)^T$ de $\mathbb{Z}^d$ no idènticament zero.

Usant l'aixecament Q podem construir el següent sistema d'equacions diferencials a partir de l'equació (B.1)

$$x'' - Q(\theta)x = ax, \qquad \theta' = \omega, \tag{B.2}$$

amb $\theta \in \mathbb{T}^d$, de manera que cada tria de condició inicial per a als angles θ dóna lloc a una equació diferencial quasiperiòdica com (B.1).

Per tal d'usar mètodes dinàmics, com pretenem en aquesta tesi, és convenient escriure el sistema (B.2) com un sistema de primer ordre a $\mathbb{R}^2 \times \mathbb{T}^d$,

$$\begin{pmatrix} x \\ x' \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ Q(\theta) - a & 0 \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}, \qquad \theta' = \omega. \tag{B.3}$$

Aquest sistema genera una dinàmica a l'espai $\mathbb{R}^2 \times \mathbb{T}^d$. Fixem-nos que l'evolució de les variables angulars θ no depèn més que del temps i de la condició inicial que fixem, però no de les variables x, x'. És per això que l'anomenarem un sistema triangular quasiperiòdic (de l'anglès "quasiperiodic skew-product"). En general un sistema triangular quasiperiòdic és un sistema d'equacions lineals de la forma

$$z' = A(\theta)z, \qquad \theta' = \omega \tag{B.4}$$

essent $z \in \mathbb{R}^n$ i $A : \mathbb{T}^d \to gl(n, \mathbb{R})$ una matriu quadrada de dimensió n. Observem que en el cas del sistema triangular quasiperòdic obtingut a partir de l'equació de Hill, (B.3), la matriu A té traça zero i, per tant, pertany a l'àlgebra de Lie $sl(2, \mathbb{R})$. En particular, qualsevol solució fonamental de (B.3) té determinant constant per tot temps. Així doncs, si triem que aquest determinant sigui 1 aquesta solució pertanyerà a $SL(2, \mathbb{R})$, el grup de Lie de matrius de dimensió 2 amb determinant 1. Això ens diu que el sistema matricial associat a (B.3),

$$X' = \begin{pmatrix} 0 & 1 \\ Q(\theta) - a & 0 \end{pmatrix} X, \qquad \theta' = \omega, \tag{B.5}$$

amb $X \in SL(2, \mathbb{R})$ genera un sistema triangular quasiperiòdic a $SL(2, \mathbb{R}) \times \mathbb{T}^d$. Aquest darrer sistema dinàmic conté tota la informació relevant del sistema ja que podem expressar qualsevol solució de (B.3) en funció de (B.5).

Aquest procediment per generar un sistema triangular per a les matrius fonamentals d'un sistema triangular quasiperiòdic també és extensible a qualsevol sistema com (B.4). En efecte, si $G \subset GL(n, \mathbb{R})$ és un grup de Lie de matrius i g és la seva corresponent àlgebra de Lie de matrius (que és una subàlgebra de $gl(n, \mathbb{R})$) aleshores

$$X' = A(\theta)X, \qquad \theta' = \omega, \tag{B.6}$$

amb $X \in G$, és un sistema triangular quasiperiòdic a $G \times \mathbb{T}^d$. Hom pot passar sempre de la formulació matricial (B.6) a la vectorial (B.4) i a la inversa.

L'equació de Hill quasiperiòdica és una generalització de l'equació de Hill clàssica en la qual la funció q és periòdica. Aquesta fou introduïda per George Hill al segle XIX en els seus estudis sobre el moviment de la lluna (veieu Barrow-Green [BG97] i les referències que hi apareixen). Tant el cas periòdic com el quasiperiòdic apareixen com a equacions variacionals en l'anàlisi de l'estabilitat de solucions periòdiques i quasiperiòdiques de dimensió baixa en sistemes hamiltonians. Per a resultats i més referències veieu Eliasson [Eli88], Jorba i Villanueva [JV97] i Bourgain [Bou97].

Un dels punts que fa més interessant l'equació de Hill quasiperiòdica és que és també l'equació de valors propis associada a l'operador de Schrödinger amb potencial quasiperiòdic,

$$\left(H_q^c x\right)(t) = -x''(t) + q(t)x(t). \tag{B.7}$$

de manera que podem mirar-nos el paràmetre a com un paràmetre espectral. Aquí el superíndex c ve de continu per distingir-lo dels operadors de Schrödinger discrets que introduirem d'aquí a un moment. Podem usar la quasiperiodicitat del potencial q per definir la següent família d'operadors de Schrödinger amb potencial quasiperiòdic

$$\left(H_{Q,\omega,\phi}^c x\right)(t) = -x''(t) + Q(\omega t + \phi)x(t). \tag{B.8}$$

Observem que (B.7) és (B.8) amb $\phi = 0$ ja que $q(t) = Q(\omega t)$.

Els operadors de Schrödinger quasiperiòdics (i les seves generalitzacions a dimensió superior) apareixen en problemes de la física quàntica, com per exemple en l'explicació de l'anomenat "efecte de Hall quàntic" que consisteix en la quantització de la conductància de Hall sota certes condicions físiques, veieu Klitzing *et al.* [vKDP80], Fröhlich [Frö94] i Osadchy & Avron [OA01]. És per aquestes interpretacions físiques que al paràmetre a també se l'anomena energia. En aquest context és convenient considerar els operadors $H^c_{Q,\omega,\phi}$ a $L^2(\mathbb{R})$. Això es pot fer gràcies a què aquests són essencialment autoadjunts i, per tant, tenen una extensió única a $L^2(\mathbb{R})$ que coincideix amb la definició donada a (B.7) per a funcions infinitament diferenciables amb suport compacte (veieu la secció III.1.2 de la tesi).

Aquestes consideracions espectrals són més senzilles en el cas d'operadors de Schrödinger discrets amb potencial quasiperiòdic, que ara introduïm. Si discretitzem l'equació de Hill amb forçament quasiperiòdic (B.1) obtenim una equació en diferències del següent tipus

$$x_{n+1} - 2x_n + x_{n-1} + v(n)x_n = ax_n,$$

on $(x_n)_{n\in\mathbb{Z}}$ és una successió de $\mathbb{R}$, a és un paràmetre i $(v(n))_{n\in\mathbb{Z}}$ és una successió quasiperiòdica. Això darrrer vol dir que existeix una funció $V : \mathbb{T}^d \to \mathbb{R}$ contínua (encara que normalment la suposarem analítica real) de manera que

$$v(n) = V(2\pi\omega n), \qquad n \in \mathbb{Z}.$$

per tot $n \in \mathbb{Z}$. Suposarem que el vector de freqüències $\omega \in \mathbb{R}^d$ és no ressonant, és a dir que

$$\langle \mathbf{k}, \omega\rangle \notin \mathbb{Z}$$

per qualsevol $\mathbf{k} \in \mathbb{Z}^d$ diferent de zero. Tal i com és habitual en la literatura dels operadors de Schrödinger amb potencial quasiperiòdic eliminarem el terme diagonal $-2x_n$ que pot ser clarament inclòs al potencial v o a l'energia a. Per tant la discretització anterior porta a estudiar equacions en diferències de la forma

$$x_{n+1} + x_{n-1} + v(n)x_n = ax_n, \tag{B.9}$$

que anomenarem de tipus Harper (l'equació de Harper s'obté quan el potencial és el cosinus). Aquesta és l'equació de valors propis de l'operador

$$(H^d_v x)_n = x_{n+1} + x_{n-1} + v(n)x_n \tag{B.10}$$

que anomenarem operador discret de Schrödinger amb potencial quasiperiòdic. Es tracta d'un operador acotat i autoadjunt a $l^2(\mathbb{Z})$. Com en el cas continu, la quasiperiodicitat del potencial porta a considerar la següent família d'operadors

$$(H^d_{V,\omega,\phi}x)_n = x_{n+1} + x_{n-1} + V(2\pi\omega n + \phi)x_n,$$

de la qual (B.10) s'obté com a cas particular en prendre $\phi = 0$. També, com en el cas continu, podem passar d'equacions del tipus Harper a un sistema dinàmic a $\mathbb{R}^2 \times \mathbb{T}^d$, ara amb temps discret, si fem

$$\begin{pmatrix} x_{n+1} \\ x_n \end{pmatrix} = \begin{pmatrix} a - V(\theta_n) & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_n \\ x_{n-1} \end{pmatrix}, \qquad \theta_{n+1} = \theta_n + 2\pi\omega. \tag{B.11}$$

Aquest és també un sistema triangular quasiperiòdic que anomenarem discret per distingir-lo dels continus que hem presentat abans. En general, un sistema discret triangular i quasiperiòdic és un sistema de la forma

$$z_{n+1} = A(\theta_n)z_n, \qquad \theta_{n+1} = \theta_n + 2\pi\omega$$

on ara $(z_n, \theta_n)_{n\in\mathbb{Z}}$ és una successió de $\mathbb{R}^n \times \mathbb{T}^d$ i $A : \mathbb{T}^d \to G$ una aplicació contínua (G és un grup de Lie de $GL(n, \mathbb{R})$). Aquest sistema dinàmic és la iteració del següent cocicle quasiperiòdic (A, ω),

$$\begin{array}{rccc} (A,\omega): & \mathbb{R}^n \times \mathbb{T}^d & \longrightarrow & \mathbb{R}^n \times \mathbb{T}^d \\ & (z,\theta) & \mapsto & (A(\theta)z, \theta + 2\pi\omega)\,. \end{array}$$

que també pot considerar-se com un endomorfisme de $G \times \mathbb{T}^d$, de manera que la seva iteració dóna lloc a un sistema quasiperiòdic triangular a $G \times \mathbb{T}^d$.

En el cas d'operadors de Schrödinger el corresponent cocicle (A^d_{a-V}, ω), que anomenarem cocicle de Schrödinger, ve donat per

$$A^d_{a-V}(\theta) = \begin{pmatrix} a - V(\theta) & -1 \\ 1 & 0 \end{pmatrix}. \tag{B.12}$$

Gràcies a la irracionalitat (o no ressonància segons s'escaigui) dels vectors de freqüència l'espectre dels operadors $H^c_{Q,\omega,\phi}$ i $H^d_{V,\omega,\phi}$ no depèn de $\phi \in \mathbb{T}^d$ i els escriurem com $\sigma^c(Q, \omega)$ i $\sigma^d(V, \omega)$ respectivament, veieu la secció III.1.

