Study of bifurcations and stability in Rayleigh-Bénard convection ⋆

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Abstract

A path-continuation Galerkin method is proposed to determine the bifurcations and stability of steady convective flows in cavities. It is based on a complete, divergent-free set of basis functions satisfying all boundary conditions. The method is applied to the Rayleigh–Bénard flow in a cubical cavity. Three bifurcations from the conductive state are identified for $Ra < 8000$. At the first bifurcation a stable x–roll and an unstable diagonal–roll are formed. The second bifurcation yields an initially unstable four–rolls structure which becomes stable later on, while the third bifurcation results in a highly unstable structure.

1. Introduction

Convection problems have been widely studied because of their fundamental interest in relation to the genesis and evolution of flow patterns, and their many technological applications[1]. Moreover, natural convection flows are frequently used as benchmark problems for Navier–Stokes solvers[2].

The study of bifurcations and stability of three–dimensional steady convective flows is usually carried out by time–forward integration of the non–stationary equations. However, time–dependent methods are affected by initial conditions and are unable to follow the evolution of unstable solutions that could lead to stable ones at secondary bifurcations. Unlike time–forward integration approaches, stability analysis methods calculate more effectively critical points and their variation with relevant parameters[3]. Furthermore, the evolution of unstable solutions and their secondary bifurcations can be determined in a straightforward way by applying a path continuation method.

The present work proposes a path-continuation method to determine the bifurcations and stability of steady convective flows in rectangular enclosures. The method is based on the Galerkin spectral method with a complete, divergent-free set of basis functions satisfying all boundary conditions. Problem of Rayleigh–Bénard flow inside a cubical cavity with six rigid and immobile walls is used to show the applicability of the current method. The cavity is heated from below and the lateral walls are chosen to be adiabatic. This geometry without preferred direction generates flow structures different from those in rectangular cavities.

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2. Governing equations

The domain covered by a parallepipedic cavity, scaled by the characteristic heigth L, is given by $\Omega = [-\frac{h_1}{2}, \frac{h_1}{2}] \times [-\frac{h_2}{2}, \frac{h_2}{2}] \times [-\frac{1}{2}, \frac{1}{2}]$. The temperature at the top and the bottom of the cubical region is kept at the constant temperatures θ_c and θ_h, respectively. The characteristic scales for velocity $\mathbf{v} = (v^{(1)}, v^{(2)}, v^{(3)})$, temperature ($\theta$), time ($t$) and pressure ($p$) are $q_0 = [\beta(\Delta\theta)gL\alpha/\nu]^{\frac{1}{2}}$, $\Delta\theta$, L^2/α and $\rho\nu q_0/L$, respectively, where ρ is the density of the fluid, g is the acceleration of gravity, $\Delta\theta = \theta_h - \theta_c$, and β, ν and α are the coefficients of thermal expansion, kinematic viscosity and thermal diffusivity, respectively. The governing dimensionless equations for the velocity $\mathbf{v}$ and the deviation θ from the static, conductive, temperature distribution are

$$\frac{1}{Pr}\frac{\partial \mathbf{v}}{\partial t} - \nabla^2 \mathbf{v} + \frac{Ra^{\frac{1}{2}}}{Pr}(\mathbf{v}\cdot\nabla\mathbf{v}) - Ra^{\frac{1}{2}}\theta\,\mathbf{K} + \nabla p = 0 \quad (1)$$

$$\frac{\partial \theta}{\partial t} - \nabla^2\theta + Ra^{\frac{1}{2}}(\mathbf{v}\cdot\nabla\theta) - Ra^{\frac{1}{2}} v^{(3)} = 0 \quad (2)$$

$$\nabla \cdot \mathbf{v} = 0 \quad (3)$$

The Rayleigh number and the Prandtl number are defined as $Ra = \beta(\Delta\theta)gL^3/\alpha\nu$ and $Pr = \nu/\alpha$, respectively, and $\mathbf{K}$ denotes the vector $(0,0,1)$.

The no-slip boundary condition ($\mathbf{v} = 0$) is prescribed at the six walls of the cavity. The thermal boundary conditions are $\partial\theta/\partial n = 0$ at the four adiabatic lateral walls and $\theta = 0$ at the top and bottom horizontal walls.

3. Numerical method

3.1. *Basis functions*

A Galerkin method is used to determine the bifurcations and stability of the steady solutions of equations (1-3). The complete set of basis functions was chosen so that each basis function satisfies all boundary conditions and the continuity equation. (see [4] for details). The velocity vector $\mathbf{v}$ and the temperature θ are approximated as a truncated series

$$\begin{pmatrix} \mathbf{v} \\ \theta \end{pmatrix} = \sum_{i=0}^{N_x}\sum_{j=0}^{N_y}\sum_{k=0}^{N_z}\left[c_{ijk}^{(x)}\mathbf{u}_{ijk}^{(x)} + c_{ijk}^{(y)}(t)\mathbf{u}_{ijk}^{(y)} + d_{ijk}\mathbf{t}_{ijk}\right] \quad (4)$$

