Computer Assisted Studies in Dynamical Systems

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Abstract

It is a well known fact that purely analytical methods are suitable to prove existence results and to get some local or semiglobal information on the dynamics of a system. If a more detailed study is desired or, in many cases, if one has to guess first the properties to be proven, we have to proceed to do a computer assisted study of the problem. In these notes I summarize several domains where a systematic use of a computer can help us to learn about a given family of dynamical systems and provide some examples.

1 Introduction

Faced to a family of Dynamical Systems (DS) of some kind (either continuous, given by a vector field f, or discrete, given by a map F, in small dimension of large dimension), there are many questions which can be studied, eventually with the help of a computer. A possible classification can take into account several aspects, depending on which is our concern.

a) **The goal**. We can have different objectives in mind:

   1. To guess properties of a DS or of some specific orbits of it,
   2. To illustrate a theoretical result or to check the physical relevance it can have,
   3. To do massive explorations of a system to distinguish the main features and how they evolve as a function of parameters (e.g. measure of regular or chaotic orbits),
   4. To complete cumbersome proofs which, in theory, can be done by hand but requiring a too long time,
   5. To obtain optimal domains of validity of a theoretical result.

b) **The models**. From a theoretical point of view we can study a system *per se*, without any concrete application in mind. But in case a DS appears as a mathematical model of a physical situation, we should proceed to some simplification of reality. It is also true that studying a model we have to introduce several *submodels* on it, to isolate difficulties. We can consider

   1. To produce a chain of systems of increasing complexity or, conversely, to obtain first a rather complete model, to analyze the effect of the different terms and proceed to the best simplifications.

2. To derive models for return maps to the vicinity of an invariant object. They can be obtained by combining local expansions in a neighbourhood $\mathcal{U}$ of the object (e.g., by using normal forms in $\mathcal{U}$) with analytic approximations of the return to $\mathcal{U}$ (the *outer map*). The coefficients of the return approximation can be obtained by numerical integration or iteration.
3. To obtain analytic approximations for f or F by fitting it to experimental data. It can happen that an analytic expression for f or F is available but to evaluate it is cumbersome. Then we can proceed as if they were experimental data, with the great advantage that we can select the points for the *experiment*.
4. To analyze in detail some paradigmatic models to learn about new phenomena and mechanisms.

c) **The objects**. In a DS we have different objects which play a key role in the dynamics. They constitute the "skeleton" of the dynamics and most of the behaviour of the system can be guessed if we know these objects, how they are related, the possible transitions from one to another and how long they take these transitions. To fix ideas we shall assume that the system is discrete. A non-exhaustive and non-disjoint list can be

1. Fixed points and periodic orbits. Invariant curves, invariant tori and other invariant manifolds having a neutral behaviour when we restrict the system to them.
2. Invariant objects which are of Cantor type or are, locally, a Cantor set times a manifold.
3. Stable and unstable invariant manifolds of the (normally, partly and, perhaps weakly) hyperbolic objects which appear among the ones of the previous items.
4. The intersections of the previous invariant manifolds: homoclinic and heteroclinic objects.
5. Other objects far from being fully identified and classified, like strange attractors and repellers.

d) **The approach**. DS are very rich and so they can be considered from different, again non-disjoint, points of view.

1. The orbital approach: we are interested in the behaviour of some orbits of the system.
2. The geometric approach: the main concern is the geometrical description of the invariant objects.
3. The functional approach: we consider orbits or invariant objects as fixed or critical points of a functional in a suitable space (e.g., the variational methods of classical mechanics).
4. The parametric approach: all the objects found can depend on parameters of the system. A key point is the possible continuation and the bifurcations found when the parameters change. It is typical that some objects exist for a set of parameters which can be, in turn, Cantorian. A good strategy is to consider phase space variables and parameters together.
5. The statistical approach: due to the complexity of the dynamics we want to consider statistical indicators. In particular we can look for measures invariant under the action of the system.

