ON THE FRACTALIZATION OF INVARIANT CURVES *

ÀNGEL JORBA AND JOAN CARLES TATJER

Departament de Matemàtica Aplicada i Anàlisi, Universitat de Barcelona, Gran Via 585, 08007 Barcelona, Spain.

Let us consider a smooth discrete dynamical system such that one of the variables is a one-dimensional angle whose dynamics is a rigid Diophantine rotation. We assume that the system has an invariant curve whose rotation number is the rotation angle of the angular variable of the system. In this note we study the fractalization of the invariant curve in the context of bifurcation theory.

1. Introduction

Consider

$$\left.\begin{array}{l}\bar{x} = f_\mu(x,\theta),\\ \bar{\theta} = \theta + \omega \pmod{2\pi},\end{array}\right\} \tag{1}$$

where $x \in \mathbb{R}^n$ $(n \geq 1)$, f_μ is smooth, $\theta \in \mathbb{T}^1$, $\mu \in \mathbb{R}^m$ is a parameter and $\omega \in (0, 2\pi)$ is such that $\frac{\omega}{2\pi}$ is Diophantine. Assume that, for a given μ_0, (1) has an attracting invariant curve, $x_{\mu_0}(\theta)$ with rotation number ω,

$$f_{\mu_0}(x_{\mu_0}(\theta),\theta) = x_{\mu_0}(\theta+\omega), \qquad \forall\, \theta \in \mathbb{T}^1.$$

We want to study the continuation (and the bifurcations) of $x_{\mu_0}(\theta)$ with respect to the parameter μ.

As a first example, let us consider the quasiperiodically forced logistic map. This is a classical model that has already been considered by many authors (see [6] for a survey). The equations are

$$\left.\begin{array}{l}\bar{x} = \alpha(1+\varepsilon\cos(\theta))x(1-x),\\ \bar{\theta} = \theta + \omega \pmod{2\pi},\end{array}\right\} \tag{2}$$

with $\omega = \pi(\sqrt{5}-1)$ and $\varepsilon = 0.5$. Fig. 1 shows the attracting set for two values of α. Note that the first set seems to be a smooth curve while

*Work supported by MCyT grants BFM2000–0623 and BFM2000–0805, the CIRIT grant 2001SGR–70 and DURSI.

the second one seems to have a fractal structure. A numerical computation shows that the Lyapunov exponent is negative in both cases. The attracting sets with fractal structure and negative Lyapunov exponents are usually called Strange Nonchaotic Attractors (SNA).

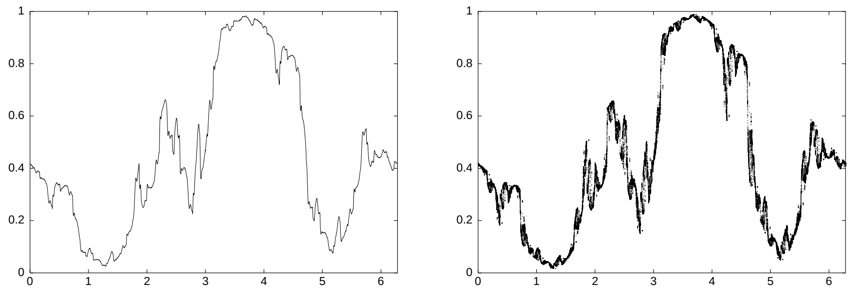

Figure 1. Left: $\alpha = 2.65$, Lyapunov exponent ≈ -0.03884. Right: $\alpha = 2.665$, Lyapunov exponent ≈ -0.00845 (horizontal axis: θ, vertical axis: x).

