CONTRIBUTION TO THE STUDY OF INVARIANT MANIFOLDS AND THE SPLITTING OF SEPARATRICES OF PARABOLIC POINTS

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Als meus pares i germà i a l'Elena

Introducció

En general, quan hom comença a explorar qualsevol camp d'una ciència s'interessa per les situacions genèriques; és a dir, es centra en els comportaments que apareixen en la "majoria" dels casos que ens trobarem a la vida real.

Aquesta metodologia permet entendre el nou problema més fàcilment ja que els casos no genèrics (o degenerats) són menyspreats (al menys a priori) en un primer estudi. D'aquesta manera la casuística és més simple i la teoria general pot ésser desenvolupada amb més facilitat.

Malgrat que aquest és un bon procediment científic, la finalitat de la ciència és explicar la realitat de la forma més completa possible. Per això, quan el cas general està ja descrit (potser no en la seva totalitat, però sí bastant desenvolupat), hom pot estudiar els casos no genèrics: les excepcions. No hem d'oblidar que en la natura no tots els processos segueixen una regla general. Aquests casos excepcionals ens aporten sovint nous tipus de comportament. Per tant, podem apendre molt de les excepcions, tant a nivell intrínsec (situacions que difereixen del comportament qualitatiu general), com per les noves tècniques que es desenvolupen per entendre-les.

Dins de certs contextes, és genéric trobar-se amb cassos degenerats. Pensem per exemple en el cas de famílies paramètriques, f_μ, les quals ens descriuen comportaments diferents segons el valor de μ. En aquesta situació és genéric (és a dir, succeeix per la majoria de les famílies) trobar-se valors del paràmetre μ_0 pels quals el comportament de f_{μ_0} és degenerate.

Molts dels processos naturals que involucren moviment es poden expressar en termes d'un sistema dinàmic, ja sigui continu: com una eqüació diferencial,

$$\frac{dx}{dt} = X(x)$$

o discret, en termes d'una aplicació:

$$x \mapsto f(x).$$

Lògicament, en l'estudi dels sistemes dinàmics apareixen també moltes degeneracions. Anem a exposar una de les més simples, de fet, possiblement la més senzilla, que ens podem trobar. Suposem que tenim un sistema dinàmic continu en $\mathbb{R}^n$ amb un punt fix x_0:

$$\frac{dx}{dt} = X(x), \qquad x \in \mathbb{R}^n, \qquad X(x_0) = 0. \tag{1}$$

Aplicant el teorema de Taylor al voltant del punt fix obtenim:

$$\frac{dx}{dt} = DX(x_0)(x - x_0) + O(\|x - x_0\|^2).$$

Per tant, prou a prop de x_0, sembla que la part dominant d'aquest sistema dinàmic ve donada per la seva part lineal. De fet, això és cert sempre i quan els valors propis de $DX(x_0)$ tinguin part real diferent del zero. En aquest cas, el teorema de Hartman ens assegura que existeix una funció bijectiva bicontínua que aplica solucions del sistema inicial (1) en un entorn del x_0, a solucions del sistema lineal

$$\frac{dx}{dt} = DX(x_0)(x - x_0)$$

el qual té solucions explícites donades per:

$$x(t) = x_0 + e^{DX(x_0)t}(x^0 - x_0) \qquad x(0) = x^0.$$

Diem que el sistema (1) i el sistema lineal anterior són topològicament conjugats. Als punts fixos d'un sistema dinàmic que compleixen que la diferencial del camp avaluada en el punt fix no té valors propis amb part real nul.la s'els anomena hiperbòlics.

La degeneració que ens ocupa és la relativa als valors propis de la part lineal del sistema dinàmic al voltant d'un punt fix (o bé d'una òrbita periòdica). Aquesta es produeix quan $DX(x_0)$ té algun valor propi amb part real igual a zero. Suposem, per exemple, que $DX(x_0)$ té un valor propi igual a 0, llavors, fent un canvi de variables si cal, tenim que si escribim $x = (x_1, x_2, \ldots, x_n)$, al voltant del punt fix x_0 la dinàmica de la variable x_1 ve donada per:

$$\frac{dx_1}{dt} = O(\|x - x_0\|^2).$$

Per tant, la informació sobre la dinàmica de la variable x_1 ve donada pels termes quadràtics (o potser d'ordre superior), és a dir l'aproximació lineal no és vàlida en aquests casos. Als punts fixos x_0, tal que $DX(x_0)$ té algun valor propi igual a 0 s'els anomena punts parabòlics (o parcialment parabòlics si n'hi ha alguna direcció hiperbòlica). Anàlogament si considerem sistemes dinàmics discrets:

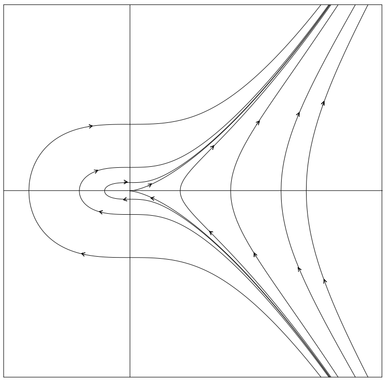
$$x \mapsto f(x)$$

ens trobarem amb la mateixa situació si la diferencial de l'aplicació f avaluada en el punt fix té valors propis 1 o -1.

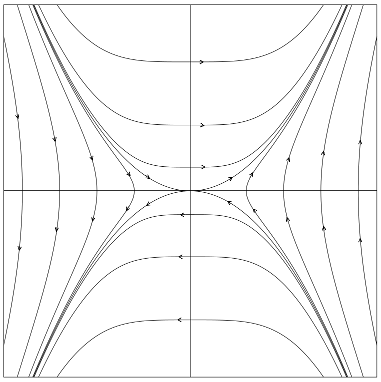
Llavors, és clar que la classificació de sistemes amb punts fixos no hiperbòlics no depèn de la part lineal de la diferencial del camp en el punt fix, sino dels primers termes no lineals del camp. Així un sistema amb la mateixa part lineal pot tenir comportaments ben diferents. Per exemple amb la part lineal igual a:

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

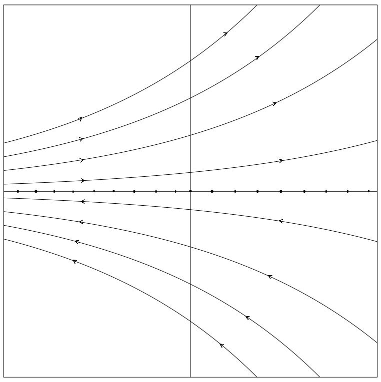
ens podem trobar comportaments com els següents:



(a) $\begin{cases} \dot{x} = y \\ \dot{y} = x^2 \end{cases}$



(b) $\begin{cases} \dot{x} = y \\ \dot{y} = x^3 \end{cases}$

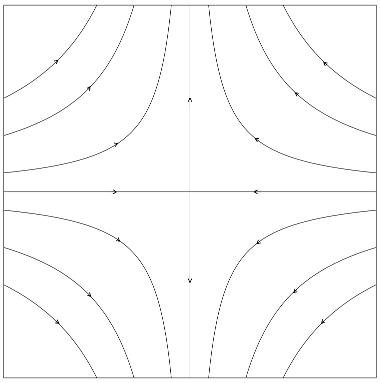


(c) $\begin{cases} \dot{x} = y \\ \dot{y} = y^2 \end{cases}$

Mentre que, per exemple, si tenim un sistema dinàmic amb un punt fix tal que la seva part lineal és:

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

amb $\lambda_1\lambda_2 < 0$ i reals, el retrat de fase al voltant del punt fix és, qualitativament, sempre el mateix:



independenment dels valors de λ_1 i λ_2 i dels termes d'ordre superior que defineixen el sistema.

Varietat central

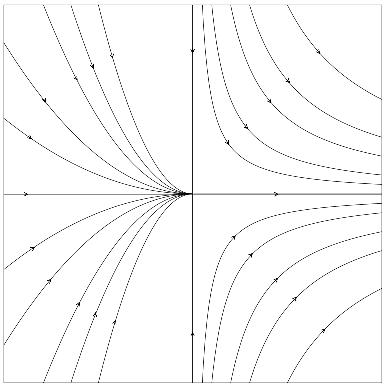
Com ja hem mencionat, un sistema dinàmic continu amb un punt fix hiperbòlic pot ser transformat (prop del punt fix) en un sistema lineal. Quan el punt fix és no hiperbòlic tenim un resultat semblant, malgrat que no tan satisfactori, per estudiar el sistema dinàmic al voltant del punt fix.

Considerem un camp X tal que l'origen és un punt fix no hiperbòlic. Sigui E^c l'espai lineal que generen els subspais propis de valors propis amb part real igual a 0 i siguin E^u i E^s els subespais lineals generats pels subspais propis de valors propis amb part real positiva i negativa respectivament. Llavors, sabem que existeixen varietats W^s_{loc} (varietat invariant estable) , W^u_{loc} (varietat invariant inestable) i W^c_{loc} (varietat invariant central), invariants pel flux generat pel camp X i tangents als espais E^s, E^u i E^c respectivament al punt fix.

Recordem que, així com es pot demostrar que les varietats estable i inestable són úniques, la varietat central no ho és en general. Fixem-nos, per exemple en el retrat de fase del sistema:

$$\begin{aligned}\dot{x} &= x^2 \\ \dot{y} &= -y\end{aligned}$$

el qual, a prop del punt fix, és



En aquest cas, E^c és el subespai generat pel vector $(1, 0)$ i per tant, ja que totes les solucions contingudes en el semiplà $x < 0$ són tangents a E^c, són varietat central.

En qualsevol cas, fixada qualsevol de les varietats centrals, podem conjugar topològicament el camp X al voltant del punt fix a un camp de la forma:

$$\begin{aligned}\dot{x}_c &= X_c(x_c) \\ \dot{x}_u &= x_u \\ \dot{x}_s &= -x_s\end{aligned}$$

on $x=(x_c,x_u,x_s)$ i $X_c = X_{|W^c_{loc}}$. Per tant, podem restringir l'estudi al voltant del punt fix a la varietat central local W^c_{loc}, ja que en la resta de direccions el comportament és ben conegut. Hem reduït doncs la dimensió del problema.

És també un fet conegut que W^u_{loc} i W^s_{loc} tenen el mateix grau de diferenciabilitat que el sistema dinàmic, això no és cert en el cas de la varietat central, la diferenciabilitat de la qual pot variar segons el domini de definició que agafem. Veure [46], [13] i [86] per més detalls.

Tornem un moment als exemples (a), (b) i (c). Observem que, en tots els casos la varietat central és tot $\mathbb{R}^2$, ja que $E^c=\mathbb{R}^2$, però en els casos (a) i (b) tenim òrbites que tendeixen a l'origen quan $t\to+\infty$ i quan $t\to-\infty$. Per tant, en alguns casos de punts fixos no hiperbòlics podem definir la varietat estable e inestable local relativa a $U\subset W^c_{loc}$ (dins de la varietat central) de manera natural:

$$\begin{aligned} W^s_{loc}(x_0) &= \{x\in U : \varphi(t,x)\in U \quad \forall t\geq 0 \text{ i } \varphi(t,x)\to x_0 \text{ quan } t\to+\infty\}\\ W^u_{loc}(x_0) &= \{x\in U : \varphi(t,x)\in U \quad \forall t\leq 0 \text{ i } \varphi(t,x)\to x_0 \text{ quan } t\to-\infty\}\end{aligned}$$

on x_0 és un punt fix del sistema i $\varphi(t,x)$ és la solució de $\dot{x}=X(x)$.

El problema de decidir si un punt fix parabòlic d'un camp o una aplicació té associades varietats estable e inestable (dins de la varietat central), no està resolt en el cas general, però es poden trobar alguns resultats d'existència i unicitat d'aquestes varietats. Per exemple, per aplicacions 2-dimensionals en les que la diferencial en el punt fix és la identitat cal mencionar els treballs de McGehee, Easton i Robinson en [63], [70], [26], [62] i [14]. També en el darrer capítol d'aquesta memòria donem condicions suficients d'existència i unicitat de varietat invariant estable per aplicacions amb la diferencial igual a la identitat en dimensió n arbritària. Per a aplicacions en dos dimensions amb la diferencial en el punt fix igual a:

$$\begin{pmatrix} 1 & 1\\ 0 & 1\end{pmatrix} \tag{2}$$

destaquem el resultats de J.Casasayas, E.Fontich i A.Nunes en [8] i de E.Fontich en [30]. Són d'aquesta darrera classe d'aplicacions les equacions en diferències de segon ordre de la forma:

$$y_{k+1}-2y_k+y_{k-1}=f(y_k),$$

on $f(0)=f'(0)=0$, si expressem la recurrència anterior de la forma

$$\begin{pmatrix} y_k\\ y_{k+1}\end{pmatrix} = F\begin{pmatrix} y_{k-1}\\ y_k\end{pmatrix} = \begin{pmatrix} 2 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix} y_{k-1}\\ y_k\end{pmatrix} + \begin{pmatrix} 0\\ f(y_k)\end{pmatrix}$$

on és clar que $DF(0,0)$ és similar a la matriu (2), per tant, via un canvi lineal de variables estem en el cas actual. També és d'aquesta classe, la "standard map" generalitzada:

$$F(x,y) = (x + y + \varepsilon V(x), y + \varepsilon V(x))$$

amb V una funció periòdica, tal que $V(0) = V'(0) = 0$.

La forma més estàndar de trobar les varietats invariants locals central, estable e inestable associades a un punt fix és descriure-les com a grafs de funcions:

$$W^*_{loc} = \operatorname{graph} \varphi^* = \{(x,y) : y = \varphi^*(x)\} \qquad\qquad * = s, u, c.$$

És també interessant saber quin tipus de regularitat té φ^*. Suposem que l'origen és un punt fix no hiperbòlic, llavors, és curiós de fer notar que, en el cas analític, les varietats invariants locals no són analítiques a l'origen en general, mentre que en el cas hiperbòlic les varietats tenen la mateixa regularitat que el camp. Per exemple, pensem en un cas molt senzill. Considerem el sistema hamiltonià associat a:

$$H(x,y) = \frac{y^2}{2} + V(x)$$

amb $V(x) = x^3 + O(x^4)$. Llavors, és clar que $y = x^{3/2}\sqrt{1 + O(x)}$ és la varietat inestable local i que $y = -x^{3/2}\sqrt{1 + O(x)}$ és la varietat estable local i a més cap de les dues és analítica a l'origen. En qualsevol cas, sí que es pot demostrar, sota certes condicions, que les varietats estable i inestable local d'un punt fix parabòlic, són analítiques en un sector complex de la forma

$$\Omega = \{x \in \mathbb{C} : 0 < |x| < r \quad \text{i} \quad |\arg(x)| < \eta\}$$

amb r i η quantitats positives.

Per últim, dir que tots els resultats d'existència i unicitat de varietats invariants associades a punts fixos parabòlics, lògicament imposen condicions sobre la part no lineal de l'aplicació, la qual cosa dificulta la comprovació de les hipòtesis.

Exemples on apareixen punts fixos degenerats

Malgrat que, com hem mencionat anteriorment, la condició d' hiperbolicitat d'un conjunt invariant: un punt fix, una òrbita periòdica, etc., d'un sistema dinàmic, és una condició genèrica (part real dels valors propis de la diferencial diferents de 0 en el cas de fluxos o de mòdul diferent de 1 el cas d'un difeomorfisme), alguns fenòmens interessants no poden explicar-se dins d'aquest contexte.

Considerem un exemple molt senzill, el cas de famílies de camps X_μ diferenciables dependents d'un paràmetre μ. Diem que la família X_μ té una bifurcació a $\mu = 0$ si i només si, per a tot entorn V de $\mu = 0$ en l'espai de paràmetres, existeixen μ_1 i μ_2, valors del paràmetre diferents que pertanyen a V, tals que les equacions associades als camps X_{μ_1} i X_{μ_2} exhibeixen diferents comportaments qualitatius. Fixem-nos en les bifurcacions locals més simples, és a dir aquelles en les que canvia el caràcter i el nombre de punts fixos. Sigui p_0 un punt fix de X_0. És conegut que si perturbem un camp vectorial amb un punt fix hiperbòlic el sistema pertorbat continua tenint un punt fix hiperbólic. Per tant, una condició necessària per obtenir bifurcacions d'aquesta mena és que p_0 sigui no hiperbòlic, pensem, per exemple, en les bifurcacions més senzilles: sella-node ($\dot{x} = \mu - x^2$), transcrítica ($\dot{x} = \mu x - x^3$), pitchfork ($\dot{x} = \mu - x^3$), etcétera. En totes elles, per $\mu = 0$, l'origen és un punt fix parabòlic. Per tant, un bon coneixement de la dinàmica d'un sistema al voltant d'un punt fix no hiperbòlic, ens ajudarà a entendre la transició entre els comportaments per $\mu < 0$ i $\mu > 0$. Igualment, en el cas d'aplicacions, les bifurcacions locals també apareixen en punts fixos no hiperbòlics.

Abans de descriure altres fenòmens que involucrin sistemes dinàmics amb objectes invariants no hiperbòlics, introduirem breument la noció de sistema integrable: suposem que tenim una equació diferencial

$$\dot{x} = f(x)$$

que descriu l'evolució d'un sistema en $\mathbb{R}^n$. Diem que una funció $F : U \subset \mathbb{R}^n \to \mathbb{R}$ és una integral primera del sistema si F és constant al llarg de les solucions del sistema (és a dir $F(x(t, x_0)) = c$, on $x(t, x_0)$ és la solució tal que $x(0, x_0) = x_0$). Suposem ara que tenim $n - 1$ integrals primeres funcionalment independents,

$$F_1(x), \dots, F_{n-1}(x)$$

llavors una solució $x(t, x_0)$ del sistema $\dot{x} = f(x)$ pot ser totalment descrita com la corba intersecció de les hipersuperfícies

$$\begin{gathered} F_1(x) = F_1(x_0) \\ \vdots \\ F_{n-1}(x) = F_{n-1}(x_0) \end{gathered}$$

excepte pel que fa a la parametrització respecte el temps. En aquest cas diem que el sistema és integrable.

Un tipus de sistemes molt important, ja que de fet molts dels fenòmens mecànics es regeixen per sistemes d'aquesta mena, són els anomenats sistemes hamiltonians. Diem que un sistema és hamiltonià si existeix una funció, que anomenem hamiltonià del sistema, $H : U \subset \mathbb{R}^{2m} \to \mathbb{R}$ tal que

$$\begin{aligned} \dot{x} &= \partial_y H(x,y) \\ \dot{y} &= -\partial_x H(x,y). \end{aligned}$$

És clar que H és una integral primera del sistema. Es diu que el sistema hamiltonià té m graus de llibertat.

La dos forma simplèctica estandard, dota $U \subset \mathbb{R}^{2m}$ d'estructura simplèctica. Aquesta estructura permet que la noció de sistema hamiltonià integrable pugui reduir-se a l'existència de m integrals primeres $F_1 = H, F_2, \dots, F_m$ (el hamiltonià n'és una) les quals estan en involució, és a dir:

$$\{F_i, F_j\} = \partial_x F_i \partial_y F_j - \partial_y F_i \partial_x F_j = 0$$

i les seves derivades són linealment independents a un obert dens. Aquest resultat és degut a Liouville-Arnorld (veure [3]).

A continuació descrivim alguns fenomens que involucren sistemes dinàmics amb objectes parabòlics.

El primer d'ells és el de les ressonàncies parabòliques. Una resonància parabòlica es produeix quan un sistema hamiltonià integrable amb 2 graus de llibertat amb un cercle de punts fixos parabòlics és perturbat. En [71] V.Rom -Kedar prova que aquesta qualitat és genèrica per a famílies 1-paramètriques (fenòmen de codimensió 1) de hamiltonians amb 2 graus de llibertat prop d'integrables, és a dir, sistemes que són petites pertorbacions de sistemes hamiltonians integrables. Experiments numèrics apunten que el moviment a prop de ressonàncies parabòliques exhibeix un nou tipus de comportament caòtic no detectat fins ara. Hi ha encara un cas més degenerat, denominat ressonància parabòlica plana, el qual es manifesta en un model que prové d'un estudi atmosfèric real, concretament de l'estudi de sondes metereològiques. Aquest model proporciona un mecanisme per transportar partícules amb velocitats inicials petites a prop de l'Equador fins a latituds altes. Veure [71] i les referències en ell per més detalls.

L'estudi fet en aquest darrer article, respecte a les ressonàncies parabòliques, és generalitzat en el cas de hamiltonians amb n graus de llibertat amb $n \geq 3$ en [56] i [57].

En [48], Hanβmann tracta torus de dimensió baixa amb una freqüència normal nul.la en sistemes hamiltonians de n graus de llibertat. Aquests torus s'anomenen normalment parabòlics. Hanβmann considera famílies de sistemes hamiltonians a prop d'integrables en un entorn de torus invariants normalment parabòlics. Sota certes condicions de transversalitat té lloc una bifurcació quasi-periodica centre-sella ($\lambda < 0$ no tenim torus invariants, $\lambda = 0$ el torus és normalment parabòlic i $\lambda > 0$ el torus és de tipus sella). L'autor demostra la persistència de la bifurcació centre-sella i dels torus normalment parabòlics parametritzats per conjunts de Cantor "grans". Hanβmann aplica aquests resultats a la dinàmica del sòlid rígid.

En el problema pla de tres cossos una òrbita parabòlica és una trajectòria d'una partícula que s'apropa a l'infinit amb velocitat zero, mentre que les trajectòries de les altres dues partícules resten acotades per temps positius. Una òrbita del problema pla de tres cossos s'anomena oscil.latòria si el límit superior (al llarg del temps) de la separació entre les partícules és infinit, però el límit inferior és finit. Sembla clar, doncs, que les òrbites oscil.latòries van i venen infinites vegades tendint (d'alguna manera) cap a infinit. Per això una bona manera d'atacar aquest problema és trobar "solucions homoclíniques a l'infinit". Per tant sembla lògic intentar portar l'infinit a algun objecte invariant. En el cas del problema pla de tres cossos, McGehee i Easton en [27] proben que l'infinit es pot veure com una tres esfera foliada de òrbites periòdiques. McGehee considera tres problemes: el problema restringit, el problema de Sitnikov (veure [78]), i quan els tres cossos es mouen sobre una recta, i en [63] demostra que (després de certs canvis de variable) l'infinit es pot reduir a una òrbita periòdica. Més tard, R. Martínez et al en [62] demostren, entre d'altres coses, que, en el problema restringit el.líptic de tres cossos, la varietat de l'infinit està també foliada d'òrbites periòdiques.

Una manera d'atacar el problema de trobar òrbites oscil.latòries és demostrant que aquestes òrbites periòdiques, les quals recordem que representen l'infinit del sistema inicial, tenen varietat estable i inestable i que aquestes es tallen transversalment. Aquesta no és una condició suficient (veure [27]), però sí que sembla necessària per demostrar l'existència d'òrbites oscil.latòries. En els problemes tractats per McGehee en [63] i en el problema el.líptic restringit [62] trobar aquestes solucions homoclíniques implica l'existència d'òrbites oscil.latòries i per tant de caos.

En tots aquests exemples, les òrbites periòdiques associades a l'infinit que s'han trobat són degenerades. Concretament la diferencial de l'aplicació de Poincaré associada a dita òrbita periòdica té valors propis 1. En els problemes considerats per McGehee i per R. Martínez et al és la identitat i en el problema pla de tres cossos té una part hiperbòlica, és a dir, en aquest cas la diferencial és de la forma

$$\begin{pmatrix} I & 0 \\ 0 & A \end{pmatrix}$$

amb $I \in \mathcal{M}_{2\times 2}$ i A és una matriu hiperbòlica. McGehee va demostrar en [63] que (sota condicions que generen una certa hiperbolicitat feble) aplicacions en dimensió 2 amb un punt fix de tipus parabòlic tal que la diferencial de l'aplicació en el punt fix és la identitat, tenen varietat estable associada la cual es pot expressar com el graf d'una funció. Aquesta funció és Lipschitz, si l'aplicació és Lipschitz, i analítica en el cas d'aplicacions analítiques. No es demostra el cas diferenciable. Més tard aquest resultat va ser generalitzat en [26] (en el cas Lipschitz) i [70] (pel cas C^∞) per fluxos de la forma

$$\begin{aligned} \dot{x} &= p_k(x,y) + O_{k+1} \\ \dot{y} &= By + q_k(x,y) + O_{k+1} \end{aligned}$$

on $x \in \mathbb{R}^2$, $y \in \mathbb{R}^3$ i p i q polinomis homogenis de grau k, $k \geq 2$. Conseqüentment l'aplicació temps unitat d'aquesta eqüació diferencial és de la forma

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x + P_k(x,y) \\ Ay + Q_k(x,y) \end{pmatrix} + O_{k+1}$$

on P_k, Q_k són polinomis homogenis de k, $k \geq 2$, i els valors propis de la matriu A són de mòdul diferent de 1.

Tot això ens indica que les òrbites parabòliques formen una varietat suau. Robinson, Xia, Moeckel i R. Martínez en [70], [88], [66] i [62] respectivament demostren l'existència d'òrbites oscil.latòries en alguns exemples del problema de tres cossos.

Aquest darrer fenomen ens va motivar a plantejar-nos el problema de donar condicions suficients per l'existència de varietat invariant estable d'una aplicació en $\mathbb{R}^n$ amb un punt fix tal que la diferencial de l'aplicació avaluada en el punt fix sigui la identitat en $\mathbb{R}^n$, així, previsiblement, podríem trobar òrbites oscil.latòries en problemes de més de tres cossos.

A part d'aquest problema, l'objetiu d'aquest treball és demostrar una fórmula asimptòtica per mesurar el trencament de separatrius,associades a punts fixos parabòlics, per una classe de sistemes hamiltonians d'un grau i mig de llibertat ràpidament forçats. Considerem sistemes hamiltonians amb un punt fix parabòlic tals que la diferencial del sistema avaluada en el punt fix és

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

i demostrem que la magnitut del trencament és exponencialment petita respecte a la freqüència de la pertorbació.

En aquest treball hem estudiat bàsicament els dos darrers problemes mencionats. Conseqüentment la tesi està dividida en dues parts. En la primera estudiem la mesura del trencament de separatrius associades a punts fixos parabòlics i en la segona donem un teorema d'existència de varietats invariants associades a un punt fix de difeomorfismes n-dimensionals amb la diferencial avaluada en el punt fix és la identitat.

Part I: Trencament de separatrius

Passem ara a introduir la noció de separatriu d'un sistema hamiltonià. Recordem que un sistema dinàmic continu en $\mathbb{R}^n$ és integrable quan té $n-1$ integrals primeres funcionalment independents. En el cas d'un sistema Hamiltonià, el teorema de Liouville-Arnold (veure [3]) ens assegura que un sistema hamiltonià és integrable si té $m = n/2$ (contant el hamiltonià) integrals primeres $F_1 = H, F_2, \dots, F_m$ en involució

$$\{F_i, F_j\} = \partial_x F_i \partial_y F_j - \partial_y F_i \partial_x F_j = 0$$

i les seves derivades són linealment independents en un obert dens.

Supossem que tenim un sistema hamiltonià integrable, la dinàmica en aquest cas és ben coneguda. En efecte, anomenem $F = (F_1, F_2, \dots, F_m)$ i $M_c = F^{-1}(c)$. Llavors, M_c és una subvarietat invariant pel flux de dimensió m. A més, si M_c és conexa i els camps

$$\begin{aligned} \dot{x} &= \partial_y F_i \\ \dot{y} &= -\partial_x F_i \end{aligned}$$

són complerts (les seves solucions estan definides per a tot temps), llavors si la varietat M_c no és compacte, és homeomorfa a $\mathbb{T}^k \times \mathbb{R}^{m-k}$ i si M_c és compacte i conexa, llavors és homeomorfa a un torus m-dimensional. En qualsevol dels dos casos, amb un canvi de variables adequat, el flux del sistema hamiltonià sobre M_c és conjugat a una traslació:

$$\varphi(t, x, y) = (x + t\omega \pmod{2\pi}, y + t\nu)$$

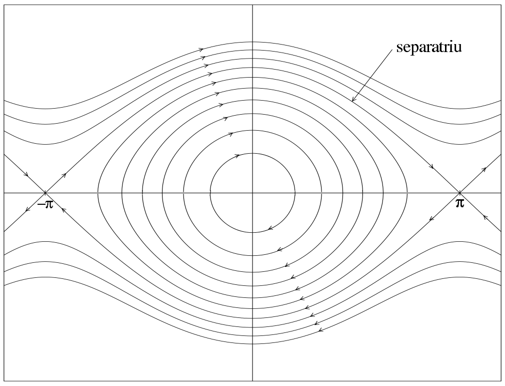
on ω i ν depenen de c. Per tant un sistema hamiltonià integrable és totalment predictible (o regular): no n'hi ha comportament estocàstic (no predictible).

Dins del contexte dels sistemes hamiltonians integrables, fixem la nostra atenció en els que, per alguna constant c_0, la varietat M_{c_0} està formada per un objecte invariant P (un punt fix, una òrbita periòdica, un torus, etc.) i les seves varietats invariants estable e inestable. Suposarem que existeixen branques de les varietats estable e inestable que coincideixen, és a dir $W^{u,+}(P) = W^{s,+}(P)$. Aquestes varietats estan foliades per solucions del sistema hamiltonià que tendeixen al objecte invariant quan el temps tendeix a $+\infty$, si estem sobre la varietat estable, o a $-\infty$ si estem sobre la inestable. Aquesta varietat invariant rep el nom de separatriu.

Aquest nom prové de considerar el cas de sistemes hamiltonians d'un grau de llibertat amb varietats invariants estable e inestable associades a punts fixos. En aquest context, les separatius "separen" diferents tipus de comportament dinàmic. Per exemple, pensem en el cas del pèndol, el moviment del qual ve regit per les equacions:

$$\dot{x} = y \qquad \dot{y} = -\sin x \qquad x \in S^1, \qquad y \in \mathbb{R}.$$

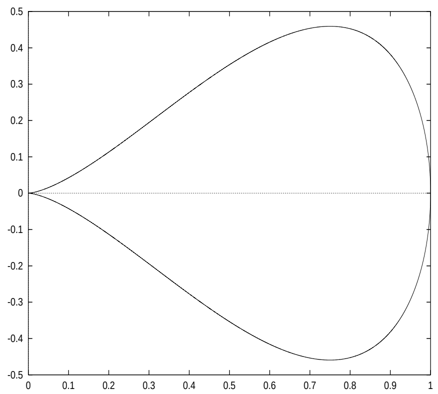
El seu retrat de fase és ben conegut:



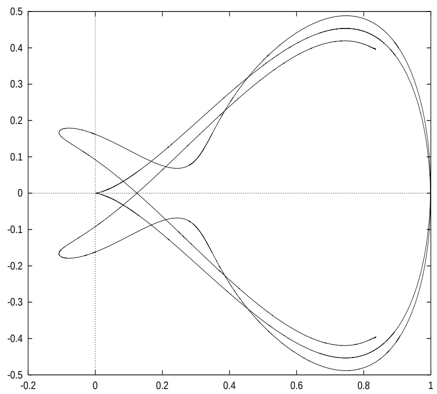
El punt de repós $(0,0)$ és estable. Si donem una empenta (una velocitat inicial) més petita que 2 al péndol situat en la posició de repós, oscil.larà al voltat del punt d'equilibri (la seva trajectória és una òrbita periòdica) i si la velocitat inicial és més gran estricte que 2, el péndol gira en el mateix sentit creuant la part de dalt del péndol representada pel punt $(\pi, 0)$. Si la velocitat inicial és igual a dos, llavors la solució tendeix al punt d'equilibre inestable, el punt $(\pi, 0)$. A aquesta darrera solució se li anomena separatriu i és clar que separa dos tipus de moviment ben diferents.

Tot i que en dimensions més altes les separatrius no separen (ja que són subvarietats de la meitat de la dimensió del espai) se les continua anomenant pel mateix nom.

Considerem, doncs, un sistema hamiltonià amb dos conjunts invariants (punts fixos, òrbites periòdiques, etc.), amb varietats estable e inestable. Supossem que $W^{s,+}(P_1) = W^{u,+}(P_2)$ i $W^{u,+}(P_1) = W^{s,+}(P_2)$, on $W^{s,u,+}(P_1)$ i $W^{s,u,+}(P_2)$ són branques de les varietats $W^{s,u}(P_1)$ i $W^{s,u}(P_2)$ respectivament. Tenim el que s'anomena connexió heteroclínica (si $P_1 \neq P_2$) o homoclínica si $P_1 = P_2$. Aquesta és una situació genérica dins dels camps hamiltonians, però quan pertorbem amb una pertorbació periòdica, en general, aquesta connexió es trenca donant lloc al fenomen anomenat trencament de separatrius:



Sistema no pertorbat



Sistema pertorbat

El fenomen del trencament de separatrius va ser descrit per Poincaré, peró va ser Melnikov (amb les idees proposades per Poincaré) qui va donar una bona eina analítica per mesurar-lo. Aquesta eina és coneguda com el métode de Poincaré-Melnikov (veure [46] i [1] per una bona introducció). Aquest métode ens dóna una fórmula asimptótica de primer ordre per mesurar aquest trencament.

Per fixar idees, suposem que tenim el sistema

$$\dot{x} = f(x) + \mu g(t, x)$$

on $x \in \mathbb{R}^2$ i per $\mu = 0$ el sistema no pertorbat té una connexió homoclínica associada a un punt fix de tipus sella. És el cas més simple. Si la pertorbació és prou petita, el sistema pertorbat té una òrbita periòdica hiperbòlica σ i per tant, existeixen varietats estable e inestable de σ. La fórmula proposada per Melnikov per mesurar la magnitud d'aquest trencament $d(t_0)$ en un temps t_0 fixat ($d(t_0)$ és la distància entre les varietats invariants del sistema perturbat) és

$$d(t_0) = \mu M(t_0) + O(\mu^2)$$

on $M(t_0)$ és la integral de Melnikov, la qual depèn de la connexió homoclínica i de g. Podem assegurar la existència de punts homoclínics (punts que pertanyen a la varietat estable e inestable de σ) transversals si existeix algun t_0 tal que $M(t_0) = 0$ i $M'(t_0) \neq 0$. És ben conegut que l'existència de punts homoclínics transversals en dimensió 2 dóna lloc a comportament caótics (teorema homoclínic d'Smale).

Suposem ara que la pertorbació té freqüència ràpida, és a dir el periode és εT amb $\varepsilon > 0$ petit. Llavors, es pot demostrar que la integral de Melnikov és $O(e^{-a/\varepsilon})$ amb $a > 0$ una certa constant. Per tant en aquest cas la funció de Melnikov, a priori, és una bona aproximació de la mesura del trencament només si $\mu = o(e^{-a/\varepsilon})$, el que ens dóna un marge molt petit de valors de μ i per tant de les perturbacions possibles pels quals poder obtenir conclussions. En aquest cas el métode de Melnikov (en la seva forma elemental) no és, a priori, una bona eina per mesurar el trencament de separatrius.

Recentment Lombardi [61] ha donat métodes rigurosos per estudiar el que ell anomena integrals oscil.latòries que són integrals del tipus

$$I(\varepsilon) = \int_{-\infty}^{+\infty} e^{it/\varepsilon} g(x_0(t,\varepsilon))dt$$

on x_0 és una solució particular del sistema $\dot{x} = F(x,t,\varepsilon)$ caracteritzada, per exemple, pel seu valor al infinit. Per exemple, si x_0 és la varietat estable d'una òrbita periòdica o d'un punt fix. Aquests tipus d'integrals són les que apareixen al aplicar el métode de Poincaré-Melnikov i per tant pot ser molt útil en el futur. Aquest treball tracta problemes en els quals apareixen fenomens exponencialment petits en sistemes reversibles. Per exemple, considerem un camp vectorial en $\mathbb{R}^4$ en forma normal i trunquem a qualsevol order. El sistema no perturbat és el sistema truncat i considerem la cua de la forma normal com una pertorbació. En [61] es prova, prop d'un cert tipus de ressonància, la persistència de connexions homoclíniques a òrbites periòdiques de tamany exponencialment petit. La ressonància és deguda al canvi de caràcter del punt fix. Dos dels valors propis són de la forma $\pm i\omega + O(\mu)$ i els altres dos λ_1, λ_2 són diferentes i reals per $\mu > 0$, $\lambda_1 = \lambda_2 = 0$ si el paràmetre $\mu = 0$ i imaginaris si $\mu < 0$. Per tant tenim aquí també un model de pertorbació en el qual apareix un punt parabòlic coexistint amb una part oscil.latòria.

Molts autors han treballat el fenomen del trencament de separatrius amb pertorbacions periòdiques de freqüència ràpida, per demostrar que, en alguns casos i sota certes hipòtesis, la funció de Melnikov dóna una bona aproximació de la mesura del trencament de separatrius.

En [54], Lazutkin estudia l'aplicació standard i raona (però no proba rigurosament) que l'angle d'intersecció entre les varietats estable e inestable és de la forma

$$\varphi = \frac{\pi}{\varepsilon}|\Theta_1|e^{-\pi^2/\sqrt{\varepsilon}}[1 + O(\varepsilon^r)]$$

amb $r > 0$. La constant $\Theta_1 \in \mathbb{C}$ ha de ser calculada numéricament. De fet, donat que és el terme dominant de l'expressió de l'angle de separació, el més important és demostrar que no és zero. En [81], Suris demostra que aquest coeficient és diferent del zero per l'aplicació standard i proposa una manera alternativa de calcular aquesta constant, però continua sent necessari computar-la numèricament. També, en [42] Gelfreich et al demostra analíticament que Θ_1 no és zero per l'aplicació d'Henón. En [55] es calcula numéricament aquesta constant per la aplicació semistandard i en [12] per l'aplicació de Hénon.

És important destacar que, en [54], s'introdueixen noves eines analítiques per l'estudi del trencament de separatrius les que han estat pioneres i han influit decisivament en el desenvolupament de l'àrea. Aquesta tècnica ha estat utilitzada fortament en aquesta memòria. (Veure [42] per una bona exposició d'aquesta tècnica).

En [34] i [33] E. Fontich i C.Simó estudien el trencament de separatrius per a famílies a prop de la identitat de difeomorfismes de classe C^r i analítiques respectivament. Sota hipòtesis bastant generals es donen cotes superiors de la distància entre varietats invariants. Concretament, en [34] es demostra que la distància entre les varietats invariants està acotada per $K\varepsilon^{r-1}$ en el cas C^r, i per famílies analítiques, en [33], es prova una cota superior exponencialment petita, de l'ordre de $e^{-k/\varepsilon}$ pel trencament de varietats invariants i es donen valors de K òptims en general.. Obviament, l'avantatge de treballar amb difeomorfismes és que es pot donar un resultat semblant en el cas de fluxos passant per l'aplicació de Poincaré (veure [31]). L'inconvenient és que és molt més difícil treballar amb sumes infinites (l'anàleg discret de la funció de Melnikov) que amb integrals.

Com hem dit abans, és més avantatgós treballar amb difeomorfismes analítics que amb fluxos, no obstant, molts dels treballs relacionats amb el trencament de separatrius són per fluxos, el que ens indica que el problema de donar expressions assimptótiques (o bé cotes superiors i inferiors) pel trencament de separatrius és substancialment més difícil en el cas d'aplicacions. En [69], R. Ramírez treballa en aplicacions simpléctiques i dóna una manera sistemàtica d'avaluar la funció de Melnikov (que en el cas d'aplicacions és una suma infinita). A més es dóna una fórmula asimptótica per l'àrea engendrada entre les varietats invariants d'aplicacions que es poden veure com perturbacions de l'aplicació de McMillan [64] i per billards.

El fenomen de trencament de separatrius ha estat estudiat ampliament en el cas dos dimensional amb sistemes de la forma

$$\dot{z} = f(z) + \mu\varepsilon^p g(x, t/\varepsilon, \varepsilon)$$

on μ i $\varepsilon > 0$ són paràmetres a priori independents i tals que l'origen és un punt fix de tipus sella. Així el sistema no perturbat és per $\mu = 0$. S'ha discutit molt en quan al grau d'optimabilitat de p. En [31] es donen cotes superiors del trencament per a valors negatius de p, concretament $p > -1/2$. Si simplifiquem el model i considerem eqüacions de segon ordre de la forma

$$\ddot{x} + f(x) = \mu\varepsilon^p g(x, t/\varepsilon, \varepsilon)$$

llavors en [32] es donen cotes superiors pel trencament de separatrius per a valors de $p > -2$. En [49], Holmes et al aconsegueixen donar cotes superiors i infereriors del trencament de separatrius per sistemes bastant generals per a valors de $p > 8$. La situació millora quan tractem sistemes concrets. L'exemple més estudiat és el pendol. En [45] i en [21] es donen expressions asimptòtiques del trencament de separatrius per l'equació

$$\ddot{x} + \sin x = \mu\varepsilon^p \sin t/\varepsilon$$

per $p > 5$ i $p > 0$ respectivament. Posteriorment A. Delshams i M.T. Seara en [20], van aconsseguir una expressió asimptótica del trencament de separatrius per sistemes més generals sempre i quan p sigui més gran que una certa quantitat que depèn de la pertorbació i de l'ordre de la singularitat de la òrbita homoclínica. Gelfreich en [43] també va donar una expressió asimptótica del trencament de separatrius, peró és díficil de saber quin p necessitem per poder aplicar-ho. Per últim en [44] Gelfreich estudia en alguns exemples concrets el cas $p < 0$. El métode proposat és la utilització d'un sistema auxiliar les varietats invariants del qual són una bona aproximació a prop de les singularitats de les varietats invariants del sistema inicial.

Tots aquests casos treballen amb sistemes hamiltonians d'un grau i mig de llibertat o bé aplicacions que preserven àrea tals que l'origen és un punt fix hiperbòlic del hamiltonià no pertorbat. Un altre contexte on apareix el fenomen de trencament de separatrius és quan considerem pertorbacions quasi-periòdiques. Com per exemple en [22], [23] i [36]. En aquest cas l'anàlisi és molt més complicat que en el cas de tenir una pertorbació periòdica en el temps.

Un altre fenomen relacionat amb el trencament de separatrius és la difussió d'Arnold. Si pertorbem un sistema hamiltonià de m graus de llibertat integrable $H_0(I)$, per una pertorbació hamiltoniana, en general, el sistema pertorbat $H(I, \varphi) = H_0(I) + \varepsilon H_1(I, \varphi)$ deixa de ser integrable i apareixen comportament no predictibles (caòtics). No obstant, la teoria KAM (veure ens assegura que "molts" dels torus invariants de dimensió màxima (en aquest cas m) que teníem en el sistema no perturbat H_0 es conserven lleugerament deformats. Les accions de les òrbites que permaneixen en aquests "forats", en els quals el teorema KAM no ens garanteix l'existència de torus invariants, podríen tenir un desplaçament d'ordre 1 independentment del tamany de la pertorbació. Si fos així, pel teorema de Nekhoroshev:

$$I_\varepsilon(t) - I_\varepsilon(0) = O(1) \qquad \text{per a } 0 \le t \le \frac{1}{\varepsilon}\exp(1/\varepsilon^a),$$

hauria de ser un desplaçament molt lent. L'exemple proposat per Arnold, [2], demostra que aquest fenomen pot succeir a través de cadenes de torus invariants parcialment hiperbòlics amb varietats estable e inestable, les quals s'intersequen transversalment. Quan considerem sistemes analítcs una de les majors dificultats és que el trencament de separatrius és exponencialment petit en general.Veure [4] per una bona introducció d'aquest fenomen.

El nostre objectiu és donar una fórmula asimptótica per mesurar el trencament de les separatrius en sistemes hamiltonians d'un grau i mig de llibertat tals que l'origen és un punt fix parabòlic del sistema no pertorbat. Concretament, la diferencial del camp en $(0,0)$ és

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Nosaltres hem seguit bàsicament l'esquema de demostració de [20]. Però degut a que molts dels seus raonaments fan servir fortament el caràcter hiperbòlic del punt fix, hem hagut d'introduir noves tècniques per tal que siguin vàlides en el cas parabólic. Per axió també hem utilitzat eines introduïdes per Lazutkin [54], [53] i utilitzades més tard per Gelfreich [43]. És important mencionar que la majoria dels nostres arguments són també vàlids per cas hiperbólic.

Presentació del problema, hipòtesis

Treballem amb sistemes Hamiltonians d' un grau i mig de llibertat, amb Hamiltonià de la forma

$$H(x,y,t/\varepsilon) = \frac{y^2}{2} + V(x) + \mu\varepsilon^p h_1(x,y,t/\varepsilon,\varepsilon,\mu)$$

on

- suposem que el potencial $V(x)$ és un polinomi d'ordre 3 com a mínim, és a dir

$$V(x) = ax^n + O(x^{n+1}) \qquad \text{amb } n \ge 3 \text{ i } a \ne 0.$$

Per tant l'origen és un punt fix parabòlic amb diferencial

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Per la pertorbació h_1, posem les següents condicions:

- suposem que és C^0, analítica en (x, y, μ) i que té ordre $k \geq 3$ en les variables (x, y), complint la condició que

$$2k - 2 \geq n. \tag{3}$$

  Observem que el cas $k < n$ (el cual és un cas de bifurcació) està permès per $n \geq 4$. A més suposem que és $2\pi\varepsilon$-periòdica i que té mitjana zero respecte la variable $\theta = t/\varepsilon$:

$$\int_0^{2\pi} h(x, y, \theta) d\theta = 0.$$

És important destacar que només necessitem que la pertorbació sigui contínua respecte t/ε.

Necessitem ara les hipòtesis relacionades amb el sistema no pertorbat. Suposem que el sistema no pertorbat ($\mu = 0$) té un òrbita homoclínica, la qual anomenem $\gamma_0 = (\alpha_0, \beta_0)$.

- Les singularitats de l'òrbita homoclínica γ_0 amb el mòdul de la part imaginària més petit són singularitats de tipus branca d'ordre $r \in \mathbb{Q}$ en punts de la forma $\pm ia$.

Amb els métodes estandards per mesurar el trencament de separatrius és totalment essencial que l'òrbita homoclínica tingui alguna mena de singularitat. De fet en el cas hiperbòlic, en [31] es demostra que existeix ρ, $0 < \rho \leq \rho_0 = \min\{\pi/\lambda_1, \pi/|\lambda_2|\}$ (on λ_1 i λ_2 són els valors propis de la diferencial avaluada al punt fix) tal que l'òrbita homoclínica és analítica en una banda complexa de la forma

$$D(\rho) = \{z \in \mathbb{C} : |\operatorname{Im} z| < \rho\}$$

i no pot extendre's a $D(\rho')$ si $\rho' > \rho_0 > 0$. (Observem, però que la nostra hipòtesi exclou la possibilitat de tenir singularitats essencials sobre la recta $\operatorname{Im} z = \pm\rho$). En canvi per trobar cotes superiors es suficient demanar que γ_0 sigui analítica en $D(\rho)$.

Aquesta darrera hipòtesi no es pot obviar, és a dir, creiem que és difícil comprovar que es compleix a partir del hamiltonià. Per veure la dificultat suposem que estem en un cas senzill. Suposem que el potencial $V(x)$ és negatiu per a $x \in (0,1)$ i és un polinomi. Llavors, és fàcil veure que sempre hi ha singularitats tipus branca (o pols si el grau del potencial és 4) en punts de la forma $\pm ia$. En efecte, (veure [52] per més detalls) sigui $V(x) = a_n x^n + \ldots + a_m x^m$ amb $n < m$. Fent el canvi $x = 1/w$, la solució del sistema no pertorbat és

$$t(w) = \int_w^1 \frac{w^{m/2-2}}{\sqrt{2(a_n + \ldots + a_m w^{m-n})}} dw.$$

Una singularitat de l'òrbita homoclínica s'assoleix a

$$t(0) = t_0 = -i \int_0^1 \frac{w^{m/2-2}}{\sqrt{-2(a_n + \ldots + a_m w^{m-n})}} dw \in i\mathbb{R}$$

ja que $a_n + \ldots + a_m w^{m-n} < 0$ si $w \in (0,1)$. A més

$$t'(w) = -w^{m/2-2} \left(\frac{1}{\sqrt{2a_n}} + O(w) \right)$$

i per tant

$$t(w) = t_0 + w^{m/2-1}(c + O(w))$$

on $c \in \mathbb{C}$ és una constant. Aïllant w en funció de $t(w) - t_0$ i desfent el canvi $w = 1/x$ obtenim que

$$x = \frac{c}{(t-t_0)^{2/(m-2)}} (1 + O(t-t_0)^{2/(m-2)})$$

a prop de $t_0 \in i\mathbb{R}$.

Però, fins i tot en aquest cas més simple, veritablement és difícil comprovar que les singularitats amb part imaginària més petita siguin precisament de la forma $\pm ia$. Pensem que l'estudi exhastiu és molt complicat ja que cal estudiar els possibles valors d'integrals del tipus

$$\int_1^h \frac{1}{\sqrt{-2V(u)}} du$$

on $|h| = +\infty$ i donar condicions sobre el potencial V per tal que, els valors amb part imaginària més petita siguin purament imaginaris. Veure [61] per una petita discussió sobre aquest tema.

El fet de tenir òrbites homoclíniques amb singularitats de tipus branca no és un fet exclusiu del cas parabòlic, per tant, podem ampliar els casos en els quals es pot aplicar el resultat donat en [20] per punts hiperbòlics.

Com hem dit abans, és important la grandessa o petitessa de p. Aquesta ve determinada bàsicament pel Lema d'extensió (el qual extén la parametrització de les varietats locals a dominis molt a prop de les singularitats de l'òrbita homoclínica).

Hem hagut de fer una assumpció relacionada amb r (l'ordre de la singularitat de l'òrbita homoclínica) i amb l'ordre de les singularitats $\pm ia$ respecte de la funció $h_1(\gamma_0(t+s), t/\varepsilon)$, el qual, seguint la notació de [20], hem anomenat ℓ.

- La condició que necessitem és

$$p - \ell \geq 0.$$

Creiem que no és óptima. Per exemple, en el cas del pèndol pertorbat, considerat per Gelfreich en [43], amb la nostra hipòtesi necessitaríem $p \geq 0$ i Gelfreich treballa amb valors de $p < 0$.

Hem considerat també un exemple particular en el cas que l'ordre del potencial és 3 i $k = 2$, el qual no estava considerat en les anteriors hipòtesis. Òbviament aquest és un cas de bifurcació perque quan el paràmetre μ és diferent del zero, (sota certes condicions sobre la pertorbació) l'origen és un punt fix hiperbòlic i quan aquest paràmetre s'anul.la, ens trobem en el cas parabòlic. A aquest cas, l'hem anomenat, cas feblement hiperbòlic. Les hipòtesis que necessitem en aquest cas són les mateixes que en el cas parabòlic. Concretament hem considerat sistemes de la forma

$$H(x, y, t/\varepsilon, \mu, \varepsilon) = h_0(x, y) + \mu\varepsilon^p h_1(x, y, t/\varepsilon, \mu, \varepsilon)$$

on

$$h_0(x, y) = \frac{y^2}{2} - x^3 + x^4 \quad \text{i} \quad h_1(x, y, t/\varepsilon, \mu, \varepsilon) = h_{12}(x, y, t/\varepsilon, \mu, \varepsilon) + h_{13}(x, y, t/\varepsilon, \mu, \varepsilon)$$

amb

$$h_{12}(x, y, t/\varepsilon, \mu, \varepsilon) = \frac{x^2}{2} g_1(t/\varepsilon, \mu, \varepsilon) + xy g_2(t/\varepsilon, \mu, \varepsilon) + \frac{y^2}{2} g_3(t/\varepsilon, \mu, \varepsilon)$$

i $h_{13}(x, y, t/\varepsilon, \mu, \varepsilon)$ és d'ordre més gran o igual que 3 en les variable (x, y).

- En el cas feblement hiperbòlic, substituim la condició donada per (3) per la següent

$$\int_0^{2\pi} g_2(\theta,\mu,\varepsilon)G_1(\theta,\mu,\varepsilon)d\theta > 0.$$

  on G_1 està determinada per l'eqüació $\partial_\theta G_1 = g_1$ i tenir mitjana zero.

  La resta d'hipòtesis enunciades anteriorment també són asumides en aquest cas.

Aquesta darrera és la condició que necessitem perque l'origen sigui un punt de tipus sella del sistema pertorbat. Recentment [37], ha considerat aquest tipus de bifurcació en el cas d'aplicacions que preserven àrea. En [38], han estat considerades ressonàncies d'ordres més alts.

Tot i que l'objectiu és l'estudi de trencament de separatrius en sistemes hamiltonians amb punts fixos parabòlics, totes les demostracions que hem fet, excepte les dels capítols tres i quatre (on trobem unes parametritzacions especials de les varietats invariants locals) són fàcilment adaptables també per punts hiperbòlics.

Hem demostrat una expressió asimptótica de l'àrea engendrada per les varietas invariants entre dos punts homoclínics consecutius s_0 i $\bar{s}_0$ donada per

$$A = \mu\varepsilon^p \left[\int_{s_0}^{\bar{s}_0} M(v,\varepsilon)dv + O(\varepsilon^b,\mu^c)e^{-a/\varepsilon} \right]$$

on b i c són constants positives les quals estan totalment especificades i depenen de r, p i ℓ. Recordem que a és el módul de les singularitats $\pm ia$ de la òrbita homoclínica.

