Cantor Spectrum for the Almost Mathieu Operator. Corollaries of localization, reducibility and duality

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Abstract

In this note we use results on reducibility, localization and duality for the Almost Mathieu operator,

$$\left(H_{b,\phi}x\right)_n = x_{n+1} + x_{n-1} + b\cos\left(2\pi n\omega + \phi\right)x_n$$

on $l^2(\mathbb{Z})$ and its associated eigenvalue equation to deduce that for $b \neq 0, \pm 2$ and ω Diophantine the spectrum of the operator is a Cantor subset of the real line. Moreover, we prove that for $|b|$ small enough all spectral gaps predicted by the Gap Labelling theorem are open.

1 Introduction. Main results

In this note we study the nature of the spectrum of the Almost Mathieu operator

$$(H_{b,\phi}x)_n = x_{n+1} + x_{n-1} + b\cos(2\pi\omega n + \phi)x_n, \qquad n \in \mathbb{Z} \tag{1}$$

on $l^2(\mathbb{Z})$, where b is a real parameter, $\omega > 0$ is an irrational number and $\phi \in \mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$. Since for each b this is a bounded self-adjoint operator, the spectrum is a compact subset of the real line which does not depend on ϕ because of the assumption on ω. This spectrum will be denoted by σ_b. The understanding of the spectrum of (1) is related to the dynamical properties of the difference equation

$$x_{n+1} + x_{n-1} + b\cos(2\pi\omega n + \phi)x_n = ax_n, \qquad n \in \mathbb{Z} \tag{2}$$

for $a \in \mathbb{R}$. In what follows we will assume that the frequency ω is Diophantine:

Definition 1 *We say that a real number ω is Diophantine whenever there exist positive constants c and $r > 1$ such that the estimate*

$$|\sin 2\pi n\omega| > \frac{c}{|n|^r}$$

holds for all $n \neq 0$.

The nature of the spectrum of this operator has been studied intensively in the last twenty years (for a review, see [Las95]) and one of the topics of interest has been to know whether the spectrum is a Cantor set or not. In this paper we derive two results on this problem. The first one is *non-perturbative*:

Corollary 2 *If ω is Diophantine, then the spectrum of the Almost Mathieu operator is a Cantor set if $b \neq 0, \pm 2$.*

Here, we prefer to call this result a corollary, rather than a theorem, because the proof requires just a combination of reducibility, point spectrum and duality developed quite recently for the Almost Mathieu operator and the related eigenvalue equation. In fact, for $|b| = 2$, Y. Last proved in [Las94] that the spectrum of the Almost Mathieu operator is a subset of the real line with zero Lebesgue measure and that it is a Cantor set for the values of ω which have an unbounded continued fraction expansion, which is a set of full measure.

The Cantor structure of the spectrum of the Almost Mathieu operator can be better understood if we make use of the concept of the rotation number ([JM82], [DS83]) which can be defined as follows. Let $(x_n)_{n\in\mathbb{Z}}$ be a non-trivial solution of (2), for some fixed a, b, ϕ. Let $S(N)$ be the number of changes of sign of such solution for $1 \leq n \leq N$, adding one if $x(N) = 0$. Then the limit

$$\lim_{N\to\infty} \frac{S(N)}{2N}$$

exists, it does not depend on the chosen solution x, nor on ϕ and it is denoted by rot (a, b).

Remark 3 *The rotation number, as defined above, belongs to the interval $[0, 1/2]$.*

For the rotation number one has the following proposition:

Proposition 4 ([JM82, AS83, DS83, Her81]) *The rotation number has the following properties:*

(i) The rotation number, rot (a, b), is a continuous function of $(a, b) \in \mathbb{R}^2$.

(ii) For a fixed b, the spectrum of (1), σ_b, is the set of $a_0 \in \mathbb{R}$, such that $a \mapsto$ rot (a, b) is not locally constant at a_0.

(iii) **(Gap labelling)** *If I is an open, non-void interval in the resolvent set of (1), $\rho_b = \mathbb{R} - \sigma_b$, then there is an integer $k \in \mathbb{Z}$ such that*

$$2rot\ (a, b) - k\omega \in \mathbb{Z}$$

for all $a \in I$. That is, rot $(a, b) = \{k\omega\}/2$ where $\{\cdot\}$ denotes the fractional part of a real number.

From this theorem, we conclude that the resolvent set is the disjoint union of countably (or finitely) many open intervals called *spectral gaps*, possibly void, and which can be uniquely labelled by an integer k called the *resonance*. If a spectral gap degenerates to a void interval we will say that it is a *collapsed gap* and otherwise that it is a *non-collapsed gap*.

