# On the centre manifold of collinear points in the planar three body problem

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**Abstract**

We study the existence of invariant tori in a neighbourhood of the collinear equilibrium points of the planar three body problem. To this end some properties of the normal form of the Hamiltonian reduced to the 4D central manifold are proved. Using this normal form, we show that the nondegeneracy conditions of KAM theorem are satisfied for all positive masses, including the $2:1$ resonance case. The evaluation of the conditions is done numerically.

## 1 Introduction

We consider the planar three body problem for positive masses $m_1, m_2, m_3$. The problem is described by a Hamiltonian system with six degrees of freedom. Using the classical integrals it can be reduced to a Hamiltonian with three degrees of freedom. Some special solutions of this problem, known as relative equilibrium solutions, play an important role in the dynamics. These are periodic solutions where the three masses rotate at constant angular velocity as a rigid body. So, they can be seen as equilibrium points in a rotating coordinate frame. There are three collinear equilibrium with the three masses on a line and two triangular equilibrium such that the three masses are in the vertex of an equilateral triangle. These special solutions are known since Euler and Lagrange.

In this paper we study the orbits in a neighbourhood of the collinear points for all positive values of the masses. It is not restrictive to consider values of the masses $m_1, m_2, m_3$ such that $m_1 + m_2 + m_3 = 1$. So, we can represent the set of admissible masses as points at the interior of a triangle where the vertex correspond to the limit cases $m_1 = 1, m_2 = 1$ and $m_3 = 1$ respectively (see figure 1). Note that the edges of this triangle correspond to restricted problems such that one of the masses is equal to zero. In a similar way, we shall use suitable normalizations for distance, time and gravitational constant. Given admissible masses $m_1, m_2, m_3$, the collinear equilibrium points are centre–centre–saddle points. For definiteness, we shall consider the collinear equilibrium with the body of mass $m_2$ in the middle. Denote by $\pm\lambda, \pm i, \pm i\omega$ the eigenvalues of the linear part. So, in a neighbourhood of the equilibrium point the quadratic part of the Hamiltonian can be written as

$$H_2(\mathbf{Q}, \mathbf{P}) = \lambda Q_1 P_1 + \frac{1}{2}(Q_2^2 + P_2^2) + \frac{\omega}{2}(Q_3^2 + P_3^2).$$

The equilibrium point has an one–dimensional unstable manifold, an one–dimensional stable one and a four–dimensional centre manifold. Two families of periodic orbits which are born at the equilibrium point live on the centre manifold: the Lyapunov family with a period approaching $2\pi/\omega$ when the periodic orbits tend to the point, and the homographic family of periodic orbits with a period tending to $2\pi$. However, for the quadratic part $H_2(\mathbf{Q}, \mathbf{P})$, these periodic orbits are surrounded by two–dimensional invariant tori. The preservation of these invariant tori for the complete Hamiltonian is guaranteed by KAM theorem under some nondegeneracy conditions.

The purpose of this paper is to study the applicability of KAM theorem in a neighbourhood of the collinear points for any values of the masses. To this end we do the following steps. First, we perform some canonical transformations to write the Hamiltonian in normal form. Then we reduce the Hamiltonian to the centre manifold. After that, we check, by numerical evaluation of the coefficients of the normal form, the nondegeneracy conditions of KAM theorem. The results presented in section 7 show that both conditions (either isoenergetic or not) are satisfied for any values of the masses in the triangle of masses.

However, we remark that for some values of the masses the eigenvalues at the collinear points are $\pm\lambda, \pm i, \pm 2i$ and a resonance $2:1$ takes place. In fact, in section 3 we prove that $1 < \omega < \sqrt{8\sqrt{2}-3} \approx 2.88335022...$ So, up to order 4, we only need to take into account the resonance $2:1$. The corresponding resonant masses describes a curve in the triangle of masses. Therefore, for resonant masses it is expected to get resonant monomials of order three in the normal form of the Hamiltonian. We prove in section 6 that this is not the case. In fact, we prove that the coefficients of these monomials are different from zero for general masses but they become zero for resonant masses, and also in the symmetrical case $m_1 = m_3$. The existence of the homographic solutions allows us to compute analytically, in an easy way, the coefficients of the resonant monomials of order three. These coefficients have $(\omega - 2)$ as a factor.

## 2 Reduction of the order

Consider three point masses $m_1, m_2, m_3$ moving in a plane under their gravitational attraction. Let $\mathbf{q}_i$ be the position vector of the $i$–th particle in an inertial reference system, $\mathbf{p}_i$ the conjugate momentum and let $\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3), \mathbf{p} = (\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3)$. The equations of motion can be written as a Hamiltonian system with six degrees of freedom with Hamiltonian function

$$\mathcal{H}(\mathbf{q}, \mathbf{p}) = \frac{1}{2}\mathbf{p}^T M^{-1}\mathbf{p} - U(\mathbf{q}), \tag{1}$$

where $M = \operatorname{diag}(m_1, m_1, m_2, m_2, m_3, m_3)$ and

$$U(\mathbf{q}) = \sum_{1 \le i < j \le 3} \frac{m_i m_j}{\|\mathbf{q}_i - \mathbf{q}_j\|}.$$

After a rescaling of the variables we can assume $m_1 + m_2 + m_3 = 1$.

We introduce new variables $\boldsymbol{\zeta}, \boldsymbol{\nu}$ in a rotating frame

$$\mathbf{q} = \Omega(t)\boldsymbol{\zeta}, \qquad \mathbf{p} = \Omega(t)\boldsymbol{\nu}$$

where

$$\Omega(t) = \operatorname{diag}(\Omega_1(t), \Omega_1(t), \Omega_1(t)), \qquad \Omega_1(t) = \begin{pmatrix} \cos(\alpha t) & -\sin(\alpha t) \\ \sin(\alpha t) & \cos(\alpha t) \end{pmatrix},$$

$\alpha$ being a positive constant to be fixed later. The equations can be written as

$$\begin{aligned} \dot{\boldsymbol{\zeta}} &= M^{-1}\boldsymbol{\nu} - \alpha J_3 \boldsymbol{\zeta}, \\ \dot{\boldsymbol{\nu}} &= \nabla U(\boldsymbol{\zeta}) - \alpha J_3 \boldsymbol{\nu}, \end{aligned} \tag{2}$$

being $J_3 = \operatorname{diag}(J, J, J)$, $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. So, we get a new Hamiltonian system with Hamiltonian function

$$\mathcal{H}(\boldsymbol{\zeta}, \boldsymbol{\nu}) = \frac{1}{2}\boldsymbol{\nu}^T M^{-1}\boldsymbol{\nu} + \alpha \boldsymbol{\zeta}^T J_3 \boldsymbol{\nu} - U(\boldsymbol{\zeta}).$$

Now, system (2) has some equilibrium points, $(\boldsymbol{\zeta}_c, \boldsymbol{\nu}_c)$, where $\boldsymbol{\zeta}_c$ is a solution of

$$M^{-1}\nabla U(\boldsymbol{\zeta}) = -\alpha^2\boldsymbol{\zeta} \tag{3}$$

and $\boldsymbol{\nu}_c = \alpha M J_3 \boldsymbol{\zeta}_c$. The solutions of the equation (3) are called central configurations. It is well known that for the planar three body problem there are three collinear central configurations for which the three masses are on a line, and two triangular configurations, where the three masses are on the vertex of an equilateral triangle.

Note that for any equilibrium point $(\boldsymbol{\zeta}_c, \boldsymbol{\nu}_c)$ of (2), $\mathbf{q}(t) = \Omega(t)\boldsymbol{\zeta}_c$, $\mathbf{p}(t) = \Omega(t)\boldsymbol{\nu}_c$ is a periodic solution of (1). These periodic orbits are called relative equilibrium solutions.

After a scaling of time, $\alpha t$, and variables, $\alpha^{2/3}\boldsymbol{\zeta}$, $\alpha^{-1/3}\boldsymbol{\nu}$, we can reduce to consider $\alpha = 1$. So, from now on we fix $\alpha = 1$. This amounts to consider $2\pi-$periodic relative equilibrium solution. In order to study a neigbourhood of the collinear equilibrium points first we reduce the dimension of the system. The phase space defined by (2) has dimension 12 but we can reduce to dimension 6 by using the classical integrals, that is, the integrals of the center of masses

$$\begin{aligned} m_1\mathbf{q}_1 + m_2\mathbf{q}_2 + m_3\mathbf{q}_3 &= 0, \\ \mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 &= 0, \end{aligned}$$

and the angular momentum

$$\mathbf{p}_1 \wedge \mathbf{q}_1 + \mathbf{p}_2 \wedge \mathbf{q}_2 + \mathbf{p}_3 \wedge \mathbf{q}_3 = \mathbf{c}.$$

To perform the reduction of the system we use the canonical transformations defined in [13].

We introduce $\boldsymbol{\xi}_i \in \mathbb{R}^2, \boldsymbol{\eta}_i \in \mathbb{R}^2, i = 1, 2, 3$.

$$\begin{aligned} \boldsymbol{\xi}_i &= \boldsymbol{\zeta}_i - \boldsymbol{\zeta}_3, & i &= 1, 2, & \boldsymbol{\xi}_3 &= \boldsymbol{\zeta}_3, \\ \boldsymbol{\eta}_i &= \boldsymbol{\nu}_i, & i &= 1, 2, & \boldsymbol{\eta}_3 &= \boldsymbol{\nu}_1 + \boldsymbol{\nu}_2 + \boldsymbol{\nu}_3. \end{aligned}$$

Now $\boldsymbol{\eta}_3 = 0$ and $\boldsymbol{\xi}_3$ is an ignorable variable. In this way we can reduce, for $\boldsymbol{\xi} = (\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \boldsymbol{\xi}_3)$ and $\boldsymbol{\eta} = (\boldsymbol{\eta}_1, \boldsymbol{\eta}_2, \boldsymbol{\eta}_3)$, to a Hamiltonian system with four degrees of freedom. Consider the canonical transformation defined by

$$\begin{aligned} \boldsymbol{\xi}_1 &= u_1\mathbf{w}, & v_1 &= \boldsymbol{\eta}_1^T\mathbf{w}, & v_4 &= \sum_{k=1}^{3} \boldsymbol{\xi}_k^T J \boldsymbol{\eta}_k, \\ \boldsymbol{\xi}_2 &= L\begin{pmatrix} u_2 \\ u_3 \end{pmatrix}, & \begin{pmatrix} v_2 \\ v_3 \end{pmatrix} &= L^{-1}\boldsymbol{\eta}_2, \end{aligned}$$

where

$$\mathbf{w} = \begin{pmatrix} \cos u_4 \\ -\sin u_4 \end{pmatrix}, \qquad L = \begin{pmatrix} \cos u_4 & \sin u_4 \\ -\sin u_4 & \cos u_4 \end{pmatrix}.$$

The new Hamiltonian can be written as

$$\mathcal{H}(\mathbf{u}, \mathbf{v}) = T(\mathbf{u}, \mathbf{v}) - U(\mathbf{u}) + v_4, \tag{4}$$

where $\mathbf{u} = (u_1, u_2, u_3, u_4)$,

$$\begin{aligned} T(\mathbf{u}, \mathbf{v}) &= \frac{1}{2m_3}\left[(v_1 + v_2)^2 + \left(v_3 + \frac{A}{u_1}\right)^2\right] + \frac{1}{2m_1u_1^2}(u_1^2v_1^2 + A^2) + \frac{1}{2m_2}(v_2^2 + v_3^2), \\ U(\mathbf{u}) &= \frac{m_1m_2}{r_{12}} + \frac{m_1m_3}{r_{13}} + \frac{m_2m_3}{r_{23}}, \\ A &= -v_4 + u_3v_2 - u_2v_3, \;\; r_{12} = [(u_1 - u_2)^2 + u_3^2]^{1/2}, \;\; r_{13} = |u_1|, \;\; r_{23} = (u_2^2 + u_3^2)^{1/2}. \end{aligned} \tag{5}$$

Note that the Hamiltonian (4) does not depend on $u_4$. So, this is an ignorable variable and fixing the value of $v_4$ we can reduce to a Hamiltonian system with three degrees of freedom

$$\begin{aligned} \dot{u}_i &= \mathcal{H}_{v_i}, \\ \dot{v}_i &= -\mathcal{H}_{u_i}, \quad i = 1, 2, 3. \end{aligned} \tag{6}$$

In fact, $v_4$ is the angular momentum, ($\|\mathbf{c}\|$), and the last reduction of the order of the system is equivalent to the classical elimination of the node.

The next lemma follows from inspection of the Hamiltonian $\mathcal{H}(\mathbf{u}, \mathbf{v})$.

