Three Dimensional p-q Resonant Orbits Close to Second Species Solutions

Esther Barrabés (barrabes@ima.udg.es)
Departament d'Informàtica i Matemàtica Aplicada, Universitat de Girona, Av. Lluís Santaló, s/n, 17071 Girona, Spain.
Tel: 34-972-418426 Fax: 34-972-418792

Gerard Gómez (gomez@cerber.mat.ub.es)
IEEC & Departament de Matemàtica Aplicada i Anàlisi, Universitat de Barcelona, Gran Via 585, 08007 Barcelona, Spain.

Abstract. The purpose of this paper is to study, for small values of μ, the three dimensional p-q resonant orbits that close to periodic Second Species Solutions (SSS) of the Restricted Three Body Problem. The work is based on an analytic study of the in– and out–maps. These maps are associated to follow, under the flow of the problem, initial conditions on a sphere of radius μ^α around the small primary, and consider the images of those initial points on the same sphere. The out–map is associated to follow the flow forward in time and the in–map backwards. For both mappings we give analytical expressions, in powers of the mass parameter. Once these expressions are obtained, we proceed to the study of the matching equations between both, obtaining initial conditions of orbits that will be "periodic" with an error of the order $\mu^{1-\alpha}$, for some $\alpha \in (1/3, 1/2)$. Since, as $\mu \to 0$, the *inner solution* and the *outer solution* will collide with the small primary, these orbits will be close to SSS.

Keywords: restricted three dimensional three-body problem, close encounters, periodic orbits, resonance

1. Introduction

Consider the 3D circular restricted three-body problem (RTBP), in which two primaries (E and M) are moving in circular orbits around their centre of masses and a third body (P), with infinitesimal mass, that moves under the gravitational force field created by the two primaries. According to Szebehely (1967), appropriate units of mass, length and time can be chosen so that the gravitational constant, the angular velocity of the primaries and their mutual distance are all equal to 1 and the masses of E and M are $1-\mu$ and μ, respectively. $\mu \in [0,1]$ is the so called mass parameter. From now on, M will be called the small primary and E the big one.

For the description of the problem, it is usual to consider a synodic reference system, $Oxyz$, in which the primaries are fixed on the x axis: E is located at $\mathbf{r}_E = (\mu, 0, 0)^T$ and M at $\mathbf{r}_M = (\mu - 1, 0, 0)^T$. If $\mathbf{r} =$

$(x, y, z)^T$ denotes the position of P in this reference frame, the equations of motion for P are (see Szebehely (1967))

$$\ddot{\mathbf{r}} + A\dot{\mathbf{r}} = \nabla\Omega(\mathbf{r}), \tag{1}$$

where

$$A = \begin{pmatrix} 0 & -2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \Omega(x, y, z) = \frac{1}{2}(x^2 + y^2) + \frac{1-\mu}{r_1} + \frac{\mu}{r_2}, \tag{2}$$

and r_1, r_2 denote the distances from P to E and M, respectively:

$$r_1^2 = (x - \mu)^2 + y^2 + z^2, \qquad r_2^2 = (x - \mu + 1)^2 + y^2 + z^2.$$

It is well known that the equations (1) have the first integral (Jacobi's integral)

$$|\dot{\mathbf{r}}|^2 = 2\,\Omega(x, y, z) \; - \; C_J, \tag{3}$$

where C_J is the Jacobi constant.

The purpose of this paper is to obtain, for small values of μ, three dimensional p-q resonant orbits close to periodic Second Species Solutions (SSS). The *p-q resonant orbits* are trajectories of the infinitesimal body P which, approximately, perform p revolutions around E while M does q revolutions around the centre of masses of the primaries. More precisely, these orbits leave at $t = t_1$ a sphere B in the configuration space, with centre at M and radius μ^α and return for the first time to the same sphere at an epoch t_2 such that

$$t_2 - t_1 \; = \; 2\pi q + \varepsilon\mu^\alpha + O(\mu^{2\alpha}) = 2\pi\tau p + \delta\mu^\alpha + O(\mu^{2\alpha}), \tag{4}$$

where $p, q \in \mathbb{N}$ are relatively prime, ϵ and δ are suitable constants and $2\pi\tau$ is the period of the elliptic orbit which approaches the motion of P far from M (see Font (2002)).

In the present study we will make use of some results obtained in a previous paper (Barrabés and Gómez (2001)), where we studied the set of initial conditions $(\mathbf{r}_i, \dot{\mathbf{r}}_i)$, with $\mathbf{r}_i \in B$, which correspond to spatial p-q resonant orbits. Some of these results will be summarised in Section 2. The rest of the paper has two parts: In the first one (Sections 3, 4 and 5), an analytic study of the so called out–map and in–map is done. These two maps correspond to the following: take initial conditions on the sphere B around M, and consider the return map to B by the flow associated to (1). The out–map is associated to follow the flow with $t > t_1$ and if $t < t_1$ we get the in–map. For both mappings we give analytical expansions, in suitable powers of the mass parameter μ. In the second part of the study (Section 6) we proceed to the study of the matching equations between both maps, obtaining initial conditions of orbits that will be "periodic" with an error of the order $\mu^{1-\alpha}$, for some $\alpha \in (1/3, 1/2)$. Since, as $\mu \to 0$, the *inner solution* and the *outer solution* will collide with the small primary, these orbits will be close to SSS.

The study of the outer solution is based on the results obtained in the above mentioned reference. There we proved that if the outer solution is approximated by the solution of the two body problem, with the same initial conditions, $(\mathbf{r}_i, \dot{\mathbf{r}}_i)$, then the error involved in the approximation is of order $O(\mu^{1-\alpha})$. This allows the derivation of an expression for the position and velocity at the return to B, $(\mathbf{r}_e, \dot{\mathbf{r}}_e)$, in terms of $(\mathbf{r}_i, \dot{\mathbf{r}}_i)$. This is the so called out-map which will be given in section 3.

In sections 4 and 5 we will study the behaviour of the inner solution, obtaining the expressions for the position and the velocity at the epoch at which the orbit leaves B, $(\mathbf{r}_f, \dot{\mathbf{r}}_f)$ (see Figure 1). Performing the composition of the inner and the outer solution (section 6), we will get the conditions under which the inner and outer solutions match up to a certain order in terms of μ^α. These will be the initial conditions of the orbits close to the Second Species Solutions.

Beyond their theoretical interest, the solutions studied can be used to perform gravity assist maneuvers. The usual ΔV-EGA" process is used to target a spacecraft to a given nominal orbit (around the large primary) at a low cost. The so called "reverse ΔV-EGA" process, in which multiple flybyes to a planet are done, can be used to reduce the Mercury approach energy, each time a spacecraft makes a near resonant return to Mercury for a gravity assist, reducing the orbit capture ΔV requirement (see Yen (1985)).

2. Summary of known results

In this section we will give the restrictions on the initial conditions and the Jacobi constant that must be fulfilled in order to have a p-q resonant orbit.

Since the initial conditions will be taken on the on the sphere B at $t_1 = 0$, they can be expressed in spherical coordinates as

$$\mathbf{r}_i = \begin{pmatrix} \mu - 1 + \mu^\alpha \cos\varphi \cos\theta \\ \mu^\alpha \cos\varphi \sin\theta \\ \mu^\alpha \sin\varphi \end{pmatrix}, \qquad \dot{\mathbf{r}}_i = v_i \begin{pmatrix} \cos\phi \cos\psi \\ \cos\phi \sin\psi \\ \sin\phi \end{pmatrix}. \tag{5}$$

There are several conditions to be imposed on them in order to get a p-q resonant orbit. First, it is necessary to guarantee that the orbit

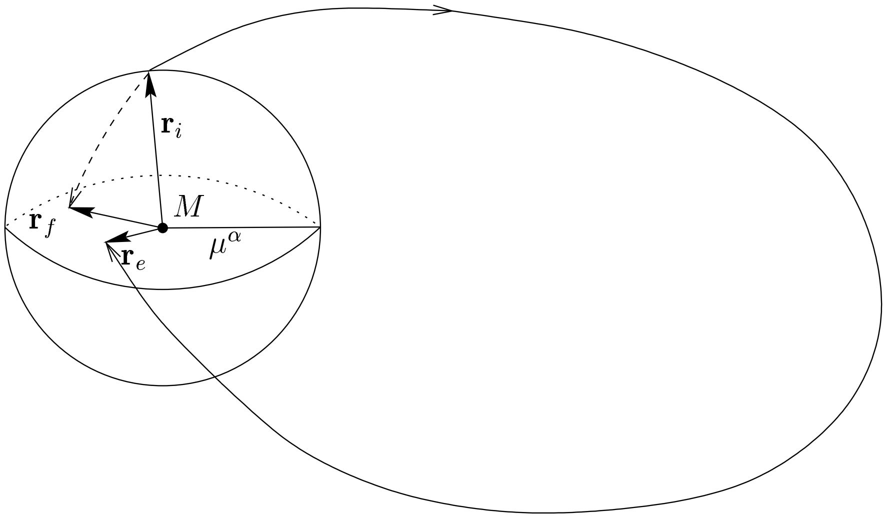

Figure 1. Qualitative representation of the outer and inner solutions. They both leave the sphere B at $\mathbf{r}_i$. The final approximated positions on the same sphere for the outer and inner solution are denoted by $\mathbf{r}_e$, $\mathbf{r}_f$, respectively.

leaves B at the initial epoch. This condition can be written as $\langle \mathbf{r}_{2i}, \dot{\mathbf{r}}_i \rangle > 0$ (where $\langle , \rangle$ denotes the scalar product and $\mathbf{r}_{2i} = \mathbf{r}_i - \mathbf{r}_M$) and is equivalent to

$$\cos a = \cos\varphi \, \cos\phi \, \cos(\theta - \psi) + \sin\varphi \, \sin\phi \, > 0,$$

where a is the angle between $\mathbf{r}_{2i}$ and $\mathbf{r}_M$. Moreover, the variables φ, θ, ϕ, ψ and v_i must be chosen in such a way that the return of P to the sphere B is guaranteed. Furthermore, the orbit should be a p-q resonant one, so we have a condition between the time of flight between $\mathbf{r}_i$ and $\mathbf{r}_e$ (equation (4)).

It is convenient to use the subscript '0' to denote the zero order term of any development in terms of μ^α. In this way

$$\begin{aligned} \phi &= \phi_0 + \Delta\phi \, \mu^\alpha + O(\mu^{2\alpha}), \\ \psi &= \psi_0 + \Delta\psi \, \mu^\alpha + O(\mu^{2\alpha}), \\ \cos a &= \cos a_0 + \Delta C \, \mu^\alpha + O(\mu^{2\alpha}), \end{aligned} \tag{6}$$

with

$$\begin{aligned} \cos a_0 &= \cos\varphi \, \cos\phi_0 \, \cos(\theta - \psi_0) + \sin\varphi \, \sin\phi_0, \\ \Delta C &= \Lambda_0 \Delta\psi + \Delta\phi \, (\sin\varphi \, \cos\phi_0 - \cos\varphi \, \sin\phi_0 \, \cos(\psi_0 - \theta)), \\ \Lambda_0 &= \cos\varphi \, \cos\phi_0 \, \sin(\theta - \psi_0). \end{aligned}$$

The above restrictions lead to the following set of equations (see Barrabés and Gómez (2001) for the details)

$$
\begin{aligned}
(\varepsilon-\delta)^2+\delta^2(3-C_J)+\,2\,(\varepsilon-\delta)\cos\varphi\,\sin\theta & \\
+\,2\delta\sqrt{3-C_J}\,\cos a_0+\,2(\varepsilon-\delta)\delta\,\sqrt{3-C_J}\cos\phi_0\,\sin\psi_0 &= 0, \qquad (7)\\
C_J-2+2\sqrt{3-C_J}\cos\phi_0\,\sin\psi_0 &= \left(\frac{p}{q}\right)^{2/3}, \qquad (8)\\
6\pi q\left(\frac{q}{p}\right)^{2/3}\Big(\sqrt{3-C_J}(\cos\varphi\,\cos\phi_0\,\sin(\psi_0-\theta) & \\
+\Delta\phi\,\sin\phi_0\,\sin\psi_0-\Delta\psi\,\cos\phi_0\,\cos\psi_0)-2\,\cos\varphi\,\cos\theta\Big) &= \varepsilon-\delta. \qquad (9)
\end{aligned}
$$

Fromequation (8) it follows that the Jacobi constant must be within the interval (C_{J1}, C_{J2}), with

$$
C_{J1}=\left(\frac{p}{q}\right)^{2/3}-2\sqrt{2-\left(\frac{p}{q}\right)^{2/3}},\quad C_{J2}=\left(\frac{p}{q}\right)^{2/3}+2\sqrt{2-\left(\frac{p}{q}\right)^{2/3}}. \qquad (10)
$$

The values C_{J1} and C_{J2} have to be excluded: if C_J was equal C_{J1} or C_{J2}, then $\cos^2\phi_0\,\sin^2\psi_0=1$ which is not acceptable (see Barrabés and Gómez (2001) for more details).

