Analytical and numerical detection of exponentially small phenomena

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Abstract

Exponentially small phenomena occur in a wide variety of problems. Always dealing with problems which are analytical, we find these phenomena both in estimates of the remainders in normal forms of vector fields and diffeomorphisms and in averaging methods and, in general, in most of the phenomena which cannot be detected by using any fixed finite order approach of the classical perturbation theory. They also appear in phenomena like the delay of the bifurcation for fixed points and periodic solutions, in the case of slowly varying parameters, in adiabatic invariance and in the corresponding bifurcation diagrams of all these problems. They can be revealed by a suitable use of complex variable, by extending the phase space and time to suitable complex strips or by using a classical method letting the order to increase, up to some optimal value, when the small parameter tends to zero. Numerically they can be detected by a careful use of high accuracy arithmetics, leading in a natural way to the problem of optimization of all the algorithms involved.

1 Introduction

Many problems in the physical sciences can be studied as the *perturbation* of *simpler problems*. If the perturbation is small, it is possible to introduce a small parameter ε and normalize the perturbation. We can write a differential equation like $\dot{x} = f_0(t,x) + \varepsilon f_1(t,x)$ and a diffeomorphism like $\bar{x} = F_0(x) + \varepsilon F_1(x)$. In what follows we shall assume that we are dealing with *analytic problems*, and we can also extend them to the case of analytical dependence in ε. Many results can be extended to different Gevrey classes too, but we shall not consider them here. The normalization of ε amounts to take $|f_1| = 1$ or $|F_1| = 1$, using suitable norms. Then, the *perturbative methods* try to look for the desired properties of the solutions of the full problem by *expanding in powers of* ε around a given solution of the unperturbed ($\varepsilon = 0$) problem. Let $\mathcal{P}(\varepsilon)$ be a measure of some property of interest.

There are many problems for which this approach leads to nothing, that is, all the coefficients in the power expansion of $\mathcal{P}$ turn out to be zero, without $\mathcal{P}(\varepsilon)$ being zero itself. They are usually denoted as *phenomena beyond all orders* (it is meant beyond all orders of this classical approach). Despite the problem is analytic and also the dependence with respect to ε, some relevant properties can be not analytical. They are related to the lack of uniform dependence when the time increases. Typically, it is realized that $\mathcal{P}(\varepsilon)$ is of the order of $\exp(-a\varepsilon^{-b})$, for some positive a and b. This is why they are denoted collectively as *exponentially small phenomena*.

Faced to these problems one can wonder why they are *relevant* in applications. Something which is exponentially small should, probably, be neglected. Beyond theoretical considerations this has a simple answer: In many cases, the exponentially small estimates give good approximations to the real behaviour not only when $\varepsilon \to 0$, but also for moderate values of ε. This allows to use the estimates to *predict*, given some tolerance δ, the *critical value* $\varepsilon(\delta)$ such that $\mathcal{P}(\varepsilon(\delta)) = \delta$. As an elementary example, consider the classical Hénon–Heiles Hamiltonian on a level of energy $h < 1/6$. Three simple hyperbolic periodic orbits appear. They have transversal heteroclinic intersections with angle of intersection essentially given by $\exp(-\frac{\pi^2/2}{h^{1/2}})$. This transversality is the responsible for the creation of a stochastic zone around the separatrices of these orbits. If we decide that values $\delta < 10^{-12}$ can be neglected (i.e., that the Hamiltonian could be considered as *integrable up to this tolerance*) we obtain the critical value $\varepsilon_{crit} \simeq 0.0319$ [6]. See [19] for an application to the J_2 (oblateness) problem of an Earth satellite, a problem known to be non–integrable [9]. In a similar way exponentially small diffusion rates gives rise to exponentially large times before the action variables change in a noticeable way. An example in the case of the Lagrange triangular relative equilibria of the restricted three–body problem can be found in [8].