**Lemma 1.** *Let be $\mathcal{H}(\mathbf{u}, \mathbf{v})$ the Hamiltonian defined in (4). Then*
*(i) For any value of the masses $\mathcal{H}(S_1(\mathbf{u}, \mathbf{v})) = \mathcal{H}(\mathbf{u}, \mathbf{v})$ where*

$$S_1 = diag(1, 1, -1, -1, -1, 1).$$

*(ii) For any value of the masses such that $m_1 = m_3$, $\mathcal{H}(S_2(\mathbf{u}, \mathbf{v})) = \mathcal{H}(\mathbf{u}, \mathbf{v})$ holds, where*

$$S_2 = \begin{pmatrix} S_{11} & 0 \\ 0 & S_{22} \end{pmatrix}, \quad \text{with} \quad S_{11} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad S_{22} = diag(1, 1, -1).$$

## 3 The collinear equilibrium points

We consider a collinear equilibrium point with the masses ordered from the left to the right as $m_3, m_2, m_1$ and we denote that point by $L_2$. We shall study that point for all values in the triangle of the masses. Other collinear points are obtained from $L_2$ by changing the values of the masses.

We denote by $(\mathbf{u}^p, \mathbf{v}^p) = (u_1^p, u_2^p, u_3^p, v_1^p, v_2^p, v_3^p)$ the coordinates of the point $L_2$. An easy calculation shows

$$\begin{aligned} u_1^p &= a(\rho + 1), & v_1^p &= 0, \\ u_2^p &= a, & v_2^p &= 0, \\ u_3^p &= 0, & v_3^p &= m_2 a(-m_1\rho + m_3), \end{aligned}$$

where $\rho$ is the solution of Euler's quintic equation

$$\begin{aligned} q(\rho) := \rho^5(m_2 + m_3) + \rho^4(2m_2 + 3m_3) + \rho^3(m_2 + 3m_3) - \rho^2(3m_1 + m_2) - \\ \rho(3m_1 + 2m_2) - (m_1 + m_2) = 0, \end{aligned} \tag{7}$$

and $a^3 = -\dfrac{m_1}{\rho^2} + \dfrac{m_1}{(\rho+1)^2} + m_2 + m_3$. We also get

$$v_4^p = -a^2[m_1 m_3(\rho + 1)^2 + m_2(m_1\rho^2 + m_3)]. \tag{8}$$

It is well known ([13]) that the characteristic polynomial of the linearized system of (6) at the point $L_2$ is

$$p(x) = (x^2 + 1)(x^4 + (1 - \beta)x^2 - \beta(2\beta + 3)) \tag{9}$$

where

$$\beta = \frac{m_1(1 + \delta^{-1} + \delta^{-2}) + m_3(1 + \sigma^{-1} + \sigma^{-2})}{m_1 + m_2(\delta^{-2} + \sigma^{-2}) + m_3}, \quad \delta = \frac{\rho}{\rho + 1}, \quad \sigma = \frac{1}{\rho + 1}. \tag{10}$$

The polynomial (9) has two real zeros $\pm\lambda$ and two pairs of imaginary ones, $\pm i$, $\pm i\omega$. Therefore $L_2$ is a centre–centre–saddle point.

From now on we fix the angular momentum equal to $v_4^p$, that is, the angular momentum at the equilibrium point and we shall consider different values of the energy.

In order to expand the Hamiltonian in a neigbourhood of the equilibrium point, first we move the equilibrium to the origin introducing $\mathbf{x} = \frac{1}{a\rho}(\mathbf{u} - \mathbf{u}^p)$, $\mathbf{y} = \mathbf{v} - \mathbf{v}^p$ which defines a canonical transformation with multiplier $1/(a\rho)$. The new Hamiltonian

$$\bar{H}(\mathbf{x}, \mathbf{y}) = \frac{1}{a\rho}\mathcal{H}(\mathbf{u}, \mathbf{v}) \tag{11}$$

is expanded using Legendre polynomials $P_n(x)$ $\left((1 - 2tx + t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)t^n\right)$. From (4) and (5) we obtain

$$\bar{H}(\mathbf{x}, \mathbf{y}) = \bar{H}_0 + \sum_{k=2}^{\infty} \bar{H}_k(\mathbf{x}, \mathbf{y}),$$

where $\bar{H}_k(\mathbf{x}, \mathbf{y})$ is an homogeneous polynomial of degree $k$. In the following we will assume the constant term $\bar{H}_0$ added to the energy $h$. In Appendix 1 we give explicitely $\bar{H}_2$ and $\bar{H}_3$.

In order to write the quadratic part of the Hamiltonian in a canonical form we perform the following transformation

$$\begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} = \mathcal{D} \begin{pmatrix} \mathbf{Q} \\ \mathbf{P} \end{pmatrix}, \tag{12}$$

where $\mathcal{D}$ is a $6 \times 6$ matrix defined by column vectors as

$$\mathcal{D} = (\bar{k}_0\mathbf{z}_1, \bar{k}_1\mathbf{e}_1, \bar{k}_2\mathbf{e}_2, \bar{k}_0\mathbf{z}_4, \bar{k}_1\mathbf{f}_1, \bar{k}_2\mathbf{f}_2). \tag{13}$$

In $\mathcal{D}$, $\mathbf{z}$ denotes an eigenvector of $D\tilde{F}(0,0)$, where $\tilde{F}$ is the vector field defined by $\bar{H}$, according to the following convention: $\mathbf{z}_1$ and $\mathbf{z}_4$ are eigenvectors corresponding to the real eigenvalues $\lambda$ and $-\lambda$ respectively, and $\mathbf{z}_2 = \mathbf{e}_1 + i\mathbf{f}_1$ and $\mathbf{z}_3 = \mathbf{e}_2 + i\mathbf{f}_2$ are eigenvectors for $i$ and $i\omega$ respectively. In (13)

$$\bar{k}_0^2 = \frac{1}{\mathbf{z}_1^T J \mathbf{z}_4}, \qquad \bar{k}_1^2 = \frac{1}{\mathbf{e}_1^T J \mathbf{f}_1}, \qquad \bar{k}_2^2 = \frac{1}{\mathbf{e}_2^T J \mathbf{f}_2},$$

where $J$ is the skew symmetric matrix $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$. It is well known (see [10]) that if $\mathbf{z}_a, \mathbf{z}_b$ are eigenvectors corresponding to eigenvalues $\lambda_a$ and $\lambda_b$ such that $\lambda_a + \lambda_b \neq 0$, then $\mathbf{z}_a^T J \mathbf{z}_b = 0$. This orthogonality property implies that the transformation (12) is canonical.

If we denote by $H(\mathbf{Q}, \mathbf{P})$ the Hamiltonian in the new variables, the quadratic part is

$$H_2(\mathbf{Q}, \mathbf{P}) = \lambda Q_1 P_1 + \frac{1}{2}(Q_2^2 + P_2^2) + \frac{\omega}{2}(Q_3^2 + P_3^2). \tag{14}$$

In order to simplify the computations needed to get the normal form it is convenient to introduce new complex variables $\boldsymbol{\xi}, \boldsymbol{\eta}$

$$\begin{aligned} \xi_1 &= \bar{k}_0 Q_1, \qquad & Q_j &= \frac{1}{\bar{k}_{j-1}}\left(\xi_j + i\frac{\bar{k}_{j-1}}{2}\eta_j\right), \\ \eta_1 &= \frac{1}{\bar{k}_0}P_1, \qquad & P_j &= i\frac{1}{\bar{k}_{j-1}}\left(\xi_j - i\frac{\bar{k}_{j-1}}{2}\eta_j\right), \quad j = 2, 3. \end{aligned} \tag{15}$$

This transformation is canonical and the Hamiltonian in $\boldsymbol{\xi}, \boldsymbol{\eta}$ can be written as

$$H(\boldsymbol{\xi}, \boldsymbol{\eta}) = \lambda \xi_1 \eta_1 + i \xi_2 \eta_2 + i\omega \xi_3 \eta_3 + \sum_{k \geq 3} H_k(\xi, \eta). \tag{16}$$

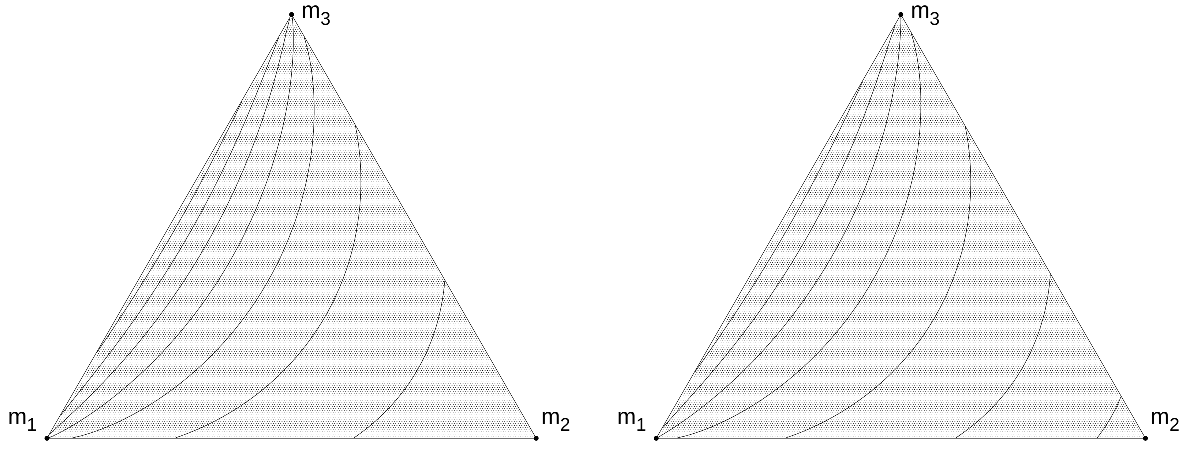
We remark that one can define a canonical transformation from $(\mathbf{x}, \mathbf{y})$ to $(\boldsymbol{\xi}, \boldsymbol{\eta})$. Anyhow we shall use the intermediate form $H(\mathbf{Q}, \mathbf{P})$ to get some properties for the homographic solutions.

The rest of the section is devoted to obtain technical properties of the eigenvalues and eigenvectors involved in the transformations described above. These properties will be used in the following sections.

First we note that if we consider $\rho$ and $\beta$ as functions of $m_1$ and $m_3$, it is clear using (7) and (10) that $\rho(m_3, m_1) = 1/\rho(m_1, m_3)$ and $\beta(m_1, m_3) = \beta(m_3, m_1)$. However $q(1) = 7(m_3 - m_1)$. Therefore, if $m_1 \leq m_3$ we have that the solution of the quintic equation satisfies $\rho \leq 1$ and the solution is $\rho = 1$ if and only if $m_1 = m_3$. Moreover $q(0) = -(m_1 + m_2)$ and $\rho = 0$ if and only if $m_1 = m_2 = 0$.

**Lemma 2.** *For any positive masses,* $1 < \omega < \omega_M = \sqrt{8\sqrt{2} - 3} \approx 2.88335022....$ *The values* $1$ *and* $\omega_M$ *are attained in the limit cases* $m_2 = 1$ *and* $m_2 = 0, m_1 = m_3 = 1/2$ *respectively. Moreover, if we consider* $\omega$ *as a function of* $m_1$ *and* $m_2$*, then* $\dfrac{\partial \omega}{\partial m_2}(m_1, m_2) < 0$.

The figure 1 shows the level sets of $\omega$ and $\lambda$ in the triangle of masses. They look similar because these curves are, in particular, level curves of $\beta$.




(a) (b)

Figure 1: Level sets of $\omega$ and $\lambda$. In (a) from left to right the values of $\omega$ are: $j/4$, $j = 11, \ldots, 5$. In (b) from left to right the values of $\lambda$ are: $j/2$, $j = 7, \ldots, 1$.

**Proof**

First we shall determine the sign of $\dfrac{\partial \omega}{\partial m_2}$. We fix the value of $m_1 \in (0, 1)$ and we take $m_3 = 1 - m_1 - m_2$. Note that $m_2 \in (0, 1 - m_1)$. We consider $\omega$ as a function of $m_2$, $\omega(m_2)$, and we have $\omega = \omega(m_2) = \sqrt{f(m_2)/2}$ where

$$f(m_2) = 1 - \beta + \sqrt{9\beta^2 + 10\beta + 1}. \tag{17}$$

Our purpose is to prove that $f(m_2)$ is a strictly decreasing function of $m_2$.

From (10) we can write

$$\beta = \frac{m_1(3\rho^2+3\rho+1)+(1-m_1-m_2)\rho^2(\rho^2+3\rho+3)}{\rho^2+m_2[(\rho+1)^2(1+\rho^2)-\rho^2]}. \tag{18}$$

In (18), $\rho$ is the solution of (7) and so, if we fix $m_1$ we can consider $\rho$ and also $\beta$ as functions of $m_2$. By derivating (17) with respect to $m_2$ we get

$$\frac{df(m_2)}{dm_2} = \left[-1+\frac{9\beta+5}{\sqrt{9\beta^2+10\beta+1}}\right]\frac{d\beta(m_2)}{dm_2}.$$

It is easy to check that the first factor in the product above is positive. Therefore $\omega$ is a strictly increasing function of $\beta$. The following claim will be proved in the appendix 2.