The next restriction follows form the fact that the elliptic orbit that approximates the p-q resonant orbit has no collisions

$$
\psi_0\neq 0, \qquad \text{or} \qquad C_J\neq\left(\frac{p}{q}\right)^{2/3}. \qquad (11)
$$

Finally, we observe that $v_i=|\dot{\mathbf{r}}_i|$ can be written in terms of μ^α using Jacobi's integral

$$
v_i^2=x_i^2+y_i^2+\frac{2(1-\mu)}{r_{1i}}+\frac{2\mu}{r_{2i}}-\,C_J.
$$

It is easy to see that $x_i^2+y_i^2=1-2\mu^\alpha\cos\varphi\,\cos\theta\,+O(\mu^{2\alpha})$ and that $r_{1i}^2=1-2\mu^\alpha\cos\varphi\,\cos\theta+\mu^{2\alpha}$, $r_{2i}=\mu^\alpha$, therefore,

$$
v_i^2=3-C_J+2\mu^{1-\alpha}+O(\mu^{2\alpha}). \qquad (12)
$$

3. Out-map

Given some initial conditions $(\mathbf{r}_i,\dot{\mathbf{r}}_i)$, with $\mathbf{r}_i\in B$, we will compute the position and velocity $(\mathbf{r}_e,\dot{\mathbf{r}}_e)$ at the return time, t_2, to the sphere

B. To this end, we will use that the solution of the RTBP can be approximated by a solution of the two body problem and that the error of the approximation is of the order $O(\mu^{1-\alpha})$.

Let $\mathbf{R}(t)$ be the solution of the RTBP problem in a sidereal reference system, $OXYZ$, with the same origin as the synodic one, and with initial conditions $(\mathbf{R}_i, \dot{\mathbf{R}}_i) = (\mathbf{R}(t_1), \dot{\mathbf{R}}(t_1))$, such that $|\mathbf{R}_i - \mathbf{R}_M(t_1)| = \mu^\alpha$, and by $\mathbf{R}_{TB}$ the solution of the two body problem

$$\ddot{\mathbf{R}} = -\frac{\mathbf{R}}{R^3},$$

with the same initial conditions. The following proposition (Barrabés (2001), Barrabés and Gómez (2001)) holds

PROPOSITION 3.1. *If there exists* $t_{\alpha/2}, t'_{\alpha/2} \in (t_1, t_2)$ *such that*

$$R_2(t) \geqslant \mu^{\alpha/2}, \quad \forall t \in (t_{\alpha/2}, t'_{\alpha/2}),$$

and $\cos a \geqslant \epsilon > 0$*, then*

$$\mathbf{R}(t) = \mathbf{R}_{TB}(t) + O(\mu^{1-\alpha}), \qquad \dot{\mathbf{R}}(t) = \dot{\mathbf{R}}_{TB}(t) + O(\mu^{1-\alpha}), \tag{13}$$

for all $t \in [t_1, t_2]$.

Using (4) and (13), we can compute the sidereal position and velocity at t_2. Since $\mu^{2\alpha} = O(\mu^{1-\alpha})$ and $(\mathbf{R}_i, \dot{\mathbf{R}}_i) = (\mathbf{R}_{TB}(t_1), \dot{\mathbf{R}}_{TB}(t_1))$

$$\begin{aligned}
\mathbf{R}(t_2) &= \mathbf{R}_{TB}(t_1 + 2\pi p\tau + \delta\mu^\alpha) + O(\mu^{1-\alpha}) \\
&= \mathbf{R}_{TB}(t_1 + 2\pi p\tau) + \dot{\mathbf{R}}_{TB}(t_1 + 2\pi p\tau)\mu^\alpha\delta + O(\mu^{1-\alpha}) \\
&= \mathbf{R}(t_1) + \dot{\mathbf{R}}(t_1)\,\delta\mu^\alpha + O(\mu^{1-\alpha}), \\
\dot{\mathbf{R}}(t_2) &= \dot{\mathbf{R}}_{TB}(t_1 + 2\pi p\tau + \delta\mu^\alpha) + O(\mu^{1-\alpha}) \\
&= \dot{\mathbf{R}}_{TB}(t_1) + \ddot{\mathbf{R}}_{TB}(t_1)\delta\,\mu^\alpha + O(\mu^{1-\alpha}) \\
&= \dot{\mathbf{R}}(t_1) - \frac{\mathbf{R}(t_1)}{R(t_1)^3}\,\delta\,\mu^\alpha + O(\mu^{1-\alpha}).
\end{aligned}$$

Since $|\mathbf{R}| = |\mathbf{r}|$ and using the Jacobi's integral (3), it can be shown that $\mathbf{R}_i = 1 + O(\mu^\alpha)$. Therefore, the out-map in sidereal coordinates is

$$\begin{aligned}
\mathbf{R}(t_2) &= \mathbf{R}_i + \dot{\mathbf{R}}_i\delta\,\mu^\alpha + O(\mu^{1-\alpha}), \\
\dot{\mathbf{R}}(t_2) &= \dot{\mathbf{R}}_i - \mathbf{R}_i\delta\,\mu^\alpha + O(\mu^{1-\alpha}).
\end{aligned} \tag{14}$$

Using the relation between sidereal and synodic coordinates

$$\mathbf{R} = G(t)\mathbf{r}, \quad G(t) = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

the above equations can be written in synodic coordinates. For the first equation in (14) we get

$$\begin{aligned}
\mathbf{r}(t_2) &= G^T(t_2)\Big(G(t_1)\mathbf{r}_i + \Big(G(t_1)\dot{\mathbf{r}}_i + \dot{G}(t_1)\mathbf{r}_i\Big)\delta\,\mu^\alpha\Big) + O(\mu^{1-\alpha}) \\
&= \begin{pmatrix} 1 & (\varepsilon-\delta)\mu^\alpha & 0 \\ -(\varepsilon-\delta)\mu^\alpha & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mathbf{r}_i + \delta\,\mu^\alpha\dot{\mathbf{r}}_i + O(\mu^{1-\alpha}).
\end{aligned}$$

For the second equation in (14) it follows that

$$\begin{aligned}
\dot{\mathbf{r}}(t_2) &= -G^T(t_2)\dot{G}(t_2)\mathbf{r}(t_2) \\
&\quad + G^T(t_2)\Big(G(t_1)\dot{\mathbf{r}}_i + \dot{G}(t_1)\mathbf{r}_i - \delta\,\mu^\alpha\,G(t_1)\mathbf{r}_i\Big) + O(\mu^{1-\alpha}) \\
&= \begin{pmatrix} 1 & (\varepsilon+\delta)\mu^\alpha & 0 \\ -(\varepsilon+\delta)\mu^\alpha & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \dot{\mathbf{r}}_i + O(\mu^{1-\alpha}).
\end{aligned}$$

Summarising, the out-map in synodic coordinates is

$$\mathbf{r}_e = \left(I - \frac{(\varepsilon-\delta)}{2}\,\mu^\alpha A\right)\mathbf{r}_i + \delta\,\mu^\alpha\dot{\mathbf{r}}_i, \tag{15a}$$

$$\dot{\mathbf{r}}_e = \left(I - \frac{(\varepsilon+\delta)}{2}\,\mu^\alpha A\right)\dot{\mathbf{r}}_i, \tag{15b}$$

where I is the 3×3 identity matrix.

It will be convenient for what follows, to have explicit expressions of the coordinates of the out-map. To this end, we define φ_e, θ_e, ϕ_e, ψ_e and v_e as the spherical coordinates of the position and the velocity, so

$$\mathbf{r}_e = \begin{pmatrix} \mu-1 \\ 0 \\ 0 \end{pmatrix} + r_{2e}\begin{pmatrix} \cos\varphi_e\,\cos\theta_e \\ \cos\varphi_e\,\sin\theta_e \\ \sin\varphi_e \end{pmatrix}, \qquad \dot{\mathbf{r}}_e = v_e\begin{pmatrix} \cos\phi_e\,\cos\psi_e \\ \cos\phi_e\,\sin\psi_e \\ \sin\phi_e \end{pmatrix},$$

where r_{2e} is the distance from P to M, that can be computed in terms of μ^α using (15a). In fact,

$$\begin{aligned}
r_{2e}^2 &= \mu^{2\alpha} - (\varepsilon-\delta)\mu^\alpha\langle\mathbf{r}_{2i}, A\mathbf{r}_i\rangle + 2\delta\mu^\alpha\langle\mathbf{r}_{2i},\dot{\mathbf{r}}_i\rangle \\
&\quad + \frac{(\varepsilon-\delta)^2}{4}\,\mu^{2\alpha}\langle A\mathbf{r}_i, A\mathbf{r}_i\rangle - \delta(\varepsilon-\delta)\mu^{2\alpha}\langle A\mathbf{r}_i,\dot{\mathbf{r}}_i\rangle + \delta^2\mu^{2\alpha}v_i^2.
\end{aligned}$$

Using (5), (6) and (12), the following identities can be obtained

$$\begin{aligned}
\langle A\mathbf{r}_i, A\mathbf{r}_i\rangle &= 4(1 - 2\mu^\alpha\cos\varphi\,\cos\theta + O(\mu^{2\alpha})), \\
\langle \mathbf{r}_{2i}, A\mathbf{r}_i\rangle &= -2\mu^\alpha\cos\varphi\,\sin\theta + O(\mu^{1+\alpha}), \\
\langle \mathbf{r}_{2i}, \dot{\mathbf{r}}_i\rangle &= \mu^\alpha\sqrt{3-C_J}\,\Big(\cos a_0 + \mu^\alpha\Delta C + O(\mu^{1-\alpha})\Big), \\
\langle A\mathbf{r}_i, \dot{\mathbf{r}}_i\rangle &= -2\sqrt{3-C_J}\,\Big(\cos\phi_0\,\sin\psi_0 \\
&\quad + \mu^\alpha(\Delta\psi\,\cos\phi_0\,\cos\psi_0 - \Delta\phi\sin\phi_0\,\sin\psi_0 + \Lambda_0)\Big) \\
&\quad + O(\mu^{1-\alpha}),
\end{aligned}$$

and, from them, it follows that

$$\begin{aligned} r_{2e}^2 &= \mu^{2\alpha}\Big[1 + 2\mu^\alpha\delta\,\sqrt{3-C_J}\Big(\Delta C + (\varepsilon-\delta)(\Delta\psi\,\cos\phi_0\,\cos\psi_0 \\ &\quad -\Delta\phi\sin\phi_0\,\sin\psi_0 + \Lambda_0)\Big) - 2\mu^\alpha(\varepsilon-\delta)^2\cos\varphi\,\cos\theta + O(\mu^{1-\alpha})\Big]. \end{aligned}$$

Thus, using (6), we get that $r_{2e} = \mu^\alpha\left[1 + \mu^\alpha\Delta r_e + O(\mu^{1-\alpha})\right]$ where

$$\begin{aligned} \Delta r_e &= \delta\sqrt{3-C_J}\,\Big[\Lambda_0(\varepsilon-\delta) + \Delta\psi\,\Big(\Lambda_0 + (\varepsilon-\delta)\cos\phi_0\,\cos\psi_0\Big) \\ &\quad +\Delta\phi\,\Big(\sin\varphi\,\cos\phi_0 - \sin\phi_0\,(\cos\varphi\,\cos(\psi_0-\theta) + (\varepsilon-\delta)\sin\psi_0)\Big)\Big] \\ &\quad -(\varepsilon-\delta)^2\cos\varphi\,\cos\theta. \end{aligned}$$

Therefore, from (15a) and the expression for r_{2e}, it can be shown (see Barrabés (2001)) that

$$\begin{aligned} \cos\varphi_e\,\cos\theta_e &= \cos\varphi\,\cos\theta + \delta\sqrt{3-C_J}\,\cos\phi_0\cos\psi_0 \\ &\quad -\mu^\alpha\Big[\delta\sqrt{3-C_J}\Big(\Delta\phi\,\sin\phi_0\cos\psi_0 + \Delta\psi\,\sin\psi_0\,\cos\phi_0\Big) \\ &\quad -(\varepsilon-\delta)\cos\varphi\,\sin\theta + \Delta r_e\Big(\cos\varphi\,\cos\theta \\ &\quad +\delta\sqrt{3-C_J}\,\cos\phi_0\cos\psi_0\Big)\Big] + O(\mu^{1-\alpha}), \quad (16a) \\ \cos\varphi_e\,\sin\theta_e &= \cos\varphi\,\sin\theta + \delta\sqrt{3-C_J}\,\cos\phi_0\,\sin\psi_0 + (\varepsilon-\delta) \\ &\quad +\mu^\alpha\Big[\delta\sqrt{3-C_J}\Big(\Delta\psi\,\cos\phi_0\,\cos\psi_0 - \Delta\phi\,\sin\psi_0\,\sin\phi_0\Big) \\ &\quad -(\varepsilon-\delta)\cos\varphi\,\cos\theta - \Delta r_e\Big(\cos\varphi\,\sin\theta + (\varepsilon-\delta) \\ &\quad +\delta\sqrt{3-C_J}\,\cos\phi_0\,\sin\psi_0\Big)\Big] + O(\mu^{1-\alpha}), \quad (16b) \\ \sin\varphi_e &= \sin\varphi + \delta\sqrt{3-C_J}\sin\phi_0 + \mu^\alpha\Big[\delta\sqrt{3-C_J}\Delta\phi\,\cos\phi_0 \\ &\quad -\Delta r_e\Big(\sin\varphi + \delta\sqrt{3-C_J}\sin\phi_0\Big)\Big] + O(\mu^{1-\alpha}). \quad (16c) \end{aligned}$$

As for the velocity it is easy to see that $v_e = v_i + O(\mu^{2\alpha})$ then, if $\cos\varphi \neq 0$, it follows from (15b) that

$$\begin{aligned} \phi_e &= \phi + O(\mu^{2\alpha}), \quad (17a) \\ \psi_e &= \psi_0 + (\Delta\psi - (\varepsilon+\delta))\mu^\alpha + O(\mu^{2\alpha}). \quad (17b) \end{aligned}$$

4. Inner Solution

As P approaches M, the influence of the big primary can be considered as a perturbation and inside the sphere B the orbit of P will be, approximately, an hyperbolic orbit. Since in this situation the orbit approaches one of the singularities of the system, it is convenient to remove it with a suitable regularization.

