Normal form for a quasi-periodic perturbation of the Sun-Jupiter RTBP

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Abstract

We make a local semi-analytical study of a quasi-periodic perturbation of the Sun-Jupiter RTBP Hamiltonian in a neighbourhood of the triangular points. First, we construct a suitable normal form of the Hamiltonian around the invariant torus that replaces L_5. Then, we use this (high order) normal form to give a description of the local non-linear dynamics.

Introduction

We are interested in the dynamics of a small particle near the triangular points of the Sun-Jupiter system. In order to perform this study, we use an specific model based on the numerical computation of a quasi-periodic solution (with two basic frequencies) of the planar four body problem Sun, Jupiter, Saturn and Uranus and we write it as a perturbation of the Sun-Jupiter RTBP. This model is a restricted five body problem that we call Tricircular Coherent Problem (TCCP). The corresponding Hamiltonian is:

$$\begin{aligned} H &= \frac{1}{2}\alpha_1(\theta_1,\theta_2)(p_x^2+p_y^2+p_z^2) + \alpha_2(\theta_1,\theta_2)(xp_x+yp_y+zp_z) + \alpha_3(\theta_1,\theta_2)(yp_x-xp_y) \\ &\quad +\alpha_4(\theta_1,\theta_2)x + \alpha_5(\theta_1,\theta_2)y - \alpha_6(\theta_1,\theta_2)\left[\frac{1-\mu}{q_S}+\frac{\mu}{q_J}+\frac{m_{sat}}{q_{sat}}+\frac{m_{ura}}{q_{ura}}\right], \end{aligned} \tag{1}$$

where $q_S^2=(x-\mu)^2+y^2+z^2$, $q_J^2=(x-\mu+1)^2+y^2+z^2$, $q_{sat}^2=(x-\alpha_7(\theta_1,\theta_2))^2+(y-\alpha_8(\theta_1,\theta_2))^2+z^2$, $q_{ura}^2=(x-\alpha_9(\theta_1,\theta_2))^2+(y-\alpha_{10}(\theta_1,\theta_2))^2+z^2$, $\theta_1=\omega_{sat}t+\theta_1^0$ and $\theta_2=\omega_{ura}t+\theta_2^0$. The masses of the bodies involved are $\mu = 9.538753600\times 10^{-4}$, $m_{sat}=2.855150174\times 10^{-4}$ and $m_{ura}=4.361228581\times 10^{-5}$. The functions $\alpha_i(\theta_1,\theta_2)_{\{i=1\div 10\}}$ are auxiliary quasi-periodic functions that can be computed by a Fourier analysis of the Tricircular solution. The concrete values of the frequencies are $\omega_{sat}=0.597039074021947$ and $\omega_{ura}=0.858425538978989$. For a description on the development of this model, see [3].

Normal Form of the TCCP Hamiltonian

The 2-D invariant torus that replaces L_5

In the TCCP system, the RTBP L_4 and L_5 points are replaced by quasi-periodic orbits with two internal frequencies and three normal ones. The two internal frequencies

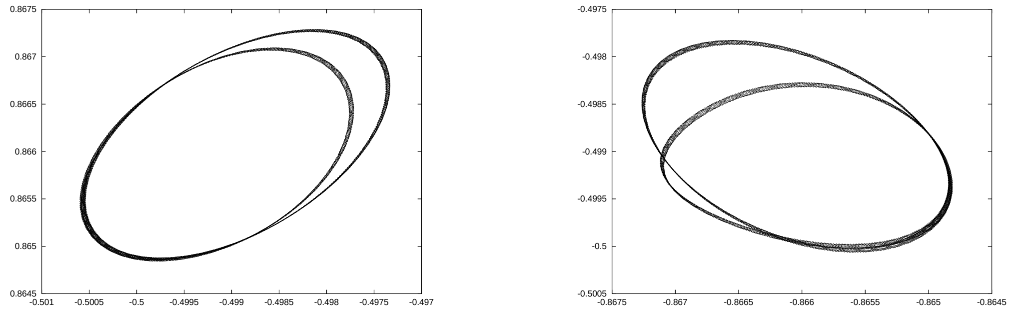

Figure 1: Planar projections of the 2-D invariant torus that replaces L_5: T_5. Left: (x, y)-projection. Right: (p_x, p_y)-projection.

