On the divergence of polynomial interpolation

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Abstract

Consider a triangular interpolation scheme on a piecewise C^1 curve of the complex plane, and let Γ be the closure of this triangular scheme. Given a meromorphic function f with no singularities on Γ, we are interested in the region of convergence of the sequence of interpolating polynomials to the function f. In particular, we focus on the case in which Γ is not fully contained in the interior of the region of convergence defined by the standard logarithmic potential. Let us call Γ_{out} the subset of Γ outside of the convergence region.

In the paper we show that the sequence of interpolating polynomials, $\{P_n\}_n$, is divergent on all the points of Γ_{out}, except on a set of zero Lebesgue measure. Moreover, the structure of the set of divergence is also discussed: the subset of values z for which there exists a partial sequence of $\{P_n(z)\}_n$ that converges to $f(z)$ has zero Hausdorff dimension (so it also has zero Lebesgue measure), while the subset of values for which all the partials are divergent has full Lebesgue measure.

The classical Runge example is also considered. In this case we show that, for all z in the part of the interval $(-5, 5)$ outside the region of convergence, the sequence $\{P_n(z)\}_n$ is divergent.

Keywords: Runge phenomenon, logarithmic potential.

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1 Introduction

The main purpose of this paper is to study the convergence, in the complex plane, of a sequence of interpolating polynomials to a given meromorphic function f. More specifically, let $\beta : [0,1] \longrightarrow \mathbb{C}$ be a simple and piecewise C^1 curve whose derivative is always different from zero, and assume that f has no poles on $\beta([0,1])$. Let us denote by $\{t_{n,j}\}_{0\le j\le n, n\ge 0}$ a triangular interpolation scheme on the interval $[0,1]$ (see Section 2). This scheme induces a triangular scheme $\{x_{n,j}\}_{0\le j\le n,n\ge 0}$ on $\beta([0,1]) \subset \mathbb{C}$, by defining $x_{n,j} = \beta(t_{n,j})$. We will denote by $\{P_n\}_{n\ge 0}$ the corresponding sequence of interpolating polynomials; in other words, P_n is the (only) polynomial of degree less than or equal to n that satisfies $P_n(x_{n,j}) = f(x_{n,j})$ for all j between 0 and n. As usual, if $x_{n,j_1} = x_{n,j_2} = \cdots = x_{n,j_r}$ for r (different) values j_k, we assume that P_n also interpolates the first $r-1$ derivatives of f on the point x_{n,j_1}. Then, given a $z \in \mathbb{C}$, a natural question is to determine if $\{P_n(z)\}_{n\ge 0}$ converges to the value $f(z)$. Classical references for this problem are [Wal35, Ste50, Dav75].

In this work we will not assume any concrete triangular scheme for the interpolating points; we will only ask them to admit a distribution.

Definition 1.1 *An interpolation scheme $\{x_{n,j}\}_{0\le j\le n,n\ge 0}$ is said to have a distribution if the limit*

$$\lim_{n\to\infty} \frac{\#\{\, j \,:\, t_{n,j} \in [0,t]\,, 0 \le j \le n\}}{n+1}$$

exists for all $t \in [0,1]$, where $\#$ is used to denote the cardinal of a set. If we denote by $\varphi(t)$ the value of this limit, then $\varphi(t)$ is known as the distribution associated to the interpolation scheme $\{x_{n,j}\}_{0\le j\le n,n\ge 0}$.

A well known interpolation scheme is obtained using equidistant nodes on a given interval, giving rise to a linear distribution φ. Another classical example is based on using a finite number of different values of $x_{n,j}$, or, in other words, applying a Taylor interpolation method on a finite number of points. In this case, φ is a piecewise constant function, and the discontinuities correspond to the position of the interpolating nodes.

Remark 1.1 *Given a distribution φ it can be proved that there exists a unique probability measure μ defined on the Borel sets of $[0,1]$ such that for $0 \le a \le b \le 1$, $\mu((a,b]) = \varphi^*(b) - \varphi^*(a)$, where φ^* is the unique function which is monotone increasing, continuous on the right, and agrees with φ wherever φ is continuous on the right (see, for instance, [Roy88], page 302). In particular, $\varphi^*(1) = \varphi(1) = 1$ and $\mu([0,1]) = 1$, and, therefore, $\mu(\{0\}) = \varphi^*(0)$.*

To study the domain of convergence of interpolation schemes with distribution φ, we denote by μ the corresponding Borel measure and we introduce the *logarithmic potential* $V : \mathbb{C} \setminus \beta([0,1]) \longrightarrow \mathbb{R}$, defined by

$$V(z) = \int_0^1 \ln|z - \beta(t)|\, d\mu. \tag{1}$$

Sometimes we will also refer to V as the *level function* of the triangular scheme. V can be extended to the values $z \in \beta([0,1])$ by defining $V(z) = -\infty$ when the integral (1) is not convergent.

Let us define $\Gamma \subset \beta([0,1])$ as the closure of the triangular scheme $\{x_{n,j}\}_{0\leq j\leq n,\, n\geq 0}$. Moreover, for simplicity, we will assume that f has a finite number of poles $\alpha_1, \ldots, \alpha_m$, that all of them are simple, and none of them belongs to Γ. We denote by α_0 a pole such that

$$V(\alpha_0) = \min_{1\leq k\leq m} V(\alpha_k).$$

Note that, as none of the α_k belongs to Γ, $|V(\alpha_0)| < \infty$. Then, if we define

$$\begin{aligned} C &= \{z \in \mathbb{C} \text{ such that } V(z) < V(\alpha_0)\}, \\ D &= \{z \in \mathbb{C} \text{ such that } V(z) > V(\alpha_0)\}, \end{aligned}$$

it is known that the sequence $\{P_n(z)\}_{n\geq 0}$ converges to $f(z)$ if $z \in C$, and that it converges to ∞ if $z \in D \setminus \Gamma$ (see, for instance, [War76, Sim77]). Note that these classical techniques do not work neither for z such that $V(z) = V(\alpha_0)$, nor for $z \in \Gamma \cap D$. This last case will be the main topic of this paper.

The situation considered here is then the following: assume that $\{x_{n,j}\}_{0\leq j\leq n,\, n\geq 0}$ is a triangular scheme on $\beta([0,1])$, having a distribution φ with a Borel measure μ. Moreover, let us write the closure of the triangular scheme Γ as $\Gamma_{in} \cup \Gamma_{out} \cup \Gamma_l$, where $\Gamma_{in} = \Gamma \cap C$ (Γ_{in} is the part of Γ inside the convergence region C), $\Gamma_{out} = \Gamma \cap D$ (Γ_{out} is outside C) and Γ_l contains the remaining points on Γ (Γ_l is the part of Γ inside the level set $V(z) = V(\alpha_0)$). We are concerned with the behaviour of the interpolating polynomial on Γ_{out}. Hence, we will focus on the case $\Gamma_{out} \neq \emptyset$. Then, although the interpolation scheme is dense on Γ_{out}, we do not expect convergence on the full region due to the well-known Runge phenomenon ([Run01]).

In this paper we will show that the subset of $z \in \Gamma_{out}$ for which the sequence $\{P_n(z)\}_{n\geq 0}$ converges to ∞ is of full Lebesgue measure[1] in Γ_{out}. Moreover, the set $z \in \Gamma_{out}$ for which there exists a subsequence of $\{P_n(z)\}_{n\geq 0}$ convergent to $f(z)$ is not only nonempty but also dense in the relative interior of Γ_{out} in $\beta([0,1])$, and with zero Hausdorff dimension (so it also has zero Lebesgue measure). In particular, if we define Γ_{out}^c as the set of points $z \in \Gamma_{out}$ for which the sequence $\{P_n(z)\}_{n\geq 0}$ converges to $f(z)$, then this set has to have zero Hausdorff dimension (these are the statements of Theorem 3.1). However, the only knowledge of the distribution φ is not enough to give more information on Γ_{out}^c: for instance, the equidistant triangular scheme $\frac{j}{n}$ $(0 < j < n)$ on $[0,1]$ has $\Gamma_{out}^c = \emptyset$ (see Theorem 3.2), while a suitable modification of this triangular scheme, that still admits the same (uniform) distribution, produces a Γ_{out}^c that is dense in Γ_{out} (see Section 3.2). We note that Theorem 3.2 can be immediately applied to the well known Runge example to prove the divergence in all the points of Γ_{out} except, of course, the endpoints of the interpolation interval since they are always interpolating nodes (see Section 3.1).

[1] We define the Lebesgue measure of a set $A \subset \beta([0,1])$ as the usual Lebesgue measure of the set $\beta^{-1}(A) \subset [0,1]$.

To present these results, the paper has been organised as follows: Section 2 contains the notations, main definitions and basic properties about triangular schemes and interpolation. Section 3 is devoted to the formal presentation of the results, including the application to the Runge example. Finally, Section 4 is devoted to the proofs of the main results and Appendix A contains some properties of the level sets of the logarithmic potential.

A natural extension of these results is to consider interpolating schemes whose closure is not contained in a curve of $\mathbb{C}$. This is work in progress.

2 Basic definitions and properties

This section introduces the main definitions used in the paper. They refer to the distribution of the nodes (Sections 2.1 and 2.2) and to the sets where the convergence is studied (Section 2.3).

2.1 Schemes and distributions

Let $\{x_{n,j}\}_{0\le j\le n, n\ge 0}$ be a triangular scheme, having a distribution φ with the associated Borel measure μ. In this paper, we will restrict ourselves to cases in which φ is continuous on $\Gamma_{out} = \Gamma \cap D$. Apparently, we are ruling out a typical situation for the distribution of an interpolation scheme: discontinuities. They usually correspond to interpolate an increasing number of derivatives of the interpolated function f at a given point, or to a very high accumulation of interpolating points in a small region. Note that the measure μ of a discontinuity point is positive so it can be represented by a suitable Dirac delta. Then, the logarithmic potential (1) goes to $-\infty$ when z goes to the discontinuity point of the distribution (see Proposition 2.1); this implies that a sufficiently small neighbourhood of this point (and, hence, the discontinuity) is included in the convergence region. Therefore, the assumption of the continuity of φ outside the convergence region seems quite natural (see Proposition 2.2).

In what follows, we will use interpolation schemes that are not triangular. The main reason is that, in some proofs, we will use schemes obtained by selecting an infinite subset of nodes of a given scheme. Hence, some lemmas and propositions are stated using non-triangular schemes. We note that the technicalities of the proofs are almost identical in the general and the triangular cases. For these reasons, we give the following definition.

Definition 2.1 *Let $\{k(n)\}_{n\ge 0}$ be a sequence of natural numbers. A sequence of complex numbers $\{x_{n,j}\}_{0\le j\le k(n), n\ge 0}$ such that $x_{n,j} = \beta(t_{n,j})$ is said to be an interpolation scheme on β if*

1. $\lim_{n\to\infty} k(n) = \infty$,
2. *if* $0 \le j \le \ell \le k(n)$ *then* $t_{n,j} \le t_{n,\ell}$.

