THE SPECTRUM OF SCHRÖDINGER OPERATORS WITH QUASI-PERIODIC POTENTIAL: A DYNAMICAL APPROACH *

JOAQUIM PUIG AND CARLES SIMÓ

Departament de Matemàtica Aplicada i Anàlisi
Universitat de Barcelona
Gran Via 585, 08007 Barcelona (Spain)

Some results on the spectrum of Schrödinger operators with quasi-periodic real analytic potential and Diophantine frequencies are presented. The eigenvalue equation associated to this operator is a quasi-periodic generalization of Hill's equation with periodic forcing. A dynamical study of this equation allows an accurate description of the structure of stability zones close to constant coefficients. This study has direct applications to the structure of the spectrum of Schrödinger operators with small quasi-periodic forcing. In particular we discuss the occurrence of Cantor spectrum.

1. Introduction

The understanding of the spectrum of quasi-periodic Schrödinger operators is important for mathematical and physical purposes. As it is well-known, the spectrum of these operators has a tendency to become a Cantor set, in contrast with the periodic case, where the spectrum is the union of bands, given by Floquet theory.

The best studied example of all quasi-periodic Schrödinger operators is the Almost Mathieu operator

$$(H_{b,\phi}x)_n = x_{n+1} + x_{n-1} + b\cos(2\pi\omega n + \phi)x_n, \qquad n \in \mathbb{Z}, \qquad (1)$$

on $l^2(\mathbb{Z})$, being ω an irrational number and b a real parameter. In this case, the spectrum has a Cantor structure, as it was shown by Puig [7]:

*This work has been partially supported by grants DGICYT BFM 2000-805 (Spain), CIRIT 2001 SGR-71 (Catalonia) and INTAS 2000-221. The first author acknowledges the PhD. grant 2000FI71UBPG.

Theorem 1.1. *If ω is Diophantine then the spectrum of the Almost Mathieu operator is a Cantor set if $b \neq 0, \pm 2$.*

Remark 1.1. That for $|b| = 2$ and ω irrational the spectrum is a Cantor set has been shown by Last [5] and Avila & Krikorian [1].

The proofs are very specific for this model. It is however interesting to consider the problem of the Cantor structure for other operators. In what follows, we will concentrate on continuous quasi-periodic Schrödinger operators. That is, we will consider the spectrum of the operator

$$\left(H_{b,\phi} x\right)(t) = -x''(t) + bQ(\omega t + \phi)x(t) \tag{2}$$

on $L^2(\mathbb{R})$, where $x \in \mathbb{R}$, $b \in \mathbb{R}$, $Q : \mathbb{T}^d \to \mathbb{R}$ is a real analytic function, $\omega \in \mathbb{R}^d$ is a Diophantine frequency vector and $\phi \in \mathbb{T}^d$ is a phase. Diophantine means that there exist constants, $K, \tau \geq d$ such that

$$|\langle \mathbf{k}, \omega \rangle| \geq \frac{K}{|\mathbf{k}|^\tau}$$

for all $\mathbf{k} \in \mathbb{Z}^d - \{0\}$. The operator $H_{b,\phi}$ is unbounded and essentially self-adjoint. Since ω is Diophantine the spectrum does not depend on ϕ.

The eigenvalue equation associated to $H_{b,\phi}$,

$$x'' + \left(a - bQ(\omega t + \phi)\right)x = 0, \tag{3}$$

is a generalization of the classical Hill's equation to the quasi-periodic case. These quasi-periodic Hill's equations were studied by Broer, Puig & Simó[2]. Many of the spectral properties of $H_{b,\phi}$ are related to dynamical properties of (3) or rather, of the skew-product flow that it defines on $\mathbb{R}^2 \times \mathbb{T}^d$,

$$\begin{pmatrix} x \\ x' \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -a + bQ(\theta) & 0 \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}', \qquad \theta' = \omega, \tag{4}$$

where $z = (x, x')^T$ and $\theta \in \mathbb{T}^d$.

A basic connection between these two approaches (the spectral and the dynamical) is given by the rotation number, which was introduced by Johnson & Moser [4] for equations of the type (3).