*Claim:* $\dfrac{d\beta(m_2)}{dm_2} < 0$.

In this way we get that $\beta(m_2)$ and also $\omega(m_2)$ are strictly decreasing functions of $m_2$.

Now we prove that $\omega \in (1, \omega_M)$. Using the symmetry of $\beta$ with respect to the line $m_1 = m_3$ in the triangle of masses, it is sufficient to consider $m_1 \leq m_3$, that is, $\rho \in (0,1)$. Moreover for a fixed value of $m_1$ the monotonicity implies that the maximum of $\beta$ is reached when $m_2 = 0$ and the minimum when $m_1 = m_3$. If $m_1 = m_3$ we get $\beta = 7(1-m_2)/(1+7m_2)$ and the minimum of that function of $m_2$ is reached at $m_2 = 1$. In this case $\beta = 0$ and so $\omega = 1$. Therefore, for positive masses $\omega > 1$.

Now we study the limit case $m_2 = 0$. The quintic equation can be written as

$$m_1(3\rho^2+3\rho+1) = m_3\rho^3(\rho^2+3\rho+3). \tag{19}$$

Then

$$\beta = (1-m_1)(1+\rho)(\rho^2+3\rho+3). \tag{20}$$

From (19) and (20) we get $\beta = (\rho^2+3\rho+3)(3\rho^2+3\rho+1)/(\rho^4+2\rho^3+\rho^2+2\rho+1)$. It is easy to check that $\beta \leq 7$ and $\beta = 7$ if and only if $\rho = 1$. This means that the maximum value of $\beta$ in the triangle of masses is 7. As $\omega$ is a strictly increasing function of $\beta$ we get the maximum value $\omega = \omega_M$ when $\beta = 7$. This ends the proof of the lemma. □

**Remark 1.** *Numerically one can see that for any fixed value of $m_2$ the function $\beta$ has its maximum for masses such that $m_1 = m_3$. Hence the same follows for $\omega$ and $\lambda$.*

Now we are interested in the eigenvectors of $D\tilde{F}(0,0)$ where $\tilde{F}$ is the vector field defined by $\bar{H}$. They will be easily obtained from the corresponding eigenvectors of $DF(\mathbf{u}^p, \mathbf{v}^p)$ where $F$ is the vector field defined by (6). After some computations we get

$$DF(\mathbf{u}^p, \mathbf{v}^p) = \begin{pmatrix} A_1 & A_2 \\ A_3 & -A_1^T \end{pmatrix} \tag{21}$$

where

$$A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ k_1 & k_2 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} \alpha_1 & 1/m_3 & 0 \\ 1/m_3 & \alpha_2 & 0 \\ 0 & 0 & k_3 \end{pmatrix}, \quad A_3 = \begin{pmatrix} k_4 & k_5 & 0 \\ k_5 & k_6 & 0 \\ 0 & 0 & k_7 \end{pmatrix}$$

with $\alpha_1 = \dfrac{m_1+m_3}{m_1 m_3}$, $\alpha_2 = \dfrac{m_2+m_3}{m_2 m_3}$ and $k_1, k_2, \ldots, k_7$ given in Appendix 3.

Let us denote by $\mathbf{w} = (u_1, u_2, u_3, v_1, v_2, v_3)^T$ an eigenvector of $DF(\mathbf{u}^p, \mathbf{v}^p)$ with eigenvalue $\mu$. Then $\mathbf{u} = (u_1, u_2, u_3)^T$ can be obtained as a non trivial solution of

$$(\mu^2 I + \mu N - L)\mathbf{u} = 0$$

where $N = A_2 A_1^T A_2^{-1} - A_1$, $L = A_2(A_3 + A_1^T A_2^{-1} A_1)$ and then $\mathbf{v} = (v_1, v_2, v_3)^T$ is given by $\mathbf{v} = -A_2^{-1}(A_1 - \mu I)\mathbf{u}$. It is easy to check that $k_3 \neq 0$ and so, $A_2$ is non singular for positive masses.

Moreover some trivial properties can be obtained for these eigenvectors. Using the symmetry $S_1$ introduced in lemma 1 we have $S_1 DF(\mathbf{u}^p, \mathbf{v}^p) S_1 = -DF(\mathbf{u}^p, \mathbf{v}^p)$. Therefore, if $\mathbf{w}$ is an eigenvector of $DF(\mathbf{u}^p, \mathbf{v}^p)$ corresponding to an eigenvalue $\mu$, $S_1\mathbf{w}$ is an eigenvector for the eigenvalue $-\mu$. It is also easy to see that if $\mu$ is imaginary and $\mathbf{w}$ is the corresponding eigenvalue, then we can take $u_1, u_2, v_3$ real and $u_3, v_1, v_2$ imaginary.

For arbitrary masses we get the following eigenvector for the eigenvalue $i$

$$\begin{aligned} \mathbf{w}_2 &= (1, k, 0, \sigma_1 i, \sigma_2 i, -\sigma_2)^T, \quad \text{where} \\ & k = \frac{1}{\rho+1}, \quad \sigma_1 = m_1(m_2\rho k + m_3), \quad \sigma_2 = m_2(m_3 - m_1\rho)k. \end{aligned} \tag{22}$$

The relation given in the following lemma will be used in section 5.

**Lemma 3.** *Let $\mathbf{w} = (u_1, u_2, u_3, v_1, v_2, v_3)^T$ be an eigenvector of $DF(\mathbf{u}^p, \mathbf{v}^p)$ corresponding to one of the eigenvalues $\lambda$ or $i\omega$. Then*

$$u_1 m_1(m_3 + \rho(1 - m_1)) + u_2 m_2(m_3 - m_1\rho) = 0. \tag{23}$$

**Proof**

We write the eigenvector (22) corresponding to the eigenvalue $i$ as $\mathbf{w}_2 = \mathbf{e}_1 + i\mathbf{f}_1$ with $\mathbf{e}_1 = (1, k, 0, 0, 0, -\sigma_2)^T$ and $\mathbf{f}_1 = (0, 0, 0, \sigma_1, \sigma_2, 0)^T$. We complete $\mathbf{e}_1, \mathbf{f}_1$ to a base of $\mathbb{R}^6$ taking vectors $\mathbf{r} \in \mathbb{R}^6$ skew–orthogonal to $\mathbf{e}_1$ and $\mathbf{f}_1$, that is, $\mathbf{e}_1^T J\mathbf{r} = 0, \mathbf{f}_1^T J\mathbf{r} = 0$, being $J$ the skew symmetric matrix $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$. We choose

$$\begin{aligned} \mathbf{r}_1 &= (\sigma_2, -\sigma_1, 0, 0, 0, 0)^T, & \mathbf{r}_3 &= (0, 0, 0, -k, 1, 0)^T, \\ \mathbf{r}_2 &= (0, 0, 1, -\sigma_2, 0, 0)^T, & \mathbf{r}_4 &= (0, 0, 0, 0, 0, 1)^T. \end{aligned}$$

We define

$$M = \begin{pmatrix} 1 & 0 & \sigma_2 & 0 & 0 & 0 \\ k & 0 & -\sigma_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & \sigma_1 & 0 & -\sigma_2 & -k & 0 \\ 0 & \sigma_2 & 0 & 0 & 1 & 0 \\ -\sigma_2 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then

$$M^{-1} DF(\mathbf{u}^p, \mathbf{v}^p) M = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & \tilde{L} \end{pmatrix}, \qquad \tilde{L} = \begin{pmatrix} 0 & a_1 & a_2 & 0 \\ a_3 & 0 & 0 & a_4 \\ a_5 & 0 & 0 & a_6 \\ 0 & a_7 & a_8 & 0 \end{pmatrix}$$

where $a_1, \dots, a_8$ are some constants. It is clear that $\tilde{L}$ has eigenvalues $\pm\lambda, \pm i\omega$. Moreover, if $(l_1, l_2, l_3, l_4)^T$ is an eigenvector of $\tilde{L}$, then $\mathbf{l} = (0, 0, l_1, l_2, l_3, l_4)^T$ is an eigenvector of the matrix $M^{-1} DF(\mathbf{u}^p, \mathbf{v}^p) M$ and so, the eigenvectors of $DF(\mathbf{u}^p, \mathbf{v}^p)$ corresponding to the eigenvalues $\lambda$ and $i\omega$ are of the following type

$$M\mathbf{l} = (\sigma_2 l_1, -\sigma_1 l_1, l_2, -\sigma_2 l_2 - k l_3, l_3, l_4)^T.$$

After an easy computation (23) follows. $\square$

In the case $m_1 = m_3$ we can compute explicitly all the eigenvalues. This case is simpler because the solution of the quintic equation is $\rho = 1$. Furthermore

$$N = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -2 \\ -1 & 2 & 0 \end{pmatrix}, \quad L = \begin{pmatrix} -1 & 0 & 0 \\ l_{21} & l_{22} & 0 \\ 0 & 0 & l_{33} \end{pmatrix}$$

where

$$l_{21} = -\frac{1}{4a^3}(8m_2 + 9m_1), \quad l_{22} = \frac{1}{4a^3}(12m_2 + 17m_1), \quad l_{33} = -\frac{1}{4a^3} 7m_1.$$

After some computations the following lemma holds

**Lemma 4.** *For positive masses such that* $m_1 = m_3$

$$\begin{aligned}
\mathbf{w}_1 &= (\mathbf{u}^T, \mathbf{v}^T)^T, \ \textit{with } \mathbf{u}^T = (0, 1, v_\lambda), \ \mathbf{v}^T = m_1 m_2(-\lambda + v_\lambda, 2(\lambda - v_\lambda), 2(1 + \lambda v_\lambda)), \\
& \qquad v_\lambda = \frac{\lambda^2 - l_{22}}{2\lambda}, \\
\mathbf{w}_2 &= (\mathbf{u}^T, \mathbf{v}^T)^T \ \textit{with } \mathbf{u}^T = (1, \frac{1}{2}, 0), \ \mathbf{v}^T = m_1(\frac{1}{2} i, 0, 0), \\
\mathbf{w}_3 &= (\mathbf{u}^T, \mathbf{v}^T)^T \ \textit{with } \mathbf{u}^T = (0, 1, v_\omega i), \ \mathbf{v}^T = a m_1 m_2((-\omega + v_\omega) i, 2(\omega - v_\omega) i, 2(1 - \omega v_\omega)), \\
& \qquad v_\omega = \frac{\omega^2 + l_{22}}{2\omega},
\end{aligned}$$

*are, respectively, eigenvectors corresponding to the eigenvalues* $\lambda, i, i\omega$.

Using (11) we get $D\tilde{F}(0,0) = \begin{pmatrix} A_1 & \frac{1}{a\rho} A_2 \\ a\rho A_3 & -A_1^T \end{pmatrix}$ where $A_1, A_2$ and $A_3$ are the matrices defined in (21). If $\mathbf{w}$ is an eigenvector of $DF(\mathbf{u}^p, \mathbf{v}^p)$ for an eigenvalue $\mu$ then $\mathbf{z} = (\mathbf{x}^T, \mathbf{y}^T)^T$ with $\mathbf{x} = \mathbf{u}$ and $\mathbf{y} = a\rho\mathbf{v}$ is the corresponding eigenvector of $D\tilde{F}(0,0)$. So, the matrix $\mathcal{D}$ in (13) can be written as

$$\mathcal{D} = \begin{pmatrix} v_{11} & e_{11} & e_{21} & v_{11} & 0 & 0 \\ v_{12} & e_{12} & e_{22} & v_{12} & 0 & 0 \\ v_{13} & 0 & 0 & -v_{13} & 0 & f_{23} \\ v_{14} & 0 & 0 & -v_{14} & f_{14} & f_{24} \\ v_{15} & 0 & 0 & -v_{15} & f_{15} & f_{25} \\ v_{16} & e_{16} & e_{26} & v_{16} & 0 & 0 \end{pmatrix}.$$

Furthermore, in $\bar{H}(\mathbf{x}, \mathbf{y})$ the monomials of degree 3 are the following (see (38) in Appendix 1):

$$x_1^3,\ x_1^2 x_2,\ x_1^2 y_3,\ x_1 x_2^2,\ x_1 x_2 y_3,\ x_1 x_3^2,\ x_1 x_3 y_2,\ x_1 y_3^2,\ x_2^3,\ x_2^2 y_3,\ x_2 x_3^2,\ x_2 x_3 y_2,\ x_2 y_3^2,\ x_3 y_2 y_3.$$

Using the particular form of $\mathcal{D}$ and the variables $\mathbf{Q}, \mathbf{P}$ defined in (12) the following lemma follows easily.

**Lemma 5.** $H_3(\mathbf{Q}, \mathbf{P})$ *does not contain the monomials* $Q_2^2 P_3, P_2^2 Q_3, P_2^2 P_3$ *and* $Q_2 P_2 Q_3$.

