KAM Theory Without Action-Angle Coordinates

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Abstract

The classical KAM methods, strongly supported on the use of canonical transformations in the action-angle context, are not efficient to be applied to a wide range of systems in which the Hamiltonian is known (for instance) written in Cartesian coordinates.

In this communication we present some ideas to deal with KAM theory using "parameterizations" instead of "transformations" and "graphs", which we think is an efficient way to work with a more general class of Hamiltonian systems than the classical methods (in particular, for systems motivated by real world problems). With the present approach, we can extend several well-known results of KAM theory to these systems, even when the classical statements are difficult to be applied.

Introduction

One of the main problems of the dynamics [1] is the study of close-to-integrable Hamiltonian systems written in the so-called action-angle variables, that is:

$$H(I,\phi) = H_0(I) + \varepsilon H_1(I,\phi), \qquad I \in U \subset \mathbb{R}^n, \qquad \phi \in \mathbb{T}^n,$$

where ε is a small parameter, see [3]. For $\varepsilon = 0$ the dynamics of this system is quite simple: the actions I are constant along the trajectories (first integrals of the system), and so, the phase space is foliated by n-dimensional tori, invariant by the flow. On any of these tori, $I = I^0$, the motion is quasi-periodic, with vector of basic frequencies $\omega(I^0) = \nabla H_0(I^0)$.

The study of the persistence of these tori for $\varepsilon \neq 0$ is one of the main objectives of the so-called KAM theory (Kolmogorov-Arnol'd-Moser, see [3] for a survey). If we restrict to the real analytic context, what it is proved is that if certain generic non-degeneracy condition is fulfilled, then "the majority" of the invariant tori of the un-perturbed system still exist for $\varepsilon \neq 0$ small enough. They are slightly deformed, but the motion inside is also analytically conjugated to a linear flow on the torus. More concretely, we have to ask for

$$\det D^2 H_0(I) \neq 0, \qquad I \in U,$$

[1] According to Poincaré, [15], this is the main problem of the dynamics.

which means non-degeneracy of the frequency map $\omega(I)$ [2]. Then, given any frequency vector $\omega^0 = \nabla H_0(I^0)$, $I^0 \in U$, for which the following Diophantine condition holds (non-resonance condition on the frequency),

$$|\langle k, \omega^0 \rangle| \geq \frac{\gamma}{|k|_1^\tau}, \qquad k \in \mathbb{Z}^n \setminus \{0\}, \tag{1}$$

for certain $\gamma > 0$ and $\tau \geq n-1$, where $|k|_1 = |k_1| + \cdots + |k_n|$, what we have is that if $|\varepsilon|$ is small enough, there is a real analytic torus of the perturbed system carrying ω_0-quasi-periodic motion, and that it is $|\varepsilon|$ close to the torus $I = I^0$. These tori fill a region of the phase space which has Cantor-like structure (with empty interior) but nearly full measure: its complementary over $U \times \mathbb{T}^n$ has Lebesgue measure controlled by $\mathcal{O}(\sqrt{\varepsilon})$. See for instance [1, 2].

To prove this result (as well other results of KAM theory), the classical approach takes advantage of the following technicalities:

(*i*) **The action-angle coordinates:** It is an important simplification, since we can write the tori as graphs $I \equiv I(\phi)$: the angles on the torus are intrinsic variables of the system. This coordinates are very useful on solving the linear differential equations (homological equations) involved in the (iterative) KAM process.

(*ii*) **The perturbative setting:** In the classical approach, the Hamiltonian is assumed to be a perturbation of an integrable one, and so, we have the simple description for the dynamics of the un-perturbed system described before. Of course, we expect that everything will be a small perturbation of the un-perturbed system if ε is small.

(*iii*) **The Lagrangian character of the tori:** In the action-angle context, any torus of the form $I = I^0$ is Lagrangian[3]. We point out that an invariant torus of a Hamiltonian system with linear irrational flow is always Lagrangian, therefore we know a-priori that the limit torus of a KAM process will be Lagrangian. However, if the sequence of approximations it is also Lagrangian, we have several simplifications along the proof.

(*iv*) **The transformation theory:** This means to perform a sequence of (canonical) transformations to the original Hamiltonian, in order to obtain a limit (but equivalent) Hamiltonian, for which the wanted family of quasi-periodic motions takes place (of course, the final transformation is only defined on a Cantor set).

Some difficulties, however, can arise when one attempts to apply directly this classical approach to general Hamiltonian systems. For instance, in some mathematical models it could be hard to establish the existence of a particular invariant torus of a given system (Kolmogorov theorem [9, 10, 4]). A very interesting (and important !) case of such situation are mathematical models of physical problems (for instance in celestial mechanics models). Some standard difficulties are:

[2] We can consider more general non-degeneracy conditions, as the iso-energetic non-degeneracy, see [12], or Rüssmann non degeneracy conditions, see [18].