In equation (4) $c_{ijk}^{(x)}$, $c_{ijk}^{(y)}$ and d_{ijk} are unknown time–dependent coefficients and $\mathbf{u}_{ijk}^{(x)}$, $\mathbf{u}_{ijk}^{(y)}$ and $\mathbf{t}_{ijk}$ are the basis functions defined as

$$\begin{aligned}
\mathbf{u}_{ijk}^{(x)}(x,y,z) &= \begin{pmatrix} f_i(\frac{x}{h_1})g_j(\frac{y}{h_2})f_k'(z) \\ 0 \\ -\frac{1}{h_1}f_i'(\frac{x}{h_1})g_j(\frac{y}{h_2})f_k(z) \\ 0 \end{pmatrix} \\
\mathbf{u}_{ijk}^{(y)}(x,y,z) &= \begin{pmatrix} 0 \\ g_i(\frac{x}{h_1})f_j(\frac{y}{h_2})f_k'(z) \\ -\frac{1}{h_2}g_i(\frac{x}{h_1})f_j'(\frac{y}{h_2})f_k(z) \\ 0 \end{pmatrix} \\
\mathbf{t}_{ijk}(x,y,z) &= \begin{pmatrix} 0 \\ 0 \\ 0 \\ h_i(\frac{x}{h_1})h_j(\frac{y}{h_2})g_k(z) \end{pmatrix}
\end{aligned} \quad (5)$$

where the prime denotes differentiation with respect to the current argument and g, f and h are the following functions

$$\begin{aligned}
g_k(x) &: \cos((2k-1)\pi x) \quad \text{or} \quad \sin(2k\pi x), \\
h_k(x) &: \cos(2(k-1)\pi x) \quad \text{or} \quad \sin((2k-1)\pi x), \\
f_k(x) &: C_k(x) = \frac{\cosh(\lambda_k x)}{\cosh(\lambda_k/2)} - \frac{\cos(\lambda_k x)}{\cos(\lambda_k/2)} \\
&\quad \text{or} \\
&\quad S_k(x) = \frac{\sinh(\mu_k x)}{\sinh(\mu_k/2)} - \frac{\sin(\mu_k x)}{\sin(\mu_k/2)}
\end{aligned} \quad (6)$$

In equation (6) the parameters λ_k and μ_k are the positive roots of equations $\tanh(\lambda_k/2) + \tan(\lambda_k/2) = 0$ and $\tanh(\mu_k/2) - \tan(\mu_k/2) = 0$, respectively.

3.2. *Linear steady problem*

Application of the Galerkin problem to the linearization of the steady part of equations (1-3) leads to the set of algebraic equations

$$M(Ra)\begin{pmatrix} \mathbf{c}^{(x)} \\ \mathbf{c}^{(y)} \\ \mathbf{d} \end{pmatrix} = 0 \quad (7)$$

where

$$M(Ra) = \begin{pmatrix} A_{xx} & A_{yx} & Ra^{\frac{1}{2}}B_x \\ A_{yx}^T & A_{yy} & Ra^{\frac{1}{2}}B_y \\ Ra^{\frac{1}{2}}B_x^T & Ra^{\frac{1}{2}}B_y^T & C \end{pmatrix} \qquad (8)$$

with

$$\begin{array}{l} A_{\alpha\beta} = \nabla^2 \mathbf{u}_{ijk}^{(\alpha)} \cdot \mathbf{u}_{lmn}^{(\beta)}, \quad B_\alpha = \mathbf{t}_{ijk} \cdot \mathbf{u}_{lmn}^{(\alpha)} \mathbf{k} \\ C = \nabla^2 \mathbf{t}_{ijk} \cdot \mathbf{t}_{lmn} \end{array} \qquad (9)$$

In equation (9) α and β are x or y, the $\cdot$ denotes the scalar product and the pairs of indices (i,l), (j,m) and (k,n) vary from 0 to N_x, N_y and N_z, respectively.

Therefore the linear problem reduces to an eigenvalue problem for the Rayleigh number. Whenever the eigenvalue is simple the associated eigenvector describes, qualitatively, the type of flow developed near the bifurcation point. The bifurcation Rayleigh numbers (Ra_c), i.e., the values of Ra for which the equation (7) has non-trivial solutions, were estimated using the power method.

3.3. *Nonlinear steady problem*

The method consists first in expanding and truncating any solution of equations (1-3) in series of eigenvectors of the linear problem

$$\mathbf{a}(x,y,z,t) = \sum_{i=1}^{L} a_i \mathbf{f}_i(x,y,z) \qquad (10)$$

where a_i are time-dependent coefficients and $\mathbf{f}_i$ are the eigenvectors of the matrix $M(Ra_0)$, defined in (8), for a Rayleigh number Ra_0 close to a bifurcation one Ra_c. Eigenvectors $\mathbf{f}_i$ are ordered by decreasing eigenvalue.

