EXISTENCE AND MEASURE OF 2-QUASIPERIODICITY IN HAMILTONIAN ONE-AND-A-HALF DEGREE OF FREEDOM SYSTEMS

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The goal of this paper is to show some properties of real analytic Hamiltonian systems of the form $H = \frac{1}{2}y^2 + V(x,t)$, where V is periodic in x and t. We prove that this system has invariant tori of rotational type for sufficiently large values of $|y|$. The measure of the set of such tori is shown to be exponentially close to full measure as $y \to \pm\infty$.

1. Introduction

In this note we consider nonautonomous Hamiltonian systems given by the equation of motion

$$\ddot{x} = -\frac{\partial}{\partial x}V(x,t),$$

where $x \in \mathbb{S}^1 = \mathbb{R}/(2\pi\mathbb{Z})$, $t \in \mathbb{S}^1$ satisfies $\dot{t} = 1$, and V is real analytic. We introduce $y = \dot{x}$, then $y \in \mathbb{R}$ and the corresponding vector field is of the form

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\frac{\partial}{\partial x} V(x,t) \\ \dot{t} &= 1. \end{aligned} \tag{1}$$

Without loss of generality we may assume that the x average of V is zero:

$$\int_0^{2\pi} V(x,t)\,\mathrm{d}x = 0.$$

Such a vector field is Hamiltonian, with $1\frac{1}{2}$ degrees of freedom, and Hamilton function

$$H(x,y,t) = \frac{1}{2}y^2 + V(x,t). \tag{2}$$

This system has a well defined area preserving Poincaré map P on the section $t = 0$, which is the restriction to this section of the flow over time 2π. Well known systems of this type are periodically forced oscillators, for example a planar pendulum with periodically varying length, given by $V(x,t) = -p(t)\cos x$ for some periodic function p. In this case the variable t is the phase of the forcing, while x and y are the angle and angular velocity of the pendulum.

In this paper we prove two results. First it is shown that the system (1) has invariant tori of rotational type, that is, of the form $y = \phi(x,t)$, where $\phi : \mathbb{S}^1 \times \mathbb{S}^1 \longrightarrow \mathbb{R}$ is real analytic. For definiteness we restrict ourselves to tori on which $y > 0$. The interesting point here is that the system may be far from integrable, depending on the choice of V. To counter this, we consider tori with a sufficiently large frequency in the x direction, compare Levi[8,9] for the cases of superquadratic potentials and pendulum type systems.

For $\gamma \in [0,1]$ and $\tau \geq 2$, define the set $\Omega_{\gamma,\tau}$ of Diophantine x frequencies of invariant tori by

$$\Omega_{\gamma,\tau} = \left\{ \omega \in \mathbb{R} : |\omega - \frac{p}{q}| > \gamma q^{-\tau} \text{ for all } \frac{p}{q} \in \mathbb{Q} \right\}. \tag{3}$$

We note that the frequency in the t variable of any invariant torus equals 1. The result is as follows:

Theorem 1.1. *Let H be the Hamiltonian given by (2). Choose γ and τ as above. Then there exists $\omega_0 > 0$, depending on γ, τ and the function V, such that for any $\omega \in \Omega_{\gamma,\tau} \cap [\omega_0, +\infty)$ there exists an invariant torus of H*

of rotational type, with frequency ω in the x direction. This torus is of the form

$$y = \omega - \omega^{-1} V(x,t) + O(\omega^{-2}),$$

as $\omega \to +\infty$.

Our second result deals with the measure of the set of rotational invariant tori. Let $\Omega = [\omega_1, \omega_2]$ be a set of frequencies (in x), with $\omega_2 > \omega_1 + L$, $\omega_1 > 0$, where $L > 0$ is any fixed constant. Define Ω' to be the set of frequencies in Ω for which the corresponding invariant torus of (2) exists. Then the relative measure μ_{rel} of invariant tori is defined by

$$\mu_{\mathrm{rel}}(\Omega) = \frac{\mu(\Omega')}{\mu(\Omega)},$$

where μ denotes Lebesgue measure.

Theorem 1.2. *Let H be the Hamiltonian defined by (2), and let $\Omega = [\omega_1, \omega_2]$ and Ω' be as above. Then*

$$\mu_{rel}(\Omega) = 1 - O\left(e^{-C\omega_1}\right) \ , \ as\ \omega_1 \to +\infty,$$

for some constant $C > 0$, depending on V.

The theorem states that the relative measure of rotational invariant tori is exponentially close to full measure. A similar result has been obtained for large classes of Hamiltonian systems by Pöschel[12], Delshams and Gutiérrez[4], Broer *et al.*[1], and Jorba and Villanueva[6,7].

Let us briefly discuss the method used to prove these results. We will first introduce a localization parameter ω that corresponds to the frequency in x of the torus under inspection, and restricts the problem in phase space to a single torus. This parameter also acts as perturbation parameter, with $\omega = +\infty$ as unperturbed case. The resulting vector field is denoted by X.

The system is averaged with respect to the fast angle x, using Neishtadt's theorem. The averaged vector field $\tilde{X}$ is an exponentially small perturbation of an integrable Hamiltonian system which does not depend on x. Since the integrable part depends explicitly on the slow angle t, we prove our results by applying a twist theorem to the corresponding Poincaré map $\tilde{P}$ on the section $t = 0$, rather than using KAM theory for the Hamiltonian vector field.

