Unstable quasi-periodic solutions for the 2-D Poiseuille problem

Pablo S. Casas[1] Àngel Jorba[2]

Abstract

In the 2-D Poiseuille problem arise several Hopf bifurcations on the branch of secondary flows, which in turn bifurcate from the laminar solution. We analyze the first Hopf bifurcation of secondary flows where the period on time of the bifurcated solution is $O(1000)$. Previous calculations of these solutions show that the Hopf bifurcation is subcritical and thus the bifurcated quasi-periodic solutions are locally stable. By improving the precision of the numerical approximation we obtain unstable quasi-periodic flows given rise to a supercritical Hopf bifurcation.

Introduction and numerical approach

An important problem in fluid mechanics is the transition from laminar to turbulent flow. In this work we analyze the first steps towards this transition in a very simple setting: the Poiseuille problem. It consists of the flow of a viscous incompressible fluid, in a channel between two parallel walls at $y = \pm h$. We simplify the problem to dimension 2 where, as we will see, there exists already a great variety of phenomenon. We suppose the fluid driven by a constant stream-wise pressure gradient G and governed by the dimensionless Navier-Stokes equations in primitive variable formulation together with the incompressibility condition

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla p + \frac{1}{Re}\Delta\mathbf{u}, \qquad \nabla\cdot\mathbf{u} = 0, \tag{1}$$

where $\mathbf{u} = (u, v)$ are the components of the velocity, p is the pressure and $Re = Gh^3\rho/(2\mu^2)$ is the Reynolds number, defined in terms of the constant density ρ and viscosity μ. As boundary conditions we suppose

$$\left.\begin{aligned} u(x,\pm 1,t) = v(x,\pm 1,t) = 0 \\ (u,v,p')(x+L,y,t) = (u,v,p')(x,y,t) \end{aligned}\right\} \qquad x\in\mathbb{R}, \quad y\in[-1,1], \quad t\geq 0, \tag{2}$$

being $p - p' = -2x/Re$ and $L = 2\pi/\alpha$ (α is the parameter wave number). There exists a steady solution for (1)–(2) known as the basic or laminar flow that has a parabolic profile, namely

$$u_b(y) = 1 - y^2, \quad v_b = 0, \quad \nabla p_b = (-\frac{2}{Re}, 0).$$

This work is strongly based on the numerical integrator employed to follow the evolution of (1)–(2): we use the one described in [2]. In that work by means of a spectral method and eliminating the velocity component v and the pressure p we reduce the numerical approximation of (1) in space variables x, y to

$$\dot{u} = \mathcal{L}(u) + \mathcal{N}(u), \tag{3}$$

for $u = u(t) = (u_1, \ldots, u_K)(t)$ a vector of $K = (2N+1)(M-2)+1$ real components arising from the evaluation of the $2N+1$ Fourier coefficients in x of the velocity component

$u(x, y, t)$ at the $M + 1$ Chebyshev collocation points in y. $\mathcal{L}(u)$ and $\mathcal{N}(u)$ correspond to linear and non-linear terms in u respectively. In (3) we have a system of ordinary differential equations which is discretized in time by using finite differences.

Given a profile of velocities (u, v) we define its amplitude A, as the distance to the laminar solution $(u_b, 0)$ in the L_2-norm

$$A = \frac{1}{2L}\|(u - u_b, v)\|_2, \qquad \|(u, v)\|_2^2 = \int_0^L \int_{-1}^1 \left[u(x, y)^2 + v(x, y)^2\right] dy\, dx. \tag{4}$$

Our starting point is the bifurcation diagram presented in figure 1. Each point in this graph corresponds to the amplitude A of a time periodic flow (a rotating wave, see [7]) for a given Reynolds number Re and $\alpha = 1.1$ fixed. This kind of flows are steady in a frame of reference moving with an appropriate speed c in the stream-wise direction x. They have been obtained previously by some authors (see [9], [4] and [6] among others) and have also been reproduced in [2] for several values of α. Besides, in figure 1 it is represented the stability to super-harmonic disturbances of each flow and two Hopf bifurcations located at the stability changes. The second Hopf bifurcation was considered in [3] for $\alpha = 1.02056$. The objective of the present work is the local study of the first Hopf bifurcation at $Re \approx 3841$.

