Resonance tongues and instability pockets in the quasi–periodic Hill–Schrödinger equation

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Abstract

This paper concerns Hill's equation with a (parametric) forcing that is real analytic, quasi-periodic with frequency vector ω, and even in the time t. Schrödinger's equation with quasi-periodic potential is a particular case. We include small parameters $b_1, \dots, b_p$ and a 'frequency' parameter a. In the parameter space $\mathbb{R}^{p+1} = \{a, b\}$, for small values of $|b|$ the following holds. The 'tongues' with rotation number $\frac{1}{2}\langle \mathbf{k}, \omega \rangle$, $\mathbf{k} \in \mathbb{Z}^d$, have C^∞-boundaries. Our arguments are based on reducibility, averaging and certain spectral properties of the Schrödinger operator with quasi-periodic potential. Analogous to the periodic case, further results are obtained, such as a criterion for transversality of the tongue boundaries at $b = 0$, the occurrence of instability pockets in $\mathbb{R}^{p+1} = \{a, b\}$, corresponding with spectral gaps for the Schrödinger operator and results on the behaviour of Lyapunov exponent and rotation number for fixed b.

Contents

1 Introduction, main results

Consider Hill's equation

$$x'' + (a + \sum_{i=1}^{p} b_i q_i(t))x = 0, \tag{1}$$

where $a, b_1, \ldots, b_p$ are real parameters and the real analytic functions $q_1, \ldots, q_p$ are even and quasi-periodic in t, with frequency vector $\omega = (\omega_1, \ldots, \omega_d) \in \mathbb{R}^d$. This means that if $\mathbb{T}^d = (\mathbb{R}/2\pi\mathbb{Z})^d$ is the d-dimensional torus, there exist real analytic functions $Q_i : \mathbb{T}^d \to \mathbb{R}$ such that, $q_i(t) = Q_i(t\omega)$, $i = 1, \ldots, p$. The frequency-vector ω, moreover, is assumed to be Diophantine with constants $c > 0$ and $\tau \geq d - 1$, i.e., $|\langle \mathbf{k}, \omega \rangle| \geq c|\mathbf{k}|^{-\tau}$, for all $\mathbf{k} \in \mathbb{Z}^d - 0$.

Objects of the study are resonance tongues. It is proved that in this analytic case, for small $|b|$, the tongue boundaries are smooth curves. Another main part of this paper deals with instability pockets, as they may occur for even q.

1.1 Background

This set-up is similar to Broer and Simó [5], where some cases with two frequencies, with golden mean ratio, were studied. For more background, mainly concerning the periodic Hill–Schrödinger equation, see [3, 6].

1.1.1 The Schrödinger equation

Replacing x by u and t by x, turns Hill's equation (1) into Schrödinger's equation

$$-\frac{\partial^2 u}{\partial x^2} - V(x, b)u = au, \tag{2}$$

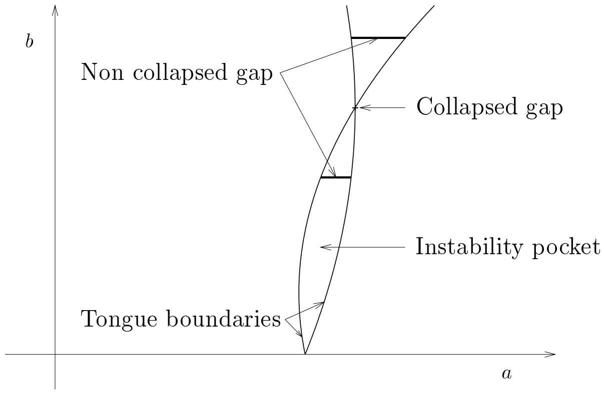

Figure 1: Resonance tongue with pocket in the (a, b)-plane giving rise to spectral gaps on each horizontal line with constant b. Note how collapses of gaps correspond to crossings of the tongue-boundaries at the extremities of an instability pocket.

with quasi-periodic potential $V(x, b) = \sum_{i=1}^{p} b_i q_i(x)$ and energy a. Compare, e.g., Eliasson [11], Dinaburg and Sinai [10], Moser and Pöschel [18] and Johnson and Moser [15]. Spectral properties of the corresponding Schrödinger operator

$$\mathcal{L}(u) = -\frac{\partial^2 u}{\partial x^2} - V(x, b)u \tag{3}$$

are important to this theory.

One difference with the latter references is that presently b is not considered as a constant, but as a multiparameter. In that respect Hill's equation (1) with quasi-periodic forcing is a natural extension of the classical Hill's equation with periodic forcing, i.e., where $d = 1$. In figure 1 the relationship between spectral gaps and resonance tongues is shown. The ends of instability pockets correspond to the collapse of a gap.

1.1.2 The periodic case

The periodic case $d = 1$ is briefly reconsidered, compare Broer and Levi [3], Broer and Simó [6] and references therein. The stability properties of the trivial solution of the periodic Hill equation are completely determined by the eigenvalues of the linear period map, also called stroboscopic or *Poincaré* map, $P_{a,b}$. The latter often can be replaced by its logarithm, the *Floquet* map. In the parameter space $\mathbb{R}^{p+1} = \{a, b\}$, the instability domains, with non-elliptic eigenvalues, correspond to *resonance tongues* that emanate from the discrete set of resonances on the a axis, given by $(a, b) = ((k/2)^2, 0)$, $k \in \mathbb{N}$. In [3, 6] the geometric phenomenon of instability pockets was studied, which occurs when the boundaries of the same tongue cross and which is also related to tangencies of these boundaries. The used methods of research are normal forms and singularity theory. It turns out that in the near–Mathieu case near the $k : 2$ resonance, one can have between 0 and $k-1$ instability pockets, with all kinds of intermediate tangencies: the whole scenario has at least the complexity of the singularity $\mathbb{A}_{2k-1}$, compare [2]. For a description and analysis of more global phenomena in the periodic case, see [4].

1.1.3 Background on the quasi-periodic case

The present case is quasi-periodic, so no appropriate Poincaré map is defined. Nevertheless resonance tongues can be defined by means of the rotation number $\mathrm{rot}(a, b)$, a concept living both in the periodic and quasi-periodic setting, compare [11, 15, 18, 21]. The rotation number of equation (1) is defined as

$$\operatorname{rot}(a,b) = \lim_{T\to+\infty} \frac{\arg\left(x'(t)+ix(t)\right)}{2\pi T},$$

where x is any non-trivial solution of this equation. This number exists and it is independent of the particular solution. In order to define resonance tongues in the quasi-periodic case, first consider the module of positive half-resonances of ω, given by

$$\mathcal{M}_+(\omega) = \left\{ \tfrac{1}{2}\langle \mathbf{k}, \omega\rangle \in \mathbb{R} \;\middle|\; \mathbf{k}\in\mathbb{Z}^d \text{ and } \langle \mathbf{k},\omega\rangle \geq 0 \right\}.$$

Notice that for Diophantine ω the set $\mathcal{M}_+(\omega)$ is dense in the positive real numbers $\mathbb{R}^+$. The main object of this paper is defined as follows

Definition 1 *Let $\mathbf{k}\in\mathbb{Z}^d$. The resonance tongue of the quasi-periodic Hill equation* (1) *associated to* $\mathbf{k}$ *is the set*

$$\mathcal{R}(\mathbf{k}) = \left\{ (a,b)\in\mathbb{R}^{p+1} \,\middle|\, \operatorname{rot}(a,b) = \tfrac{1}{2}\langle \mathbf{k},\omega\rangle \right\}.$$

Johnson and Moser [15] studied the spectrum of the Schrödinger operator $\mathcal{L}$ with quasi-periodic potential V, see (3). They found that for fixed $V(.,b)$, in particular for fixed b, the following holds. For each $\alpha_0\in\mathcal{M}_+(\omega)$ there is a unique and closed interval (possibly reduced to one point) such that a is in this interval if and only if the rotation number of (2), or equivalently (1), is α_0. Moreover, for a in the (non void) interior of this interval, the equations (2) and (1) admit exponential dichotomy.

A direct computation of the rotation number shows that the resonance tongue corresponding to a resonance $\alpha_0\in\mathcal{M}_+(\omega)$, emanates from the point $(a,b) = (\alpha_0^2, 0)$ on the b axis, just as in the periodic case. One difference is that in the quasi-periodic case, due to the denseness of $\mathcal{M}_+(\omega)$ in the positive real axis, the tongues are no longer discrete.

Broer and Simó [5], in a partly formal study, found out that, in the quasi-periodic case and for small b the geometry of each resonance tongue is quite similar to the periodic case, also compare [11]. In particular, this seems to hold for the occurrence of instability pockets in the case of even $q_1,\dots,q_p$.

Remark 1 *For larger values of b the organization of the (a,b) parameter space in the quasi-periodic case turns out to be very complicated, compare* [5, 11, 17] *and references therein, also regarding differences between close-to-constant and far-from-constant cases. Reference* [5] *also contains a lot of numerical evidence.*

1.2 Setting of the problem, results

The present paper is concerned with the geometry of the resonance tongues of the quasi-periodic Hill equation (1) in the parameter space $\mathbb{R}^{p+1} = \{a,b\}$, where the functions $q_1,\dots,q_p$ are fixed.

1.2.1 Smoothness of the tongue boundaries

Each tongue $\mathcal{R}(\alpha_0) = \mathcal{R}(\mathbf{k})$ has the form

$$\mathcal{R}(\alpha_0) = \left\{ (a,b)\in\mathbb{R}^{p+1} \mid a_-(b;\alpha_0) \leq a \leq a_+(b;\alpha_0) \right\} \tag{4}$$

with $\alpha_0 \in \mathcal{M}_+(\omega)$ and $a_+(0;\alpha_0) = a_-(0;\alpha_0) = \alpha_0^2$. Mostly the value of α_0 is fixed, in which case we suppress its occurrence in the boundary functions $a_\pm$. Note that in (4) one can ask, in general, for not more than continuity of the mappings $a_\pm : \mathbb{R}^p \to \mathbb{R}$, since it follows that $a_- \leq a_+$, compare with the periodic case [3, 6].

Nevertheless, recall that in the periodic case one can find real analytic boundary curves $a_{1,2} = a_{1,2}(b)$ such that $a_- = \min\{a_1, a_2\}$ and $a_+ = \max\{a_1, a_2\}$. The reason is that, due to Floquet theory, in the periodic case one has reducibility of the flow, implying that each boundary is the zero-set of an analytic function.

It is well-known that in the quasi-periodic context generally no reducibility holds, see [11, 18]. However, reducibility does hold inside and at boundaries of the resonance tongues. This will be used in the proof of the main result:

Theorem 1 (Smoothness of tongue boundaries) *In the above situation, assume that in Hill's equation* (1) *the functions* $q_1, \ldots, q_p$ *are real analytic and quasi-periodic with Diophantine frequency vector* $\omega \in \mathbb{R}^d$. *Then, for some constant* $C = C(q_1, \ldots, q_d, \omega)$ *and for any* $\alpha_0 \in \mathcal{M}_+(\omega)$, *there exist* C^∞*-functions* $a_1 = a_1(b)$ *and* $a_2 = a_2(b)$, *defined for* $|b| < C$, *satisfying*

$$a_- = \min\{a_1, a_2\} \text{ and } a_+ = \max\{a_1, a_2\}.$$

1.2.2 Instability pockets

In the quasi-periodic Hill equation, instability pockets can be defined as in the periodic case. A family of differential equations like (1) is said to have a *boundary crossing* at $(a, b) = (a_0, b_0)$ if there is a resonance tongue such that its two boundaries intersect at this point. This means that $\{a_0\}$ is a collapsed gap for the Schrödinger operator (2) for $b = b_0$. Compare [18] for a discussion on the lack of genericity of such collapsed gaps. The present paper also sheds some light on this matter.

As in the periodic case, two consecutive boundary crossings give rise to an *instability pocket*. The creation of instability pockets for the reversible, periodic Hill equation was studied in [3, 6]. In the present paper we treat generalizations of this to the reversible, quasi-periodic case. The computational techniques regarding normal forms and singularity theory carry over from the periodic to the quasi-periodic setting, provided that the tongue boundaries are smooth. Our Main Theorem 1 asserts that this is indeed the case for analytic forcing (potential) and small $|b|$. So for any resonance one can obtain the normal form at the tongue tip at $b = 0$ up to the necessary (finite) order. From this the tongue boundaries can be formally computed up to this order. Theorem 1 then guarantees that these formal expressions are the actual Taylor expansions of smooth tongue boundaries. As a result we find (see Section 3), that up to a subset of Diophantine frequencies of measure zero, the situation is completely similar to the periodic case as described in Section 1.1.2.

See Figure 2 for an actual instability pocket, where formal and direct numerical computations are compared. The techniques just described are useful when studying a fixed resonance. We note, however, that for investigating 'all' resonance tongues at once, even in a concrete example, certain direct computations are needed. This is similar in the periodic case, compare [6].