Using the regularising Kustaanheimo-Stiefel variables (Stiefel and Scheifele (1971)) the equations of motion are

$$\mathbf{u}'' = \frac{3 - C_J}{4}\,\mathbf{u} + O(\mu^\alpha).$$

Keeping only the linear terms of the above equation, and coming back to synodic coordinates, the error involved in the approximation is $O(\mu^{2\alpha})$ in position and $O(\mu^\alpha)$ in velocity. As the error in the approximation of the outer solution was of order $O(\mu^{1-\alpha})$, this linear approach will not be enough to match with the outer solution. Proceeding as in the planar case (Font (2002)), we will introduce, in addition to the regularization, two more changes of coordinates with which we will be able to get a suitable approximation of the inner solution.

4.1. Changes of variables and regularization

In a first step we introduce positions, $\mathbf{P} = (P_1, P_2, P_3)^T$ and momenta, $\mathbf{Q} = (Q_1, Q_2, Q_3)^T$, according to

$$\begin{aligned} \mathbf{P} &= \mathbf{r} - \mathbf{r}_M, \\ \mathbf{Q} &= \dot{\mathbf{r}} + \tfrac{1}{2} A \mathbf{r}_2, \end{aligned} \tag{18}$$

where A is given by (2). The equations of motion (1) in these coordinates become

$$\begin{aligned} \dot{\mathbf{P}} &= \mathbf{Q} - \tfrac{1}{2} A \mathbf{P}, \\ \dot{\mathbf{Q}} &= \nabla\Omega(\mathbf{r}) - A\dot{\mathbf{r}} + \tfrac{1}{2} A \dot{\mathbf{r}} \\ &= -\frac{1-\mu}{r_1^3}(\mathbf{P} - \mathbf{e}_1) - \frac{\mu}{r_2^3}\mathbf{P} - \frac{1}{2} A \mathbf{Q} + (\mu - 1)\mathbf{e}_1, \end{aligned} \tag{19}$$

being $\mathbf{e}_1 = (1,0,0)^T$, $r_1^2 = (P_1 - 1)^2 + P_2^2 + P_3^2$ and $r_2^2 = |\mathbf{P}|^2$. Furthermore, the Jacobi's integral (3) in these new coordinates is

$$|\mathbf{Q}|^2 - 2(P_1 Q_2 - P_2 Q_1) = (1-\mu)^2 - 2(1-\mu)P_1 + \frac{2(1-\mu)}{r_1} + \frac{2\mu}{|\mathbf{P}|} - C_J, \tag{20}$$

were we have used the following identities

$$
\begin{aligned}
2\Omega(\mathbf{r}) &= (1-\mu)^2 - 2(1-\mu)P_1 + P_1^2 + P_2^2 + \tfrac{2(1-\mu)}{r_1} + \frac{2\mu}{|\mathbf{P}|}, \\
\langle \dot{\mathbf{r}}, \dot{\mathbf{r}} \rangle &= \langle \mathbf{Q}, \mathbf{Q} \rangle - \langle \mathbf{Q}, A\mathbf{P} \rangle + \tfrac{1}{4}\langle A\mathbf{P}, A\mathbf{P} \rangle.
\end{aligned}
$$

In a second step we introduce an orthogonal transformation defined by

$$
\begin{aligned}
\widetilde{\mathbf{P}} &= G(t)\mathbf{P}, \\
\widetilde{\mathbf{Q}} &= G(t)\mathbf{Q},
\end{aligned} \tag{21}
$$

where $G(t)$ is the orthogonal matrix introduced in the preceding Section for performing the transformation from sidereal to synodic coordinates. Using $2\,\dot{G}(t)G(t)^T = A$ and $G(t)A = AG(t)$, the equations (19) become

$$
\begin{aligned}
\dot{\widetilde{\mathbf{P}}} &= \widetilde{\mathbf{Q}}, \\
\dot{\widetilde{\mathbf{Q}}} &= -\left(\frac{1-\mu}{r_1^3} + \frac{\mu}{|\widetilde{\mathbf{P}}|^3}\right)\widetilde{\mathbf{P}} + (1-\mu)\left(\frac{1}{r_1^3} - 1\right)G(t)\mathbf{e}_1,
\end{aligned} \tag{22}
$$

being $r_1^2 = |\widetilde{\mathbf{P}}|^2 + 1 - 2(\widetilde{P}_1 \cos t + \widetilde{P}_2 \sin t)$ and $r_2 = |\widetilde{\mathbf{P}}|$. The Jacobi's integral expression (20) becomes

$$
\begin{aligned}
|\widetilde{\mathbf{Q}}|^2 = \; & (1-\mu)^2 - 2(1-\mu)(\widetilde{P}_1 \cos t + \widetilde{P}_2 \sin t) + 2(\widetilde{P}_1\widetilde{Q}_2 - \widetilde{P}_2\widetilde{Q}_1) \\
& +2\frac{1-\mu}{r_1} + 2\frac{\mu}{|\widetilde{\mathbf{P}}|} - C_J.
\end{aligned} \tag{23}
$$

The last step is the regularization. Simultaneously, we will perform a change of scale in order to transform the neighbourhood of M of radius μ^α into a neighbourhood of radius 1. We will use the Kustaanheimo-Stiefel (KS) transformation and, as usual, we will add a zero fourth component to the vector $\widetilde{\mathbf{P}}$. The transformation is defined by

$$
\widetilde{\mathbf{P}} = \mu^\alpha L(\mathbf{u})\mathbf{u} = \mu^\alpha \begin{pmatrix} u_1 & -u_2 & -u_3 & u_4 \\ u_2 & u_1 & -u_4 & -u_3 \\ u_3 & u_4 & u_1 & u_2 \\ u_4 & -u_3 & u_2 & -u_1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}, \quad \frac{dt}{ds} = |\widetilde{\mathbf{P}}|. \tag{24}
$$

Notice that $L(\mathbf{u})^T L(\mathbf{u}) = |\mathbf{u}|^2$ and, consequently, $|\widetilde{\mathbf{P}}| = \mu^\alpha |\mathbf{u}|^2$. Therefore, $|\widetilde{\mathbf{P}}| \leqslant \mu^\alpha$ if and only if $|\mathbf{u}| \leqslant 1$, thus we want to study the solution of the RTBP in KS coordinates inside $B^* = \{\mathbf{u} \in \mathbb{R}^4;\ |\mathbf{u}| \leqslant 1\}$. Furthermore, since (1) is autonomous, we can consider $t_1 = 0$ and the relation between $|\widetilde{\mathbf{P}}|$ and $|\mathbf{u}|$ allows to write

$$
t = -\mu^\alpha \int_s^0 |\mathbf{u}(\tau)|^2 d\tau.
$$

We denote $' = d/ds$, so $\mathbf{u}' = d\mathbf{u}/ds$ and $d(L(\mathbf{u}))/ds = L(\mathbf{u}')$. To write the new equations of motion, we will make use of the following

LEMMA 4.1. (Stiefel and Scheifele (1971)) *Let* $\mathbf{u}, \mathbf{w} \in \mathbb{R}^4$ *satisfying the bilinear relation* $l(\mathbf{u}, \mathbf{w}) = u_1 w_4 - u_2 w_3 + u_3 w_2 - u_4 w_1 = 0$. *Then*

1. $L(\mathbf{u})\mathbf{w} = L(\mathbf{w})\mathbf{u}$
2. $\langle \mathbf{u}, \mathbf{u}\rangle L(\mathbf{w})\mathbf{w} - 2\langle \mathbf{u}, \mathbf{w}\rangle L(\mathbf{u})\mathbf{w} + \langle \mathbf{w}, \mathbf{w}\rangle L(\mathbf{u})\mathbf{u} = 0$

In order to use these two identities it is necessary that

$$l(\mathbf{u}(s), \mathbf{u}'(s)) = u_4' u_1 - u_3' u_2 + u_3 u_2' - u_4 u_1' = 0, \tag{25}$$

for all $\mathbf{u}(s)$, $\mathbf{u}'(s)$. In fact, since $l(\mathbf{u}, \mathbf{u}')$ is a first integral of the equations of motion, equation (25) has to be fulfilled at just one point, for instance at the initial condition ($s = 0$).

For the first equality in (22) and (24) we have $\widetilde{\mathbf{Q}} = (2/|\mathbf{u}|^2)\, L(\mathbf{u})\mathbf{u}'$. Deriving and using Lemma 4.1 we get

$$\dot{\widetilde{\mathbf{Q}}} = \frac{2\mu^{-\alpha}}{|\mathbf{u}|^4} \left(L(\mathbf{u})\mathbf{u}'' - \frac{|\mathbf{u}'|^2}{|\mathbf{u}|^2} L(\mathbf{u})\mathbf{u} \right). \tag{26}$$

As $|\mathbf{u}|$ is bounded, we can write that

$$\dot{\widetilde{\mathbf{Q}}} = -\mu^\alpha L(\mathbf{u})\mathbf{u} - \frac{\mu^{1-2\alpha}}{|\mathbf{u}|^6} L(\mathbf{u})\mathbf{u} + 3\mu^\alpha (u_x \cos t + u_y \sin t) G(t)\mathbf{e}_1 + O(\mu^{2\alpha}),$$

where $u_x = u_1^2 - u_2^2 - u_3^2 + u_4^2$ and $u_y = 2(u_1 u_2 - u_3 u_4)$. Equating this expression with the one obtained in (26), it follows that

$$L(\mathbf{u})\mathbf{u}'' - \frac{|\mathbf{u}'|^2}{|\mathbf{u}|^2} L(\mathbf{u})\mathbf{u} = -\frac{\mu^{1-\alpha}}{2|\mathbf{u}|^2} L(\mathbf{u})\mathbf{u} + O(\mu^{2\alpha}). \tag{27}$$

In order to get the equations of motion it will be necessary to introduce the Jacobi integral. Using (23), the following expression is obtained

$$\frac{|\mathbf{u}'|^2}{|\mathbf{u}|^2} = \frac{3 - C_J}{4} + \frac{\mu^{1-\alpha}}{2|\mathbf{u}|^2} + \frac{\mu^\alpha}{2|\mathbf{u}|^2}(u_x u_y' - u_x' u_y) + O(\mu^{2\alpha}), \tag{28}$$

which, together with (27), gives

$$\mathbf{u}'' = \frac{3 - C_J}{4}\,\mathbf{u} + \mu^\alpha\, \frac{u_x u_y' - u_x' u_y}{2|\mathbf{u}|^2}\,\mathbf{u} + O(\mu^{2\alpha}).$$

The term of order μ^α can be rewritten as

$$\frac{u_x u_y' - u_x' u_y}{2|\mathbf{u}|^2} = f(\mathbf{u}, \mathbf{u}') + \frac{2(u_2 u_4 + u_1 u_3) l(\mathbf{u}, \mathbf{u}')}{|\mathbf{u}|^2} = f(\mathbf{u}, \mathbf{u}'),$$

where $f(\mathbf{u}, \mathbf{u}') = -u_2 u_1' + u_1 u_2' - u_4 u_3' + u_3 u_4'$. Finally the equations of motion become

$$\mathbf{u}'' = c\,\mathbf{u} + \mu^\alpha f(\mathbf{u}, \mathbf{u}')\mathbf{u} + O(\mu^{2\alpha}), \tag{29}$$

where $c = (3 - C_J)/4$.