G. Keller has proved the existence of a SNA in some situations: consider (1) with $n = 1$, which we call T, such that $f(x, \theta) = g(x)h(\theta)$, $g : [0, \infty) \to [0, \infty)$ is bounded C^1, $h : \mathbb{T}^1 \to [0, \infty)$ is continuous, $g(0) = 0$, g increasing and strictly concave, $h(\theta_0) = 0$ for some $\theta_0 \in \mathbb{T}^1$ and we define

$$\sigma = g'(0) \exp \left(\int_{\mathbb{T}^1} \log h(\theta) d\theta \right).$$

Then, it is known[4] that there is an upper semicontinuous function $\phi : \mathbb{T}^1 \to [0, \infty)$ with an invariant graph such that:

(1) The Lebesgue measure on $\mathbb{T}^1$ lifted to the graph of ϕ is a BRS measure for (1),
(2) If $\sigma \leq 1$, then $\phi \equiv 0$ and the Lyapunov exponent of (x, θ) is $\log \sigma$ for a.e. $\theta \in \mathbb{T}^1$ and each $x \geq 0$.
(3) If $\sigma > 1$, then the Lyapunov exponent of almost any point is negative and $\{\theta : \phi(\theta) > 0\}$ has full Lebesgue measure, is a countable union of nowhere dense sets and ϕ is Lebesgue-a.e. discontinuous.
(4) If $\sigma \neq 1$ then $|x_n - \phi(\theta_n)| \to 0$ exponentially fast for Lebesgue-a.e. θ and each $x > 0$, where $(x_n, \theta_n) = T^n(x, \theta)$.

The fact that $x \equiv 0$ is always an invariant curve is essential in Keller's construction. Therefore, the mechanisms for the creation of a SNA in other examples (like (2)) should be different.

The main goal of this note is to discuss this fractalization process in the context of bifurcation theory. In the next sections, we will show how having a negative Lyapunov exponent ensures the applicability of the Implicit Function Theorem to the invariant curve in the space of continuous functions (endowed with the sup norm). This puts into question the transition to SNA in the first example: in fact, careful numerical computations seem to indicate that the fractal curve shown in the first example (and in other similar examples[6]) is, in fact, a continuous curve.[3]

An extended version of these results will appear elsewhere.

2. Continuation of an invariant curve

Assume that (1) has a continuous invariant curve $x = u_0(\theta)$ for $\mu = 0$. This curve satisfies the functional equation $F(u_0, 0) = 0$, where $F : E \times \mathbb{R}^n \to E$, $E = C^0(\mathbb{T}^1, \mathbb{R}^n)$ and, if $(u, \mu) \in E \times \mathbb{R}^n$, we have

$$F(u,\mu)(\theta) = f_\mu(u(\theta),\theta) - u(\theta+\omega).$$

The Implicit Function Theorem requires the linear map $D_uF(u_0, 0)$ to be a linear bounded operator with bounded inverse. The action of $D_uF(u, \mu)$ on an element $v \in E$ is given by

$$[D_uF(u,\mu)v](\theta) = D_x f_\mu(u(\theta),\theta)v(\theta) - v(\theta+\omega). \tag{3}$$

Defining $A(\theta) = D_x f_0(u_0(\theta), \theta)$ and $\psi(\theta) \mapsto A(\theta - \omega)\psi(\theta - \omega)$, as the *transfer operator*, it is easy to see that (3) is a homeomorphism iff 1 does not belong to the spectrum of the transfer operator. It is known[5,1] that the spectrum of a transfer operator can be written as a union of circles centered at the origin.

A standard technique to study the bifurcations of invariant curves is the reduction of its linearized normal dynamics to constant coefficients. This is the topic of the next section.

2.1. *Reducibility*

The linearized dynamics around $u_0(\theta)$ is given by[2]

$$\left.\begin{aligned} \bar{x} &= A(\theta)x, \\ \bar{\theta} &= \theta + \omega. \end{aligned}\right\} \tag{4}$$

This system is called *reducible* iff there exists a linear change of variables $x = C(\theta)y$ such that (4) becomes

$$\left.\begin{aligned} \bar{y} &= By, \\ \bar{\theta} &= \theta + \omega, \end{aligned}\right\}$$

where B does not depend on θ. An invariant curve is called reducible iff the linearised normal dynamics (4) is reducible. The reducibility can be characterized in terms of the spectrum of the transfer operator.[2] Moreover, in the 1-dimensional case $n = 1$, (4) is reducible iff $A(\theta) \neq 0$ for all θ (or $A(\theta) \equiv 0$).