Un dels objectius d'aquesta tesi és estudiar, mitjançant propietats dinàmiques, l'espectre dels operadors de Schrödinger quasiperiòdics, discrets o continus, amb potencials analítics reals i freqüències que satisfacin unes certes condicions diofàntiques que especificarem més endavant. Com que molts de resultats que obtindrem seran per a potencials "petits" ens interessarà sovint que el potencial dugui un paràmetre pertorbatiu al davant. Per això i per analogia amb l'equació de Hill clàssica (veieu Whittaker i Watson [WW62], Ince [Inc44] o Magnus i Winkler [MW79]) sovint l'equació de Hill que considerarem serà

$$x'' + (a - bq(t))\, x = 0.$$

on b és el paràmetre pertorbatiu.

Per tal d'estudiar l'espectre dels operadors de Schrödinger quasiperiòdics des d'un punt de vista dinàmic és molt útil el número de rotació que ara presentem en el cas continu (veieu la secció III.2.2). Sigui x una solució no trivial de l'equació de Hill quasiperiòdica

$$x'' + (a - bQ(\omega t + \phi))\, x = 0 \tag{B.13}$$

per a una certa Q contínua i un vector de freqüències racionalment independent. Johnson i Moser [JM82] varen demostrar que el límit

$$\lim_{t\to\infty} \frac{\arg\left(x'(t) + \mathrm{i}x(t)\right)}{t}$$

existeix i que és independent de $\phi \in \mathbb{T}^d$ i de la solució triada. El denotarem per $\mathrm{rot}^c(a - bQ, \omega)$. L'aplicació contínua

$$a \in \mathbb{R} \mapsto \mathrm{rot}^c(a - bQ, \omega)$$

és, en cert sentit, una funció de distribució suportada a l'espectre: no decreix mai i creix exactament a l'espectre de l'operador de Schrödinger associat $H^c_{bQ,\omega,\phi}$. Fixem-nos que això demostra la independència de l'espectre d'aquests operadors respecte ϕ.

Els intervals oberts de constància del número de rotació, per a un b fixat, pertanyen a la resolvent de l'operador (el complementari de l'espectre) i s'anomenen forats espectrals. En aquests forats espectrals el número de rotació no pot prendre qualsevol valor sinó que ha de ser "racional respecte ω". En efecte, el "Teorema d'etiquetatge dels forats" conclou que si I és un forat espectral de $\sigma^c(bQ,\omega)$ aleshores existeix un $\mathbf{k} \in \mathbb{Z}^d$ tal que $\langle \mathbf{k}, \omega \rangle \geq 0$ i

$$\mathrm{rot}^c(a - bQ, \omega) = \frac{1}{2}\langle \mathbf{k}, \omega \rangle$$

per qualsevol $a \in I$. No tots els números de rotació racionals tenen un forat espectral associat, ja que pot ser que aquest estigui "col·lapsat". Direm que $\{a_0\}$ és un forat espectral col·lapsat de l'espectre $\sigma^c(bQ,\omega)$ si existeix un $\mathbf{k} \in \mathbb{Z}^d$ de manera que a_0 és l'únic valor de a per al qual

$$\mathrm{rot}^c(a - bQ, \omega) = \frac{1}{2}\langle \mathbf{k}, \omega \rangle.$$

Aquest teorema d'etiquetatge dels forats té implicacions per a l'espectre dels operadors associats. En efecte, en el cas quasiperiòdic amb $d \geq 2$ els nombres de rotació racionals respecte ω, que és

$$\mathcal{M}_+(\omega) = \left\{ \langle \mathbf{k}, \omega \rangle / 2; \mathbf{k} \in \mathbb{Z}^d \text{ i } \langle \mathbf{k}, \omega \rangle \geq 0 \right\},$$

és dens a $[0, +\infty)$ degut a què ω és racionalment independent. Com que el nombre de rotació és continu i creixent exactament a l'espectre $\sigma^c(bQ,\omega)$, aquest conjunt és de Cantor si cap dels forats espectrals està col·lapsat. Això és, en qualsevol entorn d'un punt de l'espectre hi ha un interval que no pertany a l'espectre. Observem, però, que el fet que sigui un conjunt de Cantor no vol dir necessàriament que no hi pugui haver cap forat espectral col·lapsat (pensem, per exemple, en un nombre finit de forats col·lapsats). Veieu la figura B.1 per una il·lustració dels intervals de constància del número de rotació d'una equació de Hill.

Aquesta estructura de conjunt de Cantor per a $\sigma^c(bQ,\omega)$ és genèrica per a parelles (Q,ω) amb $Q : \mathbb{T}^d \to \mathbb{R}$ contínua i $\omega \in \mathbb{R}^d$ racionalment independents, com demostrà Johnson a [Joh91]. Per a una freqüència fixada i un potencial analític els resultats no són tan concloents. Quan $|b|$ és petita l'anàlisi és més senzilla ja que disposem d'una eina molt potent: la reductibilitat o conjugació a coeficients constants.

Dos sistemes triangulars quasiperiòdics

$$x' = A(\theta)x, \qquad \theta' = \omega, \tag{B.14}$$

i

$$y' = B(\theta)y, \qquad \theta' = \omega \tag{B.15}$$

amb $(x,\theta), (y,\theta) \in \mathbb{R}^n \times \mathbb{T}^d$, es diuen conjugats si existeix una aplicació $Z : \mathbb{T}^d \to GL(n,\mathbb{R})$ de manera que l'anomenada "equació homològica",

$$\partial_\omega Z(\theta) = A(\theta)Z(\theta) - Z(\theta)B(\theta),$$

es compleixi per qualsevol $\theta \in \mathbb{T}^d$. Aquí

$$\partial_\omega Z(\theta) = D_\theta Z \omega$$

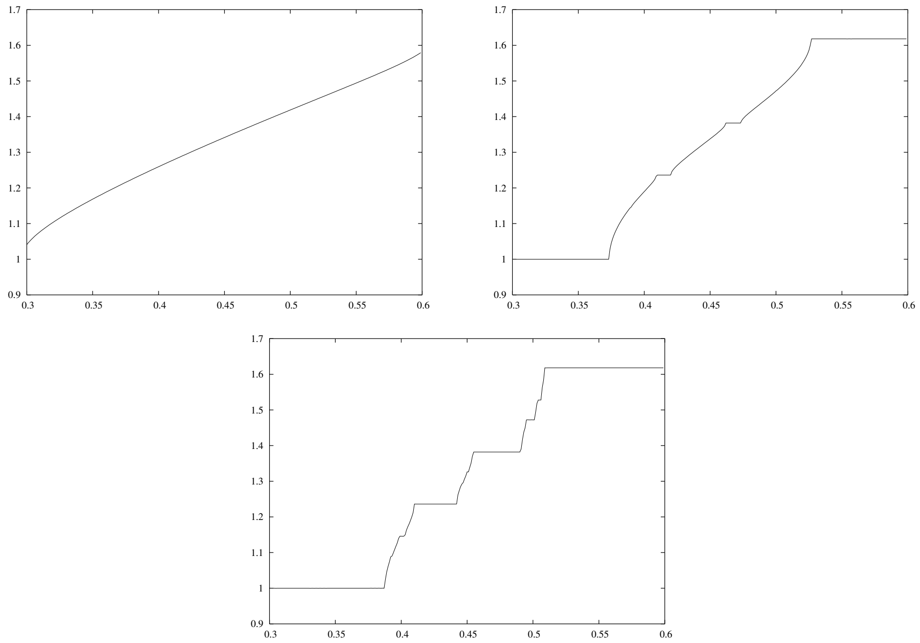

Figure B.1: Il·lustració, per mitjà d'una aproximació numèrica, del nombre de rotació de l'equació $x'' + (a + b\,(\cos t + \cos \gamma t))\,x = 0$ com a funció de a per a diferents valors del paràmetre b i $\gamma = (1+\sqrt{5})/2$. A dalt: a l'esquerra $b = 0.1$ i a la dreta $b = 0.35$. A baix: $b = 0.6$. Per a la metodologia del càlcul veieu Broer i Simó [BS98].

és la derivada en la direcció de ω. Si existeix aquesta conjugació aleshores el canvi de variables

$$(x, \theta) = (Z(\theta)y, \theta)$$

transforma (B.14) en (B.15). Els sistemes conjugats a coeficients constants s'anomenen reductibles a coeficients constants o, simplement reductibles. En aquest cas la matriu amb coeficients constants (no unívocament determinada) s'anomena la matriu de Floquet. En el cas que hi hagi alguna simetria donada per una estructura de Lie en els sistemes en qüestió demanarem que la conjugació Z pertanyi al grup de Lie, $Z : \mathbb{T}^d \to G$.

Observem que tot sistema quasiperiòdic triangular com (B.14) o (B.15) pot veure's amb freqüència ω o qualsevol múltiple enter no nul d'aquesta. Per tant, també pot considerar-se la reductibilitat a coeficients constants amb aquestes freqüències. Això no és només un pur formalisme, ja que sovint caldrà doblar la freqüència si no volem haver de complexificar el sistema. És per això que parlarem de la reductibilitat amb freqüència ω o $\omega/2$ segons s'escaigui.

Notem que l'equació de Hill (B.13) o, més aviat, al sistema triangular associat

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ bQ(\theta) - a & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \qquad \theta' = \omega, \tag{B.16}$$

és en coeficients constants quan $b = 0$. Hom voldria, per mitjà de tècniques KAM veure que per a força valors de a i per a $|b|$ petit el sistema anterior és reductible a coeficients constants. Això fou demostrat per Eliasson [Eli92] sota les hipòtesis que Q sigui analítica real i ω fortament irracional. Direm que un vector de freqüències és fortament irracional irracional si existeixen constants positives c i τ de manera que es satisfà la següent condició diofàntica

$$|\langle \mathbf{k}, \omega \rangle| \geq c|\mathbf{k}|^{-\tau} = c\left(|k_1| + \ldots + |k_d|\right)^{-\tau},$$

per qualsevol $\mathbf{k} \in \mathbb{Z}^d$ que no sigui idènticament zero, condició que d'ara en endavant denotarem per $\omega \in DC^c(c, \tau, \mathbb{R}^d)$. El conjunt de vectors de freqüències fortament independents té mesura total a $\mathbb{R}^d$.

Teorema B.1 (Eliasson [Eli92]). *Sigui $Q : \mathbb{T}^d \to \mathbb{R}^d$ analítica real amb*

$$|Q|_\rho = \sup_{|\mathrm{Im}\,\theta| < \rho} |Q(\theta)| < \infty,$$

i $\omega \in DC^c(c, \tau)$ fortament irracional. Aleshores, existeix una constant $C = C(c, \tau, \rho) > 0$ de manera que si $\mathrm{rot}^c(a - bQ, \omega)$, el número de rotació de (B.13), és racional o fortament irracional respecte ω, el sistema triangular (B.16) és reductible a coeficients constants per una transformació analítica real amb freqüència $\omega/2$ si

$$|bQ|_\rho < C.$$

A més, la matriu de Floquet, B, compleix el següent.