j	$\mathrm{Re}\,(\lambda_j)$	$\pm\mathrm{Im}\,(\lambda_j)$	$\lvert\lambda_j\rvert$	$\pm\mathrm{Arg}\,(\lambda_j)$
1	0.662315481968626	0.749225067883254	1.0	0.846891268646165
2	-0.485204809265089	0.874400533546285	1.0	2.077393707458922
3	-0.453781923686027	0.891112768248671	1.0	2.041801148412721

Table 1: Linear normal modes around the 2-D invariant torus T_5 in the TCCP system.

are ω_{sat} and ω_{ura}, the two proper frequencies of the TCCP system. The proof can be found in [8], where quasi-periodically perturbed elliptic equilibrium points are studied.

In order to compute this 2-D invariant torus (in the L_5 case, for instance), we use the method explained in [1] adapted to the non-autonomous case. The method consists, basically, in computing an invariant curve of a map. In this case, we consider the time period-of-Saturn map associated to the flow of the TCCP. Note that this map can be easily evaluated by a numerical integration of the flow associated to (1), and its differential is also obtained integrating the variational flow.

Hence, it is possible to compute, in the Poincaré section, the invariant curve that corresponds to the 2-D invariant torus that replaces L_5 for the flow of the TCCP model. Due to the smallness of the perturbation, it is enough to use, as initial guess, the coordinates of the triangular point L_5. The (x, y) and (p_x, p_y) projections of the resulting invariant torus are shown in Figure 1. From now on, we will call this 2-D invariant torus T_5.

It is also possible, using the method explained in [7], to compute the three normal modes of the invariant curve; that is, the frequencies of the three harmonic oscillators that govern the normal linear motion around T_5. They can be found as the solution of a generalized eigenvalue problem. In Table 1, the linear normal modes are shown. As all of them have modulus exactly 1, the 2-D invariant torus T_5 is linearly stable.

Second order normal form

We construct a linear change of variables (that depends on time in a quasi-periodic way) that puts the second degree terms of the Hamiltonian into a more convenient form. This is, essentially, the quasi-periodic Floquet transformation for the variational flow along the quasi-periodic orbit, but taking into account the symplectic structure of the problem. To simplify further steps in the normalizing process, we also apply a complexifying change of variables that puts the second degree terms of the Hamiltonian in the so-called diagonal form.

The symplectic quasi-periodic Floquet change The linear flow around the 2-D invariant torus T_5 is described by a linear system of differential equations, that depends quasi-periodically on time:

$$\begin{aligned} \dot{z} &= Q(\theta_1, \theta_2) z, \\ \dot{\theta}_1 &= \omega_{sat}, \\ \dot{\theta}_2 &= \omega_{ura}. \end{aligned} \tag{2}$$

Our goal is to find a real, symplectic and quasi-periodic change of variables, $z = P^r(\theta_1, \theta_2)x$, such that it reduces (2) to a linear system with real constant coefficients:

$$\dot{x} = Bx, \quad \frac{d}{dt}B \equiv 0. \tag{3}$$

We start by considering the $(\theta_1 = 2\pi)$-Poincaré section of the flow given by (2). Then, we have the following linear quasi-periodic skew product:

$$\begin{aligned} \bar{z} &= A(\theta) z, \\ \bar{\theta} &= \theta + \omega, \end{aligned}$$

where $\theta \equiv \theta_2$ and $\omega = 2\pi \left(\frac{\omega_{ura}}{\omega_{sat}} - 1 \right) = 2.75080755611202$ is the rotation number of the invariant curve corresponding to the 2-D invariant torus T_5.

