A PARALLEL METHOD TO COMPUTE QUASI-PERIODIC SOLUTIONS*

ÀNGEL JORBA AND ESTRELLA OLMEDO

Departament de Matemàtica Aplicada i Anàlisi, Universitat de Barcelona, Gran Via 585, 08007 Barcelona, Spain.
E-mails: angel@maia.ub.es, estrella@maia.ub.es.

Consider a discrete dynamical system, having a reducible invariant curve. Here we present an algorithm for the computation of the Fourier coefficients of the curve and of its reducing (Floquet) transformation, provided we have an approximation to the curve and to the Floquet transformation. The method presents a high degree of parallelism: The computing time is essentially divided by the number of processors, and the computational effort grows linearly with the number of Fourier modes needed to represents the curve. For this reason the method seems a good option to compute quasi-periodic solutions with several basic frequencies.

1. Introduction and concepts

Consider a dynamical system given by:

$$\begin{aligned} \bar{x} &= f(x,\theta), \\ \bar{\theta} &= \omega + \theta, \end{aligned} \tag{1}$$

with $\theta \in \mathbb{T}^1$, $x \in \mathbb{R}^n$ and ω is known. We say that (1) has a quasi-periodic solution with rotation number $\omega \notin 2\pi\mathbb{Q}$ when there exists a continuous parametrisation $x : \mathbb{T}^1 \to \mathbb{R}^n$ such that: $f(x(\theta),\theta) = x(\theta+\omega)$, $\forall \theta \in \mathbb{T}^1$. A quasi-periodic solution $x(\theta)$ with frequency ω is reducible iff there exists a change of variables $x = C(\theta)y$ (Floquet transformation) such that transforms the linear system:

$$\begin{aligned} \bar{x} &= D_x f(x(\theta),\theta)x, \\ \bar{\theta} &= \omega + \theta, \end{aligned} \qquad \longrightarrow \qquad \begin{aligned} \bar{y} &= By, \\ \bar{\theta} &= \omega + \theta, \end{aligned}$$

where B is a matrix that does not depend on θ (Floquet matrix).[1]

*This work is supported by the grants BFM2000–0623 (MCyT), 2001SGR–70 (CIRIT), DURSI and a predoctoral FI grant from the Generalitat de Catalunya.

Systems like (1) appear in several fields. For instance, in Dynamical Astronomy they come from considering quasi-periodic time-dependent perturbations of autonomous equations.[2] In those models, the computation of quasi-periodic solutions (and their normal behaviour) is an important step in the understanding of the dynamics.

2. An iterative method to compute quasi-periodic solutions

The method is based on a quadratically convergent scheme. Therefore, we assume that we have an approximation to the solution and an approximation to its Floquet transformation. A step of the scheme has two parts. In the first part, we compute a better approximation to the solution by using the approximated reducibility of the actual approximation. In the second part, we improve the accuracy of the Floquet transformation. This kind of scheme is very standard in KAM theory and, under suitable hypotheses, it converges to the desired solution.[1]

Let us see one step with more detail. Let $x_0(\theta)$ be an approximation to the invariant curve, and $C_0(\theta)$ an approximation to the Floquet transformation. This means that $y_0(\theta) = f(x_0(\theta), \theta) - x_0(\theta + \omega)$ is small and $C_0^{-1}(\theta + \omega) D_x f(x_0(\theta), \theta) C_0(\theta)$ is close to its average B_0. To improve the invariant curve, we look for $h(\theta)$ such that $h(\theta + \omega) = D_x f(x_0(\theta), \theta) h(\theta) + y_0(\theta)$. Note that we can approximate $h(\theta)$ by applying the transformation $h(\theta) = C_0(\theta) u(\theta)$ to this equation, and then substituting $C_0^{-1}(\theta + \omega) D_x f(x_0(\theta), \theta) C_0(\theta)$ by B_0 to obtain $u(\theta + \omega) = B_0 u(\theta) + C_0^{-1}(\theta + \omega) y_0(\theta)$. This equation is solved by expanding in Fourier series up to a given order N, and then solving $2N + 1$ linear (independent) systems of dimension n. Then, $x_1(\theta) \equiv x_0(\theta) + h(\theta)$ is a better approximation to the invariant curve. Now, let $R(\theta)$ be $C_0^{-1}(\theta + \omega) D_x f(x_1(\theta), \theta) C_0(\theta) - B_0$ and $B_1 = B_0 + \mathrm{Avg}(R(\theta))$, where $\mathrm{Avg}(R(\theta))$ denotes the average of $R(\theta)$. Let $H(\theta)$ be the solution of $H(\theta + \omega) B_1 - B_1 H(\theta) = R(\theta) - \mathrm{Avg}(R(\theta))$ ($H(\theta)$ is computed by expanding in Fourier series up to order N and solving N linear systems of dimension $2n^2$). Then, $C_1(\theta) \equiv C_0(\theta)(I + H(\theta))$ is a better approximation to the Floquet transformation for the approximated invariant curve $x_1(\theta)$. This process is repeated until the norm of the remainders, $\|y_k(\theta)\|$ and $\|C_k^{-1}(\theta + \omega) D_x f(x_k(\theta), \theta) C_k(\theta) - B_k\|$, is small enough.