In the reducible case, the bifurcations of $u_0(\theta)$ depend on the eigenvalues of B, and can be studied by means of normal forms. In the nonreducible case, there are no general techniques to study the failure of the IFT to continue invariant curves.

Regardless of the reducibility, it is not difficult to see that the logarithm of the spectral radius of the transfer operator is less than 0 iff the maximal Lyapunov exponent of (4) w.r.t. the normal directions x is less than 0. In particular, if $n = 1$ it is possible to prove, by the Birkhoff Ergodic Theorem and the principle of descent in potential theory, that the logarithm of the spectral radius is equal to the Lyapunov exponent.

Note that if the invariant curve is attracting with negative maximal Lyapunov exponent, then 0 does not belong to the spectrum of (3) and, therefore, the IFT ensures the existence of the invariant curve for a neighbourhood of μ_0.

In this setting, the SNA in Keller's example appears when the spectrum of the transfer operator touches 1. Therefore, it can be seen as a bifurcation of the nonreducible curve $x \equiv 0$.

3. A linear example

Here we will show the fractalization process in a very simple example, that allows for a quite complete analytical description. The map considered is

$$\left.\begin{aligned} \bar{x} &= \alpha \cos(\theta)\, x + 1, \\ \bar{\theta} &= \theta + \omega \pmod{2\pi}, \end{aligned}\right\} \tag{5}$$

being α a positive parameter. It is easy to see that its Lyapunov exponent is $\log\alpha - \log 2$ and, therefore, if $\alpha < 2$, the Lyapunov exponent is negative. This implies that, for $\alpha < 2$, we must have a unique and global attracting set, see Fig. 2. It is not difficult to prove that, for all $\alpha < 2$, (5) has a smooth invariant curve $x_\alpha(\theta)$, that can be written as a convergent expansion,

$$x_\alpha(\theta) = \sum_{n=0}^{\infty} \alpha^n \prod_{j=1}^{n} \cos(\theta - j\omega). \tag{6}$$

A numerical computation shows that, when α tends to 2, the length of (5) divided by its sup norm tends to infinity. This implies that the

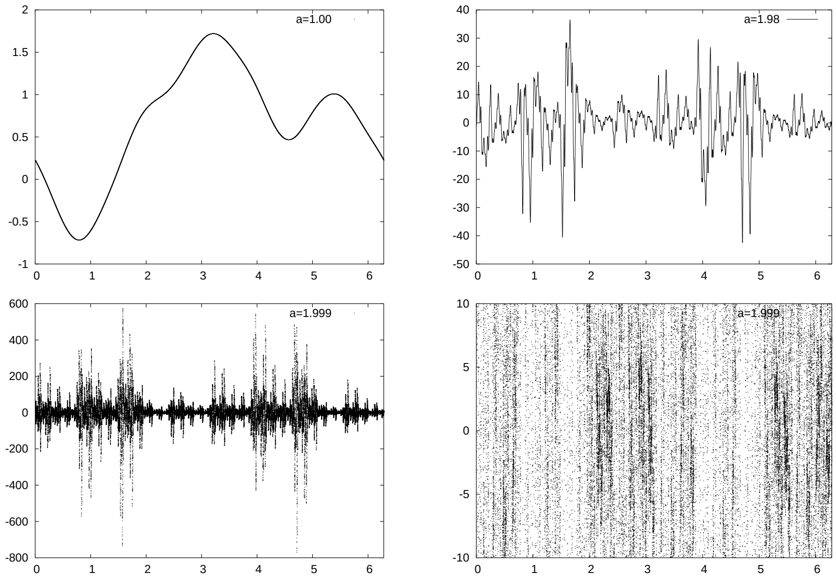

Figure 2. Attractor sets of (5), for $\alpha = 1$, $\alpha = 1.98$ and $\alpha = 1.999$. The last plot is a magnification of the $\alpha = 1.999$ case.