Passem ara a enunciar el principal corol.lari. Sigui

$$J(x,y,\theta) = \{h_0,h_1\}(x,y,\theta) = \sum_{k\neq 0} J_k(x,y)e^{ik\theta}$$

la qual avaluada a l'òrbita homoclínica té una singularitat a $u = \pm ia$ com a molt d'ordre $\ell+1$. Observem, al voltant de les singularitats $u = \pm ia$, $J_{\pm 1}(\gamma_0(u))$ té la forma:

$$J_{\pm 1}(\gamma_0(u)) = J^{\pm}_{\pm 1,0}\frac{1}{(u \mp ia)^{\ell+1}}(1 + h.o.t.).$$

Si suposem que

- $J_{\pm 1}(\gamma_0(u))$ té una singularitat d'ordre exactament $\ell+1$ a $u=\pm ai$, és a dir si $J_{1,0}^+ \neq 0$ (la qual és una condició genérica) llavors

$$A \sim \mu\varepsilon^{p-\ell+1} 8\pi |J_{1,0}^+| \frac{1}{\Gamma(\ell+1)} e^{-a/\varepsilon}$$

  on Γ és la funció Gamma.

Esquema general de la primera part

En el primer capítol, veiem quin és el comportament asimptòtic de la parametrització de l'òrbita homoclínica. Demostrem que, com sembla lògic, aquest comportament és potencial, és a dir, existeix $T > 0$ tal que, si $t \in \mathbb{C}$, $\mathrm{Re}\, t > T$, la varietat estable es comporta com $1/t^q$, amb q un cert nombre positiu i anàlogament, la varietat inestable és de la forma $1/(-t)^q$ per $t \in \mathbb{C}$, $\mathrm{Re}\, t < -T$.

En els tercer i quart capítols, hem trobat una bona parametrització $\gamma^*(t,s)$ $(* = s, u)$ de les varietats invariants en els cassos totalment parabòlic i feblement hiperbòlic respectivament. Aquesta parametrització satisfà quatre condicions importants: la primera és que γ^* és solució respecte la variable $t \in \mathbb{R}$, és analítica respecte s i que

$$\gamma^*(t + 2\pi\varepsilon, s) = \gamma^*(t, s + 2\pi\varepsilon).$$

D'aquesta manera dotem a la variable s d'un caràcter dinàmic, ja que, per exemple, la varietat estable d'una aplicació de Poincaré és doncs, $\{\gamma^*(0,s)\}$, i la dinàmica de l'aplicació de Poincaré sobre ella és simplement

$$P(\gamma^*(0,s)) = \gamma^*(0, s + 2\pi\varepsilon).$$

La darrera condició que satisfarà γ^* és que és del tipus

$$\gamma^*(t,s) - \gamma_0(t+s) = \mu\varepsilon^{p+1}\sigma^*(t,s).$$

Per trobar aquesta parametrització hem buscat una equació de punt fix per σ imposant la condició d'invariància

$$P^t(\gamma^*(t,s)) = \gamma^*(t + 2\pi\varepsilon, s)$$

on $P^t(x,y) = \varphi(t + 2\pi\varepsilon, t, x, y)$ essent φ el flux del sistema hamiltonià pertorbat i la condició

$$\gamma^*(t + 2\pi\varepsilon, s) = \gamma^*(t, s + 2\pi\varepsilon).$$

Seguidament, en el cinquè capítol, veiem que la varietat estable local del sistema hamiltonià es pot expressar com el graf d'una funció analítica. En el cas en el que la pertorbació conserva el caràcter totalment parabòlic, hem utilitzat el resultat de [30] el qual es pot aplicar gairebé directament en el nostre cas. En el cas feblement hiperbòlic, sabem que les varietats es poden escriure com el graf d'una funció analítica en un entorn de l'origen, però, lògicament, aquest entorn dependrà, a priori, dels paràmetres μ i ε, ja que els valors propis de la diferencial en depenen. Per tant, en el cas que hem anomenat feblement hiperbòlic, hem generalitzat el resultat de [30] a un entorn de l'origen que no depèn del paràmetres de pertorbació.

En el sisé capítol, hem construit coordenades de caixa de flux , és a dir coordenades en les quals el flux redreça. Aquestes coordenades estan definides en un entorn de la varietat estable que no conté el origen, però sí que és proper a ell. Les hem construit, seguint varios passos:

- Hem parametritzat les solucions del sistema perturbat amb dos paràmetres. Un és el temps i un altre és un paràmetre complex, s, de tal manera que les solucions compleixen que són analítiques respecte s i la dinàmica de l'aplicació de Poincaré és simplement un desplaçament: $s \mapsto s + 2\pi\varepsilon$. Aquest darrer fet, ens està dient, que el paràmetre s és un paràmetre dinàmic. Aquesta és potser la part més important de tot el capítol. La demostració utilitza les eines que van ser introduides per Lazutkin en [53] i han estat utilitzades també en [41].

- Després demostrem, gràcies a aquesta bona parametrització, que les solucions $w(t+s, t/\varepsilon)$ tallen a una secció (real) transversal al flux, per a certs valors (t_0, s_0), i que per tant, podem redreçar el flux en un entorn de la varietat estable. És important remarcar que aquesta entorn de la varietat estable no depèn de ε i μ. A més veiem que aquest canvi, que depèn del temps, és analític respecte x, y i $2\pi\varepsilon$ periòdic en t.

- Degut a que aquest canvi no és canònic en general, demostrem que, si el sistema és Hamiltonià, donat un canvi tal que

  $$\dot{S} = 1 \quad \text{i} \quad \dot{E} = 0$$

  podem definir un canvi de variables canònic que també redreça el flux.

En el seté capítol, si la condició $p - \ell \geq 0$ és satisfeta extenem la varietat inestable fins el domini on les variables de caixa de flux estan definides. La demostració és la mateixa que en [20]: trobar una bona aproximació de les varietats invariants i després demostrar l'existència de solucions en el domini que volem mitjançant un métode iteratiu o una aplicació del teorema del punt fix.

Per últim en el darrer capítol d'aquesta primera part es demostra que l'àrea dels lòbuls engendrats per les varietats invariants entre dos punts homoclínics consecutius és exponencialment petita. L'esquema de la demostració és el mateix que en [20], amb la salvetat que nosaltres hem considerat també òrbites homoclíniques amb singularitats de tipus branca, és a dir "pols d'ordre racional" i per tant, els càlculs són una mica més farragosos.

Part II: Varietats invariants

És de tots conegut que les varietats invariants associades a objectes invariants (un punt fix, una òrbita periòdica, etc) d'un sistema dinàmic, ens donen una informació essencial per l'anàlisi de la estructura dinàmica del sistema. Quan l'objecte invariant té alguna mena d'hiperbolicitat, hi ha resultats satisfactoris que fan referència a l'existència, regularitat i unicitat de varietats invariants en dimensions arbitràries. Veure, per exemple [50] [51] [58] [28].

Presentació del problema, hipòtesis

En la segona part d'aquesta memòria, hem considerem aplicacions de la forma

$$\begin{aligned} x &\mapsto x + p(x,y) + f(x,y) \\ y &\mapsto y + q(x,y) + g(x,y) \end{aligned}$$

on $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, p i q són polinomis homogenis de grau N_p i N_q respectivament amb N_p i N_q més gran que 2. Les funcions f i g són $o(\|(x,y)\|^{N_p})$ i $o(\|(x,y)\|^{N_q})$ respectivament.

Demanem que existeixi $V \subset \mathbb{R}^n$ un entorn de l'origen invariant per l'aplicació

$$x \mapsto x + p(x,0) + f(x,0).$$

És a dir que si $x \in V$, llavors $x + p(x,0) + f(x,0)$ també pertanyi a aquest conjunt. Concretament, una condició suficient és que

- $\forall x \in V$, $\mathrm{dist}(x + p(x,0), V(r)^c) \geq A\|x\|^{N_p}$ on $A \neq 0$.

Degut a que volem trobar una varietat estable de la forma

$$W^s_{loc} = \{(x, y) \in V \times \mathbb{R}^m : y = \varphi(x)\}$$

sembla lógic demanar que l'aplicació "contraigui" en la direcció de les x i "expandeixi" en la direcció de les y. Més concretament, demanem que

- Les matrius

  $$-Dp(x, 0), \qquad Dq(x, 0)$$

  tenen els termes diagonals positius i són estrictament diagonal dominant. A més, $D_x q(x, 0) = 0$ en un entorn de l'origen.

És relativament senzill veure que, si ens restringim a dominis de la forma $\|y\| \leq \beta\|x\|$, llavors, ja que f i g són termes d'ordre més gran que p i q respectivament

$$\|x + p(x, y) + f(x, y)\| < \|x\| \quad \text{i} \quad \|y + q(x, y) + g(x, y)\| > \|y\|.$$

Aquestes condicions són una generalització a dimensions grans de les donades en [63]. Podríem dir que ens generen una hiperbolicitat dèbil.

Hem demostrat la existència i unicitat de varietats invariants a l'origen donades com a graf d'una funció

$$\varphi : V \subset \mathbb{R}^n \to \mathbb{R}^m.$$

Hem considerat els casos en que l'aplicació és Lipschitz o analítica i veiem que φ és també Lipschitz o analítica respectivament. No considerem el cas diferenciable.

La demostració és una generalització a dimensió arbritària de les tècniques utilitzades per McGehee en [63].

Conclusions

Sobre el trencament de separatrius

Per a probar el teorema 1.2.1, hem seguit l'esquema proposat en [20]. Però hem substituit tots els arguments que involucren el fet de que el punt fix és hiperbòlic en [20], per nous arguments vàlids en el cas de que el punt fix sigui parabòlic o un punt fix el qual bifurca de parabòlic a hiperbòlic quan $\varepsilon = 0$.

Hem donat condicions suficients per a l'existència de varietats invariants en aquests cassos, representades com a grafs de funcions definides en dominis complexos independents dels paràmetres.

També hem donat parametritzacions d'aquests varietats en funció de dos paràmetres (t, s) amb bones propietats. Això inclou cassos on l'ordre respecte les variables espaials de la perturbació és més petit que l'ordre (respecte les mateixes variables) del sistema no pertorbat (hipòtesis **HP4**).

Pels sistemes que hem considerat, hem construit variables de caixa de flux (o variables temps-energia) en certs dominis complexos independents dels paràmetres. Per construir aquestes noves variables, només demanem que la perturbació sigui contínua en el temps.

Nosaltres creiem que aquestes eines, o millor dit les idees per probar-les, podríen ser útils per resoldre altres problemes.

Per exemple, donar una proba unificada de la mesura del trencament de separatrius independentment del caràcter del punt fix.

Per el moment, la nostra demostració de l'existència de coordenades de caixa de flux i el capítul destinat al càlcul efectiu de l'àrea dels lóbuls generats per les varietats estable e inestable, ja són vàlides pel cas hiperbòlic.

Passem ara a comentar algunes possibles millores d'algunes de les hipòtesis asumides en el teorema 1.2.1.

- Nosaltres hem treballat amb potencials polinòmics. D'aquesta manera ens asseguren que n'hi ha singularitats de l'òrbita homoclínica que són pols o bé singularitats tipus branca ("branching points"). Però, de fet, només necessitem que la singularitat més petita sigui d'aquests tipus. Així, treballem amb potencials i pertorbacions polinòmiques (respecte les variables x, y). El cas analític, no ha estat estudiat en aquesta memòria i seria interesant estudiar exemples d'aquest tipus en el futur.
- Respecte al tamany de la pertorbació, existeixen evidències numèriques per a pensar que la hipòtesi **HP5** ($p \geq \ell$, en particular $p \geq 0$) no és óptima. Pensem que aquesta és la hipòtesi més difícil de millorar.

Creiem que per tractar aquests dos cassos, seria necessari desenvolupar noves tècniques, sobretot en ordre a demostrar el lema d'extensió.

- Sobre la pertorbació h_1 també impossem que sigui d'ordre k en les variables x, y amb k satisfent la condició: $2k - 2 \geq n$ (recordem que n és el ordre del potencial). Pensem que aquesta hipòtesi és més tècnica que necessària. Observem que aquesta hipòtesi permet cassos de bifurcació, però, per dir-ho d'alguna manera, no massa degenerats.
- Creeim també que, usant aproximacions d'ordres més alts de la funció de "splitting", podríem tractar els cassos en els que $J^{\pm}_{1,0} = 0$, (la hipòtesis **HP7** demana $J^{\pm}_{1,0} \neq 0$)

Un problema que trobem interesant i que podria ser un proper pas al treball realitzat en aquesta memòria, és estudiar el trencament de separatrius per aplicacions amb punts fixos parabòlics.

També ens sembla interesant generalitzar les eines introduïdes en aquesta part de la memòria a dimensions altes, especialment la construcció de variables de caixa de flux en algun entorn de la varietat estable.

Sobre l'existència de varietats invariants

Hem generalitzat els resultats de varios articles, començant amb [63], a dimensions altes. Una qüestió que hem hagut de resoldre és trobar un bon conjunt de hipòtesis amb les que començar.

Voldríem mencionar que hem hagut de substituir arguments que en el cas unidimensional són pràcticament immediats, per arguments que involucren teoria del grau en el cas Lipschitz i una versió multidimensional del teorema de Rouché en el cas analític.

El primer que ens agradaria fer és probar l'optimalitat (o no) de les hipòtesis del teorema 9.4.1.

També pensem que seria interessant trobar més exemples (per exemple en mecànica celest, o en altres camps) en els que aplicar el nostre teorema.

Un problema relacionat que podria ser estudiat amb tècniques similars és l'existència de varietats invariants per a aplicacions n-dimensionals amb punts fixos parabòlics amb diferencial de la forma

$$\begin{pmatrix} J_1 & 0 \\ 0 & J_2 \end{pmatrix}$$

on J_1 i J_2 són matrius de dimensions arbitràries i tenen la forma

$$J_1 = \mathrm{Id} + N, \qquad J_2 = \mathrm{Id}$$

i N és una matriu nilpotent. Per el moment, no coneixem exemples motivadors d'aquest problema.

Agraïments

Abans de continuar amb aquesta memòria voldria recordar a les persones sense les quals aquest treball no hagués estat possible.

M'agradaria començar agraint-li al Dr. Ernest Fontich la paciència que sempre ha tingut amb mi, tant a nivell personal com a nivell científic. Sense els seus coneixements i sense les seves valuoses aportacions aquest treball no hagués estat mai possible. Li agraeixo especialment que sempre estigués disposat a respondre els meus dubtes.

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També voldria agrair-li a Pau Martín el seu interés pel meu treball i els seus valuosos suggeriments.

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Contents

Introduction

In general, when beginning to explore any scientific field, one focuses on the generic situations; that is, one centers on the behaviours that appear in "most" of the cases encountered in practice.

This methodology allows an easier understanding of the problem, since the non-generic (or degenerate) cases are left out (at least a priori) in a first approach. This way, the casuistic is simpler and the general theory can be developed more easily.

Although this is a good scientific procedure, the aim of science is to explain reality in the most complete way possible. So, when the general case has been already described (perhaps not completely, but at least in a good part), one should study the non-generic cases: the exceptions. It should not be forgotten that, in nature, not all the processes follow a general rule. The exceptional cases often provide new types of behaviour. Therefore, a lot can be learned from the exceptions, as much at an intrinsic level (situations that differ from the general qualitative behaviour) as for the new techniques that are developed in order to understand them.

In certain contexts, it is generic to encounter degenerate cases. Let us think, for instance, about the case of parametric families, f_μ, which describe different behaviours depending on the value of μ. In this situation, it is generic (that is, it occurs in most of the families) to find values of the parameter μ_0 for which the behaviour of f_{μ_0} is degenerate.

Many of the natural processes involving movement can be formulated in terms of a dynamical system, be it continuous: as a differential equation,

$$\frac{dx}{dt} = X(x)$$

or discrete, in terms of a function:

$$x \mapsto f(x).$$

Of course, in the study of dynamical systems many degenerations appear as well. Let us show one of the most elementary, in fact possibly the simplest one, that can arise. Assume that we have a continuous dynamical system in $\mathbb{R}^n$ with a fixed point x_0:

$$\frac{dx}{dt} = X(x), \qquad x \in \mathbb{R}^n, \qquad X(x_0) = 0. \tag{1}$$

Applying Taylor's theorem around the fixed point we get:

$$\frac{dx}{dt} = DX(x_0)(x - x_0) + O(\|x - x_0\|^2).$$

Therefore, close enough to x_0, it appears that the dominant part of this dynamical system is given by its linear part. In fact, this holds given that the eigenvalues of $DX(x_0)$ have nonzero real part. In such case, Hartman's theorem ensures the existence of a bijective bicontinuous function which maps solutions of the initial system (1) in a neighbourhood of x_0, to solutions of the linear system

$$\frac{dx}{dt} = DX(x_0)(x - x_0)$$

which has explicit solutions given by:

$$x(t) = x_0 + e^{DX(x_0)t}(x^0 - x_0) \qquad x(0) = x^0.$$

One says that system (1) and the linear system above are topologically conjugate. The fixed points of a dynamical system such that the differential of the field at the fixed point has no eigenvalues with zero real part are called hyperbolic points.

The degeneration that we deal with concerns the eigenvalues of the linear part of the dynamical system around a fixed point (or a periodic orbit). This arises when $DX(x_0)$ has some eigenvalue with real part equal to zero. Assume, for instance, that $DX(x_0)$ has 0 as an eigenvalue; then, after a change of variables if necessary, writing $x = (x_1, x_2, \ldots, x_n)$, around the fixed point x_0 the dynamics of the variable x_1 is given by:

$$\frac{dx_1}{dt} = O(\|x - x_0\|^2).$$

Therefore, the information about the dynamics of the variable x_1 is given by the quadratic (or maybe of higher order) terms, that is, the linear approximation is not valid in such cases. The fixed points x_0, such that 0 is an eigenvalue of $DX(x_0)$, are called parabolic points (or partially parabolic points if there exists any hyperbolic direction). Analogously, if we consider discrete dynamical systems:

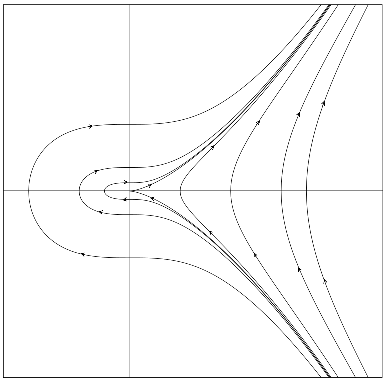
$$x \mapsto f(x)$$

we will encounter the same situation if the differential of the map f evaluated at the fixed point has 1 or -1 as eigenvalues.

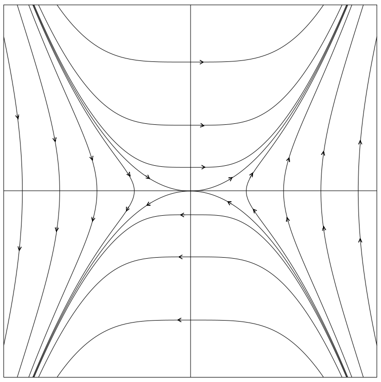
Then, it is clear that the classification of systems with non-hyperbolic fixed points does not depend on the linear part of the differential of the field at the fixed point, but on the first non-linear terms of the field. Thus, two systems with the same linear part can have very different behaviours. For instance, with linear part equal to:

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

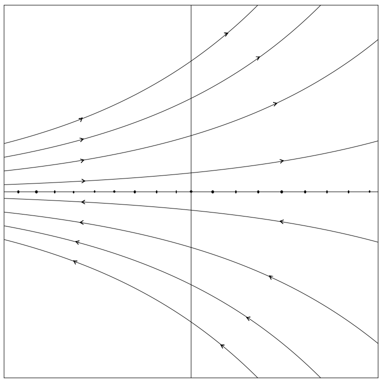
we may encounter the following behaviours:



(a) $\begin{cases} \dot{x} = y \\ \dot{y} = x^2 \end{cases}$



(b) $\begin{cases} \dot{x} = y \\ \dot{y} = x^3 \end{cases}$

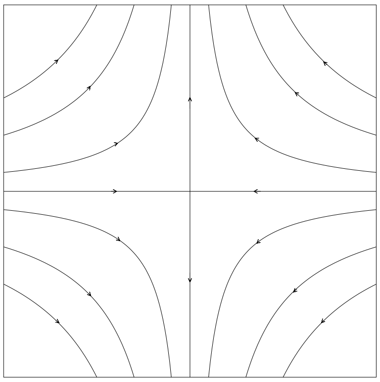


(c) $\begin{cases} \dot{x} = y \\ \dot{y} = y^2 \end{cases}$

While, for example, given a dynamical system with a fixed point such with linear part equal to:

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

with $\lambda_1\lambda_2 < 0$ and real, the phase portrait around the fixed point is, qualitatively, always the same:



independently of the values of λ_1 and λ_2 and of the higher order terms defining the system.

Central manifold

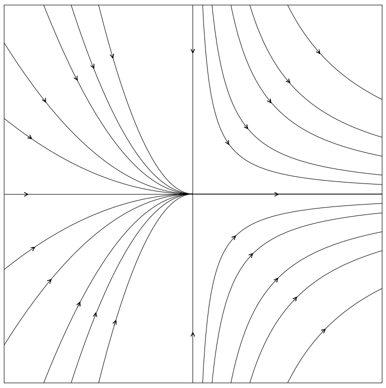
As we mentioned already, a continuous dynamical system with a hyperbolic fixed point can be transformed (near the fixed point) into a linear system. When the fixed point is not hyperbolic, there is a similar result, perhaps not so satisfactory, that allows to study the dynamical system around the fixed point.

Consider a field X such that the origin is a non-hyperbolic fixed point. Let E^c be the linear subspace generated by the eigenspaces of eigenvalues with real part equal to 0 and let E^u and E^s be the linear subspaces generated by the eigenspaces of eigenvalues with positive and negative real part, respectively. Then, it is known that there exist manifolds W^s_{loc} (stable invariant manifold), W^u_{loc} (unstable invariant manifold) and W^c_{loc} (central invariant manifold), invariant under the flow generated by the field X and tangent to the spaces E^s, E^u and E^c respectively at the fixed point.

Recall that, even though it can be shown that the stable and unstable manifolds are unique, in general the central manifold is not. Consider for instance the phase portrait of the system:

$$\begin{aligned}\dot{x} &= x^2 \\ \dot{y} &= -y\end{aligned}$$

which, near the fixed point, is



In this case, E^c is the subspace generated by the vector $(1, 0)$ and, therefore, since all the solutions contained in the half-plane $x < 0$ are tangent to E^c, they are central manifold.

In any case, fixing any of the central manifolds, the field X can be topologically conjugated around the fixed point to a field of the form:

$$\begin{aligned}\dot{x}_c &= X_c(x_c) \\ \dot{x}_u &= x_u \\ \dot{x}_s &= -x_s\end{aligned}$$

where $x = (x_c, x_u, x_s)$ and $X_c = X_{|W^c_{loc}}$. Hence, the study around the fixed point can be restricted to the local central manifold W^c_{loc}, since in the remaining directions the behaviour is well known. Thus, we have reduced the dimension of the problem.

It is also a known fact that W^u_{loc} and W^s_{loc} have the same degree of differentiability than the dynamical system. This does not hold in the case of the central manifold, whose differentiability can vary depending on the definition domain chosen. See [46], [13] and [86] for more details.

Let us go back for a moment to examples (a), (b) and (c). Notice that, in all cases, the central manifold is all of $\mathbb{R}^2$, since $E^c = \mathbb{R}^2$, but in cases (a) and (b) there are orbits tending to the origin when $t \to +\infty$ and when $t \to -\infty$. Therefore, in some instances of non-hyperbolic points one can naturally define the local stable and unstable manifold relative to $U \subset W^c_{loc}$ (inside the central manifold):

$$W^s_{loc}(x_0) = \{x \in U : \varphi(t,x) \in U \quad \forall t \geq 0 \text{ i } \varphi(t,x) \to x_0 \text{ quan } t \to +\infty\}$$
$$W^u_{loc}(x_0) = \{x \in U : \varphi(t,x) \in U \quad \forall t \leq 0 \text{ i } \varphi(t,x) \to x_0 \text{ quan } t \to -\infty\}$$

where x_0 is a fixed point of the system and $\varphi(t,x)$ is the solution of $\dot{x} = X(x)$.

The problem of deciding whether a parabolic fixed point of a field or a map has associated stable and unstable manifolds (inside the central manifold), has not been solved in general, but there are some existence and uniqueness results for these manifolds. For instance, for 2-dimensional maps having the identity as differential at the fixed point, one can mention the works of McGehee, Easton and Robinson in [63], [70], [26], [62] and [14]. Also, in the last chapter of this memoir we give necessary and sufficient conditions for the existence and uniqueness of invariant stable manifold for maps with identity differential in arbitrary dimension. For maps in two dimensions with differential at the fixed point equal to:

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \tag{2}$$

let us point out the results of J.Casasayas, E. Fontich and A. Nunes in [8] and E.Fontich in [30]. Examples of these last class of maps are the second order difference equations of the form:

$$y_{k+1} - 2y_k + y_{k-1} = f(y_k),$$

where $f(0) = f'(0) = 0$, if the recurrence above is written as

$$\begin{pmatrix} y_k \\ y_{k+1} \end{pmatrix} = F \begin{pmatrix} y_{k-1} \\ y_k \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y_{k-1} \\ y_k \end{pmatrix} + \begin{pmatrix} 0 \\ f(y_k) \end{pmatrix}$$

where it is clear that $DF(0,0)$ is similar to the matrix (2). Therefore, by means of a linear variable change we are in the current case. The generalized standard map also belongs to this case:

$$F(x,y) = (x + y + \varepsilon V(x), y + \varepsilon V(x))$$

with V a periodic function such that $V(0) = V'(0) = 0$.

The most standard way of finding the local central, stable and unstable manifolds associated with a fixed point is to describe them as graphs of maps:

$$W^*_{loc} = \text{graph}\,\varphi^* = \{(x,y) : y = \varphi^*(x)\} \qquad\qquad * = s, u, c.$$

It is also interesting to know the type of regularity of φ^*. Assume that the origin is a non-hyperbolic fixed point, then it is a remarkable fact that, in the analytic case, the local invariant manifolds are not analytic at the origin in general, while in the hyperbolic case the manifolds have the same regularity than the field. For instance, let us take into account a very simple case. Consider the Hamiltonian system associated with:

$$H(x,y) = \frac{y^2}{2} + V(x)$$

with $V(x) = x^3 + O(x^4)$. Then, it is clear that $y = x^{3/2}\sqrt{1+O(x)}$ is the local unstable manifold and that $y = -x^{3/2}\sqrt{1+O(x)}$ it the local stable manifold. Moreover, neither of them is analytic at the origin. In any case, what can be shown under certain conditions is that the local stable and unstable manifolds of a parabolic fixed point are analytic in a complex region of the form

$$\Omega = \{x \in \mathbb{C} : 0 < |x| < r \quad \text{i} \quad |\arg(x)| < \eta\}$$

with r and η positive.

Finally, let us mention that all the existence and uniqueness results for invariant manifolds associated with parabolic fixed points, reasonably impose restrictions over the nonlinear part of the map, which makes checking the hypothesis difficult.

Examples with degenerate fixed points

Although, as we already mentioned, the hyperbolicity condition of an invariant set: a fixed point, a periodic orbit, etc., of a dynamical system, is a generic condition, (nonzero real part of the eigenvalues of the differential in the case of flows, or module different from 1 in the case of a diffeomorphism), some interesting phenomena cannot be expressed in this context.

Consider a very simple example, the case of families of differentiable fields X_μ depending on a parameter μ. The family X_μ is said to have a bifurcation at $\mu = 0$ if and only if, for any neighbourhood V of $\mu = 0$ in the space parameters, there exist μ_1 and μ_2, different values of the parameter belonging to V, such that the equations associated with the fields X_{μ_1} and X_{μ_2} exhibit different qualitative behaviours. Take into account the simplest local bifurcations, that is, those in which the character and number of the fixed points changes. Let p_0 be a fixed point of X_0. It is well known that if we perturb a vector field with a hyperbolic fixed point, the perturbed system continues to have a hyperbolic fixed point. Hence, a necessary condition to obtain such bifurcations is that p_0 be non-hyperbolic, the simplest cases being: saddle-node ($\dot{x} = \mu - x^2$), transcritical

($\dot{x} = \mu x - x^3$), pitchfork ($\dot{x} = \mu - x^3$), etcetera. In all of them, for $\mu = 0$, the origin is a parabolic fixed point. Therefore, a good knowledge of the dynamics of a system around a non-hyperbolic fixed point will help us understanding the transition between the behaviours for $\mu < 0$ and $\mu > 0$. In the same way, in the case of maps, the local bifurcations also appear at non-hyperbolic fixed points.

Before describing other phenomena involving dynamical systems with non-hyperbolic invariant objects, let us briefly introduce the notion of integrable system. Assume that we have a differential equation

$$\dot{x} = f(x)$$

describing the evolution of a system in $\mathbb{R}^n$. A function $F : U \subset \mathbb{R}^n \to \mathbb{R}$ is called a first integral of the system if F is constant along the solutions of the system (that is, $F(x(t, x_0)) = c$, where $x(t, x_0)$ is the solution with $x(0, x_0) = x_0$). Assume now that we have $n - 1$ first integrals functionally independent,

$$F_1(x), \dots , F_{n-1}(x)$$

then a solution $x(t, x_0)$ of the system $\dot{x} = f(x)$ can be totally described as the curve intersection of the hypersurfaces

$$\begin{aligned} F_1(x) &= F_1(x_0) \\ &\vdots \\ F_{n-1}(x) &= F_{n-1}(x_0) \end{aligned}$$

except for the parametrization respect of the time. In this case we say that the system is integrable.

A very important class of systems, since in fact a lot of the mechanical phenomena follow systems of this kind, are the so-called Hamiltonian systems. A system is said to be Hamiltonian if there exists a function, called the Hamiltonian of the system, $H : U \subset \mathbb{R}^{2m} \to \mathbb{R}$ such that

$$\begin{aligned} \dot{x} &= \partial_y H(x, y) \\ \dot{y} &= -\partial_x H(x, y). \end{aligned}$$

It is clear that H is a first integral of the system. The Hamiltonian system is said to have m degrees of freedom.

The standard symplectic two-form endows $U \subset \mathbb{R}^{2m}$ with a symplectic structure. This structure allows the notion of integrable Hamiltonian system to be reduced to the existence of m first integrals $F_1 = H, F_2, \ldots, F_m$ (the Hamiltonian is one of them) which are pairwise in involution, that is:

$$\{F_i, F_j\} = \partial_x F_i \partial_y F_j - \partial_y F_i \partial_x F_j = 0.$$

and whose differentials are linearly independent on a dense open subset.

In what follows we describe some phenomena involving dynamical systems with parabolic objects.

The first one is that of parabolic resonances. A parabolic resonance is produced when an integrable Hamiltonian system with 2 degrees of freedom with a fixed point circle is perturbed. In [71] V.Rom -Kedar proves that this quality is generic for 1-parametric families (a codimension 1 phenomena) of Hamiltonians with 2 degrees of freedom near-integrable, that is, systems that are small perturbations of integrable Hamiltonian systems. Numerical experiments suggest that the movement near to parabolic resonances exhibits a new kind of chaotic behaviour not detected so far. There is an even more degenerate case, called planar parabolic resonance, which arises in a model coming from a real atmospheric study, specifically the study of meteorological probes. This model gives a device to transport particles with small initial speeds near to the Equator to high latitudes. See [71] and its reference list for more details.

The study carried in this last article, with respect to the parabolic resonances, is generalized to the case of Hamiltonians with n degrees of freedom with $n \geq 3$ in [56] and [57].

In [48], Hanβmann deals with low dimension tori with zero normal frequency in Hamiltonian systems with n degrees of freedom. These tori are called normally parabolic. Hanβmann considers families of near-integrable Hamiltonian systems in a neighbourhood of normally parabolic invariant tori. Under certain transversality conditions a quasi-periodic center-saddle bifurcation takes place ($\lambda < 0$ there are no invariant tori, $\lambda = 0$ the torus is normally parabolic and $\lambda > 0$ the torus is of type saddle). The author proves the persistence of the center-saddle bifurcation and of the normally parabolic tori parametrized by "big" Cantor sets. Hanβmann applies these results to the dynamics of the rigid solid.

In the planar three–body problem a parabolic orbit is the trajectory of a particle going to infinity with zero speed, while the trajectories of the other two particles remain bounded for all positive times. An orbit of the planar three–body problem is called oscillatory if the upper limit (along time) of the separation between particles is infinite, but the lower limit is finite. Thus it seems clear that the oscillatory orbits come and go infinitely often going (somehow) to infinity. Hence a good way to look at this problem is to find solutions that are "homoclinic at infinity". Therefore it seems natural to take to infinity some invariant object. In the case of the planar three–body problem McGehee and Easton prove in [27] that the infinity set may be seen as a three–sphere foliated by periodic orbits. McGehee considers three problems: the restricted problem, the Sitnikov problem (see [78]), and the 1–dimensional three–body problem, and proves in [63] that (after certain changes of variable) the infinity may be reduced to a periodic orbit. Later, R. Martínez et altr prove in [62], among other things, that in the elliptic restricted three–body problem the infinity manifold is also foliated by periodic orbits.

An approach to the search for oscillatory orbits is to prove that these periodic orbits, which we recall represent the infinity in the original system, have transversally intersecting stable and unstable manifolds. This is not a sufficient condition (see [27]), but it seems to be necessary to prove the existence of oscillatory orbits. In the problems treated by McGehee in [63] and in the elliptic restricted problem ([62]) to find these homoclinic solutions implies the existence of oscillatory orbits, hence of chaos.

In all these examples, the periodic orbits associated to infinity that have been found are degenerate. More precisely, the differential of the Poincaré mapping associated to an above mentioned periodic orbit has the eigenvalue 1. In the problems considered by McGehee and by R. Martínez et al this differential is the identity, and in the planar three-body problem it has a hyperbolic part, i.e., it has the form

$$\begin{pmatrix} I & 0 \\ 0 & A \end{pmatrix}$$

and A a hyperbolic matrix. McGehee proved in [63], under conditions generating a weak version of hyperbolicity, that 2–dimensional mappings with a fixed point of parabolic type such that their differential at the fixed point is the identity have an associated stable manifold which may be expressed as the graph of a function. This function is Lipschitz if the mapping is Lipschitz, and analytic if the mapping is so. The smooth case remains open. This result was later on generalized in [26] (in the Lipschitz case) and [70] (in the C^∞ case) to flows of the form

$$\begin{aligned} \dot{x} &= p_k(x,y) + O_{k+1} \\ \dot{y} &= BY + q_k(x,y) + O_{k+1} \end{aligned}$$

where $x \in \mathbb{R}^2$, $y \in \mathbb{R}^3$ and p, q are homogeneous polynomials of degree $k \geq 2$. Consequently the unit time mapping of this differential equation has the form

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x + P_k(x,y) \\ Ay + Q_k(x,y) \end{pmatrix} + O_{k+1}$$

where P_k, Q_k are homogeneous polynomials of degree $k \geq 2$, and the eigenvalues of the matrix A have modulus $\neq 1$.

All this indicates that parabolic orbits form a smooth manifold. Robinson, Xia, Moeckel and R. Martínez in [70], [88], [66] and [62] respectively prove the existence of oscillatory orbits in some instances of the three–body problem.

The above phenomenon was our motivation to look for sufficient conditions for the existence of a stable invariant manifold of a mapping in $\mathbb{R}^n$ with a fixed point such that the differential of the mapping in it be the identity, so, predictably, we could find oscillatory orbits in problems with more than three bodies.

Besides this problem, the goal of this work is to prove an asymptotic formula to measure the splitting of separatrices for a class of Hamiltonian systems with one and a half degrees of freedom, rapidly forced, associated to parabolic fixed points. We consider Hamiltonian systems with a parabolic fixed point such that the differential of the system at it is

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

and we prove that the magnitude of the splitting is exponentially small with respect to the frequency of the perturbation.

In this work we have studied basically the two above mentioned problems. Consequently it is divided in two parts. In the first part we study the measure of the splitting of separatrices associated to parabolic fixed points, and in the second we obtain an existence theorem for invariant manifolds associated to a fixed point of a n–dimensional diffeomorphism such that the differential at the point is the identity.

Part I: Separatrix splitting

We proceed now to introduce the notion of separatrix of a Hamiltonian system. Let us recall that a continuous dynamical system in $\mathbb{R}^n$ is integrable when it has $n-1$ functionally independent first integrals. In the case of a Hamiltonian system, the Liouville–Arnold theorem (see [3]) tells us that a Hamiltonian system is integrable if there exist $m = n/2$ first integrals: $F_1 = H, F_2, \dots, F_m$ which are pairwise in involution

$$\{F_i, F_j\} = \partial_x F_i \partial_y F_j - \partial_y F_i \partial_x F_j = 0$$

and whose differentials are linearly independent on a dense open subset.

Let us suppose that we have an integrable Hamiltonian system. Its dynamics are well known: denote $F = (F_1, F_2, \dots, F_m)$ and $M_c = F^{-1}(c)$; M_c is an m–dimensional submanifold invariant under the flow of the system. Moreover, if M_c is connected and the fields

$$\begin{aligned} \dot{x} &= \partial_y F_i \\ \dot{y} &= -\partial_x F_i \end{aligned}$$

are complete (defined for all times), then if the manifold M_c is not compact it is homeomorphic to $\mathbb{T}^k \times \mathbb{R}^{m-k}$ and if M_c is compact it is homeomorphic to an m–dimensional torus. In any of the two cases, after a suitable change of variable the flow of the system is conjugate to a translation:

$$\varphi(t, x, y) = (x + t\omega \pmod{2\pi}, y + t\nu)$$

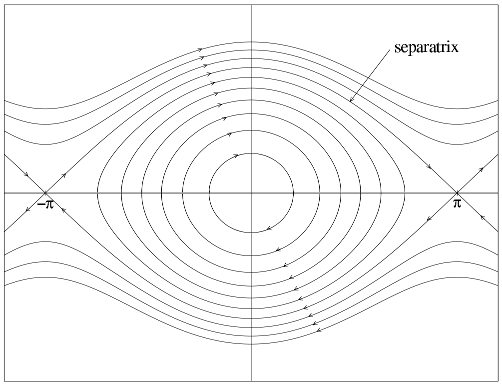
where ω and ν depend on c. Therefore an integrable Hamiltonian system is completely predictable (or regular): there is no stochastic (unpredictable) behavior.

In the context of integrable Hamiltonian systems, we focus our attention on those such that, for some constant c_0, the manifold M_{c_0} is formed by an invariant object P (a fixed point, a periodic orbit, a torus, etc.) and its stable and unstable invariant manifolds. We will assume that there exist coinciding branches of the stable and unstable manifolds, i.e. $W^u_+(P) = W^s_+(P)$. These manifolds are foliated by solutions of the Hamiltonian system converging on the invariant object when time goes to $+\infty$ if we are on the stable manifold, or to $-\infty$ if we are on the unstable one. This invariant submanifold is called the separatrix.

This name comes from the case of Hamiltonian systems with one degree of freedom with stable and unstable manifolds associated to fixed points. In this context, separatrices "split" different dynamic behaviors. For instance, let us think of the case of the pendulum, whose movement is governed by the equations:

$$\dot{x} = y \qquad \dot{y} = -\sin x \qquad x \in S^1, \qquad y \in \mathbb{R}.$$

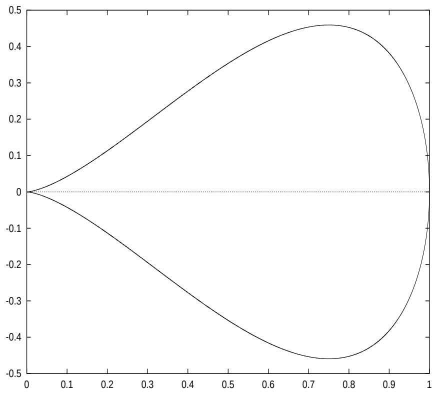
Its phase portrait is well known:



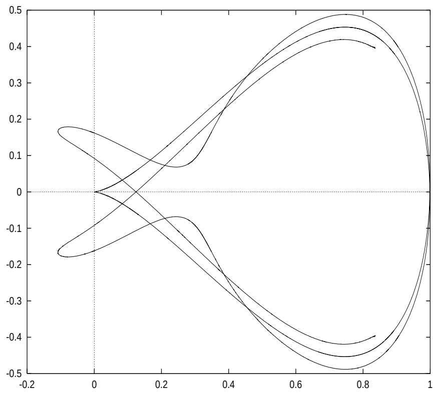
The static point $(0,0)$ is stable. A push (initial speed) of less than 2 to the static pendulum will make it oscillate around the equilibrium point (its trajectory a periodic orbit), and an initial speed strictly bigger than 2 will make the pendulum swing over the top of the stable region. If the initial speed is exactly 2, the solution converges to the unstable equilibrium point $(\pi, 0)$. The last solution is called a separatrix, and it clearly splits two very different movements.

Even though in higher dimensions separatrices, being half–dimensional submanifolds, do not split the space they receive the same name.

Let us consider then a Hamiltonian system with two invariant sets (fixed points, periodic orbits, etc.), with stable and unstable submanifolds. Suppose that $W^s_+(P_1) = W^u_+(P_2)$ and $W^u_+(P_1) = W^s_+(P_2)$, where $W^{s,u}_+(P_1)$ and $W^{s,u}_+(P_2)$ are branches of the submanifolds $W^{s,u}(P_1)$ and $W^{s,u}(P_2)$ respectively. We have either a so called heteroclinic connection (if $P_1 \neq P_2$), or a homoclinic one if $P_1 = P_2$. This situation is generic among Hamiltonian fields, but under a generic periodic perturbation this connection breaks giving rise to a phenomenon called separatrix splitting:



Unperturbed system



Perturbed system

The phenomenon of separatrix splitting was described by Poincaré, but it was Melnikov (with the ideas proposed by Poincaré) who produced a good analytic tool to measure it. This tool is known as the Poincaré-Melnikov method (see [46] and [1] for an introduction). This method yields a first order asymptotic formula to measure the splitting.

To fix ideas, let us suppose that we have the system

$$\dot{x} = f(x) + \mu g(t, x)$$

where $x \in \mathbb{R}^2$ and that for $\mu = 0$ the unperturbed system has a homoclinic connection associated to a fixed point of saddle type. This is the simplest case. If the perturbation is small enough, the perturbed system has a hyperbolic periodic orbit σ and therefore there exist stable and unstable manifolds for σ. The formula proposed by Melnikov for measuring the magnitude of this splitting $d(t_0)$ in a fixed time t_0 (where $d(t_0)$ is the distance between the invariant submanifolds of the perturbed system) is

$$d(t_0) = \mu M(t_0) + O(\mu^2)$$

where $M(t_0)$ is the Melnikov integral, which depends on the homoclinic connection and on g. We can assure the existence of homoclinic points (points belonging to both the stable and unstable manifolds of σ), and they are transversal if there exists any t_0 such that $M(t_0) = 0$ and $M'(t_0) \neq 0$. It is well known that the existence of transversal homoclinic points in dimension 2 gives rise to chaotic behaviors (Smale's homoclinic theorem).

Suppose now that the perturbation has a fast frequency, that is the period is εT with $\varepsilon > 0$ small. It can be proved then that the Melnikov integral has order $O(e^{-a/\varepsilon})$ for some constant $a > 0$. Consequently, in this case the Melnikov function is a priori a good measure of the splitting only if $\mu = o(e^{-a/\varepsilon})$, which gives us a very small range of values for μ and therefore for the perturbations under which we may still reach conclusions. In this case the Melnikov method (in its elementary form) is not, a priori, a good tool to measure separatrix splitting.

Lombardi ([61]) has recently given rigorous methods for studying what he calls oscillatory integrals, which are integrals of the type

$$I(\varepsilon) = \int_{-\infty}^{+\infty} e^{it/\varepsilon} g(x_0(t, \varepsilon))dt$$

where x_0 is a particular solution of the system $\dot{x} = F(x, t, \varepsilon)$ characterized, e.g., by its value at infinity if x_0 is the stable manifold of a periodic orbit or of a fixed point. This is the kind of integrals that appears when applying the Poincaré–Melnikov method, thus it may be very useful in the future. This work treats problems in which exponentially small phenomena in reversible systems occur. Consider for example a vector field in $\mathbb{R}^4$ in normal form, and truncate it at any order. The unperturbed system is the truncated system, and we consider as a perturbation the tail of the normal form. The persistence of homoclinic connections in periodic orbits of exponentially small size near a certain kind of resonance is proved in [61]. The resonance is due to the change of character of the fixed point. Two of the eigenvalues are of the form $\pm i\omega + O(\mu)$, and the other two λ_1, λ_2 are different and real for $\mu > 0$, $\lambda_1 = \lambda_2 = 0$ for $\mu = 0$, and imaginary for $\mu < 0$. Therefore we have obtained a model of perturbation in which a parabolic point coexists with an oscillatory part.

Many authors have studied the phenomenon of separatrix splitting with fast frequency periodic perturbations, in order to prove that in certain cases the Melnikov function yields a good approximation of the measure of separatrix splitting.

In [54] Lazutkin studies the standard mapping and reasons (without rigorously proving it) why the intersection angle between the stable and unstable manifolds has the form

$$\varphi = \frac{\pi}{\varepsilon}|\Theta_1|e^{-\pi^2/\sqrt{\varepsilon}}[1 + O(\varepsilon^r)]$$

with $r > 0$. The constant $\Theta_1 \in \mathbb{C}$ has to be numerically computed. In fact, given that it is the dominant term in the expression of the separation angle, the main goal is to prove that it is not zero. In [81] Suris proves that this coefficient is not zero for the standard mapping and proposes an alternative, although still numerical, way to compute this constant. Also, in [42] Gelfreich et al prove analytically that Θ_1 is not zero for the Henon map. In [55] this constant is numerically computed for the semistandard mapping and in [12] for the Henon map.

It is important to point out that [54] introduces pioneering new analytic tools for the study of separatrix splitting which have decisively influenced the development of this area. These techniques have been strongly used in this memoir (see [42] for an exposition of them).

In [34] and [33] E. Fontich and C. Simó study separatrix splitting in families in a neighbourhood of the identity of diffeomorphisms of class C^r and C^ω respectively. Under fairly general hypothesis upper bounds are obtained for the distance between invariant manifolds. More precisely, it is proved in [34] that the distance between invariant manifolds is bounded by $K\varepsilon^{r-1}$ in the C^r case, and in [33] an exponentially small upper bound of order $e^{-k/\varepsilon}$ for the splitting of invariant manifolds in the analytic case, plus generally optimal values of K. Of course, the advantage of working with diffeomorphisms is that one may obtain a similar result in the case of flows by means of the Poincaré mapping (see [31]). The drawback is that it is much harder to work with series (the discrete analogue of the Melnikov function) than with integrals.

As we said before, working with analytic diffeomorphisms is more advantageous than with flows. Nevertheless, many of the works concerning separatrix splitting deal with flows, which indicates that the problem of finding asymptotic expressions (or upper and lower bounds) for separatrix splitting is substantially harder in the case of mappings. In [69] R. Ramírez works with symplectic mappings and gives a systematical way to evaluate the Melnikov function (which in the case of mappings is an infinite series). Moreover an asymptotic formula is derived for the area span between the invariant manifolds for mappings that may be seen as perturbations of the McMillan mapping ([64]) and for billiards.

The separatrix splitting phenomena has been widely studied in the two-dimensional case with systems of the form

$$\dot{z} = f(z) + \mu\varepsilon^p g(x, t/\varepsilon, \varepsilon)$$

where μ and $\varepsilon > 0$ are parameters a priori independent and such that the origin is a saddle-type fixed point. Thus the perturbed system occurs at $\mu = 0$. There has been a lot of discussion about the optimal degree of p. In [31] upper bounds are given for the splitting for negative values of p, specifically $p > -1/2$. If the model is simplified, considering equations of the form

$$\ddot{x} + f(x) = \mu\varepsilon^p g(x, t/\varepsilon, \varepsilon)$$

then in [32] upper bounds are given for the splitting of separatrices for values of $p > -2$. In [49], Holmes et al are able to give upper and lower bounds for the splitting of separatrices for quite general systems and for values of $p > 8$. The situation improves when dealing with specific systems. The most studied example is the pendulum. In [45] and in [21], asymptotic expressions are given for the separatrix splitting of the equation

$$\ddot{x} + \sin x = \mu\varepsilon^p \sin t/\varepsilon$$

for $p > 5$ and $p > 0$ respectively. Later on, A. Delshams and M.T. Seara in [20], could get an asymptotic expression of the separatrix splitting for more general systems given that p is bigger than a certain quantity which depends on the perturbation and of the singularity order of the homoclinic orbit. Gelfreich in [43] also gives an asymptotic expression for the separatrix splitting, but it is difficult to find out which p is needed in order to apply it. Finally, in [44] Gelfreich studies in some specific examples the $p < 0$ case. The proposed method is the use of an auxiliary system whose invariant manifolds are a good approximation near the singularities of the invariant manifolds of the initial system.

In all these cases, one deals with Hamiltonian systems of one and a half degrees of freedom or area-preserving maps such that the origin is a hyperbolic fixed point of the non-perturbed Hamiltonian. Another situation where the separatrix splitting phenomenon appears is when one considers quasi-periodic perturbations. As, for instance, in [22], [23] and [36]. In this case the analysis becomes much more complicated than in the case of having a time periodical perturbation.

Another phenomenon related to the separatrix splitting is Arnold diffusion. If an integrable Hamiltonian system with m degrees of freedom $H_0(I)$ suffers a Hamiltonian perturbation, in general the perturbed system $H(I, \varphi) = H_0(I) + \varepsilon H_1(I, \varphi)$ is no longer integrable and non-predictable (i.e. chaotic) behaviors appear. Nevertheless, KAM theory assures that "many" of the top dimension invariant tori (m in this case) appearing in the non-perturbed system H_0 remain slightly deformed. The actions of the orbits that remain in these "holes", in which the KAM theorem does not guarantee the existence of invariant tori, could suffer an order 1 shift independently of the size of the perturbation. If so, then by Nekhoroshev's theorem:

$$I_\varepsilon(t) - I_\varepsilon(0) = O(1) \qquad \text{per a } 0 \leq t \leq \frac{1}{\varepsilon}\exp(1/\varepsilon^a),$$

it should be a very slow shift. The example proposed by Arnold, [2], shows that this situation can occur through chains of partially hyperbolic invariant tori with stable and unstable manifolds transversely intersecting each other. When considering analytic systems, one of the greatest difficulties is that the separatrix splitting is, in general, exponentially small. See [4] for a good introduction to this phenomenon.

Our aim is to give an asymptotic formula to measure the separatrix splitting in Hamiltonian systems of one and a half degrees of freedom such that the origin is a parabolic fixed point of the non-perturbed system. Specifically, the differential of the field at $(0, 0)$ is

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

We have followed basically the scheme of the proof in [20]. But, due to the fact that many of their arguments strongly use the hyperbolic character of the fixed point, we have had to introduce new techniques which are also valid in the parabolic case. To this end we have also used tools introduced by Lazutkin [54], [53] and used later by Gelfreich [43]. It is worth remarking that most of our arguments are also valid for the hyperbolic case.

Presentation of the problem, assumptions

We will work with Hamiltonian systems with one and a half degrees of freedom, with Hamiltonian of the form

$$H(x, y, t/\varepsilon) = \frac{y^2}{2} + V(x) + \mu\varepsilon^p h_1(x, y, t/\varepsilon, \varepsilon, \mu)$$

where

- we assume that the potential $V(x)$ is a polynomial of order at least 3, that is,

  $$V(x) = ax^n + O(x^{n+1}) \qquad \text{with } n \geq 3 \text{ and } a \neq 0.$$

  Therefore, the origin is a parabolic fixed point with differential

  $$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

For the perturbation h_1, the following conditions will be imposed:

- we assume that it is C^0, analytic in (x, y, μ) and that it has degree $k \geq 3$ in the variables (x, y), verifying the condition

  $$2k - 2 \geq n. \tag{3}$$

  Notice that the $k < n$ case (which is a bifurcation case) is permitted for $n \geq 4$. Moreover, we assume that h is $2\pi\varepsilon$-periodical and that it has zero average with respect to the variable $\theta = t/\varepsilon$:

  $$\int_0^{2\pi} h(x, y, \theta)d\theta = 0.$$

It is worth noticing that we only need that perturbation to be continuous with respect to t/ε.

We need now the assumptions concerning the non-perturbed system. Assume that the non-perturbed system ($\mu = 0$) has a homoclinic orbit, which we will call $\gamma_0 = (\alpha_0, \beta_0)$.

- The singularities of the second component of the homoclinic orbit β_0 with smallest module of their imaginary part are order $r \in \mathbb{Q}$ branch-type singularities at points of the form $\pm ia$.

With the standard methods used to measure the splitting of separatrices it is absolutely essential that the homoclinic orbit has some kind of singularity. In fact in the hyperbolic case, in [31] it is shown that there exists ρ, $0 < \rho < \rho_0 = \min\{\pi/\lambda_1, \pi/|\lambda_2|\}$ (where $\lambda_1 > 0$ and $\lambda_2 < 0$ are the eigenvalues at the fixed point) such that the homoclinic orbit is analytic in a complex strip of the form

$$D(\rho) = \{z \in \mathbb{C} : |\operatorname{Im} z| < \rho\}$$

and that it is not analytic in $D(\rho')$ if $\rho' > \rho > 0$. (Notice, however, that our assumption excludes the possibility of having essential singularities on the line $\operatorname{Im} z = \pm\rho$). In contrast, to find upper bounds it is enough that γ_0 be analytic in $D(\rho)$.