(i) B és nilpotent i diferent de zero si, i només si, a és l'extrem d'un forat espectral no col·lapsat de $\sigma^c(bQ, \omega)$.

(ii) B és zero si, i només si, $\{a\}$ és un forat col·lapsat.

Per a una versió més precisa d'aquest teorema veieu la secció III.3 de la tesi. Tenint en compte el teorema d'etiquetatge dels forats, si $|b|$ és petit i a és a un extrem d'un forat espectral aleshores el sistema (B.16) és reductible a coeficients constants i la matriu de Floquet és idènticament zero si, i només si, el forat és col·lapsat. Moser i Pöschel [MP84] demostraren que, en aquesta situació, un forat col·lapsat pot obrir-se per mitjà d'una pertorbació genèrica. Això portà a Eliasson, usant el resultat anterior, a demostrar la genericitat de l'espectre de Cantor per a operadors de Schrödinger amb potencial petit.

En el nostre cas volem estudiar com és l'espectre $\sigma^c(bQ, \omega)$ en funció de b per a una funció Q analítica real i un vector de freqüències ω, fortament irracional, fixats. Per això és interessant considerar el següent objecte.

Definició B.2. *Sigui $\mathbf{k} \in \mathbb{Z}^d$ amb $\alpha_0 = \langle \mathbf{k}, \omega \rangle / 2 \geq 0$. La llengua de ressonància associada a $\mathbf{k}$ es defineix com el conjunt de $(a, b) \in \mathbb{R}^2$ de manera que*

$$\mathrm{rot}^c(a - bQ, \omega) = \alpha_0$$

D'acord amb aquesta definició, per a b fixat l'interior de les llengües de ressonància correspon a forats espectrals no col·lapsats de manera que estudiar llengües de ressonància té implicacions per a l'espectre $\sigma^c(bQ, \omega)$ en funció de b. Quan les fronteres d'una llengua de ressonància es tallen per a dos valors diferents de b direm que aquesta té una butxaca d'inestabilitat (veieu la figura B.2). Observem que, a diferència del cas periòdic $d = 1$ (veieu la figura B.3) en el cas quasiperiòdic $d \geq 2$ la unió de les llengües de ressonància és densa a $\mathbb{R}^2$.

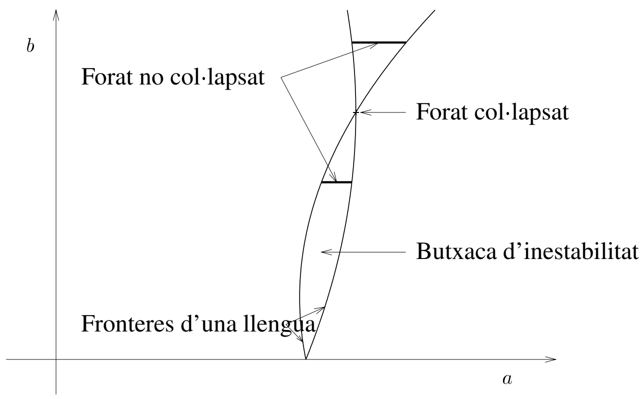

Figure B.2: Llengua de ressonància amb una butxaca en el pla de paràmetres (a, b). Aquesta dóna lloc a forats espectrals en qualsevol línia horitzontal amb b constant. Noteu com el col·lapse de forats espectrals correspon a talls de les fronteres de la llengua en els extrems de la butxaca d'inestabilitat.

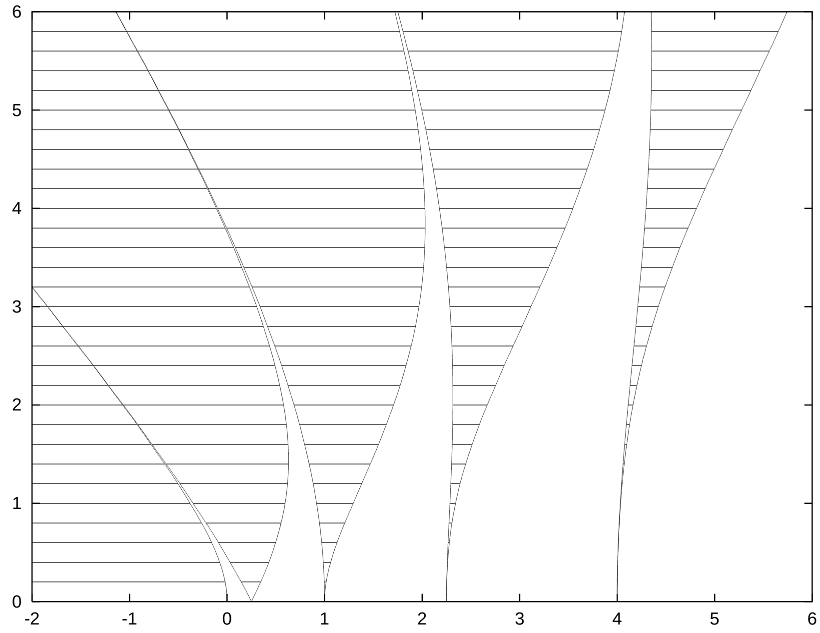

Figure B.3: Llengües de ressonància en el pla de paràmetres (a, b) per a l'equació (periòdica) de Mathieu, $x'' + (a + b\cos t)x = 0$, veieu Broer i Simó [BS00]. Les regions ombrejades corresponen a les llengües de ressonància.

En el capítol IV s'estudien les llengües de ressonància per a Q i ω fixats com abans. Per a $|b| < C$ es demostra que les fronteres de les llengües de ressonància són funcions infinitament diferenciables. En el capítol V es demostra que, de fet són analítiques reals com explicarem més endavant. Val a dir que el resultat del capítol IV és constructiu i permet obtenir el desenvolupament de Taylor de les fronteres de les llengües de ressonància al voltant de l'origen tal i com ara descriurem.

Siguin Q i ω com abans fixats i $(a_0, b_0) \in \mathbb{R}^2$ amb $|b_0| < C$, on C ve donada pel teorema d'Eliasson. Per tant, per aquest teorema, el sistema triangular associat és reductible a coeficients constants. Existeix un canvi de variables de la forma

$$\begin{pmatrix} x' \\ x \end{pmatrix} = Z\left(\frac{\omega t}{2}\right) y \tag{B.17}$$

amb $Z : \mathbb{T}^d \to SL(2, \mathbb{R})$ analítica real de manera que el sistema passa a coeficients constants

$$y' = By,$$

amb B com a matriu de Floquet. Si volem estudiar el comportament de (B.16) per a valors propers a (a, b) podem aplicar-hi el canvi de variables (B.17) de manera que obtenim un sistema

$$y' = (B + P(\theta, \mu))\, y, \qquad \theta' = \omega. \tag{B.18}$$

on $P : \mathbb{T}^d \times \mathbb{R}^2 \to sl(2, \mathbb{R})$ és analítica real i depèn d'uns nous paràmetres

$$\mu = (a - a_0, b - b_0)$$

de forma analítica real. Suposem que a_0 és a l'extrem d'un forat espectral de $\sigma^c(bQ, \omega)$. Aleshores la matriu de Floquet B és nilpotent i, per tant, el seu determinant val zero. Si P no depengués de θ aleshores l'equació

$$\det\left(B + P(\mu)\right) = 0$$

determinaria les fronteres de la llengua de ressonància de que passa per (a_0, b_0). La idea principal del capítol IV és que, aplicant un nombre finit de passos a forma normal, s'obté l'expansió de Taylor de les llengües de ressonància al voltant de (a_0, b_0) fins aquest mateix ordre. Concretament, per mitjà d'un procés de forma normal pot conjugar-se el sistema (B.18) a un del tipus

$$y' = \left(B + \sum_{k=1}^{r} B_k(\mu) + P^{r+1}(\theta, \mu)\right) y, \qquad \theta' = \omega. \tag{B.19}$$

on

(i) $B_k(\mu)$ conté termes d'ordre k en μ i té traça zero per a $k = 1, \ldots, r$.

(ii) P^{r+1} té traça zero i és analítica real per a $|\mu|$ i $|\mathrm{Im}\,\theta|$ prou petits.

El sistema (B.19) no està en coeficiens constants. Malgrat això, al capítol IV demostrem que l'equació

$$\det\left(B + \sum_{k=1}^{r} B_k(\mu)\right) = 0$$

determina el desenvolupament de Taylor fins a ordre r de les fronteres de les llengües de ressonància que passen per (a_0, b_0) (dues en el cas que $\{a_0\}$ sigui un forat espectral col·lapsat i una altrament). Observem que per a poder-les calcular explícitament cal que coneguem la matriu Z, cosa que és difícil en general. Una excepció notable s'esdevé quan $b_0 = 0$ perquè el sistema triangular (B.16) ja és en coeficients constants. Gràcies a això podem descriure qüestions com la transversalitat de les fronteres de les llengües de ressonància a l'origen o la creació de butxaques. En els propers enunciats suposem Q analítica real i ω fortament irracional.

Proposició B.3. *Les fronteres de la llengua de ressonància amb número de rotació*

$$\alpha_0 = \tfrac{1}{2}\langle \mathbf{k}, \omega\rangle \in \mathcal{M}_+^c(\omega)$$

són transversals quan $b = 0$ si, i només si, l'harmònic $\mathbf{k}$-èsim de Q és diferent de zero.

Respecte la creació de butxaques d'inestabilitat ens centrem en l'anàleg quasiperiòdic de l'equació de Mathieu, tot i que es tenen resultats similars per a altres potencials reversibles (parells respecte t).

Teorema B.4. *Considerem l'equació de Hill quasiperiòdica*

$$x'' + \left(a + b\left(\sum_{j=1}^{d} c_j \cos(\omega_j t) + \varepsilon \cos\left(\langle \mathbf{k}^*, \omega\rangle t\right)\right)\right) x = 0. \tag{B.20}$$

Aleshores,

(i) Si $\varepsilon = 0$, l'ordre de tangència a $b = 0$ de la llengua de ressonància $\mathbf{k}^*$-èsima és més gran o igual que $|\mathbf{k}^*|$ i és exactament $|\mathbf{k}^*|$ si, i només si, ω no pertany a un cert subconjunt de mesura zero dels vectors de freqüències fortament racionalment independents, que denotarem per $\mathcal{A}(\mathbf{k}^*)$.