## 4 Qualitative description of a neighbourhood of $L_2$

We consider the Hamiltonian $H(\mathbf{Q},\mathbf{P}) = \sum_{k\geq 2} H_k(\mathbf{Q},\mathbf{P})$ with $H_2(\mathbf{Q},\mathbf{P})$ given in (14).

From $H_2(\mathbf{Q},\mathbf{P})$ one can describe qualitatively the solutions in the neighbourhood of the equilibrium point. We give this description following the ideas introduced by Conley in [2] for the restricted three body problem.

We consider the solutions of the linear system of equations given by $H_2(\mathbf{Q},\mathbf{P})$. They can be written as

$$\begin{array}{rclr} Q_1(t) & = & Q_1^0 e^{\lambda t}, & z_1(t) = Q_2(t) + iP_2(t) = z_1^0 e^{-it}, \\ P_1(t) & = & P_1^0 e^{-\lambda t}, & z_2(t) = Q_3(t) + iP_3(t) = z_2^0 e^{-i\omega t}, \end{array} \tag{24}$$

where $Q_1^0, P_1^0$ are real constants and $z_1^0, z_2^0$ complex ones. Clearly for the linearised system, the stable and unstable manifolds of the origin are obtained as $Q_1^0 = 0, z_1^0 = z_2^0 = 0$ and $P_1^0 = 0, z_1^0 = z_2^0 = 0$ respectively. The central manifold corresponds to $Q_1^0 = P_1^0 = 0$.

In (24) we distinguish two families of periodic orbits $\mathcal{F}_1$ and $\mathcal{F}_2$ which are obtained by taking $Q_1^0 = P_1^0 = z_2^0 = 0$ and $Q_1^0 = P_1^0 = z_1^0 = 0$ respectively. The period is $2\pi$ for the periodic orbits in the family $\mathcal{F}_1$ and $2\pi/\omega$ in family $\mathcal{F}_2$. These families can be parametrised by the energy $h$.

We fix some value of the energy $h \in \mathbb{R}$ and some constant $c > 0$ and we consider the set

$$L(h,c) = \{(\mathbf{Q},\mathbf{P}) \in \mathbb{R}^6 \mid H_2(\mathbf{Q},\mathbf{P}) = h, |Q_1 - P_1| \leq c\}.$$

From $H_2(\mathbf{Q},\mathbf{P}) = h$ we get

$$\tau := \frac{1}{2}(Q_2^2 + P_2^2) + \frac{\omega}{2}(Q_3^2 + P_3^2) = h - \lambda Q_1 P_1 \tag{25}$$

and, hence, the motion is restricted to the region of the phase space such that $h - \lambda Q_1 P_1 \geq 0$.

We define in $L(h,c)$ the following sets $S_e = \{(\mathbf{Q},\mathbf{P}) \in L(h,c) | Q_1 - P_1 = c\}$ and $S_m = \{(\mathbf{Q},\mathbf{P}) \in L(h,c) | Q_1 - P_1 = -c\}$.

If $(\mathbf{Q},\mathbf{P}) \in L(h,c)$ using $Q_1 P_1 = [(Q_1 + P_1)^2 - (Q_1 - P_1)^2]/4$ we get

$$\frac{\lambda}{4}(Q_1 + P_1)^2 + \frac{1}{2}(Q_2^2 + P_2^2) + \frac{\omega}{2}(Q_3^2 + P_3^2) \leq h + \frac{\lambda c^2}{4}.$$

Therefore, if $h + \frac{\lambda c^2}{4} > 0$, $S_e$ is homeomorphic to a sphere $\mathbb{S}^4$. In a similar way, $S_m$ is homeomorphic to $\mathbb{S}^4$. This is the case if $h \geq 0$. Note that for a fixed value of $h$, these spheres separate the phase space which is 5–dimensional. Figure 2 shows the projection of the linear flow on the plane $Q_1, P_1$ in a neighbourhood of the origin. We note that the orbits enter in $L(h,c)$ through one of the spheres $S_e$ or $S_m$. Only in the case $h > 0$ there are orbits which go out of $L(h,c)$ through the other sphere.

We note also that for $h > 0$ if we take $Q_1 = P_1 = 0$, (25) defines a three–dimensional sphere $\mathbb{S}^3$. In fact, this sphere is the intersection of the central manifold of the equilibrium point with the corresponding energy level. The linear flow restricted to this sphere is given by the product of two harmonic oscillators with frequencies 1 and $\omega$. So, for any $h > 0$, the sphere contains two periodic orbits, corresponding to families $\mathcal{F}_1$ and $\mathcal{F}_2$ respectively, each one surrounded by two–dimensional invariant tori ([10]). The invariant unstable and stable manifolds of this center manifold reduced to the level $h$, are the objects which decide which one of the transitions $S_e \to S_e, S_e \to S_m$, $S_m \to S_e$, $S_m \to S_m$ occur. To see the relevant role of these codimension one manifolds see [15], [4].

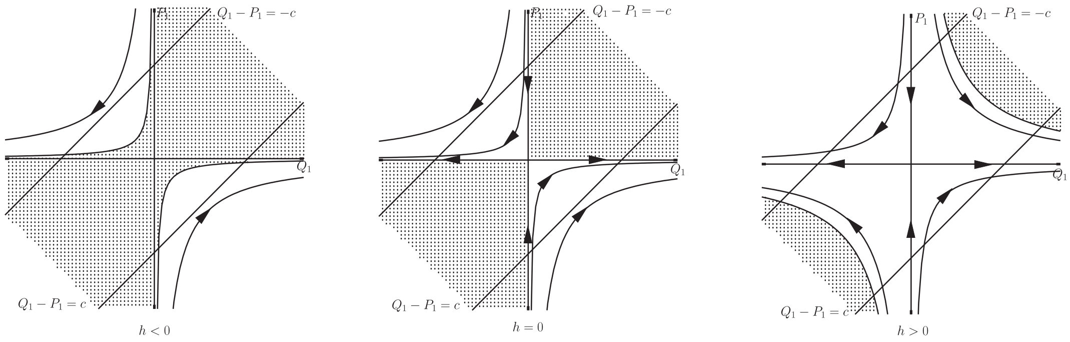



Figure 2: Projection of the linear flow on the plane $(Q_1, P_1)$ in a neighbourhood of the origin. The dashed area corresponds to the forbidden region.

Now we consider the full Hamiltonian $H(\mathbf{Q}, \mathbf{P})$. The preservation of the family of periodic orbits $\mathcal{F}_2$ associated to the eigenvalue $i\omega$ is obtained by using the Lyapunov theorem ([13]). We know that $\omega > 1$ for all masses. Therefore $1/\omega$ is not an integer and we get a family of periodic solutions of $H(\mathbf{Q}, \mathbf{P})$ with limit period $2\pi/\omega$. We shall denote again by $\mathcal{F}_2$ the family of periodic orbits of the full Hamiltonian $H(\mathbf{Q}, \mathbf{P})$. However, from lemma 2 we have $1 < \omega < 3$, then if $\omega \neq 2$, the Lyapunov theorem gives the existence of a second family, $\mathcal{F}_1$, of periodic solutions with limit period $2\pi$. In spite that Lyapunov theorem does not apply in the case $\omega = 2$, it is well known that these periodic solutions exist for arbitrary masses because it is the family of homographic solutions. In the next section we introduce the homographic solutions and we give some properties that will be useful in the next. The preservation of the 2–dimensional invariant tori will be studied in section 7.

## 5 On the collinear homographic solutions

Let $\Delta = \cup_{1 \le j,l \le 3} \{\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3) \in \mathbb{R}^6 | \mathbf{q}_j = \mathbf{q}_l\}$ be the collision set and $\mathbf{q}_c = (\mathbf{q}_{c_1}, \mathbf{q}_{c_2}, \mathbf{q}_{c_3}) \in \mathbb{R}^6 \backslash \Delta$ a central configuration of the planar three body problem, that is, $M^{-1}\nabla U(\mathbf{q}) = -\mathbf{q}$. Let $z(t) = x_1(t) + ix_2(t)$ be a solution of the Kepler problem

$$\ddot{z} = -\frac{z}{|z|^3}. \tag{26}$$

Identifying $\mathbb{R}^2$ with the complex plane, the homographic solutions can be written as

$$\mathbf{q}_j(t) = z(t)\mathbf{q}_{c_j}, \qquad j = 1, 2, 3.$$

We note that if $z(t)$ is a circular solution of (26) then $\mathbf{q}_j(t)$ is obtained from $\mathbf{q}_{c_j}$ through a rotation. In this case, the homographic solution is a relative equilibrium. In fact, it corresponds to the equilibrium point in the variables $\boldsymbol{\zeta}, \boldsymbol{\nu}$ introduced in section 2.

Let $h_k = \frac{1}{2}|\dot{z}|^2 - \frac{1}{|z|}$ be the energy of the Kepler problem (26). Using the true anomaly, $f$, we write the solutions as

$$z(t) = r_k e^{if}, \qquad r_k(t) = \frac{c_k^2}{1 + e_k \cos f},$$

where $e_k$ is the eccentricity and $c_k$ the angular momentum of the Kepler problem.

In the next we fix $c_k = 1$ and we characterise the homographic solutions by their eccentricity $e_k$. Then $h_k = \frac{1}{2}(e_k^2 - 1)$ and the period is $P_k = 2\pi(1-e_k^2)^{-3/2}$. It is not difficult to write the homographic solutions in the variables $\mathbf{Q}, \mathbf{P}$ defined in section 3. We obtain

$$
\begin{aligned}
Q_1(t) &= \bar{k}_0(A-B), & P_1(t) &= \bar{k}_0(A+B), \\
Q_2(t) &= -\frac{\bar{k}_1 v_4^p}{a^2\rho(\rho+1)}(r_k(t)-1), & P_2(t) &= -\frac{\bar{k}_1 v_4^p}{a^2\rho(\rho+1)}\dot{r}_k(t), \\
Q_3(t) &= \bar{k}_2\left\{\frac{f_{24}(\rho+1)+f_{25}}{\rho}(r_k(t)-1) - f_{23}m_2 a(m_3-m_1\rho)(r_k(t)\dot{f}(t)-1)\right\}, \\
P_3(t) &= \bar{k}_2 a\dot{r}_k(t)\left\{e_{21}m_1(m_3+\rho(1-m_1)) + e_{22}m_2(m_3-m_1\rho)\right\},
\end{aligned}
\tag{27}
$$

where

$$
\begin{aligned}
A &= -\frac{v_{14}(\rho+1)+v_{15}}{\rho}(r_k(t)-1) + v_{13}m_2 a(m_3-m_1\rho)(r_k(t)\dot{f}(t)-1), \\
B &= [v_{11}m_1 a(m_3+\rho(1-m_1)) + v_{12}m_2 a(m_3-m_1\rho)]\dot{r}_k(t).
\end{aligned}
$$

In (27) $\mathbf{z}_1 = (v_{11}, v_{12}, v_{13}, v_{14}, v_{15}, v_{16})^T$ and $\mathbf{z}_3 = \mathbf{e}_2 + i\mathbf{f}_2$ with $\mathbf{e}_2 = (e_{21}, e_{22}, 0, 0, 0, e_{26})^T, \mathbf{f}_2 = (0, 0, f_{23}, f_{24}, f_{25}, 0)^T$ are the eigenvectors of $D\tilde{F}(0,0)$, introduced in section 3, corresponding to eigenvalues $\lambda$ and $i\omega$ respectively. We recall that $a, \rho, \bar{k}_0, \bar{k}_1$ and $\bar{k}_2$ have been introduced in section 3. In (27) $\dot{f}$ means the derivative of the true anomaly, $f$, with respect $t$. Using (27) we obtain the following lemma to be used in the next section.

**Lemma 6.** *For any masses $m_1, m_2, m_3$, if $(Q_1(t), Q_2(t), Q_3(t), P_1(t), P_2(t), P_3(t))$ is a collinear homographic solution then $Q_1(t) = P_1(t), P_2(t) = \dot{Q}_2(t)$ and $P_3(t) \equiv 0$. Moreover, if $m_1 = m_3$, then $Q_1(t) = 0, P_1(t) = 0$ and $Q_3(t) = 0$.*

**Proof**

The equality $P_2(t) = \dot{Q}_2(t)$ follows directly from (27). Lemma 3 implies $Q_1(t) = P_1(t)$ and $P_3(t) \equiv 0$. The corresponding relations in the case $m_1 = m_3$ are obtained using the exact expressions of the eigenvectors given by lemma 4. □

# 6 The normal form

In this section we compute the normal form of the Hamiltonian reduced to the centre manifold of $L_2$.