[3] An n-dimensional manifold of $\mathbb{R}^{2n}$ is Lagrangian if the canonical 2-form of $\mathbb{R}^{2n}$ vanishes on its tangent bundle.

(a) **No action-angle coordinates:** For the majority of systems, there are no natural action-angle variables available, allowing to describe the invariant tori as graphs of angles that are intrinsic variables of the system.

(b) **Not perturbative setting:** Given a particular system, modeling a real problem say, a small parameter ε controlling the proximity between the given system and an integrable one is not always available. Therefore, even if the given system is "quasi-integrable" in some sense, it could be difficult, or computationally impossible, to introduce suitable action-angle coordinates in a constructive way. Moreover, it could also happen that for the actual size of the parameter ε the system written in action-angle variables the perturbation is not "small enough", hence the classical invariant KAM approach does not guarantee the existence of a certain invariant torus, even in the case it exists [4].

(c) **No Lagrangian tori:** In many cases it is possible to numerically compute an approximately invariant torus for a certain frequency vector which is also known only approximately. More concretely, it is possible to compute, with the help of a computer, a $(2n)$-dimensional trigonometric polynomial (in n variables) parameterizing the approximatelly invariant torus. We can complete the decimal expression of the frequencies to obtain a Diophantine vector (it can always be done). Moreover, the initial approximately invariant torus will be not lagrangian, in general. Even in the case it were, the refined approximations would not preserve this geometrical property.

(d) **No transformation theory:** As we only know the parametrization of the approximately invariant torus and we do not know its expression as function of intrinsic variables of the system, then it is not possible to use transformation theory: in terms of a Cartesian system of coordinates, if we perform a change of variables, we do not know how to control its effect around and object for which we only know its parameterization.

The objective of this communication is to present some ideas that allow to deal with KAM theory, without using the technicalities explained in (i), (ii), (iii) and (iv). that we have called "the classical KAM approach". Therefore, the methods proposed here are more suitable for a non-action-angle context. In what follows we outline the basic ideas of the proposed approach, and we will mention some possible generalizations. Full results will be available soon in [6].

The adapted Newton method

Let us consider a real analytic Hamiltonian $H(\zeta)$ defined for $\zeta \in U \subset \mathbb{R}^{2n}$. We will not assume that H is neither written in action-angle coordinates, nor close to integrable. We suppose known

$$\mathcal{T} : \theta \in \mathbb{T}^n \to \mathcal{T}(\theta) \in U,$$

[4] There are examples of systems that are small perturbation of integrable Hamiltonian systems, with quite simple first integrals, but with very complicate action-angle coordinates.

a real analytic parameterization of an n-dimensional torus (not necessarily Lagrangian), which is not invariant by the Hamiltonian flow of H, but "approximately invariant", with motion on the torus close to quasi-periodic, with vector of frequencies ω. We will call such an object a "quasi-torus" of the system. More concretely, we ask:

$$L_\omega \mathcal{T}(\theta) = J\nabla H(\mathcal{T}(\theta)) + R(\theta), \tag{2}$$

with $R(\theta)$ small, where L_ω is the partial derivative in the ω-direction and J is the matrix of the canonical 2-form of $\mathbb{R}^{2n}$:

$$J = \begin{pmatrix} 0_n & Id_n \\ -Id_n & 0_n \end{pmatrix}.$$

We want to compute a true ω-torus of the system, close to $\mathcal{T}$, as a limit of a sequence of approximate tori constructed by means of a Newton-like method.

If we apply the standard Newton method to $\mathcal{T}$ in order to compute a new approximation $\mathcal{T}^1 = \mathcal{T} + \Delta\mathcal{T}$, we have to solve the linear equation:

$$L_\omega \Delta\mathcal{T}(\theta) = JD^2H(\mathcal{T}(\theta))\Delta\mathcal{T}(\theta) - R(\theta). \tag{3}$$

Unfortunately, it has been impossible for us to solve this equation completely, but we are able to solve this equation except by an error which has quadratic size with respect to the size of $R(\theta)$. This suffices for our purposes. The key point to do this, and in fact, the key point for the whole process, is the following result:

Lemma 1 *With the previous notations, we define from the quasi-torus (2) the anti-symmetric matrix $\mathcal{S} = \mathcal{T}_\theta^\top J\mathcal{T}_\theta$, where $\mathcal{T}_\theta$ is the differential matrix of $\mathcal{T}$ with respect to θ. Then, if we assume that the vector $\omega \in \mathbb{R}^n$ verifies a Diophantine condition like (1), we have:*

$$\|\mathcal{S}\|_{\rho-\delta} \leq \frac{c}{\gamma\delta^{\tau+1}}\|R\|_\rho,$$

where $\|\cdot\|_\rho$ is the sup norm of an analytic function defined in the complex strip $\mathcal{D}(\rho) = \{\theta \in \mathbb{T}^n : |\mathrm{Im}(\theta_j)| \leq \rho,\ j = 1,\dots,n\}$, and c is a constant only depending on n, τ, the norm of $\mathcal{T}_\theta$, and the norm of DH on the torus.