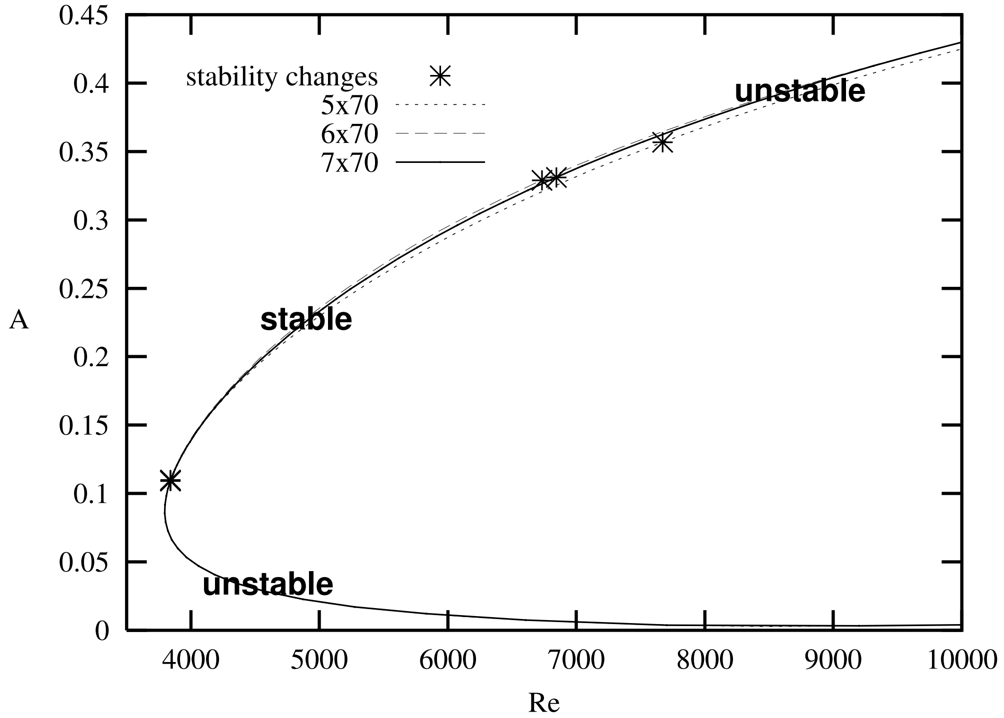

Fig. 1. Bifurcation curve of periodic flows. Each curve has specified $N \times M$. Calculations are shown for $\alpha = 1.1$ and $\Delta t = 0.02$.

Quasi-periodic solutions

In this section, we search for quasi-periodic flows that appear at the above mentioned Hopf bifurcation of rotating waves. For the values of α considered we have obtained qualitative different results than the ones reported by Soibelman & Meiron in [8]. For $\alpha = 1.1$ they found a subcritical Hopf bifurcation of quasi-periodic solutions, whereas by increasing N, the number of Fourier modes, we have achieved a supercritical Hopf bifurcation.

We take into account the spatio-temporal symmetry of our system. In [7] it is proved that, with the artificial boundaries of the channel at x and $x+L$ prescribed in (2), every solution $u(x,y,t)$ that lies on an isolated invariant 2-torus (a quasi-periodic solution), not asymptotic to a rotating wave, is a modulated wave, that is to say, there exists $\tau > 0$ and $\phi \in \mathbb{R}$ such that

$$u(x,y,n\tau+t) = u(x-n\phi,y,t) \quad \text{for every } n \in \mathbb{Z}. \tag{5}$$

As a consequence, this kind of wave has the property that, may be viewed as τ-periodic in a frame of reference moving with speed $c = (pL+\phi)/\tau$, for any integer p. In effect, defining $\tilde{x} = x - ct$ for the previous value of c, and $\tilde{u}(\tilde{x},y,t) = u(\tilde{x}+ct,y,t)$ as the velocity in the moving frame of reference at speed c, it turns out that

$$\tilde{u}(\tilde{x},y,\tau) \overset{1}{=} u(\tilde{x}+c\tau,y,\tau) \overset{2}{=} u(\tilde{x}+pL+\phi,y,\tau) \overset{3}{=} u(\tilde{x},y,0) \overset{4}{=} \tilde{u}(\tilde{x},y,0). \tag{6}$$

1, 4. By definition of $\tilde{u}$. 2. By definition of c. 3. Because u is L-periodic in x and a modulated wave. Consequently we have proved that $\tilde{u}(\tilde{x},y,t)$ is a τ-periodic function of t.