If $k(n) = n$ for all $n \ge 0$ the scheme is said to be triangular.

Next proposition is about the logarithmic potential V defined in (1). It is well known (see, for instance, [Ran95]) that V is a subharmonic function on $\mathbb{C}$ and harmonic on $\mathbb{C} \setminus \beta(\operatorname{supp} \mu)$, where $\operatorname{supp} \mu$ denotes the support of the measure μ on $[0,1]$.

Proposition 2.1 *Let V be a logarithmic potential. Then:*

1. *If $x \in \mathbb{C} \setminus \beta(\operatorname{supp} \mu)$, then V is continuous at x.*
2. *If $x \in \beta([0,1])$ and $V(x) = -\infty$ then V is continuous at x.*
3. *The set $C_M = \{x \in \mathbb{C}$ such that $V(x) < M\}$ is open and bounded for all $M \in \mathbb{R}$.*

Proof: The first statement holds because V is harmonic on $\mathbb{C} \setminus \beta(\operatorname{supp} \mu)$. Now, suppose that $x \in \beta([0,1])$ and $V(x) = -\infty$, and let $(x_n)_{n\in\mathbb{N}}$ be a sequence that converges to x when n tends to infinity. We have to prove that $\lim_{n\to\infty} V(x_n) = -\infty$. Indeed, let $f_n(t) = \ln|x_n - \beta(t)|$ and $f(t) = \ln|x - \beta(t)|$. Note that $f_n(t) \to f(t)$, for all $t \in [0,1]$, because $\ln y$, $y > 0$, is a continuous map that tends to $-\infty$ when y tends to zero. Moreover, there exists a constant $M \in \mathbb{R}$ such that $f_n(t) \leq M$, for all $t \in [0,1]$ and all $n \in \mathbb{N}$. This means that $g_n = M - f_n$ are positive functions for all $n \in \mathbb{N}$, and $g_n(t) \to g(t) = M - f(t)$, for all $t \in [0,1]$. By applying Fatou's Lemma, one gets:

$$\int_0^1 g\, d\mu \leq \liminf_{n\to\infty} \int_0^1 g_n d\mu.$$

This means that $\limsup_{n\to\infty} \int_0^1 f_n d\mu \leq \int_0^1 f d\mu = -\infty$. Therefore, $\lim_{n\to\infty} \int_0^1 f d\mu = \lim_{n\to\infty} V(x_n) = -\infty$.

Finally, suppose that there exists $M_0 \in \mathbb{R}$ such that C_{M_0} is unbounded. Let $N = \max_{t\in[0,1]} |\beta(t)| > 0$. As the set is unbounded, for all $n \in \mathbb{N}$ there exists $x_n \in \mathbb{C}$ such that $V(x_n) < M_0$ and $|x_n| > (n+1)N$. But then:

$$V(x_n) = \int_0^1 \ln|x_n - \beta(t)| d\mu \geq \ln(nN),$$

Hence, if n is large enough, $V(x_n) > M_0$ which is absurd. This means that the set C_M is bounded for all M. As V is subharmonic, C_M is also an open set. ∎

Proposition 2.2 *Let $\{x_{n,j}\}_{0\leq j\leq n, n\geq 0}$ be a triangular scheme with closure $\Gamma \subset \beta([0,1])$ and distribution φ. If $M \in \mathbb{R}$, let us define the sets $C_M = \{x \in \mathbb{C}$ such that $V(x) < M\}$ and $T_M = \{t \in [0,1]$ such that $\beta(t) \notin C_M\}$. Then the distribution φ is continuous on T_M.*

Proof: Let t_s a point of discontinuity of φ. If V is the logarithmic potential associated to the triangular scheme, then $V(\beta(t_s)) = -\infty$. As, by Proposition 2.1, $V \circ \beta$ is continuous at t_s, there exists an open neighbourhood U_s of $\beta(t_s)$ such that $\overline{U}_s \subset C_M$ and $U_s \cap \beta([0,1])$ is an open interval. Therefore, φ has no discontinuities on T_M. ∎

2.2 Regular nodes

Let $\{x_{n,j}\}_{n\geq 0, 0\leq j\leq k(n)}$ be an interpolation scheme, with closure Γ, on a parametrised curve β. Suppose that this scheme has a distribution φ. Let us define $a_{n,j}\in\mathbb{R}$ as

$$a_{n,j}=k(n)\mu([t_{n,j},t_{n,j+1}])-1,\qquad j=0,\ldots,k(n)-1, \tag{2}$$

where μ is the Borel measure associated to φ. We note that the numbers $a_{n,j}$ are zero iff the triangular scheme is equispaced with respect to the measure μ.

Definition 2.2 *If $x\in\Gamma$, the scheme $\{x_{n,j}\}_{n\geq 0,0\leq j\leq k(n)}$ is said to have regular nodes at the point $x=\beta(t)$ with respect to φ (or μ) if*

a) φ is continuous in a neighbourhood of t;

b) there exists $\delta>0$ such that, for all n, there exists $b_n\geq 0$ satisfying:

b.1) for any p and q ($p<q$) in $S_n(t,\delta)=\{j$ such that $t_{n,j}\in[t-\delta,t+\delta]\}$, we have

$$\left|\sum_{j=p}^{q-1}\frac{a_{n,j}}{k(n)}\ln(k(n))\right|\leq b_n;$$

b.2) $\lim\limits_{n\to\infty} b_n=0$.

The scheme $\{x_{n,j}\}_{n\geq 0,0\leq j\leq k(n)}$ is said to have regular nodes on a set $\Gamma'\subset\Gamma$ iff it has regular nodes at each $x\in\Gamma'$.

Given a continuous distribution φ, it is always possible to select a scheme $\{\widehat{x}_{n,j}\}_{n\geq 0,0\leq j\leq k(n)}$ such that $a_{n,j}=0$, for all n and j. Of course, the distribution of such a scheme will be again φ. In some sense, $\{\widehat{x}_{n,j}\}_{n\geq 0,0\leq j\leq k(n)}$ is a "canonical scheme" for φ. Hence, this is a local condition on the proximity of the original scheme to a "canonical scheme" in a neighbourhood of a given point t.

The condition given in Definition 2.2 is used in the proofs but only for technical reasons. We do not know whether it can be removed from the statements of the theorems.

2.3 Proper interpolation sets

Let $f:U\subset\mathbb{C}\longrightarrow\mathbb{C}$ be a meromorphic function in a simply connected region U with m simple poles $\{\alpha_k\}_{k=1}^m$. Given an interpolation scheme $\{x_{n,j}\}_{0\leq j\leq k(n),n\geq 0}$ contained in U, we denote by P_n the (unique) interpolating polynomial of f on the nodes $x_{n,0},\ldots,x_{n,k(n)}$ (note that P_n is a polynomial of degree at most $k(n)$). To fix ideas, we give now the well-known Hermite formula for the interpolating error (see [Wal35, Dav75]): let γ_n be a Jordan curve such that its interior (i.e., the bounded connected component of $\mathbb{C}\setminus\gamma_n$) contains the nodes $x_{n,0},\ldots,x_{n,k(n)}$, and f is analytic on an open neighbourhood of γ_n and its interior. Then, for all x in the interior of γ_n, we have

$$R_n(x)\equiv f(x)-P_n(x)=\frac{1}{2\pi i}\int_{\gamma_n}\frac{w_n(x)f(z)}{w_n(z)(z-x)}\,dz.$$

where $w_n(x) = \prod_{j=0}^{k(n)}(x - x_{n,j})$.

To derive information about the limit behaviour of the sequence of interpolating polynomials, we need to work on a sufficiently big region. A sufficient (but quite technical) condition for such a region is given in the next definition. This condition will be used later (in Proposition 4.1) to derive an error formula for the interpolating polynomials.

Definition 2.3 *A set $K \subset U$ is said to be a proper interpolation set with respect to f, U and $\{x_{n,j}\}_{0\le j\le k(n), n\ge 0}$ if*

1. *K is a compact set with non-empty interior.*

2. *The scheme $\{x_{n,j}\}_{0\le j\le k(n), n\ge 0}$ is contained in K.*

3. *There exists a closed, simple, regular and piecewise C^1 curve $\sigma : [0,1] \to \mathbb{C}$ such that: $\sigma([0,1]) \subset U$, none of the poles of f belongs to $\sigma([0,1])$, K is contained in the interior of σ, and for all $x \in K$,*
$$\lim_{n\to\infty} \frac{w_n(x)}{w_n(z)} = 0,$$
uniformly for $z \in \sigma([0,1])$.

4. *For any pole α_j of f belonging to the exterior of σ and any $x \in K$,*
$$\lim_{n\to\infty} \frac{w_n(x)}{w_n(\alpha_j)} = 0.$$

Remark 2.1 *Note that, if there exists a closed curve σ such that $D =$ $\operatorname{dist}(\sigma([0,1]), K) > d = \operatorname{diam}(K)$, where $\operatorname{diam}(K) = \sup_{y,z\in K}|y - z|$, then $|w_n(x)/w_n(z)| \le (d/D)^{n+1}$, for all $(x,z) \in K \times \sigma([0,1])$ and therefore, K is a proper interpolation set. This condition has already been used in [Sim77].*

Proposition 4.3 contains a suitable way of constructing proper interpolation sets using the logarithmic potential.

3 Main results

Let $\{x_{n,j}\}_{0\le j\le n, n\ge 0}$ be a triangular interpolation scheme on a piecewise C^1 curve β, with closure $\Gamma \subseteq \beta([0,1])$ and distribution φ, and let $f : U \subset \mathbb{C} \longrightarrow \mathbb{C}$ be a meromorphic function, defined on a simply connected region U, with simple poles $\{\alpha_k\}_{k=1}^m$ which do not belong to Γ. We denote by $\{P_n\}_{n\ge 0}$ the sequence of interpolating polynomials of f on the nodes $\{x_{n,j}\}_{0\le j\le n, n\ge 0}$. As before, let α_0 be a pole such that $V(\alpha_0) = \min_{1\le k\le m} V(\alpha_k)$, $C = \{z \in \mathbb{C}$ such that $V(z) < V(\alpha_0)\}$ and $D = \{z \in \mathbb{C}$ such that $V(z) > V(\alpha_0)\}$. We also define the sets $\Gamma_{in} = \Gamma \cap C$ and $\Gamma_{out} = \Gamma \cap D$. Let us split $\Gamma_{out} = \Gamma_{out}^c \cup \Gamma_{out}^d$, where Γ_{out}^c (resp. Γ_{out}^d) denotes the set of points $x \in \Gamma_{out}$ on which $\{P_n(x)\}_{n\ge 0}$ converges to $f(x)$ (resp. diverges). Finally, $\Gamma_{out}^{d,\infty}$ is defined as the set of $x \in \Gamma_{out}^d$ on which $\{P_n(x)\}_{n\ge 0}$ converges to infinity, and Γ_{out}^{sub} is the set of $x \in \Gamma_{out}$ on which $\{P_n(x)\}_{n\ge 0}$ has a subsequence convergent to $f(x)$ (note that $\Gamma_{out}^c \subset \Gamma_{out}^{sub}$).