This last hypothesis cannot be obviated, that is, we believe that it is difficult to check from the Hamiltonian that it is fulfilled. To see this difficulty, assume that we have a simple case. Suppose that the potential $V(x)$ is negative for $x \in (0,1)$ and that it is a polynomial. Then, it is easily seen that there are always branch-type singularities (or poles if the potential has degree 4) at points of the form $\pm ia$. Indeed, (see [52] for more details) let $V(x) = a_n x^n + \ldots + a_m x^m$ with $n < m$. After the variable change $x = 1/w$, the solution of the non-perturbed system is

$$t(w) = \int_w^1 \frac{w^{m/2-2}}{\sqrt{2(a_n + \ldots + a_m w^{m-n})}} dw.$$

A singularity of the homoclinic orbit is attained at

$$t(0) = t_0 = -i \int_0^1 \frac{w^{m/2-2}}{\sqrt{-2(a_n + \ldots + a_m w^{m-n})}} dw \in i\mathbb{R}$$

since $a_n + \ldots + a_m w^{m-n} < 0$ if $w \in (0,1)$. Moreover

$$t'(w) = -w^{m/2-2}\left(\frac{1}{\sqrt{2a_n}} + O(w)\right)$$

and hence

$$t(w) = t_0 + w^{m/2-1}(c + O(w))$$

where $c \in \mathbb{C}$ is a constant. Isolating w in terms of $t(w) - t_0$ and undoing the change $w = 1/x$ we obtain that

$$x = \frac{c}{(t-t_0)^{2/(m-2)}}(1 + O(t-t_0)^{2/(m-2)})$$

near $t_0 \in i\mathbb{R}$.

But, even in this simplest case, it is really difficult checking that the singularities with smallest imaginary part are precisely of the form $\pm ia$. Take into account the fact that an exhaustive study becomes very complicated, since it involves studying the possible values of integrals of the form

$$\int_1^h \frac{1}{\sqrt{-2V(u)}}du$$

where $|h| = +\infty$ and giving conditions on the potential V so that the values with smallest imaginary part are purely imaginary. See [61] for a short discussion on this subject.

Having homoclinic orbits with branch-type singularities is not an exclusive fact of the parabolic case, hence we can extend the cases to which the result given in [20] for hyperbolic points can be applied.

As we mentioned above, the bigness or smallness of p is important. This is basically determined by the extension lemma (which extends the parametrization of the local manifolds to domains very close to the singularities of the homoclinic orbit).

We have also needed to make an assumption concerning r (the order of the singularity of the homoclinic orbit) and the order of the singularities $\pm ia$ with respect to the function $h_1(\gamma_0(t+s), t/\varepsilon)$, which, following the notation of [20], we have called ℓ.

- The condition we need is

$$p - \ell \geq 0.$$

We believe that it is not optimal. For instance, in the case of the perturbed pendulum, considered by Gelfreich in [43], with our hypothesis we would need $p \geq 0$ and Gelfreich works with values of $p < 0$.

We have also considered a particular instance with potential of order 3 and $k = 2$, which was not covered by the previous hypothesis. This is obviously a bifurcation case because when the parameter μ is nonzero, and under certain conditions on the perturbation, the origin is a hyperbolic fixed point, and when μ vanishes we run into the parabolic case. We have called this the weakly hyperbolic case. The hypothesis that we need in this case are the same as in the parabolic case. Concretely we have considered systems of the form

$$H(x, y, t/\varepsilon) = h_0(x, y) + \mu\varepsilon^p h_1(x, y, t/\varepsilon, \mu, \varepsilon)$$

where

$$h_0(x, y) = \frac{y^2}{2} - x^3 + x^4 \quad \text{i} \quad h_1(x, y, t/\varepsilon, \mu, \varepsilon) = h_{12}(x, y, t/\varepsilon, \mu, \varepsilon) + h_{13}(x, y, t/\varepsilon, \mu, \varepsilon)$$

with

$$h_{12}(x, y, t/\varepsilon, \mu, \varepsilon) = \frac{x^2}{2} g_1(t/\varepsilon, \mu, \varepsilon) + xy g_2(t/\varepsilon, \mu, \varepsilon) + \frac{y^2}{2} g_3(t/\varepsilon, \mu, \varepsilon)$$

and $h_{13}(x, y, t/\varepsilon, \mu, \varepsilon)$ has order greater or equal to 3 in the variables (x, y).

- In the weakly hyperbolic case, we replace the condition given by (3) by the following one

  $$\int_0^{2\pi} g_2(\theta, \mu, \varepsilon) G_1(\theta, \mu, \varepsilon) d\theta > 0.$$

  where G_1 is determined by the equation $\partial_\theta G_1 = g_1$ and the fact its average is zero.

  The rest of the hypotheses for the parabolic case are also assumed in this case.

The last one is the condition that is required for the origin to be a saddle point of the perturbed system. Recently [37] has considered this type of bifurcation in the case of area–preserving mappings. In [38] higher order resonances have been considered.

Even tough the goal is the study of separatrix splitting in Hamiltonian systems with parabolic points, all of our proofs are easily adapted for hyperbolic points as well, except in chapters three and four, where we find special parameterization of the invariant manifolds.

We have established an asymptotic expression for the area span by the invariant manifolds between two consecutive homoclinic points s_0 and $\bar{s}_0$ of the form

$$A = \mu\varepsilon^p \int_{s_0}^{\bar{s}_0} M(v, \varepsilon) dv + O(\varepsilon^b, \mu^c)$$

where b and c are explicitly determined positive constants, depending on r, p and ℓ. Recall that a is the modulus of the singularities $\pm ia$ of the homoclinic orbit.

We state now the main corollary. Let

$$J(x,y,\theta) = \{h_0, h_1\}(x,y,\theta) \sim \sum_{k\neq 0} J_k(x,y)e^{ik\theta}$$

which, when evaluated on the homoclinic orbit, has a singularity at $u = \pm ia$ of order at most $\ell + 1$. We observe that using expansions around the singularities $u = \pm ia$, $J_{\pm 1}(\gamma_0(u))$ has the form

$$J_{\pm 1}(\gamma_0(u)) = J^{\pm}_{\pm 1,0}\frac{1}{(u \mp ia)}(1 + h.o.t.).$$

If we assume that

- $J_{\pm 1}(\gamma_0(u))$ has a singularity of order exactly $\ell + 1$ at $u = \pm ia$, i.e. if $J^{+}_{1,0} \neq 0$, which is a generic condition, then

  $$A \sim \mu\varepsilon^{p-\ell+1} 8\pi |J^{+}_{1,0}| \frac{1}{\Gamma(\ell+1)} e^{-a/\varepsilon}$$

  where Γ is Euler's Gamma function.

General scheme of the first part

In the first chapter, we get the asymptotic behavior of the parametrization of the homoclinic orbit. We prove that, as was to be expected, this behavior is potential, that is, there exists $T > 0$ such that if $t \in \mathbb{C}$, $\mathrm{Re}\, t \geq T$, the stable manifold behaves like $1/t^q$ with q a certain positive number, and, analogously, the unstable manifold has the form $1/(-t)^q$ per $t \in \mathbb{C}$, $\mathrm{Re}\, t < -T$.

In the third and fourth chapters, we have found a good parametrization $\gamma^*(t,s)$ ($* = s,u$) of the invariant manifolds in the completely parabolic and weakly hyperbolic cases respectively. This parametrization satisfies four important conditions: first, γ^* is a solution with respect to the variable $t \in \mathbb{R}$, it is analytic with respect to s and

$$\gamma^*(t + 2\pi\varepsilon, s) = \gamma^*(t, s + 2\pi\varepsilon).$$

In this way we endow the variable s with a dynamic character since, for instance, the stable manifold of a Poincaré map is $\gamma^s(0,s)$ and the dynamics of this Poincaré map on it is simply

$$P(\gamma^s(0,s)) = \gamma^s(0, s + 2\pi\varepsilon).$$

The last condition satisfied by γ^* is that it is of the type

$$\gamma^*(t,s) = \gamma_0(t+s) + \mu\varepsilon^{p+1}\sigma(t,s).$$

In order to find this parameterization, we have sought a fixed point equation for σ by imposing the invariance condition

$$P^t(\gamma^*(t,s)) = \gamma^*(t + 2\pi\varepsilon, s)$$

where $P^t(x,y) = \varphi_\mu(t + 2\pi\varepsilon, t, x, y)$, with φ_μ being the flow of the perturbed Hamiltonian system, and

$$\gamma^*(t + 2\pi\varepsilon, s) = \gamma^*(t, s + 2\pi\varepsilon).$$

Afterwards, in the fifth chapter, we show that the local stable manifold of the Hamiltonian system may be expressed as the graph of an analytic function. In the case when the perturbation preserves the completely parabolic character we have used the result of [30], which may be applied almost directly to our case. In the weakly hyperbolic case we know that the manifolds may be described by the graph of an analytic function in a neighbourhood of the origin, but, of course, this neighbourhood depends a priori on the parameters μ and ε, as the eigenvalues of the differential do. Therefore, in the case that we have called weakly hyperbolic we have generalized the result of [30] to a neighbourhood of the origin that does not depend on the perturbation parameters.

In the sixth chapter, we have built the flow box coordinates, i.e., coordinates in which the flow straightens. These coordinates are defined in a neighbourhood of the stable manifold not containing the origin, but close to it. We have built them following several steps:

- We have parametrized the solutions of the system perturbed by two parameters. One of these is time, and the other is a complex parameter s such that the solutions are analytic with respect to s and the dynamics of the Poincaré mapping are simply $s + 2\pi\varepsilon$. This last fact tells us that s is a dynamic parameter. This is possibly the main part of the chapter. The proof uses the tools introduced by Lazutkin in [53] and used as well in [41].

- We prove afterwards, thanks to this good parametrization, that the solutions $w(t+s, t/\varepsilon)$ intersect a (real) section transverse to the flow for some value (t_0, s_0), thus we are able to straighten the flow in a neighbourhood of the stable manifold. It is important to point out that this neighbourhood of the stable manifold does not depend on ε or μ. Moreover, we show that this change, which depends on time, is analytic with respect to x, y and $2\pi\varepsilon$–periodic in t.

- Due to the fact that this change is not canonical in general, we prove that if the system is Hamiltonian, given a change such that

$$\dot{S} = 1 \qquad \text{and} \qquad \dot{E} = 0$$

  we may define a canonical change of variables which also straightens the flow.

In the seventh chapter, if the condition $p - \ell \geq 0$ is fulfilled we extend the unstable manifold to the domain where the variables of the flow box are defined. The proof is the same as in [20]: to find a good approximation of the invariant manifolds and then to establish the existence of solutions in the sought domain by an iterative method or a fixed point theorem.

Finally, in the last chapter of this first part it is proved that the area of the lobes generated by the invariant manifolds between two homoclinic points is exponentially small. The scheme of proof is the same as in [20] , except that we have also considered homoclinic orbits with branch type singularities, i.e. "poles of rational order" and consequently the computation is somewhat heavier.

Part II: Invariant manifolds

It is a truth universally acknowledged that the invariant manifolds associated to invariant objects (a fixed point, a periodic orbit, etc.) of a dynamical system yield essential information for the analysis of the dynamic structure of the system. When the invariant object satisfies some hyperbolicity property there are satisfactory results concerning the existence, regularity and uniqueness of invariant manifolds in arbitrary dimensions, see for instance [50],[51],[58],[28].

Presentation of the problem, assumptions

In the second part of this memory, we have considered mappings of the form

$$
\begin{aligned}
x &\mapsto x + p(x, y) + f(x, y) \\
y &\mapsto y + q(x, y) + g(x, y)
\end{aligned}
$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, p and q are homogeneous polynomials of degrees N_p, $N_q \geq 2$ respectively. The functions f and g have orders $o(\|(x, y)\|)^{N_p}$ and $o(\|(x, y)\|)^{N_q}$ respectively.

We require that there exist a neighbourhood of the origin $V \subset \mathbb{R}^n$ invariant under the map

$$
x \mapsto x + p(x, 0) + f(x, 0)
$$

That is, if $x \in V$ then $x+p(x, 0)+f(x, 0)$ also belongs to this set. A sufficient condition is

- $\forall x \in V$, $\operatorname{dist}(x + p(x, 0), V(r)^c) \geq A\|x\|^{N_p}$, where $A \neq 0$.

As we seek to find a stable manifold of the form

$$
W^s_{loc} = \{(x, y) \in V \times \mathbb{R}^m : y = \varphi(x)\}
$$

it seems natural to demand that the mapping be “contracting” in the directions of x, and “repulsive” in the directions of y. More precisely, we require that

- The matrices

  $$
  -Dp(x, 0), \qquad Dq(x, 0)
  $$

  have the terms in the diagonal positive and are strictly diagonal dominant. Moreover $D_x q(x, 0) = 0$ in a neighbourhood of the origin.

It is relatively simple to check that, if we restrict ourselves to domains of the form $\|y\| < \beta\|x\|$, then as f and g are terms of higher order than p and q respectively

$$
\|x + p(x, y) + f(x, y)\| < \|x\| \quad \text{ i } \quad \|y + q(x, y) + g(x, y)\| > \|y\|.
$$

These conditions are a generalization to higher dimension of those given in [63]. We could characterize them as inducing a weak hyperbolicity.

We have proved the existence and uniqueness of invariant manifolds at the origin given as the graph of a function

$$\varphi : V \subset \mathbb{R}^n \to \mathbb{R}^m.$$

We have considered the cases when the mapping is Lipschitz or analytic, and we show that φ is as well Lipschitz or analytic respectively. We do not consider the differentiable case.

The proof is a generalization to arbitrary dimension of the techniques employed by McGehee in [63].

Conclusions

On the splitting of separatrices

To prove Theorem 1.2.1 we have followed the scheme of proof proposed in [20]. However we substitute all arguments which involve the fact that the fixed point is hyperbolic in [20] by new arguments valid in the case of a parabolic fixed point or a point which bifurcates from parabolic to hyperbolic just at $\varepsilon = 0$.

We have given conditions for the existence of invariant manifolds in such cases, represented as graphs of functions defined in complex domains independent of the parameters.

We also have given parameterizations of these manifolds in terms of two parameters (t, s) with good properties. This includes cases where the order of the perturbation with respect to the space variables is less than the order (with respect to the same variables) of the unperturbed system (hypothesis **HP4**).

For the systems we are considering, we have constructed flow box (or time-energy) variables in certain complex domains independent on the parameters. We do it assuming that the perturbation is just continuous with respect to time.

We believe that these tools, or rather the ideas to prove them, will be useful to other problems.

For instance to provide an unified approach to the splitting of separatrices independently of the character of the fixed point.

At the moment, our proof of the existence of flow box coordinates and the chapter devoted to the effective calculation of the area of the lobes generated by the stable and the unstable manifold, already hold in the hyperbolic case.

Let us comment some possible improvements on some of the hypotheses assumed in Theorem 1.2.1.

- We have worked with polynomial potentials and perturbations (with respect to x and y). In this way we ensure us that all the singularities of the homoclinic orbit are poles or branching points. But, in fact we only need the singularity with smaller imaginary part to be a pole or a branching point. The case of analytical potentials or analytical perturbations in x, y variables has not been studied in this memoir and could be interesting to explore some examples of this type.
- There are numerical evidences which support to think that the hypothesis **HP5** ($p \geq \ell$, in particular $p \geq 0$) is not optimal. We think that to improve this hypothesis is an interesting problem.

In order to treat these cases new techniques would have to be developed, specially a new version of the Extension Theorem.

- On the perturbation h_1 we have imposed that it has order k in the variables x, y satisfying $2k - 2 \geq n$ (we recall that n is the order of the potential). We think that this hypothesis is more technical than necessary. This hypothesis allows some cases of bifurcation, but, roughly speaking, not too much degenerated.
- We also think that, approaching the splitting function to higger orders, we could deal with cases when $J^+_{1,0} = 0$ (the hypothesis **HP7** askes $J^+_{1,0} \neq 0$).

A problem we find interesting and could be investigated as a next step of this memoir is the problem of the splitting of separatrices for maps with a parabolic fixed point.

Another interesting problem is to generalize to higger dimensions the tools introduced in this part of the memoir, specially the construction of flow box coordinates in some neighbourhoods of the stable manifold.

On the existence of invariant manifolds

We have generalized the results of several papers starting with [63] to higger dimensions. One question we had to solve is to find a right set of hypotheses to start with.

We would like to mention that we have had to substitute some simple arguments in the one dimensional setting by degree theory for the Lipschitz case and a multidimensional version of Rouché's theorem for the analytic case.

With respect to Theorem 9.4.1, a first step in order to improve it could be to investigate the optimally (or not) of the hypotheses of the Theorem.

We also think that it would be interesting to find more examples (in the field of celestial mechanics, for instance or in other applied fields) such that our theorem applies.

A related problem that could be studied with quite similar techniques, is the existence of invariant manifolds for n-dimensional maps with parabolic fixed points, whose differential is of the form

$$\begin{pmatrix} J_1 & 0 \\ 0 & J_2 \end{pmatrix}$$

where the matrices J_1 and J_2 are of arbitrary dimension and they have the form

$$J_1 = \mathrm{Id} + N \quad \text{and} \quad J_2 = \mathrm{Id}$$

where N is a nilpotent matrix. However, at present, we do not know any motivating examples.

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Part I

Splitting of separatrices

Chapter 1

Notation and main results

The goal of this first Chapter is to present the main problem we consider, the hypotheses we assume, and the rigorous statement of the main results of the first part of this memoir.

Of course we have to begin by introducing the relevant notation to be able to write precise statements.

At the end we give some examples when the above mentioned results apply.

1.1 Notation and hypotheses

We study the splitting of separatrices in two cases which we call the parabolic case and the weak hyperbolic case. Next we describe the settings of these cases and the hypotheses we will need.

1.1.1 The parabolic case

We consider Hamiltonian systems of one and a half degrees of freedom with Hamiltonian

$$H(x, y, t/\varepsilon, \mu, \varepsilon) = h_0(x, y) + \mu\varepsilon^p h_1(x, y, t/\varepsilon, \mu, \varepsilon)$$

where

$$h_0(x, y) = \frac{y^2}{2} + V(x),$$

$V(x)$ is a polynomial of degree m and order n, that is

$$V(x) = -a_n x^n - \cdots - a_m x^m$$

with $3 \leq n < m$. With these assumptions, for the unperturbed system (i.e. the system when $\mu = 0$) the origin is a parabolic fixed point and the derivative of the corresponding the vector field at $(0,0)$ is

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

The differential equations associated to the system are

$$\left\{ \begin{array}{rl} \dot{x} = & y + \mu\varepsilon^p \partial_y h_1(x, y, \frac{t}{\varepsilon}, \mu, \varepsilon) \\ \dot{y} = & -V'(x) - \mu\varepsilon^p \partial_x h_1(x, y, \frac{t}{\varepsilon}, \mu, \varepsilon). \end{array} \right. \tag{1.1.1}$$

We will assume the following hypothesis related to the unperturbed system ($\mu = 0$). Note that the unperturbed system is autonomous and independent on ε.

Hypothesis for the unperturbed system

HP1. We suppose that for $\mu = 0$ there exists and homoclinic orbit. It is not restrictive to suppose that it is in the energy level equal to zero. Moreover we suppose that the coefficient of the first term of V, $a_n > 0$. We denote the time parameterization of the homoclinic orbit by

$$\gamma_0(u) = (\alpha_0(u), \beta_0(u))$$

with some chosen (fixed) initial condition $\gamma_0(0) = (x_0, y_0)$ on the homoclinic orbit.

We suppose that $\gamma_0(u)$ is analytic in a complex strip $|\operatorname{Im} u| < a$ with branching points at $u = \pm ia$, i.e., there exists $\rho > 0$ such that for $u \in \mathbb{C}$ satisfying

$$|u - ia| < \rho \qquad \arg(u - ia) \in \left(\frac{-3\pi}{2}, \frac{\pi}{2} \right)$$

$\gamma_0(u)$ can be expressed as

$$\alpha_0(u) = \frac{c_-}{(u - ia)^{p/q}}(1 + O(u - ia)^{1/q}), \quad \beta_0(u) = \frac{d_-}{(u - ia)^{1+p/q}}(1 + O(u - ia)^{1/q}).$$

and for $u \in \mathbb{C}$ such that

$$|u + ia| < \rho \qquad \arg(u + ia) \in \left(\frac{-\pi}{2}, \frac{3\pi}{2}\right)$$

$\gamma_0(u)$ can be expressed as

$$\alpha_0(u) = \frac{c_+}{(u + ia)^{p/q}}(1 + O(u - ia)^{1/q}), \;\; \beta_0(u) = \frac{d_+}{(u + ia)^{1+p/q}}(1 + O(u + ia)^{1/q}).$$

Moreover on $u = \pm ia$ there are not other singularities of γ_0. We define

$$r = 1 + \frac{p}{q} > 1.$$

Remark 1.1.1 *We observe that, if we assume that $\beta_0(u)$ has a "potential" branching point at $u = u^*$, then for u in a neighbourhood of u^* we have that*

$$\alpha_0(u) = \frac{C}{(u - u^*)^{2/(m-2)}}(1 + O(u - u^*)^{2/(m-2)}) \tag{1.1.2}$$

$$\beta_0(u) = -\frac{2}{(u - u^*)^{2/(m-2)}}(1 + O(u - u^*)^{2/(m-2)}). \tag{1.1.3}$$

Indeed, it is clear that α_0 is a solution of the equation

$$\dot{x} = \sqrt{-2V(x)}.$$

Performing the change $w = 1/x$ we obtain the equation

$$\frac{du}{dw} = \frac{w^{m/2-2}}{\sqrt{2(a_m + wa_{m-1} + \ldots + w^{m-n}a_n)}}$$

which, in a neighbourhood of $w = 0$, can be written as

$$\frac{du}{dw} = w^{m/2-2}(c_0 + O(w)).$$

Integrating this relation we obtain

$$u - u^* = w^{(m-2)/2}(c_1 + O(w)).$$

Inverting the last equation and going back to the variable x we obtain the claimed expressions (1.1.2) and (1.1.3).

As a consequence, the exponents of $u - u^*$ are rational numbers.

Hypotheses over the perturbation

HP2. The function $h_1(x, y, \theta, \mu, \varepsilon)$ is C^0, 2π-periodic in θ, has zero mean:

$$\int_0^{2\pi} h_1(x, y, \theta, \mu, \varepsilon)\, d\theta = 0$$

and it is real analytic with respect to (x, y, μ).

HP3. The function $h_1(x, y, \theta, \mu, \varepsilon)$ is a polynomial of order k in the (x, y) variables. That is

$$h_1(x, y, \theta, \mu, \varepsilon) = \sum_{i+j=k}^{\kappa} a_{ij}(\theta, \mu, \varepsilon) x^i y^j.$$

HP4. If the order of the perturbation k is greater than 3, we assume that

$$2k - 2 \geq n \qquad \text{for } k \geq 3.$$

Remark 1.1.2 *We observe that* ***HP4*** *implies that the origin is also a parabolic fixed point of the unperturbed system and the derivative of the vector field evaluated at this point is the same as the one of the unperturbed system, that is*

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Consider the terms $a_{ij}(\theta, \mu, \varepsilon) x^i y^j$ of h_1 evaluated on γ_0. We define ℓ to be the greatest order of the branching points $\pm ia$ corresponding to $a_{ij}(\theta, \mu, \varepsilon)\alpha_0^i(u)\beta_0^j(u)$. That is:

$$\ell = \max\{i(r-1) + jr : a_{ij}(\theta, \mu, \varepsilon) \neq 0\}.$$

Also we define

$$\nu = p - \ell.$$

HP5.The constant ν is greater or equal than 0.

Remark 1.1.3 *Hypothesis **HP5** measures the growth of the perturbation term*

$$\mu\varepsilon^p h_1(x, y, t/\varepsilon, \mu, \varepsilon)$$

*evaluated at the homoclinic orbit close to the singularities. In fact, if hypothesis **HP5** is assumed:*

$$\mu\varepsilon^p \|h_1(\gamma_0(u), t/\varepsilon, \mu, \varepsilon)\|_\infty = O(\mu),$$

for $|\operatorname{Im} u| \leq a - \varepsilon$.

Remark 1.1.4 *A consequence of hypotheses **HP1-HP5**, is that if* $p < 1$*, then* $\partial_y h_1 = 0$*. Indeed, if* $\ell \geq 1$*, then by hypothesis **HP5**,* $p \geq 1$*. Therefore, we consider the case* $\ell < 1$*. By definition of* ℓ *and using that* $r \geq 1$*, we have that for any pair of positive integers,* i*,* j *such that* $a_{ij}(\theta, \mu, \varepsilon) \neq 0$*,*

$$1 > \ell \geq i(r-1) + jr \geq jr \geq j$$

Therefore, $j = 0$ *and this implies that* h_1 *has no terms with the variable* y*. Therefore* $\partial_y h_1 = 0$*.*

1.1.2 The weak hyperbolic case

What we call the weak hyperbolic case is in fact a bifurcation case, in the sense that when $\mu = 0$ the origin is a parabolic fixed point and when $\mu \neq 0$ the character of the origin becomes elliptic or hyperbolic. In this case, for the sake of concreteness, we consider a given non-perturbed system. More precisely we consider Hamiltonian systems with Hamiltonian

$$H(x, y, t/\varepsilon) = h_0(x, y) + \mu\varepsilon^p h_1(x, y, t/\varepsilon, \mu, \varepsilon)$$

where

$$h_0(x, y) = \frac{y^2}{2} + V(x)$$

and

$$V(x) = -x^3 + x^4.$$

Concerning to the perturbation, we assume that it has the form

$$h_1(x, y, t/\varepsilon, \mu, \varepsilon) = h_{12}(x, y, t/\varepsilon, \mu, \varepsilon) + h_{13}(x, y, t/\varepsilon, \mu, \varepsilon)$$

with

$$h_{12}(x, y, t/\varepsilon, \mu, \varepsilon) = \frac{x^2}{2} g_1(t/\varepsilon, \mu, \varepsilon) + xy g_2(t/\varepsilon, \mu, \varepsilon) + \frac{y^2}{2} g_3(t/\varepsilon, \mu, \varepsilon)$$

and $h_{13}(x, y, t/\varepsilon)$ is of order greater or equal than 3 in (x, y) variables.

The associated differential equations are

$$\begin{aligned} \dot{x} &= y + \mu\varepsilon^p(x g_2(t/\varepsilon, \mu, \varepsilon) + y g_3(t/\varepsilon, \mu, \varepsilon) + \partial_y h_{13}(x, y, t/\varepsilon, \mu, \varepsilon)) \\ \dot{y} &= 3x^2 - 4x^3 - \mu\varepsilon^p(x g_1(t/\varepsilon, \mu, \varepsilon) + y g_2(t/\varepsilon, \mu, \varepsilon) + \partial_y h_{13}(x, y, t/\varepsilon, \mu, \varepsilon)). \end{aligned} \tag{1.1.4}$$

We introduce the functions G_i for $i = 1, 2, 3$ determined by the conditions:

$$\partial_\theta G_i = g_i, \qquad \int_0^{2\pi} G_i(\theta, \mu, \varepsilon) d\theta = 0.$$

We assume hypotheses **HP1-HP5** of the previous subsection and moreover we impose that:

HP6. With the above introduced notation

$$\int_0^{2\pi} g_2(\theta, \mu, \varepsilon) G_1(\theta, \mu, \varepsilon) d\theta < 0.$$

Remark 1.1.5 *Hypothesis **HP6** implies that $H_1 \neq 0$ and then $k = 2$. We sill see that $r = 2$ and therefore $\ell \in \mathbb{N}$. In fact, since $g_2 \neq 0$, $\ell \geq 2r - 1 = 3$.*

Remark 1.1.6 *In Chapter 4 we will study the Poincaré map of (1.1.4) and there we will see that **HP6** implies that the origin is a saddle point when $\mu \neq 0$.*

Remark 1.1.7 *We remark that, in the weak hyperbolic case, hypothesis **HP4** does not apply.*

1.2 Main results

Using that the Poincaré map is area preserving we will prove the existence of primary homoclinic points which will be the zeros of the Melnikov function $M(s,\varepsilon)$ defined by

$$M(s,\varepsilon) = \int_{-\infty}^{\infty} \{h_0, h_1\}(\gamma_0(t+s), t/\varepsilon)\ dt.$$

We denote by A the area of the lobe generated by the stable and unstable manifold between two homoclinic points and by ϑ the angle between the stable and unstable invariant manifolds at a homoclinic point. We observe that, since the Poincaré map is area preserving, the area A will not depends on the homoclinic points.

The main results of Part I are:

Theorem 1.2.1 *Under hypotheses **HP1-HP6**, for $\varepsilon \to 0^+$, $\mu \to 0$, the following formulae hold:*

$$A = \mu\varepsilon^p \int_{s_0}^{\bar{s}_0} M(\upsilon,\varepsilon) d\upsilon + O(\mu^2\varepsilon^{2\nu+r}, \mu^2\varepsilon^{\nu+p+i_0}, \mu\varepsilon^{p+1+i_0})e^{-a/\varepsilon},$$

$$\sin\vartheta = \mu\varepsilon^p \frac{M'(s_0,\varepsilon)}{\|\dot{\gamma}_0(s_0)\|^2} + O(\mu^2\varepsilon^{2\nu+r-2}, \mu^2\varepsilon^{\nu+p+i_0-2}, \mu\varepsilon^{p-1+i_0})e^{-a/\varepsilon},$$

where $s_0 < \bar{s}_0$ are the two zeros of the Melnikov function (associated to two consecutive homoclinic points), closest to zero and

$$i_0 = \begin{cases} 1 & \text{in the parabolic case} \\ 1/2 & \text{in the weak hyperbolic case.} \end{cases}$$

We define the function

$$J(x,y,\theta) = \{h_0,h_1\}(x,y,\theta).$$

By the hypothesis on h_1, J is 2-periodic in θ and has zero average with respect to θ. Then we can consider its Fourier expansion

$$J(x,y,\theta) \sim \sum_{k\neq 0} J_k(x,y)e^{ik\theta}.$$

Moreover, for all $k \in \mathbb{Z}$, $J_k(\gamma_0(u))$ has a branching point of order at most $\ell + 1$ at $u = \pm ia$. Therefore, near the singularity $u = ia$, $J_k(\gamma_0(u))$ for $k < 0$ has the form

$$J_k(\gamma_0(u)) = \frac{1}{(u-ia)^{\ell+1}} \left(J^-_{k,0} + \sum_{m \geq 0} J^-_{k,m} (u-ia)^{m/q} \right)$$

and for $k > 0$, $J_k(\gamma_0(u))$, near the singularity $u = -ia$ has the form

$$J_k(\gamma_0(u)) = \frac{1}{(u+ia)^{\ell+1}} \left(J^+_{k,0} + \sum_{m \geq 0} J^+_{k,m} (u+ia)^{m/q} \right).$$

Here we observe that $J^+_{k,0} = \overline{J^-_{k,0}}$.

We consider the following condition:

HP7. The Fourier coefficients $J_{\pm 1}$ evaluated on $\gamma_0(u)$, that is $J_{\pm 1}(\gamma_0(u))$, have singularities of order exactly $\ell + 1$ at the points $u = \pm ai$.

Remark 1.2.2 *The hypothesis **HP7** is generic because it is equivalent to suppose that a determinate coefficient of the Laurent expansion of $J_{\pm 1}(\gamma_0(u))$ is different from zero.*

We can obtain an asymptotic expression of the Melnikov function and consequently of the area of the lobe and of the angle which can be compute explicitly.

Corollary 1.2.3 *If **HP1-HP7** holds, then for $\varepsilon \to 0^+$, $\mu \to 0$*

$$A \sim \mu \varepsilon^{\nu+1} 8\pi |J^+_{1,0}| \frac{1}{\Gamma(\ell+1)} e^{-a/\varepsilon}$$
$$|\sin \vartheta| \sim \mu \varepsilon^{\nu-1} 4\pi |J^+_{1,0}| \frac{1}{\Gamma(\ell+1)} e^{-a/\varepsilon} \frac{1}{\|\dot{\gamma}_0(s_0)\|^2}$$

where Γ is the Gamma function.

1.3 Examples

Examples of unperturbed Hamiltonian systems satisfying **HP1** are given by

$$h_0(x,y) = \frac{y^2}{2} + V(x) \tag{1.3.1}$$

where $V(x)$ is a polynomial such that has the form

$$V(x) = -x^n + x^{2(n-1)}.$$

Indeed, the Hamiltonian system has a homoclinic orbit contained in $H(x, y) = 0$. Let $\gamma(t)$ be the parameterization of the homoclinic orbit such that $\gamma(0) = (1, 0)$. To look for the singularities we look for the values of $t = t^*$ such that x becomes infinite.

We can compute exactly the homoclinic orbit which is given by

$$\gamma(t) = \left(\left(\frac{2}{2 + (t(n-2))^2} \right)^{1/(n-2)}, \frac{-2^{(n-1)/(n-2)}(n-2)t}{(2 + (t(n-2))^2)^{(n-1)/(n-2)}} \right).$$

Therefore, the homoclinic orbit has singularities at points $\pm ia$ with $a = \sqrt{2}/(n-2)$ which are branching points. It is not difficult to see that, near the singularities $\pm ia$, the first component of γ reads as

$$\frac{C_\pm}{(t \pm ia)^{1/(n-2)}} \left(1 + O(t \pm ia)^{1/(n-2)}\right)$$

with $C_- = (2a)^{-1/(n-2)} e^{-i\pi/2(n-2)}$ and $C_+ = \overline{C_-}$.

We consider a family of perturbations given by

$$h_1(x, y, t/\varepsilon) = x^k \cos(t/\varepsilon).$$

In this case, $\ell = k/(n-2)$. Of course we suppose that k satisfies the hypothesis **HP4**, that is $2k - 2 \geq n$. We observe that in this case $\ell \geq (n+2)/(2n-4)$.

Then, by Corollary 1.2.3 the area of the lobe generated between two consecutive homoclinic points satisfies the asymptotic expression

$$A \sim \mu \varepsilon^{\nu+1} 8\pi \frac{k}{2(n-2)} |C_-|^k \frac{1}{\Gamma\left(\frac{k+n-2}{n-2}\right)} e^{-a/\varepsilon}$$

where $\nu = p - k/(n-2)$ and $p \geq k(n-2)$.

Chapter 2

Analytic properties of the homoclinic orbit of the unperturbed system

2.1 Introduction and main result

The purpose of this chapter is to obtain the asymptotic behaviour of the homoclinic orbits of Hamiltonians systems of the form

$$H(x, y) = y^2/2 + V(x)$$

with $V(x)$ being an analytic function, for complex values of time in a certain domain.

We suppose that the origin is a fixed point of the corresponding Hamiltonian equation

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -V'(x). \end{aligned}$$

It is not restrictive to assume that $V(0) = 0$. We suppose that V is of the form

$$V(x) = a_n x^n + \dots$$

with $a_n \neq 0$.

If $n \geq 3$ the origin is a parabolic point, that is, the linear part of the equation at $(0, 0)$ has a double zero eigenvalue. Assuming that it has an invariant curve passing through the origin the solution on this curve has to live on the energy level $H(x, y) = 0$. Then

$$\dot{x} = y = \pm\sqrt{-2V(x)}.$$

Hence we will have that $\dot{x} = ax^k + \dots$ or $\dot{x} = ax^{k+1/2} + \dots$ according to the cases $n = 2k$ or $n = 2k + 1$, $k \in \mathbb{N}$.

For the sake of generality we consider the case $k \in \mathbb{R}$. We define the set

$$U = \mathbb{D}(0, r) - \{z \in \mathbb{C} : \operatorname{Im} z = 0, \operatorname{Re} z \leq 0\} \subset \mathbb{C}.$$

The principal result of this Chapter is the following proposition from which we derive the asymptotic representation of $x(t)$ and then $y(t)$ follows from $y(t) = \dot{x}(t)$.

Proposition 2.1.1 *Let f be an analytic function on U. Suppose that*

$$f(x) = ax^k + bx^\ell + \dots,$$

with $k, \ell \in \mathbb{R}$, $1 < k < \ell$ and $a < 0$. Consider the equation

$$\dot{x} = f(x).$$

Then, there is a solution $\varphi(t)$ defined on

$$\Omega(T, \tau) = \{t \in \mathbb{C} : \operatorname{Re} t > T, |\operatorname{Im} t| < \tau\}$$

such that

$$\varphi(t) = ct^{-p} + O(t^{-\nu})$$

with $p = 1/(k-1)$, $p < \nu < \min\{p(1 + \ell - k), p + 1/2\}$ and $c = (-p/a)^p$.

The proof of this proposition has two parts: a formal and an analytic part. Obviously the formal part is only an heuristic study of the equation $\dot{x} = f(x)$ and only gives a intuitive approach of the behavior of the solution which we want to find.

Remark 2.1.2 *From the way that T and τ enter in the estimations, see in particular conditions (2.3.4), we deduce that we can take τ big if we take T big enough.*

2.2 Formal part

We assume that φ has an asymptotic expansion of the form

$$\varphi(t) = c\frac{1}{t^p} + d\frac{1}{t^q} + e\frac{1}{t^r} + \dots, \qquad p < q < r.$$

Imposing that it is a formal solution we obtain the following equation

$$\begin{aligned}
-pc\frac{1}{t^{p+1}} - qd\frac{1}{t^{q+1}} - er\frac{1}{t^{r+1}} + \dots &= a\Big(c\frac{1}{t^p} + d\frac{1}{t^q} + e\frac{1}{t^r} + \dots\Big)^k \\
&\quad + b\Big(c\frac{1}{t^p} + d\frac{1}{t^q} + e\frac{1}{t^r} + \dots\Big)^\ell + \dots \\
&= ac^k\frac{1}{t^{pk}}\Big(1 + d\frac{1}{ct^{q-p}} + e\frac{1}{ct^{r-p}} + \dots\Big)^k + bc^\ell\frac{1}{t^{p\ell}}\Big(1 + d\frac{1}{ct^{q-p}} + e\frac{1}{ct^{r-p}} + \dots\Big)^\ell + \dots \\
&= a\Big(c^k\frac{1}{t^{pk}} + kdc^{k-1}\frac{1}{t^{kp+q-p}} + kec^{k-1}\frac{1}{t^{kp+r-p}} + \frac{k(k-1)}{2}d^2c^{k-2}\frac{1}{t^{kp+2(q-p)}} + \dots\Big) \\
&\quad + b\Big(c^\ell\frac{1}{t^{p\ell}} + \ell dc^{\ell-1}\frac{1}{t^{\ell p+q-p}} + \ell ec^{\ell-1}\frac{1}{t^{\ell p+r-p}} + \frac{\ell(\ell-1)}{2}d^2c^{\ell-2}\frac{1}{t^{\ell p+2(q-p)}} + \dots\Big)
\end{aligned}$$

The lower order terms must agree so that we have

$$p + 1 = kp, \qquad -pc = ac^k$$

that is,

$$p = \frac{1}{k-1}, \qquad c = \left(\frac{-p}{a}\right)^p = \left(\frac{1}{(1-k)a}\right)^{1/(k-1)}.$$

We observe that, if $p = 1/(k-1)$ then $kp + q - p = q + 1$, therefore, for the next order we impose the condition $\ell p = 1 + q$ to have three terms of the same order. This gives

$$q = \frac{1 + \ell - k}{k-1}.$$

Comparing the coefficients of order $q+1$ we get $-qd = kadc^{k-1} + bc^\ell$ which is equivalent to

$$d\frac{-\ell + 2k - 1}{k-1} = bc^\ell$$

which has a solution if and only if $\ell \neq 2k-1$. Note that $\ell = 2k-1$ is a kind of resonance condition.

One may think in just imposing that two terms have the same order $q+1$ and hence $\ell p > q+1$.

Comparing the corresponding coefficients we get

$$-qd = kadc^{k-1} = ka(-p/a)d = -kpd$$

so that $q = kp = k/(k-1)$ and d, for the moment, is free. Therefore, the condition $\ell p > q+1$ is equivalent to $\ell/(k-1) > k/(k-1)+1 > (2k-1)/(k-1)$. Note that this only could happen when $\ell > 2k-1$.

We finish here this heuristic study and now we give a proof of the analytic part of the Proposition 2.1.1.

2.3 Analytic part

Let $U = \mathbb{D}(0,r) - \{z \in \mathbb{C} : \operatorname{Im} z = 0,\ \operatorname{Re} z \leq 0\} \subset \mathbb{C}$ and $k, \ell \in \mathbb{R}$, $1 < k < \ell$. We consider

$$f(x) = f_0(x) + g(x)$$

with $f_0(x) = ax^k$, $a < 0$, and $g : U \to \mathbb{C}$ analytic such that $|g(x)| \leq B|x|^\ell$.

The general solution of

$$\dot{x} = ax^k$$

is $x(t) = c/(t+\alpha)^p$ with $c = (-p/a)^p$ and $p = 1/(k-1)$. The constant α takes care of the initial condition. Motivated by the discussion in the previous section we look for solutions of the form

$$\varphi(t) = \varphi_0(t) + \psi(t), \text{ with } \varphi_0(t) = c/(t+\alpha)^p.$$

We write the equation $\dot{x} = f(x)$ in the form

$$\begin{aligned} \dot{\varphi}_0(t) + \dot{\psi}(t) =& f_0(\varphi_0(t)) + Df_0(\varphi_0(t))\psi(t) \\ &+ [f(\varphi_0(t) + \psi(t)) - f_0(\varphi_0(t)) - Df_0(\varphi_0(t))\psi(t)]. \end{aligned} \tag{2.3.1}$$

First we consider the auxiliary linear equation

$$\dot{\chi}(t) = Df_0(\varphi_0(t))\chi(t) = ka\left(\frac{c}{(t+\alpha)^p}\right)^{k-1}\chi(t) = \frac{-k}{k-1}\frac{1}{t+\alpha}\chi(t)$$

which has a solution

$$\chi(t) = \frac{1}{(t+\alpha)^{p+1}}.$$

From (2.3.1), using the variation of constants formula we get the following integral equation for ψ

$$\begin{aligned} \psi(t) &= \Gamma\psi(t) \qquad (2.3.2)\\ &\equiv \frac{1}{(t+\alpha)^{p+1}} \int_T^t (s+\alpha)^{p+1} [f(\varphi_0(t)+\psi(t)) - f_0(\varphi_0(t)) - Df_0(\varphi_0(t))\psi(t)]\ ds. \end{aligned}$$

Here, we are implicitly assuming that $\psi(T) = 0$. We take $\alpha = \mu + i\lambda$ such that

$$\mu = \operatorname{Re}\alpha > 0, \qquad |\lambda| \le \tau \qquad (2.3.3)$$

are fixed and ν such that

$$p < \nu < \min\left\{p + \frac{\ell - k}{k-1}, p + 1/2\right\} = \min\{q, p+1/2\}$$

fixed. We introduce the space

$$X = \{\psi : \Omega(T,\tau) \to \mathbb{C} : \text{analytic}, \ |t+\alpha|^\nu |\psi(t)| < \infty\}.$$

We endow X with the norm

$$\|\psi\| = \sup_{t\in\Omega(T,\tau)} |t+\alpha|^\nu |\psi(t)|.$$

We call X_ρ the closed ball of radius ρ in X centered at zero. We consider $\Gamma : X_\rho \to X_\rho$ as defined by (2.3.2).

The rest of this subsection is devoted to prove that, taking suitable values for ρ and T, we have that $\Gamma : X_\rho \to X_\rho$ is a contraction.

We take

$$\begin{gathered} T \ge 1, \quad \rho \le c/2, \qquad (T+\mu)^p \ge 2c/r, \\ (T+\mu)^{\nu-p} > 1/2, \qquad (2.3.4)\\ T + \mu \ge \frac{\tau + |\lambda|}{\tan(\pi/(4p))}, \quad \text{if } p > 1, \qquad T+\mu \ge \tau + |\lambda|, \quad \text{if } p \le 1 \end{gathered}$$

and T satisfying the conditions

$$\left(1+\frac{1}{2(T+\mu)^{\nu-p}}\right)^{k-2}\le 2, \qquad \left(1+\frac{1}{2(T+\mu)^{\nu-p}}\right)^{\ell}\le 2.$$

first we check that Γ is well defined. If $s=s_1+is_2\in\Omega(T,\tau)$, it is clear that

$$|s+\alpha|\ge|\operatorname{Re}(s+\alpha)|=|s_1+\mu|\ge T+\mu.$$

Then

$$\left|\frac{c}{(s+\alpha)^p}+\psi(s)\right|\le\frac{c}{(T+\mu)^p}+\frac{1}{(T+\mu)^\nu}\|\psi\|\le\frac{1}{(T+\mu)^p}3c/2\le\frac{3r}{4}.$$

Remark 2.3.1 *Let $z\in\mathbb{C}$. If $|\arg(z)|\le\pi/4$ and $|w|<|z|$ then $|\arg(z+w)|<3\pi/4$.*

Proof. Since $|w/z|<1$, $w/z=a+ib$ with $|a|<1$. Then, $\operatorname{Re}(1+w/z)>0$ and $|\arg(1+w/z)|<\pi/2$. Therefore $|\arg(z+w)|\le|\arg(z)|+|\arg(1+w/z)|<\pi/4+\pi/2$. ■

For $s\in\Omega(T,\tau)$,

$$|\arg\varphi_0(s)|=|\arg(s+\alpha)^p|\le p\arctan\frac{\tau+|\lambda|}{T+\mu}\le\frac{\pi}{4}$$

and

$$\left|\frac{\psi(s)}{c/(s+\alpha)^p}\right|\le\frac{\|\psi\|/|s+\alpha|^\nu}{c/|s+\alpha|^p}\le\frac{\|\psi\|}{c}\frac{1}{|s+\alpha|^{\nu-p}}<1.$$

By Remark 2.3.1, this means that $|\arg(\varphi_0(s)+\psi(s))|<3\pi/4$ and hence, if $\psi\in X_\rho$, then for all $s\in\Omega(T,\tau)$, $\varphi_0(s)+\psi(s)$ belongs to the domain of f.

For $k\in\mathbb{R}$, $k>0$ it is clear that

$$|(1+w/z)^k-1|=\left|\int_0^1 k(1+sw/z)^{k-1}\frac{w}{z}\,ds\right|\le k(1+|w/z|)^{k-1}|w/z|. \tag{2.3.5}$$

Therefore, for $z,w\in\mathbb{C}$, $\operatorname{Re}z\ge0$ and $|w/z|<1$ we have that

$$|(z+w)^k-z^k-kz^{k-1}w|\le\frac{k(k-1)}{2}|z|^{k-2}|w|^2(1+|w/z|)^{k-2}.$$

To evaluate the integral in the definition of Γ we will take the path of integration $\gamma(u) = T + (t-T)u$, $u \in [0,1]$. We write $\xi = \operatorname{Re} t$ and $\eta = \operatorname{Im} t$.

We begin by bounding

$$\begin{aligned}
&\left|\int_T^t (s+\alpha)^{p+1} a\big[(\varphi_0(s)+\psi(s))^k - (\varphi_0(s))^k - k(\varphi_0(s))^{k-1}\psi(s)\big]ds\right| \\
&\le |a|\frac{k(k-1)}{2}c^{k-2}\|\psi\|^2\Big(1+\frac{\|\psi\|}{c(T+\mu)^{\nu-p}}\Big)^{k-2}\left|\int_T^t \frac{1}{|s+\alpha|^{2(\nu-p)}}\ ds\right| \\
&\le |a|\frac{k(k-1)}{2}\frac{c^{k-1}}{2}\|\psi\|\Big(1+\frac{1}{2(T+\mu)^{\nu-p}}\Big)^{k-2}\left|\int_T^t \frac{1}{|s+\alpha|^{2(\nu-p)}}\ ds\right| \qquad (2.3.6)
\end{aligned}$$

(if $k-2<0$ we have to substitute $\Big(1+\frac{1}{(T+\mu)^{\nu-p}}\Big)^{k-2}$ by 1) where we have used that $p(k-2)-(1+p)+2\nu = 2(\nu-p)$.

Now we bound the term $\int_T^t (s+\alpha)^{p+1} g(c/(s+\alpha)^p + \psi(s))ds$:

$$\begin{aligned}
&\left|\int_T^t (s+\alpha)^{p+1} g(c/(s+\alpha)^p + \psi(s))\ ds\right| \\
&\le B\left|\int_T^t |s+\alpha|^{p+1}c^\ell\left|\frac{1}{(s+\alpha)^p}+\frac{c^{-1}\|\psi\|}{(s+\alpha)^\nu}\right|^\ell ds\right| \\
&= Bc^\ell\Big(1+\frac{1}{2(T+\mu)^{\nu-p}}\Big)^\ell\left|\int_T^t \frac{1}{|s+\alpha|^{q-p}}\ ds\right| \qquad (2.3.7)
\end{aligned}$$

where we have used that $p\ell - p - 1 = q - p$.

We call I_δ the integral

$$I_\delta = \int_T^t \frac{1}{|s+\alpha|^{2\delta}}\ ds = \int_0^1 \frac{t-T}{[(T+\mu+(\xi-T)u)^2+(\eta u+\lambda)^2]^\delta}\ du.$$

We note that $(T+\mu+(\xi-T)u)^2+(\eta u+\lambda)^2 \ge (T+\mu)^2 \ge 1$. We consider two cases: $\delta < 1/2$ and $\delta \ge 1/2$. If $\delta < 1/2$

$$\begin{aligned}
|I_\delta| &\le \int_0^1 \frac{|t-T|}{[((T+\mu)+(\xi-T)u)^2+(\eta u+\lambda)^2]^\delta}\ du \\
&= \frac{|t-T|}{[((T+\mu)+(\xi-T))^2+(\eta+\lambda)^2]^\delta}\int_0^1 \frac{1}{u^{2\delta}}\ du \\
&= \frac{1}{1-2\delta}\frac{|t-T|}{|t+\alpha|^{2\delta}}. \qquad (2.3.8)
\end{aligned}$$

If $\delta \geq 1/2$, or even more generally if $\delta > 0$, let $\gamma < \delta$ be such that $0 < \gamma < 1/2$. We have

$$\begin{aligned}
|I_\delta| &\leq \int_0^1 \frac{|t-T|}{[(T+\mu+(\xi-T)u)^2+(\eta u+\lambda)^2]^\delta}\, du \\
&= \int_0^1 \frac{|t-T|}{[(T+\mu+(\xi-T)u)^2+(\eta u+\lambda)^2]^{\delta-\gamma+\gamma}}\, du \\
&\leq \frac{|t-T|}{(T+\mu)^{2(\delta-\gamma)}} \int_0^1 \frac{1}{[(T+\mu+(\xi-T)u)^2+(\eta u+\lambda)^2]^\gamma}\, du \\
&\leq \frac{1}{1-2\gamma} \frac{1}{(T+\mu)^{2(\delta-\gamma)}} \frac{|t-T|}{|t+\alpha|^{2\gamma}}. \qquad (2.3.9)
\end{aligned}$$

We observe that, by (2.3.3), $|t-T| \leq |t+\alpha|$. We recall that $2(\nu-p) < 1$. We write $q-p = q-\nu+\nu-p$, and we take $q-p = 2\delta$ and $\nu-p = 2\gamma$ in (2.3.9). Then using (2.3.8) in (2.3.6), (2.3.9) in (2.3.7) as well as assumptions (2.3.4), we have that

$$\begin{aligned}
&|\Gamma\psi(t)| \\
&\leq \frac{1}{|t+\alpha|^{p+1}} \Big[|a| c^{k-1} \frac{k(k-1)}{4} \|\psi\| \Big(1+\frac{1}{2(T+\mu)^{\nu-p}}\Big)^{k-2} \frac{1}{1-2(\nu-p)} \frac{|t-T|}{|t+\alpha|^{2(\nu-p)}} \\
&\quad + Bc^\ell \Big(1+\frac{1}{2(T+\mu)^{\nu-p}}\Big)^\ell \frac{1}{(T+\mu)^{q-\nu}} \frac{1}{1-(\nu-p)} \frac{|t-T|}{|t+\alpha|^{\nu-p}} \Big] \\
&\leq \Big[|a| c^{k-1} \frac{k(k-1)}{2} \|\psi\| \frac{1}{1-2(\nu-p)} \frac{1}{|t+\alpha|^{\nu-p}} \\
&\quad + 2Bc^\ell \frac{1}{1-(\nu-p)} \frac{1}{(T+\mu)^{q-\nu}} \Big] \frac{1}{|t+\alpha|^\nu}.
\end{aligned}$$

Clearly, if T is big enough, $|\Gamma\psi(t)| \leq \rho/|t+\alpha|^\nu$ and then $\Gamma\psi \in X_\rho$.

Next we see that Γ is a contraction. Indeed, let ψ and $\tilde{\psi}$ be two functions which belong to X_ρ,

$$\begin{aligned}
|(\Gamma\psi - \Gamma\tilde{\psi})(t)| \leq & \frac{1}{|t+\alpha|^{p+1}} \bigg| \int_T^t (s+\alpha)^{p+1} \big(f_0(\varphi_0(s)+\psi(s)) - Df_0(\varphi_0(s))\psi(s) \\
& - [f_0(\varphi_0(s)+\tilde{\psi}(s)) - Df_0(\varphi_0(s))\tilde{\psi}(s)] \big) ds \qquad (2.3.10) \\
& + \int_T^t (s+\alpha)^{p+1} [g(\varphi_0(s)+\psi(s)) - g(\varphi_0(s)+\tilde{\psi}(s))]\, ds \bigg|.
\end{aligned}$$

To evaluate the first difference we consider the function

$$\chi(z) = a[\varphi_0(s)+z)^k - k(\varphi_0(s))^{k-1} z].$$

By the mean value theorem, we have that

$$\chi(\tilde{z}) - \chi(z) = \int_0^1 \chi'(z + \zeta(\tilde{z} - z))[\tilde{z} - z]\; d\zeta$$

and, since $\chi'(w) = ak[(\varphi_0(s) + w)^{k-1} - (\varphi_0(s))^{k-1}]$ then, using (2.3.5),

$$|\chi(\tilde{\psi}(s)) - \chi(\psi(s))| \leq ak(k-1)\Big(\frac{c}{|s+\alpha|^p} + \frac{\rho}{|s+\alpha|^\nu}\Big)^{k-2} \frac{\rho}{|s+\alpha|^\nu} |\tilde{\psi}(s) - \psi(s)|.$$

(If $k - 2 < 0$ we have to substitute $\Big(\frac{c}{|s+\alpha|^p} + \frac{\rho}{|s+\alpha|^\nu}\Big)$ by $\Big(\frac{c}{|s+\alpha|^p}\Big)$).