In particular if, for a fixed b, all the spectral gaps are open and the frequency ω is irrational, then the spectrum σ_b is a Cantor set ([Las95]). In this case we will speak of *complete Cantor spectrum*. However, if non-collapsed gaps are dense in the spectrum, then this is still a Cantor set, although some (perhaps an infinite number) of collapsed gaps may also coexist.

Now we can formulate the second corollary in this paper:

Corollary 5 *Assume that $\omega \in \mathbb{R}$ is Diophantine. Then, there is a constant $C = C(\omega) > 0$ such that if $0 < |b| < C$ or $1/C < |b| < \infty$ all the spectral gaps of ρ_b are open.*

Now we shall give a short account of the existing results (to our knowledge) on the Cantor spectrum of the Almost Mathieu operator for $|b| \neq 0$ (for the case $|b| = 2$, see [Las94]). Sinai ([Sin87]), proved that for Diophantine ω's and sufficiently large (or small $|b|$), depending on ω, the spectrum σ_b is a Cantor set. Choi, Elliott and Yui [CEY90] proved that the spectrum σ_b is a Cantor set for all $b \neq 0$ when ω is a Liouville number obeying the condition

$$\left|\omega - \frac{p}{q}\right| < D^{-q},$$

for a certain constant $D > 1$ and infinitely many rationals p/q. In particular, this means that for a G_δ-dense subset of pairs (b, ω) the spectrum is a Cantor set, which is the Bellissard-Simon result [BS82]. On the other hand, collapsed gaps appear naturally in quasi-periodic Schrödinger operators, as it was shown in [BPS02]. Finally, let us mention that, if we consider the case of rational ω, all spectral gaps, apart from the middle one, are open if $b \neq 0$. This result was proved by van Mouche [vM89] and Choi, Elliott and Yui [CEY90].

Let us now outline the contents of the present paper. In section 2 we introduce the concept of reducibility of the first-order equation associated to the eigenvalue equation (2), together with the reducibility results by Eliasson to prove the corollary 5. In section 3 we introduce the non-perturbative results obtained by Jitomirskaya on the exponential localization for the Almost Mathieu operator to derive corollary 2.

2 Duality, reducibility and complete Cantor spectrum for small (and large) $|b|$

In this section we will prove that, for $0 < |b| < C$, where $C > 0$ is a suitable constant, and for $|b| > 1/C$ all spectral gaps are open. To see why this holds for both big and small values of $|b|$ we will begin with the following theorem.

Theorem 6 ([AS83]) *For every irrational ω, the rotation number of (2) satisfies the relation*

$$rot\ (a, b) = rot\ (2a/b, 4/b)$$

for all $b \neq 0$ and $a \in \mathbb{R}$.

This means that the spectrum σ_b is a Cantor set (resp. none of the spectral gaps of σ_b is collapsed) if, and only if $\sigma_{4/b}$ is a Cantor set (resp. none of the spectral gaps of $\sigma_{4/b}$ is collapsed). The theorem from which we will derive corollary 5 is due to H. Eliasson and deals with the continuous case. The adaption to our discrete case reads as follows.

Theorem 7 ([Eli92, Eli99]) *Assume that ω is Diophantine with constants c and r. Then there is a constant $C(c, r)$ such that, if $|b| < C(c, r)$ and rot (a, b) is either resonant (that is, of the form $\frac{1}{2}\{k\omega\}$ for a suitable integer) or Diophantine with respect to $\omega/2$ (that is, if the vector $(1/2, \omega/2, rot\ (a, b))$ is Diophantine) then the quasi-periodic skew-product flow defined by*

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} a - b\cos(\theta_n) & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}, \qquad \theta_{n+1} = \theta_n + 2\pi\omega \tag{3}$$

on $SL(2, \mathbb{R}) \times \mathbb{T}$ is reducible to constant coefficients, with matrix B, by means of a quasi-periodic (with frequency $\omega/2$) and analytic transformation. Moreover, if a is at an endpoint of a spectral gap of σ_b (and the rotation number is thus rot $(a, b) = \{k\omega\}/2$), then $B^2 - 2B + I = 0$, being $B = I$ if, and only if, the gap collapses.