We consider the Hamiltonian (16). To obtain the reduced Hamiltonian we shall apply the flattening method (see [11], [16]). We shall perform successive canonical transformations in order to write the Hamiltonian as

$$
H(\boldsymbol{\xi}, \boldsymbol{\eta}) = H^{(n)}(\xi_1\eta_1, \mathbf{w}) + R^{(n+1)}(\boldsymbol{\xi}, \boldsymbol{\eta}), \qquad H^{(n)}(\xi_1\eta_1, \mathbf{w}) = \sum_{k=2}^{n} H_k(\xi_1\eta_1, \mathbf{w}) \tag{28}
$$

with $n \geq 4$, $H_2 = \lambda\xi_1\eta_1 + i\xi_2\eta_2 + i\omega\xi_3\eta_3$ and $H_k$ , $k \geq 3$ an homogeneous polynomial of degree $k$ in $\xi_1\eta_1, \mathbf{w} = (\xi_2, \eta_2, \xi_3, \eta_3)$. In (28) $R^{(n+1)}$ contains terms of degree greater than or equal to $n+1$. Note that $H^{(n)}$ depends on the product $\xi_1\eta_1$ but it does not depend on $\xi_1$ nor $\eta_1$. So, for the Hamiltonian $H^{(n)}(\xi_1\eta_1, \mathbf{w})$ the centre manifold is trivially obtained as $\xi_1 = 0, \eta_1 = 0$.

To get the form (28) for the Hamiltonian we perform successive canonical transformations which eliminate the terms $\xi_1^{l_1}\eta_1^{s_1}$ with $l_1 \neq s_1$ to different orders. This process can be formally done up to any order because it does not have small divisors even if the two central frequencies are resonant. This does not mean at all that the process is convergent.

As we are interested in the applicability of KAM theorem to the reduced Hamiltonian it is convenient to put the Hamiltonian in action–angle variables. The canonical transformations will be chosen in order to cancel all the possible terms in the Hamiltonian. In particular the terms $\xi_1^{l_1}\eta_1^{s_1}$ with $l_1 \neq s_1$ will be eliminated.

As we shall see in the next we only need the normal form up to order 4. We write the generating function as $G = G_3 + G_4$ with $G_3 \in \Pi_3, G_4 \in \Pi_4$ where $\Pi_k$ stands for the space of homogeneous polynomials in the variables $(\boldsymbol{\xi}, \boldsymbol{\eta}) \in \mathbb{C}^{2n}$. We consider the canonical transformation $T_G$ defined by the time one flow for the Hamiltonian $G$ (see [3], [14]). The new Hamiltonian is $\hat{H} = \hat{H}_2 + \hat{H}_3 + \hat{H}_4 + \dots$ where

$$
\begin{aligned}
\hat{H}_2 &= H_2, \\
\hat{H}_3 &= H_3 + L_{H_2} G_3, \\
\hat{H}_4 &= H_4 + L_{H_2} G_4 + L_{H_3} G_3 - \frac{1}{2} L_{G_3} L_{H_2} G_3.
\end{aligned}
\tag{29}
$$

In (29) $L_f g = \{f, g\} = \sum_{j=1}^{3} \left( \frac{\partial g}{\partial \xi_j} \frac{\partial f}{\partial \eta_j} - \frac{\partial g}{\partial \eta_j} \frac{\partial f}{\partial \xi_j} \right)$ stands for the Poisson bracket. We take $G_3$ and $G_4$ such that

$$
\begin{aligned}
L_{H_2} G_3 &= N(F_3) \quad \text{where} \quad N(F_3) = -H_3, \\
L_{H_2} G_4 &= N(F_4) \quad \text{where} \quad N(F_4) = -H_4 - L_{H_3} G_3 + \frac{1}{2} L_{G_3} L_{H_2} G_3.
\end{aligned}
$$

In order to solve the homological equation $L_{H_2} G_k = N(F_k)$ for $k = 3, 4$ we write $G_k = \Sigma_{|\mathbf{l}+\mathbf{s}|=k} g_{\mathbf{l},\mathbf{s}} \boldsymbol{\xi}^{\mathbf{l}} \boldsymbol{\eta}^{\mathbf{s}}$ and $N(F_k) = \Sigma_{|\mathbf{l}+\mathbf{s}|=k} n_{\mathbf{l},\mathbf{s}} \boldsymbol{\xi}^{\mathbf{l}} \boldsymbol{\eta}^{\mathbf{s}}$ where $\mathbf{l} = (l_1, l_2, l_3), \mathbf{s} = (s_1, s_2, s_3)$ and $\boldsymbol{\xi}^{\mathbf{l}} \boldsymbol{\eta}^{\mathbf{s}} = \xi_1^{l_1} \xi_2^{l_2} \xi_3^{l_3} \eta_1^{s_1} \eta_2^{s_2} \eta_3^{s_3}$. Then

$$
L_{H_2} G_k = \sum_{|\mathbf{l}+\mathbf{s}|=k} \boldsymbol{\lambda} \cdot (\mathbf{s} - \mathbf{l}) g_{\mathbf{l},\mathbf{s}} \boldsymbol{\xi}^{\mathbf{l}} \boldsymbol{\eta}^{\mathbf{s}}
$$

where $\boldsymbol{\lambda} \cdot (\mathbf{s} - \mathbf{l}) := \lambda(s_1 - l_1) + i(s_2 - l_2) + i\omega(s_3 - l_3)$. If $\boldsymbol{\lambda} \cdot (\mathbf{s} - \mathbf{l}) \neq 0$ we define

$$
g_{\mathbf{l},\mathbf{s}} = \frac{n_{\mathbf{l},\mathbf{s}}}{\boldsymbol{\lambda} \cdot (\mathbf{s} - \mathbf{l})}.
$$

If there exist $\mathbf{l}, \mathbf{s} \in \mathbb{Z}^3$, $\mathbf{l}, \mathbf{s} \neq \mathbf{0}$ satisfying $\boldsymbol{\lambda} \cdot (\mathbf{s} - \mathbf{l}) = 0$ then the normal form will contain a resonant monomial $\boldsymbol{\xi}^{\mathbf{l}} \boldsymbol{\eta}^{\mathbf{s}}$.

It is clear that for any values of the masses $m_1, m_2, m_3$, $(\xi_1\eta_1)^{s_1}(\xi_2\eta_2)^{s_2}(\xi_3\eta_3)^{s_3}$ with $s_1+s_2+s_3 = 2k$, $k \geq 2$ is a resonant monomial of order $2k$. In fact these are the only resonant monomials of order 4. To get other resonant monomials we must take $s_1 = l_1$ and $(s_2 - l_2) + \omega(s_3 - l_3) = 0$. After lemma 2 we know that $\omega \in (1, 3)$. So, up to order 4 we only need to consider the resonance 2:1 reached for $\omega = 2$. In this case $\lambda = \hat{\lambda} = \frac{1}{2}\sqrt{13 + \sqrt{97}}$ and the corresponding resonant monomials are $\xi_2^2 \eta_3$ and $\xi_3 \eta_2^2$ at order 3.

The masses $m_1, m_2, m_3$ for which the frequency vector is $(\hat{\lambda}, i, 2i)$ determine a curve in the triangle of masses. We call it the resonant curve. The following lemma gives this curve in an implicit form.

**Lemma 7.** *The resonant curve is defined implicitly by*

$$
f(\nu_1, \nu_2) = 0 \quad \textit{where} \quad \nu_1 = \frac{m_1 - m_3}{2}, \quad \nu_2 = \frac{m_1 + m_3}{2}
\tag{30}
$$

*where $f$ is a polynomial in $\nu_1, \nu_2$ of degree 9, even in $\nu_1$ and such that $f(\nu_1, 0) = 0$. Moreover, $f$ can be written as $f(\nu_1, \nu_2) = \sum_{(j,k) \in J} (a_{j,k} + \sqrt{97} b_{j,k}) \nu_1^j \nu_2^k$, $J \subset \mathbb{Z}^2$, where the coefficients $a_{j,k}, b_{j,k}$ are given in table 1.*

Figure 1 shows the resonant curve in the triangle of masses as $\omega = 2$ on the left plot.

**Proof**

From (9) we have that $\omega = 2$ for $\beta = (1+\sqrt{97})/4$. We fix this value of $\beta$. Then (7) and (10) can be written in terms of the parameters $\nu_1$ and $\nu_2$ as

$$P_1(\rho, \nu_1, \nu_2) = 0, \qquad P_2(\rho, \nu_1, \nu_2) = 0$$

where $P_1$ and $P_2$ are polynomials in $\rho$ with coefficients depending on $\nu_1$ and $\nu_2$. The function $f_1(\nu_1, \nu_2)$ is the resultant of these polynomials. $f_1$ has been computed using a specific algebraic manipulator built for this purpose. We note that the coefficients $a_{j,k}, b_{j,k}$ given in the table are rational numbers. The properties of $f$ are obtained directly from the table. □

| $j$ | $k$ | $32a_{j,k}$ | $32b_{j,k}$ | $j$ | $k$ | $32a_{j,k}$ | $32b_{j,k}$ | $j$ | $k$ | $32a_{j,k}$ | $32b_{j,k}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 2 | 4672 | 1600 | 4 | 2 | –1024496 | –113904 | 4 | 4 | –676480 | 896 |
| 2 | 1 | 44316 | 4572 | 2 | 4 | 2163552 | 88416 | 2 | 6 | 3092096 | 264320 |
| 0 | 3 | –14832 | –21744 | 0 | 6 | –923248 | 319120 | 0 | 8 | –1047424 | –18816 |
| 2 | 2 | –283900 | –24508 | 6 | 1 | 95152 | 10160 | 8 | 1 | –2240 | –448 |
| 0 | 4 | –90996 | 115788 | 4 | 3 | 1451024 | 143888 | 6 | 3 | 44800 | 8960 |
| 4 | 1 | 227979 | 22779 | 2 | 5 | –4603184 | –271664 | 4 | 5 | –192640 | –38528 |
| 2 | 3 | 132658 | 23506 | 0 | 7 | 1464176 | –123536 | 2 | 7 | –313600 | –62720 |
| 0 | 5 | 416627 | –280989 | 6 | 2 | –221312 | –17024 | 0 | 9 | –109760 | –21952 |

Table 1: Coefficients of the resonant curve.

**Remark 2.** *The resonant curve is symmetrical with respect the line $m_1 = m_3$. This is a consequence of the symmetry of the function $\beta(m_1, m_3)$ given in (10). If $m_3 = 0$ then $\nu_1 = \nu_2 = m_1/2$ and from lemma 7*

$$f(\nu_1, \nu_1) = \Sigma_{(j,k)\in J}(a_{j,k} + \sqrt{97} b_{j,k})\nu_1^{j+k}.$$

*Solving the equation $f(\nu_1, \nu_1) = 0$ with $\nu_1 = m_1/2$ we get two solutions $m_1 = 1$ and $m_1^* = 0.9995998...$ (zeroes of a polynomial of degree 6 with coefficients in $\mathbb{Q}[\sqrt{97}]$). In fact, the arc of the resonant curve between $m_1 = 1$ and $m_1^*$ is outside the triangle of masses and it will not be considered.*

Assume we take non resonant masses that is, $\omega \neq 2$. Then all the terms of degree 3 in $H_3(\boldsymbol{\xi}, \boldsymbol{\eta})$ can be eliminated by the transformation $T_G$. The normal form for the Hamiltonian is the following

$$H(\mathbf{X}, \mathbf{Y}) = \lambda X_1 Y_1 + i X_2 Y_2 + i\omega X_3 Y_3 + \sum_{\substack{j_1, j_2, j_3 \in \mathbb{N}\cup\{0\} \\ j_1+j_2+j_3 = 2}} a_{j_1 j_2 j_3} (\mathbf{XY})^j + \sum_{k\geq 5} H_k(\mathbf{X}, \mathbf{Y}). \quad (31)$$

For resonant masses one could expect that the Hamiltonian $H(\boldsymbol{\xi}, \boldsymbol{\eta})$ contains resonant monomials of order 3 which give rise to the corresponding terms of order 3 in the transformed Hamiltonian $H(\mathbf{X}, \mathbf{Y})$. We will prove that this is not the case.

First we denote the resonant monomials of order 3 in $H(\boldsymbol{\xi}, \boldsymbol{\eta})$ by $H_R(\boldsymbol{\xi}, \boldsymbol{\eta}) = c\xi_2^2\eta_3 + g\xi_3\eta_2^2$ for some complex constants $c$ and $g$. Using (15) we see that $H_R(\boldsymbol{\xi}, \boldsymbol{\eta})$ is obtained from the following terms of $H_3(\mathbf{Q}, \mathbf{P})$

$$a_1 Q_2^2 Q_3 + a_2 Q_2^2 P_3 + a_3 P_2^2 Q_3 + a_4 P_2^2 P_3 + b_1 Q_2 P_2 Q_3 + b_2 Q_2 P_2 P_3.$$

The coefficients are related through

$$\begin{aligned} c &= \frac{\bar{k}_2}{2\bar{k}_1^2}\mathcal{C}, \quad \mathcal{C} = a_2 - a_4 - b_1 + i(a_1 - a_3 + b_2), \\ g &= \frac{\bar{k}_1^2}{4\bar{k}_2}\mathcal{G}, \quad \mathcal{G} = -a_1 + a_3 - b_2 + i(-a_2 + a_4 + b_1). \end{aligned}$$

We recall that $\bar{k}_1$ and $\bar{k}_2$ are constants defined in section 3. Note that $\mathcal{G} = -i\bar{\mathcal{C}}$. Moreover, from lemma 5 we have $a_2 = a_3 = a_4 = b_1 = 0$ and then $\mathcal{C} = i(a_1 + b_2)$.