We bound the first integral in (2.3.10),

$$\begin{aligned}
&\left|\int_T^t |s+\alpha|^{p+1}|a|k(k-1)\frac{c^{k-2}}{|s+\alpha|^{p(k-2)}}\Big(1 + \frac{c^{-1}\rho}{(T+\mu)^{\nu-p}}\Big)^{k-2} \|\tilde{\psi} - \psi\| \frac{\rho}{|s+\alpha|^{2\nu}}\; ds\right| \\
&= |a|k(k-1)c^{k-2}2\rho\|\tilde{\psi} - \psi\| \left|\int_T^t \frac{1}{|s+\alpha|^{p(k-2)-1-p+2\nu}}\; ds\right| \\
&= |a|k(k-1)c^{k-2}2\rho\|\tilde{\psi} - \psi\| \left|\int_T^t \frac{1}{|s+\alpha|^{2(\nu-p)}}\; ds\right|. \qquad (2.3.11)
\end{aligned}$$

Here we have used that $pk = p+1$. To deal with the second integral we use that, since g is analytic, $|g'(z)| \leq B_2|z|^{\ell-1}$ in a domain

$$\{z \in \mathbb{C} : |z| \leq r,\ |\arg z| < \theta_0\} \qquad 0 < \theta_0 < \pi.$$

Then it is bounded by

$$\begin{aligned}
&B_2 c^{\ell-1}\|\tilde{\psi} - \psi\|\Big(1 + \frac{\rho}{c(T+\mu)^{\nu-p}}\Big)^{\ell-1} \int_T^t \frac{1}{|s+\alpha|^{p(\ell-1)+\nu-1-p}}\; ds \\
&= B_2 c^{\ell-1}\|\tilde{\psi} - \psi\|\Big(1 + \frac{\rho}{c(T+\mu)^{\nu-p}}\Big)^{\ell-1} \int_T^t \frac{1}{|s+\alpha|^{q-p+\nu-p}}\; ds \qquad (2.3.12)
\end{aligned}$$

here we use that $p\ell = q+1$. We recall that $2(\nu - p) < 1$. Then, using (2.3.8) in (2.3.11) with $2\delta = 2(\nu - p)$ and using (2.3.9) in (2.3.12) with $2\gamma = \nu - p$ and $2\delta = q - p + \nu - p$

we obtain

$$\begin{aligned}
|(\Gamma\psi - \Gamma\tilde{\psi})(t)| &\leq \frac{1}{|t+\alpha|^{p+1}}\Big[|a|k(k-1)c^{k-2}2\rho\frac{1}{1-2(\nu-p)}\frac{|t-T|}{|t+\alpha|^{2(\nu-p)}}\|\tilde{\psi}-\psi\| \\
&\quad + B_2 c^{\ell-1}2\|\tilde{\psi}-\psi\|\frac{1}{|T+\mu|^{(q-p)}}\frac{1}{1-(\nu-p)}\frac{|t-T|}{|t+\alpha|^{(\nu-p)}}\Big] \\
&\leq \Big[|a|k(k-1)c^{k-2}2\rho\frac{1}{1-2(\nu-p)}\frac{1}{|T+\mu|^{\nu-p}} \\
&\quad + B_2 c^{\ell-1}2\frac{1}{1-(\nu-p)}\frac{1}{|T+\mu|^{(q-p)}}\Big]\|\tilde{\psi}-\psi\|\frac{1}{|t+\alpha|^{\nu}}.
\end{aligned}$$

Here we have used that $|t-T| \leq |t+\alpha|$. Hence, if T is big enough Γ is a contraction in X_ρ and, by the fixed point theorem, there exists a unique solution of (2.3.2) which belongs to X_ρ. This ends the proof of Proposition 2.1.1.

Chapter 3

Parameterization of local invariant manifolds

3.1 Introduction

In this Chapter we prove the existence of a special parameterization of the local stable and the local unstable manifolds which will be useful later. We only prove the existence of such parameterization for the local stable manifold, but it is clear that the result is also true for the unstable one, working with the inverse map.

In order to prove this result we need a good initial approximation of the stable (and unstable) manifold and suitable coordinates to work with.

In Section 3.3, we obtain these coordinates by canonical changes of variables using the averaging method. This method allows us to obtain two important things: remove the terms of order $\mu\varepsilon^p$ and remove the smallest degree terms (with respect to (x, y)) of h_1. We must average several times in order to obtain a high enough degree.

The initial approximation of the stable manifold is achieved as the invariant manifold of an appropriate intermediate system which is constructed in Section 3.5. It is a truncated polynomial system with coefficients chosen in a very specific way. We remark that this initial approximation is only necessary when $k < n$ and hence can be avoided if we are interested in the case $k \geq n$.

Finally we obtain a functional equation for the parameterization of the stable manifold and we prove it has a solution applying the fixed point theorem in a suitable Banach space.

It is important to say that although the system is C^0 in t/ε, we obtain a parameterization with two parameters say (t, s), which is analytic in s, considered as a complex variable, and we provide a dynamic sense for it.

3.2 Definitions and main result

We begin by introducing some notation. Given $T > 0$ and $\tau \geq a > 0$ we define the following sets:

$$D^s = D^s(T, \tau) = \{(t, s) \in \mathbb{R} \times \mathbb{C} : \; t + \operatorname{Re} s \geq T, \; |\operatorname{Im} s| \leq \tau\}$$

and

$$D^u = D^u(T, \tau) = \{(t, s) \in \mathbb{R} \times \mathbb{C} : \; t + \operatorname{Re} s \leq -T, \; |\operatorname{Im} s| \leq \tau\}.$$

Note that if $(t, s) \in D_s$, $(t + 2\pi\varepsilon, s)$ and $(t, s + 2\pi\varepsilon)$ also belong to D^s. For $k \in \mathbb{R}$, $k \geq 0$, we define the space $\mathcal{X}_k = \mathcal{X}_k^s$ of functions $h : D^s \to \mathbb{C}$ such that

(a) h is continuous,

(b) for t fixed, $s \mapsto h(t, s)$ is analytic,

(c) $h(t + 2\pi\varepsilon, s) = h(t, s + 2\pi\varepsilon)$ for all $(t, s) \in D^s$,

(d) $\|h\|_k \equiv \sup\{(t + \operatorname{Re} s)^k |h(t, s)|, \; (t, s) \in D^s\} < \infty$.

It is clear, from the definition, that $\mathcal{X}_k$ is a Banach space with the norm $\|.\|_k$ and that

$$\mathcal{X}_{k+1} \subset \mathcal{X}_k.$$

Indeed, from the definition of $\|.\|_k$ we have

$$|h_n(t, s) - h_m(t, s)| \leq \frac{1}{(t + \operatorname{Re} s)^k} \|h_n - h_m\|_k \leq \frac{1}{T^k} \|h_n - h_m\|_k.$$

Then if (h_n) is a Cauchy sequence in $\mathcal{X}_k$, (h_n) is uniformly convergent to some function $h_\infty : D^s \longrightarrow \mathbb{C}$ and therefore h_∞ satisfies the conditions (a), (b) and (c) of the definition of $\mathcal{X}_k$. If $\|h_n - h_m\|_k < \varepsilon$ for $n, m \geq n_0$, taking limit as m goes to ∞ in

$$|h_n(t,s) - h_m(t,s)| \leq \frac{1}{(t + \operatorname{Re} s)^k}\varepsilon$$

we have

$$|h_n(t,s) - h_\infty(t,s)| \leq \frac{1}{(t + \operatorname{Re} s)^k}\varepsilon \tag{3.2.1}$$

and

$$\begin{aligned}(t + \operatorname{Re} s)^k |h_\infty(t,s)| &\leq (t + \operatorname{Re} s)^k |h_n(t,s) - h_\infty(t,s)| + (t + \operatorname{Re} s)^k |h_n(t,s)| \\ &\leq \varepsilon + \|h_n\|_k \end{aligned} \tag{3.2.2}$$

From (3.2.2) it follows that h_∞ belongs to $\mathcal{X}_k$ and from (3.2.1) that $h_n \to h_\infty$ in $\mathcal{X}_k$.

The main theorem of this Chapter is the following:

Theorem 3.2.1 *Let $\tau > 0$. Assuming hypotheses **HP1-HP4** and that, in case $p < 1$ $\partial_x h_1 \partial_y h_1 = 0$, there exist $T > 0$ big enough and parameterizations $\gamma^s_{\mu,\varepsilon}(t,s)$, $\gamma^u_{\mu,\varepsilon}(t,s)$ of the local stable and unstable invariant manifolds, defined in $D^s(T,\tau)$, $D^u(T,\tau)$, respectively, such that ($*$ stands for s and u):*

1) $t \mapsto \gamma^*_{\mu,\varepsilon}(t,s)$ is a solution of system (1.1.1) and $s \mapsto \gamma^*_{\mu,\varepsilon}(t,s)$ is real analytic. Moreover the map $(t,s,\mu,\varepsilon) \mapsto \gamma^*_{\mu,\varepsilon}(t,s)$ is continuous, C^1 with respect to t and analytic with respect to (s,μ).

2) $\gamma^*_{\mu,\varepsilon}(t + 2\pi\varepsilon, s) = \gamma^*_{\mu,\varepsilon}(t, s + 2\pi\varepsilon)$ for all $(t,s) \in D^*(T,\tau)$.

3) For $\mu = 0$, $\gamma^*_{\mu,\varepsilon}(t,s)$ coincides with the restriction of the homoclinic solution $\gamma_0(t+s)$ to $D^*(T,\tau)$, and for $\mu \neq 0$ the following estimate holds:

$$\gamma^*_{\mu,\varepsilon}(t,s) = \gamma_0(t+s) + \mu\varepsilon^{p+1} G(\gamma_0(t+s), t/\varepsilon) + O(\mu\varepsilon^{p+2})$$

where $G = (G_1, G_2)$ is such that

$$\partial_\theta G(x,y,\theta) = (\partial_y h_1(x,y,\theta), -\partial_x h_1(x,y,\theta)),$$

and has zero mean.

4) $\gamma^*_{\mu,\varepsilon}(t,s) = \gamma_0(t+s) + \mu\varepsilon^{p+1}\sigma^*(t,s,\mu,\varepsilon)$ *where* $\sigma^*(t,s,\mu,\varepsilon) \in \mathcal{X}^*_\lambda \times \mathcal{X}^*_\lambda$ *with* $\lambda = \frac{2k-2}{n-2}$.

From now on, to simplify the notation, we omit the dependence on ε and μ at several places where do not play an essential role. The proof of this theorem is done in several steps.

Remark 3.2.2 *In this theorem we have introduced a new condition: if* $p < 1$ *then* $\partial_x h_1 \partial_y h_1 = 0$. *By Remark 1.1.4 hypothesis* ***HP5*** *implies this condition, and therefore, under hypotheses* ***HP1-HP5,*** *Theorem 3.2.1 applies.*

In the following sections we assume the hypotheses of Theorem 3.2.1.

3.3 Averaging of the equation

Some steps of averaging are necessary to transform the equation (1.1.1) into a suitable form. First we scale time by $\theta = \frac{t}{\varepsilon}$. The transformed system reads

$$\begin{aligned} \dot{x} &= \varepsilon y + \mu\varepsilon^{p+1}\partial_y h_1(x,y,\theta,\mu,\varepsilon) \\ \dot{y} &= -\varepsilon V'(x) - \mu\varepsilon^{p+1}\partial_x h_1(x,y,\theta,\mu,\varepsilon) \end{aligned} \tag{3.3.1}$$

where $\dot{x}$ and $\dot{y}$ now mean derivatives with respect to the new time θ. The Hamiltonian becomes $\varepsilon H(x,y,\theta,\mu,\varepsilon)$. Next we average $n+1$ times with respect to θ in order to move the contribution of the perturbation to terms of order $\mu\varepsilon^{p+2n+1}$ and $\mu^2\varepsilon^{p+3}$ in the parameters, taking care of the orders with respect to x, y.

Definition 3.3.1 *We denote by* P_l^m *with* $l, m \in \mathbb{N}$, $l \leq m$ *the set of functions which are sums of homogeneous polynomials with respect to* (x,y) *of orders between* l *and* m. *That is, we say that* $p : \mathbb{C}^2 \times \mathbb{R} \to \mathbb{C}$ *belongs to* P_l^m *if and only if* p *is* C^0 *and analytic in* (x,y) *variables and*

$$p(x,y,\theta) = \sum_{i+j=l}^{m} a_{ij}(\theta) x^i y^j$$

with the coefficients $a_{ij}(\theta)$ 2π*-periodic.*

We will also consider families of functions in P_l^m. *In such case we will have*

$$p_{\mu,\varepsilon}(x,y,\theta) = p(x,y,\theta,\mu,\varepsilon) = \sum_{i+j=l}^{m} a_{ij}(\theta,\mu,\varepsilon)x^i y^j$$

with μ *and* ε *belonging to some set.*

Moreover we will consider functions, analytic with respect to x, y *in some neighborhood of the origin, whose lower order terms will be of order* l. *We will represent their set by* $P_l = P_l^\infty$.

We will write $p = (p_{l_1}, p_{l_2}) \in P_{l_1} \times P_{l_2}$ *if* $p_{l_1} \in P_{l_1}$ *and* $p_{l_2} \in P_{l_2}$. *Moreover, if* $l_1 = l_2$ *we write that* $p \in P_{l_1}$.

For notational convenience we define

$$P_l^m = P_0^m \qquad \textit{for} \qquad l < 0, \;\; 0 \leq m \leq +\infty.$$

Let $F_\nu(x,y,\theta,\mu,\varepsilon)$ be such that it has zero mean with respect to θ and

$$F_\nu(x,y,\theta,\mu,\varepsilon) = \sum_{l=0}^{\nu} y^{2l} p_{j_l}(x,y,\theta,\mu,\varepsilon) \tag{3.3.2}$$

with $\nu \in \mathbb{Z}^+$, $j_l = \max\{0, n(\nu - l) + k - 2\nu\}$ and $p_{j_l} \in P_{j_l}$. We assume that F_ν depends on parameters μ and ε, that it is continuous (with respect to all variables) and that it is analytic with respect to (x,y,μ). We consider the Hamiltonian

$$\begin{aligned}\varepsilon\mathcal{H}_\nu(x,y,\theta,\mu,\varepsilon) =& \frac{\varepsilon}{2}y^2 + \varepsilon V(x) + \mu\varepsilon^{p+2\nu+1}F_\nu(x,y,\theta,\mu,\varepsilon) \\ &+ \mu^2\varepsilon^{2p+2}R_{2k-2}^\nu(x,y,\theta,\mu,\varepsilon)\end{aligned}$$

with V of order n and $R_{2k-2}^\nu \in P_{2k-2}$. We observe that εH has this form for $\nu = 0$ and $R_{2k-2}^0 \equiv 0$.

Lemma 3.3.2 *Under the previous conditions and assuming that* $n \geq 3$ *and* $k \geq 2$, *there exists a canonical change of variables* $(x,y,\theta) = \mathcal{C}_{\nu+1}(\bar{x},\bar{y},\theta,\mu,\varepsilon)$ *which is* C^0 *in* $(\bar{x},\bar{y},\theta,\mu,\varepsilon)$, C^1*and* 2π*-periodic in* θ *and analytic in* $(\bar{x},\bar{y},\mu)$ *and it transforms the Hamiltonian* $\varepsilon\mathcal{H}_\nu$ *to*

$$\begin{aligned}\varepsilon\mathcal{H}_{\nu+1}(\bar{x},\bar{y},\theta,\mu,\varepsilon) =& \frac{\varepsilon}{2}\bar{y}^2 + \varepsilon V(\bar{x}) + \mu\varepsilon^{p+2(\nu+1)+1}F_{\nu+1}(\bar{x},\bar{y},\theta,\mu,\varepsilon) \\ &+ \mu^2\varepsilon^{2p+2}R_{2k-2}^{\nu+1}(\bar{x},\bar{y},\theta,\mu,\varepsilon)\end{aligned} \tag{3.3.3}$$

in a neighborhood of the origin, where

$$F_{\nu+1}(\bar{x},\bar{y},\theta) = \bar{y}\partial_x S^2 - V'(\bar{x})\partial_{\bar{y}} S^2,$$

$S^2 = S^2(\bar{x},\bar{y},\theta)$ *depends on* $S^1 = S^1(\bar{x},\bar{y},\theta)$ *and they satisfy*

$$\partial_\theta S^1(x,\bar{y},\theta) = -F_\nu(x,\bar{y},\theta) \tag{3.3.4}$$
$$\partial_\theta S^2(x,\bar{y},\theta) = V'(x)\partial_{\bar{y}} S^1(x,\bar{y},\theta) - \bar{y}\partial_x S^1(x,\bar{y},\theta). \tag{3.3.5}$$

Moreover

$$F_{\nu+1} = \sum_{l=0}^{\nu+1} y^{2l} p_{i_l}(x,y,\theta)$$

with $i_l = \max\{0, n(\nu+1-l)+k-2(\nu+1)\}$, $p_{j_l} \in P_{j_l}$, $F_{\nu+1}$ *has zero mean with respect to* θ *and* $R^{\nu+1}_{2k-2} = R^{\nu+1}_{2k-2}(\bar{x},\bar{y},\theta,\mu,\varepsilon) \in P_{2k-2}$ *and*

$$R^{\nu+1}_{2k-2} = \varepsilon^{4\nu}\partial_y F_\nu \partial_x S^1 + R^{\nu}_{2k-2} + \varepsilon^{4\nu+1} r_{2k-2}$$

with $r_{2k-2} \in P_{2k-2}$. *Also* $\mathcal{H}_{\nu+1}$ *is continuous in* $(\bar{x},\bar{y},\theta,\mu,\varepsilon)$ *and analytic in* $(\bar{x},\bar{y},\mu)$.

Remark 3.3.3 *Although in the parabolic case* k *has to be greater than* 3, *we allow* $k \geq 2$ *in order to use this Lemma in the weak hyperbolic case.*

Proof. We consider a generating function $S(x,\bar{y},\theta)$ which will provide a canonical change of variables $(\bar{x},\bar{y}) \mapsto (x,y)$ implicitly through

$$\begin{aligned} \bar{x} &= \partial_{\bar{y}} S(x,\bar{y},\theta) \\ y &= \partial_x S(x,\bar{y},\theta) \end{aligned} \tag{3.3.6}$$

and then the new Hamiltonian will be

$$\varepsilon\mathcal{H}_{\nu+1}(\bar{x},\bar{y},\theta,\mu,\varepsilon) = \varepsilon H(x,y,\theta,\mu,\varepsilon) + \partial_\theta S(x,\bar{y},\theta).$$

We take

$$S(x,\bar{y},\theta) = x\bar{y} + \mu\varepsilon^{p+2\nu+1} S^1(x,\bar{y},\theta) + \mu\varepsilon^{p+2\nu+2} S^2(x,\bar{y},\theta)$$

with S^1 and S^2 satisfying (3.3.4) and (3.3.5). This choice is motivated by the next calculations. We observe that S is C^1, 2π-periodic in θ and analytic in (x,y). With it we will cancel the terms of orders $\mu\varepsilon^{p+2\nu+1}$ and $\mu\varepsilon^{p+2\nu+2}$ in the averaged Hamiltonian.

Since F_ν has zero mean with respect to θ we can choose S^1 with the additional condition that

$$\int_0^{2\pi} S^1(x, \bar{y}, \theta)\, d\theta = 0.$$

Moreover, with this choice of S^1 we have that $\partial_\theta S^2$ has zero mean and therefore we can also choose S^2 such that

$$\int_0^{2\pi} S^2(x, \bar{y}, \theta)\, d\theta = 0.$$

From (3.3.2), it is clear that $F_\nu \in P_k$. Indeed, all terms are of order

$$2l + n(\nu - l) + k - 2\nu = (n-2)(\nu - l) + k \geq k.$$

Hence S^1 is of order k and S^2 is of order greater or equal that k. From (3.3.6) we have that

$$\begin{aligned} x &= \bar{x} - \mu\varepsilon^{p+2\nu+1}\partial_{\bar{y}}S^1 - \mu\varepsilon^{p+2\nu+2}\partial_{\bar{y}}S^2 + \mu^2\varepsilon^{2p+4\nu+2}r_{2k-3} \\ y &= \bar{y} + \mu\varepsilon^{p+2\nu+1}\partial_x S^1 + \mu\varepsilon^{p+2\nu+2}\partial_x S^2 + \mu^2\varepsilon^{2p+4\nu+2}r_{2k-3} \end{aligned}$$

where the derivatives of S^1 and S^2, and r_{2k-3} are evaluated at $(\bar{x}, \bar{y}, \theta)$. Here and until the end of the proof r_j will mean a term of P_j. The averaged Hamiltonian is therefore

$$\begin{aligned} \varepsilon\mathcal{H}_{\nu+1}(\bar{x}, \bar{y}, \theta, \mu, \varepsilon) &= \varepsilon\mathcal{H}_\nu(x, y, \theta, \mu, \varepsilon) - \mu\varepsilon^{p+2\nu+1}F_\nu(x, \bar{y}, \theta) \\ &\quad +\mu\varepsilon^{p+2\nu+2}[V'(x)\partial_{\bar{y}}S^1(x, \bar{y}, \theta) - \bar{y}\partial_x S^1(x, \bar{y}, \theta)] \\ &= \frac{\varepsilon}{2}[\bar{y} + \mu\varepsilon^{p+2\nu+1}\partial_x S^1 + \mu\varepsilon^{p+2\nu+2}\partial_x S^2 + \mu^2\varepsilon^{2p+4\nu+2}r_{2k-3}]^2 \\ &\quad +\varepsilon V(\bar{x} - \mu\varepsilon^{p+2\nu+1}\partial_{\bar{y}}S^1 - \mu\varepsilon^{p+2\nu+2}\partial_{\bar{y}}S^2 + \mu^2\varepsilon^{2p+4\nu+2}r_{2k-3}) \\ &\quad +\mu\varepsilon^{p+2\nu+1}[F_\nu(x, y, \theta) - F_\nu(x, \bar{y}, \theta)] \\ &\quad -\mu\varepsilon^{p+2\nu+2}[V'(\bar{x})\partial_{\bar{y}}S^1 - \bar{y}\partial_x S^1] \\ &\quad +\mu^2\varepsilon^{2p+2}R^\nu_{2k-2} + \mu^2\varepsilon^{2p+4\nu+3}r_{2k-2} \\ &= \frac{\varepsilon}{2}\bar{y}^2 + \varepsilon V(\bar{x}) + \mu\varepsilon^{p+2\nu+3}[\bar{y}\partial_x S^2 - V'(\bar{x})\partial_{\bar{y}}S^2] \\ &\quad +\mu^2\varepsilon^{2p+4\nu+2}\partial_y F_\nu \partial_x S^1 + \mu^2\varepsilon^{2p+2}R^\nu_{2k-2} + \mu^2\varepsilon^{2p+4\nu+3}r_{2k-2}. \end{aligned}$$

Therefore we can take

$$R^{\nu+1}_{2k-2} = \varepsilon^{4\nu}\partial_y F_\nu \partial_x S^1 + R^\nu_{2k-2} + \varepsilon^{4\nu+1}r_{2k-2}. \tag{3.3.7}$$

We will need information on the orders of the terms in the Hamiltonian and the factors y they have. From (3.3.2) and (3.3.4) we write

$$S^1 = \sum_{l=0}^{\nu} y^{2l} p_{j_l}.$$

Then we have that

$$\partial_x S^1 = \sum_{l=0}^{\nu} y^{2l} p_{j_l - 1}$$

and

$$\partial_y S^1 = \sum_{l=1}^{\nu} y^{2l-1} p_{j_l} + \sum_{l=0}^{\nu} y^{2l} p_{j_l - 1} = p_{j_0 - 1} + \sum_{l=1}^{\nu} y^{2l-1} p_{j_l}$$

according to Definition 3.3.1. Moreover, since $\partial_\theta S^2 = V'(x)\partial_{\bar{y}} S^1 - \bar{y}\partial_x S^1$,

$$S^2 = p_{j_0+n-2} + \sum_{l=1}^{\nu} y^{2l-1} p_{j_l+n-1} + \sum_{l=0}^{\nu} y^{2l+1} p_{j_l-1}$$

and then

$$\begin{aligned}
\partial_x S^2 &= p_{j_0+n-3} + \sum_{l=1}^{\nu} y^{2l-1} p_{j_l+n-2} + \sum_{l=0}^{\nu} y^{2l+1} p_{j_l-2}, \\
\partial_y S^2 &= p_{j_0+n-3} + \sum_{l=1}^{\nu} y^{2l-2} p_{j_l+n-1} + \sum_{l=0}^{\nu} y^{2l} p_{j_l-1}.
\end{aligned}$$

Therefore, the function

$$F_{\nu+1}(\bar{x}, \bar{y}, \theta) = \bar{y}\partial_x S^2 - V'(\bar{x})\partial_y S^2$$

is of the form

$$\begin{aligned}
F_{\nu+1} = &p_{j_0+2n-4} + p_{j_1+2n-2} + p_{j_0+n-2} + y p_{j_0+n-3} \\
&+ \sum_{l=1}^{\nu-1} y^{2l}(p_{j_l+n-2} + p_{j_{l-1}-2} + p_{j_{l+1}+2n-2}) \\
&+ y^{2\nu}(p_{j_\nu+n-2} + p_{j_{\nu-1}-2}) + y^{2\nu+2} p_{j_\nu-2}.
\end{aligned}$$

But we observe that, if i_l are the analogous integers to j_l with $\nu+1$ instead of ν,

$$\begin{aligned} j_{l-1}-2 &= j_l+n-2 = n(\nu+1-l)+k-2(\nu+1) = i_l, \quad \text{for } 1 \le l \le \nu, \\ j_{l+1}+2n-2 &= i_l, \quad \text{for } 0 \le l \le \nu-1, \\ j_\nu - 2 &= k-2(\nu+1) = i_{\nu+1} \end{aligned}$$

thus

$$F_{\nu+1} = \sum_{l=0}^{\nu+1} y^{2l} p_{n(\nu+1-l)+k-2(\nu+1)} = \sum_{l=0}^{\nu+1} y^{2l} p_{i_l}.$$

Since S^1 and S^2 have zero mean, we also have

$$\int_0^{2\pi} F_{\nu+1}(x,y,\theta,\mu,\varepsilon)\, d\theta = 0.$$

Finally we discuss the regularity of $\mathcal{C}_{\nu+1}$ and $\mathcal{H}_{\nu+1}$. By hypothesis we have that F_ν is continuous and analytic with respect to (x,y,μ). Then S^1 and S^2 will be continuous, C^1 with respect to (x,y,μ,θ) and analytic with respect to (x,y,μ). To get the change of variables we have to apply the implicit function theorem (I.F.T.) to

$$(x,y,\bar{x},\bar{y},\theta,\mu,\varepsilon) \mapsto (\bar{x}-\partial_{\bar{y}}S(x,\bar{y},\theta,\mu,\varepsilon), y-\partial_x S(x,\bar{y},\theta,\mu,\varepsilon)).$$

This map is C^1 with respect to $(x,y,\bar{x},\bar{y},\theta,\mu)$ and continuous. A generalized version of the I.F.T. gives that we can obtain

$$(\bar{x},\bar{y}) = g(x,y,\theta,\mu,\varepsilon)$$

with g C^1 with respect to x, y, θ, μ and continuous.

A new application of the I.F.T. for analytic functions, with θ and ε fixed, gives, by uniqueness, that g also is analytic with respect to x, y, μ.

Then the result holds. ■

Now we use the previous lemma in order to perform $n+1$ steps of averaging.

Lemma 3.3.4 *There exists a canonical change of variables $(x,y,\theta) = \mathcal{C}(\bar{x},\bar{y},\theta,\mu,\varepsilon)$ which is C^0 in $(\bar{x},\bar{y},\theta,\mu,\varepsilon)$, C^1 and 2π-periodic in θ and analytic in $(\bar{x},\bar{y},\mu)$ and it transforms the Hamiltonian εH to*

$$\begin{aligned} \varepsilon\mathcal{H}(\bar{x},\bar{y},\theta,\mu,\varepsilon) =& \varepsilon h_0(\bar{x},\bar{y}) + \mu\varepsilon^{p+2n+3}F(\bar{x},\bar{y},\theta,\mu,\varepsilon) \\ &+ \mu^2\varepsilon^{2p+2}R_{2k-2}(\bar{x},\bar{y},\theta,\mu,\varepsilon) \end{aligned} \tag{3.3.8}$$

in a neighborhood of the origin, where $F \in P_{2n-2}$ *and has zero mean with respect to* θ, $R_{2k-2} \in P_{2k-2}$ *and*

$$R_{2k-2} = \partial_y h_1 \partial_x S^1 + \varepsilon r_{2k-2}$$

with S^1 *such that* $\partial_\theta S^1(x, \bar{y}, \theta) = -h_1(x, \bar{y}, \theta)$ *and has zero mean with respect to* θ*, and* $r_{2k-2} \in P_{2k-2}$*. Moreover* $\mathcal{H}$ *is continuous in* $(\bar{x}, \bar{y}, \theta, \mu, \varepsilon)$ *and analytic in* $(\bar{x}, \bar{y}, \mu)$*.*

Proof. Since $h_1 \in P_k$, we note that H has the form (3.3.2) for $\nu = 0$ and $R^0_{2k-2} = 0$. Then we begin with $F_0 = h_1$ and $R^0_{2k-2} = 0$ and we apply iteratively $n+1$ times the Lemma 3.3.2. In this way we obtain that $F \equiv F_{n+1}$ has the form

$$F_{n+1}(\bar{x}, \bar{y}, \theta) = \sum_{l=0}^{n+1} \bar{y}^{2l} p_{j_l}(\bar{x}, \bar{y}, \theta) \tag{3.3.9}$$

with $j_l = \max\{0, n(n+1-l) + k - 2(n+1)\}$ and $p_{j_l} \in P_{j_l}$. And from (3.3.7), we can write R^{n+1}_{2k-2} as

$$R^{n+1}_{2k-2} = \partial_y h_1 \partial_x S^1 + \varepsilon r_{2k-2}$$

where S^1 is the one which corresponds to the first change $\mathcal{C}_1$. Moreover the function F_{n+1} has zero mean with respect to θ. We observe that the Hamiltonians $\mathcal{H}_1, \dots, \mathcal{H}_{n-1}$ are C^0, 2π periodic in θ and analytic with respect to x, y and μ. The changes $\mathcal{C}_1, \dots, \mathcal{C}_{n+1}$ are C^1 with respect to θ.

We prove now that, if a function has the form given in (3.3.9), then it belongs to P_{2n-2}. We have that $j_l = \max\{0, n(n-1-l) + k - 2\}$, if $l \le n-1$, then $j_l > 0$ and

$$2l + j_l = n(n-1) - (n-2)l + k - 2 \ge 2(n-1).$$

And if $n \le l \le n+1$,

$$2l + j_l \ge 2n > 2n - 2.$$

Hence $F = F_{n+1} \in P_{2n-2}$. ■

Remark 3.3.5 *We observe that* $\mu^2 \varepsilon^{2p+2} R_{2k-2}$ *can be written as* $\mu^2 \varepsilon^{p+3} R_{2k-2}$*. Indeed, if* $p \ge 1$ *it follows from the comparison of powers of* ε*. And, if* $p < 1$*, by hypothesis* $\partial_y h_1 \partial_x h_1 = 0$ *which implies that* $\partial_y h_1 \partial_x S^1 = 0$*.*

Remark 3.3.6 *We observe that the change $\mathcal{C}$ is of the form*

$$\mathcal{C}(\bar{x},\bar{y},\theta) = (\bar{x},\bar{y},\theta) + \mu\varepsilon^{p+1}(G(\bar{x},\bar{y},\theta),0) + O(\mu\varepsilon^{p+2})$$

where G is such that $\partial_\theta G = (\partial_y h_1, -\partial_x h_2)$. We use this in order to prove the third property of Theorem 3.2.1.

We rename the variables $(\bar{x},\bar{y})$ by (x,y) then, as a consequence of the previous remark, the system in the new variables is

$$\begin{aligned} x' &= \varepsilon y + \mu\varepsilon^{p+2n+3}\partial_y F + \mu^2\varepsilon^{p+3}\partial_y R_{2k-2} \\ y' &= -\varepsilon V'(x) - \mu\varepsilon^{p+2n+3}\partial_x F - \mu^2\varepsilon^{p+3}\partial_x R_{2k-2}. \end{aligned} \tag{3.3.10}$$

Here $'$ means derivative with respect to θ.

3.4 Estimates for the Poincaré map

3.4.1 Notation

In this section we calculate the Poincaré map of the equation (3.3.10) defined as follows:

$$P^{\theta_0}_{\mu,\varepsilon}(x,y) = \varphi_{\mu,\varepsilon}(\theta_0 + 2\pi, \theta_0, x, y) \tag{3.4.1}$$

where $\varphi_{\mu,\varepsilon}(\theta,\theta_0,x,y)$ is the solution of the system (3.3.10) such that $\varphi_{\mu,\varepsilon}(\theta_0,\theta_0,x,y) = (x,y)$. If there is not danger of confusion, we denote $\varphi_{\mu,\varepsilon}(\theta,\theta_0,x,y)$ by $\varphi_\mu(\theta)$.

Let $U \subset \mathbb{R}^2$ be a neighborhood of the origin and let

$$V(\theta_0) = \cup_{s\in[0,1]}\varphi_\mu(\theta_0 + s2\pi, \theta_0, U)$$

and

$$\tilde{V} = \bigcup_{\theta_0\in\mathbb{R}} V(\theta_0)$$

Then, since the flow depends 2π-periodically on θ_0, the set $\tilde{V}$ is bounded.

We denote by $\mu\varepsilon^{p+2n+3}F_{\mu,\varepsilon}(x,y,\theta)$ the terms of order $\mu\varepsilon^{p+2n+3}$ in (3.3.10), we write $F_{\mu,\varepsilon} = (F^1,F^2)$ and we recall that $F_{\mu,\varepsilon} \in P_{2n-3}$. We denote

$$\mu^2\varepsilon^{p+3}R_{2k-3} = \mu^2\varepsilon^{p+3}(\partial_y R_{2k-2}, -\partial_x R_{2k-2})$$

the remaining part of (3.3.10). Moreover to simplify the notation we introduce $z = (x, y)$ and $\eta = \mu^2\varepsilon^{p+2}$.

We denote by εX_0 the vector field corresponding to the equation (3.3.10) when $\mu = 0$, that is,

$$X_0(x, y) = \begin{pmatrix} y \\ -V'(x) \end{pmatrix}$$

and we denote $\varepsilon X_\mu = \varepsilon X_0 + \mu\varepsilon^{p+2n+3}F_{\mu,\varepsilon} + \mu^2\varepsilon^{p+3}R_{2k-3}$.

It is clear that X_μ is bounded in $\tilde{V}$ and it is 2π-periodic on θ, thus there exists some constant M (independent on θ) such that, $\|X_\mu(x, y, \theta)\| \leq M$ for all $(x, y) \in \tilde{V}$ and $\theta \in \mathbb{R}$.

Moreover X_μ is Lipschitz in $\tilde{V}$. We denote by L_μ the Lipschitz constant of X_μ.

3.4.2 Some preliminary bounds

In order to determine the properties of the Poincaré map defined in (3.4.1) we need a precise knowledge of the distance between a solution and its initial condition, as well as the distance between the solutions of the unperturbed system, $\varphi_0(\theta)$, and the solutions of the perturbed one, $\varphi_\mu(\theta)$. This is studied in this subsection.

Remark 3.4.1 *As before we make the convention that if $l < 0$ in $\|(x, y)\|^l$ we understand that it represents a constant term.*

We need a simple lemma:

Lemma 3.4.2 *Let $\Omega \subset \mathbb{C}^2 \times \mathbb{R} \times \mathbb{R}^2$ be a neighborhood of $\{(0, 0)\} \times \mathbb{R} \times \{(0, 0)\}$ and let $f : \Omega \to \mathbb{R}$ be a function that is continuous, C^1 with respect to θ and analytic with respect to (x, y, μ) such that there exists a constant $c > 0$ verifying*

$$\|f(x, y, \theta, \mu, \varepsilon)\| \leq c|y|^i\|(x, y)\|^l$$

for all $(x, y, \theta, \mu, \varepsilon) \in \Omega$. Then there exists a function f_l continuous, C^1 with respect to θ, analytic with respect to (x, y, μ), $f(., ., \theta, \mu, \varepsilon) \in P_l$ such that

$$f(x, y, \theta, \mu, \varepsilon) = y^i f_l(x, y, \theta, \mu, \varepsilon).$$

Proof. We take $f_l(x,y,\theta,\mu,\varepsilon)=f(x,y,\theta,\mu,\varepsilon)/y^l$. Obviously we have to prove that f_l is analytic at points of the form $(x,0,\theta,\mu,\varepsilon)\in\Omega$. We consider $(x,0,\theta,\mu,\varepsilon)\in\Omega$ and y small enough so that the Taylor series of f with respect to y at $(x,0,\theta,\mu,\varepsilon)$ converges at y. Then by Taylor's theorem

$$\begin{aligned} f(x,y,\theta,\mu,\varepsilon) =& f(x,0,\theta,\mu,\varepsilon)+\ldots+\frac{1}{(i-1)!}\partial_y^{i-1}f(x,0,\theta,\mu,\varepsilon)y^{i-1} \\ &+\frac{1}{(i-1)!}\int_0^1(1-s)^{i-1}\partial_y^i f(x,sy,\theta,\mu,\varepsilon)y^i\,ds. \end{aligned}$$

It is clear that $\partial_y^j f(x,0)=0$ if $j<i$, and then

$$f(x,y,\theta,\mu,\varepsilon)=y^i\frac{1}{(i-1)!}\int_0^1(1-s)^{i-1}\partial_y^i f(x,sy,\theta,\mu,\varepsilon)ds. \tag{3.4.2}$$

Since f is analytic with respect to y, the derivatives $\partial_y^j f$ also are continuous, C^1 with respect to θ and analytic with respect to (x,y,μ). Therefore

$$f_l(x,y,\theta,\mu,\varepsilon)\equiv\frac{1}{(i-1)!}\int_0^1(1-s)^{i-1}\partial_y^i f(x,sy,\theta,\mu,\varepsilon)ds$$

also has the same kind of regularity. Moreover the hypotheses of the present lemma imply that $\|f_l(x,y,\theta,\mu,\varepsilon)\|\le c\|(x,y)\|^l$. Hence if $j<l$, $D^j_{(x,y)}f_l(0,0,\theta,\mu,\varepsilon)=0$ and therefore

$$f_l(x,y,\theta,\mu,\varepsilon)=\sum_{k=l}^{+\infty}\frac{1}{k!}D^k_{(x,y)}f_l(0,0,\theta,\mu,\varepsilon)(x,y)^k\in P_l.$$

■

Lemma 3.4.3 *Let $\varphi_\mu(\theta)=\varphi(\theta,\theta_0,x,y,\mu,\varepsilon)$ be the solution of*

$$\dot z=\varepsilon J(Dh_0(z)+\mu\varepsilon^{p+2n+3}DF(z,\theta,\mu,\varepsilon)+\mu^2\varepsilon^{p+3}DR(z,\theta,\mu,\varepsilon)).$$

If $\theta\in[\theta_0,\theta_0+2\pi]$ and $z=(x,y)\in U$ then there exists some constants C, C_F, μ_0 and ε_0 such that for all $|\mu|\le\mu_0$ and $|\varepsilon|\le\varepsilon_0$ the following bounds hold:

1) $\|\varphi_\mu(\theta)\|\le C\|z\|$,

2) $\|\varphi_\mu(\theta)-(x,y)\|\le\varepsilon C(|y|+\|z\|^{n-1}+\mu^2\varepsilon^{p+2}\|z\|^{2k-3})$, *3) The perturbed solution φ_μ can be expressed as*

$$\varphi_\mu(\theta) = \varphi_0(\theta) + \mu\varepsilon^{p+2n+3}\Psi_\mu(\theta, \theta_0, x, y) + \mu^2\varepsilon^{p+3}\Phi_\mu(\theta, \theta_0, x, y)$$

with

$$\|\Psi_\mu(\theta, \theta_0, x, y)\| \leq C_F\|z\|^{2n-3}$$
$$\|\Phi_\mu(\theta, \theta_0, x, y)\| \leq C\|z\|^{2k-3}$$

where, if $F_{\mu,\varepsilon} = 0$, $\Psi_\mu = 0$. Moreover, Ψ_μ and Φ_μ are C^0 with respect to ε, C^1 with respect to θ and θ_0 and analytic with respect to μ and initial conditions (x, y).

4) The functions

$$\psi_{\mu,\varepsilon}(x, y, \theta_0) \equiv \Psi_\mu(\theta_0 + 2\pi, \theta_0, x, y)$$
$$R_{2k-3}(x, y, \theta_0) \equiv \Phi_\mu(\theta_0 + 2\pi, \theta_0, x, y)$$

are such that, $\psi_{\mu,\varepsilon} \in P_{2n-3}$ and $R_{2k-3} \in P_{2k-3}$. Moreover if $F_{\mu,\varepsilon} = 0$, $\psi_{\mu,\varepsilon} \equiv 0$.

Proof. The proof is straightforward. We recall that the origin is a fixed point. In order to prove the first bound, we write the equations in the integral form. We have that

$$\begin{aligned}\|\varphi_\mu(\theta)\| &\leq \|z\| + \varepsilon\int_{\theta_0}^{\theta} \|X_\mu(\varphi_\mu(s), s) - X_\mu(0, 0, s)\|ds \\ &\leq \|z\| + \varepsilon L_\mu\int_{\theta_0}^{\theta} \|\varphi_\mu(s)\|ds\end{aligned}$$

thus, by Gronwall's lemma

$$\|\varphi_\mu(\theta)\| \leq \|z\|e^{\varepsilon L_\mu(\theta-\theta_0)} \leq \|z\|e^{\varepsilon L_\mu 2\pi}, \qquad \theta \in [\theta_0, \theta_0 + 2\pi]$$

as we want. Moreover for the second one, we note that

$$\sup_{s\in[\theta_0,\theta_0+2\pi]} \|X_\mu(x, y, s)\| \leq C(|y| + \|z\|^{n-1} + \eta\|z\|^{2k-3})$$

where the bound of $\mu\varepsilon^{p+2n+3}F_{\mu,\varepsilon}$ is included in $C(|y| + \|z\|^{n-1})$, then

$$\begin{aligned}\|\varphi_\mu(\theta) - z\| &\leq \varepsilon\int_{\theta_0}^{\theta} \|X_\mu(\varphi_\mu(s), s)\|ds \\ &\leq \varepsilon\int_{\theta_0}^{\theta} \|X_\mu(\varphi_\mu(s), s) - X_\mu(z, s)\|ds + \varepsilon\int_{\theta_0}^{\theta} \|X_\mu(z, s)\|ds \\ &\leq \varepsilon L_\mu\int_{\theta_0}^{\theta} \|\varphi_\mu(s) - z\|ds + 2\pi\varepsilon C(|y| + \|z\|^{n-1} + \eta\|z\|^{2k-3})\end{aligned}$$

therefore, by Gronwall's lemma we have that

$$\|\varphi_\mu(\theta) - z\| \le 2\pi\varepsilon C(|y| + \|z\|^{n-1} + \eta\|z\|^{2k-3})e^{2\pi\varepsilon L_\mu}.$$

This proves what we want.

To deal with the last properties, we look for solutions of (3.3.10) of the form

$$\varphi_\mu(\theta) = \varphi_0(\theta) + \mu\varepsilon^{p+2n+3}\Psi_\mu(\theta, \theta_0, x, y) + \mu^2\varepsilon^{p+3}\Phi_\mu(\theta, \theta_0, x, y).$$

We denote $\Psi_\mu(\theta, \theta_0, x, y)$ by $\Psi_\mu(\theta)$ and $\Phi_\mu(\theta, \theta_0, x, y)$ by $\Phi_\mu(\theta)$. We observe that

$$\begin{aligned} X_0(\varphi_\mu(\theta)) =& X_0(\varphi_0(\theta)) + X_0(\varphi_0(\theta) + \mu\varepsilon^{p+2n+3}\Psi_\mu(\theta)) - X_0(\varphi_0(\theta)) \\ &+ X_0(\varphi_\mu(\theta)) - X_0(\varphi_0(\theta) + \mu\varepsilon^{p+2n+3}\Psi_\mu(\theta)). \end{aligned}$$

Hence φ_μ satisfies, $\dot\varphi_\mu = \varepsilon X_\mu(\varphi_\mu, \theta)$ with initial condition $\varphi_\mu(\theta_0) = (x, y)$ and if we look for φ_μ of the form $\varphi_\mu = \varphi_0 + \mu\varepsilon^{p+2n+3}\Psi_\mu + \mu^2\varepsilon^{p+3}\Phi_\mu$ we have that

$$\dot\varphi_0 = \varepsilon X_0(\varphi_0)$$

$$\dot\Psi_\mu = \frac{1}{\mu\varepsilon^{p+2n+2}}[X_0(\varphi_0 + \mu\varepsilon^{p+2n+3}\Psi_\mu) - X_0(\varphi_0)] + F_{\mu,\varepsilon}(\varphi_\mu, \theta) \tag{3.4.3}$$

$$\dot\Phi_\mu = \frac{1}{\mu^2\varepsilon^{p+2}}[X_0(\varphi_\mu) - X_0(\varphi_0 + \mu\varepsilon^{p+2n+3}\Psi_\mu)] + R_{2k-3}(\varphi_\mu, \theta) \tag{3.4.4}$$

with initial conditions as follows:

$$\varphi_0(\theta_0) = (x, y), \qquad \Psi_\mu(\theta_0) = \Phi_\mu(\theta_0) = (0, 0).$$

We observe that the functions Ψ_μ and Φ_μ are C^1 with respect to θ and θ_0, C^0 with respect to ε and analytic with respect to μ. We deal first with the differential equation for Ψ_μ:

$$\begin{aligned} \Psi_\mu(\theta) =& \frac{1}{\mu\varepsilon^{p+2n+2}}\int_{\theta_0}^{\theta}[X_0(\varphi_0(s) + \mu\varepsilon^{p+2n+3}\Psi_\mu(s)) - X_0(\varphi_0(s))]ds \\ &+ \int_{\theta_0}^{\theta} F_{\mu,\varepsilon}(\varphi_\mu(s), s)ds \end{aligned}$$

and, since X_0 is Lipschitz and $\|\varphi_\mu(s)\| \le C\|z\|$ we have that

$$\begin{aligned} \|\Psi_\mu(\theta)\| &\le L_0\varepsilon\int_{\theta_0}^{\theta}\|\Psi_\mu(s)\|ds + \int_{\theta_0}^{\theta}\|F_{\mu,\varepsilon}(\varphi_\mu(s), s)\|ds \\ &\le L_0\varepsilon\int_{\theta_0}^{\theta}\|\Psi_\mu(s)\|ds + \tilde C_F\|z\|^{2n-3}. \end{aligned}$$

An application of the Gronwall's lemma gives the bound

$$\|\Psi_\mu(\theta)\| \le \tilde{C}_F e^{L_0\varepsilon 2\pi}\|z\|^{2n-3}, \qquad \theta \in [\theta_0, \theta_0 + 2\pi] \tag{3.4.5}$$

with $\tilde{C}_F = 0$ if $F_{\mu,\varepsilon} = 0$. It is clear that

$$\psi_{\mu,\varepsilon}(\theta_0, x, y) \equiv \Psi_\mu(\theta_0 + 2\pi, \theta_0, x, y)$$

is 2π-periodic in θ_0 and analytic with respect to initial conditions. Therefore, by Lemma 3.4.2, $\psi_{\mu,\varepsilon} \in P_{2n-3}$.

Analogously, for the equation (3.4.4) we obtain the estimate

$$\begin{aligned}\|\Phi_\mu(\theta)\| &\le L_0\varepsilon \int_{\theta_0}^{\theta} \|\Phi_\mu(\theta)\| ds + \int_{\theta_0}^{\theta} \|R_{2k-3}(\varphi_\mu(s), s)\| ds \\ &\le L_0\varepsilon \int_{\theta_0}^{\theta} \|\Phi_\mu(\theta)\| ds + C\|z\|^{2k-3}.\end{aligned}$$

As before, Gronwall's lemma gives the bound

$$\|\Phi_\mu(\theta)\| \le C\|z\|^{2k-3}, \qquad \theta \in [\theta_0, \theta_0 + 2\pi]. \tag{3.4.6}$$

It is clear that the function

$$R_{2k-3}(x, y, \theta_0) = \Phi_\mu(\theta_0 + 2\pi, \theta_0, x, y)$$

is 2π-periodic in θ_0 and analytic with respect to (x, y). Moreover by estimate (3.4.6) and Lemma 3.4.2, $R_{2k-3} \in P_{2k-3}$. ■

Now we look for the form of the Poincaré map $P_\mu^{\theta_0}$, given in (3.4.1), of the system (3.3.10).

Lemma 3.4.4 *The Poincaré map $P_\mu^{\theta_0}$ of the system (3.3.10) is*

$$\begin{aligned}P_\mu^{\theta_0}(x, y) = \begin{pmatrix} x + 2\pi\varepsilon y \\ y \end{pmatrix} + 2\pi\varepsilon \begin{pmatrix} 2\pi\varepsilon q_1(x, y, \varepsilon) \\ -V'(x) + 2\pi\varepsilon q_2(x, y, \varepsilon) \end{pmatrix} \\ + \mu\varepsilon^{p+2n+3}\psi_{\mu,\varepsilon}(x, y, \theta_0) + \mu^2\varepsilon^{p+3} R_{2k-3}(x, y, \theta_0).\end{aligned} \tag{3.4.7}$$

where q_1, $q_2 \in P_{n-1}$(independent of μ), $\psi_{\mu,\varepsilon} \in P_{2n-3}$ and $R_{2k-3} \in P_{2k-3}$.

Proof. By properties 3) and 4) of Lemma 3.4.3 we only have to compute the Poincaré map of the unperturbed system, which is independent of θ_0 since the unperturbed system is autonomous:

$$\begin{aligned}
P_0^{\theta_0}(x,y) &= \varphi_0(\theta_0+2\pi)\\
&= (x,y)+2\pi\varphi_0'(\theta_0)+(2\pi)^2\int_0^1(1-s)\varphi_0''(\theta_0+s2\pi)ds\\
&= \begin{pmatrix} x\\ y\end{pmatrix}+2\pi\varepsilon\begin{pmatrix} y\\ -V'(x)\end{pmatrix}\\
&\quad +(2\pi)^2\varepsilon^2\begin{pmatrix} \int_0^1(1-s)V'(\varphi_0^1(\theta_0+s2\pi))ds\\ -\int_0^1(1-s)V''(\varphi_0^1(\theta_0+s2\pi))\varphi_0^2(\theta_0+s2\pi)ds\end{pmatrix}
\end{aligned}$$

It is clear that

$$\begin{aligned}
|V'(\varphi_0^1(\theta_0+s2\pi))| &\leq C\|z\|^{n-1}\\
|V''(\varphi_0^1(\theta_0+s2\pi))\varphi_0^2(\theta_0+s2\pi)| &\leq C\|z\|^{n-1}
\end{aligned}$$

Hence, by Lemma 3.4.2,

$$\begin{aligned}
V'(\varphi_0^1(\theta_0+s2\pi)) &\in P_{n-1}\\
V''(\varphi_0^1(\theta_0+s2\pi))\varphi_0^2(\theta_0+s2\pi) &\in P_{n-1}.
\end{aligned}$$

Therefore, since the unperturbed system is autonomous,

$$\begin{aligned}
P_0^{\theta_0}(x,y) &= \varphi_0(\theta_0+2\pi)\\
&= \begin{pmatrix} x\\ y\end{pmatrix}+2\pi\varepsilon\begin{pmatrix} y\\ -V'(x)\end{pmatrix}+(2\pi\varepsilon)^2\begin{pmatrix} q_1(x,y,\varepsilon)\\ q_2(x,y,\varepsilon)\end{pmatrix}
\end{aligned}$$

and the results holds. ■

Remark 3.4.5 *We recall that until now we have not used the hypothesis $2k-2\geq n$ and that all the results given in this section and in the previous sections are true for $k\geq 2$. Thus, they will be applicable in the next chapter where we will deal with what we call the weak hyperbolic case.*

3.5 A useful intermediate system

In this section we will find a system such that its stable manifold is closer to the homoclinic orbit of the unperturbed system than the stable manifold of the perturbed one. We will find it by imposing that its Poincaré map contains some of the smallest degree terms of the remainder R_{2k-3} in (3.4.7). This is necessary if $k < n$. We recall that

$$2k - 2 \geq n.$$

We discompose $R_{2k-3} = (R^1_{2k-3}, R^2_{2k-3})$ in the following form

$$\begin{aligned} R^1_{2k-3}(x,y) &= \sum_{j=2k-3}^{2n-4} p^0_j x^j + y \sum_{j=2k-4}^{2n-4} p^1_j x^j + y^2 \sum_{j=2k-5}^{2n-4} p^2_j x^j \\ &\quad + y^3 r^1_{2k-6}(x,y) + p_{2n-3}(x,y) \\ &= \sum_{j=2k-3}^{2n-4} q^0_j x^j + y \sum_{j=2k-4}^{2n-4} q^1_j x^j + y^2 \sum_{j=2k-5}^{2n-4} q^2_j x^j \\ &\quad + y^3 \sum_{j=2k-6}^{2n-4} q^3_j x^j + y^4 r^2_{2k-7}(x,y) + q_{2n-3}(x,y) \end{aligned} \tag{3.5.1}$$

where $r^1_{2k-6} \in P_{2k-6}$, $r^2_{2k-7} \in P_{2k-7}$, $p_{2n-3} \in P_{2n-3}$ and $q_{2n-3} \in P_{2n-3}$. Also p^l_j and q^l_j are constants with respect to (x, y) but depend on $(\theta_0, \varepsilon, \mu)$. Since R_{2k-3} is analytic with respect to (x, y) and depends C^1 with respect to θ_0, analytically with respect to μ and continuously with respect to ε, the constants $\{p^l_j\}$ and $\{q^l_j\}$ also have the same kind of dependence.