1.2.3 Outline

Let us briefly outline the rest of this paper. In Section 2 we present the ingredients for our proof of Theorem 1. These ingredients include reducibility theorems and averaging

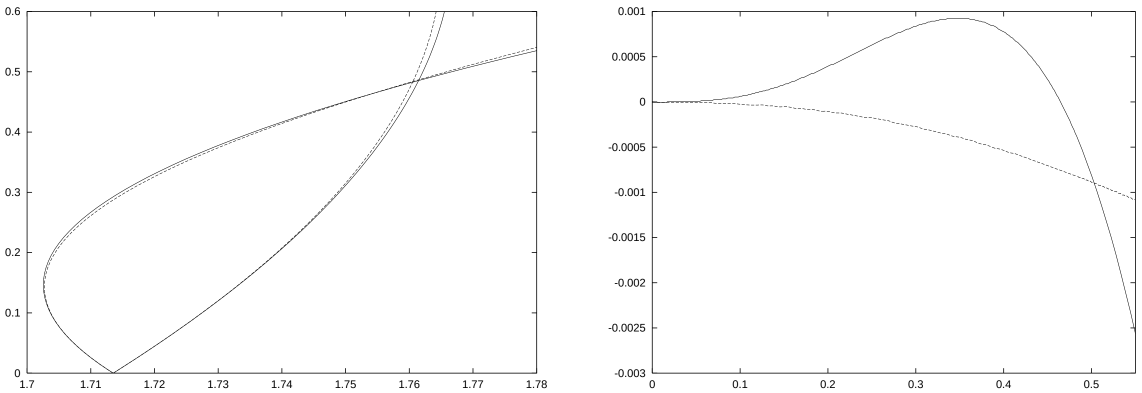

Figure 2: Left: Numerical computation of the instability pocket of the near-Mathieu equation (22) in the (a, b)-plane with $\gamma = (1+\sqrt{5})/2$, and $\varepsilon = 0.3$. Solid lines correspond to the approximation of the boundaries by second order averaging in (a, b); Dashed lines correspond to direct numerical computation. Right: Difference between the averaging and the direct numerical approximation as a function of b. Solid lines correspond to the tongue boundary that for small b turns to the left, dashed lines to the boundary as it turns to the right.

theory. Here only a sketch of this proof is presented, a full proof can be found in Section 4.

Section 3 is devoted to applications of Theorem 1. A first consequence is a a criterion for the transversality of the tongue boundaries at the tongue tip, see Section 3.1. Second, in Section 3.2 the asymptotics at the tongue tip $b = 0$ and the ensuing creation of instability pockets is studied in a suitable class of Hill equations. The techniques used for this study combine normal forms with direct computations. Since several frequencies are involved, these results are not valid for all Diophantine frequencies, but a set of zero measure must be taken out. Some properties of this set are considered in Appendix B. Following these techniques a concrete example is studied in Section (3.3) for which the tongue boundaries, up to second order, are computed explicitly. Most proofs are postponed to Section 4. Finally in Appendix A a Lipschitz property of the tongue boundaries is established, even when reducibility cannot be granted.

2 Smoothness of tongue boundaries

In this section we consider parameter values (a_0, b_0) at a tongue boundary, i.e., at an endpoint of a spectral gap, which may possibly be collapsed. At a boundary point (a_0, b_0) the rotation number $\mathrm{rot}(a_0, b_0) = \frac{1}{2}\langle \mathbf{k}, \omega\rangle$, i.e. it is 'rational'. This suggests application of a Van der Pol transformation, leading to a system of co-rotating coordinates, e.g., compare [7, 15]. In the co-rotating coordinates the tongue boundary near (a_0, b_0) gets a simpler form, that can even be further simplified by repeated time-averaging, where time-dependence is pushed to higher order in the localized parameters (a, b).

2.1 Reducibility, co-rotating coordinates

We start rewriting the quasi-periodic Hill equation (1) as a vector field on $\mathbb{T}^d \times \mathbb{R} \times \mathbb{R} = \{\theta, x, y\}$, where $\theta = (\theta_1, \ldots, \theta_d)$ are angles counted mod 2π. This yields a vector field X, in system form given by

$$
\begin{aligned}
\theta' &= \omega \\
x' &= y \\
y' &= -(a + \sum_{j=1}^{p} b_j Q_j(\theta)).
\end{aligned}
\tag{5}
$$

Note that the evenness of $Q_1, \dots, Q_d$ leads to time-reversibility, which here is expressed as follows: if $R : \mathbb{T}^d \times \mathbb{R} \times \mathbb{R} \to \mathbb{T}^d \times \mathbb{R} \times \mathbb{R}$ is given by $R(\theta, x, y) = (-\theta, x, -y)$, then $R_*(X) = -X$.

A main tool for studying (1) and (5) is the following result by Eliasson.

Theorem 2 ([11]) *Consider the quasi-periodic Hill equation* (1)*, or, equivalently the system form* (5)*. Assume that the following conditions hold*

- *The frequency vector ω is Diophantine with constants $c > 0$ and $\tau \geq d - 1$, i.e., $|\langle \mathbf{k}, \omega \rangle| \geq c|\mathbf{k}|^{-\tau}$, for all $\mathbf{k} \in \mathbb{Z}^d - \{0\}$.*
- *The functions $Q_i : \mathbb{T}^d \to \mathbb{R}, i = 1, \dots, p$ are analytic on a strip around the d–torus $\mathbb{T}^d = \mathbb{R}^d/(2\pi\mathbb{Z}^d) \subset \mathbb{C}^d/(2\pi\mathbb{Z}^d)$, given by $|\mathrm{Im}\,\theta| < \rho$.*

Then there exists a constant $C = C(\tau, \rho) > 0$ such that if for

$$
||b||_Q := \sum_{i=1}^{p} |b_i| \sup_{|\mathrm{Im}\,\theta|<\rho} |Q_i(\theta)| \ \textit{one has}\ ||b||_Q < C,
$$

while the rotation number of (1) *is either rational or Diophantine, then there exist analytic maps*

$$
\begin{aligned}
(a, b) \in \mathbb{R}^{p+1} &\mapsto B(a, b) \in sl(2, \mathbb{R}) \\
\theta \in \mathbb{T}^d &\mapsto Z(\theta) \in GL(2, \mathbb{R}),
\end{aligned}
$$

such that a fundamental solution of (1) *can be written as*

$$
X(t) = Z(\omega t/2) \exp(B(a, b)t).
$$

Moreover, in this domain,

1. *$B(a, b)$ is nilpotent and non-zero if and only if a is an endpoint of a non-degenerate spectral gap;*
2. *$B(a, b)$ is zero if and only if $\{a\}$ is a collapsed gap.*

Using the fundamental solution provided by Theorem 2, we turn to co-rotating coordinates associated to parameter values at the tongue boundaries. Let $(a_0, b_0) \in \mathbb{R}^{p+1}$ be at a tongue boundary. Then there exists a fundamental matrix of the form

$$
X((a_0, b_0))(t) = \begin{pmatrix} z_{11}(t) & z_{12}(t) \\ z_{21}(t) & z_{22}(t) \end{pmatrix} \begin{pmatrix} 1 & ct \\ 0 & 1 \end{pmatrix},
$$

where c is a constant and where $z_{ij}(t) = Z_{ij}(\omega t/2)$. Observe that $c = 0$ if and only if (a_0, b_0) is an extreme point of an instability pocket. Also observe that $z_{ij}(t)$, $1 \leq i, j \leq 2$, is quasi-periodic with frequency vector $\frac{1}{2}\omega$.

To construct the co-rotating coordinates, again consider Hill's equation (1)

$$
y'' + (a + \sum_{i=1}^{p} b_i q_i(t))y = 0
$$

and perform the linear, t-dependent change of variables

$$\begin{pmatrix} y \\ y' \end{pmatrix} = \begin{pmatrix} z_{11}(t) & z_{12}(t) \\ z_{21}(t) & z_{22}(t) \end{pmatrix} \phi,$$

where $\phi \in \mathbb{R}^2$. Observe that the change of variables in t is quasi-periodic with rotation number $\mathrm{rot}(a_0, b_0) = \frac{1}{2}\langle \mathbf{k}, \omega \rangle$. The differential equation for ϕ reads

$$\phi' = \left(\begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix} + \delta_\mu q \begin{pmatrix} z_{11}z_{12} & z_{12}^2 \\ -z_{11}^2 & -z_{11}z_{12} \end{pmatrix} \right) \phi, \tag{6}$$

where $\mu = (a - a_0, b - b_0)$ is the new local multiparameter and where $\delta_\mu q = (a - a_0) + \sum_{i=1}^{p} (b_i - b_i^0) q_i$.

Proposition 1 *In the above circumstances:*

1. *The functions $z_{11}z_{12}$, z_{11}^2 and z_{12}^2 are quasi-periodic in t with frequency vector ω.*
2. *In the case of even $q_1, \dots, q_p$ we have*

$$\begin{aligned} X(-t; (a,b)) &= X^{-1}(t; (a,b)) \text{ and, thus,} \\ z_{12}(-t) &= -z_{12}(t) \\ z_{11}(-t) &= z_{11}(t). \end{aligned}$$

3. *Let $\delta\mathrm{rot}(a,b)$ be the rotation number of (6). Then*

$$\delta\mathrm{rot}(a,b) = \mathrm{rot}(a,b) - \tfrac{1}{2}\langle \mathbf{k}, \omega \rangle.$$

Moreover, the tongue boundary is given by the equation $\delta\mathrm{rot}(a,b) = 0$.

We omit a proof, since this is elementary. The third item of the Proposition again uses [11].

2.2 Proof of Theorem 1

By repeated averaging we recursively push the time dependence of the equation (6) to higher order in the local parameter $\mu = (a - a_0, b - b_0)$. See [6, 7, 23] for details. As before, by θ we denote the angular variables in $\mathbb{T}^d = (\mathbb{R}/(2\pi\mathbb{Z}))^d$.

From Theorem 2 recall that the functions $Q_j = Q_j(\theta), j = 1, \dots, p$ are complex analytic on a strip $|\mathrm{Im}\ \theta| < \rho$. At each averaging step, for some $\rho_1 < \rho$, a linear change of variables

$$\psi = (I + R(\theta, \mu))\phi,$$

is found, that respects the time reversibility and that is complex analytic in θ on the strip $|\mathrm{Im}\ \theta| < \rho_1$ and in the local parameters μ in a neighbourhood of 0. To be more precise we have

Proposition 2 *In the above situation, after r steps of averaging system* (6) *takes the form*

$$\phi' = \left(\begin{pmatrix} 0 & c + S_2^{(r)}(\mu) \\ -S_1^{(r)}(\mu) & 0 \end{pmatrix} + M^{(r)}(\omega t, \mu) \right) \phi, \tag{7}$$

where

$$S_i^{(r)}(\mu) = \sum_{1 \le s \le r} D_{s,i}^{(r)}(\mu)$$

for $i = 1, 2$ and where the functions $D_{s,i}^{(r)}$, for $i = 1, 2$ and $1 \le s \le r$, have the following properties:

1. *$D_{s,i}^{(r)}(\mu)$ are homogeneous polynomials of degree s in μ;*
2. *$D_{s,i}^{(t)} = D_{s,i}^{(s)}$ for $s \le t \le r$;*
3. *$D_{1,i}^{(1)} = (a - a_0)[z_{1i}^2] + \sum_{j=1}^{p}(b_j - b_j^0)[q_j z_{1i}^2]$, where $[\cdot]$ denotes the time average;*
4. *The remainder $M^{(r)}(\theta, \mu)$ is complex analytic in both θ and μ, (when $|\mathrm{Im}\ \theta|$ and $|\mu|$ are sufficiently small) while it is of order $r + 1$, that is, the function*

   $$(\theta, \mu) \mapsto \left| M_{ij}^{(r)}(\theta, \mu) \right| \Big/ |\mu|^{r+1},$$

   $1 \le i, j \le 2$, is bounded on a neighbourhood of (a_0, b_0).

In the application of this result, key idea is that equation

$$\left(S_1^{(r)}(\mu)\right)\left(c + S_2^{(r)}(\mu)\right) = 0 \tag{8}$$

determines the derivatives of the tongue boundaries up to order r. In the analysis of this equation we distinguish between the cases $c \neq 0$ (non-collapsed gap) and $c = 0$ (collapsed gap). This will be done next.