Exponentially small phenomena show up in a natural way in *general tools* to study dynamical systems (like normal forms, averaging and the related problems of lack of convergence, the lack of reducibility of linear differential equations with quasi periodic coefficients), in *geometrical objects in the phase space* (like splitting of separatrices, homoclinic tangles, quasi confiners, i.e., codimension–1 manifolds almost bounding a region of the phase space, usually formed by the stable and unstable manifolds of a center manifold of codimension 2), in *parameter space* (like narrow wedges in bifurcation diagrams, width of stability channels in Hill's equations) and in the *dynamical consequences* (like measure of the lack of integrability, diffusion rates and Nekhorosev–like estimates, practical stability, adiabatic invariants). But also in many other aspects, like singular perturbations, boundary layer effects, delay of bifurcation when a parameter moves slowly with time, etc. I refer to [5, 7, 13, 17, 16] and to the talk of A. Neishtadt, given is this mini–symposium, for additional information.

To conclude this introduction I present some elementary asymptotic considerations. Assume some phenomenon is measured by a function φ depending on the small parameter ε and can be represented by an asymptotic expansion

$$\varphi(\varepsilon) \sim \sum_{m\geq 0} a_m \varepsilon^m, \qquad \text{with} \quad |\sum_{m\leq n} a_m \varepsilon^m - \varphi(\varepsilon)| < |a_{n+1}\varepsilon^{n+1}|.$$

The best bound for the error is obtained for $\frac{da_n}{dn} = |\log \varepsilon|$, where a_n is considered as extended to the reals and we assume it is monotonically increasing. Then the value of the bound, $b(\varepsilon)$, is

$$b(\varepsilon) \simeq \exp\left(-n^2 \frac{d}{dn}\frac{\log a_n}{n}\right).$$

Some examples:

- For $a_n = (n!)^\gamma$ (Gevrey classes) we obtain $n \simeq \varepsilon^{-1/\gamma}$ and $b(\varepsilon) \simeq \varepsilon^{-1/2}\exp(-\gamma\varepsilon^{-1/\gamma})$.
- For $a_n = (n^\beta!)^\gamma$, $\beta > 1$, we obtain $n \simeq \frac{|\log \varepsilon|}{\gamma\beta^2 \log|\log \varepsilon|}$ and $b(\varepsilon) \simeq \exp(-K\frac{|\log\varepsilon|}{\log|\log\varepsilon|}) = \varepsilon^{K/\log|\log\varepsilon|}$ for some $K > 0$.

- For $a_n = (\log n)^n$ we obtain $n \simeq \exp(1/\varepsilon)$ and $b(\varepsilon) \simeq \exp(-\varepsilon \exp(1/\varepsilon))$. This case shows really a quite sharp behaviour with respect to ε, as displayed in the table below:

ε	0.4	0.3	0.25	0.2	0.15	0.1
$b(\varepsilon)$	7.6×10^{-3}	2.2×10^{-4}	1.2×10^{-6}	1.3×10^{-13}	6.5×10^{-52}	4.0×10^{-956}

  This sharpness may lead to misinterpretations of numerical or experimental results.

The talk is organized as follows. In section 2 we give some results on averaging under fast periodic and quasi periodic perturbations. They have immediate consequences concerning a priori upper bounds of splitting and many other phenomena. Section 3 is devoted to present some examples where exponentially small phenomena occur. Some analytical tools for the detection of these phenomena are shown in section 4, while section 5 introduces part of the numerical methodology. The emphasis is put on the key ideas rather than on detailed technicalities.

2 Averaging. A priori bounds

One of the fundamental results is Neishtadt's theorem on averaging [15]. Let $z \in D + \Delta \subset \mathbb{C}^n$, where D is a compact in $\mathbb{R}^n$ and $D + \Delta$ denotes a Δ–neighbourhood. Consider the differential equation

$$\dot{z} = \varepsilon f(z, t, \varepsilon), \tag{1}$$

where f is analytic in z, 1–periodic and continuous in t and bounded in ε. Then, for any given ε there exists a change of variable $z = h(w, t, \varepsilon)$, with the same characteristics than f, and defined in a narrower strip, say $z \in D + \Delta/2$, such that (1) can be reduced to the form

$$\dot{w} = \varepsilon(g(w, \varepsilon) + r(w, t, \varepsilon)) \qquad \text{with} \qquad |g| < 2|f|, \quad |r| < 2|f| \exp(-c/\varepsilon), \tag{2}$$

where $c > 0$ is a constant depending on Δ and on the dimension but independent of ε. The norms are taken as the supremum in the domain of definition.