In any case it should be kept in mind that the most relevant word in all the study is *dynamics*. A collection of points belonging to some orbits carries a piece of information, but the knowledge of *how* these points appear, i.e., the *temporal evolution*, is much more important.

e) **The tools**. The mathematical tools to be used depend strongly on the chosen approach. Here I describe a rough classification of tools used in computer assisted studies.

   1. Symbolic computations. Representing (or approximating) orbits or invariant objects as functions of some class (e.g., polynomial or Fourier expansions) we can ask them to be solutions of the system or to be *formally* invariant and then derive the relations between the coefficients of the representation. The coefficients can be numbers of some class (e.g., rational) which can be kept as such or approximated by real numbers.
   2. Direct numerical methods. We want to approximate the numerical values of an orbit at some time epochs, or to describe a manifold giving the values at the points of some grid, or to find a parameter value for which a homoclinic tangency occurs, etc. They require, typically, numerical integrators, computation of zeros of functions, approximation theory, etc.
   3. Hybrid methods. Quite often it is fruitful a combination of the first two tools. An invariant unstable manifold can be approximated locally by a symbolic computation and then globalized by numerical integration. The invariance of a torus represented by Fourier series can be imposed by asking to coincide (up to some order) with the Fourier analysis of the image obtained by iteration.

   In any case it should be kept in mind that all these computations can be done with arbitrary precision (e.g. 1000 digits) and with interval arithmetics. I would like to emphasize that we can compute not only orbits, say, but also the variationals with respect to initial conditions and parameters. This is a key point to convert a numerical computation into a rigorous proof.

   Another item to be stressed here concerns the computational complexity of the algorithms used. They should be not only feasible in theory but in practice. This leads to interesting problems of optimization. For instance, when using hybrid methods to generate a very weakly unstable invariant manifold, one should decide the optimal order of the local expansion to minimize the total computation time with controlled errors.

f) **The dimension**. Unfortunately a fair description of the dynamics of a non-elementary system with the help of computers is possible only for relatively small dimension. At the beginning of the XXIth century I consider that, adding the dimensions of the phase and parameter spaces, the limit is around 7. This depends strongly on the system and the desired degree of detail. We can expect to reach a value 10 in the nearby future.

   A quite different question appears if we are interested only in a few orbits (e.g. meteorological forecast). Efforts concerning the best way (i.e., faithful and cheap) to represent solutions of evolution partial differential equations should allow to stress the dynamical systems approach for these equations.

In what follows I shall present a few examples which illustrate several aspects of the computer assisted studies in DS. For the description of numerical tools for the computation of different objects one can see [16, 18]. For massive explorations of a system and details about the required tools I just give several references like [1, 2, 3, 4, 7, 20, 22]. A few conclusions end this note.

2 Several examples

In this section I describe examples, chosen in the conservative domain, which show how symbolic and numerical explorations can lead to guess relevant properties of a system.

2.1 The Sitnikov problem

This is a well known problem restricted three-body problem (see, e.g., Moser [15]). Consider two bodies of equal mass ($= 1/2$) moving in elliptic orbits around the common centre of mass. Assume the bodies (the primaries) are on the horizontal plane through the origin in $\mathbb{R}^3$ and the eccentricity of the orbit is e. A third particle of zero mass moves under the action of the primaries on the z-axis (the vertical axis through the origin). The equations of motion are

$$\dot{z} = v, \quad \dot{v} = -z(z^2 + r^2/4)^{-3/2},$$

where r denote the distance between the primaries and the dot denotes the time derivative. The system is a non-autonomous Hamiltonian with one degree of freedom. Let u be the eccentric anomaly of the primaries. Then we can assume $r = 1 - e\cos(u)$ by normalizing distances. The state of the system is described by (z, v, u). Note that u can be taken as temporal variable. Time and u are related by Kepler's equation $t = u - e\sin(u)$.