In order to look for such periodic flows, to each value of c we associate a Poincaré section Σ_1 of the space of solutions $u(t)$ by

$$\Sigma_1 = \left\{ u(t_c) \in \mathbb{R}^K \mid t_c \geq 0, u_M(t_c) = 0, \frac{\partial u_M}{\partial t}(t_c) > 0 \right\}$$

and construct a Poincaré map $P_c : \Sigma_1 \longrightarrow \Sigma_1$ as follows: Starting from an initial condition $u = u(0) \in \Sigma_1$ we integrate (3), until a time t_c such that $u(t_c) \in \Sigma_1$ for the n_c-th time, where n_c is a positive integer which represents the minimum number of times needed by the flow to return to the initial point $u(0)$. We then set $P_c(u(0)) = u(t_c)$. In this way we have reduced the search of quasi-periodic flows to a problem of finding zeros for the map F defined as

$$0 = F(u,c) \overset{\text{def}}{=} P_c(u) - u = u(t_c) - u(0). \tag{7}$$

From (5) it is easily checked that, if $P_c(u(0)) = u(t_c) = u(0)$ then $P_c(v) = v(t_c) = v(0)$ for $v(0) = P_0^k(u(0))$ for every positive integer k. Therefore we can generate different points on the same orbit as a solution of (7). We avoid this lack of uniqueness by restricting $P_c : \Sigma_1 \cap \Sigma_2 \longrightarrow \Sigma_1$ for

$$\Sigma_2 = \left\{ u(t_c) \in \mathbb{R}^K \mid t_c \geq 0, u_p(t_c) = r, \frac{\partial u_p}{\partial t}(t_c) > 0 \right\},$$

and some suitable values of $p \in \mathbb{Z}$, $1 \leq p \leq K$, and $r \in \mathbb{R}$.

We need an initial guess for the Newton's method applied to solve (7). We have taken $\alpha = 1.1$ and $M = 70$ (the number of Chebyshev abscissas in y) the same values as the ones used in [8], but we have increased the number of Fourier modes in x using $N \geq 4$ (in [8] it was taken $N \leq 2$). For $Re < 3841$ the corresponding time periodic flow in figure 1 is unstable, but its time evolution ultimately decays to the laminar flow. According to the results in [8] concerning with the existence of a subcritical Hopf bifurcation, we tried to find quasi-periodic flows in the subcritical region, but with no success: we were not able to detect a quasi-periodic attractor solution for $Re < 3841$.

As a consequence we direct the search for quasi-periodic flows to the supercritical region, i.e. for $Re > 3841$. It is known from bifurcation theory (see [5]) that in the case of a supercritical Hopf bifurcation in which fixed points before it are unstable and after it stable, the branch of periodic solutions that emanates at the bifurcation point is locally unstable. That should be the situation for $Re > 3841$ and thus the bifurcating quasi-periodic flows are locally unstable and consequently difficult to find, because the evolution of the fluid close to them does not remains near as time evolves.