This theorem implies that, for $|b| < C$, there exists a sequence $(a^k(b))_{k\in\mathbb{Z}}$ (formed by the endpoints of the spectral gaps with rotation number $\{2l\omega\}/2$, for $l \in \mathbb{Z}$) and sequences of real analytic functions $\psi_b^k : \mathbb{T} \to \mathbb{R}$, for $k \in \mathbb{Z}$ such that the following relations are satisfied:

$$\begin{aligned}\psi_b^k(2\pi\omega(n+1)+\phi) + \psi_b^k(2\pi\omega(n-1)+\phi) + \\ b\cos(2\pi\omega n+\phi)\psi_b^k(2\pi\omega n+\phi) = a^k(b)\psi_b^k(2\pi\omega n+\phi)\end{aligned} \tag{4}$$

for all $\phi \in \mathbb{T}$ and $n \in \mathbb{Z}$. For simplicity, we drop out the dependence on b in what follows. Note that the sequences $(\psi^k(2\pi n\omega+\phi))_{n\in\mathbb{Z}}$ do not belong to $l^2(\mathbb{Z})$. We choose the ψ^k such that $\|\psi^k\|_{L^2(\mathbb{T})} = 1$. Since the quasi-periodic functions ψ^k are analytic, there exist $\beta(k) > 0$ and $A(k) > 0$ such that

$$|\hat{\psi}^k(m)| \leq A(k)\exp\left(-\beta(k)|m|\right),$$

where $\hat{\psi}^k(m)$ is the m-th Fourier coefficient of ψ^k. A substitution of the Fourier expansion in 4 shows that, for any k, these coefficients satisfy the following difference equation

$$2\cos(2\pi\omega m)\xi_m + \frac{b}{2}(\xi_{m+1}+\xi_{m-1}) = a\xi_m, \qquad m \in \mathbb{Z} \tag{5}$$

for $a = a^k$, which is equivalent to the equation

$$\xi_{m+1} + \xi_{m-1} + \frac{4}{b}\cos(2\pi\omega m)\xi_m = \frac{2a}{b}\xi_m, \qquad m \in \mathbb{Z}. \tag{6}$$

This is the eigenvalue equation for the *dual* operator $H_{4/b,0}$. Due to the exponential decay of each of the sequences $\hat{\psi}^k$ they belong to $l^2(\mathbb{Z})$ and thus, the set $(2a^k/b)_{k\in\mathbb{Z}}$ is a set of point eigenvalues for the operator $H_{4/b,0}$ with exponentially decaying eigenfunctions.

We are now going to see that all the a^k are different. Indeed, assume that there exists a $k, j \in \mathbb{Z}$, with $k \neq j$, such that $a^k = a^j$. Then, because of theorem 7, the functions ψ^k and ψ^j should be linearly independent and, therefore, the corresponding Fourier transforms $\hat{\psi}^k$ and $\hat{\psi}^j$ must also be linearly independent. However, this would mean that the difference equation

$$\xi_{m+1} + \xi_{m-1} + \frac{4}{b}\cos(2\pi\omega m)\xi_m = \frac{2a^k}{b}\xi_m$$

would have two linearly independent solutions belonging both to $l^2(\mathbb{Z})$. This is a contradiction, because for bounded potentials, like the cosine, we are always in the limit point case (see [CL55, CL90] for the continuous case). In our discrete case, this is even simpler, since any solution in $l^2(\mathbb{Z})$ of the eigenvalue equation must tend to zero at $\pm\infty$. Hence, the existence of two linearly independent solutions belonging both to $l^2(\mathbb{Z})$ would be in contradiction with the preservation of the Wronskian. Therefore, we conclude that all the gaps corresponding to rotation numbers of the form $\{2l\omega\}/2$ are open if $0 < |b| < C$ and, therefore, also for $|b| > 1/C$. A similar argument shows that the gaps with rotation number of the form $\{(2l+1)\omega\}/2$ are all open. This ends the proof of corollary 5.

3 Non-perturbative localization and Cantor spectrum for $b \neq 0$

In this section we will see how corollary 2 is a consequence of the following theorem on non-perturbative localization, due to S. Jitomirskaya:

Theorem 8 ([Jit99]) *Let ω be Diophantine. Define the set Φ of* resonant phases *as the set of those $\phi \in \mathbb{T}$ such that the relation*