(ii) Si $\varepsilon \neq 0$, $\omega \notin \mathcal{A}(\mathbf{k}^*)$ fortament irracional i $|\varepsilon|$ és prou petit, hi ha almenys una butxaca d'inestabilitat a la $\mathbf{k}^*$-èsima llengua de ressonància amb extrems $b = 0$ i $b = b(\varepsilon) \neq 0$. Aquí cal que ε tingui un signe adequat si $|\mathbf{k}^*|$ és senar.

Aquest resultat és una generalització d'un resultat de Harrell [Har79] per a l'equació de Mathieu periòdica

$$x'' + (a + b\cos t)\, x = 0.$$

i de Broer i Levi [BL95] (veieu també Broer i Simó [BS00]) per a la següent pertorbació

$$x'' + (a + b\,(\cos t + \varepsilon \cos jt))\, x = 0.$$

Aquest mètode per trobar els desenvolupaments de Taylor de les fronteres basat en formes normals (bàsicament de Birkhoff) no permet demostrar l'analiticitat de les fronteres de les llengües de ressonància. En efecte, com ja feu notar Poincaré, el pas a forma normal és genèricament divergent. Si només estem interessats en la convergència de les sèries de Taylor obtingudes per a les fronteres n'hi hauria prou que la sèrie formal

$$\sum_{k=1}^{\infty} B_k(\mu)$$

convergís en un entorn de l'origen. Això tampoc no podem esperar-ho, puix que a l'apèndix A demostrem que aquesta convergència és incompatible amb què l'espectre de l'operador de Schrödinger associat sigui un conjunt de Cantor. Aquesta és una propietat genèrica i, per tant, no podem esperar convergència en la sèrie anterior. A l'apèndix en qüestió reformulem aquests exemples com a hamiltonians quasiperiòdics la forma normal de Birkhoff dels quals és divergent. Usant tècniques de teoria del potencial i idees de Pérez-Marco [PM03] demostrem que la forma normal de Birkhoff d'un hamiltonià quasiperiòdic, per a una freqüencia fortament irracional i una part quadràtica fixades, és genèricament divergent.

La qüestió de l'analiticitat de les llengües de resssonància per a l'equació de Hill quasiperiòdica ens porta a cercar algun tipus de forma normal que sigui convergent. Això s'aconsegueix fent ús de tècniques KAM molt similars a les desenvolupades per Moser [Mos67] que formalitzem per a àlgebres de Lie de matrius qualssevol. Per concretar idees continuem en el cas de l'equació de Hill, un cop hem passat a un sistema que és pertorbació d'un altre amb coeficients constants, veieu l'equació (B.18).

La idea és intentar trobar una matriu independent del temps $M = M(\mu)$, amb traça zero, que depengui analíticament de μ en un entorn prou petit de l'origen i de manera que el sistema modificat

$$z' = (B + P(\theta,\mu) - M(\mu))\, z, \qquad \theta' = \omega,$$

sigui reductible amb matriu de Floquet exactament B. La forma de M depèn de com sigui B. En el cas de l'equació de Hill voldrem aconseguir que

$$M(\mu) = \begin{pmatrix} m_{11}(\mu) & m_{12}(\mu) \\ m_{21}(\mu) & -m_{11}(\mu) \end{pmatrix} \qquad \text{si} \qquad B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

i que

$$M(\mu) = \begin{pmatrix} 0 & 0 \\ m_{21}(\mu) & 0 \end{pmatrix} \qquad \text{si} \qquad B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

En particular, en el darrer cas (on a_0 és a l'extrem d'un forat espectral no col·lapsat) l'equació $m_{21}(a-a_0, b-b_0) = 0$ determina la frontera de la llengua de ressonància que passa per (a_0, b_0). Per tant aquesta és una funció analítica en un entorn de (a_0, b_0). El cas d'un interval col·lapsat requereix més esforç i es troba a la secció V.2.

Això té implicacions de cara a la genericitat de l'espectre de Cantor per a operadors de Schrödinger quasiperiòdics. Concretament, tenim el següent resultat, combinant B.1, B.3 i l'analiticitat de les llengües de ressonància.

Teorema B.5. *Sigui $\omega \in DC^c(c,\tau,\mathbb{R}^d)$ fortament racionalment independent i $C^a_\rho(\mathbb{T}^d,\mathbb{R})$, per a un $\rho > 0$, l'espai de funcions analítiques reals $Q : \mathbb{T}^d \to \mathbb{R}$ amb extensió analítica a $|\mathrm{Im}\,\theta| < \rho$ i tals que*

$$|Q|_\rho = \sup_{|\mathrm{Im}\,\theta|<\rho} |Q(\theta)| < \infty.$$

Aleshores, existeix una constant $C = C(c,\tau,\rho)$ de manera que, per a un potencial genèric a

$$\left\{Q \in C^a_\rho(\mathbb{T}^d,\mathbb{R}),\ |Q|_\rho < C\right\},$$

respecte la topologia induïda per la norma $|\cdot|_\rho$, l'operador $H^c_{Q,\omega,\phi}$ té tots els forats espectrals oberts i, per tant, és un conjunt de Cantor si $d \geq 2$.

Usant el teorema B.4 també podem obtenir una versió d'aquest teorema per a un Q fixat.

Teorema B.6. *Sigui $d \geq 2$. Aleshores, existeix un conjunt $\mathcal{A} \subset \mathbb{R}^d$, de mesura zero, de manera que si $\omega = (\omega_1, \ldots, \omega_d) \notin \mathcal{A}$, existeix una constant $C = C(\omega)$ tal que per gairebé tots els valors de b, amb $|b| < C$, l'espectre de l'operador*

$$Hx = -x'' + b \sum_{j=1}^{d} c_j \cos(\omega_j t) x,$$

on les constants c_j són totes diferents de zero, té tots els forats espectrals oberts.

Fins ara els resultats que hem obtingut per als operadors de Schrödinger continus eren pertorbatius: la constant C que apareixia en aquests depèn de la condició diofàntica concreta que satisfaci el vector de freqüències ω. En el cas discret i $d = 1$ la situació és força diferent degut als resultats de localització no pertorbativa desenvolupats en els darrers anys.

En els capítols VI i VII tractem amb operadors de Schrödinger quasiperiòdics i discrets. En el capítol VI s'usa un resultat de localització no pertorbativa per a l'operador "Almost Mathieu" per resoldre l'anomenat "problema dels deu martinis". En el capítol VII s'usa un resultat no pertorbatiu de localització per establir una versió no pertorbativa del teorema d'Eliasson B.1 en el cas d'una freqüència. Passem ara a descriure aquests resultats.

L'operador de Schrödinger discret quasiperiòdic més estudiat és probablement l'operador "Almost Mathieu",

$$(H_{b,\omega,\phi}x)_n = x_{n+1} + x_{n-1} + \cos(2\pi\omega n + \phi)\, x_n$$

on $b \in \mathbb{R}$ i ω és un no ressonant. L'equació de valors propis associada,

$$x_{n+1} + x_{n-1} + b\cos(2\pi\omega n + \phi)\, x_n = a x_n, \qquad n \in \mathbb{Z},$$

és l'equació de Harper. El 1981 Simon [Sim82], recollint una oferta llançada per Kac, proposà el "problema dels deu martinis": demostrar que l'espectre de l'operador "Almost Mathieu" és un conjunt de Cantor si ω és no ressonant i si $b \neq 0$ (veieu la figura B.4).

Al capítol VI resolem aquest problema per a $b \neq 0, \pm 2$ i per a valors de ω fortament no ressonants. Això darrer vol dir que existeixen unes constants positives, c i τ, de manera que la desigualtat

$$|\sin(\pi k \omega)| \geq \frac{c}{|k|^\tau}$$

es compleix per qualsevol $k \in \mathbb{Z}$ diferent de zero, condició que denotarem per $\omega \in DC^d(c, \tau, \mathbb{R})$. Notem que ω és fortament no ressonant si, i només si, $(1, \omega)$ és fortament irracional. Per a la discussió sobre les propietats diofàntiques en els casos continu i discret, veieu la secció II.2.2.

Corol·lari B.7. *Si ω és fortament no ressonant i $b \neq 0, \pm 2$, aleshores l'espectre de l'operador "Almost Mathieu" és un conjunt de Cantor.*

Per entendre per què aquest resultat és un corol·lari cal tenir en compte el següent teorema de Jitomirskaya.

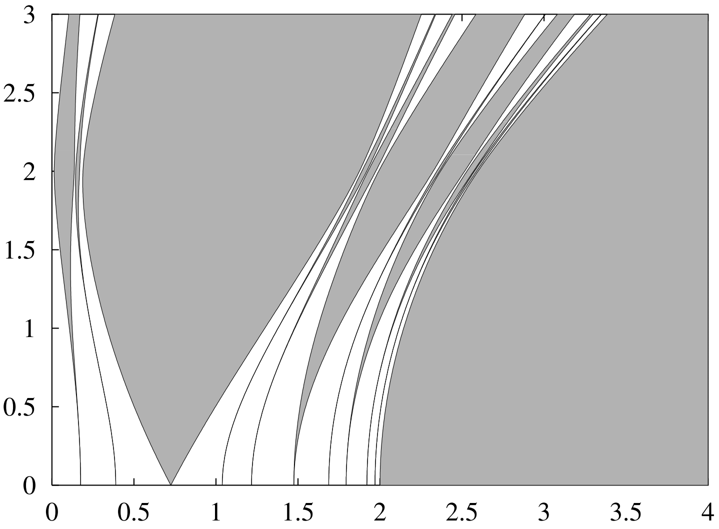

Figure B.4: Càlcul numèric dels deu forats espectrals més grans de l'operador "Almost Mathieu" per a diferents valors de b. El paràmetre espectral a és a la direcció horitzontal i b a la vertical. Observeu com es col·lapsen tots els forats quan $b = 0$.

Teorema B.8 (Jitomirkskaya [Jit99]). *Sigui ω fortament no ressonant. Definim Φ com el conjunt d'aquells valors $\phi \in \mathbb{T}$ per als quals la relació*

$$|\sin(\phi + \pi k\omega)| < \exp\left(-|k|^{\frac{1}{2\tau}}\right) \tag{B.21}$$

es compleix per a infinits valors de k. Aleshores, si $\phi \notin \Phi$ i $|b| > 2$, l'operador $H_{b,\omega,\phi}$ té només espectre purament puntual amb vectors propis que decauen exponencialment.