2.2.1 Non-collapsed gap ($c \neq 0$)

We first treat the case $c \neq 0$ of a non-collapsed gap. Without loss of generality we may assume that $c > 0$, or even $c = 1$: if necessary we just perform a linear change of variables. Due to continuity of the function $S_2^{(r)}$, the equation (8) around $\mu = 0$ is equivalent to

$$S_1^{(r)}(\mu) = 0.$$

We solve the latter equation by the Implicit Function Theorem, which provides a polynomial

$$a^{(r)}(b) = a_0 + \sum_{1 \le |\mathbf{k}| \le r,\ \mathbf{k} \in \mathbb{N}^p} \nu_{\mathbf{k}}(b - b_0)^{\mathbf{k}},$$

such that $S_1^{(r)}((a^{(r)}(b) - a_0, b - b_0)) = O_{r+1}(|b - b_0|)$, where the coefficients $\nu_{\mathbf{k}}$, $1 \le |\mathbf{k}| \le r$, are uniquely determined by the functions $D_{s,1}^{(r)}$, $1 \le s \le r$. To do so, we compute

$$\frac{\partial}{\partial a} S_1^{(r)}(\mu)_{|\mu=0} = [z_{11}^2],$$

where $[z_{11}^2] > 0$, since $z_{11}(t)$ is a solution of (1) which is not identically zero. Therefore the Implicit Function Theorem yields a unique polynomial $a^{(r)} = a^{(r)}(b)$ with the properties stated above.

Our next purpose is to show that, if $b \mapsto a(b)$ is a tongue boundary with $a(b_0) = a_0$, then

$$\lim_{b \to b_0} \frac{|a(b) - a^{(r)}(b)|}{|b - b_0|^r} = 0.$$

More precisely we shall prove the following proposition,

Proposition 3 *Consider equation* (7), *with* $c \neq 0$. *For any* $\Gamma > 0$, *let* a_{Γ_+} *and* a_{Γ_-} *be defined as*

$$a_{\Gamma_\pm}(b) = a^{(r)}(b) \pm \Gamma |b - b_0|^r .$$

Then, there exists a positive constant $\Delta = \Delta(\Gamma)$, *such that for* $0 < |b - b_0| < \Delta(\Gamma)$,

1. *The rotation number* $\mathrm{rot}(a_{\Gamma_+}(b), b)$ *is different from* $\mathrm{rot}(a_0, b_0)$,
2. *System* (7) *(or equivalently* (1) *or* (5)*) for* $\mu = (a_{\Gamma_-}(b) - a_0, b - b_0)$ *has exponential dichotomy.*

The proof is postponed to Section 4. As a direct consequence we have

Corollary 1 *Let* (a_0, b_0) *be at the tongue boundary as above and assume that* $\{a_0\}$ *is not a collapsed gap. Then, there exists a function* $b \mapsto a(b)$ *in a small neighbourhood of* $b = b_0$, *such that in this neighbourhood,*

1. $(a(b), b)$ *is at the tongue boundary of the same tongue as* (a_0, b_0),
2. *The map* $b \mapsto a(b)$ *at* b_0 *is* r*-times differentiable at* b_0 *and can be written as*

$$a(b) = a_0 + \sum_{1 \leq |\mathbf{k}| \leq r,\ \mathbf{k} \in \mathbb{N}^p} \nu_{\mathbf{k}} (b - b_0)^{\mathbf{k}} + o_r(|b - b_0|),$$

where

$$\lim_{b \to b_0} o_r(|b - b_0|)/|b - b_0|^r = 0.$$

Proof: From now on, assume that $0 < |b - b_0| < \Delta(\Gamma)$. Then, by Proposition 3, the set

$$\left\{ (a_{\Gamma_-}(b), b) : 0 < |b - b_0| < \Delta(\Gamma) \right\}$$

is an open subset of the tongue's interior. Again by Proposition 3, for each $0 < |b - b_0| < \Delta(\Gamma)$, the set $\left\{ (a_{\Gamma_+}(b), b) : 0 < |b - b_0| < \Delta(\Gamma) \right\}$ is an open subset of the complement of the tongue. Now, for each fixed b, the map $a \mapsto \delta \mathrm{rot}(a, b)$ is monotonous, while, moreover, $\mathrm{rot}(a, b)$ is continuous in a and b. Therefore, for each $0 < |b - b_0| < \Delta(\Gamma)$ there exists a unique $a(b)$ such that $(a(b), b)$ is at the tongue boundary.

Putting $a(b_0) = a_0$, the map $b \mapsto a(b)$ is continuously extended to $b = b_0$. The above argument also implies that for $0 < |b - b_0| < \Delta(\Gamma)$,

$$a_{\Gamma_-}(b) \leq a(b) \leq a_{\Gamma_+}(b) \tag{9}$$

and as $a_{\Gamma_-}(b_0) = a(b_0) = a_{\Gamma_+}(b_0) = a_0$, this inequality directly extends to $b = b_0$. Thus, due to the form of both a_{Γ_+} and a_{Γ_-}, we have that for $|b - b_0| < \Delta(\Gamma)$,

$$\left| a(b) - a^{(r)}(b) \right| \leq \Gamma |b - b_0|^r .$$

The corollary follows by letting Γ tend to zero. $\square$

Remark 2 *The similar case where* (7) *gives rise to equation* (8), *which then has the form* $(c - S_1^{(r)}(\mu)) S_2^{(r)}(\mu) = 0$, *is studied in exactly the same way.*

2.2.2 Collapsed gap ($c = 0$)

In the case $c = 0$ of a collapsed gap, system (7) reads

$$\phi' = \left(\begin{pmatrix} 0 & S_2^{(r)}(\mu) \\ -S_1^{(r)}(\mu) & 0 \end{pmatrix} + M^{(r)}(\omega t, \mu) \right) \phi. \tag{10}$$

Thus, the analogue of (8) now is

$$S_1^{(r)}(\mu) S_2^{(r)}(\mu) = 0,$$

which splits into two equations

$$S_1^{(r)}(\mu) = 0 \;\text{ and }\; S_2^{(r)}(\mu) = 0.$$

By the same arguments as in the previous subsection we find two functions

$$a_i^{(r)}(b) = \sum_{1 \le |\mathbf{k}| \le r,\, \mathbf{k} \in \mathbb{N}^p} \nu_{\mathbf{k}}^i (b - b_0)^{\mathbf{k}},$$

$i = 1, 2$, such that $S_i^{(r)}((a_i^{(r)}(b), b)) = O_{r+1}(|b - b_0|)$, and such that the coefficients $\nu_{\mathbf{k}}^i$, $1 \le |\mathbf{k}| \le r$ are uniquely determined by $D_{s,i}^{(r)}$, $1 \le s \le r$. The reason is that

$$\frac{\partial}{\partial a} S_i^{(r)}(\mu)_{|\mu=0} = [z_{1i}^2]$$

This situation is not quite so obvious as in the case of a non-collapsed gap, since here we have to deal with a crossing of surfaces. Still it can be studied essentially using the same tools as before. The proof of the following result also will be postponed to Section 4,

Proposition 4 *Under the above assumptions, for each $\Gamma > 0$ there exists a positive $\Delta = \Delta(\Gamma)$, such that if $|b - b_0| \le \Delta$, then*

$$\begin{aligned} |a_+(b) - \max_{i=1,2}\{a_i^{(r)}(b)\}| &\le 3\Gamma |b - b_0|^r \text{ and} \\ |a_-(b) - \min_{i=1,2}\{a_i^{(r)}(b)\}| &\le 3\Gamma |b - b_0|^r. \end{aligned}$$

As a direct consequence we now have

$$\lim_{b \to b_0} \frac{|a_+(b) - \max_{i=1,2}\{a_i^{(r)}(b)\}|}{|b - b_0|^r} = 0$$

and

$$\lim_{b \to b_0} \frac{|a_-(b) - \min_{i=1,2}\{a_i^{(r)}(b)\}|}{|b - b_0|^r} = 0$$

and we can choose a_+ and a_- in such a way (skipping the restriction $a_- \le a_+$) that both maps are continuous and r times differentiable at b_0. Moreover their Taylor expansions up to order r at b_0 are given by $a_1^{(r)}$ and $a_2^{(r)}$. Compare the Corollary 1, following Proposition 3.

2.2.3 Conclusion of Proof of Theorem 1

Summarizing we conclude that Theorem 1 follows from the previous subsections, since we have shown that the tongue boundaries are infinitely smooth. Indeed, by Eliasson's Theorem 2 a positive constant C exists only depending on ω and Q (see (5)), such that for any $r \in \mathbb{N}$ the following holds. For any $|b_0| < C$, polynomials $a_1^{(r)}$ and $a_2^{(r)}$ exist of degree r in $(b - b_0)$, such that

$$\lim_{b \to b_0} \frac{|a_+(b) - \max_{i=1,2}\{a_i^{(r)}(b)\}|}{|b - b_0|^r} = 0$$

and

$$\lim_{b \to b_0} \frac{|a_-(b) - \min_{i=1,2}\{a_i^{(r)}(b)\}|}{|b - b_0|^r} = 0.$$

This indeed proves Theorem 1. Moreover these subsections provide a method to compute the Taylor expansions of the tongue boundaries, provided that a certain number of harmonics of the reducing matrix Z is known, compare Theorem 2.

3 Applications and examples

In this section the methods and results of Section 2 are applied to study the geometric structure of resonance tongues in Hill's equations with quasi-periodic forcing (1). In the previous section we saw that the tongue boundaries are smooth around the origin. Also we found that their Taylor expansions around a certain point can be obtained by an averaging procedure for which one needs to know the reducing matrix at that point. In general this is not known unless $b = 0$, i.e., when the system has constant coefficients. In this section this fact will be used to obtain generalizations of results as these hold for Hill's equations with periodic coefficients, compare [3, 6].

3.1 A criterion for transversality

The first application will be a criterion for the transversality of the tongue boundaries at the origin. For simplicity assume that $p = 1$, that is, that $b \in \mathbb{R}$ and let q be the forcing. In the periodic case it is known [1, 3, 6] that the two boundaries of a certain resonance tongue are transversal at $b = 0$ if, and only if, the corresponding harmonic (or Fourier coefficient) of q does not vanish. In the quasi-periodic case, the situation is the same.

Proposition 5 *In the quasi-periodic Hill equation*

$$x'' + (a + b\ q(t))\, x = 0,$$

where $q(t) = Q(\omega t)$ and $Q : \mathbb{T}^d \to \mathbb{R}$, assume that Q is real analytic and that the frequency vector $\omega \in \mathbb{R}^d$ is Diophantine. Then the tongue boundaries of the $\mathbf{k}$th resonance, $\alpha_0 = \frac{1}{2}\langle \mathbf{k}, \omega \rangle \in \mathcal{M}_+(\omega)$, meet transversally at $b = 0$ if and only if the $\mathbf{k}$th harmonic of Q does not vanish.

Proof: Let $a_0 = \alpha_0^2$ and $\alpha_0 = \frac{1}{2}\langle \mathbf{k}, \omega \rangle \in \mathcal{M}_+(\omega)$. Then a fundamental solution for $a = a_0$ and $b = 0$ is given by

$$X(t) = \begin{pmatrix} \cos(\alpha_0 t) & \frac{1}{\alpha_0}\sin(\alpha_0 t) \\ -\alpha_0 \sin(\alpha_0 t) & \cos(\alpha_0 t) \end{pmatrix}. \tag{11}$$

Following the notation of the previous section, let

$$z_{11}(t) = \cos(\alpha_0 t) \qquad \text{and} \qquad z_{12}(t) = \frac{1}{\alpha_0}\sin(\alpha_0 t).$$

Then

$$\begin{aligned} z_{11}^2(t) &= \cos^2(\alpha_0 t) = \frac{1}{2} + \frac{1}{2}\cos(2\alpha_0 t), \\ z_{12}^2(t) &= \frac{1}{a_0}\sin^2(\alpha_0 t) = \frac{1}{2a_0} - \frac{1}{2a_0}\cos(2\alpha_0 t). \end{aligned}$$

Denoting the tongue boundaries by $a_i = a_i(b)$, for $i = 1, 2$, their derivatives at $b = 0$ are obtained averaging once

$$a_i'(0) = \frac{[z_{1i}^2 q]}{[z_{1i}^2]} \text{ for } i = 1, 2, \qquad [z_{11}^2] = \frac{1}{2}, \qquad [z_{12}^2] = \frac{1}{2a_0}.$$

If for $\mathbf{k} \in \mathbb{Z}^d$, the Fourier coefficients of Q are denoted by $Q_{\mathbf{k}}$ we have

$$[z_{11}^2 q] = \frac{1}{2} + \frac{Q_{\mathbf{k}} + Q_{-\mathbf{k}}}{4}, \qquad [z_{12}^2 q] = \frac{1}{2a_0} - \frac{Q_{\mathbf{k}} + Q_{-\mathbf{k}}}{4a_0}.$$

Therefore, since

$$\begin{aligned} a_1'(0) &= 1 + \frac{Q_{\mathbf{k}} + Q_{-\mathbf{k}}}{2}, \\ a_2'(0) &= 1 - \frac{Q_{\mathbf{k}} + Q_{-\mathbf{k}}}{2}, \end{aligned}$$

it follows that $a_1'(0) \neq a_2'(0)$ if, and only if, $Q_{\mathbf{k}} \neq 0$. This concludes our proof. □

3.2 Order of tangency at the tongue tip and creation of instability pockets

We now focus on a special class of quasi-periodic Hill equations. In analogy to the periodic case we call

$$x'' + \left(a + b\left(\sum_{j=1}^{d} c_j \cos(\omega_j t) + \varepsilon \cos\left(\langle \mathbf{k}^*, \omega\rangle t\right)\right)\right) x = 0, \tag{12}$$

a near-Mathieu equation. Here ε is a small parameter and for $\varepsilon = 0$ we speak of a Mathieu equation. In the following $\omega = (\omega_1, \dots, \omega_d)^T$ is a Diophantine frequency vector. Also we take $c_j \neq 0$ for all $j = 1, \dots, d$ and fix $\mathbf{k}^* = (k_1^*, \dots, k_d^*)^T$ a non-zero vector in $\mathbb{Z}^d$. We often abbreviate $\langle \mathbf{k} \rangle = \langle \mathbf{k}, \omega \rangle$.