$$|\sin(\phi+\pi n\omega)| < \exp\left(-|n|^{\frac{1}{2r}}\right) \tag{7}$$

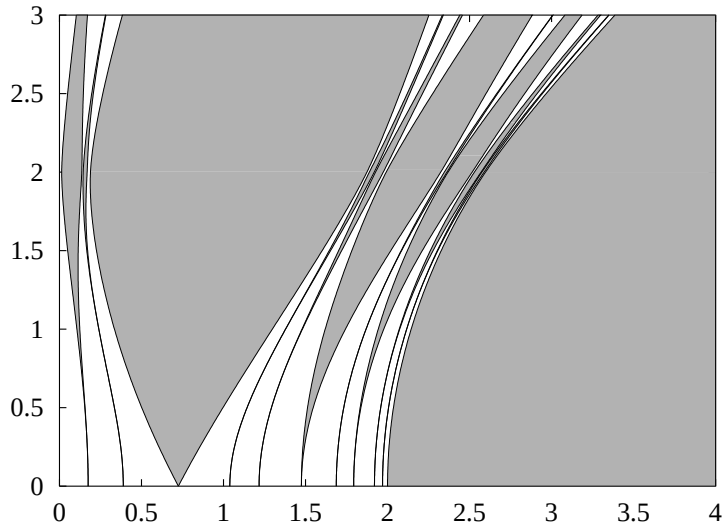

Figure 1: Numerical computation of the ten biggest spectral gaps for the Almost Mathieu operator with different values of b and $\omega = (\sqrt{5}-1)/2$. They correspond to the first integers k such that $\{k\omega\}/2$ belongs to $[1/4, 1/2]$. The coupling parameter b is in the vertical direction whereas the spectral one, a, is in the horizontal one.

holds for infinitely many values of n, being r the constant in the definition of a Diophantine number. Then, if $\phi \notin \Phi$ and $|b| > 2$ the operator $H_{b,\phi}$ has only pure point spectrum with exponentially decaying eigenfunctions. Moreover, any of these eigenfunctions $(\psi_n)_{n\in\mathbb{Z}}$ satisfies that

$$\beta(b) = -\lim_{|n|\to\infty} \frac{\log\left(\psi_n^2 + \psi_{n+1}^2\right)}{2|n|} = \log\left(\frac{|b|}{2}\right). \tag{8}$$

We are now going to see that this theorem implies corollary 2 for $|b| \neq 2$ (for $b = 0$ it is clear that the spectrum is the interval $[-2, 2]$ and, thus, not a Cantor set).

Let $|b| > 2$. Then, according to theorem 8, the operators $H_{b,0}$ and $H_{b,\pi/2}$ have only pure point spectrum with exponentially decaying eigenfunctions. The eigenvalue equations associated to these operators have the following properties:

Lemma 9 *Let $(x_n)_{n\in\mathbb{Z}}$ be a solution of the difference equation*

$$x_{n+1} + x_{n-1} + b\cos(2\pi n\omega + \phi)x_n = ax_n, \qquad n \in \mathbb{Z},$$

for some constants a, b and $\phi \in \mathbb{T}$. Then, if $\phi = 0$, $(x_{-n})_{n\in\mathbb{Z}}$ is also a solution of this equation and, if $\phi = -\pi/2$ then $(-x_{-n})_{n\in\mathbb{Z}}$ is also a solution.

Let us consider first the case of the operator $H_{b,0}$. According to theorem 8, there exists a sequence of eigenvalues $(a^k(b))_{k\in\mathbb{Z}}$ with eigenvectors $(\psi^k(b))_{k\in\mathbb{Z}}$, exponentially localized and which form a complete orthonormal basis of $l^2(\mathbb{Z})$. Moreover the set of eigenvalues $(a^k(b))_{k\in\mathbb{Z}}$ must be dense in the spectrum σ_b. Again, we do not write the dependence on b for simplicity in what follows. None of these eigenvalues can be repeated, since we are in the limit point case (see, again [CL55]). Writing each of the ψ^k as

$$\psi^k = (\psi_n^k)_{n\in\mathbb{Z}},$$

we define

$$\tilde{\psi}^k(\theta) = \sum_{k\in\mathbb{Z}} \psi_n^k e^{ik\theta},$$

for $\theta \in \mathbb{T}$. All these functions belong to $C^a_\beta(\mathbb{T},\mathbb{R})$, the set of real analytic functions of $\mathbb{T}$ with analytic extension to $|\Im\theta| < \beta$ and they are even functions of θ, because of the lemma 9 (here we have applied again that we are in the limit point case). Passing to the dual equation, we obtain that, for each $k \in \mathbb{Z}$, the sequence $(\tilde{\psi}^k(2\pi\omega n))_{n\in\mathbb{Z}}$ is a quasi-periodic solution of

$$x_{n+1} + x_{n-1} + \frac{4}{b}\cos(\theta_n)x_n = \frac{2a}{b}x_n, \qquad \theta_{n+1} = \theta_n + 2\pi\omega \qquad n \in \mathbb{Z}, \tag{9}$$

provided a is taken equal to a^k. This proves that $2a^k/b$ is at an endpoint of a spectral gap. We want to see that all these spectral gaps are open. To do so we use the following argument.