As it is shown in [7], it is possible to reduce this quasi-periodic skew-product to

$$\bar{y} = \Lambda y, \tag{4}$$

by implementing a linear change of variables $z = C(\theta)y$. Λ is a diagonal complex 6×6 matrix with constant coefficients and satisfies, jointly with the matrix $C(\theta)$, the following equation:

$$A(\theta)C(\theta) = C(\theta + \omega)\Lambda. \tag{5}$$

If we define the operator $T_\omega : \Psi(\theta) \in \mathcal{C}(\mathbf{T}^1, \mathbf{C}^n) \to \Psi(\theta + \omega) \in \mathcal{C}(\mathbf{T}^1, \mathbf{C}^n)$ and if we call $\Psi_j(\theta)$ the j-th column of the matrix $C(\theta)$, we can write the system of equations (5) by columns as a generalized eigenvalue problem:

$$A(\theta)\Psi_j(\theta) = \lambda_j T_\omega \Psi(\theta). \tag{6}$$

Remarks: **(i)** Note that $A(\theta)$ is a real matrix (it comes from $Q(\theta_1, \theta_2)$ that is also real). Thus, if $\Psi_j(\theta) \in \mathbf{C}^6$ is an eigenfunction of (6) with eigenvalue λ_j, then $\Psi_j^*(\theta)$ is also an eigenfunction of (6) with eigenvalue λ_j^* (λ_j^* and $\Psi_j^*(\theta)$ are the complex conjugates of λ_j and $\Psi_j(\theta)$, respectively). We construct the matrix $C(\theta)$ as $C(\theta) = (\Psi_1(\theta)\ \Psi_2(\theta)\ \Psi_3(\theta)\ \Psi_1^*(\theta)\ \Psi_2^*(\theta)\ \Psi_3^*(\theta))$, and, then, the matrix Λ takes the following form: $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3, \lambda_1^*, \lambda_2^*, \lambda_3^*)$. **(ii)** The eigenfunctions $\Psi_j(\theta)$ are scaled in such a way that $\|\Psi_j\|_2 = 1$.

Thus, it is possible to reduce equations (2) to complex constant coefficients in a Poincaré section of the flow (via the complex change of variables $C(\theta)$). Now, we want to extend the change of variables to the global phase space and to realify it.

Extension of the change of variables The goal is to compute a quasi-periodic change of variables $z = P^c(\theta_1, \theta_2)y$ that transforms the system (2) to

$$\dot{y} = D_B y, \tag{7}$$

where $D_B = \text{diag}(i\omega_1, i\omega_2, i\omega_3, -i\omega_1, -i\omega_2, -i\omega_3)$ and ω_j is such that $\lambda_j = \exp(i\omega_j T_{sat})$, where as usual $T_{sat} = \frac{2\pi}{\omega_{sat}}$. Note that it is possible to add to ω_j any integer multiple of $\frac{2\pi}{T_{sat}}$ (if ω_j accomplishes the condition, then $\pm(\omega_j + \frac{2k_j\pi}{T_{sat}})$, $k_j \in \mathbf{Z}$, too). We have selected a special value of $k_j \in \mathbf{Z}$ for each $j = 1, 2, 3$ in order the values of ω_j to be the closest possible to the ones in the RTBP case. This is the right choice from a perturbative point of view and it is also critical in order the change of variables to be symplectic. The proofs of the next two lemmas can be found in [4].

Lemma 1 *The solution of the Initial Value Problem (IVP)*

$$\begin{aligned} \dot{P^c}(\theta_1, \theta_2) &= Q(\theta_1, \theta_2)P^c(\theta_1, \theta_2) - P^c(\theta_1, \theta_2)D_B \\ \dot{\theta}_1 &= \omega_{sat} \\ \dot{\theta}_2 &= \omega_{ura}, \end{aligned}$$

with $\left(P^c(0) = C(\theta_2^{(0)}), \theta_1(0) = 0, \theta_2(0) = \theta_2^{(0)}\right)$, *is the linear change of variables with complex quasi-periodic coefficients that sends the system (2) to the system (7).*

Realification In order to actually implement the Floquet change, we are interested in computing the real change of variables. Thus, we will make use of the following