Theorem 3 *Consider the near-Mathieu equation with quasi-periodic forcing* (12) *as above. Then*

(i) If $\varepsilon = 0$, the order of tangency at $b = 0$ of the $\mathbf{k}^*$th resonance tongue is greater or equal than $|\mathbf{k}^*|$ and it is exactly $|\mathbf{k}^*|$ if, and only if, ω does not belong to $\mathcal{A}(\mathbf{k}^*)$, where $\mathcal{A}(\mathbf{k}^*)$ is a subset of the Diophantine frequency vectors which has measure zero.

(ii) If $\varepsilon \neq 0$, $\omega \notin \mathcal{A}(\mathbf{k}^*)$ is Diophantine and $|\varepsilon|$ is small enough, there exists at least one pocket at the $\mathbf{k}^*$th resonance tongue with ends $b = 0$ and $b = b(\varepsilon) \neq 0$. Here ε needs to have a suitable sign.

Proof: Our proof follows from an analysis of the normal form to order $|\mathbf{k}^*|$. There are several normalization techniques and any such method for arbitrary $|\mathbf{k}^*|$ can be cumbersome. Therefore we only use the format of the normal form of order $|\mathbf{k}^*|$ to find out which terms are relevant. Subsequently, the coefficients of those terms are obtained by an alternative, recurrent and simpler method.

Let us set $\alpha_0 = \langle \mathbf{k}^* \rangle / 2$ and $a = \alpha_0^2 + \alpha$. Next, a scaling

$$x = \frac{\xi}{\sqrt{\alpha_0}}, \quad y = \eta\sqrt{\alpha_0}$$

and passage to complex coordinates

$$\xi = \frac{q+ip}{\sqrt{2}}, \quad \eta = \frac{iq+p}{\sqrt{2}},$$

give the following form for the time-dependent Hamiltonian

$$H(q,p,t) = \alpha_0 iqp + \frac{q^2-p^2+2iqp}{2}\left(\frac{\alpha}{2\alpha_0}+\frac{b}{4\alpha_0}\left(\sum_{j=1}^{d} c_j\cos(\omega_j t) + \varepsilon\cos\left(\langle \mathbf{k}^* \rangle t\right)\right)\right). \quad (13)$$

Now let J be canonically conjugate to the time t, and let $\zeta_j = \exp(i\ \mathrm{sign}(k_j^*)\omega_j t)$ for $j = 1, \ldots, d$. Then (13) can be written as the sum of an integrable part H_0 and a perturbation H_1

$$H_0 = J + \alpha_0 iqp, \quad H_1 = \hat{b}\left(q^2-p^2+2iqp\right)\left(\hat{\alpha} + \sum_{j=1}^{d} c_j\left(\zeta_j + \zeta_j^{-1}\right) + \varepsilon\left(\zeta^{\mathbf{k}^*} + \zeta^{-\mathbf{k}^*}\right)\right), \quad (14)$$

where

$$\hat{\alpha} = 2\alpha/b, \qquad \hat{b} = b/(8\alpha_0)$$

act as perturbation parameters.

Analogous to [6] we apply the Giorgilli-Galgani algorithm [13] for normalization (averaging). Starting with $H_{0,0} = H_0$ and $H_{1,0} = H_1$, the terms

$$H_{j,k} = \sum_{l=1}^{k} \frac{l}{k}\left[G_l, H_{j,k-l}\right], \qquad j = 0, 1, \quad k > 0,$$

where $[\cdot,\cdot]$ denotes the Poisson bracket, are computed recurrently. A term as $H_{j,k}$ contains $\hat{b}^{j+k}$ as a factor. The functions G_n are determined for cancelling the time dependence as far as possible, i.e., if no resonances occur. To be precise, assume that $G_1, \ldots G_{n-1}$ are already computed. Then all terms in $H_{1,n-1} + H_{0,n}$ are known except the ones coming from $[G_n, H_{0,0}]$. Let K_n contain the known terms at order n. Then G_n is determined by requiring $K_n + [G_n, H_{0,0}]$ not to contain terms in the ζ_j variables. The transformed Hamiltonian then is $N = N_0 + N_1 + N_2 + \cdots$, where $N_0 = H_{0,0}$ and $N_n = H_{1,n-1} + H_{0,n}$. In particular, N_n is of order n with respect to $\hat{b}$.

It directly follows that

$$\left[H_{0,0}, q^r p^{2-r}\zeta^{\mathbf{k}}\right] = q^r p^{2-r}\zeta^{\mathbf{k}} i\left(\alpha_0(2-2r) - \langle \mathbf{k} \rangle\right), \qquad r = 0, 1, 2,$$

and this shows that all terms with $\alpha_0(2-2r) - \langle \mathbf{k} \rangle$ different from zero can be cancelled to any finite order. Proceeding by induction one observes that, if $j + k = m$ then $H_{j,k}$ has the form

$$H_{j,k} = q^2 d_1 - p^2 d_2 + iqp(d_3 + d_4),$$

with the corresponding G_m of the form

$$G_m = i(q^2 d_1 + p^2 d_2) + qp(d_3 - d_4).$$

Here d_1 contains the terms with real coefficients of the form $\hat{\alpha}_0^{m-s} \zeta^{\mathbf{k}}$ with $|\mathbf{k}| = r$, where s and r have the same parity. The terms in d_2 can be obtained from d_1 by a replacement of $\zeta^{\mathbf{k}}$ by $\zeta^{-\mathbf{k}}$. Similarly, the expression of d_3 is identical to that of d_4 but for replacing $\zeta^{\mathbf{k}}$ by $\zeta^{-\mathbf{k}}$.

Summing up, by a canonical change of variables, the Hamiltonian $H = H_0 + H_1$, up to a remainder of higher order in $\hat{b}$, can be reduced to the normal form

$$NF = J + a_0 iqp + \mathrm{coef}_1 iqp + \mathrm{coef}_2 \left(q^2 \zeta^{-\mathbf{k}^*} - p^2 \zeta^{\mathbf{k}^*} \right), \tag{15}$$

where

- $\mathrm{coef}_1 = \hat{\alpha} + r_1$, where r_1 a (real) function depending on $(\hat{b}, \hat{\alpha}, \varepsilon, c)$, and containing some power of $\hat{b}$ as a factor;
- $\mathrm{coef}_2 = \varepsilon \hat{b} f_2(\hat{b}, \hat{\alpha}, \varepsilon, c) + \hat{b}^{|\mathbf{k}^*|} \times r_2$, where r_2 a (real) function depending on $(\hat{b}, \hat{\alpha}, \varepsilon, c)$ and where $f_2(0,0,0,c) \neq 0$ does not depend on c;
- The order of the remainder in $\hat{b}$ is larger than $|\mathbf{k}^*|$.

Truncating away the remainder and passing to co-rotating coordinates (u, v) defined by

$$u = q \exp(-i\langle \mathbf{k}^* \rangle t), \qquad v = p \exp(i \langle \mathbf{k}^* \rangle t)$$

yields the system

$$u' = i\, \mathrm{coef}_1\, u - 2\, \mathrm{coef}_2\, v, \qquad v' = -2\, \mathrm{coef}_2\, u - i\, \mathrm{coef}_1\, v.$$

Therefore, up to the $|\mathbf{k}^*|$th order the tongue boundaries are given by the equation

$$\mathrm{coef}_1 = \pm 2\, \mathrm{coef}_2.$$

So if $r_2(0,0,0,c) \neq 0$ for $\varepsilon = 0$ there is a $|\mathbf{k}^*|$th order tangency at $b = 0$, while for $\varepsilon \neq 0$ there is an instability pocket. Hence, our proof of Theorem 3 is concluded by checking when $r_2(0,0,0,c)$ vanishes.

To find out whether $r_2(0,0,0,c)$ vanishes or not, it is only necessary to consider equation (12) for $\varepsilon = 0$ at the exact resonance

$$x'' + \left(\frac{\langle \mathbf{k}^* \rangle}{4} + \frac{b}{2} \left(\sum_{j=1}^{d} c_j \left(\zeta_j + \zeta_j^{-1} \right) \right) \right) x = 0. \tag{16}$$

Note that

$$r_2(0,0,0,c) = R(\omega, \mathbf{k}^*) c^{\mathbf{k}^*}$$

where now R does not depend on c. Therefore one may assume that $c_j = 1$ for $j = 1, \ldots, d$.

According to the normal form (15), any non-trivial solution $x(t)$ of (16) can be expanded in powers of b the first $K-1$ coefficients of which are quasi-periodic functions and where the Kth coefficient is also quasi-periodic if and only if $R(\omega, \mathbf{k}^*)$ vanishes. We are now going to compute this expansion directly from the differential equations, instead of using the Hamiltonian formulation.

Since we are interested in the $\mathbf{k}^*$th power in ζ, we first consider the equation (16) only for positive powers of the ζ_j :

$$x'' + \left(\frac{\langle \mathbf{k}^* \rangle^2}{4} + \frac{b}{2} \sum_{j=1}^{d} \zeta_j \right) x = 0. \tag{17}$$

Scaling time by $t = 2\tau$ turns (17) into

$$\ddot{x} + \left(\langle \mathbf{k}^* \rangle^2 + \mu \sum_{j=1}^{d} z_j^2 \right) x = 0. \tag{18}$$

where the dot denotes derivation with respect to τ and where $\mu = 2b$ is the new perturbation parameter. Also note that after this change we have $\zeta_j = z_j^2$, where $z_j = \exp(i \ \mathrm{sign}(k_j^*)\omega_j \tau)$. Any solution of equation (18) can be expanded in powers of μ as follows

$$x^{(1)} = x_0 + \mu x_1 + \mu^2 x_2 + \cdots + \mu_K x_K + O(\mu^{K+1})$$

where $K = |\mathbf{k}^*|$. Substitution of this expansion into (18) leads to the following recursive relations:

$$\ddot{x}_r + \langle \mathbf{k}^* \rangle^2 x_r = - \left(\sum_{j=1}^{d} z_j^2 \right) x_{r-1}$$

for $r = 1, \ldots, K$ and

$$\ddot{x}_0 + \langle \mathbf{k}^* \rangle^2 x_0 = 0$$

for $r = 0$. One of the two fundamental solutions of the latter equation is $x_0 = z^{-\mathbf{k}^*}$ so that the equation for x_1 becomes

$$\ddot{x}_1 + \langle \mathbf{k}^* \rangle^2 x_1 = - \sum_{j_1=1}^{d} z^{-\mathbf{k}^* + 2\mathbf{e}_{j_1}},$$

where $\mathbf{e}_j$ is the jth element of the canonical basis of $\mathbb{R}^d$. A solution of the latter equation is given by

$$x_1 = - \sum_{j_1=1}^{d} \frac{z^{-\mathbf{k}^* + 2\mathbf{e}_{j_1}}}{\langle \mathbf{k}^* \rangle^2 - \langle \mathbf{k}^* - 2\mathbf{e}_{j_1} \rangle^2}.$$

This recursive process can be continued up to any order. By induction it directly follows that at the rth step

$$x_r = (-1)^r \sum_{j_1, \ldots, j_r = 1}^{d} \frac{z^{\left(-\mathbf{k}^* + 2\left(\mathbf{e}_{j_1} + \ldots + \mathbf{e}_{j_r}\right)\right)}}{\prod_{l=1}^{r} \left(\langle \mathbf{k}^* \rangle^2 - \langle \mathbf{k}^* - 2\mathbf{s}_l \rangle^2 \right)}$$

where $\mathbf{s}_l = \mathbf{e}_{j_1} + \cdots + \mathbf{e}_{j_l}$ for $l = 1, \dots, r$. Note that when $|r| < |\mathbf{k}^*|$, the denominator of the above expression never vanishes. At the order $K = |\mathbf{k}^*|$ the equation for x_K reads

$$\ddot{x}_K + \langle \mathbf{k}^* \rangle^2 x_K + (-1)^{K-1} \sum_{j_1,\dots,j_K=1}^{d} \frac{z^{\left(-\mathbf{k}^* + 2\left(\mathbf{e}_{j_1} + \dots + \mathbf{e}_{j_K}\right)\right)}}{\prod_{l=1}^{K-1} \left(\langle \mathbf{k}^* \rangle^2 - \langle \mathbf{k}^* - 2\mathbf{s}_l \rangle^2\right)} = 0.$$

In the summation we are only interested in terms with $z^{\mathbf{k}^*}$, indeed, all other terms can be removed by a procedure similar to the one used for the previous $x_1, \dots, x_{K-1}$.