We will look for an auxiliary system of the form

$$\dot{z} = Y_\mu(z) = \varepsilon X_0 + \mu^2 \varepsilon^{p+3} Y_1 \tag{3.5.2}$$

with $z = (x, y)$, $Y_1 = (Y^1_1, Y^2_1)$ and

$$\begin{aligned} Y^1_1(x,y) &= \sum_{j=2k-3}^{2n-4} a^0_j x^j + y \sum_{j=2k-4}^{2n-4} a^1_j x^j + y^2 \sum_{j=2k-5}^{2n-4} a^2_j x^j \\ Y^2_1(x,y) &= \sum_{j=2k-3}^{2n-4} b^0_j x^j + y \sum_{j=2k-4}^{2n-4} b^1_j x^j + y^2 \sum_{j=2k-5}^{2n-4} b^2_j x^j + y^3 \sum_{j=2k-6}^{2n-4} b^3_j x^j \end{aligned}$$

where a^i_j and b^i_j are constants to be determined later. They will depend on θ_0, ε and μ. We note that $Y_1 \in P_{2k-3}$.

From now on we omit the dependence on θ_0. We recall the notation $\eta = \mu^2 \varepsilon^{p+2}$.

Now we may consider the Poincaré map of the system (3.5.2). Since it is autonomous its flow is of the form

$$\phi(\theta, x, y) = (\phi_1(\theta, x, y), \phi_2(\theta, x, y))$$

and then its Poincaré map is defined by

$$\hat{P}(x, y) = \phi(2\pi, x, y).$$

Next lemma assures that there exist coefficients a_j^l, b_j^l such that the Poincaré map of (3.5.2) and the Poincaré map (3.4.7) we are interested in coincide up to terms of order $2n-4$ in the variables (x, y) and in all terms which do not depend on μ.

Lemma 3.5.1 *There exists $\{a_j^l(\theta_0, \mu, \varepsilon)\}$ and $\{b_j^l(\theta_0, \mu, \varepsilon)\}$ such that Poincaré map of the system (3.5.2) is of the form*

$$\begin{aligned}\hat{P}_\mu^{\theta_0}(x, y) = &\begin{pmatrix} x + 2\pi\varepsilon y \\ y \end{pmatrix} + 2\pi\varepsilon \begin{pmatrix} 2\pi\varepsilon q_1(x, y, \varepsilon) \\ -V'(x) + 2\pi\varepsilon q_2(x, y, \varepsilon) \end{pmatrix} \\ &+ 2\pi\mu^2\varepsilon^{p+3} \begin{pmatrix} \displaystyle\sum_{l=0}^{2} y^l \sum_{j=2k-3-l}^{2n-4} p_j^l x^j \\ \displaystyle\sum_{l=0}^{3} y^l \sum_{j=2k-3-l}^{2n-4} q_j^l x^j \end{pmatrix} \\ &+ \mu^2\varepsilon^{p+4} y^3 H(x, y) + \mu^2\varepsilon^{p+4}\tilde{R}_{2n-3}\end{aligned} \tag{3.5.3}$$

where the functions q_1 and q_2 are the same that appear in Lemma 3.4.4, p_j^l and q_j^l are the same that appear in (3.5.1), $\tilde{R}_{2n-3} \in P_{2n-3}$, $H = (H_1, H_2)$ with $H_1 \in P_{2k-6}$ and $H_2 = y\tilde{H}_2$ with $\tilde{H}_2 \in P_{2k-7}$. Moreover all these functions are C^1 and 2π-periodic with respect to θ_0, continuous in ε and analytic in μ.

Proof. We denote by $\phi(\theta) = \phi(\theta, x, y)$ the flow of the system (3.5.2). Applying Lemma 3.4.3 to the auxiliary system (3.5.2), taking $F = 0$, we have that

$$\phi(\theta_0 + 2\pi) = \varphi_0(\theta_0) + \mu^2\varepsilon^{p+3}\phi_{2k-3}(x, y, \theta_0)$$

where φ_0 is the solution of the unperturbed system $z' = \varepsilon X_0(z)$, $z(0) = (x, y)$ and $\phi_{2k-3} \in P_{2k-3}$. Moreover, by Taylor's theorem, the Poincaré map of (3.5.2), $\hat{P}(x, y) = \phi(2\pi)$, is

$$\begin{aligned}\hat{P}(x, y) &= \phi(2\pi) = \varphi_0(2\pi) + (\phi(2\pi) - \varphi_0(2\pi)) \\ &= P_0(x, y) + 2\pi(\phi'(0) - \varphi_0'(0)) + (2\pi)^2 \int_0^1 (1 - s)[\phi''(s2\pi) - \varphi_0''(s2\pi)]ds,\end{aligned}$$

where P_0 is the Poincaré map of the equation $z' = \varepsilon X_0(z)$ which therefore is independent on θ_0. Then, $\hat{P}(x,y)$ can be written as

$$\hat{P}(x,y) = P_0(x,y) + 2\pi\varepsilon\eta Y_1(x,y) + (2\pi)^2 \int_0^1 (1-s)[\phi''(s2\pi) - \varphi_0''(s2\pi)]ds. \quad (3.5.4)$$

From $\phi' = \varepsilon X_0(\phi) + \mu^2\varepsilon^{p+3}Y_1(\phi)$ we obtain

$$\begin{aligned}
\phi'' &= \varepsilon DX_0(\phi)Y_\mu(\phi) + \mu^2\varepsilon^{p+3}DY_1(\phi)Y_\mu(\phi) \\
&= \varepsilon^2 DX_0(\varphi_0)X_0(\varphi_0) + \varepsilon^2[DX_0(\phi)X_0(\phi) - DX_0(\varphi_0)X_0(\varphi_0)] \\
&\quad +\mu^2\varepsilon^{p+4}DX_0(\phi)Y_1(\phi) + \mu^2\varepsilon^{p+3}DY_1(\phi)Y_\mu(\phi) \\
&= \varphi_0'' + \varepsilon^2[DX_0(\phi)X_0(\phi) - DX_0(\varphi_0)X_0(\varphi_0)] \\
&\quad +\mu^2\varepsilon^{p+4}DX_0(\phi)Y_1(\phi) + \mu^2\varepsilon^{p+3}DY_1(\phi)Y_\mu(\phi)
\end{aligned}$$

hence, by the mean value theorem applied to $DX_0(z)X_0(z)$ and Lemma 3.4.3

$$\begin{aligned}
\|\phi''(\theta) - \varphi_0''(\theta)\| &\leq C\varepsilon^2\|\phi - \varphi_0\| + \mu^2\varepsilon^{p+4}\|DX_0(\phi)Y_1(\phi)\| \\
&\quad +\mu^2\varepsilon^{p+3}\|DY_1(\phi)Y_\mu(\phi)\| \\
&\leq C\mu^2\varepsilon^{p+5}\|z\|^{2k-3} + \mu^2\varepsilon^{p+4}\|DX_0(\phi)Y_1(\phi)\| \\
&\quad +\mu^2\varepsilon^{p+3}\|DY_1(\phi)Y_\mu(\phi)\|
\end{aligned}$$

and since $Y_1 \in P_{2k-3}$, $Y_\mu \in P_1$ and $Y_\mu = O(\varepsilon)$ we have that

$$\|\phi''(\theta) - \varphi_0''(\theta)\| \leq C\mu^2\varepsilon^{p+4}\|z\|^{2k-3}.$$

Putting this estimate in (3.5.4) and using Lemma 3.4.2, we obtain that

$$\hat{P}(x,y) = P_0(x,y) + 2\pi\varepsilon\eta Y_1(x,y) + \varepsilon^2\eta f_{2k-3}$$

with $f_{2k-3} \in P_{2k-3}$ depending on constants $\{a_j^l\}$ and $\{b_j^l\}$. As usual, we write $f_{2k-3} = (f^1, f^2)$ and

$$\begin{aligned}
f^1(x,y) &= \sum_{j=2k-3}^{2n-4} c_j^0 x^j + y\sum_{j=2k-4}^{2n-4} c_j^1 x^j + y^2\sum_{j=2k-5}^{2n-4} c_j^2 x^j \\
&\quad +y^3 H_1(x,y) + \tilde{R}^1_{2n-3}(x,y) \\
f^2(x,y) &= \sum_{j=2k-3}^{2n-4} d_j^0 x^j + y\sum_{j=2k-4}^{2n-4} d_j^1 x^j + y^2\sum_{j=2k-5}^{2n-4} d_j^2 x^j \\
&\quad +y^3\sum_{j=2k-6}^{2n-4} d_j^3 x^j + y^4\tilde{H}_2(x,y) + \tilde{R}^2_{2n-3}(x,y)
\end{aligned}$$

where $H_1 \in P_{2k-6}$, $\tilde{H}_2 \in P_{2k-7}$ and $\tilde{R}_{2n-3} = (\tilde{R}^1_{2n-3}, \tilde{R}^2_{2n-3}) \in P_{2n-3}$. We consider now the following system with unknowns $\{a^l_j\}$ and $\{b^l_j\}$

$$\begin{aligned} a^l_j + \pi\varepsilon c^l_j &= p^l_j, \qquad l = 0,1,2, \qquad j = 2k-3-l, \dots, 2n-4 \\ b^l_j + \pi\varepsilon d^l_j &= q^l_j, \qquad l = 0,1,2,3, \;\; j = 2k-3-l, \dots, 2n-4 \end{aligned} \tag{3.5.5}$$

where p^l_j and q^j_j are defined in (3.5.1). We recall that p^l_j and q^j_j depend on μ, ε and θ_0. Obviously for $\varepsilon = 0$ the system (3.5.5) has the solution $a^l_j = p^l_j$ and $b^l_j = q^l_j$ which depends on μ and θ_0. Moreover c^l_j and d^l_j depend analytically on a^l_j and b^l_j. Thus, for ε small enough it has a solution depending on ε, μ and θ_0.

Then the Poincaré map $\hat{P}$ of the auxiliary system (3.5.2) where the coefficients a^l_j and b^l_j are chosen to be the solutions of the system (3.5.5) is:

$$\begin{aligned} \hat{P}(x,y) &= P_0(x,y) + 2\pi\varepsilon\eta \begin{pmatrix} \sum_{l=0}^{2} y^l \sum_{j=2k-3-l}^{2n-4} p^l_j x^j \\ \sum_{l=0}^{3} y^l \sum_{j=2k-3-l}^{2n-4} q^l_j x^j \end{pmatrix} \\ &\quad + \varepsilon^2 \eta y^3 H(x,y) + \varepsilon\eta \tilde{R}_{2n-3} \end{aligned} \tag{3.5.6}$$

where the functions H and $\tilde{R}_{2n-3}$ are such that $\tilde{R}_{2n-3} = (\tilde{R}^1_{2n-3}, \tilde{R}^2_{2n-3}) \in P_{2n-3}$, $H = (H_1, H_2)$ with $H_1 \in P_{2k-6}$ and $H_2 = y\tilde{H}_2$ with $\tilde{H}_2 \in P_{2k-7}$. Finally we observe that, since the coefficients p^j_j and q^l_j depend 2π-periodically on θ_0, the coefficients a^l_j and b^l_j are also 2π-periodic on θ_0. Since the coefficients a^l_j and b^l_j depend on θ_0, μ and ε we write

$$\hat{P}^{\theta_0}_\mu = \hat{P}.$$

■

3.6 The operators B and $\mathcal{B}$

The Banach spaces which we use in this section were introduced at the beginning of Section 3.2.

We will need the operator $B_k : \mathcal{X}_k \longrightarrow \mathcal{X}_k$ defined by

$$(B_k\sigma)(t,s) = \sigma(t + 2\pi\varepsilon, s) - \sigma(t,s) \tag{3.6.1}$$

with $\varepsilon > 0$. It is a well defined linear operator with $\|B_k\| \leq 2$. Indeed, it is readily seen that if $\sigma \in \mathcal{X}_k$ then $B_k\sigma \in \mathcal{X}_k$ and that

$$\begin{aligned}(t+\operatorname{Re} s)^k|(B_k\sigma)(t,s)| &\leq (t+\operatorname{Re} s)^k|\sigma(t+2\pi\varepsilon,s)| + (t+\operatorname{Re} s)^k|\sigma(t,s)| \\ &\leq (t+2\pi\varepsilon+\operatorname{Re} s)^k|\sigma(t+2\pi\varepsilon,s)|\left(\frac{t+\operatorname{Re} s}{t+2\pi\varepsilon+\operatorname{Re} s}\right)^k \\ &\quad +(t+\operatorname{Re} s)^k|\sigma(t,s)| \\ &\leq 2\|\sigma\|_k.\end{aligned}$$

Remark 3.6.1 *In fact $\|B_k\| = 2$. For the function $\sigma \in \mathcal{X}_k$ defined by*

$$\sigma(t,s) = \frac{1}{\cosh(a/(2\varepsilon))}\frac{1}{(t+s)^k}\sin\frac{t+s}{2\varepsilon}$$

we have $\|\sigma\|_k = 1$ and $\|B_k\sigma\|_k = 2$.

We will need to find a right side inverse of the operator B_k. For that we write $B_k\sigma = \psi$ from which we can obtain

$$\sigma(t,s) = -\psi(t,s) + \sigma(t+2\pi\varepsilon,s). \tag{3.6.2}$$

Applying (3.6.2) iteratively

$$\sigma(t,s) = -\sum_{j=0}^{N}\psi(t+2\pi\varepsilon j,s) + \sigma(t+2\pi\varepsilon(N+1),s). \tag{3.6.3}$$

If $\sigma \in \mathcal{X}_k$, $\lim_{t\to\infty}\sigma(t,s) = 0$ so that we are allowed to take limit as $N \to \infty$ in (3.6.3) and we obtain the formal expression

$$(B_k^{-1}\psi)(t,s) = -\sum_{j=0}^{\infty}\psi(t+2\pi\varepsilon j,s) \tag{3.6.4}$$

Lemma 3.6.2 *The operator $B_k : \mathcal{X}_k \longrightarrow \mathcal{X}_k$ has right inverses $B_k^{-1} : \mathcal{X}_\ell \longrightarrow \mathcal{X}_k$ with $\ell \geq k+1$ and*

$$\|B_k^{-1}\psi\|_k \leq \frac{1}{T^{\ell-k-1}}\left(\frac{1}{2T} + \frac{1}{2\pi\varepsilon(\ell-1)}\right)\|\psi\|_\ell.$$

In particular, if $T \geq (\ell-1)\pi/4$,

$$\|B_k^{-1}\psi\|_k \leq \frac{1+4\pi\varepsilon}{2\pi\varepsilon k}\|\psi\|_\ell.$$

Proof. We define $\psi_N(t,s) = \sum_{j=0}^{N} \psi(t + 2\pi\varepsilon j, s)$ and
$$(B_k^{-1}\psi)(t,s) = -\lim_{N\to\infty} \psi_N(t,s).$$
Let $\ell \geq k+1$. First we check that if $\psi \in \mathcal{X}_\ell$, ψ_N converges uniformly. Indeed, from
$$|\psi(t + 2\pi\varepsilon j, s)| \leq \frac{1}{(t + 2\pi\varepsilon j + \operatorname{Re} s)^\ell}\|\psi\|_\ell \leq \left(\frac{1}{T + 2\pi\varepsilon j}\right)^\ell \|\psi\|_\ell,$$
the claim follows form the M-test of Weierstrass.

One immediately shows that $B_k^{-1}\psi$ satisfies the first three conditions which define $\mathcal{X}_k$.

Moreover given $\psi \in \mathcal{X}_\ell$
$$\begin{aligned}\|B_k^{-1}\psi\|_k &= \sup_{(t,s)\in D^s} \sum_{j=0}^{\infty} (t + \operatorname{Re} s)^k |\psi(t + 2\pi\varepsilon j, s)| \\ &\leq \sup_{(t,s)\in D^s} \sum_{j=0}^{\infty} \frac{(t + \operatorname{Re} s)^k}{(t + 2\pi\varepsilon j + \operatorname{Re} s)^\ell}\|\psi\|_\ell.\end{aligned}$$
To bound the sum we introduce $u = t + \operatorname{Re} s$ and we bound
$$\sum_{j=0}^{\infty} \frac{(t + \operatorname{Re} s)^k}{(t + 2\pi\varepsilon j + \operatorname{Re} s)^\ell} = \frac{1}{2\pi\varepsilon}\frac{2\pi\varepsilon}{u^{\ell-k}} \sum_{j=0}^{\infty} \frac{1}{\left(1 + \frac{2\pi\varepsilon j}{u}\right)^\ell}.$$
Then the sum can be bounded by
$$\begin{aligned}\frac{1}{2\pi\varepsilon}\frac{1}{u^{\ell-k-1}}\left[\frac{2\pi\varepsilon}{2u} + \int_0^\infty \frac{1}{(1+x)^\ell}\,dx\right] &= \frac{1}{2\pi\varepsilon}\frac{1}{u^{\ell-k-1}}\left[\frac{2\pi\varepsilon}{2u} + \frac{1}{\ell-1}\right] \\ &= \frac{1}{2u^{\ell-k}} + \frac{1}{2\pi\varepsilon(\ell-1)u^{\ell-k-1}}.\end{aligned}$$
From the definitions of both operators we easily see that
$$B_k \circ B_k^{-1} = \mathrm{Id}_{|\mathcal{X}_\ell}.$$
■

We define $\mathcal{B} : \mathcal{X}_k \times \mathcal{X}_{k+1} \longrightarrow \mathcal{X}_k \times \mathcal{X}_{k+1}$ by
$$\mathcal{B}(\sigma_1, \sigma_2) = (B_k\sigma_1, B_{k+1}\sigma_2)$$
where B_k is defined in (3.6.1) and $\mathcal{B}^{-1} : \mathcal{X}_{k+1} \times \mathcal{X}_{k+2} \longrightarrow \mathcal{X}_k \times \mathcal{X}_{k+1}$ by
$$\mathcal{B}^{-1}(\psi_1, \psi_2) = (B_{k+1}^{-1}\psi_1, B_{k+2}^{-1}\psi_2)$$
where B_j^{-1} is defined in (3.6.4). Clearly
$$\mathcal{B}\mathcal{B}^{-1} = \mathrm{Id}_{|\mathcal{X}_{k+1}\times\mathcal{X}_{k+2}}.$$

3.7 Proof of the Theorem 3.2.1

We scale the time $t = \theta\varepsilon$ in the system (3.3.10) and we obtain

$$\begin{aligned} x' &= y + \mu\varepsilon^{p+2n+2}\partial_y F + \mu^2\varepsilon^{p+2}\partial_y R_{2k-2} \\ y' &= -V'(x) - \mu\varepsilon^{p+2n+2}\partial_x F - \mu^2\varepsilon^{p+2}\partial_y R_{2k-2} \end{aligned} \tag{3.7.1}$$

where $'$ stands for the derivative with respect to t.

It is clear that if φ and $\tilde{\varphi}$ are the general solutions of the systems (3.3.10) and (3.7.1) then for all t, t_0 for which the solutions are defined we have that

$$\tilde{\varphi}(t, t_0, x, y) = \varphi(t/\varepsilon, t_0/\varepsilon, x, y).$$

Thus the integral curves of the two systems are the same. We note that, by Definition 3.4.1 of P_μ^θ, we have that

$$\begin{aligned} P_\mu^{t/\varepsilon}(\tilde{\varphi}(t, t_0, x, y)) &= \varphi(t/\varepsilon + 2\pi, t/\varepsilon, \tilde{\varphi}(t, t_0, x, y)) \\ &= \tilde{\varphi}(t + 2\pi\varepsilon, t, \tilde{\varphi}(t, t_0, x, y)) \\ &= \tilde{\varphi}(t + 2\pi\varepsilon, t_0, x, y). \end{aligned}$$

This suggests us to look for a parameterization $\gamma_\mu^s(t, s)$ of the stable manifold of the system (3.7.1) such that $t \in \mathbb{R}$, is the time, $s \in \mathbb{C}$ is a complex parameter and the following invariance condition of the image of γ_μ^s by $P_\mu^{t/\varepsilon}$ is satisfied,

$$P_\mu^{t/\varepsilon}(\gamma_\mu^s(t, s)) = \gamma_\mu^s(t + 2\pi\varepsilon, s). \tag{3.7.2}$$

Let

$$z' = \varepsilon X_0(z) + \mu^2\varepsilon^{p+3} Y_1(z, \theta_0, \mu, \varepsilon) \tag{3.7.3}$$

be the auxiliary system (3.5.2) with the constants a_j^l and b_j^l given by Lemma 3.5.1 and let

$$\dot{z} = X_0(z) + \mu^2\varepsilon^{p+2} Y_1(z, \theta_0, \mu, \varepsilon) \tag{3.7.4}$$

be the scaled system. Let $\phi(\theta, z; \theta_0, \mu, \varepsilon)$ and $\tilde{\phi}(t, z; \theta_0, \mu, \varepsilon)$ be their respective flows. (We emphasize that (3.7.3) and (3.7.4) are autonomous and that here θ_0 is a parameter of these auxiliary systems, it is not the initial condition of the time).

We observe that, for any $\theta_0 \in \mathbb{R}$, we have that

$$\hat{P}_\mu^{\theta_0}(x,y) = \phi(2\pi, x, y; \theta_0, \mu, \varepsilon) = \tilde{\phi}(2\pi\varepsilon, x, y; \theta_0, \mu, \varepsilon).$$

Hence, since the map $\hat{P}_\mu^{\theta_0}$ satisfies the hypotheses of Proposition 5.1.1 (see the last conclusion of that proposition, it is given in Chapter 5), the stable manifold of the system (3.7.4) can be expressed as a graph of a function

$$y = -\sqrt{-2V(x)} + \mu^2\varepsilon^{p+2} g(x, \theta_0, \mu, \varepsilon) \tag{3.7.5}$$

where the function g is continuous, analytic with respect to x, μ, 2π-periodic with respect to θ_0 and $g = O(x^{n/2})$. Therefore,

$$\begin{aligned} x' &= -\sqrt{-2V(x)} + \mu^2\varepsilon^{p+2} f_1(x, \theta_0, \mu, \varepsilon) \\ &= -f_0(x) + \mu^2\varepsilon^{p+2} f_1(x, \theta_0, \mu, \varepsilon) \end{aligned} \tag{3.7.6}$$

where f_0 and f_1 are $O(x^{n/2})$. Thus we can apply the Proposition 2.1.1 given in Chapter 2 and hence, there exists some $T > 0$ big enough such that if $\operatorname{Re} u > T$ and $|\operatorname{Im} u| \leq a$, the first component of the stable manifold of the system (3.7.4), denoted by

$$\gamma = (\alpha, \beta),$$

is

$$\alpha(u, \theta_0) = \frac{c}{u^{2/(n-2)}}\left(1 + O\left(\frac{1}{u^m}\right)\right) \tag{3.7.7}$$

with c depending on $\theta_0, \mu, \varepsilon$ and satisfying

$$c^{n-2} = \frac{2}{a_n(n-2)^2} + O(\mu^2\varepsilon^{p+2}) \tag{3.7.8}$$

where a_n is the coefficient of order n of the potential $V(x)$, $p < m < \min\{4/(n-2), 2/(n-2) + 1/2\}$. Moreover, by (3.7.5) and uniqueness of the stable manifold:

$$\beta(u, \theta_0) = O\left(\frac{1}{u^{n/(n-2)}}\right). \tag{3.7.9}$$

By uniqueness of solutions, the dependence of γ on θ_0 is 2π-periodic.

Remark 3.7.1 *The constant a is the position of the singularity of the unperturbed homoclinic orbit. Having fixed a value of a, the value of T given by Proposition 2.1.1 depends on a. In fact to get the local parameterizations we can work with $|\operatorname{Im} u| \leq \tau$ with τ big, but for the results of the next Chapters we are interested in values of τ being at least a.*

Remark 3.7.2 *The dependence of γ on θ_0 comes from the dependence of a_j^l and b_j^l on θ_0 and thus γ is 2π-periodic with respect to θ_0.*

Remark 3.7.3 *We observe that $\gamma(u,\theta_0)$ is $O(\mu^2\varepsilon^{p+2})$ close of the stable manifold of the unperturbed system (system (3.7.1) when $\mu=0$) for all $\theta_0\in\mathbb{R}$. This will be useful in order to prove the third condition of Theorem 3.2.1.*

Proof of Remark 3.7.3. We define $\xi(u,\theta_0)=\alpha(u,\theta_0)-\alpha_0(u)$, (we recall that the homoclinic orbit was denoted by $\gamma_0(u)=(\alpha_0(u),\beta_0(u)))$. By (3.7.6) it is clear that

$$\dot{\xi}=f_0'(\alpha_0)\xi+[f_0(\alpha)-f_0(\alpha_0)-f_0'(\alpha_0)\xi]+\mu^2\varepsilon^{p+2}f_1(\xi+\alpha_0). \tag{3.7.10}$$

Therefore, since the equation

$$\dot{\xi}=f_0'(\alpha_0)\xi$$

has the solution $\xi=f_0(\alpha_0)$, the stable manifold of the system (3.7.10) satisfies

$$\begin{aligned}\xi(u) \;=\; & f_0(\alpha_0(u))\left[\xi(T)+\int_T^u\frac{1}{f_0(\alpha_0(s))}[f_0(\alpha(s))-f_0(\alpha_0(s))-f'(\alpha_0(s))\xi(s)]ds\right.\\ & \left.\mu^2\varepsilon^{p+2}\int_T^u\frac{1}{f_0(\alpha_0(s))}f_1(\xi(s)+\alpha_0(s),\theta_0,\mu,\varepsilon)ds\right]\end{aligned} \tag{3.7.11}$$

with $\xi(T)=\alpha(T,\theta_0)-\alpha_0(T)=c_0\mu^2\varepsilon^{p+2}$ (see Section 2.3). We denote by $B(r)$ the closed ball of radius r of $\mathcal{X}_{2/(n-2)}$ with the norm in $\mathcal{X}_{2/(n-2)}$ defined at the beginning of Section 3.2. We define the operator $\Gamma: B(r)\to B(r)$ so that $\Gamma\xi$ is given by the right side of (3.7.11). Then we are led to solve

$$\xi=\Gamma(\xi).$$

We choose $r=C\mu^2\varepsilon^{p+2}$ with c to be determined later. Let $\xi\in B(r)$ and u such that $\operatorname{Re}u\geq T$ and $|\operatorname{Im}u|\leq a$ with T big enough. Then using (3.7.7) and that $f_0,f_1=O(x^{n/2})$, we obtain that there exists a constant K, independent of θ_0, μ and ε,such that

$$\begin{aligned}\left|\frac{1}{f_0(\alpha_0(s))}[f_0(\alpha(s))-f_0(\alpha_0(s))-f_0'(\alpha_0(s))\xi(s)]\right| &\leq K\|\xi\|^2_{\mathcal{X}_{2/(n-2)}}\leq Kr^2\\ \left|\frac{1}{f_0(\alpha_0(s))}f_1(\xi(s)+\alpha_0(s),\theta_0,\mu,\varepsilon)\right| &\leq K\\ |\operatorname{Re}u|^{n/(n-2)}|f_0(\alpha_0(u))| &\leq K\end{aligned}$$

Therefore we have that

$$\begin{aligned}|\Gamma(\xi)(u)| &\le \frac{K}{|\operatorname{Re} u|^{n/(n-2)}}\big(\xi(T) + K|\operatorname{Re} u - T|[r^2 + \mu^2\varepsilon^{p+2}]\big)\\ &\le \mu^2\varepsilon^{p+2}CK\frac{1}{|\operatorname{Re} u|^{2/(n-2)}}\left[\frac{c_0}{C} + K\left(C\mu^2\varepsilon^{p+2} + \frac{1}{C}\right)\right]\\ &\le \mu^2\varepsilon^{p+2}C\frac{1}{|\operatorname{Re} u|^{2/(n-2)}}\end{aligned}$$

if C is big enough. Hence the operator Γ is well defined. Analogously we check that it is a contraction. In particular we have proved that $\xi = O(\mu^2\varepsilon^{p+2})$. ■

Another interesting property of γ is the following: for any $\theta_0 \in \mathbb{R}$ and $(t,s) \in D^s(T,a)$,

$$\begin{aligned}\hat{P}_\mu^{\theta_0}(\gamma(t+s,\theta_0)) &= \phi(2\pi,\gamma(t+s,\theta_0);\theta_0,\mu,\varepsilon) = \tilde{\phi}(2\pi\varepsilon,\gamma(t+s,\theta_0);\theta_0,\mu,\varepsilon)\\ &= \gamma(t+2\pi\varepsilon+s,\theta_0).\end{aligned}$$

In particular, for $\theta_0 = t/\varepsilon$ we have that

$$\hat{P}_\mu^{t/\varepsilon}(\gamma(t+s,t/\varepsilon)) = \gamma(t+2\pi\varepsilon+s,t/\varepsilon). \tag{3.7.12}$$

Moreover it is clear that if we define the function

$$\hat{\gamma}(t,s) = \gamma(t+s,t/\varepsilon)$$

then, since γ is 2π-periodic with respect to its second variable, we have that

$$\begin{aligned}\hat{\gamma}(t+2\pi\varepsilon,s) &= \gamma(t+2\pi\varepsilon+s,(t+2\pi\varepsilon)/\varepsilon)\\ &= \gamma(t+2\pi\varepsilon+s,t/\varepsilon)\\ &= \hat{\gamma}(t,s+2\pi\varepsilon).\end{aligned}$$

We consider the function $\hat{\gamma}(t,s)$ as a first approximation of γ_μ^s. We look for the stable manifold of the system (3.7.1) of the form

$$\gamma_\mu^s(t,s) = \hat{\gamma}(t,s) + \mu\varepsilon^{p+2}\sigma(t,s)$$

with $\sigma = (\sigma_1,\sigma_2)$ in a suitable space of functions decreasing to zero at some given rate. (See below the precise definition of the spaces.) t is the time and $s \in \mathbb{C}$.

In order to clarify the notation, we rename $\hat{\gamma}$ by γ and we denote

$$\gamma(t,s) = (\alpha(t,s),\beta(t,s)).$$

We look for the fixed point equation for the functions σ_1 and σ_2. First we summarize the properties of $\gamma(t,s)$:

- γ is continuous and analytic with respect to s,
- $\gamma(t+2\pi\varepsilon, s) = \gamma(t, s+2\pi\varepsilon)$,
- $\hat{P}_\mu^{t/\varepsilon}(\gamma(t,s)) = \gamma(t+2\pi\varepsilon, s)$ and
- By (3.7.7) and (3.7.9),

$$\gamma \in \mathcal{X}_{2/(n-2)} \times \mathcal{X}_{n/(n-2)}. \tag{3.7.13}$$

We introduce some notation

$$R(x,y) = \varepsilon^{2n}\psi_{\mu,\varepsilon}(x,y,\theta) + \mu \begin{pmatrix} y^3 r^1_{2k-6}(x,y,\theta) + p_{2n-3}(x,y,\theta) \\ y^4 r^2_{2k-7}(x,y,\theta) + q_{2n-3}(x,y,\theta) \end{pmatrix} \tag{3.7.14}$$

and

$$\begin{aligned} Q(x,y) &= (2\pi)^2 \begin{pmatrix} q_1(x,y,\varepsilon) \\ q_2(x,y,\varepsilon) \end{pmatrix} + \mu^2\varepsilon^{p+1} \begin{pmatrix} \sum_{l=0}^{2} y^l \sum_{j=2k-3-l}^{2n-4} p^l_j x^j \\ \sum_{l=0}^{3} y^l \sum_{j=2k-3-l}^{2n-4} q^l_j x^j \end{pmatrix} \\ &\quad + \mu^2\varepsilon^{p+2} y^3 H(x,y) + \mu^2\varepsilon^{p+2}\tilde{R}_{2n-3}(x,y) \end{aligned} \tag{3.7.15}$$

so that

$$P_\mu^{t/\varepsilon}(x,y) = \hat{P}_\mu^{t/\varepsilon}(x,y) + \mu\varepsilon^{p+3} R(x,y,\mu)$$

and

$$\hat{P}_\mu^{t/\varepsilon}(x,y) = \begin{pmatrix} x \\ y \end{pmatrix} + 2\pi\varepsilon \begin{pmatrix} y \\ -V'(x) \end{pmatrix} + \varepsilon^2 Q(x,y)$$

(we do not write explicitly the dependence of $\tilde{P}$, Q and R on ε).

Now we look for an equation for σ. For that we impose that $\gamma_\mu^s = \gamma + \mu\varepsilon^{p+2}\sigma$ verifies (3.7.2). By Taylor's theorem:

$$\begin{aligned} P_\mu^{t/\varepsilon}(\gamma_\mu^s(t,s)) &= P_\mu^{t/\varepsilon}(\gamma(t,s)) + \mu\varepsilon^{p+2} DP_\mu^{t/\varepsilon}(\gamma(t,s))\sigma(t,s) + \mu^2\varepsilon^{2p+4} O(|\sigma(t,s)|^2) \\ &= \hat{P}_\mu^{t/\varepsilon}(\gamma(t,s)) + \mu\varepsilon^{p+3} R(\gamma(t,s)) + \mu\varepsilon^{p+2} D\hat{P}_\mu^{t/\varepsilon}(\gamma(t,s))\sigma(t,s) \\ &\quad + \mu^2\varepsilon^{2p+5} DR(\gamma(t,s))\sigma(t,s) + \mu^2\varepsilon^{2p+4} O(|\sigma(t,s)|^2) \end{aligned}$$

Thus, using (3.7.12) we have that $P_\mu^{t/\varepsilon}(\gamma_\mu^s(t,s)) = \gamma_\mu^s(t+2\pi\varepsilon, s)$ if and only if

$$\begin{aligned}\sigma(t+2\pi\varepsilon,s) &= D\hat{P}_\mu^{t/\varepsilon}(\gamma(t,s))\sigma(t,s) + \varepsilon R(\gamma(t,s)) + \mu\varepsilon^{p+2}O(|\sigma(t,s)|^2)\\ &\quad +\mu\varepsilon^{p+3}DR(\gamma(t,s))\sigma(t,s)\\ &= \sigma(t,s) + 2\pi\varepsilon\begin{pmatrix}\sigma_2(t,s)\\ -V''(\alpha(t,s))\sigma_1(t,s)\end{pmatrix}\\ &\quad +\varepsilon^2 DQ(\gamma(t,s))\sigma(t,s)\\ &\quad +\varepsilon R(\gamma(t,s)) + \mu\varepsilon^{p+2}O(|\sigma(t,s)|^2) + \mu\varepsilon^{p+3}DR(\gamma(t,s))\sigma(t,s).\end{aligned}$$

To simplify the notation, we define

$$\begin{aligned}H(\sigma)(t,s) &= \varepsilon DQ(\gamma(t,s))\sigma(t,s)\\ &\quad +R(\gamma(t,s)) + \mu\varepsilon^{p+1}O(|\sigma(t,s)|^2)\\ &\quad +\mu\varepsilon^{p+2}DR(\gamma(t,s))\sigma(t,s),\end{aligned} \tag{3.7.16}$$

$$A(\alpha(t,s)) = \begin{pmatrix}0 & 1\\ -V''(\alpha(t,s)) & 0\end{pmatrix}$$

and

$$\mathcal{F}(\sigma)(t,s) = 2\pi\varepsilon A(\alpha(t,s))\sigma(t,s) + \varepsilon H(\sigma)(t,s). \tag{3.7.17}$$

Then the problem is reduced to find $\sigma = (\sigma_1, \sigma_2)$ such that

$$\sigma = \mathcal{B}^{-1}\mathcal{F}(\sigma). \tag{3.7.18}$$

We look for $\sigma \in \mathcal{X}_k \times \mathcal{X}_{k+1}$ for a suitable k. We define the following norm in the product space $\mathcal{X}_k \times \mathcal{X}_{k+1}$:

$$\|(h_k, h_{k+1})\|_k = L\|h_k\|_k + \|h_{k+1}\|_{k+1}$$

with

$$L = \frac{n-1}{n-2} + \frac{n^2}{(3n-4)(n-2)}$$

and we denote $B(r,k,k+1) \subset \mathcal{X}_k \times \mathcal{X}_{k+1}$ the closed ball of radius r with this norm.

We will prove that there exists $r_0 > 0$ independent of ε and μ such that the equation (3.7.18) has a solution

$$\sigma \in B\Big(r, \frac{2n-2}{n-2}, \frac{3n-4}{n-2}\Big)$$

for all $r < r_0$. For that we will apply the fixed point theorem to the operator $\mathcal{B}^{-1}\mathcal{F}$.

Lemma 3.7.4 *Let* $\ell = \frac{2n-2}{n-2}$. *The operator*

$$\mathcal{B}^{-1} \circ \mathcal{F} : B(r, \ell, \ell+1) \longrightarrow B(r, \ell, \ell+1)$$

is well defined and it is a contraction.

Proof. We recall that, by (3.7.13), the function $\gamma(t, s) \in \mathcal{X}_{2/(n-2)} \times \mathcal{X}_{n/(n-2)}$and hence we observe that, if $f_l \in P_l$,

$$f_l(\gamma(t, s)) \in \mathcal{X}_{2l/(n-2)}.$$

Let $\sigma \in B(r, \ell, \ell+1)$. Since $V''(\alpha(t, s)) \in \mathcal{X}_2$, it is clear that

$$A\sigma \in \mathcal{X}_{\ell+1} \times \mathcal{X}_{\ell+2}. \tag{3.7.19}$$

We check that

$$\begin{aligned} R(\gamma(t, s)) &\in \mathcal{X}_{\ell+2} \times \mathcal{X}_{\ell+2} \\ (DQ \circ \gamma)\, \sigma &\in \mathcal{X}_{\ell+1} \times \mathcal{X}_{\ell+2} \\ (DR \circ \gamma)\, \sigma &\in \mathcal{X}_{\ell+4} \times \mathcal{X}_{\ell+4}. \end{aligned} \tag{3.7.20}$$

Indeed, we recall that $\psi_{\mu,\varepsilon} \in P_{2n-3}$, then $\psi_{\mu,\varepsilon}(\gamma(t, s)) \in \mathcal{X}_{\ell+2} \times \mathcal{X}_{\ell+2}$. We write $R = (R_1, R_2)$. By definition (3.7.14) we have that

$$R_1(\gamma(t, s)) \in \mathcal{X}_{\ell+2} \cap \mathcal{X}_{(3n+2(2k-6))/(n-2)} \subset \mathcal{X}_{\ell+2} \cap \mathcal{X}_{(5n-8)/(n-2)} = \mathcal{X}_{\ell+2}$$

and, analogously,

$$R_2(\gamma(t, s)) \in \mathcal{X}_{\ell+2}.$$

Now we deal with $(DQ \circ \gamma)\, \sigma$. We denote

$$DQ \circ \gamma = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix}$$

and we define

$$S(x, y) = \frac{1}{(2\pi)^2} y^3 H(x, y) + \frac{1}{(2\pi)^2} \tilde{R}_{2n-3}(x, y).$$

Using that $H = (H_1, yH_2)$ with $H_1 \in P_{2k-6}$ and $H_2 \in H_{2k-7}$ and that $\tilde{R}_{2n-3} \in P_{2n-3}$ we note that

$$S(\gamma(t, s)) \in \mathcal{X}_{\ell+2} \times \mathcal{X}_{\ell+2},$$

hence $\partial_x S_1(\gamma(t,s)), \partial_x S_2(\gamma(t,s)) \in \mathcal{X}_4$ and $\partial_y S_1(\gamma(t,s)), \partial_y S_2(\gamma(t,s)) \in \mathcal{X}_3$. Moreover, by (3.7.15), using that q_1, $q_2 \in P_{n-1}$, we have that,

$$\begin{aligned} Q_{11}(t,s) &= \partial_x q_1(\gamma(t,s)) + \mu^2\varepsilon^{2p+2}\frac{1}{(2\pi)^2}\sum_{l=0}^{2}\sum_{j=2k-3-l}^{2n-4} j p_j^l \alpha^{j-1}(t,s)\beta^l(t,s) \\ &\quad + \mu^2\varepsilon^{p+2}\partial_x S_1(\gamma(t,s)) \\ &\in \mathcal{X}_{2(n-2)/(n-2)} \cap \Big(\bigcap_{l=0}^{2}\mathcal{X}_{(ln+(2k-4-l)2)/(n-2)}\Big) \cap \mathcal{X}_4 \\ &= \mathcal{X}_2 \end{aligned} \tag{3.7.21}$$

where we have used that, since the condition $2k-2 \geq n$ (hypothesis **HP4**) as well as $n \geq 3$, we have that

$$l(n-2) + (2k-4)2 \geq (2k-4)2 \geq (n-2)2. \tag{3.7.22}$$

For Q_{12}, we have that

$$\begin{aligned} Q_{12}(t,s) &= \partial_y q_1(\gamma(t,s)) + 2\pi\mu^2\varepsilon^{2p+2}\sum_{l=1}^{2}\sum_{j=2k-3-l}^{2n-4} l p_j^l \alpha^{j}(t,s)\beta^{l-1}(t,s) \\ &\quad + \mu^2\varepsilon^{p+2}\partial_y S_1(\gamma(t,s)) \\ &\in \mathcal{X}_{2(n-2)/(n-2)} \cap \Big(\bigcap_{l=1}^{2}\mathcal{X}_{((l-1)n+(2k-3-l)2)/(n-2)}\Big) \cap \mathcal{X}_3 \\ &= \mathcal{X}_2. \end{aligned} \tag{3.7.23}$$

Using again (3.7.22) we have that

$$\begin{aligned} Q_{21}(t,s) &= \partial_x q_2(\gamma(t,s)) + 2\pi\mu^2\varepsilon^{2p+2}\sum_{l=0}^{3}\sum_{j=2k-3-l}^{2n-4} j p_j^l \alpha^{j-1}(t,s)\beta^{l}(t,s) \\ &\quad + \mu^2\varepsilon^{p+2}\partial_x S_2(\gamma(t,s)) \\ &\in \mathcal{X}_{2(n-2)/(n-2)} \cap \Big(\bigcap_{l=0}^{3}\mathcal{X}_{(ln+(2k-4-l)2)/(n-2)}\Big) \cap \mathcal{X}_4 \\ &= \mathcal{X}_2. \end{aligned} \tag{3.7.24}$$

And finally, since

$$(2k-3)2 - n \geq 2(n-1) - n = n-2,$$

we have that

$$\begin{aligned} Q_{22}(t,s) &= \partial_y q_2(\gamma(t,s)) + 2\pi\mu^2\varepsilon^{2p+2}\sum_{l=1}^{3}\sum_{j=2k-3-l}^{2n-4} l p_j^l \alpha^j(t,s)\beta^{l-1}(t,s) \\ &\quad +\mu^2\varepsilon^{p+2}\partial_y S_2(\gamma(t,s)) \\ &\in \mathcal{X}_{2(n-2)/(n-2)} \cap \Big(\bigcap_{l=1}^{3}\mathcal{X}_{(n(l-1)+(2k-3-l)2)/(n-2)}\Big) \cap \mathcal{X}_3 \\ &= \mathcal{X}_2. \end{aligned} \tag{3.7.25}$$

Therefore, by (3.7.21), (3.7.23), (3.7.24) and (3.7.25) we obtain that the first component of $(DQ \circ \gamma)\sigma$

$$((DQ\circ\gamma)\sigma)_1 = Q_{11}\sigma_1 + Q_{12}\sigma_2 \in \mathcal{X}_{\ell+2}\cap\mathcal{X}_{\ell+3} = \mathcal{X}_{\ell+2}$$

and the second one

$$((DQ\circ\gamma)\sigma)_2 = Q_{21}\sigma_1 + Q_{22}\sigma_2 \in \mathcal{X}_{\ell+2}\cap\mathcal{X}_{\ell+3} = \mathcal{X}_{\ell+2}.$$

To deal with $(DR\circ\gamma)\sigma$, we observe that,

$$(\partial_x R_1\circ\gamma), (\partial_x R_2\circ\gamma) \in \mathcal{X}_4 \quad \text{and} \quad (\partial_y R_1\circ\gamma), (\partial_y R_2\circ\gamma) \in \mathcal{X}_4$$

therefore

$$\begin{aligned} (DR\circ\gamma)\sigma &= \begin{pmatrix} (\partial_x R_1\circ\gamma)\sigma_1 + (\partial_y R_1\circ\gamma)\sigma_2 \\ (\partial_x R_2\circ\gamma)\sigma_1 + (\partial_y R_2\circ\gamma)\sigma_2 \end{pmatrix} \\ &\in \mathcal{X}_{\ell+4}\times\mathcal{X}_{\ell+4}. \end{aligned}$$

Thus, by (3.7.20) and since $O(|\sigma|^2) \in \mathcal{X}_{2\ell+2}$, we have that

$$H(\sigma) \in \mathcal{X}_{\ell+1}\times\mathcal{X}_{\ell+2}. \tag{3.7.26}$$

Therefore, by (3.7.19), (3.7.26) and by definition (3.7.17) of $\mathcal{F}$ we have that

$$\mathcal{F}(\sigma) = 2\pi\varepsilon A\sigma + \varepsilon^2 H(\sigma) \in \mathcal{X}_{\ell+1}\times\mathcal{X}_{\ell+2}.$$

Finally, by Lemma 3.6.2,

$$\mathcal{B}^{-1}(\mathcal{F}(\sigma)) \in \mathcal{X}_{\ell}\times\mathcal{X}_{\ell+1}.$$

In order to prove that the operator $\mathcal{B}^{-1}\circ\mathcal{F}$ is well defined we have to check that

$$\|\mathcal{B}^{-1}(\mathcal{F}(\sigma))\|_\ell < r$$

if $\|\sigma\|_\ell < r$. We begin by bounding the norms $\|\mathcal{F}_1\|_{\ell+1}$ and $\|\mathcal{F}_2\|_{\ell+2}$. From the definitions of R, and H, (3.7.14) and (3.7.16) respectively, it is clear that there exists constants M_1 and M_2 such that

$$\begin{aligned}\|H_1(\sigma)\|_{\ell+1} &\leq M_1(\varepsilon+\mu)\\ \|H_2(\sigma)\|_{\ell+2} &\leq M_2(\varepsilon+\mu).\end{aligned}$$

From (3.7.7) and (3.7.8) we recall that

$$\alpha(t,s)=\frac{c}{(t+s)^{2/(n-2)}}+h.o.t.\quad \text{with}\quad c^{n-2}=\frac{2}{a_n(n-2)^2}+O(\mu^2\varepsilon^{p+2}).$$

Then

$$\begin{aligned}\|\mathcal{F}_1(\sigma)\|_{\ell+1} &\leq 2\pi\varepsilon\sup_{(t,s)\in D}|\sigma_2(t,s)(t+\operatorname{Re}s)^{\ell+1}|\\ &\quad+\varepsilon\sup_{(t,s)\in D}|H_1(\sigma)(t,s))(t+\operatorname{Re}s)^{\ell+1}|\\ &\leq 2\pi\varepsilon\|\sigma_2\|_{\ell+1}+\varepsilon M_1(\varepsilon+\mu)\end{aligned}$$

and

$$\begin{aligned}\|\mathcal{F}_2(\sigma)\|_{\ell+2} &\leq 2\pi\varepsilon\sup_{(t,s)\in D}|V''(\alpha(t,s))\sigma_1(t,s)(t+\operatorname{Re}s)^{\ell+2}|\\ &\quad+\varepsilon\sup_{(t,s)\in D}|H_2(\sigma)(t,s))(t+\operatorname{Re}s)^{\ell+2}|\\ &\leq 2\pi\varepsilon\left(\frac{2n(n-1)}{(n-2)^2}+o\Big(\frac{1}{T}\Big)\right)\|\sigma_1\|_\ell+\varepsilon M_2(\varepsilon+\mu).\end{aligned}$$

Therefore, using Lemma 3.6.2 we obtain

$$\begin{aligned}\|\mathcal{B}^{-1}\circ\mathcal{F}(\sigma)\|_\ell &= \|(\mathcal{B}_1^{-1}\mathcal{F}_1(\sigma),\mathcal{B}_2^{-1}\mathcal{F}_2(\sigma))\|_\ell\\ &= L\|\mathcal{B}_1^{-1}\mathcal{F}_1(\sigma_1,\sigma_2)\|_\ell+\|\mathcal{B}_2^{-1}\mathcal{F}_2(\sigma_1,\sigma_2))\|_{\ell+1}\\ &\leq L\|\mathcal{F}_1\|_{\ell+1}\frac{1+4\pi\varepsilon}{2\ell\pi\varepsilon}+\|\mathcal{F}_2\|_{\ell+2}\frac{1+4\pi\varepsilon}{2(\ell+1)\pi\varepsilon}\\ &\leq L\|\sigma_2\|_{\ell+1}\frac{n-2}{2n-2}+O(\varepsilon)+O(\mu)\\ &\quad+\left(\frac{2n(n-1)}{(3n-4)(n-2)}+o\Big(\frac{1}{T}\Big)\right)\|\sigma_1\|_\ell\\ &\quad+O(\varepsilon)+O(\mu)\\ &\leq L\frac{n-2}{2n-2}\|\sigma_2\|_{\ell+1}+\frac{2n^2}{(3n-4)(n-2)}\|\sigma_1\|_\ell\\ &\quad+O(\varepsilon)+O(\mu)\end{aligned}\tag{3.7.27}$$

if we take $o(\frac{1}{T}) \leq \frac{2n}{(3n-4)(n-2)}$. We introduce

$$\begin{aligned} a &= \frac{2(n-1)}{n-2} \\ b &= \frac{2n^2}{(3n-4)(n-2)}. \end{aligned}$$

We observe that since $a > b$ and $L = (a+b)/2$ we have $b < L < a$. Thus, $1 - La^{-1}$and $L - b$ are positive numbers. Moreover if we introduce the constant $K = \frac{2a}{a-b} > 2$ we have that

$$\begin{aligned} L - b &= \frac{a-b}{2} > \frac{L}{K} \\ 1 - La^{-1} &= \frac{a-b}{2a} = \frac{1}{K}. \end{aligned}$$

Therefore, we can bound (3.7.27) as follows

$$\begin{aligned} La^{-1}\|\sigma_2\|_{\ell+1} + b\|\sigma_1\|_\ell + O(\varepsilon+\mu) &= \|\sigma_2\|_{\ell+1} + L\|\sigma_1\|_\ell - (L-b)\|\sigma_1\|_\ell \\ &\quad -(1-La^{-1})\|\sigma_2\|_{\ell+1} + O(\varepsilon+\mu) \\ &\leq (\|\sigma_2\|_{\ell+1} + L\|\sigma_1\|_\ell)(1-\frac{1}{K}) + O(\varepsilon+\mu) \\ &\leq (1-\frac{1}{K})r + O(\varepsilon+\mu) \\ &< r \end{aligned}$$

if ε and $|\mu|$ are small enough.

Now, we prove that $\mathcal{B}^{-1} \circ \mathcal{F}$ is a contraction. Let $\sigma = (\sigma_1, \sigma_2) \in B(r, \ell, \ell+1)$ and $\bar\sigma = (\bar\sigma_1, \bar\sigma_2) \in B(r, \ell, \ell+1)$. It is easy to see that

$$\|H(\sigma) - H(\bar\sigma)\|_\ell \leq C\|\sigma - \bar\sigma\|_\ell$$

with $C > 0$ some constant. Thus, by definition (3.7.17) of $\mathcal{F}$, we obtain

$$\begin{aligned} \|\mathcal{B}^{-1}\circ\mathcal{F}(\sigma_1,\sigma_2) - \mathcal{B}^{-1}\circ\mathcal{F}(\bar\sigma_1,\bar\sigma_2)\|_\ell &\leq L\frac{n-2}{2n-2}\|\sigma_2 - \bar\sigma_2\|_{\ell+1} \\ &\quad + \frac{2n^2}{(3n-4)(n-2)}\|\sigma_1 - \bar\sigma_1\|_\ell \\ &\quad + \varepsilon C\|(\sigma_1-\bar\sigma_1, \sigma_2-\bar\sigma_2)\|_\ell \\ &\leq \left(1 - \frac{1}{K} + \varepsilon C\right)\|(\sigma_1-\bar\sigma_1, \sigma_2-\bar\sigma_2)\|_\ell \\ &< \left(1 - \frac{1}{2K}\right)\|(\sigma_1-\bar\sigma_1, \sigma_2-\bar\sigma_2)\|_\ell \end{aligned}$$

if ε is small enough.

Then, applying the fixed point theorem we obtain the existence and uniqueness of a fixed point $(\sigma_1, \sigma_2) \in B(r, \ell, \ell+1)$. This ends the proof of the lemma. ■

Next we can finish the proof of Theorem 3.2.1.