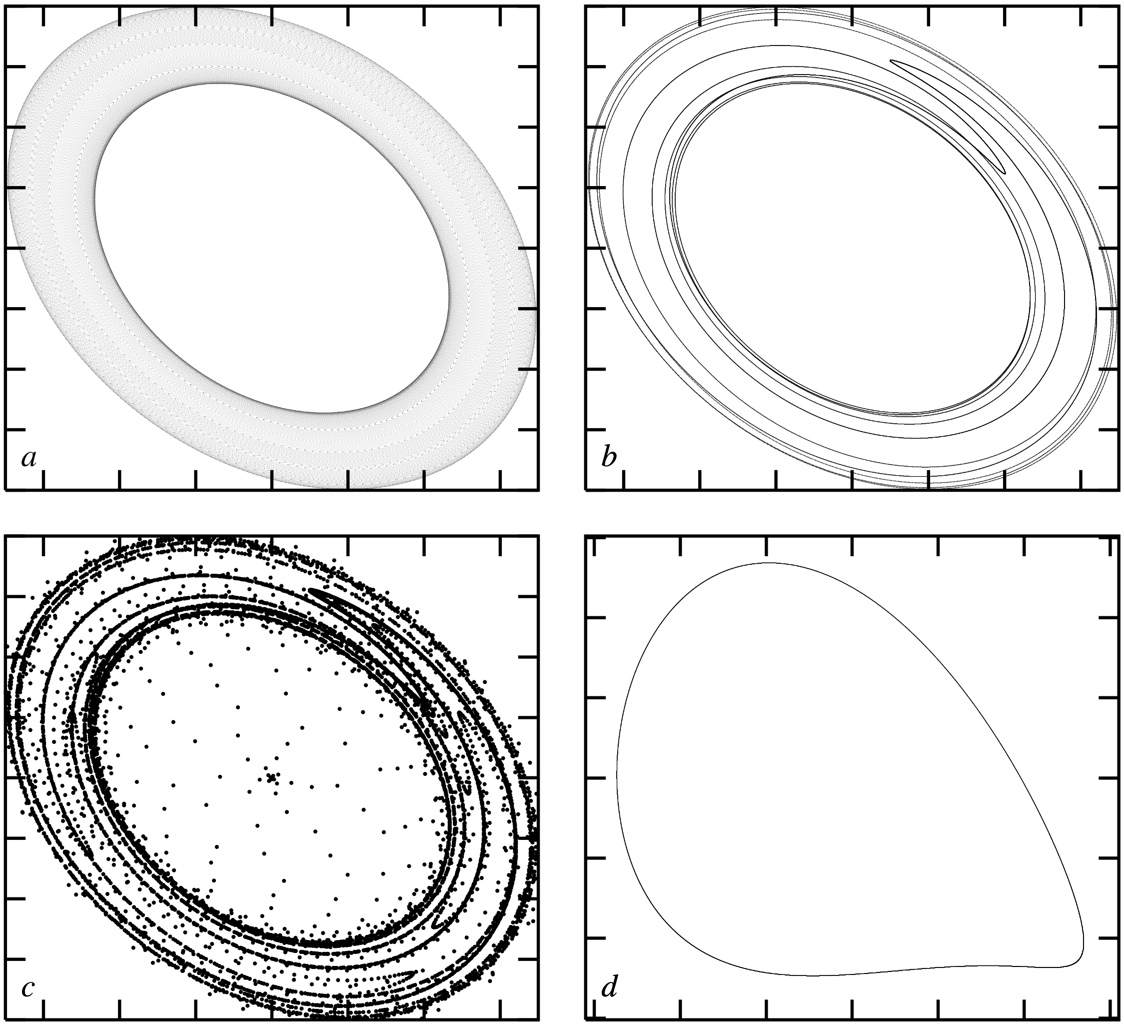

Fig. 2. Four representations of the solution vector $u(t)$ of a quasi-periodic flow projected on the plane of two arbitrary coordinates for Re $= 3853, \alpha = 1.1, N = 6, M = 70$. *The range of values for a,b,c is* $[-0.00175, 0.00175] \times [-0.0008, 0.0008]$ *and* $[-0.0011, -0.0005] \times [0.00035, 0.00038]$ *for d. a) Each* $\Delta t = 0.02$ *time units. b) Only when* $u(t) \in \Sigma_1$ *but integrating repeatedly for* $t \in [0, \tau]$*. c) As in b but without time restriction. d) As in b but on a different plane of coordinates.*

A good starting guess for the Newton's method applied to (7), was searched as follows. First we consider an unstable time periodic solution $u_P(t)$, i.e. a fixed point of (3), for

$Re < 3841$. Next we compute the eigenvalues of the linearization of (3) around the fixed point $u_P(t)$ and consider the eigenvector w (the vector along which the flow escapes in the fastest fashion from u_P) associated with the most unstable eigenvalue (complex with positive real part). Then we integrate numerically the initial condition $u_P(0)+rv$, for $|r| \ll 1$ a small constant, and the direction v is taken in the plane generated by the real and imaginary part of w. We follow the evolution of the initial condition $u_P(0) + rv$ until a time when it become as closest as possible to a quasi-periodic flow, according to (7), for a value of Re following the bifurcation and c near to the one associated with the corresponding periodic orbit for that Re. We also observe a value of the return time τ of P_c neighbouring to $2\pi/\Im(\lambda)$, being λ the only purely imaginary eigenvalue at the Hopf bifurcation.