Remark 1 *The meaning of this result can be explained as follows: if $\mathcal{S} = 0$, then $\mathcal{T}$ is a Lagrangian torus. What we have is that $\mathcal{T}$ can be not Lagrangian, but it is close to Lagrangian if $\|R\|$ is small. In particular, when $R = 0$ it is a Lagrangian torus.*

Remark 2 *In the Lagrangian case, the columns of the matrices $J\mathcal{T}_\theta\Omega^{-1}$ and $\mathcal{T}_\theta$ expand a symplectic basis of $\mathbb{R}^{2n}$ at any point of the torus, where $\Omega = \mathcal{T}_\theta^\top\mathcal{T}_\theta$. In the case of a quasi-torus, this still being a basis, but only quasi-symplectic.*

¿From Remark 2, we have that we can look for $\Delta\mathcal{T}$ as $\Delta\mathcal{T} = \mathcal{T}_\theta a + J\mathcal{T}_\theta\Omega^{-1}b$, where $a(\theta)$ and $b(\theta)$, n-dimensional and 2π-periodic in θ, are now the unknowns.

To give explicit equations for a and b, we observe that equation (3) can be written as $\mathcal{R}(\Delta\mathcal{T}) = -R$, where $\mathcal{R}$ is the linear differential operator $\mathcal{R} = L_\omega\cdot - JD^2H(\mathcal{T})\cdot$. Then, we have:

$$\mathcal{R}\mathcal{T}_\theta = R_1(\theta), \qquad \mathcal{R}(J\mathcal{T}_\theta\Omega^{-1}) = \mathcal{T}_\theta A + J\mathcal{T}_\theta\Omega^{-1}R_2,$$

where $R_1(\theta)$, $R_2(\theta)$ and $A(\theta)$ are 2π-periodic on θ, with the sizes of R_1 and R_2 of $\mathcal{O}(\|R\|)$.

Remark 3 *If $R_1 = R_2 = 0$ it implies that the torus $\mathcal{T}$ is reducible*[5]*, as it is easy to see that the dependence on θ of A can be removed.*

The equations for a an b, obtained skipping terms of quadratic size on $\|R\|$, are:

$$
\begin{aligned}
L_\omega b &= \mathcal{T}_\theta^\top JR, & (4)\\
L_\omega a &= -\Omega^{-1}\mathcal{T}_\theta^\top R - Ab. & (5)
\end{aligned}
$$

Equation (4) is always solvable if ω is Diophantine, with a free choice for $\langle b\rangle$ (the average with respect to θ). We can define $\langle b\rangle$ from (5), by asking the expression at the right-hand side to have zero mean value. This can be done if we suppose that $\det(\langle A\rangle) \neq 0$, which is how reads in this context the non-degeneracy of the frequency map for the chosen value of ω. After that, a can be obtained from (5), with free value for $\langle a\rangle$.

This process allows to define $\mathcal{T}^1$, and one can check that the new error, R^1, is $\mathcal{O}_2(\|R\|)$ (just shrinking a little bit the complex strip of definition of $\mathcal{T}$). Thus, if we apply iteratively this process, we can prove the following result:

Theorem 1 *Let us consider a real analytic Hamiltonian $H(\zeta)$ in $\mathbb{R}^{2n}$, defined in an open complex domain $\mathcal{W}$, as well a fixed vector $\omega \in \mathbb{R}^{2n}$ of Diophantine type (1). Let us also consider a real analytic n-dimensional torus of $\mathbb{R}^{2n}$, namely $\mathcal{T}(\theta)$, defined for $\theta \in \mathcal{D}(\rho)$ and 2π-periodic in θ. We assume that this torus is non-degenerate as n-dimensional manifold, in the sense that $\Omega(\theta) = \mathcal{T}_\theta^\top \mathcal{T}_\theta$ is non-singular for any $\theta \in \mathcal{D}(\rho)$, and moreover that $\mathcal{T}(\mathcal{D}(\rho)) \subset \mathcal{W}$.*

We introduce from H, ω and $\mathcal{T}$, the residue $R(\theta)$, defined from (2), when asking $\mathcal{T}$ to be an ω-torus of H, and the matrix

$$A = (\Omega)^{-1}((J\mathcal{T}_\theta)^\top D^2H(\mathcal{T})J\mathcal{T}_\theta - (\mathcal{T}_\theta)^\top D^2H(\mathcal{T})\mathcal{T}_\theta)(\Omega)^{-1},$$

for which we assume $\det\langle A\rangle \neq 0$.