**Proposition 1.** *If $m_1, m_2, m_3$ are resonant masses, then the Hamiltonian $H(\boldsymbol{\xi}, \boldsymbol{\eta})$ does not contain resonant monomials of order 3.*

**Proof**

To prove that $\mathcal{C} = 0$ we will obtain some expressions for the coefficients $a_1$ and $b_2$. To compute these expressions directly is a hard work. But using the existence of the homographic solutions we can get $a_1$ and $b_2$ in a rather simple way.

First we consider arbitrary values of the masses $m_1, m_2$ and $m_3$ such that $m_1+m_2+m_3 = 1$. We denote by $\gamma(t) = (\mathbf{Q}(t), \mathbf{P}(t))$ an homographic solution with eccentricity $e_k$ (see (27)) and we write

$$Q_j(t) = \sum_{n\geq 1} a_{jn}(t)e_k^n, \qquad P_j(t) = \sum_{n\geq 1} b_{jn}(t)e_k^n, \quad j = 1, 2, 3.$$

From lemma 6 we know that $Q_1(t) = P_1(t)$ and so $\dot{Q}_1(t) = \dot{P}_1(t)$. As $\gamma(t)$ is a solution of the differential equations associated to the Hamiltonian $H(\mathbf{Q}, \mathbf{P})$, we get

$$\lambda Q_1(t) + \frac{\partial \tilde{H}}{\partial P_1}(\gamma(t)) = -\lambda P_1(t) - \frac{\partial \tilde{H}}{\partial Q_1}(\gamma(t)),$$

where $\tilde{H}$ stands for the terms of $H(\mathbf{Q}, \mathbf{P})$ of order greater than 2. Therefore

$$2\lambda Q_1(t) = -\frac{\partial \tilde{H}}{\partial P_1}(\gamma(t)) - \frac{\partial \tilde{H}}{\partial Q_1}(\gamma(t)).$$

The right hand part of the equality above contain terms of order greater than or equal to 2 in $e_k$. Then $Q_1(t) = O(e_k^2)$ and $P_1(t) = O(e_k^2)$. From (27) we get

$$\begin{aligned} Q_2(t) &= \frac{\bar{k}_1 v_4^p}{a^2\rho(\rho+1)} e_k \cos f + O(e_k^2), \\ P_2(t) &= -\frac{\bar{k}_1 v_4^p}{a^2\rho(\rho+1)} e_k \sin f, \\ Q_3(t) &= \bar{k}_2[-f_{23} m_2 a(m_3 - m_1\rho)e_k^2 \cos^2 f] + O(e_k^3), \end{aligned}$$

By derivating $Q_3(t)$ and using $\dot{f} = (1 + e_k \cos f)^2$

$$\dot{Q}_3(t) = \bar{k}_2[2f_{23} m_2 a(m_3 - m_1\rho)e_k^2 \cos f \sin f] + O(e_k^3).$$

Now consider the differential equations for $Q_3$ and $P_3$. Using that $P_3(t) = 0$ we get

$$\dot{Q}_3 = \frac{\partial \tilde{H}}{\partial P_3}(\gamma(t)), \tag{32}$$

$$0 = -\omega Q_3(t) - \frac{\partial \tilde{H}}{\partial Q_3}(\gamma(t)). \tag{33}$$

By inspection of $\tilde{H}$ one can see that in the right hand part of the equations (32) and (33) the only terms of order 2 in $e_k$ come from the monomials $Q_2^2, P_2^2$ and $Q_2P_2$. In fact, in (32) there is only the contribution due to the term $b_2Q_2P_2$, and in (33), only $a_1Q_2^2$ contributes to terms of order $e_k^2$ in $\frac{\partial \tilde{H}}{\partial Q_3}(\gamma(t))$. So, we get

$$\begin{aligned} 2\bar{k}_2 f_{23} m_2 a(m_3 - m_1\rho)\cos f \sin f &= -b_2\left(\frac{\bar{k}_1 v_4^p}{a^2\rho(\rho+1)}\right)^2 \cos f \sin f, \\ -\bar{k}_2 \omega f_{23} m_2 a(m_3 - m_1\rho)\cos^2 f &= -a_1\left(\frac{\bar{k}_1 v_4^p}{a^2\rho(\rho+1)}\right)^2 \cos^2 f. \end{aligned}$$

Then

$$\begin{aligned} b_2 &= -\frac{2\bar{k}_2 f_{23} m_2 a^5(m_3 - m_1\rho)\rho^2(\rho+1)^2}{\bar{k}_1^2(v_4^p)^2}, \\ a_1 &= \frac{\omega\bar{k}_2 f_{23} m_2 a^5(m_3 - m_1\rho)\rho^2(\rho+1)^2}{\bar{k}_1^2(v_4^p)^2}. \end{aligned}$$

Now,

$$\mathcal{C} = i(a_1 + b_2) = i(\omega - 2)(m_3 - m_1\rho)\frac{\bar{k}_2 f_{23} m_2 a^5 \rho^2(\rho+1)^2}{\bar{k}_1^2(v_4^p)^2}. \tag{34}$$

Therefore, in the resonant case, $\omega = 2$ and $c = 0$. □

**Remark 3.** *In the symmetrical case $m_1 = m_3$ we have $\rho = 1$ and then using (34) $\mathcal{C} = 0$ independently of the value of $\omega$. Moreover for masses $m_1, m_2, m_3$ such that $m_1 \neq m_3$ and $\omega \neq 2$ we have $\mathcal{C} \neq 0$.*

We have implemented a specific algebraic manipulator to compute the normal form up to order 4 for given values of the masses. In table 2 we give the coefficients of the monomials $\xi_3\eta_2^2$ and $\xi_2^2\eta_3$ for some values of the masses. The table shows that in the symmetrical case, $m_1 = m_3$, these coefficients are "numerically zero" in agreement with remark 3. The same agreement between numerical and analytical results has also been checked for resonant masses.

Consider the term $c\xi_2^2\eta_3$ in $H(\boldsymbol{\xi}, \boldsymbol{\eta})$. Using (34) and (8) we get

$$c = \frac{\bar{k}_2}{2\bar{k}_1^2}\mathcal{C} = \frac{\bar{k}_2^2}{2\bar{k}_1^4} i(\omega - 2)(m_3 - m_1\rho)\frac{f_{23} m_2 a \rho^2(\rho+1)^2}{[m_1 m_3(\rho+1)^2 + m_2(m_1\rho^2 + m_3)]^2}.$$

If we consider non resonant masses we can define a generating function $G = G_3 + G_4$ where $G_3$ contains the monomial $g_1\xi_2^2\eta_3$ with $g_1 = \frac{c}{i(-2+\omega)}$. Therefore for positive values of $m_1, m_2, m_3$, $g_1$ is a continuous function of the masses. In a similar way and due to the monomial $\xi_3\eta_2^2$, $G_3$ contains the term $g_2\xi_3\eta_2^2$ where $g_2$ is a continuous function of the masses. So, we can define a function $G_3$ on the triangle of masses taking the limit of $g_1$ and $g_2$ in the case of resonant masses. We remark that for $\omega = 2$, the terms $\xi_2^2\eta_3$ and $\xi_3\eta_2^2$ belong to the kernel of the linear operator $L_{H_2}$. This means that they do not contribute to $\hat{H}_3$ (see (29)) . Therefore we get the following proposition.

**Proposition 2.** *Consider $m_1, m_2, m_3 \in (0,1)$ such that $m_1 + m_2 + m_3 = 1$. Then there exists a generating function $G = G(m_1, m_2, m_3)$ which depends analytically on the masses such that after the transformation $T_G$, the new Hamiltonian is of the form (31).*

| $m_1$ | $m_3$ | coefficient of $\xi_3\eta_2^2$ | coefficient of $\xi_2^2\eta_3 i$ |
|---|---|---|---|
| 0.1 | 0.8 | –1.179419348112622E-02 | 6.875177379696579E-03 |
| 0.1 | 0.7 | –9.319371384346949E-02 | 7.758543543388584E-02 |
| 0.1 | 0.6 | –1.413142569551888E-01 | 1.548904537008628E-01 |
| 0.1 | 0.5 | –1.749179227510806E-01 | 2.286007158007460E-01 |
| 0.1 | 0.4 | –2.011008770853154E-01 | 2.858515079866209E-01 |
| 0.1 | 0.3 | –2.177563331662531E-01 | 3.070251869558777E-01 |
| 0.1 | 0.2 | –2.007104469104191E-01 | 2.536095880359161E-01 |
| 0.1 | 0.1 | 9.235014541892594E-15 | –1.183014251097389E-14 |
| 0.2 | 0.7 | 1.029366603579695E-02 | –1.491208746569150E-02 |
| 0.2 | 0.6 | –1.966944788903897E-02 | 2.895185423930213E-02 |
| 0.2 | 0.5 | –4.152197150460450E-02 | 6.647693015509877E-02 |
| 0.2 | 0.4 | –5.221842494622244E-02 | 8.869746541104991E-02 |
| 0.2 | 0.3 | –4.586674997804073E-02 | 7.933390003814744E-02 |
| 0.2 | 0.2 | 6.940183773499024E-16 | –8.210080641681364E-16 |
| 0.2 | 0.1 | 1.765861018277782E-01 | –3.073164843015664E-01 |
| 0.3 | 0.6 | 6.238037823992740E-03 | –1.768228493842869E-02 |
| 0.3 | 0.5 | –4.438682094330806E-03 | 1.083995060532376E-02 |
| 0.3 | 0.4 | –9.191599656495861E-03 | 2.207101058566123E-02 |
| 0.3 | 0.3 | 2.022770770177911E-16 | 3.132560016105415E-16 |
| 0.3 | 0.2 | 4.000070097083308E-02 | –9.741018176772076E-02 |
| 0.3 | 0.1 | 1.668996116082525E-01 | –4.575591110691087E-01 |
| 0.4 | 0.5 | 1.798577883741621E-03 | –8.485012379500253E-03 |
| 0.4 | 0.4 | –1.827228383071727E-17 | 7.311853698924998E-16 |
| 0.4 | 0.3 | 7.935569740810064E-03 | –2.751323770250442E-02 |
| 0.4 | 0.2 | 3.926071923919142E-02 | –1.360534267829678E-01 |
| 0.4 | 0.1 | 1.325417655797161E-01 | –5.342349636043282E-01 |
| 0.5 | 0.4 | –1.534726145643662E-03 | 1.076465252598653E-02 |
| 0.5 | 0.3 | 3.263655003503015E-03 | –1.719301330985832E-02 |
| 0.5 | 0.2 | 2.649617500229235E-02 | –1.304106710601642E-01 |
| 0.5 | 0.1 | 9.738989522637230E-02 | –5.502476606321277E-01 |
| 0.6 | 0.3 | –3.831843227559372E-03 | 3.672812785621461E-02 |
| 0.6 | 0.2 | 1.041828802723820E-02 | –7.510521579183546E-02 |
| 0.6 | 0.1 | 6.478871988992192E-02 | –4.989471222532697E-01 |
| 0.7 | 0.2 | –4.374349135738403E-03 | 5.382987717642781E-02 |
| 0.7 | 0.1 | 3.381097753148919E-02 | –3.550369383481710E-01 |
| 0.8 | 0.1 | 3.156516816489582E-03 | –4.965627280827153E-02 |

Table 2: Coefficients of the monomials $\xi_3\eta_2^2$ and $\xi_2^2\eta_3$ for several values of the masses.

# 7 Existence of invariant tori

In this section we study the applicability of KAM theorem ([1]) to the Hamiltonian reduced to the centre manifold of the equilibrium point $L_2$.