The initial guess obtained in this way is improved through the Newton's method applied to the function $F(u, c)$ defined in (7). At this point it is worth to mention the great numerical difficulties in evaluating F because the time needed to return to Σ_1 is $\tau > 7000$. It has been necessary to adapt dynamically the minimum number of times n_c that the solution crosses Σ_1 before it returns to the starting point. For Re close to the Hopf bifurcation n_c starts at 1 and it is incremented by 1 when Re varies approximately in just 2–3 units. The approximation of the Jacobian matrix $DF(u, c)$ is implemented by finite differences in which every column is obtained evaluating F in parallel by means of a Beowulf system. This improvement, together with the use of the Broyden's "good" formula (see [1]) in the update of $DF(u, c)$, suppose a substantial increase in the speed of computations.

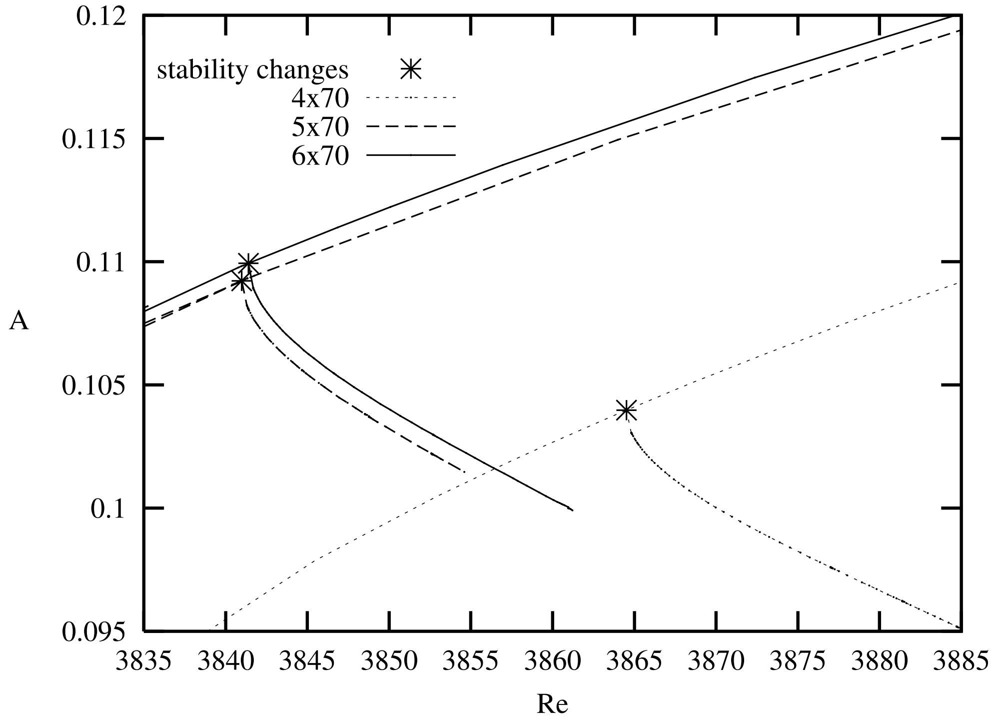

Fig. 3. Bifurcated branches of quasi-periodic flows in a supercritical Hopf bifurcation of periodic flows. Each curve has specified $N \times M$. Calculations are shown for $\alpha = 1.1$ and $\Delta t = 0.02$.

In figure 2 we present numerical evidence that solutions $u(t)$ of (7) lie in a 2-torus and are unstable on time. In a) we observe how the trajectory projected on a plane of two arbitrary coordinates of $u(t)$ fills densely a 2-torus. Likewise in b) we simply plot $u(t)$ when

$u(t) \in \Sigma_1$. Instead in this case if $t > \tau$, being τ is the period of $u(t)$ as a modulated wave, we plot $u(s)$ for s such that $0 \leq s = t - p\tau < \tau$ and p the integer part of t/τ. We observe here that the torus is reduced to a (complicated) closed curve. The restriction on time in b) is suppressed in c) where we let the flow evolve for long time and plot again $u(t)$ when $u(t) \in \Sigma_1$. In this graph it is shown that the flow is unstable because it moves away from the closed curve in b) and falls to the laminar régime: a simple point in the centre of the figure. The situation in d) is the same as in b) but the solution is projected on the plane of two other arbitrary coordinates and we observe again a (simple) closed curve.