Let x be a non-trivial solution of

$$x'' + \left(a - bQ(\omega t + \phi)\right)x = 0.$$

The *rotation number* is defined as

$$\mathrm{rot}(a, b) = \lim_{t \to \infty} \frac{\arg\left(x'(t) + ix(t)\right)}{t}.$$

The rotation number enjoys nice properties: it does not depend on the solution chosen, nor on ϕ and it is a continuous function of (a, b). Moreover, once b is fixed, the spectrum of $H_{b,\phi}$ is exactly the set of a_0's for which the rotation number is not locally constant (this shows again that the spectrum of $H_{b,\phi}$ is independent of ϕ).

The open intervals of constancy of the rotation number do not belong to the spectrum and are called the spectral gaps. These gaps can be labelled using the *Gap Labelling Theorem* [4] which shows that for every spectral gap there must exist an integer vector $\mathbf{k} \in \mathbb{Z}^d$, with $\langle \mathbf{k}, \omega \rangle \geq 0$ such that the rotation number in this gap takes the value

$$\frac{1}{2}\langle \mathbf{k}, \omega \rangle .$$

For fixed b and a given resonance as above, if the set of a's with this rotation number is a single point, this point will be called a *collapsed gap.* Otherwise the set is called a *non-collapsed gap.*

If $a \mapsto \mathrm{rot}(a, b)$ is constant in a_0, then $\mathrm{rot}(a_0, b)$ is *rational*, i.e., it is in the module of positive half-resonances of ω:

$$\mathcal{M}_+(\omega) = \left\{ \langle \mathbf{k}, \omega \rangle / 2 ; \mathbf{k} \in \mathbb{Z}^d \text{ and } \langle \mathbf{k}, \omega \rangle \geq 0 \right\} .$$

Hence, we have a link between the spectrum of $H_{b,\phi}$ and the rotation number of the associated eigenvalue equation.

Our aim is to study the structure of spectral gaps for different values of b. Having this goal in mind, it is useful to introduce the generalization of resonance tongues for Hill's equation to the quasi-periodic case:

Definition 1.1. Let $\mathbf{k} \in \mathbb{Z}^d$ with $\alpha_0 = \langle \mathbf{k}, \omega \rangle / 2 \geq 0$. The *resonance tongue* associated to $\mathbf{k}$, $\mathcal{R}(\alpha_0)$, is defined as the set of $(a, b) \in \mathbb{R}^2$ such that the rotation number of (3), $\mathrm{rot}(a, b)$, equals α_0.

For some $(a_0, b_0) \in \mathbb{R}^2$, (4) is said to be *reducible* (to constant coefficients), if there exists a change of variables of the form:

$$\begin{pmatrix} x' \\ x \end{pmatrix} = Z\left(\frac{\omega t}{2}\right) y \tag{5}$$

where $Z : \mathbb{T}^d \to SL(2, \mathbb{R})$ is an analytic matrix function, such that the system is reduced to constant coefficients:

$$y' = By,$$

where $B \in sl(2, \mathbb{R})$ is a constant matrix, called the Floquet matrix. In contrast with the periodic case, linear quasi-periodic systems are not always reducible.

In the perturbative situation, that is $|b|$ small, one can invoke a theorem by Eliasson [3] to conclude that, if Q is real analytic and the frequency vector ω Diophantine, there exists a constant C such that the system (4) is reducible to constant coefficients if (a, b) is at a tongue boundary and $|b| < C$. Moreover, the Floquet matrix has determinant zero, being identically zero if, and only if, $\{a_0\}$ is a collapsed gap of the Schrödinger operator.

To study the situation around a reducible system (4) for some (a_0, b_0) at a tongue boundary one can take the local coordinates $\mu = (a - a_0, b - b_0)$ and perform the change of variables (5) to the system (4) to obtain

$$y' = (B + P(\theta, \mu))\, y, \qquad \theta' = \omega.$$

where $P : \mathbb{T}^d \times \mathbb{R}^2 \to sl(2, \mathbb{R})$ is real analytic.

Using normal form techniques, one can obtain formal expansions for the tongue boundaries around $\mu = 0$ (equivalently for (a, b) around (a_0, b_0)). In Broer, Puig & Simó [2], these formal expansions were proved to be the actual Taylor expansions of the tongue boundaries, which are C^∞ functions [2].