length grows faster than the norm of the curve and, therefore, some kind of fractalization must be present. At this moment, we do not have analytical estimates of the growth of the length of (5). However, it is possible to prove that, if $y_\alpha(\theta) = x_\alpha(\theta)/\|x_\alpha\|_\infty$, then $y'_\alpha(\theta)$ is unbounded in any interval when $\alpha \to 2^-$.

4. The forced logistic map revisited

To apply the previous results to the forced logistic map, let us compute the normal Lyapunov exponent of the attracting set. As before, we have selected $\varepsilon = 0.5$. The results are summarized in Fig. 3, where the horizontal axis denotes the parameter α (see eq. (2)) and the vertical axis is the Lyapunov exponent. For $\alpha < \alpha_1 = 4(2-\sqrt{3})$, $x(\theta) \equiv 0$ is an attracting invariant curve that becomes unstable at α_1. Its Lyapunov exponent is represented by the curve labelled as (A). For $\alpha > \alpha_1$ there is a reducible attracting curve whose Lyapunov exponent is represented by the curve labelled as (B). This invariant curve becomes nonreducible when $\alpha = \alpha_2 \approx 2.65$ (the value α_2 has been marked in the plot with a vertical line). Note the apparent loss of smoothness of the Lyapunov exponent at

$\alpha = \alpha_2$. Then, the Lyapunov exponent starts to grow and to approach zero. We stress that, within the numerical resolution of our computations, this exponent never reaches zero. This seems to suggest that the SNA is, in fact, a continuous curve.

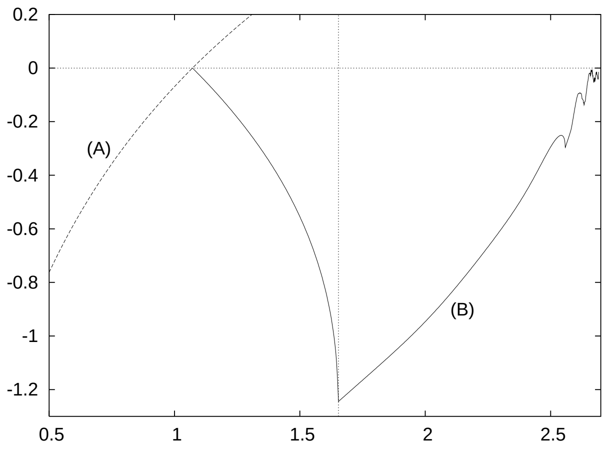

Figure 3. Lyapunov exponents of the attractor and the repeller.

5. Conclusions

We have shown simple (even linear!) examples where the process of fractalization consists of the increase of the lenght of the invariant curve, but the smoothness is preserved until the Lyapunov exponent reaches zero. This could be the case of the forced logistic map, as numerical computations seem to confirm. From the point of view of operator theory, it seems that this process of fractalization is related to the 'failure' of the IFT when 0 becomes a spectral value that is not an eigenvalue.

References

1. A. Haro, R. de la Llave, in progress.
2. À. Jorba, *Nonlinearity* **14**, 943 (2001).
3. À. Jorba, C. Simó and J.C. Tatjer, in progress.
4. G. Keller, *Fund. Math.* **151**, 139 (1996).
5. J. Mather, *Indag. Math.* **30**, 479 (1968).
6. A. Prasad, S.S. Negi and R. Ramaswamy, *Internat. J. Bifur. Chaos* **11**, 291 (2001).