STABILITY OF HOMOGRAPHIC SOLUTIONS OF THE PLANAR THREE-BODY PROBLEM WITH HOMOGENEOUS POTENTIALS

REGINA MARTÍNEZ, ANNA SAMÀ

Departament de Matemàtiques, Facultat de Ciències, Univ. Autònoma de Barcelona, 08193 Bellaterra, (Spain)i
E-mail: reginamb@mat.uab.es, sama@mat.uab.es

CARLES SIMÓ

Departament de Matemàtica Aplicada i Anàlisi, Universitat de Barcelona, Gran Via, 585, 08071 Barcelona (Spain)
E-mail: carles@maia.ub.es

Collinear and triangular homographic solutions of the planar three-body problem with potentials of the form $r^{-\alpha}$, $\alpha \in (0,2)$ are considered. Given the masses, $m_i, i = 1,2,3$ normalised by $\sum_{i=1}^{3} m_i = 1$ and a value of the energy, these solutions depend on an additional parameter, either the angular momentum c or a (generalised) eccentricity e. The purpose is to describe the stability properties and the bifurcations for any values of m_i, e and α. After several reductions, it is enough to study a three-parameter family of four-dimensional linear differential equations depending periodically on time. We are interested in the domains where the linear stability is of type EE (elliptic-elliptic), EH (elliptic-hyperbolic), HH (hyperbolic-hyperbolic) or CS (complex-saddle), as well as in the boundaries of these domains. The bifurcations emerging from $e = 0$ are studied by using normal forms. These provide also approximations for the boundaries. For e close to 1 (the value $e = 1$ corresponding to homographic solutions with a triple collision) a blow up method, applied to the variational equations, is the key tool used to study the bifurcations. For the intermediate region the bifurcation diagrams are completed numerically.

1. Homographic solutions

Let $m_j > 0, j = 1,2,3,\ m_1+m_2+m_3=1$. Consider the Hamiltonian system

$$\mathcal{H}(\mathbf{q},\mathbf{p}) = M^{-1}\|\mathbf{p}\|^2 - U(\mathbf{q}), \qquad \mathbf{q}' = M^{-1}\mathbf{p}, \qquad \mathbf{p}' = \nabla\mathcal{U}(\mathbf{q}), \qquad (1)$$

where $M = \mathrm{diag}(m_1,m_1,m_2,m_2,m_3,m_3)$, $\mathbf{q} = (\mathbf{q}_1,\mathbf{q}_2,\mathbf{q}_3)$, $\mathbf{p} = (\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3)$, $\mathbf{q}_i,\mathbf{p}_i \in \mathbb{R}^2$, and $\mathcal{U}(\mathbf{q}) = \sum_{1\le i\le j\le 3} m_i m_j \|\mathbf{q}_i - \mathbf{q}_j\|^{-\alpha}$, $0 < \alpha < 2$. Equations (1) describe the motion of the planar three-body problem with masses $\mathbf{m} = (m_1, m_2, m_3)$ under an homogeneous potential. It is not restrictive to fix $\mathcal{H} = h = -1/2$ (we are interested in $h < 0$).

We recall that an homographic solution of (1) is a solution which satisfies that the passage from a time t_1 to a time t_2 is the composition of a rotation and a change of scale. Homographic solutions are associated to central configurations (see [6]), i.e. configurations such that $\mathbf{q}$ and $M^{-1}\nabla\mathcal{U}(\mathbf{q})$ are proportional.

We are interested in the stability properties of these solutions, as mentioned in the abstract. See [3,4] for details.

The change of scale part $r(t)$ is a periodic solution of

$$r'' = -d\tilde{V}(r)/dr, \quad \text{where} \quad \tilde{V}(r) = c^2 r^{-2}/2 - r^{-\alpha}/\alpha$$

on $h = -1/2$, where $c > 0$ is a constant. We shall assume $r = r_{\min}$ at $t = 0$. The rotational part is given by the angle $f(t) = \int_0^t cr^{-2}dt$. In the Newtonian case, $\alpha = 1$, f is the true anomaly and $r(f) = c^2/(1 + e\cos f)$, where e denotes the eccentricity. In the general case ($\alpha \in (0, 2)$) we can define $e = (r_{\max} - r_{\min})/(r_{\max} + r_{\min})$ as generalised eccentricity.