Once the Galerkin method is applied, the nonlinear problem is reduced to a system of ordinary differential equations (ODEs) that can be written in the following form (the sumation convention on repeated indices is assumed)

$$B_{ij}\frac{da_i(t)}{dt} = M_{ij}(Ra)a_i(t) - \frac{Ra^{\frac{1}{2}}}{2}Q_{inj}a_i(t)a_n(t) \qquad (11)$$

where the matrices M_{ij} and Q_{inj} contain the coefficients of the linear and bilinear terms of the equations, respectively. Thus, the well-established theory and numerical methods for ODEs are applied both to obtain steady solutions and to study their stability.

The numerical process was organized in the following way: first, the Newton method was applied to determine steady solutions of equation (11) for values Ra_0 close to bifurcation ones. The Newton method was initiated with approximate solutions obtained by standard bifurcation analysis[5]. Second, a path-continuation method based on a modified Newton method[5] was used to compute steady solutions of equation (11) as a function of Ra. Finally, along the solution branches the stability properties were checked so that new secondary bifurcations were identified. At any point along the different solutions branches the stability of the flow was analysed by linearizing the equation (11) around that point. Thus, the method extends the analysis beyond the regions around bifurcation points.

4. Results for the cubical cavity

The current numerical procedure was applied to the Rayleigh–Bénard flow inside a cubical cavity heated from below. Three different bifurcations from the conductive state were identified for values of Ra up to 8000.

The smallest bifurcation Rayleigh number is 3389 and it is a double eigenvalue. Therefore, according to linear stability theory, any linear combination of the two eigenfunctions is possible. However, only a single x or y–roll, named $S1$, and a diagonal–roll, named $S2$, are found when the nonlinear procedure is applied, in agreement with the flow structures reported numerically and experimentally by Pallarès et. al.([6], [7]). At the second bifurcation Rayleigh number of 5902 a four–rolls flow structure, named $S5$ (see [6] for the notation), is identified. Structure $S4$ which develops at the third bifurcation Rayleigh number of 7458 has a toroidal configuration and it is the sum of two x–rolls and two y–rolls.

Table I shows the convergence of the bifurcation Rayleigh numbers as a function of the number of basis functions. The extrapolated values presented in Table I were determined by using a law of the form $a+b/N$ for $N=\infty$, being $N = 3\times N_x \times N_y \times N_z$ the total number of basis functions used. As the differences between the $3\times7\times7\times7$ data and the extrapolated ones are below 0.06% in all cases all current results were calculated with $3\times7\times7\times7$ functions.

The bifurcation diagram at $Pr = 130$ is depicted in Figure 1. This figure shows the variation of the absolute value of the principal component for all identified steady flow structures as a function of Ra. Broken lines in the plot indicate a change in the dominant component. Stable and unstable structures are depicted with continuous and discontinuous lines, respectively and bifurcation points are marked with dots. Flow structures that develop when $S4$ and $S5$ bifurcate are identified with addition of a and b in their respective notation.

Figure 1 indicates that $S1$ and $S5$ are the only stable flow structures. While $S1$ is stable over the whole range of Ra considered ($Ra \leq 7\times10^4$), $S5$ is unstable when it originates at $Ra_c \simeq 5902$, but becomes stable at $Ra_c \simeq 10400$ and remains so thereafter. The initial instability of $S5$ is consistent with stability theory as it appears beyond the first bifurcation $Ra_c \simeq 3389$. The later stabilization of $S5$ explains why it was experimentally observed[7].

5. Conclusions

The current numerical method shows that different stable steady flow structures can coexist for moderate values of Ra in a cubical cavity heated from below. Therefore the final assymptotic state will depend on the initial conditions. The results presented also explain the stabilization of the experimentally observed[7] four-rolls structure $S5$ beyond the transition from the conductive state, where it appears as an unstable flow structure.

Type of structure	$3\times5\times5\times5$ functions	$3\times7\times7\times7$ functions	Extrapolated values	Δ_{e-7} %
x–roll ($S1$) diagonal–roll ($S2$)	3391.17	3389.53	3388.6	0.027
four–rolls ($S5$)	5910.09	5903.97	5900.5	0.058
$2x$–rolls+ $2y$–rolls ($S4$)	7462.98	7458.35	7455.8	0.034

Table 1
Convergence of the critical Rayleigh numbers as a function of the number of trial functions used.

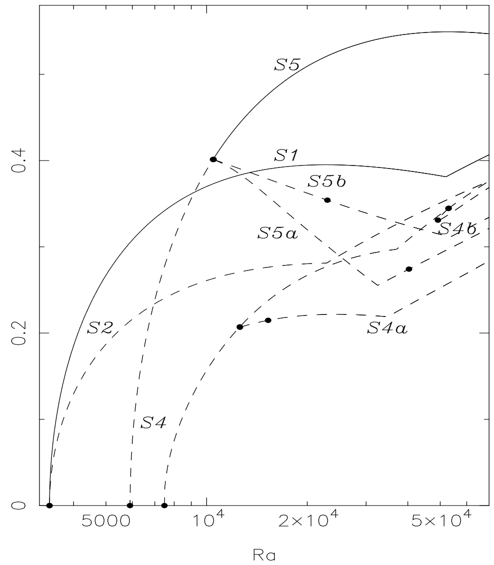

Fig. 1. Bifurcation diagram at $Pr = 130$. The component with greatest absolute value is depicted as a function of Ra.

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