Lemma 2 *Let us define the (real) matrix R as*

$$R(\theta) = \frac{1}{2}C(\theta)\left(\begin{array}{c|c} I_3 & -iI_3 \\ \hline I_3 & iI_3 \end{array}\right).$$

Then, the solution of the IVP

$$\begin{aligned} \dot{P^r}(\theta_1, \theta_2) &= Q(\theta_1, \theta_2)P^r(\theta_1, \theta_2) - P^r(\theta_1, \theta_2)B \\ \dot{\theta}_1 &= \omega_{sat} \\ \dot{\theta}_2 &= \omega_{ura}, \end{aligned} \tag{8}$$

with initial conditions $\left(P^r(0) = R(\theta_2^{(0)}), \theta_1(0) = 0, \theta_2(0) = \theta_2^{(0)}\right)$, *defines a (real) linear quasi-periodic change of variables ($z = P^r(\theta_1, \theta_2)x$) that transforms equation (2) to equation (3).*

The (real) matrix B is defined by $B = R^{-1}CD_BC^{-1}R$ and takes the form

$$B = \left(\begin{array}{ccc|ccc} 0 & 0 & 0 & \omega_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \omega_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \omega_3 \\ \hline -\omega_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\omega_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\omega_3 & 0 & 0 & 0 \end{array}\right).$$

Finally, to ensure that the transformation is canonical, we only need to check that $P^r(\theta_1, \theta_2)$ is a symplectic matrix. This can be proved analytically (analogously at how it is done in [5]) by extending the matrix P^r to the phase space of the autonomous Hamiltonian

$$H(x, y, z, \theta_1, \theta_2, p_x, p_y, p_z, p_{\theta_1}, p_{\theta_2}) = \omega_{sat} p_{\theta_1} + \omega_{ura} p_{\theta_2} + H_{TCCP}(x, y, z, \theta_1, \theta_2, p_x, p_y, p_z),$$

where H_{TCCP} is (1)) but here, to also check the correctness of the software, we have tested numerically this condition for a mesh of values of (θ_1, θ_2), with good agreement.

Implementing this quasi-periodic change of variables, the second degree terms of the Hamiltonian become:

$$H_2^r(x, y) = \frac{1}{2}\omega_1(x_1^2 + y_1^2) + \frac{1}{2}\omega_2(x_2^2 + y_2^2) + \frac{1}{2}\omega_3(x_3^2 + y_3^2), \tag{9}$$

where the frequencies take the following values: $\omega_1 = -0.080473064872369$, $\omega_2 = 0.996680625156409$ and $\omega_3 = 1.00006269133083$. We note that it is not possible to use the Lagrange-Dirichlet theorem to derive the nonlinear stability of the 2-D invariant torus, due to the different signs for the values of the frequencies.

Complexification As it is usual in these kind of computations, a complexifying change of variables is done in order to bring (9) into a diagonal form. The equations of this linear and symplectic transformation are $x_j = (q_j + ip_j)/\sqrt{2}, \quad y_j = (iq_j + p_j)/\sqrt{2}, \quad j = 1, 2, 3.$

Thus, after composing the three linear symplectic changes of variables (translation of the origin to the 2-D invariant torus, quasi-periodic symplectic transformation and complexification), the Hamiltonian of the TCCP in normal form up to order 2 takes the form:

$$H_2^c(q, p) = i\omega_1 q_1 p_1 + i\omega_2 q_2 p_2 + i\omega_3 q_3 p_3, \qquad (q, p) \in \mathbf{C}^6. \tag{10}$$

Expansion of the Hamiltonian

With the help of the recurrence of the Legendre polynomials, we are able to produce a Fourier-Taylor power expansion of the Hamiltonian (1) in these complex coordinates.