For the Jacobi's integral, using (28), we get

$$\frac{|\mathbf{u}'|^2}{|\mathbf{u}|^2} = c + \frac{\mu^{1-\alpha}}{2|\mathbf{u}|^2} + \mu^\alpha f(\mathbf{u}, \mathbf{u}') + O(\mu^{2\alpha}). \tag{30}$$

The initial conditions, $(\mathbf{u}_i, \mathbf{u}_i')$, are chosen in such a way that $\mathbf{u}_i$ is any solution of the equation $\widetilde{\mathbf{P}}_i = \mu^\alpha L(\mathbf{u}_i)\mathbf{u}_i$ and $\mathbf{u}_i'$ verifies $\mathbf{u}_i' = \frac{1}{2} L(\mathbf{u}_i)^T \widetilde{\mathbf{Q}}_i$. These initial conditions ensure that $l(\mathbf{u}_i, \mathbf{u}_i') = 0$, as required.

4.2. Approximation to the inner solution

Now, we want to find an approximated solution of the equations (29) inside B^*. The first thing we can do is to linearise. Thus, let $\boldsymbol{\zeta}(s) = (\mathbf{u}(s), \mathbf{u}'(s))^T$ be the solution of

$$\begin{cases} \boldsymbol{\zeta}' = C\boldsymbol{\zeta} + \mu^\alpha F(\boldsymbol{\zeta}) + O(\mu^{2\alpha}), \\ \qquad \boldsymbol{\zeta}(0) = \boldsymbol{\zeta}_i = \begin{pmatrix} \mathbf{u}_i \\ \mathbf{u}_i' \end{pmatrix}, \end{cases}$$

where $C = \begin{pmatrix} 0 & I_4 \\ cI_4 & 0 \end{pmatrix}$, I_4 the 4×4 identity matrix and $F(\boldsymbol{\zeta}) = \begin{pmatrix} 0 \\ f(\mathbf{u}, \mathbf{u}')\mathbf{u} \end{pmatrix}$. We also define $\boldsymbol{\zeta}_0(s) = (\mathbf{u}_0(s), \mathbf{u}_0'(s))^T$ as the solution of

$$\begin{cases} \boldsymbol{\zeta}' = C\boldsymbol{\zeta}, \\ \boldsymbol{\zeta}(0) = \boldsymbol{\zeta}_i. \end{cases}$$

We will use of the following version of Gronwall's lemma

LEMMA 4.2. *Let $g \in \mathcal{C}^1$ such that $g(t) \geqslant 0$.*

1. *If*

$$g(t) \leqslant K_0 t + K_1 \int_0^t g(\tau)\, d\tau \quad \text{for } t \geqslant 0,$$

with $K_0, K_1 > 0$, then $g(t) \leqslant K_0 t e^{K_1 t}$ for all $t \geqslant 0$.

2. *If*

$$g(t) \leqslant K_0 |t| + K_1 \int_t^0 g(\tau)\, d\tau \quad \text{for } t \leqslant 0,$$

with $K_0, K_1 > 0$, then $g(t) \leqslant K_0 |t| e^{K_1 |t|}$, for all $t \leqslant 0$.

Proof: For $\varepsilon > 0$ and $t \geq 0$, consider the function $h_\varepsilon(t) = (K_0+\varepsilon)te^{K_1 t}$. Clearly, it verifies the following properties

i) $h_\varepsilon(0) = 0 = g(0)$.

ii) $h'_\varepsilon(t) = (K_0+\varepsilon)e^{K_1 t}(1+K_1 t)$, $h'_\varepsilon(0) = K_0+\varepsilon$.

iii) $g'(0) = \lim_{t\to 0} \frac{g(t)}{t} \leqslant K_0 + \lim_{t\to 0} \frac{K_1 \int_0^t g(\tau)d\tau}{t} = K_0 + K_1 g(0) = K_0$.

In this way, $g'(0) < h'_\varepsilon(0)$ so, for small values of t $g'(t) < h'_\varepsilon(t)$ and also $g(t) < h_\varepsilon(t)$. We want to see that this inequality holds true for $t \geq 0$. Assume that there exists t_0 such that $g(t_0) = h_\varepsilon(t_0)$. Then

$$\begin{aligned} g(t_0) &= K_0 t_0 + K_1 \int_0^{t_0} g(\tau)d\tau < K_0 t_0 + K_1 \int_0^{t_0} h_\varepsilon(\tau)d\tau \\ &= K_0 t_0 + K_1(K_0+\varepsilon) \int_0^{t_0} \tau e^{K_1\tau} d\tau \\ &= K_0 t_0 + \frac{K_0+\varepsilon}{K_1}(K_1 t_0 e^{K_1 t_0} - e^{K_1 t_0} + 1) \\ &= h_\varepsilon(t_0) + K_0 t_0 + \frac{K_0+\varepsilon}{K_1}(1 - e^{K_1 t_0}). \end{aligned}$$

Fromthe Taylor expansion, it follows that $e^{K_1 t_0} > 1 + K_1 t_0$, and we get

$$g(t_0) \ < \ h_\varepsilon(t_0) + K_0 t_0 + \frac{K_0+\varepsilon}{K_1}(1 - 1 - K_1 t_0) = h_\varepsilon(t_0) - \varepsilon t_0 < h_\varepsilon(t_0),$$

which contradicts the initial hypothesis. So we have

$$g(t) < h_\varepsilon(t) = (K_0+\varepsilon)te^{K_1 t}, \quad \forall t \geqslant 0,$$

and letting $\varepsilon \to 0$ we get the desired result. To get the second assertion we have to apply this result to the function $h(-t) = g(t)$ for $t \leqslant 0$. $\square$

With the preceding notations and above Lemma we can establish the following

THEOREM 4.3. *For all s such that $|\mathbf{u}(s)| \leqslant 1$, $\boldsymbol{\zeta}(s)$ and $\boldsymbol{\zeta}_0(s)$ verify*

$$|\boldsymbol{\zeta}(s) - \boldsymbol{\zeta}_0(s)| \leqslant |s| K \mu^\alpha e^{\bar{c}|s|},$$

being $\bar{c} = \max(1, c)$.

Proof: Writing $\boldsymbol{\zeta}(s)$ and $\boldsymbol{\zeta}_0(s)$ as

$$\begin{aligned} \boldsymbol{\zeta}(s) &= \boldsymbol{\zeta}_i + \int_s^0 (C\boldsymbol{\zeta}(\tau) + \mu^\alpha F(\boldsymbol{\zeta}(\tau)) + O(\mu^{2\alpha}))\, d\tau, \\ \boldsymbol{\zeta}_0(s) &= \boldsymbol{\zeta}_i + \int_s^0 C\boldsymbol{\zeta}_0(\tau)\, d\tau, \end{aligned}$$

and subtracting both expressions, we have that

$$\begin{aligned}
|\boldsymbol{\zeta}(s)-\boldsymbol{\zeta}_0(s)| &\leqslant \int_s^0 |C(\boldsymbol{\zeta}(\tau)-\boldsymbol{\zeta}_0(\tau))|\,d\tau + \mu^\alpha \int_s^0 |F(\boldsymbol{\zeta}(\tau))|\,d\tau + K_0|s|\mu^{2\alpha} \\
&\leqslant \bar{c}\int_s^0 |\boldsymbol{\zeta}(\tau)-\boldsymbol{\zeta}_0(\tau)|\,d\tau + \mu^\alpha \int_s^0 |F(\boldsymbol{\zeta}(\tau))|\,d\tau + K_0|s|\mu^{2\alpha},
\end{aligned}$$

where $\bar{c} = \|C\| = \max(1,(3-C_J)/4)$. The term of order μ^α is bounded by

$$|F(\boldsymbol{\zeta})| \leqslant |\mathbf{u}|^2|\mathbf{u}'| \leqslant K_1,$$

since the motion takes place inside B^* and this implies (thanks to the Jacobi integral (30)) that $\mathbf{u}'$ is also bounded. Therefore

$$|\boldsymbol{\zeta}(s)-\boldsymbol{\zeta}_0(s)| \leqslant |s|K\mu^\alpha + \bar{c}\int_s^0 |\boldsymbol{\zeta}(\tau)-\boldsymbol{\zeta}_0(\tau)|\,d\tau,$$

and, using Lemma 4.2, the Theorem is proved. □

This results states that if we remain inside B^* for a bounded time, then

$$\boldsymbol{\zeta}(s) = \boldsymbol{\zeta}_0(s) + O(\mu^\alpha).$$

In order to know the magnitude of the error in synodic coordinates, it is necessary to undo the changes of coordinates. Starting from the equalities

$$\begin{aligned}
\mathbf{u}(s) &= \mathbf{u}_0(s) + O(\mu^\alpha), \\
\mathbf{u}'(s) &= \mathbf{u}_0'(s) + O(\mu^\alpha),
\end{aligned} \tag{31}$$

and assuming that $|\mathbf{u}_0|$ cannot be arbitrarily small, it is easy to see that

$$\begin{aligned}
\mathbf{r}_2(t) &= \mathbf{r}_{20}(t) + \mu^\alpha O(\mu^\alpha), \\
\dot{\mathbf{r}}(t) &= \dot{\mathbf{r}}_0(t) + O(\mu^\alpha),
\end{aligned}$$

where $(\mathbf{r}_0, \dot{\mathbf{r}}_0)$ are the vectors obtained from $(\mathbf{u}_0, \mathbf{u}_0')$. As it has already been said, we want to match the inner solution with the outer solution, which can be approximated by an elliptic orbit with an error of order $\mu^{1-\alpha}$. Therefore, the error obtained in the approximation of the inner solution is not enough to match with the approximation of the outer solution. It will be necessary to keep more terms in the equations (29). The problem is that the term of order μ^α in these equations, $f(\mathbf{u},\mathbf{u}')\mathbf{u}$, is not linear. For the purpose of writing the equations of motion in a way such that the non linear terms would be of order $\mu^{2\alpha}$, we will use the first approximation $\mathbf{u}_0$. The essential fact is the following

LEMMA 4.4. *Let* $f(\mathbf{u},\mathbf{u}') = -u_2u_1' + u_1u_2' - u_4u_3' + u_3u_4'$ *and* $(\mathbf{u}_0,\mathbf{u}_0')$ *the solution of*

$$\begin{aligned}\mathbf{u}'' &= c\mathbf{u}\\ \mathbf{u}(0) &= \mathbf{u}_i,\\ \mathbf{u}'(0) &= \mathbf{u}_i'.\end{aligned}$$

Then $f(\mathbf{u}_0(s),\mathbf{u}_0(s)) = f(\mathbf{u}_i,\mathbf{u}_i')\ \forall\, s$.

Proof: To proof the Lemma we must just use

$$\begin{aligned}\mathbf{u}_0(s) &= \cosh(\sqrt{c}\, s)\,\mathbf{u}_i + \frac{\sinh(\sqrt{c}\, s)}{\sqrt{c}}\,\mathbf{u}_i',\\ \mathbf{u}_0'(s) &= \sqrt{c}\,\sinh(\sqrt{c}\, s)\,\mathbf{u}_i + \cosh(\sqrt{c}\, s)\,\mathbf{u}_i',\end{aligned}$$

to compute $f(\mathbf{u}_0(s),\mathbf{u}_0(s))$. □

In order to use this lemma, assume that the time spent by the third body inside B^* is bounded, so we can use the equalities (31). Then

$$f(\mathbf{u}(s),\mathbf{u}'(s)) = f(\mathbf{u}_0(s),\mathbf{u}_0'(s)) + O(\mu^\alpha) = f(\mathbf{u}_i,\mathbf{u}_i') + O(\mu^\alpha),$$

and, moving back to equations (29), we can write

$$\begin{aligned}\mathbf{u}'' &= c\mathbf{u} + \mu^\alpha f(\mathbf{u},\mathbf{u}')\mathbf{u} + O(\mu^{2\alpha}) = c\mathbf{u} + \mu^\alpha f(\mathbf{u}_i,\mathbf{u}_i')\mathbf{u} + O(\mu^{2\alpha})\\ &= (c + \mu^\alpha f(\mathbf{u}_i,\mathbf{u}_i'))\mathbf{u} + O(\mu^{2\alpha}).\end{aligned}$$

Consequently, we can state that $\mathbf{u}(s)$ is a solution of

$$\begin{aligned}\mathbf{u}'' &= c_\alpha\mathbf{u} + O(\mu^{2\alpha}),\\ \mathbf{u}(0) &= \mathbf{u}_i,\\ \mathbf{u}'(0) &= \mathbf{u}_i',\end{aligned}$$

being $c_\alpha = (3 - C_J)/4 + \mu^\alpha f(\mathbf{u}_i,\mathbf{u}_i')$. As before, we denote by $\mathbf{u}_h(s)$ the solution of

$$\begin{aligned}\mathbf{u}'' &= c_\alpha\mathbf{u},\\ \mathbf{u}(0) &= \mathbf{u}_i,\\ \mathbf{u}'(0) &= \mathbf{u}_i',\end{aligned}$$

and $\boldsymbol{\zeta}_h(s) = (\mathbf{u}_h(s),\mathbf{u}_h'(s))^T$. Now, using the same arguments as in Theorem 4.3, the following result can be proved

THEOREM 4.5. *Let* $\boldsymbol{\zeta}(s)$ *and* $\boldsymbol{\zeta}_h(s)$ *defined as before, and suppose that the time spent by the third body inside* B^* *is bounded. Then*

$$|\boldsymbol{\zeta}(s) - \boldsymbol{\zeta}_h(s)| \leqslant |s| K\mu^{2\alpha} e^{\bar{c}_\alpha |s|},$$

for all s *such that* $|\mathbf{u}(s)| \leqslant 1$ *and* $\bar{c}_\alpha = \max(c_\alpha, 1)$.