Aquest teorema implica que, si $|b| > 2$ i ω és diofàntic aleshores, per a un conjunt de valors de a que és dens a l'espectre l'equació de Harper (B.9) té solucions que decauen exponencialment si $\phi \notin \Phi$. En particular, això és cert per a $\phi = 0$. Sigui doncs, a un d'aquests valors i considerem la transformada de Fourier del vector propi localitzat associat, $\psi = (\psi_k)_{k\in\mathbb{Z}}$,

$$\tilde{\psi}(\theta) = \sum_{k\in\mathbb{Z}} \psi_k e^{\mathrm{i}k\theta},$$

que és una funció analítica. Aleshores l'ona de Bloch quasiperiòdica i analítica

$$x_n = \tilde{\psi}(2\pi\omega n)$$

és solució de

$$x_{n+1} + x_{n-1} + \frac{4}{b}\cos(2\pi\omega n)\, x_n = \frac{2a}{b} x_n,$$

que és l'equació de valors propis d'un operador "Almost Mathieu" però amb uns altres paràmetres a i b. La invariància de l'operador "Almost Mathieu" per la transformació de Fourier s'anomena dualitat d'Aubry.

Al capítol VI demostrem que l'existència d'aquesta solució de Bloch implica que a és a l'extrem d'un forat espectral de $H_{4/b,\omega,\phi}$ i que aquest és col·lapsat si, i només si, té una altra ona de Bloch linealment independent com a solució de l'equació de valors propis. Ara bé, desfent el procés de dualitat d'Aubry obtindríem que l'equació de Harper original (B.9) tindria dues solucions linealment independents i que decaurien exponencialment. Això és una contradicció amb el caràcter limit-puntual dels operadors quasiperiodics de Schrödinger i, per tant, a ha de ser extrem d'un interval no col·lapsat.

En el fons aquest argument usa la reductibilitat d'un cocicle quasiperiòdic a coeficients constants. Donat un cocicle (A, ω) de $G \times \mathbb{T}^d$, direm que és reductible a coeficients constants si existeix una transformació $Z : \mathbb{T}^d \to G$ i una matriu constant a G, que també anomenarem de Floquet, de manera que es compleixi

$$A(\theta)Z(\theta) = Z(\theta + 2\pi\omega)B$$

per qualsevol $\theta \in \mathbb{T}^d$.

El teorema d'Eliasson B.1 té un anàleg per a equacions de tipus Harper amb potencials analítics reals i freqüències fortament no resssonants. Com en el cas continu, la constant C depèn de les condicions diofàntiques precises sobre ω. En el capítol VII seguim la metodologia del capítol VI per obtenir una generalització parcial no pertorbativa del teorema d'Eliasson.

Teorema B.9. *Sigui $\rho > 0$ un nombre positiu. Existeix una constant $\varepsilon_0 = \varepsilon_0(\rho)$ de manera que, per qualsevol $V \in C^a_\rho(\mathbb{T}, \mathbb{R})$ analítica real amb*

$$|V|_\rho < \varepsilon_0,$$

el cocicle de Schrödinger (A^d_{a-V}, ω) és reductible a coeficients constants per gairebé tot $a \in \mathbb{R}$, respecte la mesura de Lebesgue, i per qualsevol ω fortament no ressonant.

Aquesta és una generalització parcial ja que, mentre que el teorema d'Eliasson especifica, en funció del seu número de rotació, per a quins valors de a el sistema és reductible, el teorema anterior només diu que el conjunt de valors de a per al quals es té reductibilitat és de mesura total. Aquest resultat ha estat demostrat recentment per Avila i Krikorian [AK03] sota hipòtesis més restrictives. Notem, també, que està formulat per a $d = 1$ ja que, com mostra un exemple de Bourgain [Bou02a, Bou02b], no és cert per a $d \geq 2$ en general.

El punt més delicat en la demostració del teorema B.9 és veure que per gairebé tot punt de l'espectre hi ha solucions de Bloch analítiques i quasiperiòdiques. Per una banda, si a no pertany a l'espectre, aleshores és reductible a coeficients constants. Per altra banda, si una equació de tipus Harper té una solució de Bloch aleshores és reductible a coeficients constants.

Per veure l'existència de solucions de Bloch quasiperiòdiques de l'equació

$$x_{n+1} + x_{n-1} + V\left(2\pi\omega n + \phi\right)x_n = ax_n, \qquad n \in \mathbb{Z},$$

per gairebé tot valor de a a l'espectre de l'operador associat podem mirar d'usar el truc de la dualitat d'Aubry. En aquest cas, però, no recuperem el mateix operador, sinó que busquem solucions localitzades exponencialment de l'equació

$$\sum_{k\in\mathbb{Z}} V_k x_{n-k} + 2\cos\left(2\pi\omega n + \varphi\right)x_n = ax_n, \qquad n \in \mathbb{Z},$$

on $\varphi \in \mathbb{T}$ i $(V_k)_k$ són els coeficients de Fourier de V. Aquesta és l'equació de valors propis de l'operador

$$(L_{V,\omega,\varphi}x)_n = \sum_{k\in\mathbb{Z}} V_k x_{n-k} + 2\cos\left(2\pi\omega n + \varphi\right) x_n$$

a $l^2(\mathbb{Z})$ que, tot i que no és de Schrödinger, és autoadjunt i acotat. El paper del teorema de Jitomirskaya per a l'operador "Almost Mathieu" el juga ara el següent resultat.

Teorema B.10 (Bourgain i Jitomirskaya [BJ02b]**).** *Sigui $\rho > 0$ un nombre fixat. Existeix una constant $\varepsilon_0 = \varepsilon_0(\rho)$ de manera que, per qualsevol $V \in C^a_\rho(\mathbb{T}, \mathbb{R})$ analítica real amb*

$$|V|_\rho < \varepsilon_0,$$

i per qualsevol ω fortament no ressonant, existeix un conjunt $\Phi \subset \mathbb{T}$, de mesura zero, de manera que si $\phi \notin \Phi$ aleshores l'operador $L_{V,\omega,\phi}$ té només espectre purament puntual amb vectors propis que decauen exponencialment.

Tal i com hem intentat mostrar en aquest resum, la combinació dels punts de vista dinàmic i espectral s'ha demostrat molt fructífera i creiem que, explotant més encara aquesta interacció, es podran obtenir més resultats interessants. Des del punt de vista espectral hem vist que és possible descriure acuradament el comportament dels forats espectrals en termes de la dinàmica dels sistemes triangulars associats. Esperem que aquesta anàlisi dels forats pugui extendre's a operadors de Schrödinger més generals. Des del punt de vista dinàmic, hem estudiat una font d'exemples i mètodes molt valuosa que inclou una descripció força completa de la hiperbolicitat no uniforme, tècniques de localització (i reductibilitat) no pertorbatives i una descripció, en aquests models, de la transició del comportament regular a irregular. Esperem aplicar aquests mètodes a sistemes plenament no lineals en el futur.

Appendix C

Agraïments

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Bibliography

[AA80] S. Aubry and G. André, *Analyticity breaking and Anderson localization in incommensurate lattices*, Group theoretical methods in physics (Proc. Eighth Internat. Colloq., Kiryat Anavim, 1979) (Bristol), Hilger, Bristol, 1980, pp. 133–164.

[Afs86] Z. Afsharnejad, *Perturbation geometry for Mathieu's equation*, Indian J. Pure Appl. Math. **17** (1986), no. 11, 1284–1308.

[AK03] A. Avila and R. Krikorian, *Reducibility or non-uniform hyperbolicity for quasi-periodic Schrödinger cocycles*, Preprint (2003).

[Arn83a] V. I. Arnol′d, *Geometrical methods in the theory of ordinary differential equations*, Grundlehren der Mathematischen Wissenschaften, vol. 250, Springer-Verlag, New York, 1983.

[Arn83b] V. I. Arnol′d, *Remarks on perturbation theory for problems of Mathieu type*, Uspekhi Mat. Nauk **38** (1983), no. 4(232), 189–203.

[Arn94] V. I. Arnol′d (ed.), *Dynamical systems V: Bifurcation Theory and Catastrophe theory*, Encyclopædia of Mathematical Sciences, vol. 5, Springer-Verlag, 1994.

[Arn96] V. I. Arnol′d, *Mathematical methods of classical mechanics*, Springer-Verlag, 1996.

[AS78] J. Avron and B. Simon, *Analytic properties of band functions*, Ann. Physics **110** (1978), no. 1, 85–101.

[AS81] J. Avron and B. Simon, *The asymptotics of the gap in the Mathieu equation*, Ann. Physics **134** (1981), no. 1, 76–84.

[AS83] J. Avron and B. Simon, *Almost periodic Schrödinger operators II. The integrated density of states*, Duke Math. J. **50** (1983), 369–391.

[Azb64] M. Y. Azbel, *Energy spectrum of a conduction electron in a magnetic field*, Soviet Phys. JETP **19** (1964), 634–645.

[BB87] B. L. J. Braaksma and H. W. Broer, *On a quasiperiodic Hopf bifurcation*, Ann. Inst. H. Poincaré Anal. Non Linéaire **4** (1987), no. 2, 115–168.

[BG97] J. Barrow-Green, *Poincaré and the three body problem*, History of Mathematics, vol. 11, American Mathematical Society, Providence, RI, 1997.

[BGV03] H. W. Broer, M. Golubitsky and G. Vegter, *The geometry of resonance tongues: a singularity theory approach*, Nonlinearity **16** (2003), 1511–1538.

[BHLV98] H. W. Broer, I. Hoveijn, G. A. Lunter and G. Vegter, *Resonances in a spring-pendulum: algorithms for equivariant singularity theory*, Nonlinearity **11** (1998), no. 6, 1569–1605.

[BHLV03] H. W. Broer, I. Hoveijn, G. Lunter and G. Vegter, *Bifurcations in Hamiltonian systems*, Lecture Notes in Mathematics, vol. 1806, Springer-Verlag, Berlin, 2003, Computing singularities by Gröbner bases.

[BHN97] H. W. Broer, I. Hoveijn and M. v. Noort, *A reversible bifurcation analysis of the inverted pendulum*, Physica D **112** (1997), 50–63.

[BHS96] H. W. Broer, G. B. Huitema and M. B. Sevryuk, *Quasi-periodic motions in families of dynamical systems*, Springer-Verlag, Berlin, 1996, Order amidst chaos.

[BHvNV99] H. W. Broer, I. Hoveijn, M. van Noort and G. Vegter, *The inverted pendulum: a singularity theory approach*, J. Differential Equations **157** (1999), no. 1, 120–149.

[BHZ00] J. Bellissard, D. J. L. Herrmann and M. Zarrouati, *Hulls of aperiodic solids and gap labeling theorems*, Directions in mathematical quasicrystals (Providence, RI), CRM Monogr. Ser., vol. 13, Amer. Math. Soc., Providence, RI, 2000, pp. 207–258.

[BIST89] J. Bellissard, B. Iochum, E. Scoppola and D. Testard, *Spectral properties of one-dimensional quasi-crystals*, Comm. Math. Phys. **125** (1989), no. 3, 527–543.