The remaining terms can be indexed by the paths of length K joining $\mathbf{0}$ and $\mathbf{k}^*$ in the lattice $\mathbb{Z}^d$. The set of these paths will be denoted by $\Gamma(\mathbf{k}^*)$, and for every path $\gamma \in \Gamma(\mathbf{k}^*)$ we consider the intermediate position vectors $\mathbf{s}_r(\gamma)$. In this way the equation for x_K becomes

$$\ddot{x}_K + \langle \mathbf{k}^* \rangle^2 x_K + z^{\mathbf{k}^*} F(\omega, \mathbf{k}^*) = 0$$

where

$$F(\omega, \mathbf{k}^*) = \sum_{\gamma \in \Gamma(\mathbf{k}^*)} \frac{(-1)^{K-1}}{\prod_{l=1}^{K-1} \left(\langle \mathbf{k}^* \rangle^2 - \langle \mathbf{k}^* - 2\mathbf{s}_l(\gamma) \rangle^2\right)} = \frac{1}{4} \sum_{\gamma \in \Gamma(\mathbf{k}^*)} \frac{(-1)^{K-1}}{\prod_{l=1}^{K-1} \left(\langle \mathbf{k}^* - \mathbf{s}_l(\gamma) \rangle - \langle \mathbf{k}^* \rangle\right)},$$

which has non-vanishing denominators for all irrational frequency vectors ω. The latter equation has as a non-trivial solution

$$x_K(\tau) = -2i\tau \langle \mathbf{k}^* \rangle F(\omega, \mathbf{k}^*) z^{\mathbf{k}^*},$$

provided that $F(\omega, \mathbf{k}^*) \neq 0$, and this solution is clearly not quasi-periodic.

Next we proceed with the other fundamental solution $x^{(2)}$ of the equation (18), starting with the zero order term $z^{\mathbf{k}^*}$. However it is better to study that solution via the conjugate equation

$$\ddot{x} + \left(\langle \mathbf{k}^* \rangle^2 + \mu \sum_{j=1}^{d} z_j^{-2} \right) x = 0. \tag{19}$$

This leads to a recursive process as before for obtaining the coefficients of the expansion of $x^{(+)}$ in terms of μ. Now, taking $x^{(\pm)} = \frac{1}{2}\left(x^{(1)} \pm x^{(2)}\right)$ as fundamental solutions, we get the following equation

$$\ddot{x} + \left(\langle \mathbf{k} \rangle^2 + \mu \sum_{j=1}^{d} \left(z_j^2 + z_j^{-2} \right) \right) x = 0 \tag{20}$$

which, undoing the changes in τ and μ, can be transformed into (16).

In this way we have found two linearly independent solutions x^+ and x^- of this system, the expansion of which in powers of b have quasi-periodic coefficients in time up to order $K-1$ and where the Kth order coefficient is a function of the form $tz^{\mathbf{k}^*}$ times $F(\omega, \mathbf{k}^*)$. By comparison of coefficients it follows that $F(\omega, \mathbf{k}^*)$ and $R(\omega, \mathbf{k}^*)$ are identical except for a non-zero factor.

Note that $F(\omega, \mathbf{k}^*)$ is a rational function. We denote its numerator by $N(\omega, \mathbf{k}^*)$ and its denominator by $D(\omega, \mathbf{k}^*)$. Define $\mathcal{A}(\mathbf{k}^*)$ as the set of ω's for which $N(\omega, \mathbf{k}^*)$ is non-zero. We claim that $\mathcal{A}(\mathbf{k}^*)$ has measure zero, which follows from the fact that $N(\cdot, \mathbf{k}^*)$ is not identically zero. To check this first note that if $\omega = (1, \dots, 1)^T$, then $D(\omega, \mathbf{k}^*)$ does not vanish. Second we resort to the periodic case [6], noting that the equation now can be transformed to the classical Mathieu equation. It thereby follows that $N(\omega, \mathbf{k}^*)$ is non-zero for this value and, hence that $N(\cdot, \mathbf{k}^*)$ is not identically zero for any $\mathbf{k}^*$. Therefore the set $\mathcal{A}(\mathbf{k}^*)$, given by the zeroes of $N(\omega, \mathbf{k}^*)$, is a zero measure set and the theorem follows. $\square$

Summarizing, the tongue boundaries at the $\mathbf{k}^*$th resonance, up to order $|\mathbf{k}^*|$, are given by the equation

$$\mathrm{coef}_1 = \pm 2\,\mathrm{coef}_2$$

which, in terms of $\hat{\alpha}$, $\hat{b}$, ε and c becomes

$$\hat{\alpha} + r_1(\hat{\alpha}, \hat{b}, \varepsilon, c) = \pm 2\left(\varepsilon \hat{b} f_2(\hat{\alpha}, \hat{b}, \varepsilon, c) + \hat{b}^{|\mathbf{k}^*|} r_2(\hat{\alpha}, \hat{b}, \varepsilon, c)\right).$$

The tongue boundary crossings up to order $|\mathbf{k}^*|$ correspond to

$$\mathrm{coef}_2 = 0$$

and a further analysis requires to distinguish between the cases of even and odd K.

When K is even, then for any $0 < |\varepsilon| \ll 1$ there is a pocket ending at $b = 0$ and at

$$b_{\mathrm{tip}} = \left(\frac{-\varepsilon f_2(0,0,0,c)}{c^{\mathbf{k}^*} R_2(\omega, \mathbf{k}^*)}\right)^{\frac{1}{K-1}} + \cdots,$$

where the dots denote higher order terms in ε. If K is odd then the sign of ε must be selected such that

$$\frac{-\varepsilon f_2(0,0,0,c)}{c^{\mathbf{k}^*} R_2(\omega, \mathbf{k}^*)}$$

is positive. If this is the case, then there are two instability pockets with ends at $b = 0$ and at

$$b^{\pm}_{\mathrm{tip}} = \pm\left(\frac{-\varepsilon f_2(0,0,0,c)}{c^{\mathbf{k}^*} R_2(\omega, \mathbf{k}^*)}\right)^{\frac{1}{K-1}} + \cdots.$$

Remark 3 1. *Note that the above result only applies to quasi-periodic near-Mathieu equations of the type* (12). *For more general quasi-periodic forcings, the problem of the order of tangency of the tongue boundaries at $b = 0$ is at least as complicated as in the periodic case, see* [6].

2. *The sets $\mathcal{A}(\mathbf{k}^*)$ are not empty in general. For examples and some properties of these sets, see Appendix* B.

3. *Instead of fixing ε one can also fix $|b_0|$ sufficiently small and show that, for a suitable value of $\varepsilon = \varepsilon(b_0)$, one can create an instability pocket in equation* (12) *with ends at $b = 0$ and $b = b_0$. A suitable choice of the components of c also allows several pockets (associated to different $\mathbf{k}^*$) with ends at $b = 0$ and $b = b_0$ and for the same value of ε. In general the same complexity holds here as in the periodic case, compare with the general discussion in Section* 1.1.2.

One consequence of Theorem 3 is

Theorem 4 *Assume that in Hill's equation*

$$x'' + (a + bq(t))\, x = 0$$

the forcing q is an even function, real analytic, quasi-periodic and with Diophantine frequency vector ω. Suppose that, for some $\mathbf{k}^* \neq \mathbf{0}$, the $\mathbf{k}^*$th harmonic of q does not vanish and that $\omega \notin \mathcal{A}(\mathbf{k}^*)$. Then, the following equation

$$x'' + \left(a + b\left(\sum_{j=1}^{d} c_j \cos(\omega_j t) + q(t)\right)\right) x = 0 \tag{21}$$

has a pocket at the $\mathbf{k}^*$th resonance tongue provided that the $|c_j|$ are sufficiently large.

Proof: Let $\varepsilon > 0$ be a small parameter and define $\tilde{c}_j = c_j/\varepsilon$, for $j = 1, \ldots, d$. Writing $\tilde{b} = b/\varepsilon$, the equation (21) reads

$$x'' + \left(a + \tilde{b}\left(\sum_{j=1}^{d} \tilde{c}_j \cos(\omega_j t) + \varepsilon q(t)\right)\right) x = 0.$$

Since the $\mathbf{k}^*$th harmonic $q_{\mathbf{k}^*}$ of q does not vanish, this even function can be split into

$$q(t) = q_{\mathbf{k}^*} \cos(\langle \mathbf{k}^*, \omega \rangle t) + \tilde{q}(t)$$

where $\tilde{q}$ is an even function whose $\mathbf{k}^*$th harmonic vanishes. Let $\tilde{\varepsilon} = \varepsilon q_{\mathbf{k}^*}$. In these new parameters the equation (21) gets the form

$$x'' + \left(a + \tilde{b}\left(\sum_{j=1}^{d} \tilde{c}_j \cos(\omega_j t) + \tilde{\varepsilon} \cos(\langle \mathbf{k}^*, \omega \rangle t) + \frac{\tilde{\varepsilon}}{q_{\mathbf{k}^*}} \tilde{q}(t)\right)\right) x = 0.$$

The only difference of the latter equation with (12) is the term $\tilde{q}$. But since its $\mathbf{k}^*$th harmonic vanishes, the conclusions of Theorem 3 concerning the existence of pockets hold here, provided $\omega \notin \mathcal{A}(\mathbf{k}^*)$ is Diophantine, $\tilde{c}_j$ do not vanish and ε is sufficiently small. The latter condition is equivalent to the c_j being sufficiently large. □

3.3 A near-Mathieu example with an instability pocket

In this section the following concrete example of a near-Mathieu equation with quasi-periodic forcing is investigated:

$$x'' + (a + b(\cos t + \cos \gamma t + \varepsilon \cos(1 + \gamma)t))\, x = 0. \tag{22}$$

Here γ is a Diophantine number and ε a perturbation parameter. We consider the resonance $\alpha_0 = \frac{1}{2}(1 + \gamma)$, which means that (a, b, ε) will be near

$$\left(\left(\tfrac{1}{2}(1 + \gamma)\right)^2, 0, 0 \right).$$

Since γ is strongly incommensurable with 1 and the forcing is entire analytic, there exists a constant $C = C(\gamma)$ such that if $|b| < C$ and $|\varepsilon| < 1$, then there is reducibility at the tongue boundaries [11]. Compare Section 2. After a twofold averaging and other suitable linear transformations which do not affect the resonance domains, the system is transformed into

$$\phi' = \left(\begin{pmatrix} 0 & X(\nu) + Y(\nu) \\ -X(\nu) + Y(\nu) & 0 \end{pmatrix} + M^{(2)}(\nu) \right) \phi,$$

where $\nu = (a - a_0, b, \varepsilon)$ and where X and Y are given by the following expansions in ν

$$X(\nu) = -\frac{1}{1+\gamma} \left(\frac{a - a_0}{1+\gamma} - \frac{(a-a_0)^2}{(1+\gamma)^3} - \frac{b^2}{4(1+\gamma)^2} C(\varepsilon) \right)$$

and

$$Y(\nu) = -\frac{b}{1+\gamma} \left(\frac{-\varepsilon}{2(1+\gamma)} + \frac{\varepsilon(a-a_0)}{(1+\gamma)^3} + \frac{b}{2(1+\gamma)^2}(1 + \frac{1}{\gamma}) \right).$$

Here

$$C(\varepsilon) = \frac{\varepsilon^2}{2(1+\gamma)} + \frac{1}{2+\gamma} + \frac{1}{1+2\gamma} + 1 + \frac{1}{\gamma}.$$

Hence, in the notation of Section 2 we have $S_1^{(2)}(\nu) = X(\nu) - Y(\nu)$ and $S_2^{(2)}(\nu) = X(\nu) + Y(\nu)$. Thus $S_1^{(2)}(\nu) = 0$ (resp. $S_2^{(2)}(\nu) = 0$) if and only if $X(\nu) = Y(\nu)$ (resp. $X(\nu) = -Y(\nu)$). The Taylor expansions of the tongue boundaries up to second order in (b, ε) are given by

$$a_1^{(2)}(b,\varepsilon) = \left(\frac{1+\gamma}{2}\right)^2 - \frac{b\varepsilon}{2} + \frac{b^2}{4(1+\gamma)} \left(3 + \frac{3}{\gamma} + \frac{1}{2+\gamma} + \frac{1}{1+2\gamma} \right)$$

and

$$a_2^{(2)}(b,\varepsilon) = \left(\frac{1+\gamma}{2}\right)^2 + \frac{b\varepsilon}{2} + \frac{b^2}{4(1+\gamma)} \left(-1 - \frac{1}{\gamma} + \frac{1}{2+\gamma} + \frac{1}{1+2\gamma} \right).$$

Therefore the second order Taylor expansions of the tongue boundaries have a transversal crossing both at $(b, \varepsilon) = (0, 0)$ and at the point $(b, \varepsilon) = (\gamma\varepsilon, \varepsilon)$ if $\varepsilon \neq 0$. By Theorem 1 we know that the boundary functions $a_1^{(2)}$ and $a_2^{(2)}$ are of class C^∞ in b. With little more effort, one also establishes this same degree of smoothness in the parameter ε. Following the argument of the previous subsection, one has

Corollary 2 *For the near-Mathieu equation* (22) *there exists a positive constant C such that, if $|b| < C$ and $|\varepsilon| < C$, then the tongue boundaries of the resonance corresponding to $\alpha_0 = (1+\gamma)/2$ are C^∞ functions of (b, ε), while*

1. *For $\varepsilon \neq 0$ the tongue boundaries have two transversal crossings, one at $(a, b) = (a_0, 0)$ and the other at $(a_1^{(2)}(\gamma\varepsilon, \varepsilon) + O_3(\varepsilon), \gamma\varepsilon + O_2(\varepsilon))$,*
2. *For $\varepsilon = 0$ the tongue boundaries at $b = 0$ have a second order tangency.*

Remark 4 *Note that Corollary* 2 *exactly describes the $\mathbb{A}_3$-scenario, compare* [2]. *For the periodic analogue see* [3], *where Hill's map has a Whitney cusp singularity. Compare with Remark* 3.