We can consider passages of the particle through $z = 0$. They occur for some values of v, u. These values, together with $z = 0$ completely specify the motion. Hence, we can study some orbit by looking at the successive passages though $z = 0$. Of course the particle can escape for positive or negative time and no further passages occur. We are interested in the boundary delimiting the escape region. Due to the symmetries of the problem

$$(z, v, u) \longleftrightarrow (z, -v, -u) \quad \text{and} \quad (z, v, u) \longleftrightarrow (-z, -v, u),$$

it is enough to study what happens for positive time.

If $e = 0$ the system is autonomous and the boundary is given by some constant value of v, independent of the value of u. It is relevant to identify the boundary as follows. Consider the motion near infinity (with $z > 0$.) To this end it is useful to introduce the so-called McGehee coordinates (q, p) (see [15] for details)

$$z = 2q^{-2}, \quad v = -p.$$

When $z \to \infty$ we have $q \to 0$. In the (q, p) coordinates (or coordinates "at infinity") we have a fixed point when $(q, p) = (0, 0)$. But as u keeps moving periodically, we can consider it as a "periodic orbit at infinity". It is weakly hyperbolic and a basic result of McGehee [11] shows that it has invariant stable and unstable manifolds. These manifolds can be continued till the $z = 0$ plane. The intersection of the stable manifold gives the boundary (in (v, u)) of the escaping initial conditions.

Hence we are faced to compute an invariant manifold, and we would like to do this for arbitrary values of the eccentricity. It is convenient to represent the manifold as

$$p = \sum_{k \geq 1} a_k(u, e) q^k = \sum_{k \geq 1} \left(\sum_{j,m} a_{j,m,k} e^j g(mu) \right) q^k,$$

where g denotes either cos or sin, depending on parities. The problem is reduced to compute the coefficients $a_{j,m,k}$. An interesting point is that in [11] it is proven that the manifold is analytic except, perhaps, at $q = 0$. Hence, we cannot take for granted the convergence.

This is a typical simple example of symbolic manipulation. It follows by induction that $a_k(u, e)$ is a trigonometric polynomial in u and the coefficients of the harmonics are polynomials in e. A simple routine to determine these coefficients has been implemented. The number of coefficients to reach a maximal order $k = n$ is roughly $n^3/48$ and the computational complexity is $O(n^6)$ (with a small asymptotic constant!). So, to reach $n = 100$ takes less than 1 second of CPU time in a modern PC and order 300 (for which the number of coefficients is 562532) is attained in roughly 5 minutes. Furthermore, it is easily realized than there are relatively few relevant harmonics. If at any order we skip all the coefficients with $|a_{j,m,k}| < 10^{-30} \max\{|a_{j,m,k}|\}$ then order 300 requires a little bit more than 1 minute and only 73517 subsist.

Being able to reach easily a large order is relevant for two reasons:

- It is possible to guess a bound for the coefficients. This is indeed the case for that problem. Then it is then possible to make a reasonable conjecture and finally prove the following

  Theorem 2.1 *For the invariant manifolds of the periodic orbit at infinity in the Sitnikov problem for all $e \in [0 : 1)$ and $u \in [0, 2\pi]$ one has $|a_k(u, e)| < C(k!)^{1/3}$ for some $C > 0$.*

  In particular this shows the non-analyticity and gives us the concrete Gevrey character of the manifold.

- To obtain the boundary on $z = 0$ one proceeds as follows: first, the previous representation is used to compute $p(u, q)$ for a given e. One should take a grid on the u variable. Then numerical integration allows to reach $z = 0$. It is interesting to use a value of q as large as possible (with controlled error) to minimize errors and time in the numerical integration of the equations of motion. The asymptotic character of the representation of the manifold and explicit bounds allow to show that, uniformly in (u, e), order 80 is the optimal one if standard double precision arithmetics is used. This means that values as large as $q = 0.3$ can be taken in the representation of the manifold with small error.

For details, results and applications see [8].