Then, given a fixed $0 < \delta < \rho$, there are positive constants c_1 and c_2 depending only on n, τ, γ, $\|DH\|_\mathcal{W}$, $\|D^2H\|_\mathcal{W}$, $\|D^3H\|_\mathcal{W}$, $\|\mathcal{T}\|_\rho$, $\|\mathcal{T}_\theta\|_\rho$, $\|(\Omega)^{-1}\|_\rho$ and $|\langle A\rangle^{-1}|$, such that if $\|R\|_\rho \leq c_1$, then there exists $\mathcal{T}^*$, a true ω-torus of the system, defined on $\mathcal{D}(\rho-\delta)$, with $\|\mathcal{T}^* - \mathcal{T}\|_{\rho-\delta} \leq c_2\|R\|_\rho$.

Remark 4 *We can give explicit expressions for c_1 and c_2, and they are not extremely complicate: they can be computed for particular examples.*

Remark 5 *The Hamiltonian H has never been modified along the iterative process. So, no transformation theory has been used.*

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[5]In this context, reducibility refers to the reduction to constant coefficients of the variational equations around the torus, by means of a linear change of variables, depending in time in ω-quasi-periodic way (quasi-periodic Floquet theory, see [5]).

Other results and Generalizations

We can extend the previous methodology to other contexts related to KAM theory. Here, we display some of the most natural generalizations. Part of these results have been achieved, and other ones are work in progress.

1. **Lindstedt series around an invariant torus:** They can be used to approximate the close-to-integrable dynamics around a given invariant and Diophantine torus, as a substitute of normal forms, which require to use adapted coordinates (see [3] for Lindstedt series in the action-angle context). ¿From them, we can establish that the proximity to integrability of a general Hamiltonian around an invariant and Diophantine torus, behaves in exponentially small way as function of the distance to the torus.

2. **KAM tori are very "sticky":** Classical Nekhoroshev estimates can be proved for trajectories with initial conditions close to an invariant torus, proving that such trajectories remain close to the invariant torus for extremely big times. The usual techniques are based on normal form computations (see [13, 14]), but we can also obtain this result from estimates on Lindstedt series.

3. **Exponentially small estimates in the KAM theorem:** For any Diophantine frequency vector ω' close enough to ω and with Diophantine constant γ (see (1)) "not too small", there exists an invariant torus with frequency vector ω'. The measure of the holes in the phase space not filled by invariant tori, behaves in exponentially small way with respect to the size of $|\omega - \omega'|$. See [11, 8] for previous results in the classical setting.

4. **The finite differentiable case:** It refers to two contexts: **(a)** In the analytic case, to prove that the Cantor family of KAM tori is $\mathcal{C}^r$-smooth, for any r, in the sense of Whitney (see [17]). See [5] for related literature. **(b)** If $H \in \mathcal{C}^r$, for r big enough, then the results "also work" (with statements adapted to the context). See, for instance, [19, 16, 5].

5. **More general non-degeneracy conditions:** The standard non-degeneracy condition $\det(\langle A \rangle) \neq 0$ can be relaxed and replaced by suitable higher order non-degeneracies (see [5]).

6. **Lower-dimensional tori:** We also can generalize these ideas to study invariant tori which are not n-dimensional (maximal dimensional tori) but of dimension r, with $r < n$. Of course, the technical difficulties increase. (see [5, 8]).

7. **Other Generalizations:** The previous results can be generalized to Hamiltonian systems depending on time in quasi-periodic way (see [7]), Hamiltonian systems of $\mathbb{R}^{2n}$ related to a symplectic structure different from the canonical one, or more in general, to general $(2n)$-dimensional symplectic and analytic manifolds. However, these ideas also works (suitable adapted, this is a different context that requires a separate approach) for exact symplectic maps, with all the generalizations previously mentioned.

Concussions

In spite our new approach do not give rise to new result, it allows to simplify the proofs of some well-known results (the main reason of this fact is because we do not use transformation theory), and also allows to generalize these results to contexts that now are outside the range of applicability of the actual results.

A good test to prove the efficiency of these ideas could be if they allow to prove, for instance, existence of invariant tori in "real world problems" where we have numerical evidences of this fact, but we are unable to prove it rigorously using classical methods (for instant, by checking the hypotheses of our results using computer assisted proofs). We are also working in this line.

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