End of the proof of Theorem 3.2.1.. Going back to the original variables we obtain the result we had stated in Theorem 3.2.1. Indeed, as we have said in Remark 3.3.6 the change $\mathcal{C}$ has the form

$$(x, y, \theta) = \mathcal{C}(\bar{x}, \bar{y}, \theta) = (\bar{x}, \bar{y}, \theta) + \mu\varepsilon^{p+1}(G(\bar{x}, \bar{y}, \theta), 0) + O(\mu\varepsilon^{p+2})$$

where (x, y, θ) are the original variables and $(\bar{x}, \bar{y}, \theta)$ are the variables for which we have proved the suitable parameterization of the invariant local stable curve. Therefore, the local stable manifold, $\gamma^s_{\mu,\varepsilon}$, of the Poincaré map of the original system (1.1.1) has the form

$$\begin{aligned}\tilde{\gamma}^s_{\mu,\varepsilon}(t,s) &= \gamma^s_\mu(t,s) + \mu\varepsilon^{p+1}G(\gamma^s_\mu(t,s), t/\varepsilon) + O(\mu\varepsilon^{p+2})\\ &= \gamma_0(t+s) + \mu\varepsilon^{p+1}G(\gamma_0(t+s), t/\varepsilon) + O(\mu\varepsilon^{p+2}).\end{aligned}$$

and it satisfies all the conditions of Theorem 3.2.1 except that it is not a solution of the system (1.1.1) with respect to t. Condition 2) follows because the change $\mathcal{C}$ is 2π-periodic in θ.

In order to find a parameterization of the local stable manifold which is a solution with respect to t for any $s \in \mathbb{C}$ such that $|\operatorname{Im} s| \le \tau$ we define $t_0 = T - \operatorname{Re} s$ and

$$\gamma^s_{\mu,\varepsilon}(t,s) = \varphi(t, t_0, \tilde{\gamma}^s_{\mu,\varepsilon}(t_0, s))$$

where $\varphi(t, t_0, x, y)$ is the general solution of system (1.1.1). We observe that, for all $(t,s) \in D^s$, $\tilde{\gamma}^s_{\mu,\varepsilon}(t,s)$ belongs to the local stable manifold of the system (1.1.1), hence $\gamma^s_{\mu,\varepsilon}(t,s)$ is a parameterization of the stable manifold. It is clear that the properties 1), 3) and 4) of Theorem 3.2.1 are satisfied by $\gamma^s_{\mu,\varepsilon}$. Moreover

$$\begin{aligned}\gamma^s_{\mu,\varepsilon}(t, s+2\pi\varepsilon) &= \varphi(t, t_0, \tilde{\gamma}^s_{\mu,\varepsilon}(t_0, s+2\pi\varepsilon)) = \varphi(t, t_0, \tilde{\gamma}^s_{\mu,\varepsilon}(t_0+2\pi\varepsilon, s))\\ &= \varphi(t+2\pi\varepsilon, t_0+2\pi\varepsilon, \varphi(t_0+2\pi\varepsilon, t_0, \tilde{\gamma}^s_{\mu,\varepsilon}(t_0, s)))\\ &= \varphi(t+2\pi\varepsilon, t_0, \tilde{\gamma}^s_{\mu,\varepsilon}(t_0, s))\\ &= \gamma^s_{\mu,\varepsilon}(t+2\pi\varepsilon, s).\end{aligned}$$

Therefore, $\gamma^s_{\mu,\varepsilon}$ is the parameterization that we look for. We observe that $\gamma^s_{\mu,\varepsilon}$ is defined for all $(t,s) \in D^s$. ■

Remark 3.7.5 *Assuming hypotheses **HP1**-**HP5**, we have the same conclusions as in Theorem 3.2.1. This is a immediate consequence of Remark 1.1.4.*

Chapter 4

The case of weak hyperbolic points

4.1 Introduction

In this Chapter we consider systems such that the unperturbed system has a parabolic fixed point but for the perturbed system the fixed point becomes hyperbolic (of course the corresponding eigenvalues tend to zero). In this case we assume that the unperturbed system is a concrete explicit Hamiltonian. This is a bifurcation case.

We prove the existence of a special parameterization of the stable and unstable manifold in a domain independent of the parameters ε and μ. In fact, we prove the existence of such parameterization for the stable manifold but it is easy to see that, with slight changes, the proof works for the unstable one. As in the previous Chapter, we need a good initial approximation for the stable manifold as well as a good coordinates.

Since the time parameterization of the homoclinic orbit near the fixed point (that is, when $t \to \pm\infty$) has a potential behaviour, and we know that the parameterization of the stable manifold near a hyperbolic fixed point (which will be the case for the perturbed system) is exponential in time, it seems natural to suspect that the homoclinic orbit of the unperturbed system is not a good approximation of the stable curve of the perturbed one. Actually, for μ small, there is a competition between the potential and the exponential character.

The structure of the proof is similar to the one of the previous Chapter.

More precisely, in this Chapter we consider the Hamiltonian

$$H(x, y, t/\varepsilon) = h_0(x, y) + \mu\varepsilon^p h_1(x, y, t/\varepsilon, \mu, \varepsilon)$$

where

$$h_0(x,y) = \frac{y^2}{2} + V(x), \qquad V(x) = -(x^3 - x^4)$$

and

$$h_1(x,y,t/\varepsilon,\mu,\varepsilon) = h_{12}(x,y,t/\varepsilon,\mu,\varepsilon) + h_{13}(x,y,t/\varepsilon,\mu,\varepsilon)$$

with

$$h_{12}(x,y,t/\varepsilon) = \frac{x^2}{2} g_1(t/\varepsilon,\mu,\varepsilon) + xyg_2(t/\varepsilon,\mu,\varepsilon) + \frac{y^2}{2} g_3(t/\varepsilon,\mu,\varepsilon)$$

and $h_{13}(x,y,t/\varepsilon)$ is of order 3 in the (x,y) variables. Below we will not write the dependence of g_i on μ and ε. The associated equations are

$$\begin{aligned} \dot{x} &= y + \mu\varepsilon^p(xg_2(t/\varepsilon) + yg_3(t/\varepsilon) + \partial_y h_{13}(x,y,t/\varepsilon,\mu,\varepsilon)) \\ \dot{y} &= -V'(x) - \mu\varepsilon^p(xg_1(t/\varepsilon) + yg_2(t/\varepsilon) + \partial_x h_{13}(x,y,t/\varepsilon,\mu,\varepsilon)). \end{aligned} \tag{4.1.1}$$

We observe that the unperturbed system has a homoclinic orbit given by $\gamma_0 = (\alpha_0,\beta_0)$ where

$$\alpha_0(t) = \frac{2}{2+t^2}, \qquad \beta_0(t) = -\frac{4t}{(2+t^2)^2}. \tag{4.1.2}$$

4.2 Definitions and main result

As in Chapter 3, we introduce some notation. We define G_i by the conditions $\partial_\theta G_i = g_i$ and $\int_0^{2\pi} G_i(\theta) = 0$ for $i = 1,2,3$.

Given $T > 0$ and $\tau = \sqrt{2}$ we define the sets

$$D^s = D^s(T,\tau) = \{(t,s) \in \mathbb{R}\times\mathbb{C} : \ t + \operatorname{Re} s \geq T, \ |\operatorname{Im} s| \leq \tau\}$$

and

$$D^u = D^u(T,\tau) = \{(t,s) \in \mathbb{R}\times\mathbb{C} : t + \operatorname{Re} s \leq -T, |\operatorname{Im} s| \leq \tau\}$$

and for $\rho(t)$, a $2\pi\varepsilon$-periodic positive function, $k,l \in \mathbb{Q}$, $(k > 1, l \geq 1)$ we define the space $\mathcal{Y}_k^l = \mathcal{Y}_k^l(\rho)$ of the functions $h : D^s \to \mathbb{C}$ such that

(a) h is continuous,

(b) for t fixed, $s \mapsto h(t,s)$ is analytic,

(c) $h(t, s + 2\pi\varepsilon) = h(t + 2\pi\varepsilon, s)$ for all $(t,s) \in D^s$,

(d) $\|h\|_k^l = \sup\{(t + \operatorname{Re} s)^k e^{\rho(t)l(t+\operatorname{Re} s)}|h(t,s)|, (t,s) \in D^s\} < \infty$.

In a similar way as in Chapter 3 we can prove that $\mathcal{Y}_k^l$ is a Banach space with the norm $\|.\|_k^l$ and that

$$\mathcal{Y}_{k+1}^l \subset \mathcal{Y}_k^l.$$

The next result gives the existence and properties of a special parameterization of the stable and the unstable invariant manifolds.

Theorem 4.2.1 *Assuming hypotheses **HP1-HP6**, there exist $T > 0$ big enough and parameterizations $\gamma_{\mu,\varepsilon}^s(t,s)$, $\gamma_{\mu,\varepsilon}^u(t,s)$ of the local stable and unstable invariant manifolds, defined in $D^s(T,\tau)$, $D^u(T,\tau)$, respectively, such that ($*$ stands for s and u):*

1) $t \mapsto \gamma_{\mu,\varepsilon}^*(t,s)$ is a solution of system (4.1.1) and $s \mapsto \gamma_{\mu,\varepsilon}^*(t,s)$ is real analytic. Moreover the map $(t,s,\mu,\varepsilon) \mapsto \gamma_{\mu,\varepsilon}^*(t,s)$ is continuous, C^1 with respect to t and analytic with respect to (s,μ).

2) $\gamma_{\mu,\varepsilon}^*(t + 2\pi\varepsilon, s) = \gamma_{\mu,\varepsilon}^*(t, s + 2\pi\varepsilon)$ for all $(t,s) \in D^*(T,\tau)$

3) For $\mu = 0$, $\gamma_{\mu,\varepsilon}^*(t,s)$ coincides with the restriction of the homoclinic solution $\gamma_0(t+s)$ to $D^*(T,\tau)$, and for $\mu \neq 0$ the following estimate holds:

$$\gamma_{\mu,\varepsilon}^*(t,s) = \gamma_0(t+s) + \mu\varepsilon^{p+1}G(\gamma_0(t+s), t/\varepsilon) + O(\mu\varepsilon^{p+1+\lambda})$$

where $0 < \lambda < 1/2$ and

$$\partial_\theta G(x,y,\theta) = (\partial_y h_1(x,y,\theta), -\partial_x h_1(x,y,\theta))$$

and has zero mean.

4) $\gamma_{\mu,\varepsilon}^*(t,s) = \gamma_0(t+s) + \mu\varepsilon^{p+1}\sigma^*(t,s)$ where $\sigma^*(t,s) \in \mathcal{Y}_2^0 \times \mathcal{Y}_2^0$.

The proof of this theorem is similar to that of Theorem 3.2.1, but we have to be more explicit in some computations and we must perform another kind of change of coordinates in order to look for the suitable approximation of the homoclinic orbit. Concretely, since the fix point is hyperbolic, we must look for an approximation of the homoclinic orbit with an exponential behavior in $t+s$ near the fix point.

Remark 4.2.2 *We note that the second component, β_0, of the homoclinic orbit has two poles of order 2 at $\pm i\sqrt{2}$, thus in this case $r=2$ and $a=\sqrt{2}$ (see (4.1.2)). We observe that, in the weak hyperbolic case the hypotheses imply that $p \geq 1$.*

Proof. The hypotheses **HP6** and **HP5** as well as $r=2$ imply that $\ell \geq 2r-1$. Indeed, by hypothesis **HP6**, we have that $g_2 \neq 0$ for some $\theta \in \mathbb{R}$, therefore, by definition of ℓ, the order of the pole of the term xy evaluated at the homoclinic orbit is smaller than ℓ, hence

$$\ell \geq r+r-1 = 2r-1 \geq 3.$$

■

4.3 Averaging of the equation

First we scale the time by $\theta = t/\varepsilon$. The transformed systems reads

$$\begin{aligned}
\dot{x} &= \varepsilon y + \mu\varepsilon^{p+1}(xg_2(\theta) + yg_3(\theta) + \partial_y h_{13}(x,y,\theta)) \\
\dot{y} &= -\varepsilon V'(x) - \mu\varepsilon^{p+1}(xg_1(\theta) + yg_2(\theta) + \partial_x h_{13}(x,y,\theta))
\end{aligned}$$

where now $\dot{x}$ and $\dot{y}$ mean derivatives with respect to the new time θ. The new Hamiltonian is $\varepsilon H(x,y,\theta,\mu,\varepsilon)$. We apply Theorem 3.3.4 in order to remove the contribution of the perturbation until orders $\mu\varepsilon^{p+9}$ and $\mu^2\varepsilon^{2p+2}$ in the parameters.

Lemma 4.3.1 *There exists a canonical change of variables $(x,y,\theta) = \tilde{\mathcal{C}}(\bar{x},\bar{y},\theta,\mu,\varepsilon)$ which is C^0 in $(\bar{x},\bar{y},\theta,\mu,\varepsilon)$, $C^1$and 2π-periodic in θ and analytic in $(\bar{x},\bar{y},\mu)$ and it transforms the Hamiltonian εH to*

$$\varepsilon\tilde{\mathcal{H}}(\bar{x},\bar{y},\theta,\mu,\varepsilon) = \varepsilon h_0(\bar{x},\bar{y}) + \mu\varepsilon^{p+9}F(\bar{x},\bar{y},\theta,\mu,\varepsilon) + \mu^2\varepsilon^{2p+2}R_2(\bar{x},\bar{y},\theta,\mu,\varepsilon)$$

with

$$\begin{aligned} R_2(\bar{x},\bar{y},\theta) &= -\big[\bar{x}^2 g_2(\theta)G_1(\theta) + \bar{x}\bar{y}[g_3(\theta)G_1(\theta) + g_2(\theta)G_2(\theta)] + \bar{y}^2 g_3(\theta)G_2(\theta)\big] \\ &\quad + R_3(\bar{x},\bar{y},\theta) + \varepsilon r_2(\bar{x},\bar{y},\theta) \end{aligned} \tag{4.3.1}$$

in a neighborhood of the origin, where $r_2 \in P_2$, $R_3 \in P_3$, $F \in P_4$ and has zero mean with respect to θ. Moreover $\tilde{\mathcal{H}}$ is continuous in $(\bar{x},\bar{y},\theta,\mu,\varepsilon)$ and analytic in $(\bar{x},\bar{y},\mu)$.

Proof. We apply Lemma 3.3.4 and we obtain a new Hamiltonian

$$\begin{aligned} \varepsilon\tilde{\mathcal{H}}(\bar{x},\bar{y},\theta,\mu,\varepsilon) &= \varepsilon h_0(\bar{x},\bar{y}) + \mu\varepsilon^{p+9}F_4(\bar{x},\bar{y},\theta) \\ &\quad + \mu^2\varepsilon^{2p+2}R_2(\bar{x},\bar{y},\theta,\mu,\varepsilon) \end{aligned}$$

where $F_4 \in P_4$ and has zero mean with respect to θ, $R_2 \in P_2$ and

$$R_2 = \partial_{\bar{y}}h_1\partial_{\bar{x}}S^1 + \varepsilon r_2$$

with S^1 such that $\partial_\theta S^1(x,\bar{y},\theta) = -h_1(x,\bar{y},\theta)$ and has zero mean with respect to θ and $r_2 \in P_2$. Moreover $\tilde{\mathcal{H}}$ is continuous in ε and analytic in μ. Next we look for a more detailed expression of R_2. Since

$$h_1 = h_{12} + h_{13}$$

with $h_{13} \in P_3$, $\partial_{\bar{y}}h_{13}\partial_{\bar{x}}S^1 \in P_3$, therefore it is clear that we can write R_2 as

$$R_2 = \partial_{\bar{y}}h_{12}\partial_{\bar{x}}S^1 + \varepsilon r_2 + r_3$$

with $r_2 \in P_2$, $r_3 \in P_3$. Finally we compute

$$\begin{aligned} \partial_{\bar{y}}h_{12}(\bar{x},\bar{y},\theta)\partial_{\bar{x}}S^1(\bar{x},\bar{y},\theta) &= -[\bar{x}^2 g_2(\theta)G_1(\theta) + \bar{x}\bar{y}(g_3(\theta)G_1(\theta) + g_2(\theta)G_2(\theta)) \\ &\quad + \bar{y}^2 g_3(\theta)G_2(\theta)] + r_3(\bar{x},\bar{y},\theta). \end{aligned}$$

with $r_3 \in P_3$ and we define $F \equiv F_4$. Then the statement holds. ■

We write R_3 from definition (4.3.1) of the form

$$R_3(\bar{x},\bar{y},\theta,\mu,\varepsilon) = \bar{R}_3(\bar{x},\bar{y},\mu,\varepsilon) + \tilde{R}_3(\bar{x},\bar{y},\theta,\mu,\varepsilon) \tag{4.3.2}$$

where $\tilde{R}_3$ has zero mean with respect to θ. We rename the variables $(\bar{x},\bar{y})$ by (x,y).

Lemma 4.3.2 *There exists a canonical change of variables* $(x,y,\theta)=\bar{\mathcal{C}}(\bar{x},\bar{y},\theta,\mu,\varepsilon)$ *which is* C^0 *in* $(\bar{x},\bar{y},\theta,\mu,\varepsilon)$*,* C^1 *and* 2π*-periodic in* θ *and analytic in* $(\bar{x},\bar{y},\mu)$ *and it transforms the Hamiltonian* $\varepsilon\tilde{\mathcal{H}}$ *to*

$$\begin{aligned}\varepsilon\mathcal{H}(\bar{x},\bar{y},\theta,\mu,\varepsilon) \;&=\; \varepsilon h_0(\bar{x},\bar{y})+\mu\varepsilon^{p+9}F(\bar{x},\bar{y},\theta,\mu,\varepsilon)+\mu^2\varepsilon^{2p+2}\bar{R}_3(\bar{x},\bar{y},\mu,\varepsilon)\\ &\quad+\mu^2\varepsilon^{2p+2}R_2(\bar{x},\bar{y},\theta,\mu,\varepsilon)\end{aligned}$$

with

$$\begin{aligned}R_2(\bar{x},\bar{y},\theta) \;&=\; -\left[\bar{x}^2g_2(\theta)G_1(\theta)+\bar{x}\bar{y}[g_3(\theta)G_1(\theta)+g_2(\theta)G_2(\theta)]+\bar{y}^2g_3(\theta)G_2(\theta)\right]\\ &\quad+\varepsilon r_2(\bar{x},\bar{y},\theta)\end{aligned} \tag{4.3.3}$$

in a neighborhood of the origin, where $r_2\in P_2$*,* $\bar{R}_3\in P_3$ *(defined in (4.3.2)),* $F\in P_4$ *and has zero mean with respect to* θ*. Moreover* $\mathcal{H}$ *is continuous in* $(\bar{x},\bar{y},\theta,\mu,\varepsilon)$ *and analytic in* $(\bar{x},\bar{y},\mu)$*.*

Proof. As in Lemma 3.3.2, we consider a generating function $S(x,\bar{y},\theta)$ which will provide a canonical change of variables $(\bar{x},\bar{y})\mapsto(x,y)$ implicitly through

$$\begin{aligned}\bar{x}&=\partial_{\bar{y}}S(x,\bar{y},\theta)\\ y&=\partial_xS(x,\bar{y},\theta)\end{aligned} \tag{4.3.4}$$

and then the new Hamiltonian will be

$$\varepsilon\mathcal{H}(\bar{x},\bar{y},\theta,\mu,\varepsilon)=\varepsilon\tilde{\mathcal{H}}(x,y,\theta,\mu,\varepsilon)+\partial_\theta S(x,\bar{y},\theta).$$

We take

$$S(x,\bar{y},\theta)=x\bar{y}+\mu^2\varepsilon^{2p+2}S^1(x,\bar{y},\theta)$$

with S^1 satisfying

$$\partial_\theta S^1=-\tilde{R}_3.$$

We observe that S is C^1, 2π-periodic in θ and analytic in (x,y). Since $\tilde{R}_3$ has zero mean with respect to θ we can choose S^1 such that it has also zero mean

From (4.3.4) we have that

$$\begin{aligned}x&=\bar{x}-\mu^2\varepsilon^{2p+2}\partial_{\bar{y}}S^1+\mu^4\varepsilon^{4p+4}r_2\\ y&=\bar{y}+\mu^2\varepsilon^{2p+2}\partial_xS^1+\mu^4\varepsilon^{4p+4}r_2\end{aligned}$$

where the derivatives of S^1 and S^2, and $r_2 \in P_2$ are evaluated at $(\bar{x}, \bar{y}, \theta)$. Since the terms of order $\mu^2\varepsilon^{2p+2}$ are not modify, the averaged Hamiltonian is therefore:

$$\begin{aligned} \varepsilon\mathcal{H}(\bar{x},\bar{y},\theta,\mu,\varepsilon) &= \varepsilon\tilde{\mathcal{H}}(x,y,\theta,\mu,\varepsilon) - \mu^2\varepsilon^{2p+2}\tilde{R}_3(x,\bar{y},\theta) \\ &= \varepsilon h_0(\bar{x},\bar{y}) + \mu\varepsilon^{p+9}F(\bar{x},\bar{y},\theta,\mu,\varepsilon) + \mu^2\varepsilon^{2p+2}\bar{R}_3(\bar{x},\bar{y},\mu,\varepsilon) \\ &\quad +\mu^2\varepsilon^{2p+2}R_2(\bar{x},\bar{y},\theta,\mu,\varepsilon) \end{aligned}$$

where $R_2 \in P_2$ has the form (4.3.3).

Finally an analogous argument as in Lemma 3.3.2 gives the regularity of $\bar{\mathcal{C}}$ and $\bar{\mathcal{H}}$. ■

We rename the variables $(\bar{x}, \bar{y})$ by (x, y) and then, in the new variables the equations are

$$\begin{aligned} \dot{x} &= \varepsilon y + \mu\varepsilon^{p+9}\partial_y F + \mu^2\varepsilon^{2p+2}(\partial_y\bar{R}_3 + \partial_y R_2) \\ \dot{y} &= -\varepsilon V'(x) - \mu\varepsilon^{p+9}\partial_x F - \mu^2\varepsilon^{2p+2}(\partial_x\bar{R}_3 + \partial_x R_2). \end{aligned} \tag{4.3.5}$$

Now we perform the last change of variables in order to put the system (4.3.5) in the definitive suitable form. For this we write

$$\bar{R}_3(x,y,\mu,\varepsilon) = g(\mu,\varepsilon)x^2y + \bar{r}_3(x,y,\mu,\varepsilon)$$

Lemma 4.3.3 *The canonical change of variables given by*

$$(u,v) = \Phi(x,y) = (x, y + \mu^2\varepsilon^{2p+1}g(\mu,\varepsilon)x^2)$$

which is C^0 in $(\bar{x},\bar{y},\mu,\varepsilon)$ and analytic in $(\bar{x},\bar{y},\mu)$, transforms the system (4.3.5) into

$$\begin{aligned} \dot{u} &= \varepsilon v + \mu\varepsilon^{p+9}\partial_v F + \mu^2\varepsilon^{2p+2}(\partial_v f_3 + \partial_v R_2) \\ \dot{v} &= -\varepsilon V'(u) - \mu\varepsilon^{p+9}\partial_u F - \mu^2\varepsilon^{2p+2}(\partial_u f_3 + \partial_u R_2) \end{aligned}$$

where $f_3 \in P_3$ and it has the form

$$f_3 = \bar{f}_3 + \varepsilon\tilde{f}_3$$

with $\bar{f}_3$ has not terms in u^2v and it does not depend on θ. Moreover R_2 is of the form (4.3.3).

Proof. The proof is straightforward. We perform the change Φ, and then in the new variables (u,v) the first equation of (4.3.5) reads as

$$\begin{aligned} \dot{u} &= \varepsilon y + \mu\varepsilon^{p+9}\partial_y F(x,y,\theta,\mu,\varepsilon) + \mu^2\varepsilon^{2p+2}g(\mu,\varepsilon)x^2 \\ &\quad \mu^2\varepsilon^{2p+2}(\partial_y\bar{r}_3(x,y,\theta,\mu,\varepsilon) + \partial_y R_2(x,y,\theta,\mu,\varepsilon)) \\ &= \varepsilon v + \mu\varepsilon^{p+9}\partial_v F + \mu^2\varepsilon^{2p+2}(\partial_v f_3 + \partial_v R_2) \end{aligned}$$

and

$$\begin{aligned}
\dot{v} &= \dot{y} + 2\mu\varepsilon^{2p+1} g(\mu,\varepsilon) x\dot{x} \\
&= -\varepsilon V'(x) - \mu\varepsilon^{p+9}\partial_x F - \mu^2\varepsilon^{2p+2}(2g(\mu,\varepsilon)xy + \partial_x r_3 + \partial_x R_2)) \\
&\quad +2\mu\varepsilon^{2p+2} g(\mu,\varepsilon) x (y + \mu\varepsilon^{p+9}\partial_y F + \mu^2\varepsilon^{2p+2}(\partial_y \bar{R}_3 + \partial_y R_2)) \\
&= -\varepsilon V'(u) - \mu\varepsilon^{p+9}\partial_u F - \mu^2\varepsilon^{2p+2}(\partial_u f_3 + \partial_u R_2)
\end{aligned}$$

where all the functions are evaluated in (u, v) and f_3 has the desired form. ■

Remark 4.3.4 *We observe that the change $\mathcal{C} = \Phi \circ \bar{\mathcal{C}} \circ \tilde{\mathcal{C}}$ is of the form*

$$\mathcal{C}(\bar{x},\bar{y},\theta) = (\bar{x},\bar{y},\theta) + \mu\varepsilon^{p+1}(G(\bar{x},\bar{y},\theta),0) + O(\mu\varepsilon^{p+2})$$

where G is such that $\partial_\theta G = (\partial_y h_1, -\partial_x h_1)$. We will use this form in order to prove the third property of Theorem 4.2.1.

We rename the variables (u, v) by (x, y) and then in the new variables the equations are

$$\begin{aligned}
\dot{x} &= \varepsilon y + \mu\varepsilon^{p+9}\partial_y F + \mu^2\varepsilon^{2p+2}(\partial_y f_3 + \partial_y R_2) \\
\dot{y} &= -\varepsilon V'(x) - \mu\varepsilon^{p+9}\partial_x F - \mu^2\varepsilon^{2p+2}(\partial_x f_3 + \partial_x R_2)
\end{aligned} \tag{4.3.6}$$

4.4 Estimates for the Poincaré map

In this section we provide with an expression of the Poincaré map of system (4.3.6). For any fixed $\theta_0 \in \mathbb{R}$, we consider the Poincaré map defined by

$$P_\mu^{\theta_0}(x,y) = \varphi(\theta_0 + 2\pi, \theta_0, x, y, \mu, \varepsilon) \tag{4.4.1}$$

where $\varphi(\theta, \theta_0, x, y, \mu, \varepsilon)$ is the solution of the system (4.3.6) such that

$$\varphi(\theta_0, \theta_0, x, y, \mu, \varepsilon) = (x, y).$$

We denote it by $\varphi_\mu(\theta)$ if the initial conditions do not play an essential role.

We observe that, in the proof of Lemma 3.4.3, we have not used hypothesis **HP4** (which it is satisfied in the weak hyperbolic case). Moreover neither we have used that the order k in (x, y) variables of the perturbation h_1 be bigger than 3. Hence we can use Lemma 3.4.3 in this Section. We only have to change $\mu^2\varepsilon^{p+3}$ by $\mu^2\varepsilon^{2p+2}$. We denote by

$$X_\mu(x,y,\theta) = X_0(x,y) + \mu\varepsilon^{p+9}F(x,y,\theta) + \mu^2\varepsilon^{2p+2}R(x,y,\theta)$$

the vector field of equations (4.3.6) where $F = (\partial_y F, -\partial_x F)$ and

$$R = (\partial_y f_3 + \partial_y R_2, -(\partial_x f_3 + \partial_x R_2)).$$

The next lemma gives a formula for the Poincaré map $P_\mu^{\theta_0}$.

Lemma 4.4.1 *The Poincaré map $P_\mu^{\theta_0}$ of the system (4.3.6) is*

$$\begin{aligned} P_\mu^{\theta_0}(x,y) \;=\; & \begin{pmatrix} 1+\mu^2\varepsilon^{2p+2}(c_{13}+\varepsilon c_1) & 2\pi\varepsilon + c_4\mu^2\varepsilon^{2p+2} \\ c_2\mu^2\varepsilon^{2p+2} & 1+\mu^2\varepsilon^{2p+2}(-c_{13}+\varepsilon c_3) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \\ & +2\pi\varepsilon\begin{pmatrix} 2\pi\varepsilon q_1(x,y,\varepsilon) \\ -V'(x)+2\pi\varepsilon q_2(x,y,\varepsilon) \end{pmatrix} + \mu\varepsilon^{p+9}\psi_{\mu,\varepsilon}(x,y,\theta_0,\mu,\varepsilon) \qquad (4.4.2) \\ & +\mu^2\varepsilon^{2p+2}R_2(x,y,\theta_0,\mu,\varepsilon) \end{aligned}$$

where c_{13} does not depend on ε and μ, $c_i = c_i(\theta_0,\mu,\varepsilon)$, $i = 1,2,3,4$, are 2π-periodic functions with respect to θ_0 and

$$c_2 > 0 \quad \textit{and} \quad c_1 + c_3 > 0$$

for all $\theta_0 \in \mathbb{R}$ and $q_1 \in P_2$, $q_2 \in P_2$, $\psi_{\mu,\varepsilon} \in P_3$ and $R_2 = (R_2^1, R_2^2) \in P_2$ is of the form

$$\begin{aligned} R_2^1(x,y,\theta_0,\mu,\varepsilon) &= \varepsilon f_{20}(\theta_0,\mu,\varepsilon)x^2 + f_{11}(\theta_0,\mu,\varepsilon)xy + f_{02}(\theta_0,\mu,\varepsilon)y^2 + r_3^1 \\ R_2^2(x,y,\theta_0,\mu,\varepsilon) &= g_{20}(\theta_0,\mu,\varepsilon)x^2 + \varepsilon g_{11}(\theta_0,\mu,\varepsilon)xy + g_{02}(\theta_0,\mu,\varepsilon)y^2 + r_3^2 \end{aligned}$$

where r_3^1, $r_3^2 \in P_3$. Moreover all functions are C^0, C^1 and 2π-periodic with respect to θ_0 and analytic with respect to (x,y,μ).

Proof. We note that, since h_1 is continuous with respect to θ, the Poincaré map

$$P_\mu^{\theta_0}(x,y) = \varphi(\theta_0+2\pi,\theta_0,x,y,\mu,\varepsilon)$$

is $C^1$and 2π-periodic with respect to θ_0. We omit the dependence in this variable.

With a small modification in Lemma 3.4.3 ($\mu^2\varepsilon^{2p+2}$ instead of $\mu^2\varepsilon^{p+3}$) we can prove that the solutions of system (4.3.6) can be expressed of the form

$$\begin{aligned} \varphi(\theta,\theta_0,x,y,\mu,\varepsilon) \;=\; & \varphi_0(\theta,\theta_0,x,y) + \mu\varepsilon^{p+9}\psi(\theta,\theta_0,x,y,\mu,\varepsilon) \\ & +\mu^2\varepsilon^{2p+2}\phi(\theta,\theta_0,x,y,\mu,\varepsilon) \end{aligned}$$

where φ_0 is the flow of the unperturbed system ($\mu = 0$) the functions ψ and ϕ satisfy

$$\begin{aligned} \|\psi\| &\leq C\|(x,y)\|^3 \\ \|\phi\| &\leq C\|(x,y)\| \end{aligned}$$

and ϕ satisfies the equation

$$\dot{\phi} = \frac{1}{\mu^2\varepsilon^{2p+1}}[X_0(\varphi) - X_0(\varphi_0 + \mu\varepsilon^{p+9}\psi)] + R.$$

Therefore, it is clear that, for all $\theta \in [\theta_0, \theta_0 + 2\pi]$

$$\phi(\theta, \theta_0, x, y, \mu, \varepsilon) = \int_{\theta_0}^{\theta} R(\varphi(s, \theta_0, x, y, \mu, \varepsilon), s)ds + O(\varepsilon).$$

We observe that, by property 2) of Lemma 3.4.3,

$$\|\varphi(\theta, \theta_0, x, y, \mu, \varepsilon) - (x,y)\| \leq C\varepsilon\|(x,y)\|,$$

therefore,

$$\phi(\theta, \theta_0, x, y, \mu, \varepsilon) = \int_{\theta_0}^{\theta} R(x, y, s)ds + O(\varepsilon).$$

We define

$$\begin{aligned} \psi_{\mu,\varepsilon}(x, y, \theta_0) &= \psi(\theta_0 + 2\pi, \theta_0, x, y, \mu, \varepsilon) \\ \phi_{\mu,\varepsilon}(x, y, \theta_0) &= \phi(\theta_0 + 2\pi, \theta_0, x, y, \mu, \varepsilon) \end{aligned}$$

and then the Poincaré map of system (4.3.6) has the form

$$P_\mu^{\theta_0}(x,y) = P_0^{\theta_0}(x,y) + \mu\varepsilon^{p+9}\psi_{\mu,\varepsilon}(x,y,\theta_0) + \mu^2\varepsilon^{2p+2}\phi_{\mu,\varepsilon}(x,y,\theta_0)$$

with $\psi_{\mu,\varepsilon} \in P_3$ and $\phi_{\mu,\varepsilon} \in P_1$. Moreover$(\partial_y f_3 + \partial_y R_2, -(\partial_x f_3 + \partial_x R_2))$

$$\begin{aligned} \phi_{\mu,\varepsilon}(x, y, \theta_0) &= 2\pi \int_{\theta_0}^{\theta_0+2\pi} R(x, y, s)ds + O(\varepsilon) \\ &= 2\pi \begin{pmatrix} \partial_y f_3(x,y) \\ -\partial_x f_3(x,y) \end{pmatrix} + \begin{pmatrix} \int_{\theta_0}^{\theta_0+2\pi} \partial_y R_2(x,y,s)ds \\ -\int_{\theta_0}^{\theta_0+2\pi} \partial_x R_2(x,y,s)ds \end{pmatrix} + O(\varepsilon). \end{aligned} \tag{4.4.3}$$

Therefore, in order to prove that $P_\mu^{\theta_0}$ has the desired formula (4.4.2), we have to compute the linear terms of the Poincaré map of order $\mu^2\varepsilon^{2p+2}$ given by the linear terms of order $\mu^2\varepsilon^{2p+2}$ of the expression

$$\begin{pmatrix} \int_{\theta_0}^{\theta_0+2\pi} \partial_y R_2(x,y,s)ds \\ -\int_{\theta_0}^{\theta_0+2\pi} \partial_x R_2(x,y,s)ds \end{pmatrix}. \tag{4.4.4}$$

Because $\int_{\theta_0}^{\theta_0+2\pi} g_2(\theta)G_2(\theta) = 0$, it is clear that the linear terms of the first component of (4.4.4) are

$$\begin{aligned} -\int_{\theta_0}^{\theta_0+2\pi} x(g_3(s)G_1(s) + g_2(s)G_2(s)) - \int_{\theta_0}^{\theta_0+2\pi} 2yg_3(s)G_2(s)ds + O(\varepsilon) \\ = -x\int_{\theta_0}^{\theta_0+2\pi} g_3(s)G_1(s)ds - 2y\int_{\theta_0}^{\theta_0+2\pi} g_3(s)G_2(s)ds + O(\varepsilon) \end{aligned}$$

and the linear terms of the second one are

$$\begin{aligned} \int_{\theta_0}^{\theta_0+2\pi} 2xg_2(s)G_1(s)ds + \int_{\theta_0}^{\theta_0+2\pi} y(g_3(s)G_1(s) + g_2(s)G_2(s))ds + O(\varepsilon) \\ = 2x\int_{\theta_0}^{\theta_0+2\pi} g_2(s)G_1(s)ds + y\int_{\theta_0}^{\theta_0+2\pi} g_3(s)G_1(s)ds + O(\varepsilon) \end{aligned}$$

We denote

$$\begin{aligned} c_{13} &= -\int_{\theta_0}^{\theta_0+2\pi} g_3(\theta)G_1(\theta)d\theta = -\int_0^{2\pi} g_3(\theta)G_1(\theta)d\theta \\ \bar{c}_2 &= 2\int_{\theta_0}^{\theta_0+2\pi} g_2(\theta)G_1(\theta)d\theta = 2\int_0^{2\pi} g_2(\theta)G_1(\theta)d\theta. \end{aligned}$$

Then the linear terms of the Poincaré map $P_\mu^{\theta_0}$ are of the form

$$\begin{pmatrix} x + 2\pi\varepsilon y + \mu^2\varepsilon^{2p+2}x(c_{13} + \varepsilon c_1) + c_4\mu^2\varepsilon^{2p+2}y \\ y + c_2\mu^2\varepsilon^{2p+2}x + \mu^2\varepsilon^{2p+2}y(-c_{13} + \varepsilon c_3) \end{pmatrix}$$

with

$$c_i = c_i(\theta_0, \mu, \varepsilon) = \bar{c}_i + O(\varepsilon) + O(\mu) \qquad \text{for} \quad i = 1, 2, 3$$

Moreover, by **HP6** we have that $\bar{c}_2 > 0$ therefore, if ε and $|\mu|$ are small enough, the function c_2 is positive for all $\theta_0 \in \mathbb{R}$.

Finally, the Poincaré map is area preserving, hence

$$\det DP_\mu^{\theta_0}(0,0) = 1 + \mu^2\varepsilon^{2p+3}(c_1 + c_3 - c_2 2\pi + O(\mu^2\varepsilon^{2p+2})) = 1.$$

This implies that

$$c_1 + c_3 - c_2(2\pi + O(\mu^2\varepsilon^{2p+2})) = 0$$

hence, since c_2 is a positive function of θ_0, $c_1 + c_3$ is also a positive function. ■

4.5 A useful intermediate system

We observe that the origin is a fixed point of the Poincaré map $P_\mu^{\theta_0}$, and, by hypothesis **HP6**, the origin is a saddle point of $P_\mu^{\theta_0}$. Therefore, there exist local unstable and stable local invariant manifolds for the system (4.3.6) to the origin.

We note that if $\bar{c}_2 > 0$, then for μ and ε small enough, $DP_\mu^{\theta_0}(0,0)$ has two conjugate eigenvalues λ, $\bar{\lambda}$, which implies that, since the Poincaré map is area preserving, do not exist invariant manifolds in this case. In case that $\bar{c}_2 = 0$, to decide if the origin has local invariant stable and unstable manifolds, the terms of order $\mu^2\varepsilon^{2p+3}$ of the linear part of the Poincaré map must be studied . We have not studied this case.

We recall that the homoclinic curve of the origin in the unperturbed system is

$$\alpha(t) = \frac{2}{2+t^2} \qquad \beta(t) = -\frac{4t}{(2+t^2)^2}.$$

Thus, the convergence to the origin of the homoclinic orbit when t goes to $\pm\infty$ is of the order of a power of t^{-1}. We know that if the fixed point is a saddle point, the convergence of the orbits on the invariant manifolds to the origin is exponential. This suggests us that we will need a more accurate approximation of the stable manifold of the perturbed system so that the orbits on it have an exponential behaviour when t goes to $+\infty$. This section is devoted to find this suitable approximation of the local stable invariant curve of the perturbed system.

For this we perform a linear change of coordinates in order to put the linear part of $P_\mu^{\theta_0}$ in a more suitable form. We denote by $G_\mu^{\theta_0}$ the transformed map. Then we find a system (what we will call auxiliary system) by imposing that its Poincaré map contains the terms in x^2 of the remainder of $G_\mu^{\theta_0}$, its linear part and all the terms which do not depend on μ of $G_\mu^{\theta_0}$.

4.5.1 A suitable linear change of variables

We perform a linear change of variables in order to the linear part of the transformed Poincaré map be of the form

$$\begin{pmatrix} \cosh(2\pi\varepsilon\rho) & \sinh(2\pi\varepsilon\rho)/\rho \\ \rho\sinh(2\pi\varepsilon\rho) & \cosh(2\pi\varepsilon\rho) \end{pmatrix}$$

with ρ defined implicitly by

$$2\cosh(2\pi\varepsilon\rho(\theta_0)) = 2 + \mu^2\varepsilon^{2p+3}(c_1(\theta_0) + c_3(\theta_0)) \tag{4.5.1}$$

(the trace is an invariant magnitude). We observe that, since $c_1 + c_3 > 0$, ρ is well defined and it is a 2π-periodic function with respect to θ_0.

Remark 4.5.1 *If ρ satisfies (4.5.1) then*

$$\rho = \frac{\sqrt{c_1 + c_3}}{2\pi}\mu\varepsilon^{p+1/2}(1 + O(\mu^2\varepsilon^{2p+3})) \tag{4.5.2}$$

Proof. We denote

$$K = \frac{c_1 + c_3}{2},$$

then, $\cosh(2\pi\varepsilon\rho) = 1 + \mu^2\varepsilon^{2p+3}K$ and we have that

$$\begin{aligned} 2\pi\varepsilon\rho &= \operatorname{arc}\cosh(1 + \mu^2\varepsilon^{2p+3}K) \\ &= \log\left(1 + \mu^2\varepsilon^{2p+3}K + ((1 + \mu^2\varepsilon^{2p+3}K)^2 - 1)^{1/2}\right) \\ &= \log\left(1 + \mu^2\varepsilon^{2p+3}K + \sqrt{2K}\mu\varepsilon^{p+3/2}(1 + \mu^2\varepsilon^{2p+3}K/2)^{1/2}\right) \\ &= \log\left(1 + \mu^2\varepsilon^{2p+3}K + \sqrt{2K}\mu\varepsilon^{p+3/2} + O(\mu^3\varepsilon^{3p+9/2})\right) \\ &= \sqrt{2K}\mu\varepsilon^{p+3/2} + \mu^2\varepsilon^{2p+3}K - \mu^2\varepsilon^{2p+3}K + O(\mu^3\varepsilon^{3p+9/2}) \\ &= \sqrt{2K}\mu\varepsilon^{p+3/2}(1 + O(\mu^2\varepsilon^{2p+3})). \end{aligned}$$

Thus the statement holds. ∎

In order to simplify the notation we introduce

- $c = \cosh(2\pi\varepsilon\rho)$ and $s = \sinh(2\pi\varepsilon\rho)$ then $c^2 - s^2 = 1$.

- The transformed linear part,

$$A(\rho) = \begin{pmatrix} \cosh(2\pi\varepsilon\rho) & \sinh(2\pi\varepsilon\rho)/\rho \\ \rho\sinh(2\pi\varepsilon\rho) & \cosh(2\pi\varepsilon\rho) \end{pmatrix}.$$

- The change matrix $C_\mu^{\theta_0}$ will be of the form

$$C_\mu^{\theta_0}(\theta_0) = \begin{pmatrix} 1 & 0 \\ e(\theta_0) & d(\theta_0) \end{pmatrix}. \tag{4.5.3}$$

- We denote

$$DP_\mu^{\theta_0}(0,0) = \begin{pmatrix} d_1 & d_4 \\ d_2 & d_3 \end{pmatrix}$$

  and we note that $d_1 d_3 - d_2 d_4 = 1$ and that

$$2\cosh(2\pi\varepsilon\rho) = d_1 + d_3 \tag{4.5.4}$$

From now on, if there is not danger of confusion, we omit the dependence on θ_0.

Lemma 4.5.2 *If the function ρ satisfies (4.5.1), there exists a linear change of variables continuous in ε, $C^1$and 2π-periodic in θ_0 and analytic in μ of the form*

$$(x, y) = C_\mu^{\theta_0}(u, v)$$

with $C_\mu^{\theta_0} = \mathrm{Id} + O(\mu^2\varepsilon^{2p+1})$, such that the Poincaré map in the new variables (u, v), given by

$$G_\mu^{\theta_0} = C_\mu^{\theta_0} \circ P_\mu^{\theta_0} \circ (C_\mu^{\theta_0})^{-1}$$

is of the form:

$$\begin{aligned} G_\mu^{\theta_0}(u, v) &= A(\rho)\begin{pmatrix} u \\ v \end{pmatrix} + 2\pi\varepsilon\begin{pmatrix} 2\pi\varepsilon q_1(u, v, \varepsilon) \\ -V'(x) + 2\pi\varepsilon q_2(u, v, \varepsilon) \end{pmatrix} \\ &\quad + \mu\varepsilon^{p+9}\psi_{\mu,\varepsilon}(u, v, \theta_0) + \mu^2\varepsilon^{2p+2}\bar{R}_2(u, v, \theta_0) \end{aligned} \tag{4.5.5}$$

with ρ satisfying (4.5.1), q_1, q_2, $\psi_{\mu,\varepsilon}$ are the same that in (4.4.2) and, $\bar{R}_2 \in P_2$ and has the form

$$\begin{aligned} \bar{R}_2^1(x, y, \theta_0, \mu, \varepsilon) &= \varepsilon f_{20}(\theta_0, \mu, \varepsilon)x^2 \\ &\quad + f_{11}(\theta_0, \mu, \varepsilon)xy + f_{02}(\theta_0, \mu, \varepsilon)y^2 + r_3^1 \\ \bar{R}_2^2(x, y, \theta_0, \mu, \varepsilon) &= \varepsilon g_{11}(\theta_0, \mu, \varepsilon)xy \\ &\quad g_{20}(\theta_0, \mu, \varepsilon)x^2 + g_{02}(\theta_0, \mu, \varepsilon)y^2 + r_3^2 \end{aligned} \tag{4.5.6}$$

where r_3^1 and r_3^2 belong to P_3.

Proof. We impose that the linear part of the new system have the form

$$\begin{pmatrix} c & s/\rho \\ \rho s & c \end{pmatrix}$$

and we obtain the matrix equation:

$$\begin{pmatrix} c & s/\rho \\ \rho s & c \end{pmatrix}\begin{pmatrix} 1 & 0 \\ e & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ e & d \end{pmatrix}\begin{pmatrix} d_1 & d_4 \\ d_2 & d_3 \end{pmatrix}. \tag{4.5.7}$$

This system is overdetermined, for this, first we check that, if d and e satisfy the equations

$$\begin{aligned} c + e\frac{s}{\rho} &= d_1 \\ d\frac{s}{\rho} &= d_4 \end{aligned} \tag{4.5.8}$$

then the equation (4.5.7) is satisfied. Indeed, only we have to see that, if equations (4.5.8) are satisfied, then

$$\begin{aligned} \rho s + ce &= ed_1 + dd_2 \\ cd &= ed_4 + dd_3. \end{aligned}$$

We deal with the first equation. Using (4.5.4) and that $d_1 d_3 - d_2 d_4 = 1$, we have that

$$\begin{aligned} \rho s + e(c - d_1) - dd_2 &= \rho s - (c - d_1)^2 \frac{\rho}{s} - d_4 d_2 \frac{\rho}{s} \\ &= \frac{\rho}{s}(s^2 - c^2 - d_1^2 + 2d_1 c - d_4 d_2) \\ &= \frac{\rho}{s}(-1 - d_1^2 + d_1^2 + d_1 d_3 - d_4 d_2) \\ &= 0. \end{aligned}$$

And the second one

$$\begin{aligned} cd - ed_4 - dd_3 &= \frac{\rho}{s} d_4 (c - (d_1 - c) - d_3) \\ &= \frac{\rho}{s} d_4 (2c - (d_1 + d_3)) = 0. \end{aligned}$$

Consequently, it is enough to find d and e satisfying the equations (4.5.8). First we note that

$$\frac{s}{\rho} = \left(2\pi\varepsilon + \sum_{k=1}^{\infty} \frac{(2\pi\varepsilon)^{2k+1}}{(2k+1)!}\rho^{2k}\right) = 2\pi\varepsilon(1 + O(\mu^2\varepsilon^{2p+3}))$$

and then

$$
\begin{aligned}
e = -\frac{(c-d_1)}{2\pi\varepsilon(1+O(\mu^2\varepsilon^{2p+3}))} &= \frac{d_1-d_3}{2}\frac{1}{2\pi\varepsilon(1+O(\mu^2\varepsilon^{2p+3}))} \\
= \frac{O(\mu^2\varepsilon^{2p+2})}{2\pi\varepsilon(1+O(\mu^2\varepsilon^{2p+3}))} &= O(\mu^2\varepsilon^{2p+1})
\end{aligned}
$$

and

$$
\begin{aligned}
d = \frac{d_4}{2\pi\varepsilon(1+O(\mu^2\varepsilon^{2p+3}))} &= \frac{2\pi\varepsilon + c_4\mu^2\varepsilon^{2p+2}}{2\pi\varepsilon(1+O(\mu^2\varepsilon^{2p+3}))} \\
= \frac{1+O(\mu^2\varepsilon^{2p+1})}{1+O(\mu^2\varepsilon^{2p+3})} &= 1+O(\mu^2\varepsilon^{2p+1}).
\end{aligned}
$$

Thus the linear part of G^{θ_0} has the form we have prescribed. Now we perform the change of variables

$$
\begin{pmatrix} u \\ v \end{pmatrix} = (C_\mu^{\theta_0})^{-1}\begin{pmatrix} x \\ y \end{pmatrix}
$$

with $C_\mu^{\theta_0}$ given by (4.5.3) with d and e determined by (4.5.8). Then

$$
\begin{aligned}
G_\mu^{\theta_0}(u,v) &= ((C_\mu^{\theta_0})^{-1}\circ P_\mu^{\theta_0}\circ C_\mu^{\theta_0})(u,v) = ((C_\mu^{\theta_0})^{-1}\circ DP_\mu^{\theta_0}(0,0)\circ C_\mu^{\theta_0})(u,v) \\
&\quad +2\pi\varepsilon(C_\mu^{\theta_0})^{-1}\begin{pmatrix} 2\pi\varepsilon q_1(u,v,\varepsilon) \\ -V'(u)+2\pi\varepsilon q_2(u,v,\varepsilon)\end{pmatrix} + \mu\varepsilon^{p+9}(C_\mu^{\theta_0})^{-1}\psi_{\mu,\varepsilon}(u,v,\theta_0) \\
&\quad +\mu^2\varepsilon^{2p+2}(C_\mu^{\theta_0})^{-1}R_2(u,v,\theta_0) \\
&= \begin{pmatrix} c & s/\rho \\ \rho s & c \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} + 2\pi\varepsilon\begin{pmatrix} 2\pi\varepsilon q_1(u,v,\varepsilon) \\ -V'(u)+2\pi\varepsilon q_2(u,v,\varepsilon)\end{pmatrix} \\
&\quad +\mu\varepsilon^{p+9}\psi_{\mu,\varepsilon}(u,v,\theta_0) + \mu^2\varepsilon^{2p+2}\bar{R}_2(u,v,\theta_0)
\end{aligned}
$$

with $\bar{R}_2 \in P_2$ having the form (4.5.6). ■

4.5.2 The auxiliary system

We decompose $\bar{R}_2 = (\bar{R}_2^1, \bar{R}_2^2)$ (the remainder term of $G_\mu^{\theta_0}$ given in (4.5.5)) and we decompose $\bar{R}_2^2$ in the form

$$
\bar{R}_2^2(u,v,\theta_0) = a_1(\theta_0)u^2 + r_2(u,v,\theta_0) \tag{4.5.9}
$$

where $r_2 \in P_2$ and has not terms in u^2. Since $\bar{R}_2$ is analytic with respect to (u, v) and depends C^1 with respect to θ_0, continuously with respect to ε and analytically with respect to μ, $a_1(\theta_0)$ has the same kind of dependence.

We will look for an auxiliary autonomous system of the form

$$\dot{w} = Y_\mu(w) = \varepsilon X_0(w) + \mu^2 \varepsilon^{2p+2} Y_1(w, \theta_0, \mu, \varepsilon) \tag{4.5.10}$$

with $w = (u, v)$, εX_0 the vector field corresponding to the unperturbed case, that is

$$X_0(u, v) = \begin{pmatrix} v \\ -V'(u) \end{pmatrix}$$

and

$$Y_1(w) = \begin{pmatrix} 0 \\ b_1 u + b_2 u^2 \end{pmatrix}$$

with b_1, b_2, depending on θ_0, μ and ε, to be determined by imposing the condition that the Poincaré map of $\dot{w} = Y_\mu(w, \theta_0, \mu, \varepsilon)$ is close, in a certain sense to be made precise later, to the Poincaré map $P_\mu^{\theta_0}$.

Note that the linearized equation of equation (4.5.10) is $\dot{w} = Bw$ with

$$B = \varepsilon \begin{pmatrix} 0 & 1 \\ \mu^2 \varepsilon^{2p+1} b_1 & 0 \end{pmatrix}.$$

Since the system (4.5.10) is autonomous, its flow has the form

$$\phi(\theta, u, v, \mu, \varepsilon) = (\phi_1(\theta, u, v, \mu, \varepsilon), \phi_2(\theta, u, v, \mu, \varepsilon)).$$

We denote by F_μ the Poincaré map of (4.5.10), that is

$$F_\mu(u, v) = \phi(2\pi, u, v, \mu, \varepsilon).$$

The next lemma proves that there exist constants b_1 and b_2 such that the coefficient of u^2, the linear part and the terms which do not depend on μ of the Poincaré map of (4.5.10), F_μ, are the same as the corresponding ones of the map $G_\mu^{\theta_0}$.