Note that the dimension of the linear systems in the method only depends on the dimension of the phase space, and their number only depends on the number of Fourier modes used in the approximation. For this reason, the method seems suitable for computing tori of higher dimensions.

2.1. *Parallelism*

This algorithm can be easily parallelised by solving all the linear systems "at once", and by doing all the evaluations of f and $D_x f$ inside each step "at once". This last point is specially important when the map is hard to compute numerically, for instance, when f is a Poincaré map of a flow.

3. Applications

Consider the motion of a particle in the Earth-Sun system, modelled as the Restricted Three Body Problem (RTBP) with primaries Earth and Sun, and perturbed by two planets (Jupiter and Saturn), that move in circular orbits around the centre of masses of Earth and Sun, in such a way that they only affect the particle. The equations of the system, using the usual RTBP axis and units, are given by the Hamiltonian

$$\begin{aligned} H = \frac{p_x^2}{2} + \frac{p_y^2}{2} + \frac{p_z^2}{2} + yp_x - xp_y - \frac{1-\mu}{r_s} - \frac{\mu}{r_t} - \frac{m_1}{r_1} - \frac{m_2}{r_2} - \\ -\frac{m_1}{a_1^2}(\cos(\theta_1)x + \sin(\theta_1)y) - \frac{m_2}{a_2^2}(\cos(\theta_2)x + \sin(\theta_2)y), \end{aligned}$$

where $\theta_1 = \omega_1 t$, $\theta_2 = \omega_2 t + \phi$ (ϕ is an initial phase), μ is the mass of the Earth, m_1, m_2 are the masses of the planets, a_1, a_2 are their semi-major axis, ω_1, ω_2 denote their frequencies, and $r_s^2 = (x-\mu)^2 + y^2 + z^2$, $r_t^2 = (x-(\mu-1))^2 + y^2 + z^2$, $r_1^2 = (x - a_1\cos(\theta_1))^2 + (y - a_1\sin(\theta_1))^2 + z^2$ and $r_2^2 = (x - a_2\cos(\theta_2))^2 + (y - a_2\sin(\theta_2))^2 + z^2$.

To apply the method, we consider the Poincaré section $\theta_1 = 0$ and take the diffeomorphism given by the evaluation of the flow at time $2\pi/\omega_1$. We compute an invariant curve replacing one of the triangular points of the Earth-Sun system, with an accuracy of 10^{-12}. The corresponding quasi-periodic solution is normally elliptic, and the normal frequencies (the argument of the eigenvalues of the Floquet matrix) are equal to $2.9465422596978760 \times 10^{-2}$, $2.2077010842674819 \times 10^{-1}$ and $2.2091117967580665 \times 10^{-1}$. The total computing time spent to look for the solution is $104s$, $55s$, $30s$ and $21s$ with 1, 2, 4 and 6 processors respectively. This means that we have parallelised the 98.3% of the computing time.

References

1. H.W. Broer, G.B. Huitema and M.B. Sevryuk. Lecture Notes in Math., Vol. 1645 (1996), Springer Verlag.
2. E. Castellà and À. Jorba. The Lagrangian Points of the Real Earth-Moon System. Proceedings Equadiff03.