We also autonomize the complete Hamiltonian by adding the fictitious momenta corresponding to the angular variables $\theta = (\theta_1, \theta_2) \in \mathbf{T}^2$. Let us denote them $p_\theta = (p_{\theta_1}, p_{\theta_2}) \in \mathbf{C}^2$. In this way, it is possible to write the expanded Hamiltonian as:

$$H(q, p, \theta, p_\theta) = \langle \varpi, p_\theta \rangle + H_2(q, p) + \sum_{n \geq 3} H_n(q, p, \theta), \qquad (q, p) \in \mathbf{C}^6\,,\ \theta \in \mathbf{T}^2, \tag{11}$$

where $H_2 \neq H_2(\theta)$ is given by (10), $(H_n)_{n\geq 2}$, denotes an homogeneous polynomial of degree n in the variables q and p, $\varpi = (\omega_{sat}, \omega_{ura})$ and $\langle \cdot, \cdot \rangle$ is the standard scalar product.

Normal form of order higher than 2

We compute a high order normal form for the expansion of the TCCP Hamiltonian, supposing that up to degree 2 is already in the normal form (10). The goal of the

k_1	k_2	k_3	Re (h_k)	Im (h_k)
1	0	0	-8.0473064872368966e-02	0.0000000000000000e+00
0	1	0	9.9668062515640865e-01	0.0000000000000000e+00
0	0	1	1.0000626913308270e+00	0.0000000000000000e+00
2	0	0	5.6008074695424814e-01	9.9022635266146223e-14
1	1	0	-1.5539627415430354e-01	1.9737284347219547e-14
0	2	0	5.5093985824138381e-03	-3.4515403004164990e-16
1	0	1	5.4161903856716140e-02	2.6837558280952768e-15
0	1	1	6.6103538676104013e-03	-2.3704452239135327e-16
0	0	2	-3.4144388415478051e-04	1.3980906058550924e-20

Table 2: Coefficients of the normal form, up to degree 2 in the actions for the TCCP case. The first three columns contain the exponents of the actions, and the fourth and fifth columns are the real and imaginary parts of the coefficients. Imaginary parts must be zero, but they are not due to the different accumulation errors.

normalizing transformation is to autonomize and to suppress the maximum number of terms of the Hamiltonian expansion. We use, basically, the Lie series method as described in [6], but introducing the necessary modifications in order to deal with quasi-periodic coefficients.

After performing all the changes up to a suitable degree $n = N$, the Hamiltonian takes the form

$$H = \langle \varpi, p_\theta \rangle + \mathcal{N}(q_1p_1, q_2p_2, q_3p_3) + \mathcal{R}(q_1, q_2, q_3, p_1, p_2, p_3, \theta_1, \theta_2), \tag{12}$$

where $\mathcal{N}$ denotes the normal form (that only depends on the products q_jp_j) and $\mathcal{R}$ is the remainder (of order greater than N).

We write the normal form $\mathcal{N}$ in real action-angle coordinates. It is not difficult to see that $\mathcal{N}$ does not depend on the angles φ_j but only on the actions I_j:

$$\mathcal{N} = \sum_{|k|=1}^{[N/2]} h_k I_1^{k_1} I_2^{k_2} I_3^{k_3}, \qquad k \in \mathbf{Z}^3, \quad h_k \in \mathbf{R}. \tag{13}$$

Values for the coefficients h_k can be found in Table 2.

Changes of variables

We have also computed explicit expressions for the transformation from the initial variables of (10) to the final ones. This change of variables is a Taylor truncated power series up to degree N with Fourier coefficients (with two angles) also truncated, in their turn, at orders (N_{f_1}, N_{f_2}). We will use this changes of variables to send information from the normal form coordinates to the initial ones, and vice versa.

Local non-linear dynamics

If we are close enough to the 2-D invariant torus T_5 and we obviate the rotators $\langle \varpi, p_\theta \rangle$, the (non-linear) dynamics can be described accurately by the truncated normal