Rotating and pulsating coordinates $\zeta(t)$ are introduced through

$$\mathbf{q}(t) = r(t)\tilde{\Omega}(f(t))\zeta(t),\ \tilde{\Omega} = \mathrm{diag}(\Omega, \Omega, \Omega),\ \Omega = \begin{pmatrix} \cos f & -\sin f \\ \sin f & \cos f \end{pmatrix}.$$

Let η be the conjugated momenta and use f as new time. Then $\mathcal{H}(\zeta, \eta, f)$ is periodic in f. Next step is to do a center of mass reduction: $\mathbf{u}_i = \zeta_i - \zeta_3$, $\mathbf{v}_i = \eta_i$, $i = 1, 2$, $\mathbf{u}_3 = \zeta_3$, $\mathbf{v}_3 = \sum_{i=1}^3 \eta_i$, which allows to restrict the study to the equations for $\mathbf{u}_i, \mathbf{v}_i, i = 1, 2$. Let $(\mathbf{u}^*, \mathbf{v}^*)$ be an equilibrium for them, $\mathbf{u}^*$ giving rise to central configurations. The linearisation of the system around $(\mathbf{u}^*, \mathbf{v}^*)$ is of the form

$$\dot{\mathbf{y}} = \mathcal{A}\mathbf{y}, \quad \mathbf{y} \in \mathbb{R}^8, \quad \mathcal{A} \in \mathcal{L}(\mathbb{R}^8, \mathbb{R}^8), \quad \dot{} = d/df. \tag{2}$$

Lemma 1.1. *System (2) splits as two 4×4 systems, the first one having 1 as eigenvalue of multiplicity 4.*

After a linear change of variables the second part in (2) is of the form

$$\dot{\mathbf{x}} = A(f)\mathbf{x}, \quad A(f) = \begin{pmatrix} 0 & I_2 \\ \tilde{A} & -2J_2 \end{pmatrix}, \qquad \tilde{A} = g^{\alpha-2}\begin{pmatrix} \gamma_{11} + 1 & \gamma_{12} \\ \gamma_{12} & \gamma_{22} + 1 \end{pmatrix}, \tag{3}$$

where γ_{ij} are constant coefficients depending on $\mathbf{u}^*$ (and hence on $\mathbf{m}$) and $g = c^{2/(2-\alpha)}/r$, a periodic solution of $\ddot{z} = -U_z(z)$, $U(z) = z^2/2 - z^\alpha/\alpha$ on the level $-(1/2)c^{2\alpha/(2-\alpha)}$ of $E = \frac{1}{2}\dot{z}^2 + U(z)$. System (3) approaches a singularity at $g_{\min}$ when $c \to 0$. An orthogonal change reduces $\tilde{A}$ to $g^{\alpha-2}\mathrm{diag}(\lambda_1, \lambda_2)$. parameters λ_1, λ_2 depend on $\mathbf{m}$ only through a function $\beta(\mathbf{m})$ ranging in

$(0, \beta_{\max})$. One has $\beta = 3(\alpha+2)(m_1m_2+m_1m_3+m_2m_3)$ in the triangular case. It is a rational function on the m_i and on the solution of the generalization of Euler's quintic equation in the collinear one. See [3] for explicit expressions.

Hence we should study the stability of a system

$$\ddot{x} = \lambda_1 g^{\alpha-2}x - 2\dot{y}, \quad \ddot{y} = \lambda_2 g^{\alpha-2}y + 2\dot{x} \tag{4}$$

with $\lambda_1, \lambda_2 \in \mathbb{R}$ and g as before.