Once we have obtained the first solution of (7) by the Newton's method, we take Re as a continuation parameter to trace the bifurcating branch of quasi-periodic flows. If $u(x, y, t)$ is a modulated wave then

$$\int_0^L \int_{-1}^1 u(x, y, \tau)^2 dy\, dx = \int_0^L \int_{-1}^1 u(x - \phi, y, 0)^2 dy\, dx = \int_0^L \int_{-1}^1 u(x, y, 0)^2 dy\, dx$$

and therefore its amplitude $A(t)$, as defined in (4), is a τ-periodic function of t. This is not the case for the rotating waves in figure 1 for which A is constant in time. Thus, in order to consider a concrete value for $A(t)$ in the case of a quasi-periodic flow $u(t)$, we evaluate it at the concrete time t when $u(t) \in \Sigma_1 \cap \Sigma_2$.

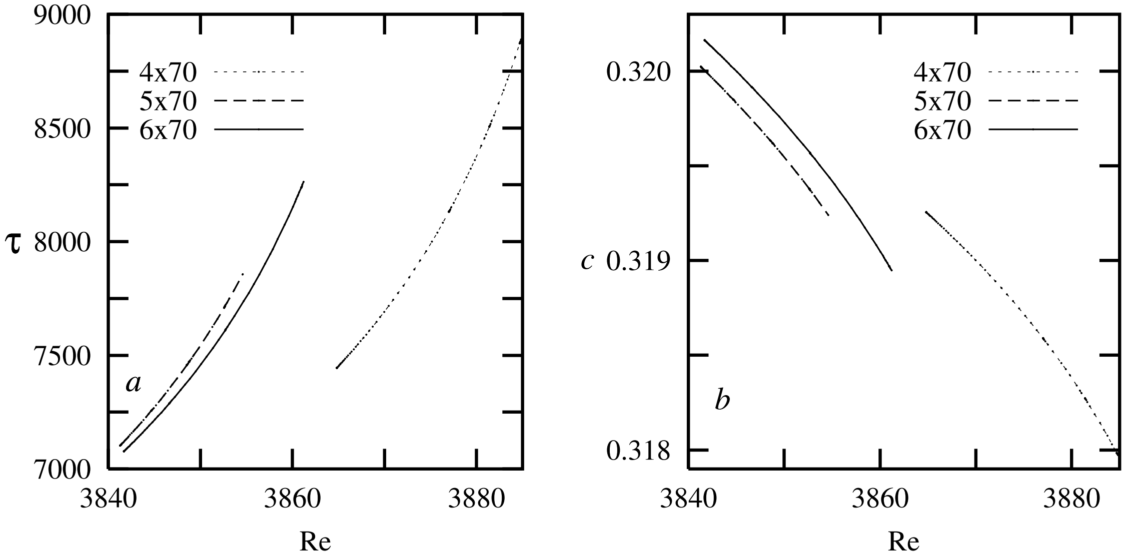

Fig. 4. Different curves for $N \times M$ points. a) τ represents the period on time for a quasi-periodic flow when the observer sees it as periodic. b) The appropriate value of c that converts a quasi-periodic flow in periodic.

In figure 3 we plot the amplitude A for each quasi-periodic solution as a function of Re. It seems that we have achieved both qualitative and quantitative convergence because we obtain a similar graph on increasing the values of N, M. It is also clear from this plot that the Hopf bifurcation is supercritical, so at least locally solutions on the bifurcating branch are unstable. The upper branch, not presented in this plot, consists of the same quasi-periodic solutions as on the lower branch, but they cross Σ_2 in opposite sense. We can easily do the analysis of the stability of a quasi-periodic solution by means of the eigenvalues of the Poincaré map P_c at a fixed point $u(t)$ such that $P_c(u(0)) = u(0)$. In the range of Re presented in figure 3