Then, under this notation and conditions, we have the following results.

Theorem 3.1 *If $K \subset U$ is a proper interpolation set, then*

1. *If $x \in K \cap C$ then $\{P_n(x)\}_{n\geq 0}$ converges to $f(x)$; if $x \in \overset{\circ}{K} \cap C$ then $\{P_n(x)\}_{n\geq 0}$ converges locally uniformly to $f(x)$.*
2. *If $x \in K$, but $x \notin C \cup \Gamma \cup \{\alpha_1, \dots, \alpha_m\}$, then $\{P_n(x)\}_{n\geq 0}$ converges to ∞.*
3. *If the relative interior of Γ_{out} in $\beta([0,1])$ is non-empty, then the set Γ_{out}^{sub} is uncountable and dense in the relative interior of Γ_{out}.*
4. *Assume that the triangular scheme has regular nodes on the set Γ_{out}. Then we have that $\lambda(\beta^{-1}(\Gamma_{out}^{d,\infty})) = \lambda(\beta^{-1}(\Gamma_{out}))$, where λ denotes the Lebesgue measure defined in $[0,1]$. Moreover, Γ_{out}^{sub} has Hausdorff dimension equal to zero.*

Remark 3.1 *Items 1 and 2 are well-known results (see, for instance, [War76]), that we have included for completeness.*

Remark 3.2 *The distribution φ is continuous on the points $t \in [0,1]$ for which $\beta(t)$ does not belong to the convergence region. This follows from Proposition 2.2 and Theorem 3.1.*

Remark 3.3 *Note that the assumption on the finiteness of the number of poles can be easily satisfied shrinking, if necessary, the domain U without altering the compact set K. The hypothesis that all the poles have to be simple seems stronger than necessary; see also Remark 4.1.*

A particular but very important case corresponds to the use of equidistant nodes for the interpolation of a meromorphic function with a finite number of poles, all of them simple. In this case, the previous result can be more specific.

Theorem 3.2 *We select $x_{n,j} = t_{n,j} = \frac{j}{n}$, $j = 0, \dots, n$, $n > 0$ (this means that, with the previous notation, β is the identity, $\Gamma = [0,1]$ and $V(z) = \int_0^1 \ln|z-t|\,dt$). Let us denote by C_{max} the level curve of V such that $C_{max} \cap [0,1] = \{0,1\}$, and we assume that f is a meromorphic function with a finite number of simple poles, in an open neighbourhood of C_{max} and its interior. If $\Gamma_{out} \neq \emptyset$ we have*

1. *Γ_{in} is an interval centred at $\frac{1}{2}$, and Γ_{out} is the union of two disjoint intervals.*
2. *The interpolation diverges for all $x \in \Gamma_{out} \setminus \{0,1\}$.*
3. *The set Γ_{out}^{sub} is uncountable and dense in Γ_{out}. The Hausdorff dimension of Γ_{out}^{sub} is zero.*

Remark 3.4 *Obviously, the values 0 and 1 have been excluded in item 2 since they are interpolating nodes for all n.*

Remark 3.5 *Theorem 3.2 can be trivially extended to equidistant nodes on any interval of $\mathbb{R}$.*

Theorem 3.2 can be directly applied to a well-known example introduced by C. Runge in [Run01], where he showed the divergence of the interpolating polynomials on some points of the set Γ_{out}.

3.1 Application to an example by C. Runge

Consider the interpolation of the function

$$f(x) = \frac{1}{1+x^2}, \tag{3}$$

using equidistant abscissas on the interval $[-5,5]$. In this case, the distribution associated to the triangular scheme is given by $\varphi(t) = \frac{1}{10}(t+5)$, and the corresponding Borel measure is the (normalised) Lebesgue measure. The logarithmic potential for this case is given by

$$V(z) = \int_{-5}^{5} \ln|z-t|\, dt. \tag{4}$$

As $V(z)$ is symmetric with respect to the real axis, and the poles of (3) are $x = \pm i$, the convergence region is given by

$$C_R = \{z \in \mathbb{C} \text{ such that } V(z) < V(i)\}. \tag{5}$$

The boundary of the convergence region (the curve $V(z) = V(i)$) cuts the real line in two points, $t_R \approx 3.633384302388$ and $-t_R$. Hence, the convergence is assured inside the region (5) and, for $z \notin [-5,5] \cup \overline{C}_R$, the interpolation is divergent. The behaviour on the part of $[-5,5]$ outside C_R has not been previously studied using the logarithmic potential (4), due to its singular character. By using specific techniques for this example, several authors have shown the lack of convergence for some values $z \in [-5,5] \setminus C_R$ (see [Run01, Mon10, Ste50, IK66, Die68]). The results in this paper are based on the use of the logarithmic potential (4), and imply divergence (to ∞) on a full Lebesgue measure subset of $[-5,-t_R] \cup [t_R,5]$. A more detailed study (see the proof of Theorem 3.2) shows that the convergence set is not only of zero measure, but it only contains two points: ± 5 (note that these points are always interpolating abscissas, so the convergence follows trivially). If $z \in (-5,-t_R) \cup (t_R,5)$ then we distinguish two categories: i) "convergence to infinity" (full measure), or ii) there are partials convergent to $f(z)$ (zero Hausdorff dimension but uncountable and dense). We are not aware of similar results in the literature, and we refer to the proof of Theorem 3.2 for the details. We stress that these results only depend on the location of the poles of f, and not on the function itself.

3.2 A counterexample

A natural question is whether the results in Theorem 3.1 for general interpolating schemes can be improved in the direction of Theorem 3.2. In other words, we can ask if, under the hypotheses of Theorem 3.1, the divergence takes place on all the points of Γ_{out} and not only on a full measure subset.

This section contains an example showing that such a result cannot be true in general. The example uses a triangular scheme with a uniform distribution (and without equidistant nodes!), and the subset of points z of Γ_{out} on which $\{P_n(z)\}_n$ converges to $f(z)$ is dense on Γ_{out}.

The triangular scheme $\{x_{n,j}\}_{0\le j\le n, n\ge 0}$ will be taken on $[0,1]\subset\mathbb{C}$ or, in other words, $\beta(t)=t$. As before, we assume that we are interpolating a meromorphic function f, such that the convergence region $C=\{z\in\mathbb{C}\,/\,V(z)<V_0\}$ does not contain the whole interval $[0,1]$; that is, we assume that we can split $[0,1]$ as $[0,t_0]\cup(t_0,t_1)\cup[t_1,1]$ such that $C\cap[0,1]=(t_0,t_1)$, being $0<t_0<t_1<1$. Let us start by defining the usual equidistant nodes on $[0,1]$, $t^{\star}_{n,j}=j/n$, $j=0,\dots,n$, $n\ge 1$, and let $\nu:\mathbb{N}\to\mathbb{Q}\cap[0,1]$ be a one to one map. Let us select a real value $\alpha\in(0,1)$, and let us define the values $\hat{t}_{n,j}$ as follows:

$$\hat{t}_{n,j}=\left\{\begin{array}{lll} t^{\star}_{n,j} & \text{if} & j>n^{\alpha}, \\ \nu(j) & \text{if} & j\le n^{\alpha}. \end{array}\right.$$

Finally, let us define the triangular scheme $t_{n,j}$ as the result of sorting the values $\hat{t}_{n,j}$ for each n (that is, $t_{n,j}\le t_{n,k}$ if $j\le k$). Of course, this sorting is only necessary to match the previous definition of triangular scheme.

Let us now compute the distribution of the scheme $\{t_{n,j}\}_{n,j}$.

$$\lim_{n\to\infty}\frac{\#\{j\le n\,/\,t_{n,j}\le t\}}{n+1}=\lim_{n\to\infty}\frac{[nt]+O(n^{\alpha})}{n+1}=t,$$

where $[nt]$ denotes the integer part of nt. So, the scheme $\{t_{n,j}\}_{n,j}$ is uniformly distributed for any $\alpha\in(0,1)$.

To apply the previous results, we have to check that the triangular scheme has regular nodes. In this case, we have that $a_{n,j}$ is zero except for a number of j values of $O(n^{\alpha})$, for each n. On the other hand, it is not difficult to see that,

$$\mu([t_{n,j},t_{n,j+1}])\le\frac{n^{\alpha}+1}{n}.$$

This estimate corresponds to the worst case, namely $t_{n,0}=0$ and $t_{n,1}=[n^{\alpha}+1]/n$. Then, we can easily bound the value $|a_{n,j}|$:

$$|a_{n,j}|\le n\mu([t_{n,j},t_{n,j+1}])+1\le n^{\alpha}+2.$$

Then, if we add the assumption $\alpha<1/2$, we have that, for any indices p and q,

$$0\le\left|\sum_{j=p}^{q}\frac{a_{n,j}}{n}\ln n\right|\le\sum_{j=0}^{n-1}\frac{|a_{n,j}|}{n}\ln n\le O(n^{\alpha})\frac{n^{\alpha}+2}{n}\ln n=b_n\to 0 \text{ when } n\to\infty.$$

This implies that the nodes are regular on $[0,1]$. Hence, we are under the assumptions of Theorem 3.1. So, we can also ensure that there exists a full measure subset of $[0,t_0]\cup[t_1,1]$ for which the interpolation is divergent. On the other hand, it is clear that any rational number r inside $[0,1]$ appears in the triangular scheme, for all n bigger than some $n_0(r)$. This implies that we have a dense subset of $[0,t_0]\cup[t_1,1]$ for which the interpolation is convergent.

Note that if we define ν as a one to one map from $\mathbb{N}$ on $\mathbb{Q}$ (instead of $\mathbb{Q}\cap[0,1]$), what we obtain is that the interpolation converges on a dense subset of $\mathbb{R}$, while the support of the measure associated to the nodes is still uniform, with support on the interval $[0,1]$. With this idea we can construct examples in which the polynomials are convergent on a dense subset of *any* subset of the domain of f, without changing the initial (uniform or not) distribution.

4 Proofs

To facilitate the reading, the proof has been divided in several parts.

4.1 A formula for the interpolation error

The next proposition gives an expression of the error of interpolation for meromorphic functions. This result extends a previous one contained in [Sim77].