An equation like (2) has a real analytic and quasi-periodic solution if, and only if, the two-dimensional system (3) is reducible to constant coefficients. This is the contents of the following lemma

Lemma 10 *Let $A : \mathbb{T} \to SL(2,\mathbb{R})$ be a real analytic map, with analytic extension to $|\Im\theta| < \delta$. Assume that there is a nonzero real analytic map $v : \mathbb{T} \to \mathbb{R}^2$, with analytic extension to $|\Im\theta| < r$ such that*

$$v(\theta + 2\pi\omega) = A(\theta)v(\theta), \qquad \theta \in \mathbb{T}.$$

Then, the quasi-periodic skew-product flow

$$(u,\theta) \in \mathbb{R}^2 \times \mathbb{T} \mapsto (A(\theta)u, \theta + 2\pi\omega) \in \mathbb{R}^2 \times \mathbb{T}, \tag{10}$$

is reducible to constant coefficients by means of a quasi-periodic transformation which is analytic in $|\Im\theta| < \delta$. Moreover the Floquet matrix satisfies $B^2 - 2B + I = 0$.

Proof: Since $v = (v_1, v_2)^T$ does not vanish, we can define the transformations

$$Z(\theta) = \left(v_1^2(\theta) + v_2^2(\theta)\right)^{-1} \begin{pmatrix} v_1(\theta) & -v_2(\theta) \\ v_2(\theta) & v_1(\theta) \end{pmatrix},$$

whose determinant is always different from zero. Let

$$d(\theta) = v_1^2(\theta) + v_2^2(\theta)$$

and $f(\theta) = d(\theta)/d(\theta+\omega)$. The transformation Z defines a conjugation of A with B, being

$$Z(\theta+\omega)A(\theta) = B(\theta)Z(\theta),$$

which means that B is

$$Z(\theta+\omega)A(\theta)Z(\theta)^{-1} = \begin{pmatrix} 1 & b_{12}(\theta) \\ 0 & f(\theta) \end{pmatrix},$$

since $\det Z(\theta+\omega)A(\theta)Z(\theta)^{-1} = f(\theta)$. Let now

$$Y(\theta) = \begin{pmatrix} 1 & 0 \\ 0 & d(\theta) \end{pmatrix}.$$

This transformation conjugates the skew-product defined by B to the one with matrix

$$Y(\theta+\omega)B(\theta)Y(\theta)^{-1} = \begin{pmatrix} 1 & \tilde{b}_{12}(\theta) \\ 0 & 1 \end{pmatrix}.$$

This system is reducible to constant coefficients by means of a quasi-periodic transformation since the system is analytic in θ and the frequency ω is Diophantine. □

Now, write the dual Schrödinger equation (9) as a quasi-periodic skew-product on $SL(2,\mathbb{R})$, like (3), whose matrix we denote by $A(\theta)$. Writing

$$v(\theta) = (\tilde{\psi}^k(\theta), \tilde{\psi}^k(\theta - 2\pi\omega))^T,$$

we can apply the lemma to conclude that the dual system is reducible.

Since the duality transformation sends endpoints of spectral gaps to endpoints of spectral gaps with the same rotation number, we conclude that, whenever there is a $k \in \mathbb{Z}$ such that the corresponding a^k is at an endpoint of a collapsed spectral gap, there must exist also another analytic and quasi-periodic solution of the dual equation (9).

Passing again to the dual and using the limit point character of the operators $H_{b,\phi}$ we conclude that all of the a^k are endpoints of non-collapsed gaps, because otherwise equation (2) would have two linearly independent solutions belonging to $l^2(\mathbb{Z})$. Moreover the set $(a^k)_{k\in\mathbb{Z}}$ is dense in σ_b, since the spectrum is pure-point. Therefore the spectra σ_b, for $b \neq 0$ are Cantor sets. More concretely this shows that for any $|b| \neq 2, 0$, there is an infinite number of open spectral gaps in the spectrum σ_b whose rotation number is of the form $\{2k\omega\}/2$, for some $k \in \mathbb{Z}$, and that these gaps are dense in the spectrum. In a similar way, using the pure point hypothesis for the operator $H_{b,\pi/2}$, it can be shown that there is an infinite number of open spectral gaps in the spectrum σ_b whose rotation number is of the form $\{(2k+1)\omega\}/2$, for some $k \in \mathbb{Z}$, and that these gaps are dense in the spectrum. This ends the proof of the corollary 2.

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