We note that, in fact, L^1 in t is enough to prove the theorem and that (1) is equivalent to the *fast periodic* equation $\dot{z} = f(z, t/\varepsilon, \varepsilon)$. Formula (2) tells us that one can *autonomize* the equation up to an exponentially small remainder. To prove the result one has to do a finite sequence of changes of variable of the form $z = w + \varepsilon v(w, t, \varepsilon)$, chosen to cancel the dominant part of the time dependence, and then selecting the appropriate number of steps in the averaging process, which is $O(1/\varepsilon)$. Note also that if ε is not small enough it can happen that no step is possible. Than $z = w$ and $r = f$.

An extension to the quasi periodic case can be found in [20]. Now the differential equation is

$$\dot{z} = \varepsilon f(z, \theta, \varepsilon), \quad \dot{\theta} = \omega, \tag{3}$$

with z as before and $\theta \in \mathbb{T}^d + \Delta$, a Δ–neighbourhood of the d–dimensional torus (i.e., $|\mathrm{Im}\ \theta_j| < \Delta$, $j = 1, \ldots, d$). Of course, the two Δ's can be different. Assume that the frequencies $\omega \in \mathbb{R}^d$ satisfy the Diophantine condition

$$|(k, \omega)| > \frac{\gamma}{|k|^{\tau - 1}}, \ \forall k \in \mathbb{Z}^d \setminus \{0\}, \ \tau > d - 1. \tag{4}$$

In that case we must require analyticity both with respect to z and θ in the respective domains. Then, for any given ε there exists a change of variable $z = h(w, \theta, \varepsilon)$, also analytical in (z, θ) in a narrower domain, such that (3) can be reduced to the form

$$\dot{w} = \varepsilon(g(w, \varepsilon) + r(w, \theta, \varepsilon)), \quad \dot{\theta} = \omega \qquad \text{with} \qquad |g| < 2|f|, \quad |r| < 2|f| \exp(-c\varepsilon^{-1/\tau}), \tag{5}$$

where $c > 0$ is a constant depending on Δ, on the dimensions (n and d), on γ and τ, but independent of ε.

Some comments are in order: (i) In both cases, if the initial system is Hamiltonian the changes can be done symplectically and the final system is also Hamiltonian; (ii) Also in both cases, if $f = f_0 + \varepsilon^p f_1$, $p > 0$ and f_0 is autonomous, then $|r|$ contains ε^p as additional factor; (iii) In the quasi periodic case, if $\tau = d - 1$ then in the exponentially small bound of $|r|$ one should replace $\varepsilon^{-1/\tau}$ by $(\varepsilon|\log\varepsilon|)^{-1/\tau}$; (iv) Also in the quasi periodic case, if $f = f_0 + \varepsilon^p f_1$ as before and f is an entire function, then Δ can be chosen quite large, of the order of $|\log\varepsilon|$. Hence, the exponent in the bound is of the form $-c|\log\varepsilon|\varepsilon^{-1/\tau}$ (if $\tau > d-1$, and modified as before otherwise).

Let us see a simple application to obtain a priori bounds. Let $z \to z + \varepsilon F(z)$ a near the identity analytic map. It is easy to do a suspension to obtain a flow, 1–periodic in t, having the map as time 1 map. Indeed, it is enough to construct an interpolating function taking the value z at $t = 0$ and $F(z)$ at $t = 1$. Furthermore we can require that both at $t = 0$ and $t = 1$ the derivatives are equal to 0. A cubic Hermite interpolation will do. Let $H(t, z, F(z))$ be this function. The vector field is given by $\frac{d}{dt}H$. Note that given t^* and a value z^* we must find the value of z such that the flow passing through z at $t = 0$ reaches z^* at $t = t^*$. This is obtained by using the implicit function theorem. By construction the vector field is piecewise analytic and just $\mathcal{C}^0$ at $t = 0$ mod 1. Higher regularity can be obtained (but it is not needed for our purposes) by asking for several derivatives of H being 0 at $t = 0$ and $t = 1$.