In this Theorem we have assumed that the time spent by the third body inside B^* is bounded. If this was not the case, then the error of the first approximation should not be of order $O(\mu^\alpha)$ and the equations of motion could not be written as $\mathbf{u}'' = c_\alpha \mathbf{u} + O(\mu^{2\alpha})$. In what follows we will find which conditions are required in order to satisfy this condition.

If the time spend in B^* by the third body is bounded, then the time spend by the hyperbolic approximation, $\mathbf{u}_h$, will also be bounded. We know that this approximation can be written in terms of the initial condition as

$$\begin{aligned} \mathbf{u}_h(s) &= \cosh(\sqrt{c_\alpha}\, s)\, \mathbf{u}_i + \frac{\sinh(\sqrt{c_\alpha}\, s)}{\sqrt{c_\alpha}}\, \mathbf{u}'_i, \\ \mathbf{u}'_h(s) &= \sqrt{c_\alpha}\, \sinh(\sqrt{c_\alpha}\, s)\, \mathbf{u}_i + \cosh(\sqrt{c_\alpha}\, s)\, \mathbf{u}'_i. \end{aligned} \tag{32}$$

The first equality allows the computation of a solution, s_f, of the equation $|\mathbf{u}_h(s)| = 1$, in terms of $(\mathbf{u}_i, \mathbf{u}'_i)$. Denoting by η_i the angle between the vectors $\mathbf{u}_i$ and $\mathbf{u}'_i$ and using the following notation:

$$V_i = |\mathbf{u}'_i|, \qquad m_\alpha = 1 + \frac{V_i^2}{c_\alpha}, \qquad n_\alpha = 2\frac{V_i \cos\eta_i}{\sqrt{c_\alpha}},$$

it can be shown that

$$s_f = \frac{1}{2\sqrt{c_\alpha}} \ln\left(\frac{m_\alpha - n_\alpha}{m_\alpha + n_\alpha}\right). \tag{33}$$

Recall that $s_f < 0$, thus, necessarily, $\frac{m_\alpha - n_\alpha}{m_\alpha + n_\alpha} < 1$. For this it will be enough that $n_\alpha > 0$. In order to check this condition we will compute $\langle \mathbf{u}_i, \mathbf{u}'_i\rangle$. Recall that, by the definitions of η_i, V_i and as $|\mathbf{u}_i| = 1$, we have that $\langle \mathbf{u}_i, \mathbf{u}'_i\rangle = V_i \cos\eta_i$. On the other hand, using the changes of coordinates, it can be seen that

$$\begin{aligned} \langle \mathbf{u}_i, \mathbf{u}'_i\rangle &= \langle \mathbf{u}_i, \frac{1}{2} L(\mathbf{u}_i)^T \widetilde{\mathbf{Q}}_i\rangle = \frac{1}{2}\langle L(\mathbf{u}_i)\mathbf{u}_i, \widetilde{\mathbf{Q}}_i\rangle = \frac{1}{2}\langle \widetilde{\mathbf{P}}_i, \widetilde{\mathbf{Q}}_i\rangle \\ &= \frac{1}{2\mu^\alpha}\langle \mathbf{P}_i, \mathbf{Q}_i\rangle = \frac{1}{2\mu^\alpha}\langle \mathbf{r}_{2i}, \dot{\mathbf{r}}_i + \frac{1}{2}A_3 \mathbf{r}_{2i}\rangle \\ &= \frac{1}{2\mu^\alpha}\langle \mathbf{r}_{2i}, \dot{\mathbf{r}}_i\rangle = \frac{1}{2} v_i \cos a, \end{aligned}$$

or, equivalently, $V_i \cos\eta_i = \frac{1}{2} v_i \cos a$. As one of the restrictions on the initial conditions is $\cos a > 0$, we obtain that $n_\alpha > 0$.

To ensure that s_f is bounded, we write (33) in terms of μ^α. The next expressions will be useful:

– from the definition of c_α:

$$\frac{1}{\sqrt{c_\alpha}} = \frac{1}{\sqrt{c}}\left(1 - \frac{f(\mathbf{u}_i, \mathbf{u}'_i)}{2c}\,\mu^\alpha + O(\mu^{2\alpha})\right),$$

– from the Jacobi's integral:

$$V_i = \sqrt{c}\left(1 + \frac{f(\mathbf{u}_i, \mathbf{u}_i')}{2c}\,\mu^\alpha + \frac{\mu^{1-\alpha}}{4c} + O(\mu^{2\alpha})\right),$$

– from the relation between $\cos\eta_i$ and $\cos a$:

$$\cos\eta_i = \cos a\left(1 - \frac{f(\mathbf{u}_i, \mathbf{u}_i')}{2c}\,\mu^\alpha + O(\mu^{2\alpha})\right).$$

Thus, we obtain:

$$\begin{aligned} m_\alpha + n_\alpha &= 2(1+\cos a) - 2\cos a\,\frac{f(\mathbf{u}_i, \mathbf{u}_i')}{2c}\,\mu^\alpha + 2(1+\cos a)\,\frac{\mu^{1-\alpha}}{4c} + O(\mu^{2\alpha}),\\ m_\alpha - n_\alpha &= 2(1-\cos a) + 2\cos a\,\frac{f(\mathbf{u}_i, \mathbf{u}_i')}{2c}\,\mu^\alpha + 2(1-\cos a)\,\frac{\mu^{1-\alpha}}{4c} + O(\mu^{2\alpha}). \end{aligned}$$

Since the expression $1+\cos a$ is bounded ($1 \leqslant 1+\cos a \leqslant 2$), the only problems can appear in the numerator of (33), because $1-\cos a$ could take values close to zero. To ensure that s_f is bounded, it will be necessary to require that

$$1 - \cos a \geqslant \epsilon' > 0,$$

for some positive ϵ' independent of μ. With this hypothesis, we can conclude that

$$s_f = \frac{1}{2\sqrt{c}}\ln\left(\frac{1-\cos a}{1+\cos a}\right) + O(\mu^\alpha). \tag{34}$$

Moreover, when the error involved in the approximation using KS coordinates was computed in synodic coordinates, it was required that the distance to the origin had not to be too small. If we only worry about the approximation at time s_f, then this hypothesis is not necessary. However, if we want to ensure that

$$\begin{aligned} \mathbf{r}(t) &= \mathbf{r}_h(t) + O(\mu^{3\alpha}),\\ \mathbf{r}'(t) &= \mathbf{r}_h'(t) + O(\mu^{2\alpha}), \end{aligned}$$

for all the time P is inside B, the distance $|\mathbf{u}_h(s)|$ cannot be close to zero. The minimum distance to the origin can be written as

$$|\mathbf{u}_h(s_m)|^2 = 1 - \frac{1}{2}\left(m_\alpha - \sqrt{m_\alpha^2 - n_\alpha^2}\right).$$

Again, if $1-\cos a \geqslant \epsilon' > 0$, then this minimum distance will not be near zero. Consequently, from now on, we assume that $|a| \geqslant \epsilon > 0$ for some ϵ independent of μ. With this hypothesis, we are avoiding the case in which the third body leaves the sphere B in the direction of the radius vector. Observe that if we choose this direction, when we move back in time we go towards the centre of the sphere.

5. In-Map

We want to describe the position and the velocity vectors after passing through B (going back in time) in terms of the initial conditions $(\mathbf{r}_i, \dot{\mathbf{r}}_i)$. For this, we are going to use the hyperbolic approximation already introduced. We define the in-map as

$$\begin{pmatrix} \mathbf{r}_f \\ \dot{\mathbf{r}}_f \end{pmatrix} = \begin{pmatrix} \mathbf{r}_h(t_f) \\ \dot{\mathbf{r}}_h(t_f) \end{pmatrix},$$

where $(\mathbf{r}_h, \dot{\mathbf{r}}_h)$ are the synodic coordinates associated to the approximation $(\mathbf{u}_h, \mathbf{u}'_h)$ and t_f is defined by

$$t_f = -\mu^\alpha \int_{s_f}^{0} |\mathbf{u}(s)|^2 \, ds.$$

In order to write $(\mathbf{r}_f, \dot{\mathbf{r}}_f)$ in terms of $(\mathbf{r}_i, \dot{\mathbf{r}}_i)$, we have to compute $(\mathbf{u}_h(s_f),\ \mathbf{u}'_h(s_f))$ and undo the changes of variables introduced in the previous section. Using (32) and (33), we obtain

$$\begin{aligned} \mathbf{u}_f = \mathbf{u}_h(s_f) &= \frac{1}{\Delta_\alpha}\left(\left(1+\frac{V_i^2}{c_\alpha}\right)\mathbf{u}_i - 2\frac{V_i\cos\eta_i}{c_\alpha}\mathbf{u}'_i\right), \\ \mathbf{u}'_f = \mathbf{u}'_h(s_f) &= \frac{1}{\Delta_\alpha}\left(-2V_i\cos\eta_i\,\mathbf{u}_i + \left(1+\frac{V_i^2}{c_\alpha}\right)\mathbf{u}'_i\right), \end{aligned} \tag{35}$$

being

$$\Delta_\alpha = \sqrt{m_\alpha^2 - n_\alpha^2} = \sqrt{\left(1+\frac{V_i^2}{c_\alpha}\right)^2 - 4\frac{V_i^2\cos^2\eta_i}{c_\alpha}}.$$

Let us start with the third change of variables defined in (24) and denote by $(\widetilde{\mathbf{P}}_f, \widetilde{\mathbf{Q}}_f)$ the vectors associated to $(\mathbf{u}_f, \mathbf{u}'_f)$. Since $|\mathbf{u}_f| = 1$, these vectors can be written as

$$\widetilde{\mathbf{Q}}_f \;=\; 2L(\mathbf{u}_f)\mathbf{u}'_f.$$

Using (35), the fact that $L(\cdot)$ is linear and Lemma 4.1, the following expressions can be derived:

$$\begin{aligned} \widetilde{\mathbf{P}}_f &= \frac{\mu^\alpha}{\Delta_\alpha^2}\Big(T_1 L(\mathbf{u}_i)\mathbf{u}_i + T_2 L(\mathbf{u}_i)\mathbf{u}'_i\Big) &= \frac{1}{\Delta_\alpha^2}\Big(T_1\mathbf{P}_i + \frac{\mu^\alpha}{2}T_2\mathbf{Q}_i\Big), \\ \widetilde{\mathbf{Q}}_f &= \frac{2}{\Delta_\alpha^2}\Big(T_3 L(\mathbf{u}_i)\mathbf{u}_i + T_1 L(\mathbf{u}_i)\mathbf{u}'_i\Big) &= \frac{1}{\Delta_\alpha^2}\Big(2\mu^{-\alpha}T_3\mathbf{P}_i + T_1\mathbf{Q}_i\Big), \end{aligned}$$

where the coefficients T_i, $i=1,2,3$ are defined by

$$
\begin{aligned}
T_1 &= \left(1+\frac{V_i^2}{c_\alpha}\right)^2 - 4\,\frac{V_i^4\cos^2\eta_i}{c_\alpha^2},\\
T_2 &= 4\,\frac{V_i\cos\eta_i}{c_\alpha}\left(\frac{2V_i^2\cos^2\eta_i}{c_\alpha}-1-\frac{V_i^2}{c_\alpha}\right),\\
T_3 &= -2V_i\cos\eta_i\left(1-\frac{V_i^4}{c_\alpha^2}\right).
\end{aligned}
$$

Now, let us undo the rotation introduced in (21). We denote

$$
\begin{aligned}
\mathbf{P}_f &= G(t_f)^T\widetilde{\mathbf{P}}_f,\\
\mathbf{Q}_f &= G(t_f)^T\widetilde{\mathbf{Q}}_f.
\end{aligned}
$$

It will be necessary to determine the matrix $G(t_f)$. As t_f depends explicitly on $\mathbf{u}(s)$ (the solution of the RTBP, which it is unknown) we will have to use the approximation $\mathbf{u}_h(s)$ and (34) to give an estimation of the final time t_f. After some computations, one gets

$$t_f = -\mu^\alpha\frac{\cos a}{\sqrt{c}} + O(\mu^{2\alpha}).$$

Therefore, we can write

$$
\begin{aligned}
\mathbf{P}_f &= \left(I+\frac{\cos a}{2\sqrt{c}}\,\mu^\alpha A\right)\widetilde{\mathbf{P}}_f + O(\mu^{3\alpha}),\\
\mathbf{Q}_f &= \left(I+\frac{\cos a}{2\sqrt{c}}\,\mu^\alpha A\right)\widetilde{\mathbf{Q}}_f + O(\mu^{2\alpha}).
\end{aligned}
\tag{36}
$$