2.2 Collisions in Newtonian N-body problems.

It is well known that one of the main difficulties in Newtonian N-body problems is the possible existence of collisions, when two of the bodies tend to the same position when $t \to t^*$ for some t^*. This presents difficulties because the equations of motion are no longer valid. But even if not an exact collision occurs but a close approach, numerical difficulties show up and a strong dependence on initial conditions.

Both from a theoretical and a practical point of view it is relevant to study if eventual collisions can be regularized in some sense. A natural condition is to ask to recover continuity with respect to initial conditions if the collision orbits are excluded.

Then it is well known that single binary collisions (i.e., there are exactly two bodies, i and j, such that the mutual distance $r_{i,j}$ tends to 0 when $t \to t^*$) are regularizable. On the other hand collisions involving three or more bodies are non-regularizable in general. They can be regularizable in some subproblems.

A natural question is to decide about the remaining problem: several couples of bodies i_k, j_k, $k=1,\dots,m$ have binary collisions simultaneously at different places, the other bodies not being involved in any collision. This problem was known to be regularizable, but no information on the degree of differentiability of the regularization was available.

This lead to make careful numerical experiments in [9] (where details on the background can also be found). The concrete problem studied was a subproblem of the four-body one, of the so-called bi-isosceles type. Assume the masses satisfy $m_1 = m_2$ while m_3 and m_4 are arbitrary. It is required that the bodies move on a plane with coordinates of the form (x_1, y_1), $(-x_1, y_1)$, $(0, y_3)$, $(0, y_4)$ for all time. It is not restrictive to assume $2m_1y_1 + m_3y_3 + m_4y_4 = 0$. Hence the system is a Hamiltonian with 3 degrees of freedom. Simultaneous binary collisions occur if $x_1 = 0$, $y_3 = y_4 \neq y_1$.

Assume some initial conditions w_0 for $t = 0$ in the phase space lead to a simultaneous binary collision for some time $t^* > 0$. It is possible to take an analytic arc of initial conditions $w(s)$, parametrized by s in a neighbourhood of $s = 0$ with $w(s = 0) = w_0$ and such that no simultaneous binary collision occurs for $0 \leq t \leq t^{**}$ for some $t^{**} > t^*$ if $s \neq 0$. Then the image of the arc can be computed numerically. The result is somewhat surprising: The image of the arc seems to be exactly $\mathcal{C}^{8/3}$, with a behaviour of the form $s^{8/3}$ around $s = 0$.

This result has been later proved [10] obtaining

Theorem 2.2 *Simultaneous binary collisions in the planar four-body problem are $\mathcal{C}^{8/3}$ regularizable. For most of the masses the result is sharp.*

From the proof seems clear that the result can be extended to any number of bodies and dimensions.

2.3 Integrability criteria based on higher order variationals

In this example we consider analytic Hamiltonians. Most Hamiltonian systems are non-integrable. However integrable ones play an important role due to the clean structure of the phase space and to the fact that they can be used as a starting point to study nearby non-integrable ones. It is relevant to have criteria to decide if a system is integrable.

A celebrated theorem by Ziglin [24] (and many other versions of it) gives a necessary criterion based of the study of the first order variational equations along some particular families of solutions. The first difficulty in very simple examples appears when we consider the family of Hénon-Heiles (HH) potentials

$$H = \frac{1}{2}(x_1^2 + x_2^2 + y_1^2 + y_2^2) + \frac{1}{3}x_1^3 + \lambda x_1 x_2^2,$$

where λ is a real parameter. The "classical" system is obtained when $\lambda = -1$. From the results of Ziglin follows that the classical HH system is non-integrable and, in fact, it is proved that for all $\lambda \in \mathbb{R}$, except four values, the system is non-integrable.

What happens for the exceptional four values? For three of them it is immediately checked that they are integrable (by displaying the additional first integral). The fourth one is $\lambda = 1/2$. A direct numerical simulation, by using a suitable Poincaré section and values of the energy not too close to zero (say, at least $h = 0.2$) makes evident that the system is non-integrable. This evidence prompts us for theoretical results allowing to prove the non-integrability when we cannot decide from Ziglin's theorem.