Now we are in condition to apply Neishtadt theorem. By averaging the time dependence in the constructed vector field we obtain an autonomous vector field, εg, plus an exponentially small remainder. Hence

$$F(z) = \varphi^{\varepsilon g}_{t=1}(z) + \text{exponentially small}\ .$$

Of course, one has to be very careful about the domain of validity, because the change of variable can magnify the exponentially small terms in some domains, or we can lose control on some relevant parameters, like normal hyperbolicity or the smallness of the perturbation of a twist map.

3 Several examples and some analytic tools

Here I collect several examples found in different situations:

1. Analysis of bifurcations. Consider a Bogdanov–Takens bifurcation of a planar diffeomorphism
$$F : (x, y) \longrightarrow (x + y, y) + \text{ higher order terms.}$$
Under generic conditions (related to the coefficients of the quadratic terms) the map can be written as the time 1 map of a vector field having a Bogdanov–Takens bifurcation plus exponentially small terms. Doing the unfolding the bifurcation diagram is similar to the one found for the vector field case. There appears a Hopf line, but the loop (or homoclinic connection) line is replaced by an exponentially small region in the parameter plane where homoclinic phenomena occur. Instead of the loop line there are curves of “first” and “last” homoclinic tangencies. Beyond the domain of homoclinic phenomena and the Hopf line there are $\mathcal{C}^r$ normally hyperbolic invariant circles, which replace the limit cycles appearing in the vector field case. The degree of differentiability, r, of these invariant circles has

a bound which decreases when approaching the domain of homoclinic phenomena. Full details can be found in [2]. For further discussion on the changes of differentiability of the invariant circles one can see [3].

Similar phenomena in many different cases can be analyzed by using essentially the same tools. This applies also to higher dimensions, e.g., to the case of families of volume preserving maps unfolding a fixed point with eigenvalues 1, $\exp(\pm i\alpha)$.

2. Splitting of separatrices for area–preserving maps (APM). Following ideas of Lazutkin in the case of the standard map, in [6] the general case of a family of APM close to the identity is analyzed. It is assumed that the maps are ε–perturbations of the identity and tend to behave as the time ε flow of a Hamiltonian having an hyperbolic fixed point with an homoclinic connection. Let $\sigma(t)$ be the expression of the homoclinic connection as a function of time. This function is analytic on a complex strip around the real axis. Let δ be the smallest imaginary part of the singularities of σ and let $h(\varepsilon) = \ln(\lambda(\varepsilon))$, where $\lambda(\varepsilon)$ is the dominant eigenvalue at the hyperbolic fixed point. Then for any $\eta > 0$ there exists a constant $N(\eta)$, independent of ε, such that the splitting of the separatrices of the APM is bounded as
$$\alpha < N(\eta) \exp\left(-\frac{2\pi(\delta-\eta)}{h}\right).$$
Here α denotes any natural measure of the splitting, either the angle of the invariant manifolds at a homoclinic point, the area of the lobes between both manifolds, the homoclinic invariant (see [7]), etc.

The proof is based on the contruction of an uniform (in ε) Birkhoff Normal Form, the use of a natural parameterization for the manifolds, an extension theorem to control the manifolds on the complex plane when the parameter is close to the singularities of σ, Cauchy estimates and the APM character of the map. Note that this approach allows to obtain estimates on the constant which appears in the exponentially small bounds, in constrast with a direct application of Neishtadt theorem. However see [16] and references therein for the estimate of the best constants to be found in general.

One should also remark that no lower bounds can be given in general. They depend of the family considered. For instance, it is well known that in the case of the standard map there are asymptotic formulas for the splitting angle. It is different from zero. But replacing the sin term by a trigonometric polynomial with suitable chosen coefficients can produce that the angle at a given homoclinic point passes through zero for an infinite sequence of values of ε tending to zero. At these points the intersection of the manifolds passes from simple to triple and then, moving again ε, two additional homoclinic points are created. In the case that the limit Hamiltonian flow has a separatrix such that σ has several singularities with imaginary part equal to the minimum of the imaginary parts (in absolute value), a complicated pattern of homoclinics can develop. This depends, then, on the real parts of the singularities.