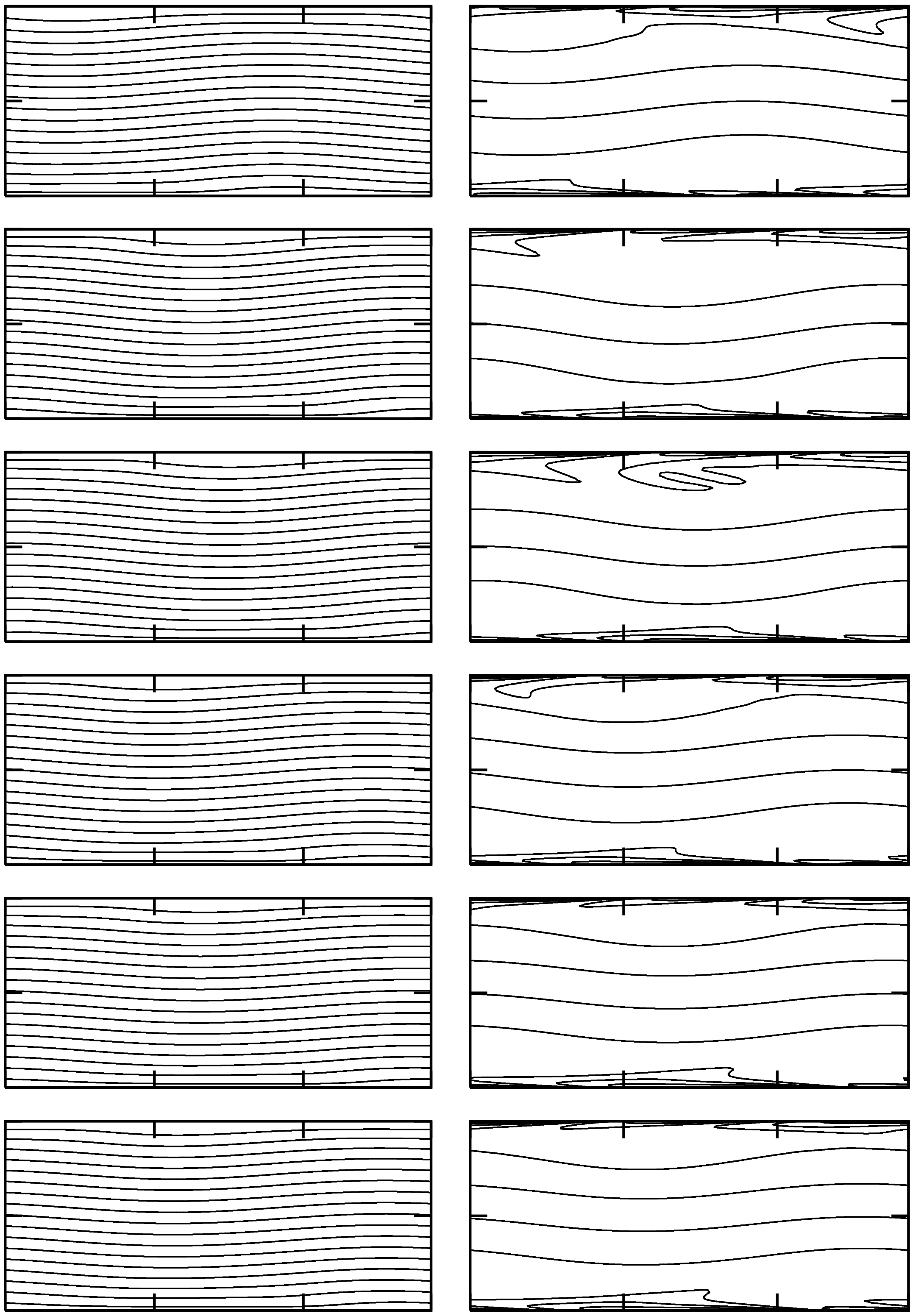

Fig. 5. Stream lines and levels of vorticity for a quasi-periodic flow at $Re = 3853, \alpha = 1.1, N = 6, M = 70, \Delta t = 0.02$. *Surface levels are plotted for* $0 \leq x \leq L, -1 \leq y \leq 1$. *The represented time instants are* $0, \tau/6, 2\tau/6, 3\tau/6, 4\tau/6$, *and* $5\tau/6$ *for* $\tau = 7608.78$.

the quasi-periodic solutions are unstable. An indicator of the numerical effort involved in the evaluation of the map P_c is plotted in figure 4 again for different values of N. The big slope of the Re-τ curve gives rise to high computational cost as Re is increased. Finally in figure 5 the stream lines and levels of vorticity of a quasi-periodic flow are presented, at six equidistant values of time in the interval $[0, \tau]$ for $\tau = 7608.78$.

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1. Departamento de Matemática Aplicada I. Universidad Politécnica de Cataluña.
Diagonal, 647; 08028 Barcelona. Spain.
e-mail: `pablo@casas.upc.es`

2. Departamento de Matemática Aplicada y Análisis. Universidad de Barcelona.
Gran Via, 585; 08007 Barcelona. Spain.
e-mail: `angel@maia.ub.es`