Now we must expand the coefficients T_i of $\widetilde{\mathbf{P}}_f$ and $\widetilde{\mathbf{Q}}_f$, in powers of μ^α. Inserting these expansions in (36) (see Barrabés (2001) for the details) one gets

$$
\begin{aligned}
\mathbf{P}_f = {} & \left(I+\frac{\cos a}{2\sqrt{c}}\,\mu^\alpha A-\frac{\cot^2 a}{2c}\,\mu^{1-\alpha}I\right)\mathbf{P}_i\\
&+\frac{\cos a}{\sqrt{c}}\,\mu^\alpha\left(-I+\Big(\frac{f(\mathbf{u}_i,\mathbf{u}_i')}{c}\,I-\frac{\cos a}{2\sqrt{c}}\,A\Big)\mu^\alpha+\frac{\cot^2 a}{4c}\,\mu^{1-\alpha}I\right)\mathbf{Q}_i\\
&+O(\mu^{3\alpha}),\\
\mathbf{Q}_f = {} & \left(\frac{\cos a}{\sin^2 a}\,\frac{\mu^{1-\alpha}}{\sqrt{c}}\right)(\mu^{-\alpha}\mathbf{P}_i)+\left(I+\frac{\cos a}{2\sqrt{c}}\,\mu^\alpha A-\frac{\cot^2 a}{2c}\,\mu^{1-\alpha}I\right)\mathbf{Q}_i\\
&+O(\mu^{2\alpha}).
\end{aligned}
$$

At this point, it remains only one change of coordinates to deal with. Using (18) and the fact that $|\mathbf{r}_{2i}|=\mu^\alpha$, finally we arrive to the expressions

$$
\begin{aligned}
\mathbf{r}_{2f} = & \left(1 - \frac{\cot^2 a}{2c}\,\mu^{1-\alpha}\right)\,\mathbf{r}_{2i} \\
& + \frac{\cos a}{\sqrt{c}}\,\mu^{\alpha}\left(-1 + \frac{f(\mathbf{u}_i, \mathbf{u}_i')}{c}\,\mu^{\alpha} + \frac{\cot^2 a}{4c}\,\mu^{1-\alpha}\right)\dot{\mathbf{r}}_i \\
& - \frac{\cos^2 a}{2c}\,\mu^{2\alpha} A\dot{\mathbf{r}}_i + O(\mu^{3\alpha}), \\
\dot{\mathbf{r}}_f = & \left(\frac{\cos a}{\sin^2 a}\,\frac{\mu^{1-\alpha}}{\sqrt{c}}\right)(\mu^{-\alpha}\mathbf{r}_{2i}) \\
& + \left(I + \frac{\cos a}{\sqrt{c}}\,\mu^{\alpha} A - \frac{\cot^2 a}{2c}\,\mu^{1-\alpha} I\right)\dot{\mathbf{r}}_i + O(\mu^{2\alpha}),
\end{aligned}
\tag{37}
$$

and it can be shown that

$$
r_{2f} = |\mathbf{r}_f - \mathbf{r}_M| = \mu^{\alpha}\Big(1 + O(\mu^{2\alpha})\Big), \qquad v_f = 2\sqrt{c}\left(1 + \frac{\mu^{1-\alpha}}{4c} + O(\mu^{2\alpha})\right).
$$

Let us conclude this section specifying each coordinate of the in-map. We define, as for the out-map, the spherical coordinates φ_f, θ_f, ϕ_f, ψ_f and v_f

$$
\mathbf{r}_f = \begin{pmatrix} \mu - 1 \\ 0 \\ 0 \end{pmatrix} + r_{2f}\begin{pmatrix} \cos\varphi_f\,\cos\theta_f \\ \cos\varphi_f\,\sin\theta_f \\ \sin\varphi_f \end{pmatrix}, \quad \dot{\mathbf{r}}_f = v_f\begin{pmatrix} \cos\phi_f\,\cos\psi_f \\ \cos\phi_f\,\sin\psi_f \\ \sin\phi_f \end{pmatrix}.
$$

From(37) we can establish the relations between the angles at the arrival point and the spherical coordinates of the initial conditions given in (5). For the positions, these relations are given by the following equalities

$$
\begin{aligned}
\cos\varphi_f\ \cos\theta_f = &\ \cos\varphi\ \cos\theta\ - 2\cos a_0\ \cos\phi_0\ \cos\psi_0 \\
&+2\mu^\alpha\Big[\cos a_0\ (\Delta\psi\ \cos\phi_0\sin\psi_0 + \Delta\phi\ \sin\phi_0\cos\psi_0) \\
&-\Delta C\ \cos\phi_0\cos\psi_0 + \frac{\cos a_0}{\sqrt{c}}\ \cos\phi_0\ (\cos a_0\sin\psi_0 \\
&-\Lambda_0\cos\psi_0)\Big] + O(\mu^{1-\alpha}), \\
\cos\varphi_f\ \sin\theta_f = &\ \cos\varphi\ \sin\theta\ - 2\cos a_0\ \cos\phi_0\ \sin\psi_0 \\
&-2\mu^\alpha\Big[\cos a_0\ (\Delta\psi\ \cos\phi_0\cos\psi_0 - \Delta\phi\ \sin\phi_0\sin\psi_0) \\
&+\Delta C\ \cos\phi_0\sin\psi_0 + \frac{\cos a_0}{\sqrt{c}}\ \cos\phi_0\ (\cos a_0\cos\psi_0, \\
&+\Lambda_0\sin\psi_0)\Big] + O(\mu^{1-\alpha}), \\
\sin\varphi_f = &\ \sin\varphi\ - 2\cos a_0\sin\phi_0 \\
&-2\mu^\alpha\Big[\Delta\phi\ \cos a_0\cos\phi_0 + \Delta C\ \sin\phi_0 + \frac{\cos a_0}{\sqrt{c}}\,\Lambda_0\sin\phi_0\Big] \\
&+O(\mu^{1-\alpha}),
\end{aligned}
\tag{38}
$$

where Λ_0 and ΔC are defined in (6). For the velocities, the relations obtained are

$$
\begin{aligned}
v_f\ &= v_i + O(\mu^{2\alpha}), \\
\phi_f\ &= \phi + O(\mu^{2\alpha}), \\
\psi_f\ &= \psi + \Big(\Delta\psi + 2\,\frac{\cos a_0}{\sqrt{c}}\Big)\mu^\alpha + O(\mu^{2\alpha}).
\end{aligned}
\tag{39}
$$

6. Matching maps

The aim of this section is to get initial conditions of spatial p-q resonant orbits close to periodic orbits. For this goal we will use the in and the out maps, which give the return positions and velocities to the sphere B after passing inside and outside B, respectively (see Figure 1). To get periodic orbits it would be necessary that both maps were equal. As we have their explicit expressions up to order $\mu^{1-\alpha}$, we will only match the terms of order zero and order μ^α of both maps.

We will find restrictions on the variables θ, φ, ψ_0, ϕ_0, $\Delta\psi$, $\Delta\phi$ and the Jacobi's constant C_J in order to fulfill the matching equations. To ensure the orbit is p-q resonant, the condition (7) will be added and we will also discard those initial that do not verify $0 < \cos a_0 < 1$.

Thus, let us take the expressions (16a), (16b), (16c) and (17a) for the out map and (38), (39) for the in map (notice that it is sufficient to lead with these expressions because the final distances from P to M, r_{2e} and r_{2f}, and the modulus of the final velocities, v_e and v_f, are equal up to order $\mu^{2\alpha}$). First, we will match the zero-order terms from the final positions. We get, denoting by $C = \delta\sqrt{3 - C_J}$

$$\begin{gathered}
\cos\varphi\,\cos\theta + C\,\cos\phi_0\cos\psi_0 = \cos\varphi\,\cos\theta\,-2\cos a_0\,\cos\phi_0\,\cos\psi_0,\\
\cos\varphi\sin\theta + C\cos\phi_0\sin\psi_0 + \varepsilon - \delta = \cos\varphi\sin\theta - 2\cos a_0\cos\phi_0\,\sin\psi_0,\\
\sin\varphi + C\sin\phi_0 = \sin\varphi\,-2\cos a_0\sin\phi_0.
\end{gathered}$$

Fromthem we infer that $\varepsilon - \delta = 0$ and

$$\varepsilon = \delta = -\frac{\cos a_0}{\sqrt{c}}. \tag{40}$$

Observe that ε and δ verify the p-q resonant condition (7). On another hand, using $\varepsilon - \delta = 0$ in (9) we get the equation

$$-\cos\phi_0\cos\psi_0\,\Delta\psi + \sin\phi_0\sin\psi_0\,\Delta\phi = \frac{\cos\varphi\,\cos\theta}{\sqrt{c}} + \Lambda_0, \tag{41}$$

being $\Lambda_0 = \cos\varphi\,\cos\phi_0\sin(\theta - \psi_0)$.

From(39) and (17a) we see that the zero-order terms are equal and the terms of order μ^α match if

$$\varepsilon + \delta = -2\,\frac{\cos a_0}{\sqrt{c}},$$

which is true according to (40). So, the expressions of the velocities do not bring any new information.

Let us take the terms of order μ^α of the coordinates of the final positions (16a), (16b), (16c) and (38). We denote by O_1, O_2 and O_3 the coefficients of the terms of order μ^α in the developments of the out map and I_1, I_2 and I_3 the coefficients of the in map. Then, using the values founded for ε and δ we get

$$
\begin{aligned}
O_1 = \;& 2\cos a_0\Big(\Delta\psi\,\big(\sin\psi_0\cos\phi_0 + \Lambda_0(\cos\varphi\,\cos\theta\, - 2\cos a_0\cos\phi_0\cos\psi_0)\big)\\
&+\Delta\phi\,\big(\sin\phi_0\,\cos\psi_0 + (\sin\varphi\,\cos\phi_0 - \cos\varphi\,\sin\phi_0\cos(\psi_0-\theta))\\
&\times(\cos\varphi\,\cos\theta\, - 2\cos a_0\cos\phi_0\cos\psi_0)\big)\Big),\\
O_2 = \;& -2\cos a_0\Big(\Delta\psi\,\big(\cos\psi_0\cos\phi_0 - \Lambda_0(\cos\varphi\,\sin\theta\, - 2\cos a_0\cos\phi_0\sin\psi_0)\big)\\
&-\Delta\phi\,\big(\sin\phi_0\,\sin\psi_0 + (\sin\varphi\,\cos\phi_0 - \cos\varphi\,\sin\phi_0\cos(\psi_0-\theta))\\
&\times(\cos\varphi\,\sin\theta\, - 2\cos a_0\cos\phi_0\sin\psi_0)\big)\Big),\\
O_3 = \;& -2\cos a_0\Big(\Delta\phi\,\big(\cos\phi_0 - (\sin\varphi\,\cos\phi_0 - \cos\varphi\,\sin\phi_0\cos(\psi_0-\theta))\\
&\times(\sin\varphi\, - 2\cos a_0\sin\phi_0) - \Delta\psi\,\Lambda_0(\sin\varphi\, - 2\cos a_0\sin\phi_0)\Big).\\
I_1 = \;& 2\Delta\psi\,\big(\cos a_0\cos\phi_0\sin\psi_0 - \Lambda_0\cos\phi_0\cos\psi_0\big)\\
&+2\Delta\phi\,\Big(\cos a_0\sin\phi_0\cos\psi_0 - \cos\phi_0\cos\psi_0\big(\sin\varphi\,\cos\phi_0\\
&\qquad - \cos\varphi\,\sin\phi_0\cos(\psi_0-\theta)\big)\Big)\\
&+c^{-1/2}2\cos a_0\,\cos\phi_0(\cos a_0\sin\psi_0 - \Lambda_0\cos\psi_0),\\
I_2 = \;& -2\Delta\psi\,\big(\cos a_0\cos\phi_0\cos\psi_0 + \Lambda_0\cos\phi_0\sin\psi_0\big)\\
&+2\Delta\phi\,\Big(\cos a_0\sin\phi_0\sin\psi_0\\
&\qquad - \cos\phi_0\sin\psi_0\big(\sin\varphi\,\cos\phi_0 - \cos\varphi\,\sin\phi_0\cos(\psi_0-\theta)\big)\Big)\\
&-c^{-1/2}2\cos a_0\,\cos\phi_0(\cos a_0\cos\psi_0 + \Lambda_0\sin\psi_0),\\
I_3 = \;& -2\Delta\phi\,\Big(\cos a_0\cos\phi_0 + \sin\phi_0\big(\sin\varphi\,\cos\phi_0 - \cos\varphi\,\sin\phi_0\cos(\psi_0-\theta)\big)\Big)\\
&-2\Delta\psi\,\Lambda_0\sin\phi_0 - c^{-1/2}2\cos a_0\,\Lambda_0\sin\phi_0.
\end{aligned}
$$