3. Measures of the lack of integrability. Most of the Hamiltonian systems appearing as models of physical problems are non integrable. However, to prove non integrability can be a non trivial problem. Several criteria have been developed, assuming that one knows a solution of the problem, $\varphi(t)$, to be considered as a Riemann surface, when the monodromy group given by the solutions of the first order variational equations along such surface is considered. They rely on Ziglin's theorem and on the different applications made by several

researchers. An alternative approach is based on the use of differential Galois theory (see [14] and recent work by Morales–Ramis). Work on the use of higher order variational equations is in progress.

However this algebraic approach do not gives (up to now, this may be different in the near future) quantitative information about "how to measure" the lack of integrability. A natural measure can be the size of the largest stochastic zone, the metric entropy or some other indicator. These in turn are related to the splitting of separatrices. If the system is an ε–perturbation of an integrable one (or we consider the vicinity of a totally elliptic fixed point and then the variation of the energy with respect to the one at the fixed point plays the role of ε) the lack of integrability turns out to be exponentially small with respect to the parameter. In [19] it is shown that the so called J_2 problem (the effect of the oblateness of a planet on a satellite moving around), which is a non integrable problem, produces effects that can be completely neglected from a quantitative point of view if we consider the case of the Earth.

4. Practical stability of totally elliptic fixed points. Consider a Hamiltonian around a totally elliptic fixed point. Assuming non resonance of the frequencies, a Normal Form up to some order (the order depending on the size of the domain considered, the smaller the domain the higher the order), can give exponentially small estimates to the rate of diffusion. Note that, in general, due to the non definite character of the quadratic part of the Hamiltonian around the fixed point one has to make use of suitable combinations of the action variables introduced by the Normal Form process. An application to the triangular points of the 3D circular restricted three–body problem appears in [8]. Note also that in the case of the 3D elliptic problem there is always a resonance between the mean motion and the vertical oscillations. However a resonant Normal Form still allows to obtain bounds on the rate of diffusion, but we should allow for some fast oscillations rather close to the libration point. See [11] for details.

5. A boundary layer effect on the triple collision manifold. The planar three–body problem can not be regularized in general (see [12]). That is, there is no way to pass from an orbit ending at triple collision to another one starting at a triple ejection in such a way that we recover continuity with respect to initial conditions. To study the problem one introduces a suitable blow up which gives rise to the so called "triple collision manifold", a manifold which replaces the triple collision singularity. The vector field given by Newton's law can be extended to the manifold (in an analytic way) and the fixed points of this vector field (all of them on the triple collision manifold) are related to possible limit shapes of the bodies when approaching to or escaping from collision. In fact they are central configurations.

   A natural question is, then, how we can "match" an approach to triple collision with an escape from it (or ejection). The possible passages are completely determined if we know about the existence (or not) of an heteroclinic connection between collisions of triangular type and ejections of the same type. These manifolds depend on the masses of the bodies. In [22, 23] It is shown that, in the case that the masses are of the form $m_1 = \varepsilon$, $m_2 = \varepsilon$, $m_3 = \varepsilon$ such an heteroclinic connection exists for a countable set of values of $\varepsilon = \varepsilon_n$, with $\varepsilon_n \to 0$ when $n \to \infty$. But, in fact, these connections are very difficult to detect numerically. Indeed, the connecting orbits leave from a vicinity of one of the triangular points (say, L_4^i), move to very close to the collinear subproblem (which is an invariant subproblem) and then they move quickly away from the vicinity of the collinear subproblem towards another triangular point (say, L_4^s). The minimum distance to the collinear subproblem is of the form $\exp(-c/\varepsilon)$, where c is given by a definite integral. Hence, most of the heteroclinic orbit sits on a "boundary layer" close to the collinear problem.