Lemma 4.5.3 *There exist $b_1(\theta_0, \mu, \varepsilon)$ and $b_2(\theta_0, \mu, \varepsilon)$, such that the Poincaré map of the system (4.5.10) is of the form*

$$\begin{aligned} F_\mu^{\theta_0}(u, v) \;&=\; A(\rho) \begin{pmatrix} u \\ v \end{pmatrix} + 2\pi\varepsilon \begin{pmatrix} 2\pi\varepsilon q_1(u, v, \varepsilon) \\ -V'(x) + 2\pi\varepsilon q_2(u, v, \varepsilon) \end{pmatrix} \\ &\quad + \mu^2 \varepsilon^{2p+2} \begin{pmatrix} 0 \\ a_1 u^2 \end{pmatrix} + \mu^2 \varepsilon^{2p+2} \tilde{R}_2(u, v, \theta_0) \end{aligned} \tag{4.5.11}$$

where a_1 is defined in (4.5.9), q_1 and q_2 are the same as the ones in Lemma 4.4.2, ρ satisfies (4.5.1) and the remaining term $\tilde{R}_2 \in P_2$ is of the form given in (4.5.6) and its second component has not terms in u^2. Moreover all these functions are C^0, C^1 and 2π-periodic with respect to θ_0 and analytic with respect to (u, v, μ).

Proof. The proof is similar to that of Lemma 4.4.1. We sketch it. Using Lemma 3.4.3 (putting $\mu^2\varepsilon^{2p+2}$ instead of $\mu^2\varepsilon^{p+3}$) we can prove that

$$F_\mu(x, y, \theta_0) = P_0^{\theta_0}(x, y) + \mu^2\varepsilon^{2p+2}\phi_{\mu,\varepsilon}(x, y, \theta_0)$$

with $\phi_{\mu,\varepsilon} \in P_1$ and such that

$$\phi_{\mu,\varepsilon}(x, y, \theta_0) = 2\pi Y_1(x, y, \theta_0, \mu, \varepsilon) + \varepsilon f_1(x, y, \theta_0, \mu, \varepsilon).$$

with $f_1 \in P_1$ (see proof of Lemma 4.4.1, concretely (4.4.3)). We write $f_1 = (f_1^1, f_2^1)$ and

$$f_1^2(u, v) = e_1 u^2 + g_1(u, v)$$

with $g_1 \in P_1$. Obviously since the auxiliary system is analytic, c_1 is an analytic function of b_1, b_2 and ε^{2p+1}. We consider the equation obtained equating the coefficients of u^2 in both sides of the second components of F_μ and $P_\mu^{\theta_0}$:

$$b_2 + \varepsilon e_1 = a_1. \tag{4.5.12}$$

(a_1 is defined in (4.5.9)). The implicit function theorem applied to (4.5.12) when $\varepsilon = 0$ shows that we can isolate b_2 as a continuous function of $(\theta, \mu, \varepsilon)$, which is C^1 in θ and analytic in μ. Finally we observe that, since the coefficient a_1 depends 2π-periodically with respect to θ_0, b_2 is also 2π-periodic on θ_0.

Now we deal with the linear terms, We consider the linear system

$$\dot{w} = Bw.$$

It is clear that the linear term of F_μ is

$$e^{2\pi B}w = \begin{pmatrix} \cosh(2\pi b\varepsilon) & \sinh(2\pi b\varepsilon)/b \\ b\sinh(2\pi b\varepsilon) & \cosh(2\pi b\varepsilon) \end{pmatrix} w$$

with $b = \mu\varepsilon^{p+1/2}\sqrt{b_1}$. Hence in order to check that the linear part of F_μ is $A(\rho)$ it is sufficient to choose $b = \rho$ and

$$b_1 = b_1(\theta_0) = \frac{b^2}{\mu^2\varepsilon^{2p+1}} = \frac{c_1 + c_3}{(2\pi)^2}(1 + O(\mu^2\varepsilon^{2p+3}))$$

where ρ satisfies (4.5.1) and c_1 and c_3 are defined in (4.4.2).

Since the coefficients b_1 and b_2 depend on θ_0 we write

$$F_\mu = F_\mu^{\theta_0}.$$

■

4.6 The operators $\mathcal{B}$ and $\mathcal{C}$

The Banach spaces which we use in this Section were fixed at the beginning of Section 4.2. For any $\varepsilon > 0$ and $\rho = \rho(t/\varepsilon) > 0$ a $2\pi\varepsilon$-periodic function of t, we define the operators

$$\mathcal{B} : \mathcal{Y}_k^l \times \mathcal{Y}_k^l \to \mathcal{Y}_k^l \times \mathcal{Y}_k^l$$

and, for $\eta > 0$ and $l > 1$,

$$\mathcal{C} : \mathcal{Y}_k^l \times \mathcal{Y}_{k+1}^l \to \mathcal{Y}_k^l \times \mathcal{Y}_{k+1}^l$$

by the expressions

$$\begin{aligned}(\mathcal{B}\sigma)(t,s) &= \sigma(t+2\pi\varepsilon, s) - A(\rho(t/\varepsilon))\sigma(t,s)\\ (\mathcal{C}\sigma)(t,s) &= \sigma(t+2\pi\varepsilon, s) - A(\rho(t/\varepsilon))\sigma(t,s)\end{aligned}$$

where

$$A(\rho(t/\varepsilon)) = \begin{pmatrix} \cosh(2\pi\varepsilon\rho(t/\varepsilon)) & \sinh(2\pi\varepsilon\rho(t/\varepsilon))/\rho(t/\varepsilon) \\ \rho(t/\varepsilon)\sinh(2\pi\varepsilon\rho(t/\varepsilon)) & \cosh(2\pi\varepsilon\rho(t/\varepsilon)) \end{pmatrix}.$$

These operators are well defined. The operators $\mathcal{B}$ and $\mathcal{C}$ are formally equal. The difference is their domain of definition.

Let k_1, k_2, l_1 and l_2 be positive real numbers. We endow the product space

$$\mathcal{Y} = \mathcal{Y}_{k_1}^{l_1} \times \mathcal{Y}_{k_2}^{l_2}$$

with the norm

$$\|\psi\|_{\mathcal{Y}} = \alpha_1\|\psi_1\|_{k_1}^{l_1} + \alpha_2\|\psi\|_{k_2}^{l_2} \tag{4.6.1}$$

with α_1, $\alpha_2 > 0$ will be chosen later on. We note that the product space becomes a Banach spaces and that the operators $\mathcal{B}$ and $\mathcal{C}$ are well defined linear operators.

We will need a right inverses of $\mathcal{B}$ and $\mathcal{C}$. First we look for a formal right inverse of $\mathcal{B}$. Since $\mathcal{C}$ is formally equal to $\mathcal{B}$, the formal expression obtained also will work for $\mathcal{C}^{-1}$. For that we write $\mathcal{B}\sigma = \psi$ and we obtain

$$\sigma(t,s) = -A^{-1}(\rho(t/\varepsilon))\psi(t,s) + A^{-1}(\rho(t/\varepsilon))\sigma(t+2\pi\varepsilon, s). \tag{4.6.2}$$

Since $\rho(t/\varepsilon)$ is a $2\pi\varepsilon$-periodic function, applying (4.6.2) iteratively we have that

$$\begin{aligned}\sigma(t,s) &= -\sum_{j=0}^{N} A^{-(j+1)}(\rho(t/\varepsilon))\psi(t+2\pi\varepsilon j, s) \\ &\quad + A^{-(N+1)}(\rho(t/\varepsilon))\sigma(t+2\pi\varepsilon(N+1), s).\end{aligned} \tag{4.6.3}$$

It is not difficult to see that

$$A^{-(j+1)}(\rho(t/\varepsilon)) = \begin{pmatrix} \cosh(2\pi\varepsilon\rho(t/\varepsilon)(j+1)) & -\sinh(2\pi\varepsilon\rho(t/\varepsilon)(j+1))/\rho(t/\varepsilon) \\ -\rho\sinh(2\pi\varepsilon\rho(t/\varepsilon)(j+1)) & \cosh(2\pi\varepsilon\rho(t/\varepsilon)(j+1)) \end{pmatrix}.$$

If $\sigma \in \mathcal{Y}_k^l \times \mathcal{Y}_k^l$, σ goes to zero when t goes to ∞ fast enough ($l \geq 1$), so that when we take limit as $N \to \infty$ in (4.6.3), $A^{-(N+1)}(\rho(t/\varepsilon))\sigma(t+2\pi\varepsilon(N+1), s) \to 0$ we obtain the formal expression for $\mathcal{B}^{-1}$ (and $\mathcal{C}^{-1}$):

$$\sigma(t,s) = -\sum_{j=0}^{\infty} A^{-(j+1)}(\rho(t/\varepsilon))\psi(t+2\pi\varepsilon j, s).$$

The following two lemmas give useful bounds for the right inverse of the operators $\mathcal{B}$ and $\mathcal{C}$. The formal expression in both of them is the same, but the spaces where they are defined are different. This fact gives essential different bounds of the norm of $\mathcal{B}^{-1}$ and $\mathcal{C}^{-1}$.

We omit the dependence of ρ and A on t/ε.

Lemma 4.6.1 *Let $k > 2$ and $l \geq 1$. The operator $\mathcal{B}$ has a right inverse $\mathcal{B}^{-1} : \mathcal{Y}_k^l \times \mathcal{Y}_k^l \to \mathcal{Y}_{k-2}^l \times \mathcal{Y}_{k-1}^l$ with*

$$\|(\mathcal{B}^{-1}\psi)_1\|_{k-2}^l \leq \frac{1}{2\pi\varepsilon}\left[\frac{1}{(k-1)(k-2)}\|\psi_2\|_k^l + O\Big(\frac{1}{T}\Big)\|\psi\|_{\mathcal{Y}}\right]$$

and

$$\|(\mathcal{B}^{-1}\psi)_2\|_{k-1}^l \leq \frac{1}{2\pi\varepsilon}\left[\frac{1}{(k-1)}\|\psi_2\|_k^l + O\Big(\frac{1}{T}\Big)\|\psi\|_{\mathcal{Y}}\right]$$

Proof. Along the proof we will use that, $T > 0$ is big enough. Also we will assume that ε is small enough so that

$$\varepsilon \leq \frac{1}{T}.$$

We define $\psi_N(t,s) = \sum_{j=0}^{N} A^{-(j+1)}\psi(t+2\pi\varepsilon j, s)$ and

$$(\mathcal{B}^{-1}\psi)(t,s) = \lim_{N\to\infty} \psi_N(t,s).$$

First we prove that if $\psi \in \mathcal{Y}_k^l \times \mathcal{Y}_k^l$, ψ_N converges uniformly. Indeed from

$$|(A^{-(j+1)}\psi(t+2\pi\varepsilon j, s))_1| \leq \left(\frac{1}{(T+2\pi\varepsilon j)^k}\|\psi_1\|_k^l + \frac{1}{\rho}\frac{1}{(T+2\pi\varepsilon j)^k}\|\psi_2\|_k^l\right)\|A^{-1}\|_{\mathcal{Y}}$$

and

$$|(A^{-(j+1)}\psi(t+2\pi\varepsilon j, s))_2| \leq \left(\frac{\rho}{(T+2\pi\varepsilon j)^k}\|\psi_1\|_k^l + \frac{1}{(T+2\pi\varepsilon j)^k}\|\psi_2\|_k^l\right)\|A^{-1}\|_{\mathcal{Y}}$$

the claim follows from the M-test of Weierstrass. Therefore $\mathcal{B}^{-1}\psi$ satisfies the first three conditions which define $\mathcal{Y}_{k-2}^l \times \mathcal{Y}_{k-1}^l$. For $u \geq 0$, we define

$$S_1^k(u) = \sum_{j=0}^{+\infty} \frac{u^{k-1}}{(u+2\pi\varepsilon j)^k} = \frac{1}{2\pi\varepsilon}\sum_{j=0}^{+\infty}\frac{2\pi\varepsilon}{u}\frac{1}{\left(1+\frac{2\pi\varepsilon j}{u}\right)^k}$$

$$S_2^k(u) = \sum_{j=0}^{+\infty} \frac{u^{k-2}}{(u+2\pi\varepsilon j)^k}2\pi\varepsilon j = \frac{1}{2\pi\varepsilon}\sum_{j=0}^{+\infty}\frac{2\pi\varepsilon}{u}\frac{1}{\left(1+\frac{2\pi\varepsilon j}{u}\right)^k}\frac{2\pi\varepsilon j}{u}$$

and we observe that, for $u > 0$ and $k > 1$, we have that

$$\begin{aligned} S_1^k(u) &\leq \frac{1}{2\pi\varepsilon}\left[\frac{2\pi\varepsilon}{u} + \int_0^{+\infty}\frac{1}{(1+x)^k}dx\right] = \frac{1}{2\pi\varepsilon}\left[\frac{2\pi\varepsilon}{u} + \frac{1}{k-1}\right] \\ &\leq \frac{1}{2\pi\varepsilon}\left[\frac{1}{k-1} + O\Big(\frac{1}{T}\Big)\right] \end{aligned} \tag{4.6.4}$$

and, for $k > 2$

$$\begin{aligned} S_2^k(u) &\leq \frac{1}{2\pi\varepsilon}\left[\frac{2\pi\varepsilon}{u}\frac{1}{(k-1)e} + \int_0^{+\infty}\frac{x}{(1+x)^k}dx\right] \\ &= \frac{1}{2\pi\varepsilon}\left[\frac{2\pi\varepsilon}{u}\frac{1}{(k-1)e} + \frac{1}{(k-1)(k-2)}\right] \\ &= \frac{1}{2\pi\varepsilon}\left[\frac{1}{(k-1)(k-2)} + O\Big(\frac{1}{T}\Big)\right]. \end{aligned} \tag{4.6.5}$$

Let $\psi \in \mathcal{Y}_k^l \times \mathcal{Y}_k^l$. We denote $u = t + \mathrm{Re}\, s$. Then we obtain the following bound:

$$\begin{aligned}
\|(\mathcal{B}^{-1}\psi)_1\|_{k-2}^l &\leq \sup_{D^s} \sum_{j=0}^{\infty} u^{k-2} e^{\rho u l} \cosh(2\pi\varepsilon\rho(j+1)) |\psi_1(t + 2\pi\varepsilon j, s)| \\
&\quad + \sup_{D^s} \sum_{j=0}^{\infty} u^{k-2} e^{\rho u l} \frac{\sinh(2\pi\varepsilon\rho(j+1))}{\rho} |\psi_2(t + 2\pi\varepsilon j, s)| \\
&\leq \sup_{D^s} \sum_{j=0}^{\infty} \frac{u^{k-2}}{(u+2\pi\varepsilon j)^k} e^{-2\pi\varepsilon\rho l j} \cosh(2\pi\varepsilon\rho(j+1)) \|\psi_1\|_k^l \\
&\quad + \sup_{D^s} \sum_{j=0}^{\infty} \frac{u^{k-2}}{(u+2\pi\varepsilon j)^k} e^{-2\pi\varepsilon\rho l j} 2\pi\varepsilon j \cosh(2\pi\varepsilon\rho(j+1)) \|\psi_2\|_k^l.
\end{aligned}$$

Using that for $x \geq 0$, $e^{-x}\cosh x \leq 1$, $\sinh x \leq x \cosh x$ and the bounds (4.6.4) and (4.6.5) we bound the first component of $\mathcal{B}^{-1}\psi$ by

$$\begin{aligned}
\|(\mathcal{B}^{-1}\psi)_1\|_{k-2}^l &\leq e^{2\pi\varepsilon\rho} \left[\sup_{D^s} \frac{1}{u} S_1^k(u) \|\psi_1\|_k^l + S_2^k(u) \|\psi_2\|_k^l \right] \\
&\leq \frac{1}{2\pi\varepsilon} \left[\frac{1}{(k-1)(k-2)} \|\psi_2\|_k^l + O(\frac{1}{T}) \|\psi\| \right].
\end{aligned}$$

Here we have used that $\varepsilon \leq \frac{1}{T}$. Analogously the second component can be bounded by

$$\begin{aligned}
\|(\mathcal{B}^{-1}\psi)_2\|_{k-1}^l &\leq \sup_{D^s} \sum_{j=0}^{\infty} u^{k-1} e^{\rho u l} \rho \sinh(2\pi\varepsilon\rho(j+1)) |\psi_1(t + 2\pi\varepsilon j, s)| \\
&\quad + \sup_{D^s} \sum_{j=0}^{\infty} u^{k-1} e^{\rho u l} \cosh(2\pi\varepsilon\rho(j+1)) |\psi_2(t + 2\pi\varepsilon j, s)| \\
&\leq \sup_{D^s} \sum_{j=0}^{\infty} \frac{u^{k-1}}{(u+2\pi\varepsilon j)^k} e^{-2\pi\varepsilon\rho l j} \rho \sinh(2\pi\varepsilon\rho(j+1)) \|\psi_1\|_k^l \\
&\quad + \sup_{D^s} \sum_{j=0}^{\infty} \frac{u^{k-1}}{(u+2\pi\varepsilon j)^k} e^{-2\pi\varepsilon\rho l j} \cosh(2\pi\varepsilon\rho(j+1)) \|\psi_2\|_k^l
\end{aligned}$$

and using that $\varepsilon \leq \frac{1}{T}$, $\sinh x \leq \cosh x$ and that $e^{-x}\cosh x \leq 1$ it can be bounded by

$$\begin{aligned}
\|(\mathcal{B}^{-1}\psi)_2\|_{k-1}^l &\leq e^{2\pi\varepsilon\rho} \sup_{D^s} S_1^k(u) (\rho \|\psi_1\|_k^l + \|\psi_2\|_k^l) \\
&\leq \frac{1}{2\pi\varepsilon} \left[\frac{1}{k-1} \|\psi_2\|_k^l + O(\frac{1}{T}) \|\psi\| \right].
\end{aligned}$$

■

The following lemma gives a useful bound of a right inverse of $\mathcal{C}$.

Lemma 4.6.2 *Let $\eta > 0$, $k \geq 1$ and $l > 1$. Then, the operator $\mathcal{C}$ has a right inverse $\mathcal{C}^{-1} : \mathcal{Y}_k^l \times \mathcal{Y}_{k+1}^l \to \mathcal{Y}_k^{l-\eta} \times \mathcal{Y}_{k+1}^{l-\eta}$. Moreover there exists a constant K independent of ε and ρ such that*

$$\|(\mathcal{C}^{-1}\psi)_1\|_k^{l-\eta} \leq \frac{K}{2\pi\varepsilon\rho}\|\psi\|_{\mathcal{Y}}$$

and

$$\|(\mathcal{C}^{-1}\psi)_2\|_{k+1}^{l-\eta} \leq \frac{K}{2\pi\varepsilon\rho}\|\psi_1\|_{\mathcal{Y}}.$$

Proof. Formally, the operator $\mathcal{C}$ is the same than $\mathcal{B}$, but the definition domain is different. For this the proof of that the three first conditions of the definition of $\mathcal{Y}_k^{l-\eta} \times \mathcal{Y}_{k+1}^{l-\eta}$ are satisfied by $\mathcal{C}^{-1}\psi$ if $\psi \in \mathcal{Y}_k^l \times \mathcal{Y}_{k+1}^l$ is similar to that of the previous lemma. Analogously that in Lemma 4.6.1 for $u \geq 0$, we define

$$S(u) = \sum_{j=0}^{\infty} e^{-\rho 2\pi\varepsilon jl} \cosh(2\pi\varepsilon\rho(j+1))$$

and, since $l > 1$, we observe that:

$$S(u) = \frac{1}{2}\sum_{j=0}^{\infty}\left[e^{-\rho 2\pi\varepsilon(jl-j-1)} + e^{-\rho 2\pi\varepsilon(jl+j-1)}\right] \leq \frac{e^{2\pi\varepsilon\rho(1+l)/2}}{\sinh(2\pi\varepsilon\rho(1-l)/2)} \leq \frac{K}{2\pi\varepsilon\rho}. \quad (4.6.6)$$

Let $\psi \in \mathcal{Y}_k^l \times \mathcal{Y}_{k+1}^l$. Now we bound $\|(\mathcal{C}^{-1}\psi)_1\|_k^{l-\eta}$ and $\|(\mathcal{C}^{-1}\psi)_2\|_{k+1}^{l-\eta}$:

$$\begin{aligned}
\|(\mathcal{C}^{-1}\psi)_1\|_k^{l-\eta} &\leq \sup_{D^s}\sum_{j=0}^{\infty} u^k e^{\rho u(l-\eta)}\cosh(2\pi\varepsilon\rho(j+1))|\psi_1(t+2\pi\varepsilon j, s)| \\
&\quad + \sup_{D^s}\sum_{j=0}^{\infty} u^k e^{\rho u(l-\eta)}\frac{\sinh(2\pi\varepsilon\rho(j+1))}{\rho}|\psi_2(t+2\pi\varepsilon j, s)| \\
&\leq \sup_{D^s}\sum_{j=0}^{\infty}\frac{u^k}{(u+2\pi\varepsilon j)^k}e^{-\rho u\eta}e^{-\rho 2\pi\varepsilon jl}\cosh(2\pi\varepsilon\rho(j+1))\|\psi_1\|_k^l \\
&\quad + \sup_{D^s}\sum_{j=0}^{\infty}\frac{u^k 2\pi\varepsilon(j+1)}{(u+2\pi\varepsilon j)^{k+1}}e^{-\rho u\eta}e^{-\rho 2\pi\varepsilon jl}\cosh(2\pi\varepsilon\rho(j+1))\|\psi_2\|_{k+1}^l \\
&= K\sup_{D^s}S(u)\|\psi\|_{\mathcal{Y}} \leq \frac{K}{2\pi\varepsilon\rho}\|\psi\|_{\mathcal{Y}}
\end{aligned}$$

and the second one

$$\begin{aligned}
\|(\mathcal{C}^{-1}\psi)_2\|_{k+1}^{l-\eta} &\leq \sup_{D^s}\sum_{j=0}^{\infty} u^{k+1}e^{\rho u(l-\eta)}\rho\sinh(2\pi\varepsilon\rho(j+1))|\psi_1(t+2\pi\varepsilon j,s)| \\
&\quad + \sup_{D^s}\sum_{j=0}^{\infty} u^{k+1}e^{\rho u(l-\eta)}\cosh(2\pi\varepsilon\rho(j+1))|\psi_2(t+2\pi\varepsilon j,s)| \\
&\leq \sup_{D^s}\rho u\sum_{j=0}^{\infty}\frac{u^k}{(u+2\pi\varepsilon j)^k}e^{-\rho u\eta}e^{-2\pi\varepsilon\rho l j}\cosh(2\pi\varepsilon\rho(j+1))\|\psi_1\|_k^l \\
&\quad + \sup_{D^s}\sum_{j=0}^{\infty}\frac{u^{k+1}}{(u+2\pi\varepsilon j)^{k+1}}e^{-\rho u\eta}e^{-2\pi\varepsilon\rho l j}\cosh(2\pi\varepsilon\rho(j+1))\|\psi_2\|_{k+1}^l \\
&= \sup_{D^s}(\rho u e^{-\rho u\eta}+1)S(u)\|\psi\|_{\mathcal{Y}} \leq \frac{K}{2\pi\varepsilon\rho}\|\psi\|_{\mathcal{Y}}.
\end{aligned}$$

Here we have used that the function $e^{-\rho u\eta}u\rho$ is bounded by a constant independent of ε and ρ (but depending on η). This finishes the proof. ∎

4.7 Proof of the Theorem 4.2.1

As in Section 3.7 of Chapter 3, we scale the time $t = \varepsilon\theta$ in the system (4.3.6) and we obtain

$$\begin{aligned}
x' &= y + \mu\varepsilon^{p+8}\partial_y F + \mu^2\varepsilon^{2p+1}(\partial_y f_3 + \partial_y R_2) \\
y' &= -V'(x) - \mu\varepsilon^{p+8}\partial_x F - \mu^2\varepsilon^{2p+1}(\partial_x f_3 + \partial_x R_2).
\end{aligned} \tag{4.7.1}$$

Here $'$ stands for the derivative with respect to t. We denote by φ and $\tilde{\varphi}$ the flows of system (4.3.6) and its scaled system (4.7.1). For any t, t_0 for which the solutions are defined we have that

$$\tilde{\varphi}(t,t_0,x,y,\mu,\varepsilon) = \varphi(t/\varepsilon, t_0/\varepsilon, x, y, \mu, \varepsilon).$$

We recall that the definition (4.4.1) of P_μ^θ: $P_\mu^\theta(x,y) = \varphi(\theta+2\pi,\theta,x,y,\mu,\varepsilon)$. Then

$$\begin{aligned}
P_\mu^{t/\varepsilon}(\tilde{\varphi}(t,t_0,x,y)) &= \varphi(t/\varepsilon+2\pi, t/\varepsilon, \tilde{\varphi}(t,t_0,x,y),\mu,\varepsilon) \\
&= \tilde{\varphi}(t+2\pi\varepsilon, t, \tilde{\varphi}(t,t_0,x,y),\mu,\varepsilon) \\
&= \tilde{\varphi}(t+2\pi\varepsilon, t_0, x, y, \mu, \varepsilon).
\end{aligned}$$

As in Chapter 3, we look for a parameterization $\tilde{\gamma}^s_\mu(t,s)$ of the stable manifold of the system (4.7.1) such that $t \in \mathbb{R}$ is the time, $s \in \mathbb{C}$ a complex parameter. For this we look for γ^s_μ satisfying the invariance condition:

$$P^{t/\varepsilon}_\mu(\gamma^s_\mu(t,s)) = \gamma^s_\mu(t+2\pi\varepsilon, s). \tag{4.7.2}$$

In fact, we will find the local stable manifold of $G^{t/\varepsilon}_\mu$ instead of the local stable manifold of $P^{\theta_0}_\mu$, γ^s_μ. For this we observe that, if γ^s_μ satisfies (4.7.2), then the function $\gamma^s_{G,\mu}$ defined by

$$\gamma^s_{G,\mu}(t,s) = \left(C^{t/\varepsilon}_\mu(t)\right)^{-1}\gamma^s_\mu(t,s), \tag{4.7.3}$$

where $C^{t/\varepsilon}_\mu(t/\varepsilon)$ is the change of Lemma 4.5.2, satisfies that

$$\begin{aligned}\gamma^s_{G,\mu}(t+2\pi\varepsilon,s) &= \left(C^{t/\varepsilon}_\mu(t)\right)^{-1}(t+2\pi\varepsilon)\gamma^s_\mu(t+2\pi\varepsilon,s)\\ &= \left(C^{t/\varepsilon}_\mu(t)\right)^{-1}P_\mu(\gamma^s_\mu(t,s))\\ &= G^{t/\varepsilon}_\mu(\gamma^s_{G,\mu}(t,s)).\end{aligned} \tag{4.7.4}$$

Here we have used that $C^{t/\varepsilon}_\mu(t)$ is $2\pi\varepsilon$-periodic in t. Therefore, if we find $\gamma^s_{G,\mu}$ satisfying the invariance condition given in (4.7.4), then $\gamma^s_\mu = C^{t/\varepsilon}_\mu(t/\varepsilon)\gamma^s_{G,\mu}$ satisfies (4.7.2) what is that we want to prove.

We note that, since $p \geq 1$ (Remark 4.2.2), $C^{t/\varepsilon}_\mu(t/\varepsilon)$ satisfies

$$C^{t/\varepsilon}_\mu(t/\varepsilon) = \operatorname{Id} + O(\mu^2\varepsilon^{2p+1}) = \operatorname{Id} + O(\mu^2\varepsilon^{p+2}). \tag{4.7.5}$$

Let

$$w' = \varepsilon X_0(w) + \mu^2\varepsilon^{2p+2}Y_1(w,\theta_0,\mu,\varepsilon)$$

the autonomous auxiliary system (4.5.10) with the constants b_1 and b_2 given in Lemma 4.5.3 and

$$w' = X_0(w) + \mu^2\varepsilon^{2p+1}Y_1(w,\theta_0,\mu,\varepsilon) \tag{4.7.6}$$

its scaled system. Let $\phi(\theta,w;\theta_0,\mu,\varepsilon)$ and $\tilde{\phi}(\theta,w;\theta_0,\mu,\varepsilon)$ be the flows of the auxiliary system (4.5.10) and its scaled system ($t = \varepsilon\theta$). (We recall that for such systems θ_0 is not an initial condition, it is a parameter of the system).

In order to achieve the third property of Theorem 4.2.1 we must prove that the homoclinic orbit of the scaled auxiliary system (4.7.6) is $O(\mu\varepsilon^{p+2})$ close to the homoclinic connexion of the unperturbed system. The following elementary lemma check this.

Lemma 4.7.1 *There exists a parameterization of the stable invariant manifold $\gamma(u,\theta_0)$ of the system (4.7.6), where b_1 and b_2 are given in Lemma 4.5.3, and it satisfies that*

$$\gamma(u,\theta_0) - \gamma_0(u) = O(\mu\varepsilon^{p+2})$$

for all $\theta_0 \in \mathbb{R}$ and for all u such that $\operatorname{Re} u \geq T$ and $|\operatorname{Im} u| \leq \sqrt{2}$.

We recall that γ_0 is the homoclinic orbit for the unperturbed system (system (4.3.6) with $\mu = 0$).

Proof. A direct substitution in system (4.7.6) checks that the curve defined by $\gamma(u,\theta_0) = (\alpha(u,\theta_0), \beta(u,\theta_0))$ with

$$\alpha(u,\theta_0) = \frac{k_1\rho^2}{(k_2\cosh(\rho u) - 1)} \qquad \beta(u,\theta_0) = -\frac{k_1k_2\rho^3\sinh(\rho u)}{(k_2\cosh(\rho u)-1)^2} \tag{4.7.7}$$

where

$$\rho = \mu\varepsilon^{p+1/2}\sqrt{b_1}, \quad k_1 = \frac{3}{3 + b_2\mu^2\varepsilon^{2p+1}} \quad \text{and} \quad k_2 = \sqrt{1+2\rho^2k_1^2}$$

is a homoclinic solution of system (4.7.6). The dependence on θ_0 proceed of the dependence in this variable of b_1 and b_2. We recall here that the homoclinic orbit of the unperturbed system is given by

$$\alpha_0(u) = \frac{2}{2+u^2} \qquad \beta_0(u) = -\frac{4u}{(2+u^2)^2}.$$

We deal with the first component of the homoclinic orbit γ. By the maximum principle, it is clear that the function $|\alpha(u) - \alpha_0(u)|$ takes the maximum value in points of the form $u = t + i\sqrt{2}$ with $t \in \mathbb{R}$. We denote $c_1 = \cos\rho\sqrt{2}$ and $s_1 = \sin\rho\sqrt{2}$, then we have that

$$\begin{aligned} |\alpha(u) - \alpha_0(u)| &= \frac{1}{|k_2\cosh(\rho u) - 1||2+u^2|}|k_1\rho^2(2+u^2) - 2(k_2\cosh(\rho u) - 1)| \\ &\leq \frac{|k_1\rho^2t^2 - 2(k_2c_1\cosh(\rho t) - 1)|}{|k_2c_1\cosh(\rho t) - 1|t^2} \\ &\quad + \frac{|k_1\rho^2 2\sqrt{2}t - 2k_2s_1\sinh(\rho t)|}{k_2s_1\sinh(\rho t)2\sqrt{2}t}. \end{aligned} \tag{4.7.8}$$

We observe that, if $t \geq T$ with T big enough (but independent of ε and μ) then the functions

$$k_1\rho^2t^2 - 2(k_2c_1\cosh(\rho t) - 1) \quad \text{and} \quad k_1\rho^2 2\sqrt{2}t - 2k_2s_1\sinh(\rho t)$$

are negatives. Indeed, we observe that

$$k_2 c_1 = (1 + k_1^2 \rho^2 + O(\rho^4))(1 - \rho^2) = 1 + O(\rho^4)$$

therefore,

$$\begin{aligned} k_1 \rho^2 t^2 - 2(k_2 c_1 \cosh(\rho t) - 1) &\leq (k_1 - k_2 c_1)\rho^2 t^2 - 2(k_2 c_1 - 1) - k_2 c_1 \rho^4 t^4 \frac{1}{12} \\ &= -\frac{b_2}{3}\rho^4 (1 + O(\rho^2)) t^2 + O(\rho^4) - \rho^4 t^4 \frac{1}{12}(1 + O(\rho^2)) \\ &= \rho^4 t^2 \left(-\frac{b_2}{3} + t^2 \frac{1}{12} + O(\rho^2) \right) \leq 0 \end{aligned}$$

if $t \geq T$ big enough. Analogously, since

$$k_1 \rho 2\sqrt{2} - 2k_2 s_1 = \rho^2 2\sqrt{2}(1 + O(\rho^2)) - 2\sqrt{2}\rho^2 + O(\rho^4) = O(\rho^4)$$

then

$$\begin{aligned} k_1 \rho^2 2\sqrt{2} t - 2k_2 s_1 \sinh(\rho t) &\leq (k_1 \rho^2 2\sqrt{2} - 2k_2 s_1 \rho) t - 2k_2 s_1 \rho^3 t^3 \frac{1}{6} \\ &\leq t \left(1 - \frac{1}{3} t^2 \right) O(\rho^4) \end{aligned}$$

which is negative if $t \geq T$ big enough.

Bounding the terms in (4.7.8), we obtain

$$\begin{aligned} \frac{|k_1 \rho^2 t^2 - 2(k_2 c_1 \cosh(\rho t) - 1)|}{|k_2 c_1 \cosh(\rho t) - 1| t^2} &= \frac{k_2 c_1 (2(\cosh(\rho t) - 1) - \rho^2 t^2)}{|k_2 c_1 \cosh(\rho t) - 1| t^2} \\ &\quad + \frac{2(k_2 c_1 - 1) + \rho^2 t^2 (k_1 - k_2 c_1)}{|k_2 c_1 \cosh(\rho t) - 1| t^2} \\ &\leq O(\rho^2) \end{aligned}$$

and, in the same way

$$\frac{|k_1 \rho^2 2\sqrt{2} t - 2k_2 s_1 \sinh(\rho t)|}{k_2 s_1 \sinh(\rho t) 2\sqrt{2} t} \leq O(\rho^2).$$

In order to bound $|\beta(u) - \beta_0(u)|$ we write

$$\begin{aligned} \beta(u) - \beta_0(u) &= \frac{k_2}{k_1 \rho} \sinh(\rho u) \alpha^2(u) - u \alpha_0^2(u) \\ &= u(\alpha^2(u) - \alpha_0^2(u)) + \alpha^2(u) \left(\frac{k_2}{k_1 \rho} \sinh(\rho u) - u \right) \end{aligned}$$

and it is straightforward that

$$|\beta(u) - \beta_0(u)| = O(\rho^2).$$

Moreover, by Remark 4.2.2, γ satisfies that

$$\gamma(u,\theta_0) - \gamma_0(u) = O(\rho^2) = O(\mu^2\varepsilon^{2p+1}) = O(\mu^2\varepsilon^{p+2}). \tag{4.7.9}$$

■

The following remark is elementary but provides a useful property of $\gamma(u,\theta_0)$. Since the auxiliary system (4.5.10) and its scaled system are autonomous, for all $\theta_0 \in \mathbb{R}$, we have that

$$\begin{aligned} F^{t_0/\varepsilon}(\gamma(t+s,\theta_0)) &= \phi(2\pi, \gamma(t+s,\theta_0); \theta_0, \mu, \varepsilon) \\ &= \tilde{\phi}(2\pi\varepsilon, \gamma(t+s,\theta_0); \theta_0, \mu, \varepsilon) = \gamma(t+2\pi\varepsilon, \theta_0). \end{aligned}$$

In particular, for $\theta_0 = t/\varepsilon$ we have that

$$F^{t/\varepsilon}(\gamma(t+s,t/\varepsilon)) = \gamma(t+2\pi\varepsilon+s, t/\varepsilon). \tag{4.7.10}$$

We define the function

$$\hat{\gamma}(t,s) = \gamma(t+s, t/\varepsilon)$$

and we observe that, since γ is 2π-periodic with respect to its second variable, we have that

$$\hat{\gamma}(t+2\pi\varepsilon, s) = \hat{\gamma}(t, s+2\pi\varepsilon).$$

We consider $\hat{\gamma}$ as a first approximation of $\gamma^s_{G,\mu}$ defined in (4.7.3). We look for $\gamma^s_{G,\mu}$ of the form

$$\gamma^s(t,s) = \hat{\gamma}(t,s) + \mu\varepsilon^{p+1+\lambda}\sigma(t,s)$$

where $0 < \lambda < 1/2$ and $\sigma = (\sigma_1, \sigma_2) \in \mathcal{Y}^1_4 \times \mathcal{Y}^1_5$.

In order to simplify the notation we rename $\hat{\gamma}$ by γ, we write

$$\gamma(t,s) = (\alpha(t,s), \beta(t,s))$$

and $\gamma^s = \gamma^s_{G,\mu}$. We will impose the invariance condition given by

$$G^{t/\varepsilon}_\mu(\gamma^s(t,s)) = \gamma^s(t+2\pi\varepsilon, s)$$

with

$$\gamma^s(t,s) = \gamma(t,s) + \mu\varepsilon^{p+1+\lambda}\sigma(t,s)$$

and we will look for a suitable fixed point equation for σ.

First we summarize the main properties of γ:

- γ is continuous and analytic with respect to s,
- $\gamma(t+2\pi\varepsilon, s) = \gamma(t, s+2\pi\varepsilon)$
- For $(t,s) \in D^s$,

$$F_\mu^{t/\varepsilon}(\gamma(t,s)) = \gamma(t+2\pi\varepsilon, s). \tag{4.7.11}$$

- Making some elementary operations in the definition (4.7.7) we can check that

$$\gamma(t,s) \in \mathcal{Y}_2^{2/3} \times \mathcal{Y}_3^{2/3} \tag{4.7.12}$$

In order to simplify the notation we introduce

$$\bar{R}(u,v), \qquad \tilde{R}(u,v) \quad \text{and} \quad R_3(u,v)$$

where $\bar{R} = (\bar{R}_2^1, \bar{R}_2^2)$ is of the form

$$\begin{aligned}
\bar{R}_2^1(x,y,\theta_0,\mu,\varepsilon) &= \varepsilon f_{20}(\theta_0,\mu,\varepsilon)x^2 \\
\bar{R}_2^2(x,y,\theta_0,\mu,\varepsilon) &= \varepsilon g_{11}(\theta_0,\mu,\varepsilon)xy
\end{aligned}$$

and R_3 has the form

$$R_3(x,y,\theta_0,\mu,\varepsilon) = \begin{pmatrix} f_{11}(\theta_0,\mu,\varepsilon)xy + f_{02}(\theta_0,\mu,\varepsilon)y^2 \\ g_{02}(\theta_0,\mu,\varepsilon)y^2 \end{pmatrix} + r_3(x,y,\theta_0,\mu,\varepsilon)$$

with $r_3 \in P_3$. Moreover we introduce

$$Q(u,v) = (2\pi)^2 \begin{pmatrix} q_1(u,v,\varepsilon) \\ q_2(u,v,\varepsilon) \end{pmatrix} + \mu^2\varepsilon^{2p} \begin{pmatrix} 0 \\ a_1 u^2 \end{pmatrix} + \mu^2\varepsilon^{2p}\tilde{R}(u,v)$$

so that

$$G^{t/\varepsilon}(u,v) = F^{t/\varepsilon}(u,v) + \mu\varepsilon^{p+9}\psi_{\mu,\varepsilon}(u,v) + \mu^2\varepsilon^{2p+3}\bar{R}(u,v) + \mu^2\varepsilon^{2p+2}R_3(u,v)$$

and

$$F^{t/\varepsilon}(u,v) = A(\rho)\begin{pmatrix} u \\ v \end{pmatrix} + 2\pi\varepsilon\begin{pmatrix} 0 \\ -V'(x) \end{pmatrix} + \varepsilon^2 Q(u,v)$$

(we do not write explicitly the dependence of Q, $\bar{R}$ and $\tilde{R}$ on θ_0, ε and μ). We recall that

$$V(x) = -(x^3 - x^4).$$

Then, by Taylor's theorem

$$\begin{aligned}
G^{t/\varepsilon}(\gamma^s(t,s)) &= G^{t/\varepsilon}(\gamma(t,s)) + \mu\varepsilon^{p+1+\lambda} DG^{t/\varepsilon}(\gamma(t,s))\sigma(t,s) + \mu^2\varepsilon^{2p+2+2\lambda} O(|\sigma(t,s)|^2) \\
&= F^{t/\varepsilon}(\gamma(t,s)) + \mu\varepsilon^{p+9}\psi_{\mu,\varepsilon}(\gamma(t,s)) + \mu^2\varepsilon^{2p+3}\bar{R}(\gamma(t,s)) \\
&\quad +\mu^2\varepsilon^{2p+2}R_3(\gamma(t,s)) + \mu\varepsilon^{p+1+\lambda}DF^{t/\varepsilon}(\gamma(t,s))\sigma(t,s) \\
&\quad +\mu^3\varepsilon^{3p+4+\lambda}D\bar{R}(\gamma(t,s))\sigma(t,s) + \mu^2\varepsilon^{2p+10+\lambda}D\psi_{\mu,\varepsilon}(\gamma(t,s))\sigma(t,s) \\
&\quad +\mu^3\varepsilon^{3p+3+\lambda}DR_3(\gamma(t,s))\sigma(t,s) + \mu^2\varepsilon^{2p+2+2\lambda}O(|\sigma(t,s)|^2)
\end{aligned}$$

Thus using (4.7.11), $G^{t/\varepsilon}(\gamma^s(t,s)) = \gamma^s(t+2\pi\varepsilon, s)$ if and only if

$$\begin{aligned}
\sigma(t+2\pi\varepsilon, s) &= DF^{t/\varepsilon}(\gamma(t,s)\sigma(t,s) + \varepsilon^{8-\lambda}\psi_{\mu,\varepsilon}(\gamma(t,s)) + \mu\varepsilon^{p+2-\lambda}\bar{R}(\gamma(t,s)) \\
&\quad +\mu\varepsilon^{p+1-\lambda}R_3(\gamma(t,s)) + \mu\varepsilon^{p+9}D\psi_{\mu,\varepsilon}(\gamma(t,s))\sigma(t,s) \\
&\quad +\mu^2\varepsilon^{2p+3}D\bar{R}(\gamma(t,s))\sigma(t,s) + \mu^2\varepsilon^{2p+2}DR_3(\gamma(t,s))\sigma(t,s) \\
&\quad +\mu\varepsilon^{p+1+\lambda}O(|\sigma(t,s)|^2) \\
&= A(\rho)\sigma(t,s) + 2\pi\varepsilon\begin{pmatrix} 0 \\ (6\alpha(t,s) - 12\alpha^2(t,s))\sigma_1 \end{pmatrix} \\
&\quad +\varepsilon^2 DQ(\gamma(t,s))\sigma(t,s) + \varepsilon^{8-\lambda}\psi_{\mu,\varepsilon}(\gamma(t,s)) + \mu\varepsilon^{p+2-\lambda}\bar{R}(\gamma(t,s)) \\
&\quad +\mu\varepsilon^{p+1-\lambda}R_3(\gamma(t,s)) + \mu\varepsilon^{p+9}D\psi_{\mu,\varepsilon}(\gamma(t,s))\sigma(t,s) \\
&\quad +\mu^2\varepsilon^{2p+2}[DR_3(\gamma(t,s)) + D\tilde{R}(\gamma(t,s)) + \varepsilon D\bar{R}(\gamma(t,s))]\sigma(t,s) \\
&\quad +\mu\varepsilon^{p+1+\lambda}O(|\sigma(t,s)|^2).
\end{aligned}$$

We define

$$\begin{aligned}
H^1(\sigma)(t,s) &= DQ(\gamma(t,s))\sigma(t,s) + \varepsilon^{6-\lambda}\psi_{\mu,\varepsilon}(\gamma(t,s)) + \mu\varepsilon^{p+7}D\psi_{\mu,\varepsilon}(\gamma(t,s))\sigma(t,s) \\
&\quad +\mu^2\varepsilon^{2p}[DR_3(\gamma(t,s)) + D\tilde{R}(\gamma(t,s)) + \varepsilon D\bar{R}(\gamma(t,s)]\sigma(t,s) \\
&\quad +\mu\varepsilon^{p-1+\lambda}O(|\sigma(t,s)|^2), \\
H^2(\sigma)(t,s) &= \bar{R}(\gamma(t,s)),
\end{aligned}$$

the vector

$$B(w) = \begin{pmatrix} 0 \\ (6w - 12w^2) \end{pmatrix} \tag{4.7.13}$$

and

$$\mathcal{F}(\sigma) = 2\pi\varepsilon B(\alpha)\sigma_1 + \varepsilon^2 H^1(\sigma). \tag{4.7.14}$$

We can reduce the problem to to find $\sigma = (\sigma_1, \sigma_2)$ such that

$$\sigma = \mathcal{B}^{-1}\mathcal{F}(\sigma) + \mu\varepsilon^{p+2-\lambda}\mathcal{C}^{-1}H^2(\sigma). \tag{4.7.15}$$

(here we distinguish between $\mathcal{B}^{-1}$ and $\mathcal{C}^{-1}$ because as we will see below the functions $\mathcal{F}(\sigma)$ and $H^2(\sigma)$ belong to different spaces).

We look for $\sigma \in \mathcal{Y}_4^1 \times \mathcal{Y}_5^1$. For this we endow the product space $\mathcal{Y}_4^1 \times \mathcal{Y}_5^1$ with the norm

$$\|\psi\|_{\mathcal{Y}} = \|\psi_1\|_4^1 + \frac{1}{7}\|\psi_2\|_5^1$$

for $\psi = (\psi_1, \psi_2) \in \mathcal{Y}_4^1 \times \mathcal{Y}_5^1$ and it becomes a Banach space. We denote by $B(r) \subset \mathcal{Y}_4^1 \times \mathcal{Y}_5^1$ the closed ball of radius r with this norm.

Lemma 4.7.2 *The operator $\mathcal{G} : B(r) \to B(r)$ given by*

$$\mathcal{G}(\sigma) = \mathcal{B}^{-1}\mathcal{F}(\sigma) + \mu\varepsilon^{p+2-\lambda}\mathcal{C}^{-1}H^2(\sigma) \tag{4.7.16}$$

is well defined and it is a contraction.

Proof. We recall that

$$\gamma \in \mathcal{Y}_2^{2/3} \times \mathcal{Y}_3^{2/3}.$$

(see (4.7.12)). Let $\sigma = (\sigma_1, \sigma_2) \in \mathcal{Y}_4^1 \times \mathcal{Y}_5^1$. First, we check that

$$\mathcal{F}(\sigma) \in \mathcal{Y}_6^1 \times \mathcal{Y}_6^1$$
$$H^2(\sigma) \in \mathcal{Y}_4^{4/3} \times \mathcal{Y}_5^{4/3}.$$

First we deal with the terms involving $\mathcal{F}(\sigma)$. It is clear that

$$B(\alpha)\sigma \in \{0\} \times \mathcal{Y}_6^{5/3}$$
$$\begin{pmatrix} 0 \\ 2a_1\alpha(t,s)\sigma_1(t,s) \end{pmatrix} \in \{0\} \times \mathcal{Y}_6^{5/3}.$$

Since $Q \in P_2$, we have that

$$DQ(\gamma(t,s))\sigma(t,s) \in \mathcal{Y}_6^{5/3} \times \mathcal{Y}_7^{5/3}$$

and since $\psi_{\mu,\varepsilon} \in P_3$

$$\begin{aligned} \psi_{\mu,\varepsilon}(\gamma(t,s)) &\in \mathcal{Y}_6^2 \times \mathcal{Y}_6^2, \\ D\psi_{\mu,\varepsilon}(\gamma(t,s))\sigma(t,s) &\in \mathcal{Y}_8^{7/3} \times \mathcal{Y}_8^{7/3}. \end{aligned}$$

Finally, using that the second component of $\tilde{R}$ and $\bar{R}$ have not terms in u^2 and belong to P_2 we have that

$$\begin{aligned} D\tilde{R}(\gamma(t,s))\sigma(t,s) &\in \mathcal{Y}_6^{5/3} \times \mathcal{Y}_7^{5/3} \\ D\bar{R}(\gamma(t,s))\sigma(t,s) &\in \mathcal{Y}_6^{5/3} \times \mathcal{Y}_7^{5/3}. \end{aligned}$$

Hence, from definition (4.7.14) of $\mathcal{F}$ we have that

$$\mathcal{F}(\sigma) \in \mathcal{Y}_6^{5/3} \times \mathcal{Y}_6^{5/3} \subset \mathcal{Y}_6^1 \times \mathcal{Y}_6^1.$$

We recall that $\bar{R}$ has not terms in u^2, thus,

$$H^2(\sigma) \in \mathcal{Y}_4^{4/3} \times \mathcal{Y}_5^{4/3}.$$

We deal with the terms involving $\mathcal{G}$. By Lemma 4.6.1,

$$\mathcal{B}^{-1}\mathcal{F}(\sigma) \in \mathcal{Y}_4^1 \times \mathcal{Y}_5^1$$

and by Lemma 4.6.2

$$\mathcal{C}^{-1}H^2(\sigma) \in \mathcal{Y}_4^1 \times \mathcal{Y}_5^1.$$

Therefore $\mathcal{G}(\sigma) \in \mathcal{Y}_4^1 \times \mathcal{Y}_5^1$. In order to prove that the operator $\mathcal{G}$ is well defined, we must see that $\|\mathcal{G}(\sigma)\|_{\mathcal{Y}} < r$ if $\|\sigma\|_{\mathcal{Y}} \leq r$.