Proposition 4.1 *Suppose that $f : U \subset \mathbb{C} \longrightarrow \mathbb{C}$ and $\{x_{n,j}\}_{0\le j\le n, n\ge 0}$ satisfy the hypotheses of Theorem 3.1, and K is a proper interpolation set with respect to f, U and $\{x_{n,j}\}_{0\le j\le n,n\ge 0}$. Then, for all $x \in K \setminus \{\alpha_k\}_{k=1}^m$, we have*

$$\lim_{n\to\infty}\left[R_n(x) - \sum_{k=1}^{m} \frac{w_n(x)}{w_n(\alpha_k)} \frac{\operatorname{Res}(f,\alpha_k)}{x-\alpha_k}\right] = 0.$$

Proof: First of all, we choose a point $x \in K \setminus \{\alpha_k\}_{k=1}^m$ and define $D = d(x, \sigma([0,1])) > 0$. Consider the map

$$g_n(x,z) = \frac{f(z)w_n(x)}{(z-x)w_n(z)}.$$

Then, we claim that

$$\lim_{n\to\infty}\int_\sigma g_n(x,z)\,dz = 0. \tag{6}$$

Indeed, it is easy to see that we only need to prove (6) when σ is a C^1 map (then, the piecewise C^1 case is straightforward):

$$\left|\int_\sigma g_n(x,z)\,dz\right| \le \frac{M}{D}\int_0^1 \frac{|w_n(x)|}{|w_n(\sigma(t))|}|\sigma'(t)|\,dt \le \frac{MN}{D}\int_0^1 \frac{|w_n(x)|}{|w_n(\sigma(t))|}\,dt,$$

where $M = \sup_\sigma |f(z)|$ and $N = \sup_{t\in[0,1]}|\sigma'(t)|$. As K is a proper interpolation set, (6) follows.

On the other hand, the map $g_n(x,\cdot)$ has the only poles $\{\alpha_k\}_{k=1}^m$, x and $x_{n,j}$, $0 \le j \le n$. By the Residue Theorem:

$$\frac{1}{2\pi i}\int_\sigma g_n(x,z)\,dz = \sum_{j\in J} \operatorname{Res}(g_n(x,\cdot),\alpha_j) + \frac{1}{2\pi i}\int_{\gamma_n} g_n(x,z)\,dz, \tag{7}$$

where $j \in J$ iff α_j belongs to the interior of the closed curve σ and $\gamma_n : [0,1] \longrightarrow \mathbb{C}$ is a closed C^1 curve such that a) $\gamma_n([0,1])$ is contained in the interior of σ; b) $x_{n,0},\ldots,x_{n,n}$ is contained in the interior of γ_n; c) f has no poles neither in $\gamma_n([0,1])$ nor in the interior of γ_n; d) x belongs to the interior of γ_n (see Figure 1).

By computing all the residues, and taking into account the expression of the interpolating polynomial, we obtain:

$$\frac{1}{2\pi i}\int_\sigma g_n(x,z)\,dz = -\sum_{j\in J}\frac{w_n(x)}{w_n(\alpha_j)}\frac{\operatorname{Res}(f,\alpha_j)}{(x-\alpha_j)} + f(x) - P_n(x). \tag{8}$$

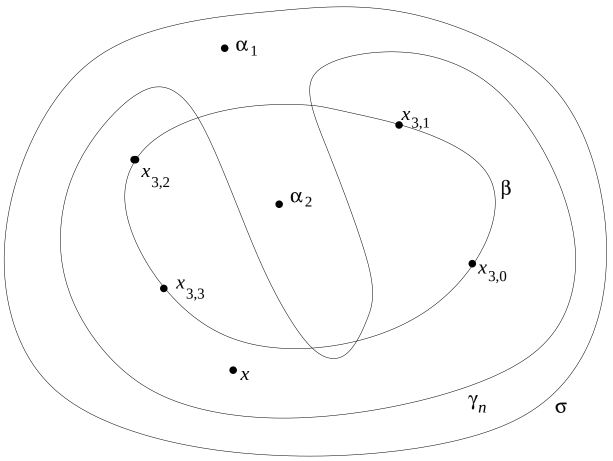

Figure 1: A possible configuration for β, σ, x, γ_n, $x_{n,0}, \ldots, x_{n,n}$ $(n = 3)$ and the poles α_j.

Finally, as K is a proper interpolation set, if α_j is a pole such that $j \notin J$, we have

$$\lim_{n\to\infty} \frac{w_n(x)}{w_n(\alpha_j)} = 0. \tag{9}$$

If we make n tend to infinity in equation (8) and take into account (6) and (9), we easily find the formula for the interpolation error of the proposition. ■

Remark 4.1 *The only place where we use the condition that the poles are simple is in Proposition 4.1, to derive formula (8) from formula (7). Similar formulas can be derived for poles of different multiplicities, and the results of this paper should follow in a similar way.*

4.2 The asymptotic behaviour of w_n

Next propositions describe the behaviour of $w_n(x)$ when n tends to infinity.

Proposition 4.2 *Let $\{x_{n,j}\}_{n\geq 0, 0\leq j\leq k(n)}$ be an interpolation scheme on a parametrised curve $\beta : [0,1] \longrightarrow \mathbb{C}$, having a distribution φ. Denote by μ the Borel measure associated to φ and by Γ the closure of the scheme. Then:*

1. *If $x \in \mathbb{C} \setminus \Gamma$ then:*
$$\lim_{n\to\infty} \frac{\ln|w_n(x)|}{k(n)+1} = \int_0^1 \ln|x - \beta(t)|\, d\mu.$$

2. *If $x \in \Gamma$ then:*

   (a) *If $\int_0^1 \ln|x - \beta(t)|\, d\mu = -\infty$:*
$$\lim_{n\to\infty} \frac{\ln|w_n(x)|}{k(n)+1} = \int_0^1 \ln|x - \beta(t)|\, d\mu = -\infty.$$

(b) If $\int_0^1 \ln|x-\beta(t)|\,d\mu$ *is finite:*

$$\limsup_{n\to\infty} \frac{\ln|w_n(x)|}{k(n)+1} \le \int_0^1 \ln|x-\beta(t)|\,d\mu.$$

Proof:

1. We distinguish the cases $x \notin \beta([0,1])$ and $x \in \beta([0,1]) \setminus \Gamma$. First, suppose that $x \notin \beta([0,1])$. We define, for all $p \ge 0$ and $0 \le i \le 2^p$, values $s_{i,p} \in [0,1]$ such that

a) $s_{i,p} < s_{j,p}$ if $i < j$, for all $p \ge 0$.

b) $s_{0,p} < 1/2^{p-1}$ and $1 - s_{p,p} < 1/2^{p-1}$ for all $p \ge 1$.

c) $s_{i,p}$ is not a point of discontinuity of φ, for all $p \ge 0$ and $0 \le i \le 2^p$ (we recall that the set of discontinuities of φ is countable).

d) For all $p \ge 1$ and all $0 \le j \le 2^{p-1}$ there exists $0 \le i \le 2^p$ such that $s_{i,p} = s_{j,p-1}$.

e) $|s_{i,p} - s_{i+1,p}| \le 1/2^{p-1}$ for all $p \ge 1$ and all $0 \le i \le 2^p$.

If φ is continuous, we can simply take $s_{i,p} = i/2^p$. Otherwise, if some of the values $s_{i,p} = i/2^p$ coincide with a discontinuity of φ, we can slightly modify these values to meet the conditions above.

Let us fix n. Then, for $1 \le i \le 2^p$,

$$\#\{j : t_{n,j} \in (s_{i-1,p}, s_{i,p}]\,,\, 0 \le j \le k(n)\} = (k(n)+1)(\varphi(s_{i,p}) - \varphi(s_{i-1,p})) + r_{i,p,n},$$

where $r_{i,p,n}/(k(n)+1) \to 0$ when $n \to \infty$ (see Remark 1.1). Let us now define the values

$$M_{i,p} = \sup_{t\in[s_{i-1,p},s_{i,p}]} \ln|x-\beta(t)|, \qquad m_{i,p} = \inf_{t\in[s_{i-1,p},s_{i,p}]} \ln|x-\beta(t)|,$$

for $1 \le i \le 2^p$. We easily see that:

$$\ln|w_n(x)| = \sum_{j=0}^{k(n)} \ln|x-\beta(t_{n,j})| \le \sum_{i=1}^{2^p} M_{i,p}((k(n)+1)(\varphi(s_{i,p}) - \varphi(s_{i-1,p})) + r_{i,p,n})$$

and

$$\ln|w_n(x)| \ge \sum_{i=1}^{2^p} m_{i,p}((k(n)+1)(\varphi(s_{i,p}) - \varphi(s_{i-1,p})) + r_{i,p,n}).$$

Now, we define the functions $F_p : [0,1] \longrightarrow \mathbb{R}$ and $f_p : [0,1] \longrightarrow \mathbb{R}$, such that:

$$F_p = \sum_{i=1}^{2^p} M_{i,p}\chi_{(s_{i,p},s_{i+1,p}]}, \qquad f_p = \sum_{i=1}^{2^p} m_{i,p}\chi_{(s_{i,p},s_{i+1,p}]},$$

where χ_E represents the characteristic function of a set $E \subset [0,1]$. Therefore, taking lower and upper limits when n tends to infinity, we obtain:

$$\begin{aligned}\int_0^1 f_p\,d\mu = \sum_{i=1}^{2^p} m_{i,p}(\varphi(s_{i,p}) - \varphi(s_{i-1,p})) \quad &\le \quad \liminf_{n\to\infty} \frac{\ln|w_n(x)|}{k(n)+1} \\ \le \quad \limsup_{n\to\infty} \frac{\ln|w_n(x)|}{k(n)+1} \quad &\le \quad \sum_{i=1}^{2^p} M_{i,p}(\varphi(s_{i,p}) - \varphi(s_{i-1,p})) = \int_0^1 F_p\,d\mu.\end{aligned}$$

As $\ln|x-\beta(t)|$ is a continuous map with respect to t, $\{F_p(t)\}_{p\in\mathbb{N}}$ and $\{f_p(t)\}_{p\in\mathbb{N}}$ converge to $\ln|x-\beta(t)|$ for all $t\in[0,1]$. By the Lebesgue's dominated convergence theorem, we obtain the desired result.

On the other hand, if $x=\beta(t_0)\notin\Gamma$ then there is an open interval I_0 such that $t_0\in I_0$ and $\beta(I_0)\cap\Gamma=\emptyset$. Then, using the same argument as before, replacing the interval $[0,1]$ by $[0,1]\setminus I_0$, we finish the proof of item 1.