[BIT91] J. Bellissard, B. Iochum and D. Testard, *Continuity properties of the electronic spectrum of* 1*D quasicrystals*, Comm. Math. Phys. **141** (1991), no. 2, 353–380.

[BJ02a] J. Bourgain and S. Jitomirskaya, *Absolutely continuous spectrum for 1D quasiperiodic operators*, Invent. Math. **148** (2002), no. 3, 453–463.

[BJ02b] J. Bourgain and S. Jitomirskaya, *Continuity of the Lyapunov exponent for quasiperiodic operators with analytic potential*, J. Statist. Phys. **108** (2002), no. 5-6, 1203–1218, Dedicated to David Ruelle and Yasha Sinai on the occasion of their 65th birthdays.

[Bje03] K. Bjerklov, *Dynamical properties of quasi-periodic schrödinger equations*, Ph.D. thesis, KTH, 2003.

[BL95] H. W. Broer and M. Levi, *Geometrical aspects of stability theory for Hill's equations*, Arch. Rational Mech. Anal. **131** (1995), no. 3, 225–240.

[BLS04] H. W. Broer, M. Levi and C. Simó, *Geometry of stability zones of Hill's equations*, In progress (2004).

[BLV97] H. W. Broer, G. Lunter and G. Vegter, *Equivariant singularity theory with distinguished parameters: two case-studies of resonant hamiltonian systems*, Physica D **112** (1997), 64–80.

[BMS76] N. N. Bogoljubov, J. A. Mitropoliskii and A. M. Samoĭlenko, *Methods of accelerated convergence in nonlinear mechanics*, Hindustan Publishing Corp., Delhi, 1976.

[Boc01] F.-P. Boca, *Rotation C^*-algebras and almost Mathieu operators*, Theta Series in Advanced Mathematics, vol. 1, The Theta Foundation, Bucharest, 2001.

[Boh47] H. Bohr, *Almost Periodic Functions*, Chelsea Publishing Company, New York, N.Y., 1947.

[Bou97] J. Bourgain, *On Melnikov's persistency problem*, Math. Res. Lett. **4** (1997), no. 4, 445–458.

[Bou00] J. Bourgain, *Hölder regularity of integrated density of states for the almost Mathieu operator in a perturbative regime*, Lett. Math. Phys. **51** (2000), no. 2, 83–118.

[Bou02a] J. Bourgain, *On the spectrum of lattice Schrödinger operators with deterministic potential*, J. Anal. Math. **87** (2002), 37–75, Dedicated to the memory of Thomas H. Wolff.

[Bou02b] J. Bourgain, *On the spectrum of lattice Schrödinger operators with deterministic potential. II*, J. Anal. Math. **88** (2002), 221–254, Dedicated to the memory of Tom Wolff.

[Bou02c] J. Bourgain, *New results on the spectrum of lattice Schrödinger operators and applications*, Mathematical results in quantum mechanics (Taxco, 2001) (Providence, RI), Contemp. Math., vol. 307, Amer. Math. Soc., Providence, RI, 2002, pp. 27–38.

[Bou04a] J. Bourgain, *Green's function estimates for lattice schrödinger operators and applications*, Princeton University Press, Princeton, N. J., 2004, Annals of Mathematics Studies, to appear.

[Bou04b] J. Bourgain, *On quasi-periodic lattice Schrödinger operators*, Discrete Contin. Dyn. Syst. **10** (2004), no. 1-2, 75–88.

[BPS03] H. W. Broer, J. Puig and C. Simó, *Resonance tongues and instability pockets in the quasi-periodic Hill-Schrödinger equation*, Comm. Math. Phys **241** (2003), no. 2–3, 467–503.

[Bry89] A. D. Bryuno, *Normalization of a Hamiltonian system near an invariant cycle or torus*, Uspekhi Mat. Nauk **44** (1989), no. 2(266), 49–78.

[BS82] J. Bellissard and B. Simon, *Cantor spectrum for the almost Mathieu equation*, J. Funct. Anal. **48** (1982), no. 3, 408–419.

[BS98] H. W. Broer and C. Simó, *Hill's equation with quasi-periodic forcing: resonance tongues, instability pockets and global phenomena*, Bol. Soc. Brasil. Mat. (N.S.) **29** (1998), no. 2, 253–293.

[BS00] H. W. Broer and C. Simó, *Resonance tongues in Hill's equations: a geometric approach*, J. Differential Equations **166** (2000), no. 2, 290–327.

[BV92] H. W. Broer and G. Vegter, *Bifurcational aspects of parametric resonance*, Dynamics reported: expositions in dynamical systems (Berlin), Springer, Berlin, 1992, pp. 1–53.

[CEY90] M. D. Choi, G. A. Elliott and N. Yui, *Gauss polynomials and the rotation algebra*, Invent. Math. **99** (1990), no. 2, 225–246.

[CFKS87] H. L. Cycon, R. G. Froese, W. Kirsch and B. Simon, *Schrödinger operators with application to quantum mechanics and global geometry*, study ed., Springer-Verlag, Berlin, 1987.

[Chi86] L. Chierchia, *Quasi-periodic Schrödinger operators in one dimension, absolutely continuous spectra, Bloch waves and integrable Hamiltonian systems.*, Ph.D. thesis, University of New York. 132 p. , 1986.

[Chi87] L. Chierchia, *Absolutely continuous spectra of quasiperiodic Schrödinger operators*, J. Math. Phys. **28** (1987), no. 12, 2891–2898.

[Chu89] V. A. Chulaevsky, *Almost periodic operators and related nonlinear integrable systems*, Nonlinear Science: Theory and Applications, Manchester University Press, Manchester, 1989, With a foreword by Ya. G. Sinaĭ, Translated from the Russian.

[CL55] E. Coddington and N. Levinson, *Theory of ordinary differential equations*, McGraw-Hill Book Company, Inc., New York-Toronto-London, 1955.

[CL90] R. Carmona and J. Lacroix, *Spectral theory of random schrödinger operators*, Probability and its Applications, Birkhäuser, 1990.

[CL99] C. Chicone and Y. Latushkin, *Evolution semigroups in dynamical systems and differential equations*, Mathematical Surveys and Monographs, vol. 70, American Mathematical Society, Providence, RI, 1999.

[CLS92] S.-N. Chow, K. Lu and Y. Q. Shen, *Normal form and linearization for quasiperiodic systems*, Trans. Amer. Math. Soc. **331** (1992), no. 1, 361–376.

[Cop78] W. A. Coppel, *Dichotomies in stability theory*, Springer-Verlag, Berlin, 1978, Lecture Notes in Mathematics, Vol. 629.

[CS90] V. A. Chulaevsky and Y. G. Sinaĭ, *Anderson localization and KAM-theory*, Analysis, et cetera (Boston, MA), Academic Press, Boston, MA, 1990, pp. 237–249.

[CS91] V. A. Chulaevsky and Y. G. Sinaĭ, *The exponential localization and structure of the spectrum for* $1D$ *quasi-periodic discrete Schrödinger operators*, Rev. Math. Phys. **3** (1991), no. 3, 241–284.

[DCJ87] C. De Concini and R. A. Johnson, *The algebraic-geometric AKNS potentials*, Ergodic Theory Dynam. Systems **7** (1987), no. 1, 1–24.

[Din97] E. I. Dinaburg, *Some problems in the spectral theory of discrete operators with quasiperiodic coefficients*, Uspekhi Mat. Nauk **52** (1997), no. 3(315), 3–52.

[DK70] Y. L. Daletskiĭ and M. G. Kreĭn, *Ustoichivost reshenii differentsialnykh uravnenii v banakhovom prostranstve*, Izdat. "Nauka", Moscow, 1970, Nonlinear Analysis and its Applications Series.

[DS75] E. Dinaburg and Y. Sinai, *The one-dimensional Schrödinger equation with quasi-periodic potential*, Funkt. Anal. i. Priloz. **9** (1975), 8–21.

[DS83] F. Delyon and B. Souillard, *The rotation number for finite difference operators and its properties*, Comm. Math. Phys. **89** (1983), no. 3, 415–426.

[Eli88] L. H. Eliasson, *Perturbations of stable invariant tori for Hamiltonian systems*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **15** (1988), no. 1, 115–147 (1989).

[Eli92] L. H. Eliasson, *Floquet solutions for the one-dimensional quasi-periodic Schrödinger equation*, Comm. Math. Phys. **146** (1992), 447–482.

[Eli98a] L. H. Eliasson, *One-dimensional quasi-periodic Schrödinger operators—dynamical systems and spectral theory*, European Congress of Mathematics, Vol. I (Budapest, 1996) (Basel), Birkhäuser, Basel, 1998, pp. 178–190.

[Eli98b] L. H. Eliasson, *Reducibility and point spectrum for linear quasi-periodic skew-products*, Proceedings of the International Congress of Mathematicians, Vol. II (Berlin, 1998), no. Extra Vol. II, 1998, pp. 779–787 (electronic).

[Eli99] L. H. Eliasson, *On the discrete one-dimensional quasi-periodic Schrödinger equation and other smooth quasi-periodic skew products*, Hamiltonian systems with three or more degrees of freedom (S'Agaró, 1995) (Dordrecht), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., vol. 533, Kluwer Acad. Publ., Dordrecht, 1999, pp. 55–61.

[Eli01] L. H. Eliasson, *Almost reducibility of linear quasi-periodic systems*, Smooth ergodic theory and its applications (Seattle, WA, 1999) (Providence, RI), Amer. Math. Soc., Providence, RI, 2001, pp. 679–705.

[Eli02a] L. H. Eliasson, *Ergodic skew-systems on* $\mathbb{T}^d \times \mathrm{SO}(3,\mathbb{R})$, Ergodic Theory Dynam. Systems **22** (2002), no. 5, 1429–1449.

[Eli02b] L. H. Eliasson, *Perturbations of linear quasi-periodic system*, Dynamical systems and small divisors (Cetraro, 1998) (Berlin), Lecture Notes in Math., vol. 1784, Springer, Berlin, 2002, pp. 1–60.

[Fas89] F. Fassò, *On a relation among Lie series*, Celestial Mech. Dynam. Astronom. **46** (1989), no. 2, 113–118.

[Fin74] A. Fink, *Almost periodic differential equations*, Lecture Notes in Mathematics, no. 377, Springer, 1974.

[FJ00] R. Fabbri and R. Johnson, *Genericity of exponential dichotomy for two-dimensional differential systems*, Ann. Mat. Pura Appl. (4) **178** (2000), 175–193.