An alternative approach based on differential Galois theory has been seen to be fruitful to study integrability. For background and statement of basic results I refer to [12]. Assume H is a Hamiltonian and $z(t)$ a solution different from an equilibrium point. By considering $t \in \mathbb{C}$, the solution z defines a Riemann surface Γ. We can consider the first order variationals of the Hamiltonian vector field along Γ. Associated to these (linear) equations there exists a Galois group, G, of the Picard-Vessiot extension of the basic field of functions (defined by the coefficients of the variationals in Γ). In [13] the following key result, known as Morales-Ramis theorem, is obtained

Theorem 2.3 *Let H be a completely integrable with meromorphic first integrals in a neighbourhood of Γ. Then the identity component, G^0, of G is abelian.*

It seems natural to use the information contained in higher order variationals. It has been found convenient to check first the natural candidate to proceed with the study. To this end we considered higher order variational equations along a solution Γ of the HH system with $\lambda = 1/2$. The solution selected is quite simple, being given by elliptic integrals, and Γ turns out to be a torus with one point removed.

Independent checks have been done, by numerical integration along complex paths on time and by symbolic manipulation of the expansions of elliptic functions, to look for the solutions of second order variationals around the singularity in Γ. These checks give no obstruction to integrability. Proceeding to third order it is found (with perfect agreement by both methods) a lack of commutativity which prevents from integrability.

This evidence pushed us to prove that, under a suitable definition of the Galois groups, G^k, associated to the variational equations of order k, $k \in \mathbb{N}$, the identity components $(G^k)^0$ of all these Galois groups must be abelian. I refer to [14] for background, detailed statements and proofs.

In particular this result supplies a countable set of necessary criteria to be satisfied by an integrable Hamiltonian. A standing question is if one can assure that the system is integrable in case all these conditions are met.

2.4 Choreographic solutions of the N-body problem

In this example we shall restrict our attention to planar motions. Very few solutions of the N-body problem are explicitly known. They belong to the so-called relative equilibrium solutions (*res*) and solutions derived from these ones. In a *res* the mutual distances between the bodies remain unchanged and all of them rotate around the centre of masses as if they were a rigid body. Among the *res* there are particularly simple solutions, when the N masses are equal and they occupy the vertices of a regular N-gon. In that case it is obvious that all bodies move along the same path. Furthermore, if we label the bodies as $0, 1, \ldots, N-1$ according to the order in which they pass through a given point of the path, the position of body number k is $\varphi(t - kT/N)$ when body number 0 is at $\varphi(t)$, where T denotes the period of the solution. That is, the time shift from one body to the next one is constant.

We can ask ourselves if there other periodic solutions with the same property: the bodies move along the same path with constant time shift. These solutions are named as *choreographies*. The N-gon *res* are the trivial choreographies.

Only recently the first non-trivial choreography was proved to exists for the 3-body problem by Chenciner and Montgomery [6] by using variational methods. The path is a figure eight curve and it lives on the zero angular momentum level. Surprisingly enough this orbit is totally elliptic inside that level (a 3 degrees of freedom problem). Furthermore, sufficient conditions for the applicability of KAM theorem have been checked from a local normal form whose coefficients are obtained numerically [21].

If the N-body case for $N > 3$, an existence proof is missing despite a large amount of examples has been obtained numerically [5, 19]. As usual in that kind of problems the difficulties in the proofs come from the fact that it is hard to show that the process of minimizing the action to find these solutions do not leads to orbits with collisions. On the other hand the existence of choreographic solutions is simple if the Newtonian potential is replaced by a "strong force" potential (of the form $1/r^\alpha$, $\alpha \geq 2$ instead of $1/r$).