In what follows we deal first with g close to constant (small e). Then with g having a minimum $g_{\min}$ approaching 0 when $\delta := 1 - e \to 0$.

2. The near constant case

We assume an even more general case than (3) and (4) with

$$\tilde{A}(t) = \begin{pmatrix} \lambda_1 G_1(t;\epsilon) & 0 \\ 0 & \lambda_2 G_2(t;\epsilon) \end{pmatrix}, \ \lambda_i \in \mathbb{R}\setminus\{0\}, \ G_i(t;\epsilon) = 1 - \sum_{j\geq 1} \epsilon^j c_{i,j}(t), \tag{5}$$

G_1, G_2 analytic in (t, ϵ) with $c_{i,j}(t)$ even periodic functions of t with period T. T can depend on ϵ but a change of scale of t reduces it to a constant.

Remark 2.1. In general $c_{i,1}$ in (5) contain all harmonics. But if (5) comes from the variational equations along periodic orbits emanating from a fixed point, then (as follows from Lindstedt-Poincaré method) G_i satisfy d'Alembert property: an harmonic of order k contains at least ϵ^k as factor.

In the case $\epsilon = 0$ the monodromy matrix is $\mathcal{M} = \exp(AT)$ with eigenvalues $\mu_i, \mu_i^{-1}, i = 1, 2$. Let $\mathrm{tr}_i = \mu_i + \mu_i^{-1}$. The different possibilities, depending on (λ_1, λ_2), are shown in Fig. 1 left, and are EE (in open domain B1), EH (in A), HH (in B2) and CS (in C).

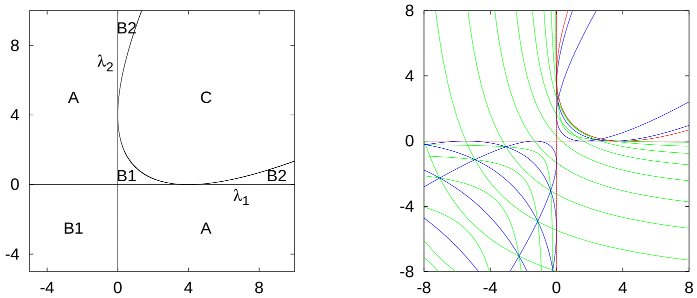

Figure 1. Stability domains for $\epsilon = 0$ and locus of possible bifurcations.

When ϵ moves away from 0 bifurcations can only appear if some $|\mathrm{tr}_i| = 2$ or $\mathrm{tr}_1 = \mathrm{tr}_2$. Let $\nu = T/\pi$. In A $\cup$ B1 possible bifurcations appear if some eigenvalue of $\mathcal{M}$ is of the form $\pm \mathrm{i}\,\omega$ with $\omega = k/\nu$, $k \in \mathbb{N}$. In B1 also if $\omega_1 \pm \omega_2 = 2k/\nu$, $k \in \mathbb{N}$. In C if the imaginary part of the eigenvalues, β, satisfies $|\beta| = k/\nu$, $k \in \mathbb{N}$. No bifurcations can occur in B2. Fig. 1 right shows the possible locations of the bifurcations for $\nu = 2.6$.

The steps for the bifurcation conditions are:
1) Reduction to diagonal form; 2) Normal Form (NF) up to the order, n, at which the time-depending terms cannot be eliminated; 3) Use of co-rotating coordinates (see [1]). In the new variables and skipping the terms of order $> n$ the matrix of the system is constant and the conditions are obtained from their coefficients. The richer case occurs in B1 when the frequencies satisfy $\nu\omega_1 \in \mathbb{N}$, $\nu\omega_2 \in \mathbb{N}$, $\nu\omega_1 \equiv \nu\omega_2 \bmod 2$. See [4] for full discussion.