[FJP02] R. Fabbri, R. Johnson and R. Pavani, *On the nature of the spectrum of the quasi-periodic Schrödinger operator*, Nonlinear Anal. Real World Appl. **3** (2002), no. 1, 37–59.

[Frö94] J. Fröhlich, *Mathematical aspects of the quantum Hall effect*, First European Congress of Mathematics, Vol. II (Paris, 1992) (Basel), Birkhäuser, Basel, 1994, pp. 23–48.

[FSW90] J. Fröhlich, T. Spencer and P. Wittwer, *Localization for a class of one-dimensional quasi-periodic Schrödinger operators*, Comm. Math. Phys. **132** (1990), no. 1, 5–25.

[Ful69] W. Fulton, *Algebraic curves. An introduction to algebraic geometry*, W. A. Benjamin, Inc., New York-Amsterdam, 1969, Notes written with the collaboration of Richard Weiss, Mathematics Lecture Notes Series.

[Gab03] F. Gabern, *On the dynamics of the Trojan asteroids*, Ph.D. thesis, Universitat de Barcelona, 2003.

[GG78] A. Giorgilli and L. Galgani, *Formal integrals for an autonomous Hamiltonian system near an equilibrium point*, Celestial Mech. **17** (1978), no. 3, 267–280.

[GG85] A. Giorgilli and L. Galgani, *Rigorous estimates for the series expansions of Hamiltonian perturbation theory*, Celestial Mech. **37** (1985), no. 2, 95–112.

[GJLS97] A. Y. Gordon, S. Jitomirskaya, Y. Last and B. Simon, *Duality and singular continuous spectrum in the almost Mathieu equation*, Acta Math. **178** (1997), no. 2, 169–183.

[GJSM01] G. Gómez, À. Jorba, C. Simó and J. Masdemont, *Dynamics and mission design near libration points. Vol. IV*, World Scientific Monograph Series in Mathematics, vol. 5, World Scientific Publishing Co. Inc., River Edge, NJ, 2001, Advanced methods for triangular points.

[GMC01] G. Gómez, J. Mondelo and S. C., *Refined fourier analysis: procedures, error estimates and applications.*, Preprint (2001).

[GS01] M. Goldstein and W. Schlag, *Hölder continuity of the integrated density of states for quasi-periodic Schrödinger equations and averages of shifts of subharmonic functions*, Ann. of Math. (2) **154** (2001), no. 1, 155–203.

[Hal92] J. K. Hale, *Oscillations in nonlinear systems*, Dover Publications Inc., New York, 1992, Corrected reprint of the 1963 original.

[Har55] P. Harper, *Single band motion of conduction electrons in a uniform magnetic field*, Proc. Phys. Soc. London **A68** (1955), 874–892.

[Har79] E. M. Harrell, *The band-structure of a one-dimensional, periodic system in a scaling limit*, Ann. Physics **119** (1979), no. 2, 351–369.

[HdlL03a] A. Haro and R. de la Llave., *A parameterization method for the computation of whiskers in quasi periodic maps: numerical implementation*, Preprint (2003).

[HdlL03b] A. Haro and R. de la Llave., *A parameterization method for the computation of whiskers in quasi periodic maps: rigorous results*, Preprint (2003).

[HdlL03c] A. Haro and R. de la Llave., *Spectral theory of transfer operators (I): General results.*, Preprint (2003).

[HdlL03d] A. Haro and R. de la Llave., *Spectral theory of transfer operators (II): Vector bundle maps over rotations.*, Preprint (2003).

[HdlL03e] A. Haro and R. de la Llave., *Spectral theory of transfer operators (III): Push forward operators for anosov systems.*, Preprint (2003).

[Her83] M. Herman, *Une méthode pour minorer les exposants de Lyapunov et quelques exemples montrant le caractère local d'un théorème d'Arnold et de Moser sur le tore de dimension 2*, Comment. Math. Helvetici **58** (1983), no. 3.

[Hoc86] H. Hochstadt, *The functions of mathematical physics*, second ed., Dover Publications Inc., New York, 1986.

[IK94] E. Isaacson and H. B. Keller, *Analysis of numerical methods*, Dover Publications Inc., New York, 1994, Corrected reprint of the 1966 original [Wiley, New York; MR **34** #924].

[Inc44] E. L. Ince, *Ordinary Differential Equations*, Dover Publications, New York, 1944.

[Jan92] C. Janot, *Quasicrystals : a primer*, Clarendon Press, 1992.

[Jit95] S. Y. Jitomirskaya, *Almost everything about the almost Mathieu operator. II*, XIth International Congress of Mathematical Physics (Paris, 1994) (Cambridge, MA), Internat. Press, Cambridge, MA, 1995, pp. 373–382

[Jit99] S. Y. Jitomirskaya, *Metal-insulator transition for the almost Mathieu operator*, Ann. of Math. (2) **150** (1999), no. 3, 1159–1175.

[Jit02] S. Jitomirskaya, *Nonperturbative localization*, Proceedings of the International Congress of Mathematicians, Vol. III (Beijing, 2002) (Beijing), Higher Ed. Press, 2002, pp. 445–455.

[JM82] R. Johnson and J. Moser, *The rotation number for almost periodic potentials*, Comm. Math. Phys. **84** (1982), 403–438.

[JM03] R. Johnson and F. Mantellini, *Non-autonomous differential equations*, Dynamical Systems (Cetraro, 2000) (Berlin), Lecture Notes in Math., vol. 1822, Springer, Berlin, 2003, pp. 173 – 229.

[Joh80] R. Johnson, *Analyticity of spectral subbundles*, J. Diff. Eq. **35** (1980), 366–387.

[Joh82] R. Johnson, *The recurrent Hill's equation*, J. Diff. Eq. **46** (1982), 165–193.

[Joh83] R. Johnson, *A review of recent work on almost periodic differential and difference operators*, Acta Appl. Math. **1** (1983), no. 3, 241–261.

[Joh91] R. Johnson, *Cantor spectrum for the quasi-periodic Schrödinger equation*, J. Diff. Eq. **91** (1991), 88–110.

[JS81] R. Johnson and G. Sell, *Smoothness of spectral subbundles and reducibility of quasi-periodic linear differential systems*, J. Diff. Eq. **41** (1981), 262–288.

[JS92] À. Jorba and C. Simó, *On the reducibility of linear differential equations with quasiperiodic coefficients*, J. Differential Equations **98** (1992), no. 1, 111–124.

[JS96] À. Jorba and C. Simó, *On quasi-periodic perturbations of elliptic equilibrium points*, SIAM J. Math. Anal. **27** (1996), no. 6, 1704–1737.

[JV97] À. Jorba and J. Villanueva, *On the persistence of lower-dimensional invariant tori under quasi-periodic perturbations*, J. Nonlinear Sci. **7** (1997), no. 5, 427–473.

[Kat70] S. B. Katok, *Linear prolongations of dynamical systems and the problem of reducibility*, Mat. Zametki **8** (1970), 451–462.

[Kat76] T. Kato, *Perturbation theory for linear operators*, second ed., Springer-Verlag, Berlin, 1976, Grundlehren der Mathematischen Wissenschaften, Band 132.

[KH95] A. Katok and B. Hasselblatt, *Introduction to the modern theory of dynamical systems*, Encyclopedia of Mathematics and its Applications, vol. 54, Cambridge University Press, Cambridge, 1995, With a supplementary chapter by Katok and Leonardo Mendoza.

[Kri] R. Krikorian, *Reducibility, differentiable rigidity and Lyapunov exponents for quasi-periodic cocycles on* $T \times SL(2, R)$, Preprint.

[Kri99a] R. Krikorian, *Réductibilité des systèmes produits-croisés à valeurs dans des groupes compacts*, Astérisque **259** (1999).

[Kri99b] R. Krikorian, *Réductibilité presque partout des flots fibrés quasi-périodiques à valeurs dans les groupes compacts*, Ann. scient. Éc. Norm. Sup. **32** (1999), 187–240.

[Las94] Y. Last, *Zero measure spectrum for the almost Mathieu operator*, Comm. Math. Phys. **164** (1994), no. 2, 421–432.

[Las95] Y. Last, *Almost everything about the almost Mathieu operator. I*, XIth International Congress of Mathematical Physics (Paris, 1994) (Cambridge, MA), Internat. Press, Cambridge, MA, 1995, pp. 366–372.

[Las99] J. Laskar, *Introduction to frequency map analysis*, Hamiltonian systems with three or more degrees of freedom (S'Agaró, 1995) (Dordrecht), Kluwer Acad. Publ., Dordrecht, 1999, pp. 134–150.

[LFC92] J. Laskar, C. Froeschlé and A. Celletti, *The measure of chaos by the numerical analysis of the fundamental frequencies. Application to the standard mapping*, Phys. D **56** (1992), no. 2-3, 253–269.

[LK63] D. Levy and J. Keller, *Instability intervals of Hill's equation*, Comm. Pure Appl. Math. **16** (1963), 469–479.

[LZ82] B. M. Levitan and V. V. Zhikov, *Almost periodic functions and differential equations*, Cambridge University Press, Cambridge, 1982, Translated from the Russian by L. W. Longdon.

[Mos67] J. Moser, *Convergent series expansions for quasi-periodic motions*, Math. Ann. **169** (1967), 136–176.

[Mos81] J. Moser, *An example of Schrödinger equation with almost periodic potential and nowhere dense spectrum*, Comment. Math. Helvetici **56** (1981), 198–224.

[Mos98] N. G. Moshchevitin, *Differential equations with almost periodic and conditionally periodic coefficients: recurrence and reducibility*, Mat. Zametki **64** (1998), no. 2, 229–237, Translation in Math. Notes 64 (1998), no. 1-2, 194–201 (1999).

[MP84] J. Moser and J. Pöschel, *An extension of a result by Dinaburg and Sinai on quasi-periodic potentials*, Comment. Math. Helvetici **59** (1984), 39–85.

[MS66] J. L. Massera and J. J. Schäffer, *Linear differential equations and function spaces*, Pure and Applied Mathematics, Vol. 21, Academic Press, New York, 1966.

[MvM75] H. P. McKean and P. van Moerbeke, *The spectrum of Hill's equation*, Invent. Math. **30** (1975), no. 3, 217–274.

[MW79] W. Magnus and S. Winkler, *Hill's equation*, Dover Publications Inc., New York, 1979, Corrected reprint of the 1966 edition.

[Nn95] C. Núñez, *Extension of a Moser-Pöschel theorem for the Schrödinger equation with ergodic potential*, XIV CEDYA/IV Congress of Applied Mathematics (Spanish)(Vic, 1995) (Barcelona), Univ. Barcelona, Barcelona, 1995, p. 10 pp. (electronic).