6. Center manifolds for Hamiltonian systems. Consider a fixed point of a Hamiltonian system which has an hyperbolic plane and all the other eigenvalues are totally elliptic. That is, the quadratic part is of the form

$$H_2 = \lambda x_1 y_1 + \frac{1}{2}\sum_{j=2}^{n} \omega_j (x_j^2 + y_j^2).$$

   It is clear that the fixed point has a center manifold of codimension 2. We can look for that center manifold by doing a canonical transformation which cancels all the terms of the form $x_1^{k_1} y_1^{l_1}$ with $k_1 + l_1 = 1$. This is possible (no problem of small divisors arises). Then, if we denote the new variables again as x and y, the center manifold is easily obtained as $x_1 = y_1 = 0$ (see [21]). We can ask about the convergence of the procedure. It leads to an analytical center manifold in the case of $n = 2$.

   The following result has been recently obtained [10]. Assume that the ω vector satisfies a diophantine condition. Let

$$H|_{W^c} = H_2 + H_3 + H_4 ...,$$

   where H_n denotes the homogeneous part of degree n, be the Hamiltonian restricted to the center manifold. Let $a_n = ||H_n||$ the norm of H_n obtained as sum of the moduli of the coefficients. Then a bound of the form

$$a_n < (\gamma \ln n)^n,$$

   where γ is some positive constant, follows. According to the elemntary considerations made at the Introduction, this results in an approximation to a center manifold which is doubly exponentially small .

7. The case of several frequencies. In [20] a very simple example is proposed to study things that can happen when studying the splitting in the case of two frequencies. The perturbation can be considered at a trigonometric polynomial, but a suitable reformulation and the lucky fact that the first step of the averaging procedure can be carried out explicitely, allows to show that, despite a few harmonics are present at the beginning, all harmonics show up (with coefficients explicitely computable) after the first averaging step.

   In general, the key idea to compute splitting in the multifrequency case is to compute formally the Melnikov integral (that is, to obtain the effect of the first order variational equations along the separatrix of the unperturbed problem) and then the big effort is to show that the result obtained in this way represents, indeed, the dominant part of the splitting.

   This approach is successful if the perturbation contains "all" the relevant harmonics from the very beginning or they can be made to appear by a simple transformation. Then the Melnikov computation gives a sum of products of terms of two different types: a factor which decreases when the order of the harmonic increases, because it carries higher powers of ε or because of Cauchy estimates, and another factor, which comes from the computation of the integrals. This term is exponentially small in ε, but the coefficient in the numerator of the exponential term is a small divisor. So, for a fixed value of ε low order harmonics cannot be dominant because the small divisors are not so small and the exponentially small term makes things small. Higher order harmonics cannot be dominant because despite the exponentially small can be close to 1 if the small divisor is smaller than ε (or a power of it), the coefficient or the high powers of ε appearing as the other factor make the term again negligible.

Hence, on should "optimize" to locate the current dominant terms. This approach was first introduced (as far as the author knows) in [20] and later on used systematically in papers as [4] and many other.

In the classical example of diffusion of Arnold, which can be presented as

$$H(y, I_2, I_3, x, \theta_2, \theta_3) = \frac{1}{2}\left(y^2 + I_2^2\right) + I_3 + \varepsilon(\cos x - 1)\left(1 + \mu(\cos\theta_2 + \cos\theta_3)\right) \qquad (6)$$

there are two small parameters. If $\mu = 0$ the system is integrable for all ε. Setting $\mu \neq 0$ the perturbation destroyes the integrability and a splitting shows up. The problem is that the perturbation contains only two harmonics. If μ is exponentially small with respect to ε (with the correct coefficients) the prediction of Melnikov formula is correct. The small divisors play no role. However, if bot parameters are equal or μ is a power of ε Melnikov prediction is wrong. Using a "higher order" variational (or Melnikov) approach, allows many more harmonics to show up, and they contribute to the splitting more than the initial term.

In [24] the system (6) is studied with $\mu = \varepsilon$. The approach is as follows: An averaging procedure is started until a given resonant terms appears as first contribution to the remainder. At this point the Melnokov formula is used to detect the effect of this term. To make things simple the ratio of frequencies is assumed to be a constant type number, e.g., the golden mean. Then the effects of different harmonics, related to small divisors, are compared to detect the dominant one. In [25] the procedure is fully justified.