We write the normal form (31) as

$$\begin{aligned} H(\mathbf{X},\mathbf{Y}) &= \lambda X_1Y_1 + iX_2Y_2 + i\omega X_3Y_3 - a_{020}(X_2Y_2)^2 - a_{002}(X_3Y_3)^2 - a_{011}X_2Y_2X_3Y_3 - \\ & a_{200}(X_1Y_1)^2 - a_{110}iX_1Y_1X_2Y_2 - a_{101}iX_1Y_1X_3Y_3 + \sum_{k\geq 5} H_k(\mathbf{X},\mathbf{Y}). \end{aligned} \tag{35}$$

To get action–angle variables we introduce a canonical transformation $(\mathbf{X},\mathbf{Y}) \mapsto (X_1, \boldsymbol{\varphi}, Y_1, \mathbf{I})$ where

$$X_k = \sqrt{I_k}e^{i\varphi_k}, \qquad Y_k = -i\sqrt{I_k}e^{i\varphi_k}, \quad k = 2,3.$$

The new Hamiltonian is

$$H(X_1,\boldsymbol{\varphi},Y_1,\mathbf{I}) = H_2(X_1Y_1,\mathbf{I}) + H_4(X_1Y_1,\mathbf{I}) + R_5(X_1,\boldsymbol{\varphi},Y_1,\mathbf{I}),$$

where

$$\begin{aligned} H_2(X_1Y_1,\mathbf{I}) &= \lambda X_1Y_1 + I_2 + \omega I_3, \\ H_4(X_1Y_1,\mathbf{I}) &= -a_{020}I_2^2 - a_{011}I_2I_3 - a_{002}I_3^2 - a_{200}(X_1Y_1)^2 - a_{110}iX_1Y_1I_2 - a_{101}iX_1Y_1I_3. \end{aligned}$$

To obtain the Hamiltonian reduced to the centre manifold, $H_c$, we take $X_1 = 0$ and $Y_1 = 0$. Then we get $H_c(\mathbf{I},\boldsymbol{\varphi}) = H_{c0}(\mathbf{I}) + H_{c1}(\mathbf{I},\boldsymbol{\varphi})$ where

$$H_{c0}(\mathbf{I}) = I_2 + \omega I_3 - a_{020}I_2^2 - a_{011}I_2I_3 - a_{002}I_3^2. \tag{36}$$

The Hamiltonian system associated to $H_{c0}$ is

$$\begin{aligned} \dot I_2 &= 0, \qquad \dot\varphi_2 = -(1 - 2a_{020}I_2 - a_{011}I_3), \\ \dot I_3 &= 0, \qquad \dot\varphi_3 = -(\omega - a_{011}I_2 - 2a_{002}I_3). \end{aligned} \tag{37}$$

Clearly (37) is an integrable system. So, we look at $H_c(\mathbf{I},\boldsymbol{\varphi})$ as a perturbation of $H_{c0}(\mathbf{I})$.

The nondegeneracy conditions of KAM theorem are the following

$$\text{(a)} \qquad D_1 = \det\left(\frac{\partial^2 H_{c0}}{\partial \mathbf{I}^2}\right) \neq 0,$$

$$\text{(b)} \qquad D_2 = \det\begin{pmatrix} \dfrac{\partial^2 H_{c0}}{\partial \mathbf{I}^2} & \dfrac{\partial H_{c0}}{\partial \mathbf{I}} \\ \left(\dfrac{\partial H_{c0}}{\partial \mathbf{I}}\right)^T & 0 \end{pmatrix} \neq 0.$$

If the condition (a) is satisfied the theorem says that for almost every frequency vector there exists a 2–dimensional invariant torus near the unperturbed one. The condition (b) or isoenergetic condition gives the existence of invariant tori on every energy level.

From (36) we have that $D_1 = 4a_{020}a_{002} - a_{011}^2$ is constant. $D_2$ depends on $I_2$ and $I_3$. Anyhow it is sufficient to check the condition (b) at the origin, that is, $D_2 = 2a_{020}\omega^2 - 2a_{011}\omega + 2a_{002} \neq 0$.

We have evaluated $D_1$ and $D_2$ numerically for different values of the masses in the triangle of masses. Using a specific manipulator we compute the coefficients of the normal form up to order 4 for given values of the masses $m_1, m_2, m_3$. Once the normal form is known, $D_1$ and $D_2$ are obtained immediately. The values of $D_1$ and $D_2$ for a set of masses are given in table 3. Figures 3 and 4 show, respectively, $D_1$ and $D_2$ as functions of $m_2$. In figure 3 every continuous line corresponds to $D_1(m_2)$ for a fixed value of $m_1$. Note that once $m_1$ is fixed, $m_2 \in (0, 1-m_1)$ and $m_3 = 1 - m_1 - m_2$. The same holds for figure 4. These results show that $D_1 > 0$ and $D_2 > 0$ for positive masses. Then KAM theorem ensures the existence of 2–dimensional invariant tori in a neighbourhood of $L_2$ for any energy level.

| $m_1$ | $m_2$ | $D_1$ | $D_2$ |
|---|---|---|---|
| 0.1 | 0.1 | 123.192444126190900 | 17.788785832091770 |
| 0.1 | 0.2 | 57.981216051192120 | 9.276140574712066 |
| 0.1 | 0.3 | 40.728298686281160 | 6.589952363214971 |
| 0.1 | 0.4 | 34.682311867438230 | 5.341187287868248 |
| 0.1 | 0.5 | 33.684503156474210 | 4.677695909919546 |
| 0.1 | 0.6 | 36.745287504522810 | 4.332526838718575 |
| 0.1 | 0.7 | 45.813213441577230 | 4.225259225545290 |
| 0.1 | 0.8 | 71.089285311101490 | 4.457041208199403 |
| 0.2 | 0.1 | 103.393517316096200 | 18.218294888530650 |
| 0.2 | 0.2 | 50.824193245420020 | 9.300570286444902 |
| 0.2 | 0.3 | 36.743296393792050 | 6.516670411071603 |
| 0.2 | 0.4 | 32.138454012679390 | 5.240962280420760 |
| 0.2 | 0.5 | 32.257632210686840 | 4.589675857720991 |
| 0.2 | 0.6 | 37.081352918128770 | 4.316558741864279 |
| 0.2 | 0.7 | 52.071994619538740 | 4.504639989181360 |
| 0.3 | 0.1 | 102.073743778695200 | 19.298538564906000 |
| 0.3 | 0.2 | 51.744613621402370 | 9.781560417457513 |
| 0.3 | 0.3 | 38.491722199751520 | 6.832744606754595 |
| 0.3 | 0.4 | 34.883295985464800 | 5.515028693081639 |
| 0.3 | 0.5 | 36.988170596588670 | 4.914702092384127 |
| 0.3 | 0.6 | 47.942107180607540 | 4.948789669577259 |
| 0.4 | 0.1 | 110.485328727285100 | 20.671363273493780 |
| 0.4 | 0.2 | 57.759778937352510 | 10.473316682437390 |
| 0.4 | 0.3 | 44.584133478133890 | 7.353628431238850 |
| 0.4 | 0.4 | 42.745509539056030 | 6.044269112305969 |
| 0.4 | 0.5 | 51.108291030540560 | 5.761858546700239 |
| 0.5 | 0.1 | 129.480083003787500 | 22.377846871814720 |
| 0.5 | 0.2 | 70.374438999527100 | 11.407304656778590 |
| 0.5 | 0.3 | 57.580164145066870 | 8.157801766505628 |
| 0.5 | 0.4 | 62.291451633215250 | 7.158104149041286 |
| 0.6 | 0.1 | 166.170648618560100 | 24.623203692516640 |
| 0.6 | 0.2 | 95.954711362321600 | 12.779312315293600 |
| 0.6 | 0.3 | 88.834742771266920 | 9.732525836537608 |
| 0.7 | 0.1 | 243.304387577001600 | 27.947054572142850 |
| 0.7 | 0.2 | 159.814511483404300 | 15.400391716639330 |
| 0.8 | 0.1 | 460.303431252741100 | 34.385581785583930 |

Table 3: Determinants $D_1$ and $D_2$ for several values of the masses.

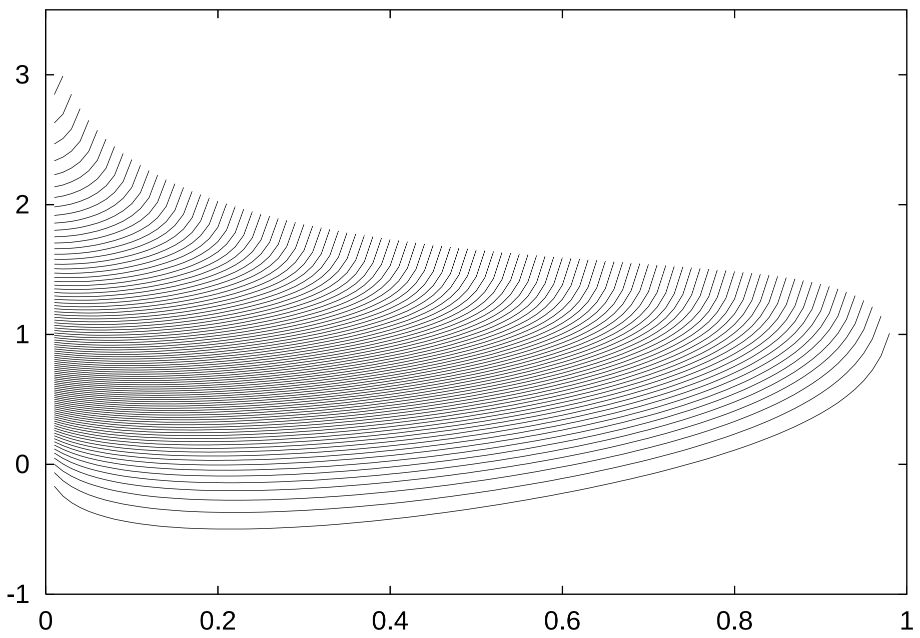



Figure 3: Representation of $D_1$. Every continuous line is the plot of $\log 10(m_1 m_2 D_1)$ as a function of $m_2$ for a fixed value of $m_1$.

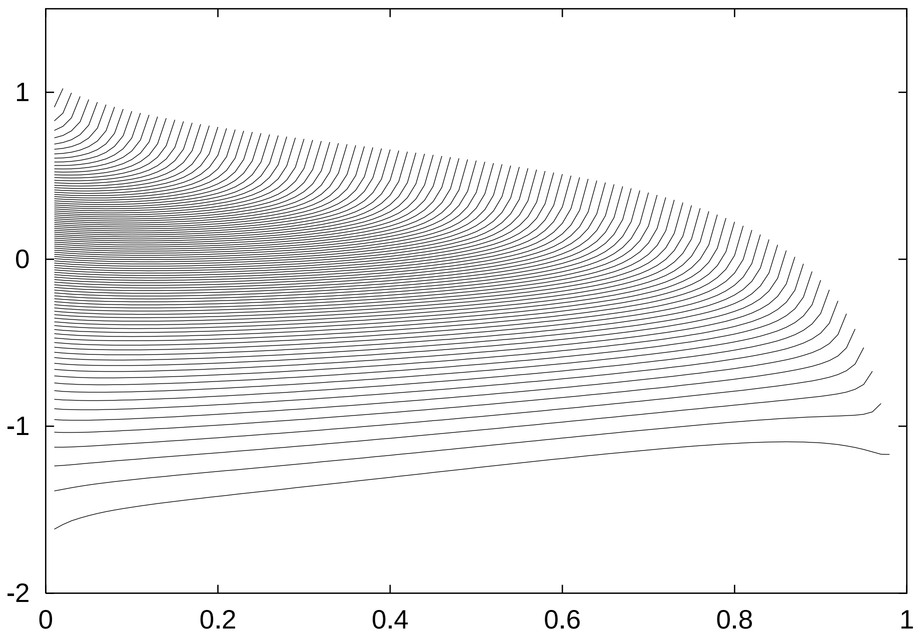



Figure 4: Representation of $D_2$. Every continuous line is the plot of $\log 10(m_1 m_2 D_2)$ as a function of $m_2$ for a fixed value of $m_1$.

## 8 Conclusions

We have considered the planar three body problem in a small neighbourhood of one collinear equilibrium point, with fixed angular momentum. So, the phase space has dimension 6. The intersection of the centre manifold with a fixed level of energy is an invariant set, $\mathcal{S}$, homeomorphic to a 3–dimensional sphere. In $\mathcal{S}$ there are two remarkable families of periodic orbits, the Lyapunov family and the homographic one. According to the results of section 7, KAM theorem can be applied to obtain invariant tori in $\mathcal{S}$ even in the resonant case. Moreover, using a theorem of Graff [6] these tori have 3–dimensional stable and unstable invariant manifolds. The unstable (stable) manifold goes out of some neighbourhood of the equilibrium point for positive (negative) time. The intersections, if they exist, of these invariant manifolds will give homoclinic or heteroclinic orbits to the tori and so, the well known symbolic dynamics methods can be applied.

We can compare with the planar, circular, restricted three body problem which can be described by a Hamiltonian system with two degrees of freedom. In this problem there are also three collinear equilibrium points which are saddle–centre points. It is known, ([9],[8]), that the stable and unstable invariant manifolds of the periodic Lyapunov orbits intersect for a large set of values of the Jacobi constant. The homoclinic and heteroclinic orbits give a rich dynamics with nice applications to mission design orbits ([8]).

Furthermore, for the planar three body problem, as Moeckel pointed out in ([12]), the sphere $\mathcal{S}$ is a normally hyperbolic invariant set which has 4D dimensional invariant manifolds. It is expected that these invariant manifolds play a role of "quasi–confiner" for the dynamics in a similar way as the invariant manifolds of points $L_{1,2}$ do in Hill's problem ([17]) or the centre manifolds do in the three dimensional restricted three body problem ([4], [5], [7]).