[NY01] A. Nunes and X. Yuan, *A note on the reducibility of linear differential equations with quasiperiodic coefficients*, Preprint. To appear in International Journal of Mathematics and Mathematical Sciences (2001).

[OA01] D. Osadchy and J. Avron, *Hofstadter butterfly as quantum phase diagram*, J. Math. Phys. **42** (2001), no. 12, 5665–5671.

[OP92] R. Obaya and M. Paramio, *Directional differentiability of the rotation number for the almost periodic Schrödinger equation.*, Duke Math. J. **66** (1992), 521–552.

[OS89] A. Olvera and C. Simó, *Normal forms close to invariant circles of twist maps*, European Conference on Iteration Theory (ECIT 87) (Singapore) (C. Alsina, J. Llibre, C. Mira, C. Simó, G.Targonski and R. Thibault, eds.), World Scientific, Singapore, 1989, pp. 438–443.

[OS98] A. Olvera and C. Simó, *Normal forms of twist maps in a resonant annulus*, Preprint, 1998.

[PF92] L. Pastur and A. Figotin, *Spectra of random and almost-periodic operators*, Springer-Verlag, Berlin, 1992.

[PM03] R. Pérez-Marco, *Convergence or generic divergence of the Birkhoff normal form*, Ann. of Math. (2) **157** (2003), no. 2, 557–574.

[Poi92] H. Poincaré, *Les méthodes nouvelles de la méchanique céleste*, Gauthier-Villars, Paris, 1892.

[Pos86] M. Postnikov, *Lie groups and Lie algebras*, "Mir", Moscow, 1986, Lectures in geometry. Semester V.

[PS03] J. Puig and C. Simó, *Analytic families of reducible quasi-periodic equations*, In preparation (2004).

[Pui03] J. Puig, *Reducibility of linear differential equations with quasi-periodic coefficients: a survey*, Preprint University of Barcelona (2003), Available at `http://www.maia.ub.es/~puig/preprints/qpred.ps`.

[Pui04] J. Puig, *Cantor spectrum for the Almost Mathieu operator*, Comm. Math. Phys **244** (2004), no. 2, 297–309.

[Ran95] T. Ransford, *Potential theory in the complex plane*, London Mathematical Society Student Texts, vol. 28, Cambridge University Press, Cambridge, 1995.

[Rel69] F. Rellich, *Perturbation theory of eigenvalue problems*, Assisted by J. Berkowitz. With a preface by Jacob T. Schwartz, Gordon and Breach Science Publishers, New York, 1969.

[RS75] M. Reed and B. Simon, *Methods of modern mathematical physics. II. Fourier analysis, self-adjointness*, Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1975.

[RS78] M. Reed and B. Simon, *Methods of modern mathematical physics. IV. Analysis of operators*, Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1978.

[RS80] M. Reed and B. Simon, *Methods of modern mathematical physics. I*, second ed., Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, 1980, Functional analysis.

[Rüs75] H. Rüssmann, *On optimal estimates for the solutions of linear partial differential equations of first order with constant coefficients on the torus*, Dynamical systems, theory and applications (Rencontres, Battelle Res. Inst., Seattle, Wash., 1974) (Berlin), Springer, Berlin, 1975, pp. 598–624.Lecture Notes in Phys., Vol. 38.

[Rüs80] H. Rüssmann, *On the one-dimensional Schrödinger equation with a quasiperiodic potential*, Nonlinear dynamics (Internat. Conf., New York, 1979) (New York), New York Acad. Sci., New York, 1980, pp. 90–107.

[Sim82] B. Simon, *Almost periodic Schrödinger operators: a review*, Adv. in Appl. Math. **3** (1982), no. 4, 463–490.

[Sim94] C. Simó, *Averaging under fast quasiperiodic forcing*, Hamiltonian mechanics (Toruń, 1993) (New York), Plenum, New York, 1994, pp. 13–34.

[Sim00a] B. Simon, *Schrödinger operators in the twentieth century*, J. Math. Phys. **41** (2000), no. 6, 3523–3555.

[Sim00b] B. Simon, *Schrödinger operators in the twenty-first century*, Mathematical physics 2000 (London), Imp. Coll. Press, London, 2000, pp. 283–288.

[Sin87] Y. G. Sinai, *Anderson localization for one-dimensional difference Schrödinger operator with quasiperiodic potential*, J. Statist. Phys. **46** (1987), no. 5-6, 861–909.

[Sok85] J. B. Sokoloff, *Unusual band structure, wave functions and electrical conductance in cr ystals with incommensurate periodic potentials*, Physics Reports **126** (1985), no. 4, 189–244.

[SS74] R. J. Sacker and G. R. Sell, *Existence of dichotomies and invariant splittings for linear differential systems. I*, J. Differential Equations **15** (1974), 429–458.

[SS76a] R. J. Sacker and G. R. Sell, *Existence of dichotomies and invariant splittings for linear differential systems. II*, J. Differential Equations **22** (1976), no. 2, 478–496.

[SS76b] R. J. Sacker and G. R. Sell, *Existence of dichotomies and invariant splittings for linear differential systems. III*, J. Differential Equations **22** (1976), no. 2, 497–522.

[SS78] R. J. Sacker and G. R. Sell, *A spectral theory for linear differential systems*, J. Diff. Eq. **27** (1978), 320–358.

[Sto92] J. J. Stoker, *Nonlinear vibrations in mechanical and electrical systems*, Wiley Classics Library, John Wiley & Sons Inc., New York, 1992, Reprint of the 1950 original, A Wiley-Interscience Publication.

[VdPS28] B. Van der Pol and M. Strutt, *On the stability of the solutions of Mathieu's equation*, The London, Edinburgh and Dublin Phil. Mag. 7th series **5** (1928), 18–38.

[vKDP80] K. v. Klitzing, G. Dorda and M. Pepper, *New method for high-accuracy determination of the fine-structure constant based on quantized hall resistance*, Phys. Rev. Lett. **45** (1980), 494–497.

[vM89] P. van Mouche, *The coexistence problem for the discrete Mathieu operator*, Comm. Math. Phys. **122** (1989), no. 1, 23–33.

[WW62] E. T. Whittaker and G. N. Watson, *A course of modern analysis. An introduction to the general theory of infinite processes and of analytic functions: with an account of the principal transcendental functions*, Cambridge University Press, New York, 1962.

[YS75] V. A. Yakubovich and V. M. Starzhinskii, *Linear differential equations with periodic coefficients. 1, 2*, Halsted Press [John Wiley & Sons] New York-Toronto, Ont.,, 1975.

List of Notation

$gl(n,\mathbb{R})$	Square real matrices of dimension n	9
$[\cdot,\cdot]$	Lie bracket of matrices	11
ad	Adjoint operator	11
$GL(n,\mathbb{R})$	Real invertible matrices of dimension n	11
$sl(n,\mathbb{R})$	Real matrices with dimension n and trace zero	11
$SL(n,\mathbb{R})$	Special linear group	11
$sp(n,\mathbb{R})$	Infinitessimally symplectic matrices	11
$Sp(n,\mathbb{R})$	Symplectic matrices	11
$so(n,\mathbb{R})$	Skew-symmetric matrices	12
$SO(n,\mathbb{R})$	Special orthogonal group	12
$\mathbb{T}$	The torus $\mathbb{R}/(2\pi\mathbb{Z})$	13
$\lvert\cdot\rvert_\rho$	The supremum norm on $\lvert\mathrm{Im}\,\theta\rvert < \rho$	15
$C^a_\rho(\mathbb{T}^d)$	Real analytic functions on $\mathbb{T}^d$ with $\lvert\cdot\rvert_\rho < \infty$	15
$[f]$	Average of a quasi-periodic function f	17
$\bar{f}$	Average of a quasi-periodic function f	17
∂_ω	Directional derivative along the ω direction	21
$DC^d(C,\tau,\mathbb{R}^d)$	Strongly nonresonant frequency vectors	27
$DC^c(C,\tau,\mathbb{R}^d)$	Strongly rationally independent frequency vectors	28
$\mathcal{S}(A,\omega)$	Stable subbundle of the cocyle (A,ω)	31
$\mathcal{U}(A,\omega)$	Unstable subbundle of the cocyle (A,ω)	31
Σ^d	Sacker-Sell spectrum of a quasi-periodic cocycle	32
Σ^c	Sacker-Sell spectrum of a continuous quasi-periodic skew-product	33
H^c_q	Continuous Schrödinger operator with potential q	40
$H^c_{Q,\omega,\phi}$	Continuous Schrödinger operator with potential $Q(\omega\cdot+\phi)$	40
H^d_v	Discrete Schrödinger operator with potential v	40
$L^2(\mathbb{R})$	The space of square integrable functions on $\mathbb{R}$	40
$l^2(\mathbb{R})$	The space of square sequences functions on $\mathbb{Z}$	42
$H^d_{V,\omega,\phi}$	Discrete Schrödinger operator with potential $Q(\omega\cdot+\phi)$	49
$\sigma^c(Q,\omega)$	Spectrum of $H^c_{Q,\omega,\phi}$	50
$\sigma^d(V,\omega)$	Spectrum of $H^d_{V,\omega,\phi}$	50
$\mathrm{rot}^c(a-Q,\omega)$	Rotation number of $x''+(a-Q(\omega t)x=0$	51
$\mathrm{rot}^d_s(a-V,\omega)$	Sturmian rotation number of $x_{n+1}+x_{n-1}+V\,(2\pi\omega n+\phi)\,x_n = ax_n$	54

$\mathrm{rot}_f^d(A,\omega)$	Fibered rotation number of the cocyle (A,ω)	56
$\mathrm{rot}_f^d(a-V,\omega)$	Fibered rotation number of $x_{n+1}+x_{n-1}+V\left(2\pi\omega n+\phi\right)x_n=ax_n$	56
$\{\cdot\}$	Fractional part of a real number	59
$DC_\omega^c(K,\tau,\mathbb{R}^d)$	Strongly rationally independent rotation numbers with respect to ω	62
$DC_\omega^d(K,\tau,\mathbb{R}^d)$	Strongly nonresonant rotation numbers with respect to ω	62
$L_{V,\omega,\varphi}$	Long-range operator $\left(L_{V,\omega,\varphi}\psi\right)_n=\sum V_k\psi_{n-k}+\cos\left(2\pi\omega n+\varphi\right)\psi_n$	144
$\sigma^L(V,\omega)$	The spectrum of $L_{V,\omega,\varphi}$	144
IDS	Integrated density of states	147
$\mathcal{H}$	Real analytic quasi-periodic Hamiltonians	154
$\mathcal{H}_{(\omega,\alpha)}$	Hamiltonians in $\mathcal{H}$ with fixed quadratic part	154

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