Returning to the 3-body problem it is natural to inquire about the existence of other choreographies beyond the triangle (the Lagrange solution) and the figure eight. This problem has been studied in [21] and 344 additional choreographies have been found. These are essentially different from the figure eight (i.e., not obtained as *satellites* of the eight or as *relative* choreographies obtained from the eight by changing the angular momentum to have commensurable frequencies; see [21] for precise definitions and background). It is clear from this numerical evidence, from the shape of the different paths and the organization of these choreographies in families, that the set of choreographic *classes* for the 3-body problem is unbounded.

The methodology to find them can be easily described. Let (x_j, y_j) be the position of the mass b_j of mass $m_j = 1/3$, $j = 1, 2, 3$. Assume we take initial conditions in isosceles configuration and suitable velocities:

$$x_3 = x_2, \quad y_3 = -y_2, \quad \dot{x}_3 = -\dot{x}_2, \quad \dot{y}_3 = \dot{y}_2.$$

Then $(x_1, y_1, \dot{x}_1, \dot{y}_1)$ are determined from the centre of mass integrals. This isosceles triangle has a symmetry axis passing through b_1. Assume that at some time $\tau > 0$ we are in another isosceles configuration with b_2 in the symmetry axis. Let β be the angle between the initial and the present axes of symmetry. If a rotation of angle $-\beta$ applied to positions and velocities at time τ leads to the initial conditions, then we have a *relative* choreography of period 6τ.

Hence, one can scan a grid of initial conditions of isosceles type (on a fixed level of energy, say h=-1/2, this is a 3D submanifold). We restrict the maximal time of integration to a moderate value $\tau^* = 5$ and discard conditions leading to too close approaches to collision. If for some $\tau < \tau^*$ we are again in isosceles configuration and the rotation described above leads to conditions close to the initial ones, a refinement is done to obtain a relative choreography by Newton's method. As we are interested in *true* choreographies, i.e., in a fixed frame, the previous solutions are continued with respect to the angular momentum until a true choreography (if any) is found in the continuation process. See [21] for additional details.

It has to be noted that, among all these choreographies, only the figure eight solution lives on the zero angular momentum level.

2.5 Splitting of separatrices in the multifrequency case

Consider an integrable (analytic) Hamiltonian system with one degree of freedom having a homoclinic connection to a saddle, i.e., a separatrix (think about a pendulum, for instance). Assume we add to it a fast periodic perturbation. A relevant question concerning the dynamics is which is the fate of the separatrix, i.e., how the separatrix is *splitted* by the perturbation. In an equivalent way we can consider (by changing time scale) that the frequency is finite and the size of the perturbation is small. This problem is important because it is a model of the behaviour of a system with two degrees of freedom near a resonance.

Going to higher dimensions we can try to analyze a similar problem but replacing a pendulum by a pendulum cross a rotor. This will give a model of the behaviour, near a simple resonance, of a system with three or more degrees of freedom. This is what we denote as the *multifrequency* case. Note that the problem of multiple resonances is harder, and very scarce information is available on it.

A priori bounds in the present problem can be obtained by applying general results [17]. The relevant frequencies are the ones of the rotor part (say ν,) and the external frequency (say, equal to 1). Let $\omega = (\nu, 1)^{\mathrm{T}} \in \mathbb{R}^d$. If ω satisfies a Diophantine condition $|(k, \omega)| \geq b|k|^{-\tau}$, $\forall k \in \mathbb{Z}^p \setminus \{0\}$ with $b > 0$, $\tau < d-1$, then a bound of the splitting has the form $\exp(-c_1/\varepsilon^{c_2})$, where $c_1 > 0$, $c_2 = 1/(\tau+1)$. Note that the information that one can obtain about the constant c_1 is very rough. This is in contrast with the one-frequency case, where it is known that c_1 is related to the (complex) singularities of a parametrization of the unperturbed separatrix.