2. In this case, there exists $t_0\in[0,1]$ such that $x=\beta(t_0)$. We also distinguish two cases:

i) If $\mu(\{t_0\})=0$ then we can proceed in the same way as before, but using only supremum and upper limit. We observe that $\{F_p(t)\}_{p\in\mathbb{N}}$ converges to $\ln|x-\beta(t)|$ for all $t\neq t_0$. Let $M=\sup_{t\in[0,1]}\ln|x-\beta(t)|$. If $G_p(t)=M-F_p(t)$ then $0\le G_p\le G_{p+1}$ for all $p\ge 0$ and $\{G_p(t)\}_{p\in\mathbb{N}}$ tends to $M-\ln|x-\beta(t)|$ for all $t\in[0,1]\setminus\{t_0\}$. As $\mu(\{t_0\})=0$, by the Lebesgue's Monotone Convergence Theorem we obtain that:

$$\limsup_{n\to\infty}\frac{\ln|w_n(x)|}{k(n)+1}\le\int_0^1\ln|x-\beta(t)|d\mu.$$

This result does not depend on the integrability of $\ln|x-\beta(t)|$, but if $\int_0^1\ln|x-\beta(t)|d\mu=-\infty$, we have that the upper and lower limit are both equal to $-\infty$. Hence, item 2 follows.

ii) If $\mu(\{t_0\})>0$, on one hand we have that

$$\int_0^1\ln|x-\beta(t)|d\mu=-\infty,$$

and, on the other hand,

$$\frac{\ln|w(\beta(t_0))|}{k(n)+1}=\frac{1}{k(n)+1}\sum_{j\in S_{n,\epsilon}}\ln|\beta(t_0)-\beta(t_{n,j})|+\frac{1}{k(n)+1}\sum_{j\in T_{n,\epsilon}}\ln|\beta(t_0)-\beta(t_{n,j})|,$$

where

$$T_{n,\varepsilon}=\{j\ :\ 0\le j\le k(n)\text{ and }t_0-\varepsilon<t_{n,j}\le t_0+\varepsilon\},$$

and

$$S_{n,\varepsilon}=\{0,1,\dots,k(n)\}\setminus T_{n,\varepsilon}.$$

As φ is discontinuous at t_0, if $\varepsilon>0$ is small enough, there exists $a_0>0$ such that

$$\lim_{n\to\infty}\frac{\#T_{n,\varepsilon}}{k(n)+1}\ge a_0.$$

Taking upper limits we obtain:

$$\limsup_{n\to\infty}\frac{1}{k(n)+1}\sum_{j\in T_{n,\varepsilon}}\ln|\beta(t_0)-\beta(t_{n,j})|\le\limsup_{n\to\infty}\frac{\#T_{n,j}}{k(n)+1}\ln(L\varepsilon)\le a_0\ln(L\varepsilon),$$

where $L > 0$ is a suitable constant that exists if ε is small enough. Moreover, there exists a constant $M \in \mathbb{R}$ such that:

$$\limsup_{n\to\infty} \frac{1}{k(n)+1} \sum_{j\in S_{n,\varepsilon}} \ln|\beta(t_0) - \beta(t_{n,j})| \leq M.$$

This means that

$$\limsup_{n\to\infty} \frac{\ln|w(\beta(t_0))|}{k(n)+1} \leq M + a_0 \ln(L\varepsilon),$$

for all $\varepsilon > 0$ small enough. This implies that

$$\lim_{n\to\infty} \frac{\ln|w(\beta(t_0))|}{k(n)+1} = -\infty,$$

which finishes the proof. ■

Now, we have an easy way to find proper interpolation sets.

Proposition 4.3 *Let $U \subset \mathbb{C}$ be an open and simply connected set, $f : U \to \mathbb{C}$ a meromorphic function with a finite number of poles that are simple and $\beta : [0,1] \to \mathbb{C}$ a parametrised curve such that $\beta([0,1]) \subset U$. Let V be the level function associated to the interpolation scheme $\{x_{n,j}\}_{0\leq j\leq n, n\geq 0} \subset \beta([0,1])$, and assume that there exists $V_1 \in \mathbb{R}$ such that the set $C_{V_1} = \{z \in \mathbb{C} : V(z) < V_1\}$ satisfies $\overline{C}_{V_1} \subset U$. If $K \subset U$ is a compact set satisfying*

1. $\overset{\circ}{K} \neq \emptyset$.
2. $\{x_{n,j}\}_{0\leq j\leq n, n\geq 0} \subset K$.
3. *K is contained in a connected component of the set C_{V_1}.*

Then, C_{V_1} is simply connected and K is a proper interpolation set.

Proof: As $\Gamma \subset K \subset C_{V_1}$, Lemma A.2 implies that C_{V_1} is simply connected. Let V_2 be a real value such that $V_1 < V_2$ and $C_{V_2} \subset U$. Lemma A.2 implies the existence of a closed and simple curve $\sigma : [0,1] \to \mathbb{C}$ such that: a) for all $t \in [0,1]$, $V_1 < V(\sigma(t)) < V_2$; and b) C_{V_1} is contained in the interior of σ (therefore, K is also contained in the interior of σ). We note that this curve can be selected so that it does not contain any pole of f (we recall that the number of singularities of f is finite). Now we can check that the requirements of Definition 2.3 are fulfilled. It is clear that the only points that need to be verified are the two limits in items 3 and 4.

Hence, we take $x \in K$ and $z \in \sigma([0,1])$, and define $a_n(z) = |w_n(x)|^{1/n}/|w_n(z)|^{1/n}$ and $V_\sigma = \min_{t\in[0,1]} V(\sigma(t))$. Taking logarithms, we can apply Proposition 4.2 to obtain

$$\limsup_{n\to\infty} a_n(z) \leq \exp(V_\sigma - V_1) < 1,$$

and this implies the limit in item 3 of Definition 2.3. The limit in item 4 can be proved in a similar way. ■

The next proposition, jointly with Proposition 4.5, extends Lemma 3.4 in [LS94] to more general interpolating schemes.

Proposition 4.4 *Let $\{x_{n,j}\}_{n\geq 0, 0\leq j\leq k(n)}$ be an interpolation scheme on a parametrised curve $\beta : [0,1] \longrightarrow \mathbb{C}$. Denote by μ the Borel measure associated to the interpolation scheme. Suppose that $t_0 \in [0,1]$ satisfies that there exist $c_0 > 0$, $\tau > 2$ and an infinite set $N(t_0) \subset \mathbb{N}$ such that*

$$|\beta(t_{n,j}) - \beta(t_0)| \geq \frac{c_0}{k(n)^\tau}, \tag{10}$$

for all $n \in N(t_0)$ and all $0 \leq j \leq k(n)$. Then, if $\{x_{n,j}\}_{n\geq 0, 0\leq j\leq k(n)}$ has regular nodes at t_0, we have

$$\lim_{\substack{n \to \infty \\ n \in N(t_0)}} \frac{\ln|w_n(\beta(t_0))|}{k(n)+1} = \int_0^1 \ln|\beta(t_0) - \beta(t)| d\mu,$$

where $w_n(x) = \prod_{j=0}^{k(n)} (x - x_{n,j})$.

Proof: Note that, re-numbering the elements of $N(t_0)$ and suppressing the rows of the interpolating scheme whose index does not belong to $N(t_0)$, the problem is reduced to the case $N(t_0) = \mathbb{N}$. Hence, we will only focus on this case. Moreover, we define $x = \beta(t_0)$ and $f(t) = \ln|x - \beta(t)|$, $t \in [0,1]$. We can also assume that f is integrable and that $\beta(t_0) \in \Gamma$ (if it is not, then we can apply Proposition 4.2 and obtain the desired result).

From (10), it is clear that, for all $n \geq 0$ and $0 \leq j \leq k(n)$, t_0 is different from $t_{n,j}$. Let $j_0 = j_0(n)$ denote the maximum index j for which $t_{n,j} < t_0$. If $t_{n,j} > t_0$ for all $0 \leq j \leq n$ we define $j_0 = -1$. Let δ_S be the value of δ that appears in Definition 2.2. Now, we select $\delta > 0$ such that the following conditions are met:

1. φ is continuous on $[t_0 - \delta, t_0 + \delta] \cap [0,1]$;
2. $\delta \leq \delta_S$;
3. $f(t) < 0$ if $t \in [t_0 - \delta, t_0 + \delta] \cap [0,1]$ and is strictly decreasing if $t \in (t_0 - \delta, t_0) \cap [0,1]$ and strictly increasing if $t \in (t_0, t_0 + \delta) \cap [0,1]$.

Note that item 3 follows from the fact that the map $t \mapsto |\beta(t_0) - \beta(t)|$ has an isolated minimum at $t = t_0$ (we recall that β is a regular parametrisation).

Let us define the functions f_n as

$$\begin{aligned} f_n(t) &= f(t)\chi_{[0,t_0-\delta]}(t) + f(t)\chi_{[t_0+\delta,1]}(t) \\ &\quad + \sum_{j=p(n)}^{j_0-1} f(t_{n,j})\chi_{[t_{n,j},t_{n,j+1})}(t) + \sum_{j=j_0+2}^{q(n)} f(t_{n,j})\chi_{(t_{n,j-1},t_{n,j}]}(t), \end{aligned}$$

where $p(n)$ is the minimum index j such that $t_{n,j} \in [t_0 - \delta, t_0 + \delta]$, $q(n)$ is the maximum value of j such that $t_{n,j} \in [t_0 - \delta, t_0 + \delta]$, and that the terms $f(t)\chi_{[0,t_0-\delta]}(t)$ and $f(t)\chi_{[t_0+\delta,1]}(t)$ vanish if $t_0 - \delta < 0$ and $t_0 + \delta > 1$, respectively.

Lemma 4.1 *The functions f_n satisfy*

1. *f_n is majorised by a μ-integrable function F, that is $|f_n| \leq F$, for all $n \geq 0$.*
2. *The sequence $\{f_n(t)\}_{n\geq 0}$ tends to $f(t)$ for almost all $t \in [0,1]$, with respect to the measure μ.*

Proof: Let us start by the case $t_0 \notin \{0,1\}$. Note that $f(t) < 0$ if $t \in [t_0-\delta, t_0+\delta]$, and that $f(t)$ is strictly decreasing if $t_0 - \delta < t < t_0$ and strictly increasing if $t_0 < t < t_0+\delta$. This implies that $|f_n(t)| \leq |f(t)|$ for all $t \in [0,1]$. Therefore, as we are assuming that f is integrable on $[0,1]$, we can take $F(t) = |f(t)|$ and this proves the first item of the lemma.

To prove the second item, let us define $T_\delta \subset [t_0-\delta, t_0+\delta]$ as the set of values $t \in [t_0-\delta, t_0+\delta] \setminus \{t_0\}$ for which there exists $n_t \in \mathbb{N}$ such that

a) for all $n \geq n_t$ there exists $j(n)$ with $t_{n,j(n)} < t < t_{n,j(n)+1}$,

b) $\lim\limits_{n\to\infty} t_{n,j(n)} = \lim\limits_{n\to\infty} t_{n,j(n)+1} = t$.

It is clear that $f_n(t)$ tends to $f(t)$ if $t \in [0,1] \setminus [t_0-\delta, t_0+\delta]$. We will finish the proof by showing that $\{f_n(t)\}_{n\geq 0}$ tends to $f(t)$ for $t \in T_\delta$, and that $\mu(T_\delta) = \mu([t_0-\delta, t_0+\delta])$.