Generic bifurcations occur at surfaces in the $\lambda_1, \lambda_2, \epsilon$ variables (exceptionally at the intersection of surfaces). The process is justified by the

Lemma 2.1. *Generically (i.e., if some coefficient is non-zero) the bifurcation diagram given by the NF to order n persists for the full system.*

Detailed studies in the d'Alembert case (see Remark 2.1) are relevant for the homographic solutions. For instance, assume that in the domain B1

$$\omega_j = n_j\pi/T, \quad n_j \in \mathbb{N}, \quad j = 1, 2, \quad n_1 \equiv n_2 (\mathrm{mod}\ 2) \tag{6}$$

holds for $\epsilon = 0$. Let (a_1, a_2) be the corresponding values of (λ_1, λ_2).

Theorem 2.1. *In the d'Alembert case, under the assumptions in (6) and if non-degeneracy conditions are satisfied, one has*

- *(i) If $n_1 = 3n_2$, then around $(\lambda_1, \lambda_2, \epsilon) = (a_1, a_2, 0)$, regions EE, EH and CS exist and HH has either 0, 1 or 2 connected components.*
- *(ii) If $n_1 \neq 3n_2$, then regions EE, EH, HH, CS are always present and no local changes in the topology of these domains occur.*

3. Near the singular case

Consider again (4), but now assume $g = g(t;\delta)$ periodic in t and depending on $\delta \in (0, \delta_0]$, δ_0 small enough. Suppose $g(t;\delta) > 0\ \forall t$ and $g(0;\delta) \to 0$ for $\delta \to 0$. System (4) has a singularity at $t = 0$ for $\delta = 0$. Our purpose is to study the stability for small $\delta > 0$.

Let $U(z) = z^\alpha V(z)$ where V is smooth for $z > 0$, such that: (A1) There exists $z_a > 0$ with $V(z_a) = 0, V_z(z_a) > 0$ and $V(z) < 0, V_z(z) > 0\ \forall z \in (0, z_a)$; (A2) $V(z) = \gamma + V_1(z)$, with $\gamma < 0$, $V_1(z) = O(z^s)$ for some $s > (2-\alpha)/2$.

Moreover we shall assume: (B) for $\delta > 0$, $g(t;\delta)$ is the periodic solution of $\ddot{z} = -U_z(z)$ with U satisfying (A1)+(A2), on the energy level $E = -\delta$ such that $g(0;\delta) = g_0$, the minimum of $g(t;\delta)$.

Theorem 3.1. *Consider system (4) where g satisfies (B). Let us define $\hat{\lambda} = \gamma(2-\alpha)^2/8$, γ as in (A2). Assume $\lambda_1 > \lambda_2 > \hat{\lambda}$ or, $\lambda_1 > \hat{\lambda} > \lambda_2$. Let $\beta_j = (1 - \lambda_j/\hat{\lambda})^{1/2}$, $j = 1, 2$. Then we have the following asymptotic behaviour for the stability parameters when δ goes to 0*

$$\begin{aligned}
\log tr_1 &= k_1 - \frac{2-\alpha}{2\alpha}\beta_1 \log\delta(1+o(1)) + \ldots, \\
\log tr_2 &= k_2 - \frac{2-\alpha}{2\alpha}\beta_2 \log\delta(1+o(1)) + \ldots, && \text{if} \quad \lambda_2 > \hat{\lambda}, \\
tr_2 &= k_3 + k_4 \cos[k_5 - \gamma_2(1+o(1))\log(\delta)] + \ldots, && \text{if} \quad \lambda_2 < \hat{\lambda},
\end{aligned}$$

where $k_j, j = 1, \ldots, 5$ are constants, $\gamma_2 = \beta_2/i$, provided that some coefficient, d_1, is different from zero.

The proof uses blow up techniques and the analysis of a neighbourhood of an heteroclinic chain (see [3]).

4. Application and numerical completion of results

The results of Secs. 2 and 3 are applied to Eq. (4). For intermediate values of e numerical methods have been used. Fig. 2 shows some results for the triangular solutions. Partial results in the Newtonian case can be found in [5]. This case shows some degeneracies, as proven in [4], related to the fact that g is a trigonometrical polynomial and to the isocronicity. Some candidates to bifurcations do not give rise to changes of stability.