A model example of this situation was studied in [17], including symbolic and numeric studies. In particular it is presented how to generate local approximations to the invariant manifolds of an invariant torus and how to globalize them to measure the splitting. If there are several action variables which split, we can look for the splitting in each one of them or, simply, the *splitting vector*. These are functions defined on a torus. A simpler approach can be to obtain the so-called *splitting potential*, whose gradient is the splitting vector. From the local approximations the splitting vector is computed by numerical integration (using multi-precision arithmetics when required).

A classical model for the problem at hand is Arnold's example of diffusion. The standing Hamiltonian (equivalent to the original example) is

$$H(x, I_2, I_3, y, \theta_2, \theta_3) = \frac{1}{2}\left(x^2 + I_2^2\right) + I_3 + \varepsilon(\cos y - 1)\left(1 + \mu(\cos\theta_2 + \cos\theta_3)\right),$$

where the pendulum cross rotor part is obtained if $\mu = 0$. This example was used as a model for diffusion, but the presence of two small parameters, ε, μ, is non natural. Furthermore, to be able to measure the splitting produced when $\mu \neq 0$, one has to require μ to be exponentially small with respect to ε.

In [23] this model with $\mu = \varepsilon$ is considered. The usual approach to measure splitting, i.e., Melnikov's method, does not work. The reason is that there are "too few" harmonics in the perturbation. Instead, one has to modify the method to try to capture the contributions of the different harmonics to the splitting. Hence an *adapted Melnikov method* is introduced in [23]. It should be remarked that not only the initial harmonics present in the perturbation are relevant. The higher order ones which appear, when higher order variationals are included, as linear combinations of terms of lower order, are essential. In an equivalent way, the type of terms which can be expected to give important contributions to the splitting can be found by averaging. As usual, one has then to "optimize" to determine which are the dominant terms as a function of ε.

However, due to the fact that higher order coefficients depend strongly on combinatorics of the previous ones, it can happen (and it has been explored that it *do happens*) that the terms candidate to be the dominant ones have a too small coefficient.

The suggested approach is tested against symbolic computations and accurate direct numerical evaluation of the splitting. Once more the computer is a valuable tool to detect several complicated properties. In particular it allows to guess an optimal value for the upper bound of the splitting when we consider, for the present model, the rotor part with frequency β, i.e., we study the splitting of the separatrix which appears, for $\mu = 0$, if $I_2 = \beta$. More concretely we can state the

Conjecture 2.4 *Let β be a constant type number, that is, the quotients q_n of the Continued Fraction Expansion are bounded. Let $\underline{c}$ be the best Diophantine constant of β, that is,* $\liminf \{|q\beta - p|q \text{ for } (p,q) \in \mathbb{Z}^2\}$. *Then the 2D invariant tori of Arnold's Hamiltonian have a splitting whose upper bound has a sharp estimate of the form*

$$\exp\left(-(8\pi\underline{c}(1+\beta))^{1/2}\left(\frac{|\ln\eta|}{\eta}\right)^{1/2}\right),$$

where $\eta^2 = \varepsilon = \mu$.

3 Conclusions

In the examples above there are no details about the different algorithm used and their implementation on computer. They are intended to show how this "not so widely used" tool is, in fact, extremely valuable for theoretical and practical problems. It requires, however, a good knowledge of numerical analysis, computer science and complexity.

Furthermore, the design of all algorithms (either symbolic or numeric) should be guided, if possible, by a priori knowledge of properties of the system. In this way the algorithms can be tailored to the concrete needs, gaining in accuracy and efficiency.

Many algorithms, specially concerning existence, properties and bifurcations of fixed points, periodic orbits, homoclinic connections, etc, can be turned into rigorous mathematical proofs in a systematic way, from the numeric information. A typical theorem which applies "when $\varepsilon < \varepsilon_0$", can be quantified to obtain precise values of ε_0 (even if they are quite pessimistic) and then this can be checked using as many digits as needed, with a correct control of errors. I foresee that this will be a routine task (even an automatized task) in the near future.

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