Finally, we have to require that $O_i = I_i$, for $i = 1, 2, 3$. We get the system of linear equations in the variables $\Delta\psi$, $\Delta\phi$,

$$
\begin{aligned}
(\Lambda_0\,G_1)\Delta\psi + (F\,G_1)\Delta\phi = C_1\\
(\Lambda_0\,G_2)\Delta\psi + (F\,G_2)\Delta\phi = C_2\\
(\Lambda_0\,G_3)\Delta\psi + (F\,G_3)\Delta\phi = C_3,
\end{aligned}
\tag{42}
$$

being Λ_0 defined in (6), $F = \sin\varphi\,\cos\phi_0 - \cos\varphi\,\sin\phi_0\cos(\psi_0 - \theta)$ and

$$
\begin{aligned}
G_1 &= \cos a_0(\cos\varphi\,\cos\theta - 2\cos a_0\cos\phi_0\cos\psi_0) + \cos\phi_0\cos\psi_0,\\
G_2 &= \cos a_0(\cos\varphi\,\sin\theta - 2\cos a_0\cos\phi_0\sin\psi_0) + \cos\phi_0\sin\psi_0,\\
G_3 &= \cos a_0(\sin\varphi\, - 2\cos a_0\sin\phi_0) + \sin\phi_0,\\
C_1 &= c^{-1/2}\cos a_0\,\cos\phi_0(\cos a_0\sin\psi_0 - \Lambda_0\cos\psi_0),\\
C_2 &= -\,c^{-1/2}\cos a_0\,\cos\phi_0(\cos a_0\cos\psi_0 + \Lambda_0\sin\psi_0),\\
C_3 &= -\,c^{-1/2}\cos a_0\,\Lambda_0\sin\phi_0.
\end{aligned}
\tag{43}
$$

In conclusion, in order to match the in map and the out map (up to terms of order μ^α) it is necessary that there exists c, θ, φ, ψ_0, ϕ_0, $\Delta\psi$, $\Delta\phi$ verifying (41) and (42). Next, we will study the values of c, θ, φ, ψ_0 and ϕ_0 that ensure that there is solution for the four equations.

First of all, let us consider the compatibility of (42). We observe that the matrix of the system

$$M = \begin{pmatrix} \Lambda_0 \, G_1 & F \, G_1 \\ \Lambda_0 \, G_2 & F \, G_2 \\ \Lambda_0 \, G_3 & F \, G_3 \end{pmatrix}$$

can only have rang 0 or 1 and that G_1, G_2 and G_3 cannot be simultaneously zero because $G_1^2 + G_2^2 + G_3^2 = 1 - \cos^2 a_0$ and $\cos a_0 \neq \pm 1$. Thus, M will be of rang 0 if and only if $F = \Lambda_0 = 0$. This situation can only happen in one of the following situations:

1. $\phi_0 = \pm\pi/2$ and $\varphi = \pm\pi/2$.
2. $\phi_0 = \pm\pi/2$ and $\psi_0 - \theta = \pi/2 + k\pi$, for $k \in \mathbb{Z}$.
3. $\theta - \psi_0 = k\pi$ for $k \in \mathbb{Z}$ and $\varphi = (-1)^k \phi_0$.

The first and the third do not fulfill the condition $\cos a_0 = \pm 1$. The second case implies that $C_1 = C_2 = C_3 = 0$, so the system disappears.

If $\Lambda_0 \neq 0$ or $F \neq 0$, M has rang 1. Then, system (42) will be compatible if the determinants

$$\Delta_{ij} = \begin{vmatrix} G_i & C_i \\ G_j & C_j \end{vmatrix}, \qquad i < j,$$

are all different from zero. The explicit expressions for them are

$$\begin{aligned}
\Delta_{12} &= \Delta\Big(\cos\varphi \,(\cos a_0 \cos(\theta - \psi_0) - \Lambda_0 \sin(\theta - \psi_0)) \\
&\qquad + \cos\phi_0 (1 - 2\cos^2 a_0)\Big), \\
\Delta_{23} &= \Delta\Big(\cos\varphi \, \sin(\theta - \psi_0) \,(\sin\phi_0 \cos\varphi \, \sin\theta \, - \sin\varphi \, \sin\psi_0 \cos\phi_0) \\
&\qquad + \cos\psi_0 (2\sin\phi_0 \cos^2 a_0 - \sin\varphi \, \cos a_0 - \sin\phi_0)\Big), \\
\Delta_{13} &= \Delta\Big(\cos\varphi \, \sin(\theta - \psi_0) \,(\sin\phi_0 \cos\varphi \, \cos\theta \, - \sin\varphi \, \cos\psi_0 \cos\phi_0) \\
&\qquad - \sin\psi_0 (2\sin\phi_0 \cos^2 a_0 - \sin\varphi \, \cos a_0 - \sin\phi_0)\Big),
\end{aligned}$$

where

$$\Delta = -\frac{\cos^2 a_0 \cos\phi_0}{\sqrt{c}},$$

and where we know that $\cos^2 a_0 \neq 0$. Let us study them in the case of the planar and spatial RTBP separately. In the planar case $\varphi = \phi_0 = 0$ and, substituting in the expressions above, we get that $\Delta_{ij} = 0$, for all i,j. Thus, in the planar restricted problem the system (42) has solution with one degree of freedom.

In the spatial case (when φ or ϕ_0 do not vanish), the unique solution of $\Delta_{ij} = 0$ for all i,j which is acceptable ($0 < \cos a_0 < 1$) is $\phi_0 = \pm\pi/2$. Then, we have the following cases:

1. if $\phi_0 = \pm\pi/2$ and $\theta - \psi_0 \neq \pi/2 + k\pi$ for $k \in \mathbb{Z}$, the system (42) has solution with one degree of freedom.

2. if $\phi_0 = \pm\pi/2$ and $\theta - \psi_0 = \pi/2 + k\pi$ for $k \in \mathbb{Z}$, the system (42) reduces to the system $0 = 0$.

The final step is find which solutions of (42) verifies the equation (41). We will continue this study separately for the planar and spatial case.

6.1. Planar case

In this subsection we will take $\varphi = 0$ and $\phi_0 = 0$. In this case, we have seen that the outer and the inner maps match up to terms of order μ^α, which is is equivalent to say that the system (42) has solution. In fact, this system reduces to one equation (the system has rang 1) and we have to add the equation (41). After substituting the values $\varphi = 0$ and $\phi_0 = 0$ in both equations we get

$$\begin{aligned} \sin^2(\theta - \psi_0)\,\Delta\psi &= \frac{-1}{\sqrt{c}}\cos(\theta - \psi_0) \\ -\cos\psi_0\,\Delta\psi &= \frac{\cos\theta}{\sqrt{c}} + \sin(\theta - \psi_0). \end{aligned} \tag{44}$$

(Observe that $\Delta\phi$ has disappeared. We are dealing with plane orbits.) In order to ensure that the system (44) has solution we have to add the condition

$$E(\theta, \psi_0) = \frac{1}{\sqrt{c}}\Big(\cos\theta\,\sin^2(\theta-\psi_0) - \cos\psi_0\,\cos(\theta-\psi_0)\Big) + \sin^3(\theta-\psi_0) = 0. \tag{45}$$

We observe the following properties:

1. $E(\theta + \pi, \psi_0) = -E(\theta, \psi_0)$, thus if (θ, ψ_0) is a solution of (45), $(\theta + \pi, \psi_0)$ too.

2. If $C_J = -1$ (or $c = 1$), then $\psi_0 = \pi/2$ are solutions of (45).

3. The values

   a) $(\theta, 0)$, with $\tan^3\theta = 1/\sqrt{c}$,

   b) $(0, \psi_0)$, $\sin^3\psi_0 - \dfrac{2}{\sqrt{c}}\sin^2\psi_0 + \dfrac{1}{\sqrt{c}} = 0$

are also solution of $E(\theta, \psi_0) = 0$.

This properties allow us to find, numerically, the solutions of (45), which can be seen in Figure 2.

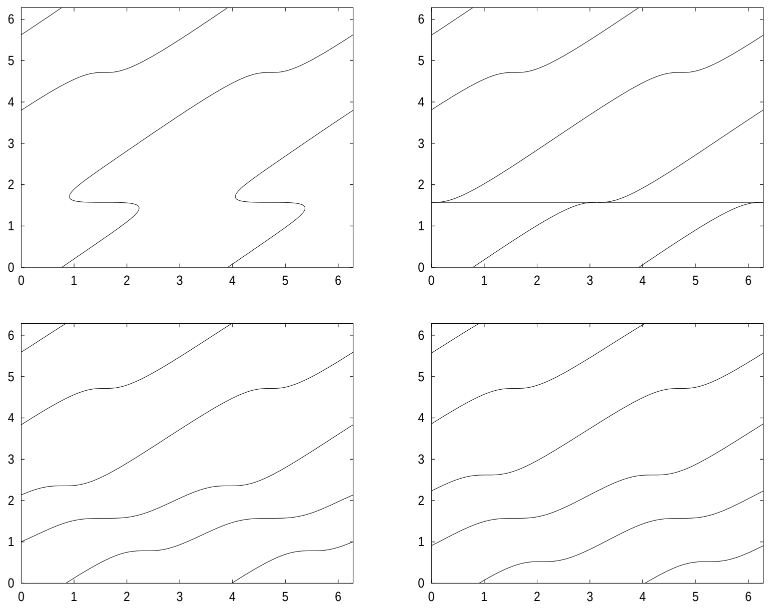

Figure 2. Curves $E(\theta, \psi_0) = 0$. The figures on the top are for the values of Jacobi's constant $C_J = -2$ for the figure in the left side and $C_J = -1$ for the figure in the right. The figures on the bottom are for $C_J = 1$ for the figure in the left and $C_J = 2$ for the figure in the right.

At this point, it is necessary to remember that ψ_0 and θ are subjected to the restrictions (11) and (8) which, in the present case, are written as

$$\sin\psi_0 = \frac{2 - C_J + (p/q)^{2/3}}{2\sqrt{3 - C_J}}, \qquad C_J \neq \left(\frac{p}{q}\right)^{2/3}, \tag{46}$$

for $C_J \in (C_{J1}, C_{J2})$ (see (10)). Therefore, depending on p and q we get the following cases:

- If $p/q = 1$, then $\sin\psi_0 = \sqrt{3 - C_J}/2 = \sqrt{c}$. Thus $\psi_0 \in [0, \pi]\backslash\{\pi/2\}$ (if $\psi_0 = \pi/2$, from (8) it follows that $C_J = C_{Ji}$, for $i = 1$ or 2, which is not acceptable), and there is only one value for C_J for every ψ_0.

- If $p/q < 1$, then fixed $\psi_0 \in [0, 2\pi] \setminus \{\pi/2, 3\pi/2\}$, there is one value for C_J given by $\sqrt{3 - C_J} = 2\sqrt{c} = \sin\psi_0 + \sqrt{\sin^2\psi_0 + 1 - (p/q)^{2/3}}$.

- If $p/q > 1$, it will be necessary that $\sin\psi_0 \geqslant \sqrt{(p/q)^{2/3}-1}$ and there are two values for C_J: $2\sqrt{c} = \sin\psi_0 \pm \sqrt{\sin^2\psi_0 + 1 - (p/q)^{2/3}}$.

Any value of C_J and ψ_0 verifying (46) is called an admissible value. Then, fixed an admissible value for C_J and ψ_0, it is necessary see if there exists any value for θ verifying (45). First of all, it can be proved that there exists two values. This follows from the fact that we can write

$$E(\theta,\psi_0) = \frac{r}{4}(3\cos(\theta-\sigma_1) + \cos(3\theta-\sigma_2)),$$

where $r = (3 - 2c - (p/q)^{2/3})/(2c)$ and σ_1, σ_2 depend on ψ_0 and c. Fixed ψ_0, the number of solutions of $E(\theta,\psi_0) = 0$ is the same that the number of solutions of the equation $3\cos x + \cos(3x+s) = 0$, which have two solutions.

Concluding, we get that

- If $p/q = 1$, $\sin\psi_0 = \sqrt{c}$ and, fixed C_J, the solutions of (45) are $\theta = \psi_0 \pm \pi/2$. But for this values, $\cos a_0 = \cos(\theta-\psi_0) = 0$ so they are not admissible.

- If $p/q \neq 1$, then $\theta = \psi_0 \pm \pi/2$ are not solution of (45). Using the first property of the function $E(\theta,\psi_0)$, we get two solutions θ_1 and θ_2 such that $\theta_2 - \theta_1 = \pi$. This implies that $\cos(\theta_1-\psi_0)\cos(\theta_2-\psi_0) < 0$. Therefore we only get one admissible value for θ (which verify $\cos a_0 = \cos(\theta-\psi_0) > 0$).