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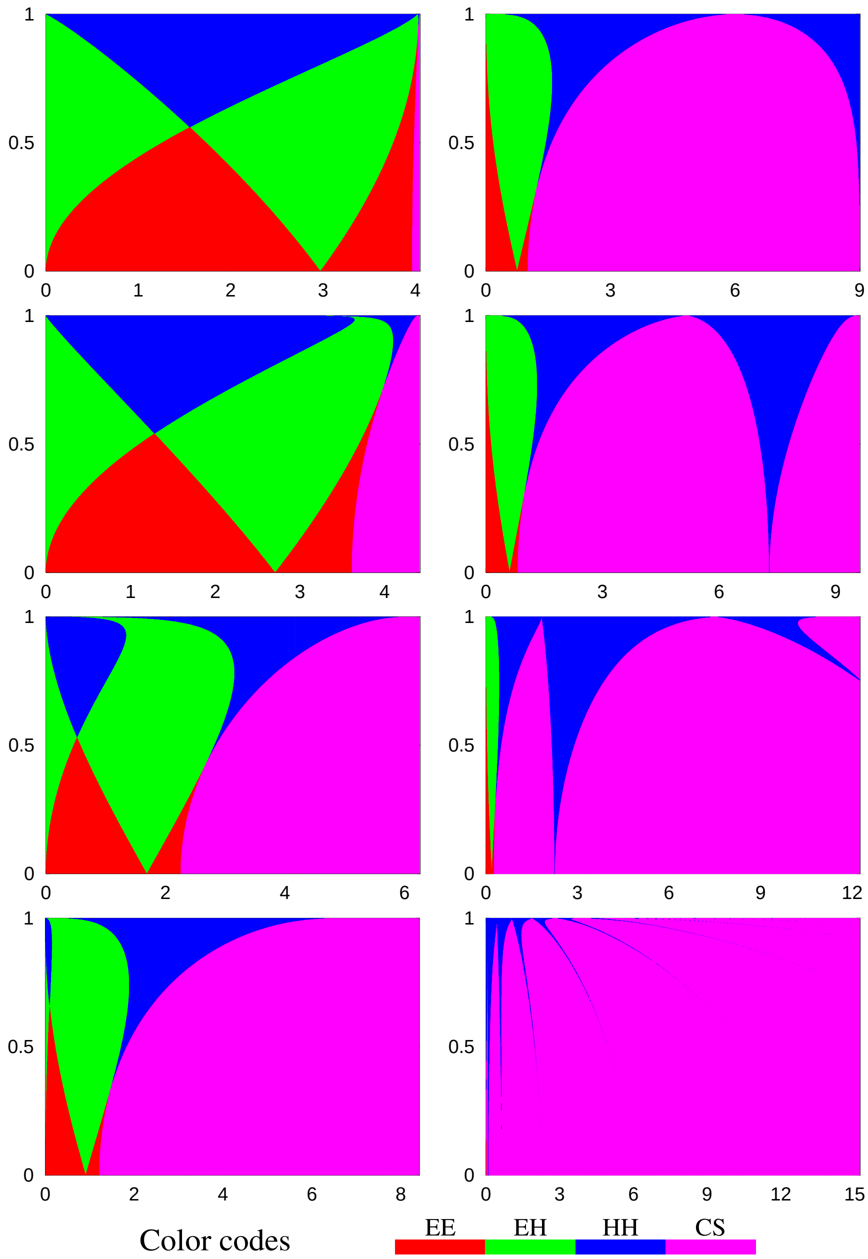

Figure 2. Domains with different linear behaviour in the triangular case. The horizontal (vertical) variable is β (e). Left column: $\alpha = 0.01, 0.1, 0.5, 0.9$ (from top to bottom). Right column; $\alpha = 1, 1.1, 1.5, 1.9$.