So, let us select a fixed value $t \in T_\delta$. As $t \neq t_0$, there exists $n_1 \geq n_t$ such that $t_0 \notin (t_{n,j(n)}, t_{n,j(n)+1})$. Therefore, if $n \geq n_1$, we have that $j(n) \neq j_0$. Now we distinguish two cases: $j(n) < j_0$ or $j(n) > j_0$. For the first case we have:

$$f_n(t) = \ln|x - \beta(t_{n,j(n)})| = f(t_{n,j(n)}),$$

and, for the second one,

$$f_n(t) = \ln|x - \beta(t_{n,j(n)+1})| = f(t_{n,j(n)+1}).$$

As f is continuous at t, we conclude that $f_n(t) \to f(t)$.

To see that $\mu(T_\delta) = \mu([t_0-\delta, t_0+\delta])$, let us define $A \subset (t_0-\delta, t_0+\delta)$ as the set of values t for which $\varphi(t)$ is locally constant. Clearly, A is an open set and, hence, there exist disjoint open intervals I_m, $m \in \mathbb{N}$, such that $A = \cup_m I_m$ and φ is constant on each I_m. Then, as $\mu(I_m) = \mu(\overline{I}_m) = 0$ (this follows from the continuity of φ on $[t_0-\delta, t_0+\delta]$), we have that $\mu(\cup_m \overline{I}_m) = 0$. Let us define the set B as

$$B = (t_0-\delta, t_0+\delta) \setminus \left[\left(\bigcup_{n,j} \{t_{n,j}\} \right) \bigcup \left(\bigcup_m \overline{I}_m \right) \bigcup \{t_0\} \right]$$

As $\mu(B) = \mu([t_0-\delta, t_0+\delta])$, we will finish the proof by showing that $B \subset T_\delta$. Hence, let us select a value $t \in B$. We will proceed by steps.

i) φ is strictly increasing at t: if it is not, it means that there exists $t_1 \neq t$ such that $\varphi(t_1) = \varphi(t)$. This implies that the closed interval with endpoints t_1 and t has zero measure, so it is contained in one of the $\overline{I}_m$ defined above. Therefore, $t \notin B$.

ii) There exists n_t such that, for each $n \geq n_t$, there is a value $j(n)$ satisfying $t_{n,j(n)} < t < t_{n,j(n)+1}$: Note that, for all t_1 and t_2 in $[t_0-\delta, t_0+\delta]$ such that $t_1 < t < t_2$, we have that

$\mu([t_1,t])$ and $\mu([t,t_2])$ are both strictly positive. This implies that both intervals $[t_1,t]$ and $[t,t_2]$ must contain, from some $n=n_t$ on, several values $t_{n,j}$.

iii) $\lim_{n\to\infty} t_{n,j(n)} = \lim_{n\to\infty} t_{n,j(n)+1} = t$: Apply the last paragraph for a sequence of values t_1 and t_2 converging to t.

Finally, note that the proof for the case $t_0 \in \{0,1\}$ can be obtained with minor modifications of the previous proof. ∎

Following with the proof of the proposition, we have that:

$$\int_{t_0-\delta}^{t_0+\delta} f_n \, d\mu = \sum_{j=p(n)}^{j_0-1} f(t_{n,j})\mu([t_{n,j},t_{n,j+1}]) + \sum_{j=j_0+2}^{q(n)} f(t_{n,j})\mu([t_{n,j-1},t_{n,j}])$$
$$= \sum_{j=p(n)}^{j_0-1} \frac{1}{k(n)} f(t_{n,j}) + \sum_{j=j_0+2}^{q(n)} \frac{1}{k(n)} f(t_{n,j}) + \sum_{j=p(n)}^{j_0-1} \frac{a_{n,j}}{k(n)} f(t_{n,j}) + \sum_{j=j_0+2}^{q(n)} \frac{a_{n,j-1}}{k(n)} f(t_{n,j}),$$

where the values $a_{n,j}$ are defined in (2). Let us now define

$$w_n^{(\delta)}(x) = \prod_{j=p(n)}^{q(n)} (x - x_{n,j}),$$

that is, the "part" of w_n corresponding to $[t_0-\delta, t_0+\delta]$. Therefore,

$$\frac{\ln |w_n^{(\delta)}(x)|}{k(n)} = S_1 + S_2 - S_3,$$

where:

$$S_1 = \int_{t_0-\delta}^{t_0+\delta} f_n \, d\mu, \qquad S_2 = \frac{1}{k(n)} (f(t_{n,j_0}) + f(t_{n,j_0+1})),$$

and

$$S_3 = \sum_{j=p(n)}^{j_0-1} \frac{a_{n,j}}{k(n)} f(t_{n,j}) + \sum_{j=j_0+2}^{q(n)} \frac{a_{n,j-1}}{k(n)} f(t_{n,j}).$$

Now we will study the limit of S_1, S_2 and S_3 when n tends to infinity. By the hypotheses of the proposition, $|\beta(t_0) - \beta(t_{n,j})| \geq c_0/k(n)^\tau$ for $n \geq 0$ and $0 \leq j \leq k(n)$. Then, as f is negative on $[t_0-\delta, t_0+\delta] \cap [0,1]$, we have that

$$\frac{\ln c_0 - \tau \ln k(n)}{k(n)} \leq \frac{f(t_{n,j_0})}{k(n)} \leq 0,$$

and

$$\frac{\ln c_0 - \tau \ln k(n)}{k(n)} \leq \frac{f(t_{n,j_0+1})}{k(n)} \leq 0.$$

Hence,

$$\lim_{n\to\infty} S_2 = 0.$$

Using Lemma 4.1 and Lebesgue's Dominated Convergence Theorem we obtain

$$\lim_{n\to\infty} S_1 = \int_{t_0-\delta}^{t_0+\delta} f\, d\mu.$$

Next, we define

$$s_{n,j} = \sum_{\ell=p(n)}^{j} \frac{a_{n,\ell}}{k(n)}, \quad p(n) \le j \le j_0 - 1, \qquad \hat{s}_{n,j} = \sum_{\ell=j}^{q(n)-1} \frac{a_{n,\ell}}{k(n)}, \quad j_0 + 1 \le j \le q(n) - 1.$$

Now, using Abel's summation formula, we can write

$$\begin{aligned}
\sum_{j=p(n)}^{j_0-1} \frac{a_{n,j}}{k(n)} f(t_{n,j}) &= \sum_{j=p(n)}^{j_0-2} s_{n,j}(f(t_{n,j}) - f(t_{n,j+1})) + s_{n,j_0-1} f(t_{n,j_0-1}), \\
\sum_{j=j_0+2}^{q(n)} \frac{a_{n,j-1}}{k(n)} f(t_{n,j}) &= \sum_{j=j_0+2}^{q(n)-1} \hat{s}_{n,j}(f(t_{n,j+1}) - f(t_{n,j})) + \hat{s}_{n,j_0+1} f(t_{n,j_0+2}).
\end{aligned}$$

Then, using that f is decreasing on $[t_0-\delta, t_0)$, increasing on $(t_0, t_0+\delta]$, and negative on both intervals, we have

$$\begin{aligned}
|S_3| \le\ & \sum_{j=p(n)}^{j_0-2} |s_{n,j}|(f(t_{n,j}) - f(t_{n,j+1})) + |s_{n,j_0-1}||f(t_{n,j_0-1})| \\
& + \sum_{j=j_0+2}^{q(n)-1} |\hat{s}_{n,j}|(f(t_{n,j+1}) - f(t_{n,j})) + |\hat{s}_{n,j_0+1}||f(t_{n,j_0+2})|.
\end{aligned}$$

As the nodes $t_{n,j}$ are regular at t_0 (see Definition 2.2), there exist values $b_n \to 0$ such that

$$|s_{n,j}| \le \frac{b_n}{\ln(k(n))}, \quad p(n) \le j \le j_0 - 1, \qquad |\hat{s}_{n,j}| \le \frac{b_n}{\ln(k(n))}, \quad j_0 + 1 \le j \le q(n) - 1.$$

Therefore, we can bound $|S_3|$ as

$$\begin{aligned}
|S_3| &\le \frac{b_n}{\ln(k(n))} \left[f(t_{n,p(n)}) - 2f(t_{n,j_0-1}) + f(t_{n,q(n)}) - 2f(t_{n,j_0+2}) \right] \\
&\le 6 \frac{b_n}{\ln(k(n))} \left| \ln(c_0) - \tau \ln(k(n)) \right|,
\end{aligned}$$

Hence,

$$\lim_{n\to\infty} S_3 = 0.$$

With all this we have proved that, if $x = \beta(t_0)$,

$$\lim_{n\to\infty} \frac{\ln |w_n^{(\delta)}(x)|}{k(n)+1} = \int_{t_0-\delta}^{t_0+\delta} f\, d\mu.$$

To finish the proof, we will show a similar identity on the set $D_\delta = [0,1] \setminus [t_0 - \delta, t_0 + \delta]$. Note that we can assume that δ is small enough such that $\mu(D_\delta) > 0$ (this follows from the continuity of φ at t_0). For each n, let us define J_n as the set of indices j such that $t_{n,j} \in D_\delta$ (in other words, $J_n = \{0, \ldots, p(n) - 1, q(n) + 1, \ldots, k(n)\}$). Then, it is not difficult to check that

$$\lim_{n\to\infty} \frac{\# J_n}{k(n) + 1} = \mu(D_\delta). \tag{11}$$

Let us now consider the subscheme $\{x_{n,j}\}_{n\geq 0, j\in J_n}$, and let $\hat{\varphi}$ and $\hat{\mu}$ be the corresponding distribution and Borel measure. We note that

$$\begin{aligned} \hat{\mu}(I) &= \frac{1}{\mu(D_\delta)}\mu(I) \quad \text{if} \quad I \subset D_\delta, \\ \hat{\mu}(I) &= \quad 0 \qquad\quad\; \text{if} \quad I \subset [t_0 - \delta, t_0 + \delta]. \end{aligned} \tag{12}$$

As t_0 does not belong to the closure of the subscheme, we can apply Proposition 4.2 to obtain

$$\lim_{n\to\infty} \sum_{j\in J_n} \frac{f(t_{n,j})}{\# J_n} = \int_0^1 f(t)\, d\hat{\mu}.$$

Now, taking into account equations (11) and (12), we can derive

$$\lim_{n\to\infty} \sum_{j\in J_n} \frac{f(t_{n,j})}{k(n) + 1} = \int_{D_\delta} f\, d\mu.$$