4 Proofs

Main aim of this section is to prove the Propositions 3 and 4, see Section 2. To this end we here give a proposition regarding the tongue boundaries, compare with the methods of [9, 18, 20, 21]. This result is of interest for its own sake as well: at the end of the section we draw some further conclusions on rotation number and maximal Lyapunov exponent.

Proposition 6 *In the context of Section* 2*, consider the differential equation*

$$\phi' = \left(\begin{pmatrix} 0 & R_1(b) \\ -R_2(b) & 0 \end{pmatrix} + M(\omega t, b) \right) \phi, \tag{23}$$

where R_1*,* R_2 *are continuous, and where positive constants* K_1*,* K_2 *and* Δ_1 *exist such that*

1. $|M(\theta, b)| \leq K_1 |b - b_0|^{r+1}$ *for* $|b - b_0| < \Delta_1$ *and* $\theta \in \mathbb{T}^d$*,*
2. $\min(|R_1(b)|, |R_2(b)|) \geq K_2 |b - b_0|^r$ *for* $|b - b_0| < \Delta_1$*.*

Then, there exists a constant $\Delta = \Delta(K_1, K_2)$ *such that*

(Elliptic case) *If* $R_1(b)R_2(b) > 0$ *and* $0 < |b - b_0| < \Delta$*, then* (23) *has rotation number strictly different from zero;*

(Hyperbolic case) *If* $R_1(b)R_2(b) < 0$ *and* $0 < |b - b_0| < \Delta$*, then* (23) *has exponential dichotomy.*

Proof: We start with the second alternative.

Hyperbolic case: Let $E = \sqrt{-R_1(b)R_2(b)}$ and assume that $R_1(b) > 0, R_2(b) < 0$ (the other case is treated similarly). Then the eigenvalues of the time-independent matrix in equation (23) are $\pm E$. We first perform a change of variables, defined for $b \neq b_0$, which puts this time-independent part into diagonal form.

To this end consider the transformation $\psi = P(b)\phi$ where

$$P(b) = \begin{pmatrix} \sqrt{|R_2|} & -\sqrt{|R_1|} \\ \sqrt{|R_2|} & \sqrt{|R_1|} \end{pmatrix}.$$

Then ψ satisfies the equation

$$\psi' = \left(\begin{pmatrix} -E & 0 \\ 0 & E \end{pmatrix} + \frac{1}{2E} \begin{pmatrix} \sqrt{|R_2|} & -\sqrt{|R_1|} \\ \sqrt{|R_2|} & \sqrt{|R_1|} \end{pmatrix} M_{r+1}(\omega t, b) \begin{pmatrix} \sqrt{|R_1|} & \sqrt{|R_1|} \\ -\sqrt{|R_2|} & \sqrt{|R_2|} \end{pmatrix} \right) \psi,$$

which can be written in the form

$$\psi' = E \left(\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} + N(\omega t, b) \right) \psi. \tag{24}$$

To (24), in order to prove exponential dichotomy, we aim to apply Coppel [9]. For this it is sufficient that, for $0 < |b - b_0| < \Delta_2(\Gamma, K_1, K_2)$, the eigenvalues of the above matrix are bounded away from zero.

In fact, we can bound any element $n_{ij}(\theta, b)$ of the new time-dependent part N in the following way

$$|n_{ij}| \leq 2 \frac{\max\{|R_1(b)|, |R_2(b)|\}}{2|R_1(b)R_2(b)|} K_1 |b - b_0|^{r+1} \leq \frac{K_1 |b - b_0|^{r+1}}{\min\{|R_1(b)|, |R_2(b)|\}} \leq \frac{K_1}{K_2} |b - b_0|$$

for $0 < |b - b_0| < \Delta_1$. Now let $\Delta = \frac{K_2}{2K_1}$. Then, using Gerschgorin lemma (see, e.g., [14]) for the eigenvalues $\lambda(b), -\lambda(b)$ of the matrix in (24) we obtain bounds

$$\lambda(b) \geq E(b)\left(1 - \frac{K_1}{K_2}|b - b_0|\right) \geq \frac{E(b)}{2} \geq C|b - b_0|^r$$

for $|b - b_0| < \Delta$. We conclude that the system has exponential dichotomy for $0 < |b - b_0| < \Delta$.

Elliptic case: The proof is similar to the hyperbolic case. First perform a linear change of variables (only defined for $b \neq b_0$) to transform the time-independent part of the system to a convenient form. After this write down the equation for the rotation number of a general differential equation in $SL(2, \mathbb{R})$, compare [11, 15, 20, 21] for definitions and similar results.

In the elliptic case the eigenvalues of the time-independent matrix are $\pm iE(b)$, where now $E = \sqrt{R_1(b)R_2(b)}$. Assume $R_1(b), R_2(b) > 0$ (the other case is treated similarly). Consider the linear change of variables $\psi = P(b)\phi$, defined for $b \neq b_0$, where

$$P(b) = \begin{pmatrix} 0 & \sqrt{|R_1|} \\ -\sqrt{|R_2|} & 0 \end{pmatrix}.$$

Then ψ satisfies the equation

$$\psi' = \left(\begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix} + \frac{1}{E}\begin{pmatrix} 0 & \sqrt{|R_1|} \\ -\sqrt{|R_2|} & 0 \end{pmatrix} M_{r+1}(\omega t, b) \begin{pmatrix} 0 & -\sqrt{|R_1|} \\ \sqrt{|R_2|} & 0 \end{pmatrix}\right)\psi,$$

which can be written as

$$\psi' = E\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + N(\omega t, b)\right)\psi. \tag{25}$$

Proceeding as before we get bounds

$$|n_{ij}(\theta, b)| \leq K|b - b_0| \text{ for all } \theta \in \mathbb{T}^d \text{ and } |b - b_0| < \Delta_1.$$

The argument of a solution $\psi = \varphi_b(t)$ (25) satisfies the following first order equation (compare [11])

$$\varphi_b'(t) = -E\left(1 + (n_{11} - n_{22})\cos\varphi_b \sin\varphi_b + n_{12}\sin^2\varphi_b - n_{21}\cos^2\varphi_b\right). \tag{26}$$

where n_{ij} denotes $n_{ij}(\theta, b)$. Thus,

$$\frac{\varphi_b(t)}{t} = -E\left(1 + \frac{1}{t}\int_0^t \left((n_{11} - n_{22})\cos\varphi_b \sin\varphi_b - (n_{12} + n_{21})\cos^2\varphi_b + n_{12}\right) ds\right) \leq$$

$$\leq -E(b)(1 - K|b - b_0|)$$

for $|b - b_0| < \Delta_1$ and, passing to the limit, which equals the rotation number of equation (25),

$$\delta\mathrm{rot}(b) \leq -E(b)(1 - K|b - b_0|).$$

In particular, the rotation number is different from zero if $0 < |b - b_0| < \Delta$. □

4.1 Non-collapsed gap

Suppose we are in the case of a non-collapsed gap. Present aim is to apply Proposition 6 in order to prove Proposition 3.

Using the notation of Proposition 6 it follows that

$$R_1(b) = S_1^{(r)}(a^{(r)}(b) - a_0 \pm \Gamma|b-b_0|^r, b-b_0) = \pm\Gamma[z_{11}^2]|b-b_0|^r + O_{r+1}(|b-b_0|)$$

and that

$$R_2(b) = 1 + S_2^{(r)}(a^{(r)}(b) - a_0 \pm \Gamma|b-b_0|^r, b-b_0),$$

where, by a linear change of coordinates, we obtained that $c = 1$. From now on write $\Gamma_1 = \Gamma[z_{11}^2]$. The remainder matrix M is of order $r+1$ and therefore the first condition of Proposition 6 is fulfilled for suitable constants $K_1(\Gamma)$ and Δ_1.

Next we pass to the second condition of Proposition 6. By continuity of $R_2(b)$, and since $R_2(b_0) = 1$, there exists a positive constant (again to be named Δ_1) such that if $|b-b_0| < \Delta_1$ then $R_2(b) > \frac{1}{2}$. By the same continuity argument, then also $|R_1(b)| \geq \frac{\Gamma_1}{2}|b-b_0|^r$ for $|b-b_0| < \Delta_1$. Thus, in this case

$$|R_1(b)|, |R_2(b)| \geq \Gamma_1|b-b_0|^r/2.$$

Therefore, there exists $\Delta = \Delta(\Gamma, K_1(\Gamma)) > 0$ such that if $0 < |b-b_0| < \Delta$

1. Equation (7) for $(a,b) = (a^{(r)}(b) + \Gamma|b-b_0|^r, b)$ has rotation number strictly different from zero.
2. Equation (7) for $(a,b) = (a^{(r)}(b) - \Gamma|b-b_0|^r, b)$ has exponential dichotomy.

This proves Proposition 4 and so the smoothness of the tongue boundaries when we are not in a collapsed gap.

4.2 Collapsed gap

Present aim is to prove Proposition 4, following ideas similar to the case of a non-collapsed gap. We shall see that the tongue boundaries can be divided over sectors, determined by whether the modulus of the modified rotation number or the Lyapunov exponent are greater than some constants, as shown in Figure 3. From this we obtain the tangency of the required order.

Note that in the case of a collapsed gap, by a suitable scaling, the fundamental matrix can be chosen such that

$$[z_{11}^2] = [z_{12}^2] = 1.$$

We have to show that for each $\Gamma > 0$ there exists a constant $\Delta = \Delta(\Gamma)$ such that for $|b-b_0| < \Delta$

$$|a_+(b) - \max_{i=1,2}\{a_i^{(r)}(b)\}| \leq 2\Gamma|b-b_0|^r \quad \text{and} \quad |a_-(b) - \min_{i=1,2}\{a_i^{(r)}(b)\}| \leq 2\Gamma|b-b_0|^r.$$

To do so consider equation (10) where the parameter values are given by

$$\left(\max_{i=1,2}\{a_i^{(r)}(b)\} \pm \Gamma|b-b_0|^r, b\right) \quad \text{and} \quad \left(\min_{i=1,2}\{a_i^{(r)}(b)\} \pm \Gamma|b-b_0|^r, b\right).$$

We need the following

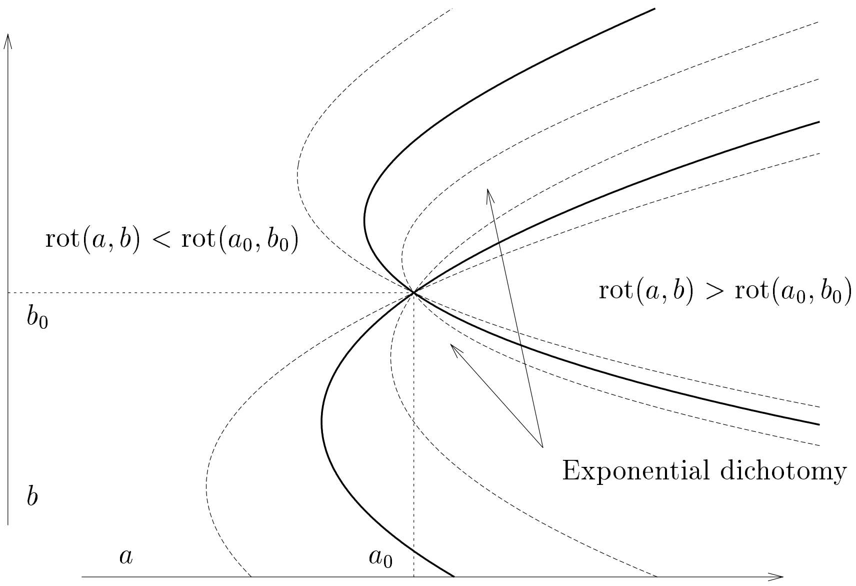

Figure 3: Areas of exponential dichotomy inside a resonance tongue as guaranteed by Lemma 2 and areas with rotation number different from $\mathrm{rot}(a_0, b_0)$ outside the tongue as guaranteed by Lemma 2. Solid lines denote tongue boundaries.