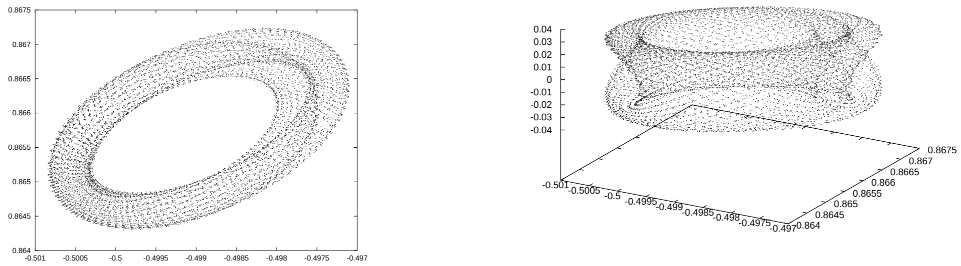

Figure 2: Projection on the (x, y) plane (left) and on the configuration space (right) of an elliptic 3-D invariant torus of the vertical family near T_5. The intrinsic frequencies are ω_{sat}, ω_{ura} and $\omega_3 = 1.000062350$. The normal ones are $\omega_1 = -0.08044599352$ and $\omega_2 = 0.9966839283$.

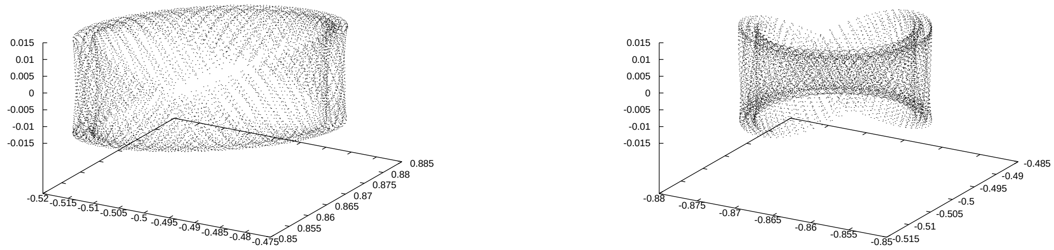

Figure 3: Projections on the (x, y, z) (left) and (p_x, p_y, p_z) spaces of a four-dimensional invariant torus of the vertical family near T_5. The intrinsic frequencies are ω_{sat}, ω_{ura}, $\omega_2 = 0.9966811761$ and $\omega_3 = 1.000063022$, and the normal one is $\omega_1 = -0.08048083168$.

form of the TCCP Hamiltonian (13). As this is an integrable normal form, the dynamics is very simple: the phase space is completely foliated by families of invariant tori. We can easily compute lower and maximal invariant tori using the truncated normal form and send them, via the changes of variables, to the initial synodical coordinates of the TCCP system. This changes of variables add two additional frequencies (the system's intrinsic frequencies, ω_{sat} and ω_{ura}) to the invariant tori. Thus, in our case, the invariant tori are of dimensions three, four and five. Figures 2, 3 and 4 are examples of these computations. See the captions for more details.

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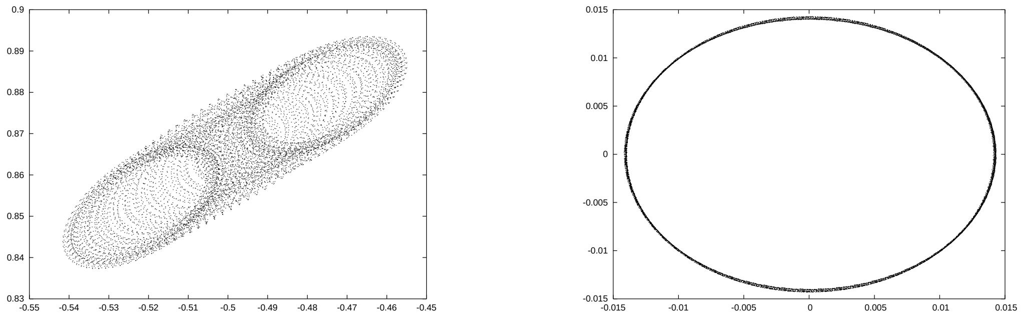

Figure 4: Projections of an elliptic 5-D invariant torus of the vertical family near T_5. The proper frequencies are ω_{sat}, ω_{ura}, $\omega_1 = -0.08046420047$, $\omega_2 = 0.9966802858$ and $\omega_3 = 1.000063496$. Left plot: (x, y)-projection. Right plot: (z, p_z)-projection.

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