This finishes the proof of the Proposition. ■

4.3 Estimates on measures and dimensions

Let $\{x_{n,j}\}_{n\geq 0, 0\leq j\leq n}$ be a triangular scheme on a piecewise C^1 curve $\beta : [0,1] \to \mathbb{C}$. In what follows, $c > 0$ and $\tau > 2$ will denote real numbers, and $n_0 \geq 1$ will be a natural number. We define the following sets:

$$D(c,\tau,n_0) = \left\{ t \in [0,1] \;/\; \forall n \geq n_0,\ \forall j \in \{0,\ldots,n\} \;:\; |\beta(t) - \beta(t_{n,j})| \geq \frac{c}{n^\tau} \right\},$$

and

$$D = \bigcup_{c>0} \bigcup_{\tau>2} \bigcup_{n_0\geq 1} D(c,\tau,n_0).$$

Proposition 4.5 *The above-defined set $D \subset [0,1]$ has Lebesgue measure equal to 1. The set $[0,1] \setminus D$ has Hausdorff dimension equal to 0.*

Proof: As $D(c,\tau,n_0) \subset [0,1]$, but the condition used to define this set is stated on $\beta([0,1]) \subset \mathbb{C}$, we need to "translate" it to the parameter space $[0,1]$. Hence, we define the sets

$$B(c,\tau,n_0) = \left\{ t \in [0,1] \;/\; \forall n \geq n_0,\ \forall j \in \{0,\ldots,n\} \;:\; |t - t_{n,j}| \geq \frac{c}{n^\tau} \right\},$$

and

$$B=\bigcup_{c>0}\bigcup_{\tau>2}\bigcup_{n_0\geq 1}B(c,\tau,n_0).$$

We choose $0=d_0<d_1<\cdots<d_{s-1}<d_s=1$ such that, for any interval $[d_r,d_{r+1}]$, we have

a) either $\mathrm{Re}\,\dfrac{d}{dt}\beta(t)\neq 0$ or $\mathrm{Im}\,\dfrac{d}{dt}\beta(t)\neq 0$ for all $t\in[d_r,d_{r+1}]$;

b) β is C^1 on $[d_r,d_{r+1}]$.

We will use a couple of lemmas.

Lemma 4.2 *$B\setminus\{d_0,\ldots,d_s\}\subset D$.*

Proof: We define $I_r=(d_r,d_{r+1})$. Given $t_0\in B\setminus\{d_0,\ldots,d_s\}$ there exists an index r such that $t_0\in I_r$. We will focus on the case $\mathrm{Re}\,\frac{d}{dt}\beta(t)\neq 0$ for all $t\in\overline{I_r}$ (the case $\mathrm{Im}\,\frac{d}{dt}\beta(t)\neq 0$ is similar). Then there exist $n_1\geq 1$, $c_1>0$ and $\tau>2$ such that, for all $n\geq n_1$ and all $0\leq j\leq n$, we have

$$|t-t_{n,j}|\geq\frac{c_1}{n^\tau},$$

As $\inf_{t\in I_k}|\mathrm{Re}\,\dfrac{d}{dt}\beta(t)|=c_2>0$ (the parametrization β is regular), by the mean value theorem we obtain

$$|\beta(t_0)-\beta(t_{n,j})|\geq|\mathrm{Re}\,\beta(t_0)-\mathrm{Re}\,\beta(t_{n,j})|\geq c_2|t_0-t_{n,j}|\geq\frac{c_1c_2}{n^\tau},$$

for all $t_{n,j}\in I_r$. On the other hand, there exists a constant $c_3>0$ such that for all $t\notin I_r$, $|\beta(t_0)-\beta(t)|>c_3$. Then we can select n_0 such that $\frac{c_1c_2}{n^\tau}<c_3$, for all $n\geq n_0$. If we define $c_0=c_1c_2$, we have

$$|\beta(t_0)-\beta(t_{n,j})|\geq\frac{c_0}{n^\tau},$$

for all $n\geq n_0$ and $0\leq j\leq n$. This finishes the proof of the lemma. ■

Lemma 4.3 *The set $[0,1]\setminus B$ has Hausdorff dimension equal to 0. The set $B\subset[0,1]$ has Lebesgue measure equal to 1.*

Proof: Let us define the sets

$$\begin{aligned}
I_{n_0,\tau,c,n,j} &= \left\{t\in[0,1]\,/\,|t-t_{n,j}|<\frac{c}{n^\tau}\right\},\\
I_{n_0,\tau,c} &= \bigcup_{n\geq n_0}\bigcup_{j=0}^{n}I_{n_0,\tau,c,n,j},\\
B_0 &= \bigcap_{c>0}\bigcap_{\tau>2}\bigcap_{n_0\geq 1}I_{n_0,\tau,c}.
\end{aligned}$$

Note that, with these definitions, $B_0=[0,1]\setminus B$.

We recall ([Fal90]) that the s-dimensional Hausdorff measure of B_0, $\mathcal{H}^s(B_0)$, is defined as

$$\mathcal{H}^s(B_0) = \lim_{\delta\to 0} \mathcal{H}^s_\delta(B_0),$$

being

$$\mathcal{H}^s_\delta(B_0) = \inf\left\{ \sum_i \operatorname{diam}(U_i)^s \text{ such that } \{U_i\}_i \text{ is a } \delta\text{-cover of } B_0 \right\},$$

where a δ-cover is a cover by a finite or countable number of sets of diameter not bigger than δ. The Hausdorff dimension of B_0, $\dim_{\mathrm{H}}(B_0)$, is then defined as

$$\dim_{\mathrm{H}}(B_0) = \inf\{s \geq 0 : \mathcal{H}^s(F) = 0\} = \sup\{s \geq 0 : \mathcal{H}^s(F) = \infty\}.$$

The idea is to show that the Hausdorff measure of B_0, is zero for all $s > 0$. This will prove that the Hausdorff dimension of B_0 is 0.

Let us select $s \in (0,1)$. Note that $\{I_{n_0,\tau,c,n,j}\}_{n\geq n_0, 0\leq j\leq n}$ is a cover of B_0, for all $n_0 \geq 1$, $\tau > 2$ and $c > 0$. Now we select a value $\delta \in (0,1)$. As

$$\operatorname{diam}(I_{n_0,\tau,c,n,j}) = \frac{2c}{n^\tau},$$

if we take $c = \delta/2$ we have that $\{I_{n_0,\tau,c,n,j}\}_{n\geq n_0, 0\leq j\leq n}$ is a δ-cover of B_0 for all $n_0 \geq 1$ and $\tau > 2$. Hence,

$$\mathcal{H}^s_\delta(B_0) \leq \sum_{n\geq n_0} \sum_{j=0}^{n} \operatorname{diam}(I_{n_0,\tau,c,n,j})^s = \sum_{n\geq n_0} (n+1)\frac{\delta^s}{n^{s\tau}}.$$

Now, selecting $\tau = 3/s$ (note that $\tau > 2$) we have

$$\mathcal{H}^s_\delta(B_0) \leq \delta^s \sum_{n\geq n_0} \frac{n+1}{n^3} \leq M\delta^s,$$

where $M > 0$ does not depend on n_0. This implies that $\mathcal{H}^s(B_0) = 0$ (in fact, this also shows that $\mathcal{H}^s_\delta(B_0) = 0$, see [Fal90]) and, hence, that the Hausdorff dimension of B_0 is zero. In particular, this implies that the Lebesgue measure of B_0 is also zero and, therefore, that B has total measure on $[0,1]$. ■

The proof of the proposition is now an immediate consequence of these two lemmas. ■

Remark 4.2 *Note that, by Proposition 4.4, if $t_0 \in D$ and the scheme has regular nodes at $\beta(t_0)$ then*

$$\lim_{n\to\infty} \frac{\ln|w_n(\beta(t_0))|}{n+1} = \int_0^1 \ln|\beta(t_0) - \beta(t)|\, d\mu.$$

4.4 Proof of Theorem 3.1

We will proceed item by item.

1.- By Proposition 4.1, to have convergence of the polynomial interpolation in a point $x \in K$ it is enough that

$$\limsup_{n\to\infty} \frac{\ln|w_n(x)|}{n+1} < V(\alpha_0).$$

Moreover, by Proposition 4.2, this inequality is true if $V(x) < V(\alpha_0)$. It is clear that, in the case $x \in \overset{\circ}{K}$, this convergence is locally uniform.

2.- This is an immediate consequence of Propositions 4.1 and 4.2.

3.- We select a value y in the relative interior of Γ_{out} in $\beta([0,1])$, and let A_y be a closed interval such that: a) A_y is a neighbourhood of y; and b) A_y is contained in the relative interior of Γ_{out}. Let us define I_{n_0} as

$$I_{n_0} = \bigcup_{n\geq n_0} \bigcup_{j=0}^{n} \{x \in A_y \,/\, |x - x_{n,j}| < \psi(n)\}\,.$$

where $\psi(n)$ is a strictly positive function –to be selected later on– that goes to zero when n goes to infinity. Moreover, let us define

$$I = \bigcap_{n_0>0} I_{n_0}.$$

As I_{n_0} are open and dense sets (with respect to the induced topology in the closed set A_y), and $I_{n_0+1} \subset I_{n_0}$, we have that I is an uncountable dense subset of A_y.

Now, we select $x \in I$ and we will show that there exists a partial of $\{P_n(x)\}_n$ convergent to $f(x)$. From Proposition 4.1, it is enough to show that, for any $x \in I \subset \Gamma$, the sequence

$$r_n = \sum_{k=1}^{m} \frac{w_n(x)}{w_n(\alpha_k)} \frac{\mathrm{Res}\,(f,\alpha_k)}{x-\alpha_k}$$

has a subsequence convergent to zero. We define positive numbers $M_{1,2,3}$ such that

$$\max_{1\leq k\leq m} \left|\frac{\mathrm{Res}(f,\alpha_k)}{x-\alpha_k}\right| \leq M_1, \qquad \sup_{n,j} |x - x_{n,j}| \leq M_2, \qquad \inf_{n,j,k} |\alpha_k - x_{n,j}| \geq M_3 > 0.$$

Let $j_0 = j_0(n)$ be such that $|x - x_{n,j_0}| = \min_{0\leq j\leq n} |x - x_{n,j}|$. As $x \in I$ we have that, for all $n_0 > 0$, there exist $n \geq n_0$ such that $|x - x_{n,j_0}| < \psi(n)$. Then:

$$|r_n| \leq m M_1 M_2^n M_3^{n+1} |x - x_{n,j_0}| \leq M_4 M_5^n \psi(n),$$

where $M_4 = m M_1 M_3$ and $M_5 = M_2 M_3$. Now, let us select the function ψ in such a way that $M_5^n \psi(n)$ goes to zero when n goes to infinity (for instance, it is enough to take $\psi(n) = 1/n!$). So far, we have proved that for all n_0, there exist values $n \geq n_0$ such that $|r_n| \leq M_4 M_5^n \psi(n)$. Hence, there exists a subsequence of $\{r_n\}_n$ that is convergent to zero.

Now, as $I \subset A_y \subset \Gamma_{out}$, we have that $I \subset \Gamma_{out}^{sub}$. This implies that Γ_{out}^{sub} is uncountable, and that $y \in \overline{\Gamma_{out}^{sub}}$. This finishes the proof of this item.