Overview

The localization parameter is introduced in the next section, while section 3 deals with the averaging procedure. Finally, in section 4 we complete the proofs of theorems 1.1 and 1.2.

2. Localization

To prove our main results we need to show the existence of a Cantor family of invariant tori, that can be parametrized by the frequency in x. We denote this frequency by ω, and perform the following symplectic coordinate transformation:

$$(x, y, t) \mapsto (x, \tilde{y}, t) = (x, y - \omega, t).$$

In the new coordinates the system is given as a vector field by

$$X = (\omega + \tilde{y})\frac{\partial}{\partial x} + \frac{\partial}{\partial t} - \left(\frac{\partial}{\partial x}V(x,t)\right)\frac{\partial}{\partial \tilde{y}}. \tag{4}$$

In the original setting we have a single vector field with a family of invariant tori. In the new setting, there is a Cantor family of vector fields, parametrized by ω, and for each member only one invariant torus is considered, namely the one with frequency ω in x. Because of this, ω is sometimes called a localization parameter, cf. Broer *et al.*[2] and Huitema[5]. From here on we drop the tilde on y for convenience. The vector field X is a perturbation of the integrable vector field $(\omega + y)\frac{\partial}{\partial x} + \frac{\partial}{\partial t}$, with an invariant torus $y = 0$. In the next section, this perturbation is made small by averaging.

3. Averaging

We embed the phase space $\mathbb{S}^1 \times \mathbb{R} \times \mathbb{S}^1$ in $E = (\mathbb{C}/(2\pi\mathbb{Z})) \times \mathbb{C} \times (\mathbb{C}/(2\pi\mathbb{Z}))$ in the natural way. For $D \subset E$ and $d > 0$, define

$$D + d = \{z \in E : \exists\, x \in D \text{ with } |z - x| \leq d\}.$$

The supremum norm on D is denoted by $||\ldots||_D$. For vector valued functions it is the maximum of the supremum norms of the components.

We assume $\omega \gg 1$, and thus the vector field X can be averaged with respect to the fast angle x. This leads to the following result.

Theorem 3.1. *Let D_2 be an open neighborhood of $\mathbb{S}^1 \times \{0\} \times \mathbb{S}^1$. Let X be as in (4), defined on $D_3 = D_2 + d$, for some $d > 0$. Then for some constant $\omega_0 > 0$ and all $\omega > \omega_0$ there exists a symplectic analytic transformation*

$\Psi = \Psi_\omega : D_2 \longrightarrow D_3$, piecewise analytic in ω, such that $\tilde{X} = \Psi_*^{-1} X$ is a vector field on D_2 given by

$$\tilde{X}(x, y, t) = (\omega + y + g(y, t)) \frac{\partial}{\partial x} + \frac{\partial}{\partial t} + R(x, y, t),$$

where $||R||_{D_2} \leq e^{-\tilde{c}\omega}$ for some $\tilde{c} > 0$. Moreover, Ψ preserves t and is near identity in the sense that $\Psi(x, y, t) = (x, y - \omega^{-1} V(x, t), t) + O(\omega^{-2})$. The function g satisfies $\int_0^{2\pi} g(0, t)\, \mathrm{d}t = 0$ and $||\frac{\partial}{\partial y} g||_{D_2} \leq \frac{1}{2}$.

This is just Neishtadt's theorem in the present context, with slight adaptations. For a proof see Neishtadt[11], Broer *et al.*[3] and Simó and Valls[14].

4. The twist theorem

Let $D_1 \subset D_2$ be a small open neighborhood of $\mathbb{S}^1 \times \{0\} \times \mathbb{S}^1$ in E. For ω_0 sufficiently large and $\omega > \omega_0$, the return map $\tilde{P}$, corresponding to the averaged vector field $\tilde{X}$, maps D_1 into D_2. Furthermore, $\tilde{P} = \tilde{F} + \tilde{R}$, where

$$\tilde{F}(x, y) = (x + 2\pi(\omega + y + h(y)), y) \text{ and } ||\tilde{R}||_{D_1} \leq e^{-c\omega},$$

for some $c > 0$. Here h is analytic, with $h(0) = 0$ and $|h'(0)| \leq \frac{1}{2}$. Thus the map $\tilde{F}$ is a twist map, having an invariant circle $y = 0$ with rotation number ω. The twist depends nontrivially on y.

By the twist theorem, cf. Moser[10] and Rüssmann[13], the perturbed map $\tilde{P}$ has an invariant circle of rotation number ω near $y = 0$ if the perturbation $\tilde{R}$ is sufficiently small and ω satisfies the Diophantine condition

$$|\omega - \frac{p}{q}| \geq \gamma |q|^{-\tau} \text{ for all } \frac{p}{q} \in \mathbb{Q},$$

where $\tau > 2$ and $\gamma > 0$. We also need to assume that D_1 contains $\mathbb{S}^1 \times [-\gamma, \gamma] \times \mathbb{S}^1$. This proves the existence of invariant circles of $\tilde{P}$ for sufficiently large ω. The smallness condition on $\tilde{R}$ amounts to $||\tilde{R}||_{D_1} \leq C\gamma^2$ for some constant $C > 0$. Since the perturbation is exponentially small as $\omega \to +\infty$, this shows that we can take γ also exponentially small, and hence the relative measure of invariant circles is

$$\mu_{\text{rel}} = 1 - O(\gamma) = 1 - O\left(e^{-c\omega}\right)$$

for some $c > 0$, as claimed in theorem 1.2.

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