Thus we have seen that for $p \neq q$, there is a family of values for C_J, ψ_0 and θ that guarantee the orbit is p-q resonant and the inner and outer maps match up to order μ^α. However, we cannot say the same for $p = q = 1$. This can be explained as follows: the generating orbits (for $\mu = 0$) associated to p-q resonant orbits are bifurcation orbits of 1st species-2ond species (see (Hénon, 1997)). When $p = q = 1$, a generating orbit of 1st species only can be an ellipse intersecting the unit circle twice (type 1) or a retrograde circle of radius 1 and period 2π (type 3). Hénon (page 102) proofs that there cannot exists orbits of type 1 of 1st species-2ond species. If the generating orbit was of type 3 then the third body P and the secondary M would have a previous encounter at a half period, which cannot be possible because we have supposed that $|r(t) - r_{_M}(t)| > \mu^\alpha$ for $t \in (t_1, t_2)$.

In Figure 3 there are represented several plane p-q resonant orbits with initial conditions such that the outer and inner maps match.

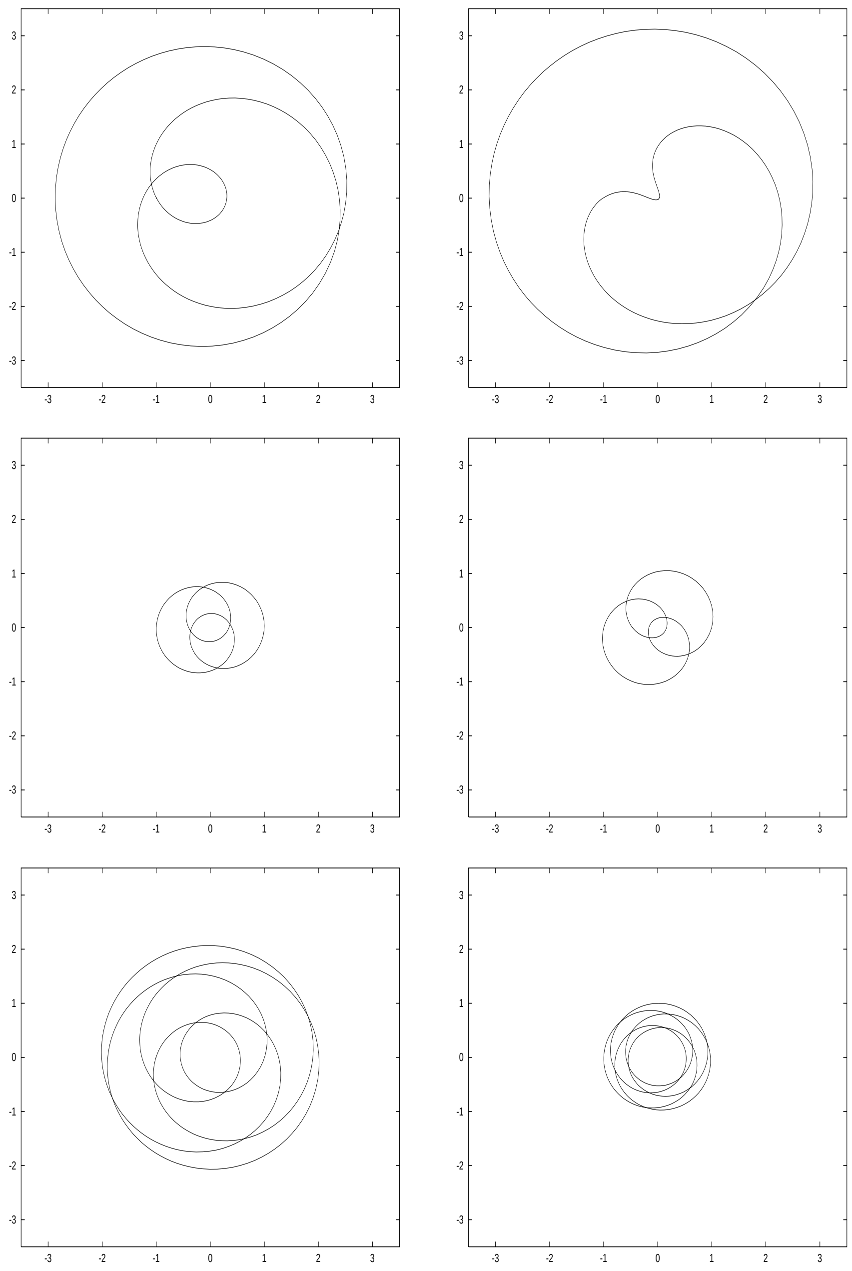

Figure 3. Examples of plane p-q resonant orbits close to periodic Second Species Solutions. The figures on the top are orbits 1-2 resonant for $C_J = -0.850431$ and $C_J = 1.059752$ (the orbits on the left and on the right respectively). The figures on the middle are 2-1 resonant orbits for $C_J = 0.303724$ and $C_J = 0.565699$. The figures on the bottom are 2-3 resonant orbits (left) and 3-2 resonant orbits (right).

6.2. Spatial case

As we have seen, for spatial orbits, the system (42) has solution only for $\phi_0 = \sigma\pi/2$, with $\sigma = \pm 1$. For this values, we have to study which solutions of (42) are also a solution of (41). There are two possibilities:

– $\psi_0 - \theta = \pi/2 + k\pi$ for $k \in \mathbb{Z}$. Then, the system (42) reduces to the system 0=0 and the solution of (41) are

$$\Delta\phi = \begin{cases} \sigma(-1)^k \dfrac{\cos\varphi}{\sqrt{c}} & \text{if } \cos\theta \neq 0, \\ \text{indefinite} & \text{if } \cos\theta = 0, \end{cases}$$

and $\Delta\psi$ can take any value.

– $\psi_0 - \theta \neq \pi/2 + k\pi$ per a $k \in \mathbb{Z}$. Then, the system (42) reduces to one equation:

$$F\, G_i\, \Delta\phi = 0.$$

As G_i cannot vanish simultaneously for all i, the solution of (42) will be $\Delta\phi = 0$. Moreover, as $\phi_0 = \sigma\pi/2$ the equation (41) will be

$$\sigma \sin\psi_0\, \Delta\phi = \frac{\cos\varphi\, \cos\theta}{\sqrt{c}}.$$

As $\Delta\phi = 0$, we get $\cos\varphi\, \cos\theta = 0$. But $\cos\varphi \neq 0$ (if not, $\cos a_0 = \pm 1$), so $\cos\theta = 0$ and, then, $\theta = \pi/2 + m\pi$ for $m \in \mathbb{Z}$.

Again we have to remember that this solutions have to verify the restrictions (11) and (8). The first one is obviously true and the first one reduces to

$$C_J = 2 + \left(\frac{p}{q}\right)^{2/3}.$$

This will imply that p must be less than q.

Finally, in Figure 4 there are represented several spatial p-q resonant orbits with initial conditions $\Delta\phi = 0$ and $\theta = \pi/2$. This implies that the initial conditions are (up to order μ^α)

$$\begin{array}{lll} x = \mu - 1, & y = \mu^\alpha \cos\varphi, & z = \mu^\alpha \sin\varphi, \\ \dot{x} = 0, & \dot{y} = 0, & \dot{z} = v \sin\phi_0. \end{array}$$

The arc of orbit represented is obtained by integrating the equations of motion from $t = 0$ to $t = 2\pi q + \varepsilon\mu^\alpha$.

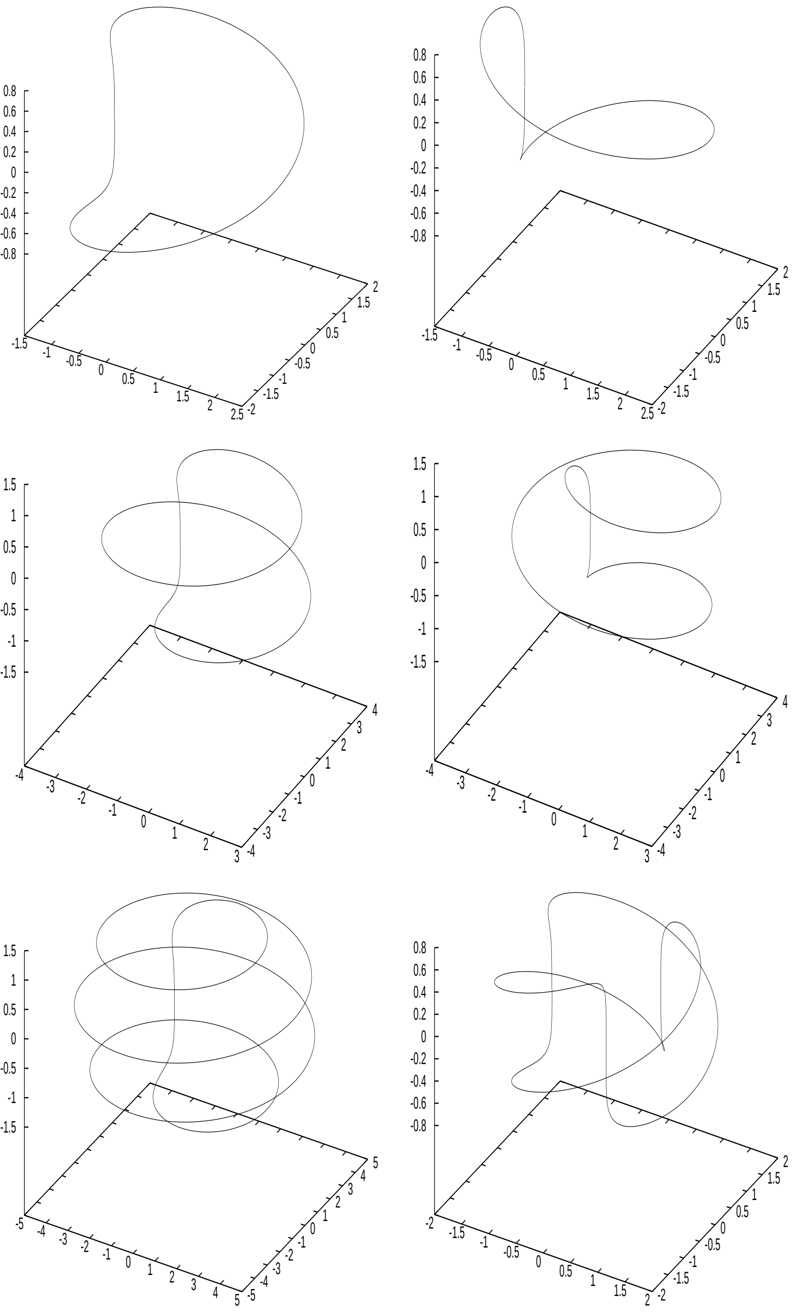

Figure 4. Examples of spatial p-q resonant orbits close to periodic Second Species Solutions. For all cases we have taken $\theta = \pi/2$ and $\Delta\phi = 0$. The figures on the top are 1-2 resonant orbits. The figures on the middle are 1-3 resonant orbits. The figures on the bottom are 1-5 resonant (left) and 3-5 resonant orbits (right). For the figures on the left we have taken $\phi_0 = \pi/2$ and for the figures on the right $\phi_0 = -\pi/2$.

7. Conclusions

In this paper we have proved the existence of solutions of the spatial restricted three-body problem close to Second Species Solutions. The study is done analyzing the conditions under which the inner and outer solutions, both on a sphere of radius μ^α around the small primary, match up to a certain order in terms of μ^α. The matching is analyzed carefully in the planar and the spatial case.

8. Acknowledgments

The researchers have been supported by grant CIRIT 2000SGR-27 (Catalonia) and the second author by grant DGICYT BFM2000-805 (Spain).

References

Barrabés, E., 'Òrbites de segona espècie del problema espacial de 3 cossos'. Ph.D. thesis, Universitat Autònoma de Barcelona. 2001.

Barrabés, E. and Gómez, G., 'Spatial p-q resonant orbits of the RTBP'. *Celestial Mechanics and Dynamical Astronomy (to appear).*

Font, Q., Nunes, A. Simó, C., 'Consecutive quasi-collisions in the planar circular RTBP'. *Nonlinearity, 15 (2002) 115–142.*.

Hénon, M., *Generating Families in the Restricted Three-Body Problem* Springer–Verlag (1997).

Stiefel, E. and Scheifele, G. *Linear and regular celestial mechanics.* Die Grundlehren der mathematischen Wissenschaften. Band 174. Berlin- Heidelberg-New York: Springer-Verlag. IX, (1971).

Szebehely, V., *Theory of Orbits. The Restricted Problem of Three Bodies.* Academic Press, Inc. (1967).

Yen, C.L., 'Ballistic Mercury orbiter mission via Venus and Mercury gravity assist' *AAS/AIAA Astrodynamics Specialist Conference*, Paper AAS 85-346.