Lemma 1 *There exists a positive constant* $\Delta = \Delta(\Gamma)$ *such that system* (10) *for the parameter values*

$$\mu = \left(\max_{i=1,2}\{a_i^{(r)}(b)\} - a_0 + \Gamma|b-b_0|^r, b-b_0\right) \text{ and } \left(\min_{i=1,2}\{a_i^{(r)}(b)\} - a_0 - \Gamma|b-b_0|^r, b-b_0\right)$$

has rotation number strictly different from zero for $0 < |b - b_0| < \Delta$. *In particular*

$$\min_{i=1,2}\{a_i^{(r)}(b)\} - \Gamma|b-b_0|^r \leq a_-(b) \leq a_+(b) \leq \max_{i=1,2}\{a_i^{(r)}(b)\} + \Gamma|b-b_0|^r.$$

Proof: The proof follows the same scheme as in Section 4.1. We just prove the result for the maximum, since the case of the minimum runs similarly. Given these choices of μ, in the notation of Proposition 6 we have

$$R_1(b) = S_1^{(r)}\left(\max_{i=1,2}\{a_i^{(r)}(b)\} + \Gamma|b-b_0|^r - a_0, b-b_0\right) =$$

$$= S_1^{(r)}\left(\max_{i=1,2}\{a_i^{(r)}(b)\} - a_0, b-b_0\right) + \Gamma[z_{11}^2]|b-b_0|^r + O_{r+1}(|b-b_0|)$$

and

$$R_2(b) = S_2^{(r)}\left(\max_{i=1,2}\{a_i^{(r)}(b)\} + \Gamma|b-b_0|^r - a_0, b-b_0\right) =$$

$$= S_2^{(r)}\left(\max_{i=1,2}\{a_i^{(r)}(b)\} - a_0, b-b_0\right) + \Gamma[z_{12}^2]|b-b_0|^r + O_{r+1}(|b-b_0|),$$

where we recall that $[z_{11}^2] = [z_{12}^2] = 1$. Due to the definition of $a_1^{(r)}$ and $a_2^{(r)}$ we have

$$\min\{S_1^{(r)}(\max_{i=1,2}\{a_i^{(r)}(b)\} - a_0, b-b_0),\ S_2^{(r)}(\max_{i=1,2}\{a_i^{(r)}(b)\} - a_0, b-b_0)\} = O_{r+1}(|b-b_0|)$$

and hence

$$\min\{R_1(b), R_2(b)\} = \Gamma|b-b_0|^r + O_{r+1}(|b-b_0|).$$

By continuity

$$\min\{R_1(b), R_2(b)\} \geq \frac{\Gamma}{2}|b-b_0|^r$$

for $|b-b_0| < \Delta_1(\Gamma)$ (in particular this means that both $R_1(b)$ and $R_2(b)$ are positive). Therefore, we can apply Proposition 6 to find the existence of a positive constant $\Delta = \Delta(\Gamma) > 0$ such that for $0 < |b-b_0| < \Delta$ equation (10) has rotation number strictly different from zero. $\square$

A naive attempt to reproduce the arguments of the previous case of the non-collapsed gap fails, since our methods give little control over the set $\{(a,b) \mid a_1^{(r)}(b) = a_2^{(r)}(b)\}$. However, the following lemma proves to be sufficient for our purposes.

Lemma 2 *There exists a positive constant $\Delta = \Delta(\Gamma)$ such that system* (10) *for the parameter values*

$$\begin{aligned} \mu &= \left(\max_{i=1,2}\{a_i^{(r)}(b)\} - a_0 - \Gamma|b-b_0|^r, b-b_0\right) \text{ and} \\ &\quad \left(\min_{i=1,2}\{a_i^{(r)}(b)\} - a_0 + \Gamma|b-b_0|^r, b-b_0\right) \end{aligned}$$

has exponential dichotomy if $0 < |b-b_0| < \Delta$ and $|a_1^{(r)}(b) - a_2^{(r)}(b)| > 2\Gamma|b-b_0|^r$. In particular, we then have

$$a_-(b) \leq \min_{i=1,2}\{a_i^{(r)}(b)\} + \Gamma|b-b_0|^r \leq \max_{i=1,2}\{a_i^{(r)}(b)\} - \Gamma|b-b_0|^r \leq a_+(b).$$

Proof: The proof has the same set-up as before. Note that the condition

$$|a_1^{(r)}(b) - a_2^{(r)}(b)| > 2\Gamma|b-b_0|^r$$

is just another way of saying that

$$\min_{i=1,2}\{a_i^{(r)}(b)\} + \Gamma|b-b_0|^r \leq \max_{i=1,2}\{a_i^{(r)}(b)\} - \Gamma|b-b_0|^r.$$

We restrict to the maximum-case, since the results for the minimum are completely analogous. If we consider the parameters

$$(a,b) = (\max_{i=1,2}\{a_i^{(r)}(b)\} - \Gamma|b-b_0|^r, b)$$

then, in the notation of Proposition 6,

$$R_1(b) = S_1^{(r)}\left(\max_{i=1,2}\{a_i^{(r)}(b)\} - \Gamma|b-b_0|^r - a_0, b-b_0\right) =$$

$$= S_1^{(r)}\left(\max_{i=1,2}\{a_i^{(r)}(b)\} - a_0, b-b_0\right) - \Gamma[z_{11}^2]|b-b_0|^r + O_{r+1}(|b-b_0|)$$

and

$$R_2(b) = S_2^{(r)}\left(\max_{i=1,2}\{a_i^{(r)}(b)\} - \Gamma|b-b_0|^r - a_0, b-b_0\right) =$$

$$= S_2^{(r)}\left(\max_{i=1,2}\{a_i^{(r)}(b)\} - a_0, b-b_0\right) - \Gamma[z_{12}^2]|b-b_0|^r + O_{r+1}(|b-b_0|).$$

Hence, due to the definition of $a_1^{(r)}$ and $a_2^{(r)}$,

$$\min\{S_1^{(r)}(\max_{i=1,2}\{a_i^{(r)}(b)\} - a_0, b-b_0),\ S_2^{(r)}(\max_{i=1,2}\{a_i^{(r)}(b)\} - a_0, b-b_0)\} = O_{r+1}(|b-b_0|)$$

whereas

$$\max\{S_1^{(r)}(\max_{i=1,2}\{a_i^{(r)}(b)\} - a_0, b-b_0),\ S_2^{(r)}(\max_{i=1,2}\{a_i^{(r)}(b)\} - a_0, b-b_0)\} =$$
$$|a_1^{(r)}(b) - a_2^{(r)}(b)| + O_{r+1}(|b-b_0|).$$

Therefore we have the following bounds

$$\min\{R_1(b), R_2(b)\} \leq -\Gamma|b-b_0|^r + O_{r+1}(b-b_0)$$

and

$$\max\{R_1(b), R_2(b)\} \geq |a_1^{(r)}(b) - a_2^{(r)}(b)| - \Gamma|b-b_0|^r + O_{r+1}(b-b_0) \geq$$
$$\Gamma|b-b_0|^r + O_{r+1}(b-b_0).$$

By continuity and the dominance of O_r-terms

$$\min\{R_1(b), R_2(b)\} \leq -\frac{\Gamma}{2}|b-b_0|^r \text{ and } \max\{R_1(b), R_2(b)\} \geq \frac{\Gamma}{2}|b-b_0|^r$$

if $|b-b_0| < \Delta_1(\Gamma)$. This shows that the conditions of Proposition 6 for exponential dichotomy are fulfilled. □

In the set $\{(a,b) \mid |a_1^{(r)}(b) - a_2^{(r)}(b)| \leq 2\Gamma|b-b_0|^r\}$ we cannot perform the analysis of the above lemma since neither in the set $\{(a,b) \mid S_1^{(r)}(\mu) = S_2^{(r)}(\mu)\}$ nor in some nearby points, the required information can be obtained a priori by our methods. However this knowledge is not needed if we just want to prove Proposition 4. Indeed, assume that

$$|a_1^{(r)}(b) - a_2^{(r)}(b)| \leq 2\Gamma|b-b_0|^r$$

or, equivalently,

$$\min_{i=1,2}\{a_i^{(r)}(b)\} + \Gamma|b-b_0|^r \geq \max_{i=1,2}(a_i^{(r)}(b)) - \Gamma|b-b_0|^r.$$

We already know by Lemma 1 that there exists a positive constant $\Delta = \Delta(\Gamma)$ such that for $0 < |b-b_0| < \Delta$

$$\min_{i=1,2}\{a_i^{(r)}(b)\} - \Gamma|b-b_0|^r \leq a_-(b) \leq a_+(b) \leq \max_{i=1,2}\{a_i^{(r)}(b)\} + \Gamma|b-b_0|^r.$$

In particular, for such values of b

$$a_+(b) \geq \min_{i=1,2}\{a_i^{(r)}(b)\} - \Gamma|b-b_0|^r \geq \max_{i=1,2}\{a_i^{(r)}(b)\} - 3\Gamma|b-b_0|^r$$

and

$$a_-(b) \leq \max_{i=1,2}\{a_i^{(r)}(b)\} + \Gamma|b-b_0|^r \leq \min_{i=1,2}\{a_i^{(r)}(b)\} + 3\Gamma|b-b_0|^r.$$

Thus, taking $|b-b_0| < \Delta$ (where Δ is determined from the Lemmata 1 and 2), we find

$$|a_+(b) - \max_{i=1,2}\{a_i^{(r)}(b)\}| \leq 3\Gamma|b-b_0|^r \text{ and } |a_-(b) - \min_{i=1,2}\{a_i^{(r)}(b)\}| \leq 3\Gamma|b-b_0|^r,$$

as was to be shown. □

4.3 Differentiability of rotation number and Lyapunov exponent for a fixed potential

In this section we fix the multiparameter $b_0 \in \mathbb{R}^p$ in a sufficiently small neighbourhood of the origin, to ensure reducibility according to [11], see Section 2. In this case we study rotation number $\rho = \rho(a)$ and (maximal) Lyapunov exponent $\lambda = \lambda(a)$ of the quasi-periodic Hill equation (1), or equivalently (5), in dependence of the parameter a. The results in this setting are completely analogous to the periodic case, and proofs can be obtained from those of the previous section, especially that of Proposition 6.

Corollary 3 *In the above situation, let a_0 be an endpoint of a spectral gap. Then*

1. *If a_0 is in the left (resp. right) endpoint of a non-degenerate spectral gap, then the functions*

$$\alpha \in (-1,1) \mapsto \rho(a_0 - \alpha^2) \text{ and } \alpha \in (-1,1) \mapsto \lambda(a_0 + \alpha^2)$$

   (resp $\alpha \in (-1,1) \mapsto \rho(a_0 + \alpha^2)$ and $\alpha \in (-1,1) \mapsto \lambda(a_0 - \alpha^2)$) are differentiable at zero.

2. *If $\{a_0\}$ is a collapsed spectral gap, then the functions*

$$a \mapsto \rho(a) \text{ and } a \mapsto \lambda(a)$$

   are differentiable at a_0.

In particular, in any non-collapsed spectral gap $[a_-, a_+]$ the function $a \mapsto w(a) := -\lambda(a)^2$ is analytic in (a_-, a_+) and has lateral derivatives at $a = a_-, a_+$.

The same result was obtained in [20, 21] in more general contexts (e.g., for the Schrödinger equation with almost periodic or ergodic potential). Our method of proof is very similar to [20].

Remark 5 *With a little more effort, one can recover the fact that for fixed, small potential the function $a \mapsto w(a)$ in a gap $[a_-, a_+]$ is of class $C^\omega((a_-, a_+)) \cap C^\infty([a_-, a_+])$, see Moser and Pöschel* [18].

5 Conclusions and outlook

Summarizing, this paper studies the geometric structure of resonance tongues in a class of Hill equations with quasi-periodic forcing $\sum_{i=1}^p b_i Q_i(\omega t)$. Several results were obtained, analogous to the periodic case, regarding smoothness of tongue boundaries and the occurrence of instability pockets. Here we used reducibility of the equations. However it is well-known that in general no reducibility holds.