4.- As the triangular scheme $\{x_{n,j}\}_{0\leq j\leq n, n\geq 0}$ has regular nodes on the set Γ_{out}, we can apply Propositions 4.4 and 4.5 (see also Remark 4.2) to derive the existence of a total Lebesgue measure set $D \subset [0,1]$ such that, if $x \in \beta(D)$,

$$\lim_{n\to\infty} \frac{\ln|w_n(x)|}{n+1} = \int_0^1 \ln|x-\beta(t)|\,d\mu.$$

Finally, as $V(x) > V(\alpha_0)$, the divergence (to infinity) of the sequence $R_n(x)$ follows.

As the Hausdorff dimension of $\beta^{-1}(\Gamma_{out}^{sub})$ is zero (see Proposition 4.5), the bi-Lipschitz character of β implies that the Hausdorff dimension of Γ_{out}^{sub} is also zero ([Fal90]). ∎

4.5 Proof of Theorem 3.2

It is clear that $\{t_{n,j}\}_{0\leq j\leq n, n\geq 1}$ is uniformly distributed with regular nodes. Next lemma shows an interesting property of this triangular scheme.

Lemma 4.4 *Let $t \in (0,1)$, and let us define the set $N_t^0 \subset \mathbb{N}$ as*

$$N_t^0 = \left\{ n \in \mathbb{N} \text{ such that } |t_{n,j} - t| > \frac{c}{n^\tau}, \ \ \forall j \in \{1,\ldots,n-1\} \right\}$$

Then, if $c \leq \frac{1}{2}$ and $\tau \geq 3$, N_t has infinite elements.

Proof: For simplicity, we select the values $c = \frac{1}{2}$ and $\tau = 3$ (note that if the result holds for these values, it then holds for any $c \leq \frac{1}{2}$ and $\tau \geq 3$).

The statement of the lemma follows immediately from the fact that, if $n \notin N_t^0$, then $n+1 \in N_t^0$. Hence, we assume that $n \notin N_t^0$ so there exists a natural number ℓ $(0 < \ell < n)$ such that

$$|t_{n,\ell} - t| \leq \frac{c}{n^\tau}, \tag{13}$$

Then, for any $0 < j < n+1$,

$$\left|\frac{j}{n+1} - t\right| \geq \left|\left|\frac{j}{n+1} - \frac{\ell}{n}\right| - \left|\frac{\ell}{n} - t\right|\right|. \tag{14}$$

Moreover,

$$\left|\frac{j}{n+1} - \frac{\ell}{n}\right| = \frac{1}{n(n+1)}\,|jn - \ell(n+1)|,$$

and using that, for all n, ℓ and j such that $0 < \ell < n$ and $0 < j < n+1$ we have $jn \neq \ell(n+1)$, it follows that

$$\left|\frac{j}{n+1} - \frac{\ell}{n}\right| \geq \frac{1}{n(n+1)} \geq \frac{c}{n^2}.$$

Now, putting this lower bound in (14) and using (13), we obtain

$$\left|\frac{j}{n+1} - t\right| \geq \frac{c}{n^2} - \frac{c}{n^3} \geq \frac{c}{(n+1)^3}, \text{ if } n \geq 2.$$

This shows that $n+1 \in N_t^0$. ■

Remark 4.3 *The values for c and τ used in this lemma are not optimal.*

For technical reasons, in the last lemma we have excluded the values $j=0$ and $j=n$ in the definition of the set N_t^0. So, we define

$$N_t = \left\{ n \in \mathbb{N} \text{ such that } |t_{n,j} - t| > \frac{c}{n^\tau}, \ \ \forall j \in \{0, \ldots, n\} \right\}.$$

Corollary 4.1 *$N_t \subset N_t^0$ is an infinite set.*

Proof: If $m \in N_t^0 \setminus N_t$, we must have either

$$|t| \leq \frac{c}{m^\tau} \quad \text{ or } \quad |1-t| \leq \frac{c}{m^\tau},$$

and it is clear that there is only a finite set of values m satisfying this condition. Therefore, $N_t^0 \setminus N_t$ must be finite and then N_t must be infinite. ■

Remark 4.4 *These results are valid, in particular, for any $t \in \mathbb{Q} \cap (0,1) = \bigcup_{n\geq 0} \bigcup_{j=1}^{n-1} \{t_{n,j}\}$.*

Let us now start with the proof of the theorem. As item 1 is a well known result (see, for instance, [War76]) and item 3 follows from Theorem 3.1 (in this case, Γ_{out} is relatively open in $[0,1]$), we only focus on the proof of item 2.

From Proposition 4.4 and Corollary 4.1, it follows that, for all $t \in (0,1)$,

$$\lim_{\substack{n \to \infty \\ n \in N_t}} \frac{\ln |w_n(t)|}{n+1} = \int_0^1 \ln |t-s| \, ds.$$

We define now $K = \overline{C}_{max}$. To apply Proposition 4.3, we select V_1 such that $C_{max} \subset C_{V_1}$ and f is still defined on a neighbourhood of the closure of C_{V_1}. It turns out that K is a proper interpolation set with respect to f and C_{V_1}.

Hence, if $R_n(t)$ denotes the interpolating error at t, and $t \in \Gamma_{out}$, using Proposition 4.1 and that $V(t) > V(i)$, we have

$$\lim_{\substack{n \to \infty \\ n \in N_t}} R_n(t) = \infty.$$

This is, the sequence of interpolating polynomials $\{P_n(t)\}_{n\in\mathbb{N}}$ has a subsequence that diverges to infinity.

Remark 4.5 *As a side point, note that, if $t \in \mathbb{Q} \cap [0,1]$, there exists a subsequence of $\{P_n(t)\}_{n\in\mathbb{N}}$ that converges to the value $f(t)$. This subsequence can be easily obtained as follows: if $t = \frac{k}{m}$, then we take the values n that are multiple of m. In this way, the point t is always contained in the interpolating nodes. This shows that the sequence $\{P_n(t)\}_{n\in\mathbb{N}}$ can also have convergent subsequences.*

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A Technical lemmas

Here we have included some technical lemmas used along the proofs. We keep the same notation as in the paper, that is, $\beta : [0,1] \to \mathbb{C}$ is a piecewise C^1 curve, $\{x_{n,j}\}_{0\le j\le n, n\ge 0} \subset \beta([0,1])$ is a triangular scheme with closure Γ, distribution φ, and associated Borel measure μ. V will be the corresponding logarithmic potential (1).

Lemma A.1 *Let V_1 be a real number, and let $A \neq \emptyset$ be one of the connected components of $\{z \in \mathbb{C} \ : \ V(z) < V_1\}$. Then, $\mu(\beta^{-1}(A \cap \Gamma)) > 0$.*

Proof: Assume that $\mu(\beta^{-1}(A \cap \Gamma)) = 0$. If $A \cap \Gamma = \emptyset$ then, as there are no singularities of (1) inside A, V is harmonic on A. As V takes the constant value V_1 on the boundary of the bounded set A, by the maximum principle, V must take the constant value V_1 on A. As the points $x \in A$ satisfy $V(z) < V_1$ this forces $A = \emptyset$, which is absurd.

Suppose now that there exists $x \in A \cap \Gamma$. Then, there exists a connected open neighbourhood $B(x)$ of x such that $B(x) \subset A$ and that there exists t_1 and t_2 inside $[0,1]$ ($t_1 < t_2$) such that $\beta([t_1,t_2]) \subset B(x) \cap \Gamma$. As $\mu([t_1,t_2]) = 0$, (1) can be written as

$$V(x) = \int_0^{t_1} \ln|x-\beta(t)|\,d\mu + \int_{t_2}^1 \ln|x-\beta(t)|\,d\mu.$$

Therefore, V is harmonic on $B(x)$. As in the previous case, it can be seen that $B(x) = \emptyset$, which is again absurd. Hence, $\mu(\beta^{-1}(A \cap \Gamma)) > 0$. ■

Lemma A.2 *Let V_0 be a real number such that Γ is contained in a connected component of $C_{V_0} = \{z \in \mathbb{C} \ : \ V(z) < V_0\}$. Then,*

a) the set C_{V_0} is simply connected,

b) for all $V_1 > V_0$, the set C_{V_1} is simply connected,

c) for all $V_1 > V_0$, there exists a piecewise C^1, closed and simple curve $\sigma : [0,1] \to \mathbb{C}$ such that C_{V_0} is contained in the interior of σ and $V_0 < V(\sigma(t)) < V_1$, $\forall\, t \in [0,1]$.

Proof: By Lemma A.1, any connected component of C_{V_0} intersects Γ. As Γ is contained in one of these connected components, there is a unique connected component for C_{V_0}. This proves that C_{V_0} is connected. Let us select a value $V_1 > V_0$. As $C_{V_0} \subset C_{V_1}$ and Γ is contained in a connected component of C_{V_0}, we have that Γ is contained in a connected component of C_{V_1}. Then, as C_{V_0} is connected, C_{V_1} is connected.

For each (bounded) set C_{V_j}, $j = 1, 2$, we denote by A_j the (only) unbounded connected component of $\mathbb{C} \setminus C_{V_j}$, and we define $B_j = \mathbb{C} \setminus A_j$. Therefore, B_j, $j = 1, 2$ are open subsets of $\mathbb{C}$ such that

i) $\Gamma \subset B_j$, $j = 1, 2$.

ii) $C_{V_j} \subset B_j$, $j = 1, 2$.

iii) $\partial B_j \subset \partial C_{V_j}$, $j = 1, 2$.

Using Theorem 2.2 (p. 202) of [Con78] we obtain that B_j is simply connected. Using the Maximum Principle for subharmonic functions ([Ran95]), we have that the maximum values of $V(z)$, $z \in B_j$, are attained on the boundary of B_j. Now, using iii) above and that $V(z)$ is not constant (this follows immediately from its asymptotic behaviour when $z \to \infty$, see [Ran95]), we obtain that $V(z) < V_j$ for any $z \in B_j$. This shows that $B_j \subset C_{V_j}$ and, from i), that $B_j = C_{V_j}$. Hence, a) and b) are proved.

From the Riemann Mapping Theorem, there exists a one-to-one analytic function $h : C_{V_1} \to \mathbb{C}$ such that $h(C_{V_1}) = D$, where D denotes the open unit disc of $\mathbb{C}$. From the definition of the sets C_{V_j} and the continuity of $V(z)$ for $z \notin \Gamma$, we have that $\overline{C}_{V_0} \subset C_{V_1}$. Then, as $\overline{h(C_{V_0})} = h(\overline{C}_{V_0}) \subset h(C_{V_1}) = D$, there exists a circle $\alpha : [0, 1] \to D$ enclosing $\overline{h(C_{V_0})}$. Therefore, the curve $\sigma = h^{-1} \circ \alpha : [0, 1] \to C_{V_1}$ satisfies item c). ■

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