As a result is turns out that, for any value of ε small enough, there are at most three different harmonics which play a key role. When two of them have a coefficient with the same modulus, some kind of bifurcation, giving rise to the creation and destruction of homoclinic points, shows up. The effect of the third (in order of magnitude) harmonic is more subtle. Assume that there is one dominant harmonic and that the next two have essentially the same order of magnitude. Then another kind of bifurcation appears and the homoclinic points become multiple and, in particular, non transversal. Both kinds of bifurcations appear, generically, for two sets of values of ε which tend to zero.

If we look for the variation of two momenta (say I_2 and I_3) an homoclinic point is found when $\Delta I_2 = 0$ and $\Delta I_3 = 0$. On a suitable Poincaré section (say, $x = \pi$) the conditions $\Delta I_2 = 0$ and $\Delta I_3 = 0$ define two curves. The bifurcations are related to the changes in the topology and intersections of these curves.

4 Numerical detection

Many subtle problems of exponentially small phenomena are worth to be investigated numerically, either before a systematic analytic study is carried out, to gain insight, or as a check and valuable confirmation of analytical results. Here some hints are given about how to proceed. For some of the methods I refer also to [5]. If we are interested in fine details one should work with multiple precision arithmetics, but it is worth always to proceed first with a version in standard double or quadruple precision, to detect possible mistakes in the algorithms or shortcuts to spped up the process. In any case one should keep in mind that it is extremely important how to analyze the results of the computations. For instance, the homoclinic invariant for the Hénon conservative map is of the form

$$\alpha = 10^6 \frac{4\pi}{h^6} \times \exp(-2\frac{\pi^2}{h}) \sum_{m\geq 0} \omega_m h^{2m}$$

where

$$\begin{aligned}\omega_0 = \; & 2.4744255935532510538408547495280197564980756378387855615663361394710252 \\ & 8273274948808418290910851169946433391445382830578763584069806693870578 \\ & 2428807388264012993781781027201833804313458132788909135263774014373885 \\ & 3037285651676805636028962808376746723918504793506206988599550672673741 \\ & 7853775173429774044625853973810145019928678619780817351345642853337845\end{aligned}$$

and where the ω_m are the coefficients of the correction term. I recall that h is $\ln \lambda(\varepsilon)$. It turns out that the sum $\sum_{m\geq 0} \omega_m h^{2m}$ is not convergent, but its Borel transform is, with a radius of convergence equal to 2δ, where δ is the smallest imaginary part of the singularities of $\sigma(t)$, the separatrix of the limit Hamiltonian. To be able to detect this, one has to compute the splitting for different values of ε, but also to organize things in such a way that it is easy to obtain, later on, the values of the ω_m.

In general the procedure to compute some splitting (either in the one frequency case or in the multifrequency one) is

- To obtain a representation of the invariant object for which the manifolds have to be computed. This is trivial if we want the invariant manifolds of a fixed point, either for a discrete map or for a flow. For a periodic orbit of a flow, use it as fixed point of a Poincaré map. For an invariant curve of a map or an invariant torus of a flow or map, use, e.g., a Lindstedt–Poincaré representation.
- To obtain a local representation of the invariant manifolds. This is simple in the case of 1D manifolds, asking for a representation as a graph or a natural representation. See [18] for several cases. Eventually this requires the computation of higher order derivatives of the Poincaré map. This can be done by integration of higher order variational equations or by numerical differentiation (with optimal step size, changing with the order of the differential). For higher dimensional manifolds local representations as Taylor–Fourier series and formal computation can be useful. See [22, 20, 24] for different examples.
- The homoclinic orbit can be then found as a solution for the local parameters and a fixed number of forward (along W^u) and backward (along W^s) iterates. A Newton method with a good initial guess is suitable. The guess can be obtained by continuation with respect the parameter.
- Working with multiple precision, all the process should be optimized. Higher order local expansions are expensive, but a too low order is also expensive, because then the number of iterates increases. In case one has to integrate a flow with multiple precision, definitely a ecurrent Taylor method is the most efficient way. It requires the computation of higher order derivatives. This is efficiently done by recurrences based on Leibnits rule. Remind also that the initial steps of Newton method can be done with less digits. Only at the last steps full accuracy is needed.

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