5.1 Larger $|b|$

According to numerical ([5]) and analytical work ([12]), in the case where b is one-dimensional, it seems that for each tongue boundary there exists a critical value b_{crit}, such that for $|b| < b_{\mathrm{crit}}$ reducibility holds, while for $|b| > b_{\mathrm{crit}}$ not even continuous reduction (to Floquet form) seems possible.

Resonance tongues, however, can be defined by the rotation number. Since this definition is independent of reducibility, we can still speak of tongue boundaries for larger $|b|$. Therefore the problem remains, whether and to what extent the possible non-reducibility of the system affects the regularity of the tongue boundaries. In Appendix A we show that tongue boundaries are always globally Lipschitz, but is this the best possible result in general? Is it possible that the tongue boundaries be continuated with some degree of regularity above the critical value?

5.2 Reversibility or not

Theorem 1 does not need the forcing $\sum_{i=1}^{p} b_i Q_i(\omega t)$ to be even. However, for the results in Section 3.3 on the existence of instability pockets, this evenness (or reversibility of the system) is essential, compare the periodic analogues in [3, 6]. In fact in non-reversible situations all gaps can be opened by an arbitrarily small perturbation, also compare [18].

5.3 Analyticity?

It should be noted that our Main Theorem 1 on the regularity of the tongue boundaries for small analytic potentials just expresses infinite differentiability, whereas in the periodic case analyticity holds.

In the present quasi-periodic case, analyticity of the tongue boundaries would follow if the averaging process of Section 2.2 were convergent. However generically this is not the case. Indeed, such convergence of the averaging process in a point at a tongue boundary would imply the existence of a spectral interval (that is, a whole interval in the spectrum) besides the point in the tongue boundary, where generically we have Cantor spectrum, see [19, 16, 11].

Of course these considerations do not forbid analyticity of the tongue boundaries and it is interesting to know whether analyticity is the case or not. Work in this direction, in a more general context, can be found in [22].

A Lipschitz property of tongue boundaries in the large

In the paper we approached the regularity of the tongue boundaries using reducibility. However, there exists numerical evidence [5], that in cases far from constant coefficients this approach can't be used. Presently we reconsider the quasi-periodic Hill equation (1), or equivalently (5), where we only assume the components of ω to be rationally independent (i.e., not necessarily Diophantine) and where the functions $Q_1, \ldots, Q_p : \mathbb{T}^d \to \mathbb{R}$ are just continuous.

Proposition 7 *In the above situation, let*

$$C_i = \|Q_i\|_{C(\mathbb{T}^d,\mathbb{R})}, \quad i = 1, \ldots, p.$$

Let $b \in \mathbb{R}^p \mapsto a(b) \in \mathbb{R}$ be a (left or right) tongue boundary. Then for all $b, b' \in \mathbb{R}^d$ we have

$$\left|a(b) - a(b')\right| \leq \sum_{i=1}^{p} C_i \left|b_i - b_i'\right|.$$

Our proof is based on Sturm-like arguments for the oscillation of the zeroes of a second order linear differential equation. First we review some useful properties of the rotation number [15, 21].

For any non-trivial solution $x(t)$ of equation (1) we define a function $N(T; x)$, which is the number of zeroes of $x(t)$ in the interval $[0, T]$. From the form that (1) takes in polar coordinates (by the so-called Prüfer transformation), it follows that all the zeroes of $x(t)$ are simple and that the limit

$$\lim_{T\to\infty} \frac{\pi N(T; x)}{T}$$

exists for all initial conditions, agreeing with the rotation number of (1), or equivalently, the system (5).

Our idea to prove Proposition 7 is to use a suitable Sturm Oscillation Theorem to control the zeroes of a variation $Q + \delta Q$ of the original potential Q, with the property that $\delta Q(\theta)$ is either positive or negative for all $\theta \in \mathbb{T}^d$.

Lemma 3 *Assume that the maps $Q, \delta Q : \mathbb{T}^d \to \mathbb{R}^d$ are continuous and that $\delta Q(\theta) > 0$ for all $\theta \in \mathbb{T}^d$. Let ρ_1 be the rotation number of*

$$x'' + Q(\theta)x = 0, \quad \theta' = \omega$$

and ρ_2 the rotation number of

$$y'' + \left(Q(\theta) + \delta Q(\theta)\right) y = 0, \quad \theta' = \omega.$$

Then $\rho_1 \leq \rho_2$.

Lemma 3 is a direct consequence of the Sturm Comparison Theorem, see, e.g., [8]. Indeed, by this result, the number of zeroes $N(T; x)$ in the interval $[0, T]$ is less than or equal to the number of zeroes $N(T, y)$ of y in the same interval, assuming that we have the same initial conditions $x(0) = y(0)$, $x'(0) = y'(0)$. Therefore, by the above considerations

$$\rho_1 = \lim_{T\to\infty} \frac{\pi N(T; x)}{T} \leq \lim_{T\to\infty} \frac{\pi N(T; y)}{T} = \rho_2,$$

as was to be shown.

We proceed showing how Lemma 3 can be used to check the Lipschitz condition stated in Proposition 7. First, note that, if the condition $\delta Q > 0$ is replaced by $\delta Q < 0$, then we have $\rho_1 \geq \rho_2$.

In the setting of Proposition 7, condition

$$\delta a - \sum_{i=1} C_i |\delta b_i| > 0 \tag{27}$$

implies that $\delta Q(\theta) = \delta a + \sum_{i=1}^{p} \delta b_i Q_i(\theta) > 0$ for all $\theta \in \mathbb{T}^d$ and thus, by Lemma 3, that

$$\mathrm{rot}(a, b_1, \ldots, b_p) \leq \mathrm{rot}(a + \delta a, b_1 + C_1 \delta b_1, \ldots, b_p + C_p \delta b_p).$$

Now, if (a,b) is in the boundary of a certain tongue (for simplicity assume a is the right endpoint of the corresponding spectral gap), this means that for arbitrarily small perturbations in the a direction, the rotation number is *strictly* larger than that of the original equation. That is, for any $\delta' a > 0$,

$$\mathrm{rot}(a,b) < \mathrm{rot}(a+\delta' a, b).$$

The Lemma then yields that, if $(\delta a, \delta b)$ satisfies (27), also

$$\mathrm{rot}(a,b) < \mathrm{rot}(a+\delta' a+\delta a, b_1 + C_1\delta b_1, \ldots, b_p + C_p \delta b_p).$$

As $\delta' a$ may be arbitrarily small the perturbations $(\delta a, \delta b)$ in the half cone defined by condition (27) do not contain any point in the boundary of the same tongue as (a,b). Therefore, in our proof of Proposition 7 we have

$$a_1(b) - a_2(b) \leq \sum_{i=1}^{p} C_i \left| b_1 - b_2 \right|.$$

In order to prove the remaining inequality, observe that perturbations in the half cone

$$\delta a + \sum_{i=1} C_i |\delta b_i| < 0 \tag{28}$$

contain no points in the left boundary of the tongue of (a,b). By contradiction assume that such a point in the left boundary exists and let $(\delta a, \delta b)$ satisfying (28) be the corresponding perturbation. Then, due to the openess of the above condition, there exists a positive constant $\delta' a$ such that $(\delta a + \delta' a, \delta b)$ still satisfies (28). Moreover, as we are assuming $(a+\delta a, b+\delta b)$ to be in the endpoint of the left spectral gap and $\delta' a > 0$

$$\mathrm{rot}(a,b) = \mathrm{rot}(a+\delta a, b+\delta b) < \mathrm{rot}(a+\delta' a+\delta a, b+\delta b).$$

On the other hand, the perturbation $(-\delta a - \delta' a, -\delta b)$ satisfies condition (27) and therefore

$$\mathrm{rot}(a+\delta' a+\delta a, \delta b) \leq \mathrm{rot}(a,b),$$

which implies $\mathrm{rot}(a,b) < \mathrm{rot}(a,b)$. This is the desired contradiction, whereby Proposition 7 is proved.

Remark 6 *The Lipschitz property in Proposition* 7 *regarding tongue boundaries also holds in the periodic case, where the proof runs exactly the same, and where this is referred to as the* directional convexity *of stability and instability domains, see* [24]. *The property also provides a bound on the derivatives of the tongue boundaries whenever they exist. This bound coincides with the one obtained in the averaging process of Section* 2.2.

B Structure of the sets $\mathcal{A}(\mathbf{k})$

An interesting question related to Theorem 3 is whether the set of Diophantine frequency vectors in $\mathcal{A}(\mathbf{k})$, for a fixed resonance $\mathbf{k}$, is empty or not.

When $d=2$, we can always assume that $\omega = (1,\gamma)$, where γ is a real number. Note that any real irrational γ for which $N(\omega,\mathbf{k}) = 0$ for some $\mathbf{k}$, is Diophantine, since it is algebraic. Direct computations, performed on $F(\omega,\mathbf{k})$, yield that if the order of the resonance is less than 5, all the roots of $N((1,\gamma),\mathbf{k})$ are either rational or complex (i.e., nonreal). However, for $\mathbf{k}=(3,2)$,

$$N\left((1,\gamma),(3,2)\right)=24+172\gamma+454\gamma^2+505\gamma^3+232\gamma^4+49\gamma^5+4\gamma^6$$

which has real irrational zeroes. Direct computation also shows that the same happens for all resonances $6\le|\mathbf{k}|\le 9$ with $k_1\ne 1$, $k_2\ne 1$. For $d\ge 3$ the situation is even simpler, since for $\mathbf{k}^*=(1,1,1)$ the polynomial $N(\omega,\mathbf{k}^*)$ has real Diophantine zeroes.

There is one case when $|\mathbf{k}^*|$th order tangency at the $\mathbf{k}^*$th resonance always can be granted:

Proposition 8 *In the Mathieu equation*

$$x''+\left(a+b\left(\cos(t)+\cos(\gamma t)\right)\right)x=0, \tag{29}$$

where $2\gamma\ne 0$ is not a negative integer, the order of tangency at $b=0$ of the resonance tongue boundaries corresponding to $\mathbf{k}^*=(K,1)$, for any K, exactly is $|\mathbf{k}^*|=K+1$.

Proof: In this case the number of paths of minimal length in $\mathbb{Z}^2$ joining $(0,0)$ and $\mathbf{k}^*$ exactly is $K+1$ and any of these can be labelled by an integer between 0 and K. These paths will be denoted by $\sigma_0,\dots,\sigma_K$. To show that the order of tangency is exactly $K+1$ we must compute $F((1,\gamma),(K,1))=:f(\gamma,K)$, which amounts to

$$f(\gamma,K)=\frac{(-1)^K}{4}\sum_{j=0}^{K}\frac{1}{\prod_{l=1}^{K}\left(\langle\mathbf{k}^*-\mathbf{s}_l(\sigma_j)\rangle-\langle\mathbf{k}^*\rangle\right)} \tag{30}$$

and show that for $\gamma\notin\mathbb{Z}$, this does not vanish. For each of the paths σ_j, $j=0,\dots,K$, let α_j be the contribution to the sum in (30). Then

$$\alpha_j=\frac{1}{(K-1+\gamma)1}\frac{1}{(K-2+\gamma)2}\cdots\frac{1}{(K-j+\gamma)j}\frac{1}{(K-j)(j+\gamma)}\cdots\frac{1}{1(K-1+\gamma)},$$

where the total number of factors is K. Using the Gamma-function it follows that

$$\sum_{j=0}^{K}\alpha_j=\sum_{j=0}^{K}\frac{\Gamma(K-j+\gamma)\Gamma(j+\gamma)}{j!(K-j)!\Gamma(K+\gamma)\Gamma(K+\gamma)}.$$

Since

$$d(\gamma,K)=D((1,\gamma),(K,1))=(K-1+\gamma)1\cdot\ldots\cdot(\gamma+1)(K-1)\cdot(\gamma)(K)$$

it is clear that

$$d(\gamma,K)=\frac{\Gamma(K+\gamma)K!}{\Gamma(\gamma)}$$

implying that

$$d(\gamma,K)\sum_{j=0}^{K}\alpha_j=\sum_{j=0}^{K}\binom{K}{j}\frac{\Gamma(K-j+\gamma)\Gamma(j+\gamma)}{\Gamma(K+\gamma)\Gamma(\gamma)}=\frac{1}{\Gamma(K+\gamma)\Gamma(\gamma)}\frac{\Gamma(K+2\gamma)\Gamma(\gamma)^2}{\Gamma(2\gamma)},$$

where the last identity is an application of Pochhammer's formula, see [25]. Therefore, the relevant coefficient is

$$f(\gamma,K)=\frac{(-1)^K}{4}\sum_{j=0}^{K}\alpha_j=\frac{1}{D}\frac{\Gamma(\gamma)^2}{\Gamma(K+\gamma)\Gamma(\gamma)\Gamma(2\gamma)}\Gamma(K+2\gamma)$$

which, if 2γ is not a negative integer, is different from zero. $\square$

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