In Puig & Simó [8], this problem was put on a more general context and the following theorem was obtained:

Theorem 1.2. *Assume that in Hill's equation with quasi-periodic forcing (3),*

$$x'' + (a - bQ(\omega t))\, x = 0,$$

$Q : \mathbb{T}^d \to \mathbb{R}$ *is real analytic,* $\omega \in \mathbb{R}^d$ *is Diophantine,* (a_0, b_0) *is at a tongue boundary and the first-order system associated to (3) is reducible at* (a_0, b_0)*. Then there is a real analytic function* f*, with* $f(a_0, b_0) = 0$*, defined in a neighbourhood of* (a_0, b_0) *such that* $f(a, b) = 0$ *if, and only if* (a, b) *is at a tongue boundary.*

Remark 1.2. In particular, every tongue boundary is real analytic in a suitable neighbourhood of $b = 0$. Using Eliasson's reducibility results [3], the tongue boundaries are real analytic in the real domain $|b| < C$.

To prove Theorem 1.2, we use a KAM scheme adapted to the formalism of Lie algebras similar to Moser [6] to show that, if $|\mu|$ is small enough, ω is Diophantine and P is real analytic in both θ and μ, with $\mathrm{tr}\, P = 0$, there exist real analytic functions $Z(\theta, \mu)$ (with $\det Z = 1$) and $M(\mu)$ (with $\mathrm{tr}\, M = 0$), for $|\mu|$ and $|\mathrm{Im}\, \theta|$ small enough, such that the modified system

$$z' = (B + P(\theta, \mu) - M(\mu))\, z, \qquad \theta' = \omega,$$

can be reduced to constant coefficients, with Floquet matrix exactly B. Moreover

$$M(\mu) = \begin{pmatrix} m_{11}(\mu) & m_{12}(\mu) \\ m_{21}(\mu) & -m_{11}(\mu) \end{pmatrix} \qquad \text{if} \qquad B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$M(\mu) = \begin{pmatrix} 0 & 0 \\ m_{21}(\mu) & 0 \end{pmatrix} \qquad \text{if} \qquad B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

2. Application to Cantor spectrum

The analyticity of tongue boundaries has direct applications to the structure of the spectrum of the operators $H_{b,\phi}$, as in (2), for small $|b|$. The generic situation for these potentials is that "all gaps are open":

Theorem 2.1. *Let $\omega \in \mathbb{R}^d$ be Diophantine and let $C^a_r(\mathbb{T}^d)$, for a fixed $r > 0$, be the space of real analytic functions $Q : \mathbb{T}^d \to \mathbb{R}$ with analytic extension to $|Im\ \theta| < r$ and such that*

$$|Q|_r = \sup_{|Im\ \theta|<r} |Q(\theta)| < \infty.$$

Then, there exists a constant $C = C(\omega, r)$ such that for a generic potential in

$$\left\{Q \in C^a_r(\mathbb{T}^d, \mathbb{R}),\ |Q|_r < C\right\},$$

with respect to the $|\cdot|_r$-topology, the operator

$$H_{Q,\phi} = -\frac{d^2}{dt^2} + Q(\omega t + \phi)$$

on $L^2(\mathbb{R})$ has all spectral gaps open and, thus, it is a Cantor set.

Remark 2.1. Under the same hypothesis Eliasson [3] proved the genericity of Cantor spectrum.

The idea is that tongue boundaries $a_1(b)$ and $a_2(b)$ corresponding to the $\mathbf{k}$-th resonance at the origin satisfy that

$$a_1'(0) = Q_0 + |Q_{\mathbf{k}}| \text{ and } a_2'(0) = Q_0 - |Q_{\mathbf{k}}|,$$

being $Q_{\mathbf{k}}$ the Fourier coefficients of Q and $a_i' = da_i/db$.

It is also possible to derive a similar result for a specific class of potentials (a generalization of Mathieu equation to the quasi-periodic case):

Theorem 2.2. *Let $d \geq 2$. Then, there is a set $A \subset \mathbb{R}^d$, of zero measure, such that if $\omega = (\omega_1, \ldots, \omega_d) \notin A$, then, there is a constant $C = C(\omega)$ such that for almost all values of b, with $|b| < C$, the spectrum of the operator*

$$H_{b,0}x = -x'' + b \sum_{j=1}^{d} c_j \cos(\omega_j t)x,$$

where the constants c_j are all different from zero, has all gaps open.

The key tool is the Theorem 3 in Broer, Puig & Simó [2] which describes the asymptotics of the tongue boundaries at $b = 0$ for these quasi-periodic Mathieu equations. The exceptional set A is algebraic.

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