## Acknowledgements

The authors are very grateful to Carles Simó for his useful comments. This work has been supported by the DGICYT grant BFM 2000–805 (Spain) and the CIRIT grant 2000 SGR–27 (Catalonia). The second author is pleased also to acknowledge support by a grant of the Departament de Matemàtiques, Universitat Autònoma de Barcelona.

## Appendix 1

Once the equilibrium point has been translated to the origin we expand the Hamiltonian of the problem using the Legendre polynomials. We obtain the following expressions for the quadratic and cubic terms of the Hamiltonian $\bar{H}(\mathbf{x},\mathbf{y})$ defined in (11)

$$\begin{aligned}
a\rho\bar{H}_2(\mathbf{x},\mathbf{y}) = {} & \frac{1}{a}\left(-m_1m_2\rho^{-1} - m_1m_3\rho^2\sigma^3 + \frac{A_p}{m_3}v_3^p\rho^2\sigma^3 + \frac{3\alpha_1}{2a}A_p^2\rho^2\sigma^4\right)x_1^2 + \\
& \left(\rho^2\sigma^2v_3^p\left(\frac{v_3^p}{m_3} + \frac{2\alpha_1}{a}A_p\sigma\right) + \frac{2m_1m_2}{a}\rho^{-1}\right)x_1x_2 + \rho\sigma^2\left(-\frac{A_p}{m_3a} + \frac{v_3^p}{m_3} + \frac{2\alpha_1}{a}A_p\sigma\right)x_1y_3 + \\
& \left(\frac{\alpha_1}{2}\rho^2\sigma^2(v_3^p)^2 - \frac{m_1m_2}{a}\rho^{-1} - \frac{m_2m_3}{a}\rho^2\right)x_2^2 + \rho\sigma\left(-\frac{2}{m_3}v_3^p - \frac{\alpha_1}{a}A_p\sigma + \alpha_1v_3^p\sigma\right)x_2y_3 + \\
& \frac{m_2}{2a}(m_1\rho^{-1} + m_3\rho^2)x_3^2 + \rho\sigma\left(\frac{\alpha_1}{a}A_p\sigma + \frac{v_3^p}{m_3}\right)x_3y_2 + \frac{\alpha_1}{2}y_1^2 + \frac{\alpha_2}{2}y_2^2 + \frac{1}{m_3}y_1y_2 + \\
& \left(\frac{\alpha_2}{2} - \frac{1}{m_3}\sigma + \frac{\alpha_1}{2}\sigma^2\right)y_3^2,
\end{aligned}$$

$$
\begin{aligned}
a\rho\bar{H}_3(\mathbf{x},\mathbf{y}) = &\frac{1}{a}\left(-\frac{1}{m_3}\rho^3 A_p v_3^p\sigma^4 - \frac{2\alpha_1}{a}\rho^3 A_p^2\sigma^5 + m_1 m_2\rho^{-1} + m_1 m_3\rho^3\sigma^4\right) x_1^3 + \\
&\left(-\frac{1}{m_3}\rho^3\sigma^3(v_3^p)^2 - \frac{3\alpha_1}{a}\rho^3\sigma^4 v_3^p A_p - \frac{3m_1m_2}{a}\rho^{-1}\right) x_1^2 x_2 + \\
&\rho^2\sigma^3\left(\frac{A_p}{m_3 a} - \frac{v_3^p}{m_3} - \frac{3\alpha_1}{a}A_p\sigma\right) x_1^2 y_3 + \\
&\rho^2\sigma^2\left(\frac{2v_3^p}{m_3} + \frac{2\alpha_1}{a}A_p\sigma - 2\alpha_1 v_3^p\sigma\right) x_1x_2y_3 + \rho\sigma^2\left(\frac{1}{m_3} - \alpha_1\sigma\right) x_1 y_3^2 + \\
&\rho^2\sigma^2\left(-\frac{v_3^p}{m_3} - \frac{2\alpha_1}{a}A_p\sigma\right) x_1x_3y_2 + \left(-\alpha_1\rho^3\sigma^3(v_3^p)^2 + \frac{3m_1m_2}{a}\rho^{-1}\right) x_1x_2^2 + \\
&\alpha_1\rho^2\sigma^2 v_3^p x_2^2 y_3 - \frac{3m_1m_2}{2a}\rho^{-1}x_1x_3^2 + \frac{m_2}{a}\left(m_3\rho^3 - m_1\rho^{-1}\right)x_2^3 + \rho\sigma\left(\frac{1}{m_3} - \alpha_1\sigma\right) x_3y_2y_3 + \\
&\rho\sigma\left(\alpha_1\sigma - \frac{1}{m_3}\right) x_2y_3^2 - \alpha_1\rho^2\sigma^2 v_3^p x_2x_3y_2 + \frac{3m_2}{2a}\left(m_1\rho^{-1} - m_3\rho^3\right) x_2x_3^2 \qquad (38)
\end{aligned}
$$

where the constants $a, \rho, \sigma, v_3^p, v_4^p$ have been introduced in section 3, $A_p = -v_4^p - av_3^p$ and $\alpha_1, \alpha_2$ are the scaled masses defined at the end of (21).

# Appendix 2

In this appendix we prove the claim made in lemma 2 that is, $\frac{d\beta}{dm_2} < 0$ for any fixed value of $m_1 \in (0,1)$.

After some computations and using (7) we get

$$
\frac{d\beta}{dm_2} = \frac{\partial\beta}{\partial m_2} + \frac{\partial\beta}{\partial\rho}\rho'(m_2) = \frac{-2p_4(\rho)}{\bar{p}_4(\rho)\rho^4[m_2C(\rho)+1]^2(1-m_1)^5}
$$

where $C(\rho) = \frac{1}{\rho^2}(\rho^4 + 2\rho^3 + \rho^2 + 2\rho + 1)$ and

$$
\begin{aligned}
\bar{p}_4(\rho) &= 5\rho^4(1-m_1) + 4\rho^3(3-3m_1-m_2) + 3\rho^2(3-3m_1-2m_2) - 2\rho(3m_1+m_2) - \\
&\quad (3m_1+2m_2), \\
p_4(\rho) &= \sum_{i=0}^{4} a_i\rho^i, \qquad i = 0,\dots,4,
\end{aligned}
$$

for some coefficients $a_i$.

Using (7) we write

$$
\begin{aligned}
\rho\bar{p}_4(\rho) &= \rho\bar{p}_4(\rho) - 2q(\rho) = 3\rho^5(1-m_1) + 2\rho^4(3-3m_1-m_2) + \\
&\quad \rho^3(3-3m_1-2m_2) + \rho(3m_1+2m_2) + 2(m_1+m_2) > 0.
\end{aligned}
$$

So we only need to prove that $p_4(\rho) > 0$ when $\rho$ satisfies the quintic equation $q(\rho) = 0$. We define $P_4(\rho) := p_4(\rho) + 15q(\rho)$. It is clear that $P_4(\rho) = p_4(\rho)$. However if $\rho = 0$ then $m_1 = m_2 = 0$. Moreover $P_4(0) = 0$. We shall prove that $P_4'(\rho) > 0$ for any fixed value of $m_1 \in (0,1), m_2 < 1 - m_1$. We define a new polynomial of degree 5 as $P_5(\rho) = \rho P_4'(\rho) - 30q(\rho)$ and we write $P_5(\rho) = \sum_{i=0}^{5} b_i\rho^i$ where

$$
b_5 = 45(1-m_1), \qquad b_i = \sum_{j=0}^{6} c_{ij}m_2^j, \quad i = 1,\dots,4, \qquad b_0 = 30(m_1+m_2).
$$

The coefficients $c_{ij}$ are given at the end of this appendix. Clearly $b_0, b_5 > 0$. Also $c_{2j} > 0$ for $j = 0, \ldots, 6$, then $b_2 > 0$.

We claim that $b_1, b_3$ and $b_4$ are positive. First we consider $b_1$. We have $c_{16} > 0$ and

$$\begin{aligned} & c_{12}m_2^2 + c_{11}m_2 = \\ & \quad m_2(m_1-1)^2[30m_1^2m_2 + 140m_1m_2 + 8(1-m_1^2-m_2) + 20m_1(1-m_1)] + 30 > 0, \\ & c_{15}m_2^5 + c_{14}m_2^4 + c_{13}m_2^3 = m_2^3\{18m_1m_2^2 + \\ & \quad (m_1-1)[(7m_1+14m_2)(m_2+m_1-1) - 8m_2^2 - 61(1-m_1^2-m_1m_2) - 56m_1m_2]\} > 0. \end{aligned}$$

Therefore $b_1 > 0$.

The sign of $b_3$ and $b_4$ is determined taking into account the following trivial remark: consider a function $g(m_2) = am_2^2 + bm_2 + c$ such that $a < 0$ and $c > 0$. If $a(1-m_1)^2 + b(1-m_1) + c \geq 0$ then $g(m_2) > 0$ for $0 < m_2 < 1 - m_1$.

For $b_4$ we have $c_{46} > 0, c_{43} > 0$ and $c_{45}m_2^5 + c_{44}m_2^4 = (1-m_1)m_2^4(4m_2 + 52m_3) > 0$. Using the remark it is not difficult to check that $c_{42}m_2^2 + c_{41}m_2 + c_{40} > 0$. A similar argument proves that $b_3 > 0$. □

The coefficients $c_i$ involved in the proof above are the following

$$\begin{aligned} & c_{46} = 24,\ \ c_{45} = 48(m_1-1),\ \ c_{44} = 52(m_1-1)^2,\ \ c_{43} = 8(17m_1+19)(m_1-1)^2, \\ & c_{42} = 12(5m_1+13)(m_1-1)^3,\ \ c_{41} = 90 - 400m_1 + 400m_1^2 - 200m_1^4 + 80m_1^5, \\ & c_{40} = 2(m_1-1)(64m_1^5 - 136m_1^4 + 12m_1^3 + 140m_1^2 - 92m_1 - 33), \\ & c_{36} = 36,\ \ c_{35} = 72(m_1-1),\ \ c_{34} = 6(m_1-1)(13m_1-22),\ \ c_{33} = 6(34m_1+29)(m_1-1)^2, \\ & c_{32} = 9(10m_1+17)(m_1-1)^3,\ \ c_{31} = 159 - 348m_1 - 174m_1^2 + 720m_1^3 - 471m_1^4 + 84m_1^5, \\ & c_{30} = 9m_1(m_1-1)(28m_1^4 - 71m_1^3 + 40m_1^2 + 26m_1 - 28), \\ & c_{26} = 12,\ \ c_{25} = 24m_1 + 12,\ \ c_{24} = 26(m_1-1)^2,\ \ c_{23} = 2(34m_1+47)(m_1-1)^2, \\ & c_{22} = 6(5m_1^2 + 38m_1 + 11)(m_1-1)^2,\ \ c_{21} = -12(2m_1^2 + 15m_1 + 1)(m_1-1)^3, \\ & c_{20} = 2m_1(104m_1+37)(m_1-1)^4, \\ & c_{16} = 12,\ \ c_{15} = 24m_1 - 6,\ \ c_{14} = 2(13m_1-7)(m_1-1),\ \ c_{13} = (68m_1+61)(m_1-1)^2, \\ & c_{12} = 2(15m_1^2 + 70m_1 - 4)(m_1-1)^2,\ \ c_{11} = 38 + 4m_1 - 60m_1^2 + 76m_1^3 - 28m_1^4, \\ & c_{10} = m_1(62 + 2m_1 - 178m_1^2 + 352m_1^3 - 263m_1^4 + 70m_1^5). \end{aligned}$$

# Appendix 3

The constants $k_1, \ldots, k_7$ defined in (21) are the following

$$\begin{aligned} k_1 &= \frac{\beta_4}{m_3}\sigma^2 + 2\sigma, \quad k_2 = \beta_3\sigma(\alpha_1\sigma - 1/m_3) - 1, \quad k_3 = \alpha_2 - \frac{2}{m_3}\sigma + \alpha_1\sigma^2, \\ k_4 &= \frac{1}{a^3}\left[2m_1(m_2\rho^{-3} + m_3\sigma^3)\right] + (\beta_3 + \beta_4)(3\sigma^2 - \frac{\beta_3}{m_3}\sigma^3), \\ k_5 &= -\frac{2m_1m_3}{a^3}\rho^{-3} + \frac{\beta_3^2}{m_3}\sigma^2 - 2\beta_3\sigma, \\ k_6 &= \frac{2m_2}{a^3}(m_1\rho^{-3} + m_3) - \alpha_1\beta_3^2\sigma^2, \quad k_7 = -\frac{m_2}{a^3}(m_1\rho^{-3} + m_3), \end{aligned}$$

where $\beta_3 = m_2(m_3 - m_1\rho)$, $\beta_4 = -m_1m_3(\rho+1)^2 - m_2(m_1\rho^2 + m_3)$ and $\sigma = \dfrac{1}{\rho+1}$.

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