Let $\sigma \in \mathcal{Y}_4^1 \times \mathcal{Y}_5^1$ be with norm $\|\sigma\| \leq r$ with r small enough, but independent of ε and μ. We observe that from Lemma 4.7.1, if $(t,s) \in D^s$ we have that

$$\begin{aligned} \left| \frac{k_1\rho^2}{(k_2\cosh(\rho(t+s)) - 1)} \right| &\leq \frac{2}{(t+\operatorname{Re} s)^2}\left(1 + O\Big(\frac{1}{T}\Big) + O(\rho^2)\right) \\ &= \frac{2}{(t+\operatorname{Re} s)^2}\left(1 + O\Big(\frac{1}{T}\Big)\right) \end{aligned}$$

Moreover from the definition of $H^1 = (H_1^1, H_2^1)$, there exist two constants M_1 and M_2 independent of ε and μ such that

$$\|H_1^1(\sigma)\|_6^1 \leq M_1 \qquad \text{and} \qquad \|H_2^1(\sigma)\|_6^1 \leq M_2.$$

We denote $u = t + \operatorname{Re} s$, then by definition (4.7.13) of B and definition (4.7.14) of $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2)$, and $H^1 = (H_1^1, H_2^1)$ we have that

$$\begin{aligned}
\|\mathcal{F}_1\|_6^1 &\leq M_1\varepsilon^2 \\
\|\mathcal{F}_2\|_6^1 &\leq 2\pi\varepsilon \sup_{D^s} u^6 \left[\frac{12}{u^2} + \frac{48}{u^4}\right] |\sigma_1(t,s)| + u^6\varepsilon^2 |H_2^1(\sigma)(t,s)| \\
&\leq 2\pi\varepsilon 12 \|\sigma_1\|_4^1 + \varepsilon^2 \left[O\left(\frac{1}{T}\right) + O(\varepsilon)\right] \|\sigma\|_{\mathcal{Y}}.
\end{aligned}$$

Therefore by Lemma 4.6.1

$$\begin{aligned}
\|\mathcal{B}^{-1}\mathcal{F}(\sigma)\|_{\mathcal{Y}} &= \|(\mathcal{B}^{-1}\mathcal{F}(\sigma))_1\|_4^1 + \frac{1}{7}\|(\mathcal{B}^{-1}\mathcal{F}(\sigma))_2\|_5^1 \\
&\leq \frac{1}{2\pi\varepsilon}\left[\frac{1}{20}\|\mathcal{F}_2(\sigma)\|_6^1 + O\left(\frac{1}{T}\right)\|\mathcal{F}(\sigma)\|_{\mathcal{Y}}\right] \\
&\quad + \frac{1}{7}\frac{1}{2\pi\varepsilon}\left[\frac{1}{5}\|\mathcal{F}_2(\sigma)\|_6^1 + O\left(\frac{1}{T}\right)\|\mathcal{F}(\sigma)\|_{\mathcal{Y}}\right] \\
&\leq \left[\frac{12}{20} + \frac{12}{35}\right]\|\sigma_1\|_4^1 + \left[O\left(\frac{1}{T}\right) + O(\varepsilon)\right]\|\sigma\|_{\mathcal{Y}} \\
&= \frac{33}{35}\|\sigma_1\|_4^1 + \left[O\left(\frac{1}{T}\right) + O(\varepsilon)\right]\|\sigma\|_{\mathcal{Y}}.
\end{aligned}$$

Finally, we deal with the last term. Since

$$\rho = \frac{\sqrt{c_1 + c_3}}{2\pi}\mu\varepsilon^{p+1/2}(1 + O(\mu^2\varepsilon^{2p+3}))$$

we have

$$\begin{aligned}
\mu\varepsilon^{p+2-\lambda}\|\mathcal{C}^{-1}H^2(\sigma)\|_{\mathcal{Y}} &\leq \mu\varepsilon^{p+2-\lambda}\frac{K}{2\pi\varepsilon\rho}(\|H_1^2(\sigma)\|_4^{4/3} + \|H_2^2(\sigma)\|_5^{4/3}) \\
&= O(\varepsilon^{1/2-\lambda}).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|\mathcal{G}(\sigma)\|_{\mathcal{Y}} &\leq \frac{33}{35}\|\sigma_1\|_4^1 + \left[O\left(\frac{1}{T}\right) + O(\varepsilon)\right]\|\sigma\|_{\mathcal{Y}} + O(\varepsilon^{1/2-\lambda}) \\
&\leq \frac{33}{35}r + \left[O\left(\frac{1}{T}\right) + O(\varepsilon)\right]\|\sigma\|_{\mathcal{Y}} + O(\varepsilon^{1/2-\lambda}) \\
&\leq r
\end{aligned}$$

if T is big enough, ε is small enough and $0 < \lambda < 1/2$. Then the application

$$\mathcal{G} : B_r \to B_r$$

is well defined. Since the linear terms in σ are the dominant terms, $\mathcal{G}$ is a contraction. Then, by the fixed point theorem, there exists $\sigma \in \mathcal{Y}_4^1 \times \mathcal{Y}_5^1$ such that

$$G^{t/\varepsilon}(\gamma(t,s) + \mu\varepsilon^{p+1+\lambda}\sigma(t,s)) = \gamma(t + 2\pi\varepsilon, s) + \mu\varepsilon^{p+1+\lambda}\sigma(t + 2\pi\varepsilon, s).$$

This proves the lemma. ■

End of the proof of Theorem 4.2.1.. We go back to the original variables. We observe that, since the matrix $C_\mu^{t/\varepsilon}(t/\varepsilon)$ given in Lemma 4.5.2 is $2\pi\varepsilon$-periodic and satisfies (4.7.5) we have that

$$\begin{aligned}
\gamma_\mu^s(t,s) &= C_\mu^{t/\varepsilon}(t/\varepsilon)\gamma^s(t,s) = \gamma^s(t,s) + O(\mu^2\varepsilon^{p+2}) \\
&= \gamma(t,s) + \mu\varepsilon^{p+1+\lambda}\sigma(t,s) + O(\mu^2\varepsilon^{p+2}) \\
&= \gamma(t,s) + O(\mu\varepsilon^{p+1+\lambda})
\end{aligned}$$

and by Lemma 4.7.1

$$\gamma_\mu^s(t,s) = \gamma_0(t+s) + O(\mu\varepsilon^{p+1+\lambda}).$$

Moreover, as we have said in Remark 4.3.4 the change $\mathcal{C}$ has the form

$$(x,y,\theta) = \mathcal{C}(\bar{x},\bar{y},\theta) = (\bar{x},\bar{y},\theta) + \mu\varepsilon^{p+1}(G(\bar{x},\bar{y},\theta),0) + O(\mu\varepsilon^{p+2})$$

where (x,y,θ) are the original variables and $(\bar{x},\bar{y},\theta)$ are the variables for which we have proved the existence of the suitable parameterization of local invariant stable manifold. Therefore, the local stable manifold, $\gamma_{\mu,\varepsilon}^s$, of the original system (4.1.1) has the form

$$\begin{aligned}
\gamma_{\mu,\varepsilon}^s(t,s) &= \gamma_\mu^s(t,s) + \mu\varepsilon^{p+1}G(\bar{\gamma}_\mu^s(t,s), t/\varepsilon) + O(\mu\varepsilon^{p+1+\lambda}) \\
&= \gamma_0(t+s) + \mu\varepsilon^{p+1}G(\gamma_0(t+s), t/\varepsilon) + O(\mu\varepsilon^{p+1+\lambda}).
\end{aligned}$$

In order to find a parameterization of the local stable manifold which be a solution with respect to t we define $t_0 = T - \operatorname{Re} s$ and for all $t \geq t_0$

$$\tilde{\gamma}_{\mu,\varepsilon}^s(t,s) = \varphi(t, t_0, \gamma_{\mu,\varepsilon}^s(t_0,s))$$

where $\varphi(t,t_0,x,y)$ is the general solution of system (4.1.1). We observe that, for all $(t,s) \in D^s$, $\gamma_{\mu,\varepsilon}^s(t,s)$ belongs to the local stable manifold of the system (4.1.1), hence

$\tilde{\gamma}^s_{\mu,\varepsilon}(t,s)$ is a parameterization of the stable manifold. It is clear that the properties 1), 3) and 4) of Theorem 4.2.1 are satisfied by $\gamma^s_{\mu,\varepsilon}$. Moreover

$$\begin{aligned}
\tilde{\gamma}^s_{\mu,\varepsilon}(t, s+2\pi\varepsilon) &= \varphi(t, t_0, \gamma^s_{\mu,\varepsilon}(t_0, s+2\pi\varepsilon)) = \varphi(t, t_0, \gamma^s_{\mu,\varepsilon}(t_0+2\pi\varepsilon, s)) \\
&= \varphi(t+2\pi\varepsilon, t_0+2\pi\varepsilon, \varphi(t_0+2\pi\varepsilon, t_0, \gamma^s_{\mu,\varepsilon}(t_0, s))) \\
&= \varphi(t+2\pi\varepsilon, t_0, \gamma^s_{\mu,\varepsilon}(t_0, s)) \\
&= \tilde{\gamma}^s_{\mu,\varepsilon}(t+2\pi\varepsilon, s).
\end{aligned}$$

Therefore, $\tilde{\gamma}^s_{\mu,\varepsilon}$ is the parameterization that we look for. We observe that $\tilde{\gamma}^s_{\mu,\varepsilon}$ is defined for all $(t,s) \in D^s$. ■

Chapter 5

Invariant manifolds as graphs

5.1 Introduction and main results

The goal of this Chapter is to prove that the local stable invariant curve of the averaged system (3.3.10) (written after scaling by ε):

$$\begin{aligned} \dot{x} &= y + \mu\varepsilon^{p+2n+2}\partial_y F(x,y,\theta) + \mu^2\varepsilon^{p+2}\partial_y R_{2k-2}(x,y,\theta) \\ \dot{y} &= -V'(x) - \mu\varepsilon^{p+2n+2}\partial_x F(x,y,\theta) - \mu^2\varepsilon^{p+2}\partial_x R_{2k-2}(x,y,\theta) \\ \dot{\theta} &= 1/\varepsilon, \end{aligned} \tag{5.1.1}$$

in the parabolic case, the one of its auxiliary system (3.7.4) and the one of the system (4.3.6) (written after scaling by ε):

$$\begin{aligned} \dot{x} &= y + \mu\varepsilon^{p+8}\partial_y F(x,y,\theta) + \mu^2\varepsilon^{2p+1}(\partial_y f_3 + \partial_y R_2)(x,y,\theta) \\ \dot{y} &= -V'(x) - \mu\varepsilon^{p+8}\partial_x F(x,y,\theta) - \mu^2\varepsilon^{2p+1}(\partial_x f_3 + \partial_x R_2)(x,y,\theta) \\ \dot{\theta} &= 1/\varepsilon \end{aligned} \tag{5.1.2}$$

in the weak hyperbolic case, given in Chapters 3 and 4 respectively, can be writen as the graph of a function φ which will depend analytically on x, μ, C^1 with respect to θ and continuously with respect to ε. Concretely we prove the following results:

For the parabolic fixed point case:

Proposition 5.1.1 *The local stable manifold of the system (5.1.1) is the graph of a function $y = f(x) + \mu\varepsilon^{p+2}g(x,\theta,\mu,\varepsilon)$ where*

1) $f(x) = -\sqrt{-2V(x)}$

2) $f(x)$ *and* $g(x,\theta,\mu,\varepsilon)$ *are analytic with respect to* (x,μ) *in*

$$\Omega(\delta,\mu_0) = \left\{ x \in \mathbb{C} \,:\, 0 < |x| < \delta \;,\; |\arg(x)| < \frac{\pi}{5(\beta-1)} \right\} \times \{\mu \in \mathbb{R} : |\mu| \le \mu_0\}$$

where $\beta = \frac{n}{2}$ *and* μ_0 *small enough.*

3) $g(x,\varepsilon,\theta,\mu)$ *is* C^0 *and* C^1 *in* θ*. Moreover it is* 2π*-periodic on* θ*.*

4) $f(x), g(x,\varepsilon,\theta,\mu) = O(x^\beta)$*.*

Analogously, the local stable curve of the auxiliary system (3.7.4), can be expressed as the graph of a function $y = f(x) + \mu^2\varepsilon^{p+2}h(x,\theta,\mu,\varepsilon)$ *where* h *has the same properties as* g*.*

And for the weak hyperbolic fixed point case:

Proposition 5.1.2 *There exist the local stable manifold of the system (5.1.2) and it is the graph of a function* $y = f(x) + \mu\varepsilon^{p+1/2}g(x,\theta,\mu,\varepsilon)$ *such that*

1) $f(x) = -\sqrt{-2V(x)}$

2) f *is analytic in* $\Omega(\delta) = \left\{x \in \mathbb{C} : 0 < |x| < \delta \;,\; |\arg(x)| < \frac{\pi}{3}\right\}$*.*

3) $g(x,\theta,\mu,\varepsilon)$ *is* C^0 *in* $\varepsilon \in (0,\varepsilon_0)$ *with* ε_0 *small enough,* C^1 *and* 2π*-periodic in* θ*.*

4) $g(x,\theta,\mu,\varepsilon)$ *is analytic in* $\Omega(\delta) \times \{\mu \in \mathbb{R} : |\mu| \le \mu_0\}$ *with* μ_0 *small enough.*

5) $f(x) = O(x^{3/2})$ *and* $g(x,\theta,\mu,\varepsilon) = O(x)$*.*

Proposition 5.1.1 follows from results in [30] concerning stable curves for maps associated to parabolic points which can be realized as Poincaré maps of equation (5.1.1). Proposition 5.1.2, follows from the results in Section 5.2, where we develop an analogous theory to the one in [30] for what we call the weak hyperbolic case.

5.2 The case of weak hyperbolic fixed point

5.2.1 Introduction

We study the existence and analyticity of invariant curves asymptotic to a fixed point of a family of two dimensional maps $F_{\mu,\varepsilon} : U \subset \mathbb{R}^2 \to \mathbb{R}^2$ having a fixed point, which we may assume that it is the origin, with linear part

$$\begin{pmatrix} c & b \\ \mu & c \end{pmatrix} \tag{5.2.1}$$

where $b = b(\mu, \varepsilon) \neq 0$ and c has the form

$$c = c(\mu, \varepsilon) = 1 + e\mu + O(\mu^2), \qquad e = e(\varepsilon) \tag{5.2.2}$$

and μ, ε are small parameters. Moreover we suppose that the nonlinear part of $F^2_{\mu,\varepsilon}(x, y)$ (the second component of $F_{\mu,\varepsilon}$) is of the form $a_2x^2 + h.o.t.$ with $a_2 \neq 0$.

To simplify a little bit the notation we note that the linear change of variables $C_1(x, y) = (bx, y)$ transforms the linear part (5.2.1) into

$$\begin{pmatrix} c & 1 \\ \mu b & c \end{pmatrix}. \tag{5.2.3}$$

We shall rename the small parameter μb by μ. Of course, the value of e will change. The eigenvalues of the matrix (5.2.3) are

$$\lambda = c \pm \sqrt{\mu}.$$

Therefore for $\mu > 0$ the fixed point is a saddle point and has stable and unstable one dimensional invariant curves through it, tangent to the vectors $(1, \pm\sqrt{\mu})$ respectively. We perform the linear change of variables $C_2(x, y) = (x, y - \sqrt{\mu}x)$, which transform the linear part (5.2.3) into

$$\begin{pmatrix} c + \sqrt{\mu} & 1 \\ 0 & c - \sqrt{\mu} \end{pmatrix}.$$

For $\mu = 0$ the origin is a parabolic point and the only eigendireccion of its linear part is generated by the vector $(1, 0)$.

When $\mu > 0$ the origin is a saddle point. Therefore the classical stable and unstable manifold theorems, guarantee the existence and properties of the local invariant manifolds in a neighborhood of the origin, V_μ, which depends on μ. In general, V_μ may shrink to $\{0\}$ when μ goes to zero. We can represent the manifolds as graphs of functions with suitable regularity conditions in this domain. We could extend the invariant manifold to a larger domain by iteration of the map but we would not be able to guarantee that the invariant manifolds can be expressed as graphs of functions in domains independent of μ.

Our purpose is to find a neighborhood of the origin, independent of the parameter μ, such that a branch of the local stable and unstable curves can be expressed as the graph of a function (depending on μ) and to give regularity conditions of this function.

We follow the same scheme of proof of [30]. In this work, E.Fontich deals with the case $\mu = 0$, where the origin is a parabolic point. First we look for a normal form which must have a suitable dependence on μ. This suggests us to look for a normal form for the weak hyperbolic case, which when $\mu = 0$ must be the normal form of the parabolic case given in [30].

First we deal with the Lipschitz case. In the following section we prove the existence and uniqueness of a branch of the stable manifold in the half right plane $(x > 0)$. First, we prove the existence of a suitable set such that the stable curve that we look for belongs to it. Then, we prove the existence of a stable manifold which can be expressed as a graph of a Lipschitz function φ, defined in a open set which does not depend on μ and ε. Since the origin is a saddle point, the uniqueness can be proved easily. Finally, we prove the continuity of the stable curve with respect to the parameter ε using the asymptotic behaviour when $x \to 0$ of the function φ.

The analytic case is more technical. In fact we already know that, for fixed values of the parameters, the stable curves are analytic, but here they are found to be analytic, as graphs in some complex domain independent on the parameters.

We also would like to mention that from the results of Chapter 4 we can obtain that, locally, the curves are graphs, but here the (complex) domain we obtain for the graph is much bigger.

First we perform a change of variables in order to move the stable curve closer to the x-axis. Then we prove some suitable bounds in order to describe the behaviour of the stable manifold in a complex domain. Finally, an standard argument using the Rouche's Theorem, gives the analyticity of the function φ.

5.2.2 Normal form

We look for a normal form with a good behavior near $\mu = 0$. For this reason we will not change the linear part. We shall give the normal form until order two.

By Taylor's theorem,

$$F_{\mu,\varepsilon}(x,y) = \begin{pmatrix} (c+\sqrt{\mu})x + y \\ (c-\sqrt{\mu})y \end{pmatrix} + \begin{pmatrix} f_{20}x^2 + f_{11}xy + f_{02}y^2 \\ g_{20}x^2 + g_{11}xy + g_{02}y^2 \end{pmatrix} + \begin{pmatrix} r_3^1(x,y) \\ r_3^2(x,y) \end{pmatrix}. \tag{5.2.4}$$

We look for a change

$$C(\xi,\eta) = \begin{pmatrix} \xi \\ \eta \end{pmatrix} + \begin{pmatrix} \phi(\xi,\eta) \\ \psi(\xi,\eta) \end{pmatrix} = \begin{pmatrix} \xi + \phi_{20}\xi^2 + \phi_{11}\xi\eta + \phi_{02}\eta^2 \\ \xi + \psi_{20}\xi^2 + \psi_{11}\xi\eta + \psi_{02}\eta^2 \end{pmatrix}$$

such that $C \circ N_{\mu,\varepsilon} = F_{\mu,\varepsilon} \circ C$ with

$$N_{\mu,\varepsilon}(\xi,\eta) = \begin{pmatrix} (c+\sqrt{\mu})\xi + \eta \\ (c-\sqrt{\mu})\eta \end{pmatrix} + \begin{pmatrix} h^1(\xi,\eta) \\ h^2(\xi,\eta) \end{pmatrix} + \begin{pmatrix} r_3^1(x,y) \\ r_3^2(x,y) \end{pmatrix}$$

with (h^1, h^2) as simple as possible, compatible with the condition of being continuous with respect to μ. At it is usual we compare the terms of order two in the equation $C \circ N_{\mu,\varepsilon} = F_{\mu,\varepsilon} \circ C$. We find the equations

$$\begin{aligned}
f_{20} - h^1_{20} &= ((c+\sqrt{\mu})^2 - c - \sqrt{\mu})\phi_{20} - \psi_{20} \\
g_{20} - h^2_{20} &= ((c+\sqrt{\mu})^2 - c + \sqrt{\mu})\psi_{20} \\
f_{11} - h^1_{11} &= 2(c+\sqrt{\mu})\phi_{20} + (c^2 - \mu - c - \sqrt{\mu})\phi_{11} - \psi_{11} \\
g_{11} - h^2_{11} &= 2(c+\sqrt{\mu})\psi_{20} + (c^2 - \mu - c + \sqrt{\mu})\psi_{11} \\
f_{02} - h^1_{02} &= \phi_{20} + (c-\sqrt{\mu})\phi_{11} + ((c-\sqrt{\mu})^2 - c - \sqrt{\mu})\phi_{02} - \psi_{02} \\
g_{02} - h^2_{02} &= \psi_{20} + (c-\sqrt{\mu})\psi_{11} + ((c-\sqrt{\mu})^2 - c + \sqrt{\mu})\psi_{02}.
\end{aligned}$$

There are several choices for a normal form. We choose $h^1_{20} = h^1_{11} = h^1_{02} = h^2_{02} = 0$ and $\phi_{02}(\mu) = \psi_{02}(\mu) = 0$. Then the other coefficients are determined uniquely as the solutions of the following linear system:

$$\begin{pmatrix}
a_1(\mu) & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & a_2(\mu) & 0 & 1 & 0 \\
2(c+\sqrt{\mu}) & a_3(\mu) & 0 & -1 & 0 & 0 \\
0 & 0 & 2(c+\sqrt{\mu}) & a_4(\mu) & 0 & 1 \\
1 & c-\sqrt{\mu} & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & (c-\sqrt{\mu}) & 0 & 0
\end{pmatrix}
\begin{pmatrix} \phi_{20} \\ \phi_{11} \\ \psi_{20} \\ \psi_{11} \\ h^2_{20} \\ h^2_{11} \end{pmatrix}
=
\begin{pmatrix} f_{20} \\ g_{20} \\ f_{11} \\ g_{11} \\ f_{02} \\ g_{02} \end{pmatrix}$$

where

$$
\begin{aligned}
a_1(\mu) &= (c+\sqrt{\mu})^2 - (c+\sqrt{\mu})\\
a_2(\mu) &= (c+\sqrt{\mu})^2 - (c-\sqrt{\mu})\\
a_3(\mu) &= c^2 - \mu - (c+\sqrt{\mu})\\
a_4(\mu) &= c^2 - \mu - (c-\sqrt{\mu}).
\end{aligned}
$$

It is not difficult to check that the determinant of the matrix of the linear system is $2+O(\sqrt{\mu})$. Then, if μ is small enough, the linear system has a unique solution and therefore we find ϕ_{20}, $\phi_{11}, \psi_{20}, \psi_{11}, h^2_{20}$ and h^2_{11} as functions of μ.

If we assume that F is jointly C^n with respect to (x, y, μ) then $c = c(\mu)$, $f_{jk} = f_{jk}(\sqrt{\mu})$ and $g_{jk} = g_{jk}(\sqrt{\mu})$, for $j, k = 0, 1, 2$, are C^{n-1}, C^{n-2} and C^{n-2} respectively, then ϕ_{20}, $\phi_{11}, \psi_{20}, \psi_{11}, h^2_{20}$ and h^2_{11} depend C^{n-2} on $\sqrt{\mu}$. In the analytic case, obviously, the dependence on $\sqrt{\mu}$ is analytic.

Moreover, if $F_{\mu,\varepsilon}$ is a family of analytic maps with respect to (x, y), C^m with respect to another parameter, η, and analytic with respect to $\sqrt{\mu}$, then all the coefficients of the normal form are C^m with respect to η and analytic with respect to $\sqrt{\mu}$.

Other kinds of regularity may be assumed. For instance that $\mu \mapsto F_\mu$ is C^n from $\mathbb{R}$ to $C^n(U)$. In this case the coefficients ϕ_{20}, $\phi_{11}, \psi_{20}, \psi_{11}, h^2_{20}$ and h^2_{11} will be C^n.

We summarize this result in the following theorem.

Theorem 5.2.1 *Let $F_{\mu,\varepsilon} : U \subset \mathbb{R}^2 \to \mathbb{R}^2$ be a family of diffeomorphisms of class C^n, $n \geq 2$, depending continuously on μ and ε, having the form*

$$
F_{\mu,\varepsilon}(x,y) = \begin{pmatrix} (c+\sqrt{\mu})x + y \\ (c-\sqrt{\mu})y \end{pmatrix} + \begin{pmatrix} f_{20}x^2 + f_{11}xy + f_{02}y^2 \\ g_{20}x^2 + g_{11}xy + g_{02}y^2 \end{pmatrix} + \begin{pmatrix} r^1_3(x,y) \\ r^2_3(x,y) \end{pmatrix}
$$

where $f_{jk} = f_{jk}(\sqrt{\mu}, \varepsilon)$, $g_{jk} = g_{jk}(\sqrt{\mu}, \varepsilon)$ with $j, k = 0, 1, 2$, $c = 1 + e\mu + O(\mu^2)$ and $g_{20}(0, \varepsilon) \neq 0$. Then, if $\mu \geq 0$ is small enough there exists a polynomial change of variables, continuous in μ and ε, such that in the new variables the map has the form

$$
F_{\mu,\varepsilon}(x,y) = \begin{pmatrix} (c+\sqrt{\mu})x + y \\ (c-\sqrt{\mu})y \end{pmatrix} + \begin{pmatrix} 0 \\ a_{20}x^2 + a_{11}xy \end{pmatrix} + \begin{pmatrix} r^1_3(x,y) \\ r^2_3(x,y) \end{pmatrix} \tag{5.2.5}
$$

with $a_{jk} = a_{jk}(\sqrt{\mu}, \varepsilon)$ for $j, k = 0, 1, 2$, $a_{20}(0, \varepsilon) = g_{20}(0, \varepsilon) \neq 0$ and $a_{11}(0, \varepsilon) = g_{11}(0, \varepsilon) - 2f_{20}(0, \varepsilon)$.

Remark 5.2.2 *Later on we will deal with families $F_{\mu,\varepsilon}$ depending analytically on μ. In such case, following the previous argument we easily check that the normal form, and in particular a_{20} and a_{11}, depends analytically on $\sqrt{\mu}$.*

Remark 5.2.3 *We do not assume any condition on the sign of e defined in (5.2.2).*

5.2.3 Existence and uniqueness of stable curves

In this section we consider families of maps in the normal form described in the previous section and we look for sufficient conditions for the existence of a local stable invariant manifolds in the right-hand plane. Considering the inverse map F_μ^{-1} we can deal with the local unstable invariant curve. We will denote this local stable curve by W_δ^{s+}. A more precise definition of W_δ^{s+} is

$$W_\delta^{s+} = \{z \in U : \; F_\mu^n(z) \in B_\delta(0) \cap \{x > 0\} \text{ for } n \geq 0,\, F_\mu^n(z) \to 0 \text{ when } n \to +\infty\}.$$

If $\mu > 0$ the first condition in the definition of W_δ^{s+} characterizes the set, but if $\mu = 0$ the whole space is a central manifold and it is convenient to add the second condition. In general we will not explicitly write the dependence on μ of the objects we will deal with. We look for the invariant manifold as a graph of a function φ of the form

$$y = \varphi(x) = -c_1 x - c_2 x^\alpha + \ldots, \qquad \alpha = 3/2.$$

We put the minus sign for notational convenience. If we impose the invariance condition of graph φ,

$$F^2_{\mu,\varepsilon}(x, \varphi(x)) = \varphi(F^1_{\mu,\varepsilon}(x, \varphi(x))) \tag{5.2.6}$$

for $\mu > 0$ we will obtain the Taylor expansion of φ. Since we want an expression of φ uniformly valid for $\mu \geq 0$, we add the term $-c_2 x^\alpha$ with $\alpha = 3/2$ which comes from the asymptotic expression of φ when $\mu = 0$. The following values for c_1 and c_2 are obtained comparing terms of order 1 and 2 in (5.2.6) respectively:

- $c_1 = 2\sqrt{\mu}$
- $c_2^2 = \dfrac{a_{20} - c_1 a_{11}}{\alpha(c - \sqrt{\mu})^{\alpha-1}} > 0$ if μ is small enough.

To have real values for c_2 (at least for μ small) we have to assume that $a_{20}(0,\varepsilon) > 0$. Looking at the dynamics of $F_{\mu,\varepsilon}$ on graph(φ) we see that to get the stable curve we have to choose c_2 positive. Of course, this is a formal calculation but serves to foresee a domain in $\mathbb{R}^2$ where the local stable curve belongs to. Therefore we introduce the set

$$\begin{aligned} &A(\delta) \\ &= \{(x,y) \in \mathbb{R}^2 :\ 0 < x < \delta,\ \ -2\sqrt{\mu}x - 2c_2x^\alpha < y < -\sqrt{\mu}x - d\mu x - \frac{c_2}{2}x^\alpha\} \end{aligned} \quad (5.2.7)$$

with $d = \frac{c-1}{\mu}$ if $\mu > 0$ and $d = e$ if $\mu = 0$, and we concentrate on the study of the dynamics of F in it. We denote by O_l and o_l terms of the form $O(x^l)$ and $o(x^l)$ respectively.

The main result of this section is:

Theorem 5.2.4 *Let $F_\eta : U \subset \mathbb{R}^2 \to \mathbb{R}^2$, $\eta = (\mu,\varepsilon) \in V_\eta = V_\mu \times V_\varepsilon \subset [0,+\infty)\times\mathbb{R}^m$, V_η open, $0 \in V_\eta$, be a family of C^n maps, $n \geq 2$, depending continuously with respect to μ and ε, of the form (5.2.5), satisfying that $a_{20}(0,\varepsilon) > 0$. Then, there exists $\delta > 0$ (independent on μ and ε) such that $W_\delta^{s+} \cap A(\delta)$ is the graph of a function φ_η, Lipschitz with respect to x and continuous with respect to μ and ε.*

Remark 5.2.5 *Considering F_η^{-1} and its normal form, we obtain results for the unstable manifold.*

We introduce some notation. From now on we denote

$$(x_k, y_k) = F_\eta^k(x,y), \quad k \in \mathbb{Z}.$$

We define the sets

$$a_+(\delta) = \{(x,y) \in \mathbb{R}^2 :\ 0 < x < \delta,\ y = -\sqrt{\mu}x - d\mu x - \frac{c_2}{2}x^\alpha\}$$

and

$$a_-(\delta) = \{(x,y) \in \mathbb{R}^2 :\ 0 < x < \delta,\ y = -2\sqrt{\mu}x - 2c_2x^\alpha\}.$$

Of course, these sets depend continuously with respect to μ and ε. From now on we omit the dependence on ε.

Lemma 5.2.6 *If $\delta > 0$ is small enough we have*

1) if $(x,y) \in A(\delta)$, $0 < x_1 < x$,

2) if $(x,y) \in a_+(\delta)$, $y_1 > -\sqrt{\mu}x_1 - d\mu x_1 - \frac{c_2}{2}x_1^\alpha$,

3) if $(x,y) \in a_-(\delta)$, $y_1 < -2\sqrt{\mu}x_1 - 2c_2x_1^\alpha$.

Proof. We recall that $d\mu = c - 1$. Given $\rho > 0$, if δ is small enough and $(x,y) \in A(\delta)$ it is clear that $|r_3^1(x,y)| \leq \rho|x|^2$. In the following we will use ρ as small as necessary. Hence if $(x,y) \in A(\delta)$, the following holds:

$$\begin{aligned} x_1 &= (c+\sqrt{\mu})x + y + r_3^1(x,y) > (c+\sqrt{\mu})x - 2\sqrt{\mu}x - 2c_2x^\alpha + r_3^1(x,y) \\ &\geq x(c - \sqrt{\mu} - 2c_2x^{\alpha-1} - \rho x) > 0. \end{aligned}$$

Also,

$$\begin{aligned} x - x_1 &= x(1 - c - \sqrt{\mu}) - y - r_3^1(x,y) \\ &> x(1 - c - \sqrt{\mu}) + (\sqrt{\mu} + d\mu)x + \frac{c_2}{2}x^\alpha - r_3^1(x,y) \\ &= (1 - c + d\mu)x + \frac{c_2}{2}x^\alpha - r_3^1(x,y) \geq x^\alpha\left(\frac{c_2}{2} - \rho|x|^{2-\alpha}\right) > 0 \end{aligned}$$

and the first property is proved. Now we demonstrate the second one. We recall that

$$c_2^2 = \frac{a_{20} - c_1 a_{11}}{\alpha(c - \sqrt{\mu})^{\alpha-1}} \quad \text{with} \quad c_1 = 2\sqrt{\mu}$$

and that for all $A \in \mathbb{R}$

$$(1 + Ax)^\alpha = 1 + \alpha A x^\alpha + O(x^{2\alpha}).$$

Then, if $(x,y)\in a_+(\delta)$ we have

$$\begin{aligned}
y_1 + (\sqrt{\mu}+d\mu)x_1 + \frac{c_2}{2}x_1^\alpha &= (c-\sqrt{\mu})y + a_{20}x^2 + a_{11}xy + r_3^2(x,y)\\
&\quad + (\sqrt{\mu}+d\mu)[(c+\sqrt{\mu})x + y + r_3^1(x,y)] + \frac{c_2}{2}[(c+\sqrt{\mu})x + y + r_3^1(x,y)]^\alpha\\
&= -(c-\sqrt{\mu})\Big((\sqrt{\mu}+d\mu)x + \frac{c_2}{2}x^\alpha\Big) + a_{20}x^2 - a_{11}x\Big((\sqrt{\mu}+d\mu)x + \frac{c_2}{2}x^\alpha\Big)\\
&\quad + (\sqrt{\mu}+d\mu)\Big((c-d\mu)x - \frac{c_2}{2}x^\alpha\Big) + \frac{c_2}{2}\Big((c-d\mu)x - \frac{c_2}{2}x^\alpha\Big)^\alpha + o_2\\
&= x(\sqrt{\mu}+d\mu)(\sqrt{\mu}-d\mu) - \frac{c_2}{2}\Big(x^\alpha(c+d\mu) - \Big((c-d\mu)x - \frac{c_2}{2}x^\alpha\Big)^\alpha\Big)\\
&\quad + x^2(a_{20} - a_{11}(\sqrt{\mu}+d\mu)) + o_2\\
&= x\mu(1-d^2\mu) - \frac{c_2}{2}\Big(x^\alpha[(c+d\mu) - (c-d\mu)^\alpha] + \alpha x^{2\alpha-1}(c-d\mu)^{\alpha-1}\frac{c_2}{2}\Big)\\
&\quad + x^2(a_{20} - a_{11}(\sqrt{\mu}+d\mu)) + o_2\\
&= x\mu(1-d^2\mu + O_{\alpha-1}) + x^2\Big(a_{20} - a_{11}(\sqrt{\mu}+d\mu) - \alpha(c-d\mu)^{\alpha-1}\frac{c_2^2}{4}\Big)\\
&= x\mu(1-d^2\mu + O_{\alpha-1}) + x^2\Big(\frac{3}{4}a_{20} + O(\sqrt{\mu}) + o_1\Big) > 0
\end{aligned}$$

if μ and δ are small enough.

Finally we check the third property. Let $(x,y)\in a_-(\delta)$. Then,

$$\begin{aligned}
y_1 + 2\sqrt{\mu}x_1 + 2c_2x_1^\alpha &= (c-\sqrt{\mu})y + a_{20}x^2 + a_{11}xy + r_3^2(x,y)\\
&\quad +2\sqrt{\mu}[(c+\sqrt{\mu})x + y + r_3^1(x,y)]\\
&\quad +2c_2[(c+\sqrt{\mu})x + y + r_3^1(x,y)]^\alpha\\
&= -(c-\sqrt{\mu})(2\sqrt{\mu}x + 2c_2x^\alpha) + a_{20}x^2 - a_{11}x(2\sqrt{\mu}x + 2c_2x^\alpha)\\
&\quad +2\sqrt{\mu}[(c-\sqrt{\mu})x - 2c_2x^\alpha]\\
&\quad +2c_2[(c-\sqrt{\mu})x - 2c_2x^\alpha]^\alpha + o_2\\
&= -2c_2x^\alpha[c+\sqrt{\mu} - (c-\sqrt{\mu})^\alpha]\\
&\quad +x^2[a_{20} - 2a_{11}\sqrt{\mu} - 4c_2^2(c-\sqrt{\mu})^{\alpha-1}\alpha] + o_2\\
&= -2c_2x^\alpha\sqrt{\mu}[1+\alpha+O(\sqrt{\mu})] - 3x^2[a_{20}+O(\sqrt{\mu})] + o_2
\end{aligned}$$

which is strictly negative if μ and δ are small enough. ■

We define the sectors

$$S_1 = \{(\xi,\eta)\in\mathbb{R}^2 : 0<\eta<\xi\},$$

$$S_2 = \{(\xi, \eta) \in \mathbb{R}^2 : \xi < \eta < 0\},$$

$$S = S_1 \cup S_2$$

and

$$S'_1 = \{(\xi, \eta) \in \mathbb{R}^2 : 0 < \eta < \frac{5}{6}\xi\},$$

$$S'_2 = \{(\xi, \eta) \in \mathbb{R}^2 : \frac{5}{6}\xi < \eta < 0\},$$

$$S' = S'_1 \cup S'_2$$

Lemma 5.2.7 *If δ and μ are small enough, $(x, y) \in A(\delta)$ and $\zeta \in \overline{S_i} - \{(0,0)\}$, then*

$$DF_\eta(x, y)\zeta \in S_i, \qquad i = 1, 2. \tag{5.2.8}$$

Moreover, if $\zeta \in \overline{S'_i} - \{(0,0)\}$, then

$$DF_\eta(x, y)\zeta \in S'_i, \qquad i = 1, 2. \tag{5.2.9}$$

Proof. By linearity of $DF_\eta(x, y)$, it is sufficient to prove the first part of the lemma for $\zeta = (\xi, \eta) \in \overline{S_1} - \{(0,0)\}$. We write

$$\begin{aligned} DF_\eta(x, y) &= \begin{pmatrix} c + \sqrt{\mu} & 1 \\ 2a_{20}x + a_{11}y & c - \sqrt{\mu} + a_{11}x \end{pmatrix} + O_2 \\ &\equiv \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \end{aligned}$$

which means that

$$DF_\eta(x, y) \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} A_{11}\xi + A_{12}\eta \\ A_{21}\xi + A_{22}\eta \end{pmatrix}.$$

First we note that if $(x, y) \in A(\delta)$, $|y| \le 2\sqrt{\mu}x + 2c_2x^\alpha$. Then

$$\begin{aligned} A_{21} &= 2a_{20}x + a_{11}y + O_2 \\ &> x[2a_{20} - a_{11}(2\sqrt{\mu} + 2c_2x^{\alpha-1}) + O_1] > 0 \end{aligned}$$

since a_{20} is a positive number. Clearly $A_{22} > 0$. Then if, $(\xi_1, \eta_1)^T = DF(x,y)(\xi,\eta)^T$,

$$\eta_1 = A_{21}\xi + A_{22}\eta > (A_{21} + A_{22})\eta \geq 0.$$

Moreover

$$\begin{aligned}
\xi_1 - \eta_1 &= (A_{11} - A_{21})\xi + (A_{12} - A_{22})\eta \\
&> (A_{11} - A_{21} + A_{12} - A_{22})\eta \\
&= (1 + O(\sqrt{\mu}) + O_1)\eta \geq 0.
\end{aligned}$$

Notice that $A_{11} > A_{21}$ and $\xi > 0$.

Let $\zeta = (\xi, \eta) \in \overline{S_1'} - \{0\}$, then it is clear that $\eta_1 \geq 0$ and

$$\begin{aligned}
\frac{5}{6}\xi_1 - \eta_1 &= \Big(\frac{5}{6}A_{11} - A_{21}\Big)\xi + \Big(\frac{5}{6}A_{12} - A_{22}\Big)\eta \\
&> \Big(A_{11} - \frac{6}{5}A_{21} + \frac{5}{6}A_{12} - A_{22}\Big)\eta \\
&= \Big(\frac{5}{6} + O(\sqrt{\mu}) + O_1\Big)\eta \geq 0.
\end{aligned}$$

■

Lemma 5.2.8 *Let* $0 < r < 2$. *If* δ *is small enough,* $(x,y) \in A(\delta)$ *and* $0 < x \leq 1/j^r$ *then* $0 < x_1 < 1/(j+1)^r$.

Proof. If $x < 1/(j+1)^r$, by Lemma 5.2.6, we have that $0 < x_1 < x$ and the result holds. Thus we can suppose that $1/(j+1)^r < x \leq 1/j^r$. Then,

$$\begin{aligned}
x_1 - \frac{1}{(j+1)^r} &= (c + \sqrt{\mu})x + y + r_3^1(x,y) - \frac{1}{(j+1)^r} \\
&\leq (c - d\mu)x - \frac{c_2}{2}x^\alpha + r_3^1(x,y) - \frac{1}{(j+1)^r} \\
&= x - \frac{c_2}{2}x^\alpha + r_3^1(x,y) - \frac{1}{(j+1)^r} \\
&\leq \frac{1}{j^r} - \frac{1}{(j+1)^r} - \frac{c_2}{2}x^\alpha + r_3^1(x,y) \\
&< \frac{r}{j^{r+1}} + O\Big(\frac{1}{j^{r+2}}\Big) - \frac{c_2}{2}\frac{1}{(j+1)^{r\alpha}} + o\Big(\frac{1}{j^{2r}}\Big)
\end{aligned}$$

and, since $r+1>\alpha r$,

$$\frac{r}{j^{r+1}}-\frac{c_2}{2}\frac{1}{(j+1)^{r\alpha}}+o\Big(\frac{1}{j^{2r}}\Big)=\frac{-1}{(j+1)^{r\alpha}}\Big(\frac{c_2}{2}-\frac{r(j+1)^{r\alpha}}{j^{r+1}}\Big)+o\Big(\frac{1}{j^{2r}}\Big)<0.$$

if j is big enough. ■

From now on we fix

$$r=3/2.$$

For the next lemma we need an expression of F_η^{-1}. It is not difficult to see that

$$\begin{aligned}F_\eta^{-1}(x,y) &= \frac{1}{c^2-\mu}\begin{pmatrix}(c-\sqrt{\mu})x-y\\ (c+\sqrt{\mu})y\end{pmatrix}\\ &+\frac{1}{(c^2-\mu)^3}a_{20}\begin{pmatrix}[(c-\sqrt{\mu})x-y]^2\\ -(c+\sqrt{\mu})[(c-\sqrt{\mu})x-y]^2\end{pmatrix}\\ &+\frac{c+\sqrt{\mu}}{(c^2-\mu)^3}a_{11}\begin{pmatrix}((c-\sqrt{\mu})x-y)y\\ -(c+\sqrt{\mu})[(c-\sqrt{\mu})x-y]y\end{pmatrix}+O_3.\end{aligned}$$

Lemma 5.2.9 *If δ is small enough, $(x,y)\in A(\delta)$ and $\zeta=(\xi,\eta)\in S$, then there exists a constant $M>0$ (independent of μ and ε) such that*

$$\left|\pi_1 DF_\eta^{-1}(x,y)\zeta\right|\le\Big(\frac{1}{c+\sqrt{\mu}}+Mx\Big)|\xi|. \tag{5.2.10}$$

Moreover, if $(\xi,\eta)\in S'$,

$$\left|\pi_2 DF_\eta^{-1}(x,y)\zeta\right|\le\Big(1+Mx\Big)|\xi| \tag{5.2.11}$$

Proof. We write

$$DF_\eta^{-1}(x,y)\equiv\begin{pmatrix}B_{11}&B_{12}\\ B_{21}&B_{22}\end{pmatrix}$$

so that

$$\pi_1 DF_\eta^{-1}(x,y)\begin{pmatrix}\xi\\ \eta\end{pmatrix}=B_{11}\xi+B_{12}\eta.$$

We have that

$$\begin{aligned} B_{11} &= \frac{1}{c+\sqrt{\mu}} + \frac{1}{(c^2-\mu)^3} 2a_{20}[(c-\sqrt{\mu})x - y](c+\sqrt{\mu}) \\ &\quad + \frac{c+\sqrt{\mu}}{(c^2-\mu)^2} a_{11} y + O_2 \\ &\le \frac{1}{c+\sqrt{\mu}} + Mx \end{aligned}$$

where M can be taken independent of μ and ε, and

$$\begin{aligned} B_{12} &= -\frac{1}{c^2-\mu} - \frac{1}{(c^2-\mu)^3} 2a_{20}((c-\sqrt{\mu})x - y) \\ &\quad + \frac{c+\sqrt{\mu}}{(c^2-\mu)^3} a_{11}[(c-\sqrt{\mu})x - 2y] + O_2 \end{aligned}$$

which is negative. Thus if $(\xi, \eta) \in S_1$

$$B_{11}\xi + B_{12}\eta < B_{11}\xi$$

and then

$$\pi_1 DF_\eta^{-1}(x,y) \begin{pmatrix} \xi \\ \eta \end{pmatrix} \le \Big(\frac{1}{c+\sqrt{\mu}} + Mx \Big) \xi. \tag{5.2.12}$$

Moreover,

$$\begin{aligned} \pi_1 DF_\eta^{-1}(x,y) \begin{pmatrix} \xi \\ \eta \end{pmatrix} &= B_{11}\xi + B_{12}\eta \ge (B_{11} + B_{12})\xi \\ &= \Big(\frac{1}{c+\sqrt{\mu}} - \frac{1}{c^2-\mu} + O_1 \Big) \xi \\ &\ge -\Big(\frac{1}{c+\sqrt{\mu}} + Mx \Big) \xi. \end{aligned} \tag{5.2.13}$$

Therefore by (5.2.12) and (5.2.13) the bound (5.2.10) holds.

Now we prove (5.2.11). Let $\zeta = (\xi, \eta) \in S'$, it is clear that

$$\begin{aligned} |\pi_2 DF^{-1}(x,y)\zeta| = |B_{21}\xi + B_{22}\eta| &= \left| \frac{1}{c-\sqrt{\mu}}\eta + O_1\xi \right| \\ &\le \frac{1}{c-\sqrt{\mu}}|\eta| + Mx|\xi| \\ &\le \left(\frac{1}{c-\sqrt{\mu}} \frac{5}{6} + Mx \right) |\xi| \\ &\le (1+Mx)|\xi| \end{aligned}$$

if μ is small enough. ■

Lemma 5.2.10 *If δ is small enough and $z_i = (x_i, y_i)$, $i = 1, 2$, are two different points such that z_i, $F_\eta(z_i) \in A(\delta)$, $x_i \leq 1/j^r$, $i = 1, 2$, and $z_1 - z_2 \in S$, then, there exists a constant $M > 0$ (independent of μ and ε) such that*

$$|\pi_1(F_\eta(z_2) - F_\eta(z_1))| \geq \frac{c + \sqrt{\mu}}{1 + M(j+1)^{-3/2}} |\pi_1(z_2 - z_1)|.$$

Proof. We recall that, since $z_1 - z_2 \in S$ and z_1, $z_2 \in A(\delta)$, the segment between z_1 and z_2 belongs to $A(\delta)$. Therefore, by the mean value theorem in integral form, we can write the difference $F_\eta(z_2) - F_\eta(z_1)$ in the following way,

$$F_\eta(z_2) - F_\eta(z_1) = \int_0^1 DF_\eta(z_1 + t(z_2 - z_1))(z_2 - z_1)\, dt.$$

It is clear that $F_\eta(z_2) - F_\eta(z_1) \in S$ and therefore the segment $\overline{F_\eta(z_1)F_\eta(z_2)}$ is contained in $A(\delta)$. Now applying the mean value theorem and Lemma 5.2.9, concretely, using estimate (5.2.10) we have

$$\begin{aligned} |\pi_1(z_2 - z_1)| &= |\pi_1[F_\eta^{-1}(F_\eta(z_2)) - F_\eta^{-1}(F_\eta(z_1))]| \\ &\leq \sup_{\zeta \in \overline{F_\eta(z_1)F_\eta(z_2)}} |\pi_1 DF_\eta^{-1}(\zeta)(F_\eta(z_2) - F_\eta(z_1))| \\ &\leq \sup_{\zeta \in \overline{F_\eta(z_1)F_\eta(z_2)}} \Big(\frac{1}{c + \sqrt{\mu}} + M\pi_1(\zeta)\Big) |\pi_1(F_\eta(z_2) - F_\eta(z_1))|. \end{aligned} \tag{5.2.14}$$

Let $\zeta \in \overline{F_\eta(z_1)F_\eta(z_2)}$. By Lemma 5.2.8, we have that

$$\pi_1(F_\eta(z_1)),\ \ \pi_1(F_\eta(z_2)) < 1/(j+1)^r.$$

Thus $\zeta_1 \leq 1/(j+1)^r$. Then, using (5.2.14) we obtain

$$|\pi_1(z_2 - z_1)| \leq \Big(\frac{1}{c + \sqrt{\mu}} + \frac{M}{(j+1)^r}\Big) |\pi_1(F_\eta(z_2) - F_\eta(z_1))|.$$

The inequality of the statement holds if we denote again by M a bound of the expression $M(c + \sqrt{\mu})$ for all values of μ. ■

Lemma 5.2.11 *If δ is small enough and $z_i = (x_i, y_i)$, $i = 1, 2$, are two different points such that z_i, $F_\eta(z_i) \in A(\delta)$, $x_i \le 1/j^r$, $i = 1, 2$, and $z_1 - z_2 \in S'$, then, there exists a constant $M > 0$ (independent of μ) such that*

$$\begin{aligned}
|\pi_2(z_2 - z_1)| &\le \left(1 + \frac{M}{(j+1)^r}\right) |\pi_1(F_\eta(z_2) - F_\eta(z_1))| \\
&\le \left(1 + \frac{M}{(j+1)^r}\right) \|F_\eta(z_2) - F_\eta(z_1)\|.
\end{aligned}$$

Proof. As in the previous lemma, we know that $F_\eta(z_2) - F_\eta(z_1) \in S$ and therefore the segment $\overline{F_\eta(z_1)F_\eta(z_2)}$ is contained in $A(\delta)$. Applying the mean value theorem and estimate (5.2.11) we have that

$$\begin{aligned}
|\pi_2(z_2 - z_1)| &= |\pi_2[F_\eta^{-1}(F_\eta(z_2)) - F_\eta^{-1}(F_\eta(z_1))]| \\
&\le \sup_{\zeta \in \overline{F_\eta(z_1)F_\eta(z_2)}} |\pi_2 DF_\eta^{-1}(\zeta)(F_\eta(z_2) - F_\eta(z_1))| \\
&\le \sup_{\zeta \in \overline{F_\eta(z_1)F_\eta(z_2)}} \Big(1 + M\pi_1(\zeta)\Big)|\pi_1(F_\eta(z_2) - F_\eta(z_1))|. \qquad (5.2.15)
\end{aligned}$$

Clearly, if $\zeta \in \overline{F_\eta(z_1)F_\eta(z_2)}$, by Lemma 5.2.8, we have that $\zeta_1 \le 1/(j+1)^r$. Then, using (5.2.15) we obtain

$$|\pi_2(z_2 - z_1)| \le \Big(1 + \frac{M}{(j+1)^r}\Big)|\pi_1(F_\eta(z_2) - F_\eta(z_1))|$$

and the inequality of the statement holds. ∎

5.2.4 Proof of the existence of local stable manifold

Once we have established the previous lemmas the proof is the same as the one in [30]. We sketch it. We define the set

$$H(A) = \{\Gamma, \text{ differentiable arcs connecting } a_- \text{ with } a_+ \text{ such that } T_p\Gamma \subset S,\ \forall p \in \Gamma\}.$$

We fix any $\Gamma_0 \in H(A)$ and we define

$$\Gamma_k = F_\eta^{-k}(F_\eta^k(\Gamma_0) \cap A(\delta)).$$

The intersection

$$\Gamma_\infty = \bigcap_{k\geq 0} \Gamma_k$$

is non empty because it is the intersection of a nested sequence of compact non-empty sets. We want to prove that Γ_∞ is a single point. Assume that Γ_∞ has two different points $z_1 \neq z_2$. Note that this means that $\pi_1(z_1 - z_2) \neq 0$. Then there exists j_0 such that $0 < x_i < 1/j_0^r$ $i = 1, 2$. If we apply the Lemma 5.2.8 inductively we have that

$$0 < \pi_1 F_\eta^j(z_i) < \frac{1}{(j + j_0)^r}. \tag{5.2.16}$$

Moreover, by definition of $H(A)$, $z_1 - z_2 \in S$, and by Lemma 5.2.7, $F^j(z_1) - F^j(z_2) \in S$. Applying Lemma 5.2.10 inductively we obtain

$$\begin{aligned} |\pi_1(F_\eta^j(z_2) - F_\eta^j(z_1))| &\geq \frac{c + \sqrt{\mu}}{1 + M(j + j_0)^{-3/2}} |\pi_1(F_\eta^{j-1}(z_2) - F_\eta^{j-1}(z_1))| \\ &\geq \prod_{i=1}^{j} \frac{c + \sqrt{\mu}}{1 + M(i + j_0)^{-3/2}} |\pi_1(z_2 - z_1)|. \end{aligned}$$

When $\mu > 0$ the product diverges when $j \to +\infty$. When $\mu = 0$ the product converges to some value different from zero. Thus $\{F_\eta^j(z_1)\}_j$ and $\{F_\eta^j(z_2)\}_j$ cannot converge both to zero. But this is a contradiction with (5.2.16) . Therefore Γ_∞ is a single point.

This argument is valid for any $\Gamma_0 \in H(A)$, therefore,as in [30], we conclude that $W_\delta^{s+} \cap A(\delta)$ is the graph of a function φ_μ.

To see that φ_μ is Lipschitz we suppose that there are points $z_i = (x_i, \varphi_\mu(x_i))$, $i = 1, 2$, $x_1 \neq x_2$, such that

$$\frac{|\varphi_\mu(x_1) - \varphi_\mu(x_2)|}{|x_1 - x_2|} \geq \frac{(c + \sqrt{\mu})^2}{2c - (c - \sqrt{\mu})^2}.$$

We claim that $F_\eta^2(z_2) - F_\eta^2(z_1) \in S$ and hence they should coincide. Indeed

$$\begin{aligned} F_\eta^2(z_2) - F_\eta^2(z_1) &= \int_0^1 DF_\eta(F_\eta(z_1 + t(z_1 - z_2)))DF_\eta(z_1 + t(z_1 - z_2))(z_1 - z_2)\, dt \\ &= \begin{pmatrix} ((c + \sqrt{\mu})^2 + O_1)(x_1 - x_2) + (2c + O_1)(\varphi(x_1) - \varphi(x_2)) \\ ((c - \sqrt{\mu})^2 + O_1)(\varphi(x_1) - \varphi(x_2)) \end{pmatrix}. \end{aligned} \tag{5.2.17}$$

It is not difficult to check that, if $\varphi(x_1) - \varphi(x_2) > 0$ then $F_\eta^2(z_2) - F_\eta^2(z_1) \in S_1$ and if $\varphi(x_1) - \varphi(x_2) < 0$ we have that $F_\eta^2(z_2) - F_\eta^2(z_1) \in S_2$.

5.2.5 Uniqueness of the local stable invariant manifold

We know that, if μ is different from zero, the origin is a saddle point. Hence there exists a neighborhood of the origin which we denote by $U_{\mu,\varepsilon} \subset \mathbb{R}^2$, depending on μ and ε, such that, the local stable invariant manifold $W^{s+}_{U_{\mu,\varepsilon}}$, is unique and can be represented as the graph of a function which we denote by φ'. Let $B_{\mu,\varepsilon} \subset \mathbb{R}$ the domain of φ'. We denote $B^+_{\mu,\varepsilon} = B_{\mu,\varepsilon} \cap \{x > 0\}$.

Proposition 5.2.12 *Let F_η be a C^n, $n \geq 2$, map of the form (5.2.5). Then there exists $\delta > 0$ (independent of μ and ε) such that $W^{s+}_\delta = \text{graph } \varphi'$.*

Proof. We consider first the case $\mu > 0$. We suppose that there exists a Lipschitz function $\tilde{\varphi}$ defined on the open set

$$D = \{x \in \mathbb{R} : 0 < x < \delta\}$$

such that if $(x, y) \in \text{graph } \tilde{\varphi}$, $F^k_\eta(x, y)$ goes to $(0, 0)$ when k goes to infinity. For k big enough, we have that $\pi_1 F^k_\eta(x, \varphi'(x)) \in B^+_{\mu,\varepsilon}$. Therefore, by uniqueness of the local stable manifold

$$\text{graph}\, \varphi' = W^{s+}_{U_{\mu,\varepsilon}} = \text{graph}\, \tilde{\varphi}_{|B^+_{\mu,\varepsilon}},$$

and since the sets graph φ' and graph $\tilde{\varphi}_{|B^+_{\mu,\varepsilon}}$ are invariant, $\tilde{\varphi} = \varphi'$ for $0 < x < \delta$. This implies that $W^{s+}_\delta = \text{graph } \varphi'$.

For $\mu = 0$ this result is consequence of a result given in [30]. ■

5.2.6 Proof of the continuity with respect to the parameter ε

Let $y = \varphi(x, \mu, \varepsilon)$ be the stable curve of the map F_η. For convenience we write $F_\eta(x, y) = F(x, y, \mu, \varepsilon)$. We fix $\varepsilon_0 \in V_\varepsilon$, $\mu \in V_\mu$ and $x \in (0, \delta)$.

We will see that, for any $\nu > 0$ there exists $\delta_0 > 0$ such that, if $\|\varepsilon - \varepsilon_0\| < \delta_0$, then

$$|\varphi(x, \mu, \varepsilon) - \varphi(x, \mu, \varepsilon_0)| < \nu. \tag{5.2.18}$$

From now on we omit the dependence on μ in the notation.

Given x, let j_x be such that $0 < x < 1/j_x^r$, with $r = 3/2$. There exist $c_1(\varepsilon)$ and $c_2(\varepsilon)$ depending continuously with respect to ε and μ such that

$$|\pi_2 F^j(x, \varphi(x, \varepsilon), \varepsilon)| \leq c_1(\varepsilon)|\pi_1 F^j(x, \varphi(x, \varepsilon), \varepsilon)| + c_2(\varepsilon)|\pi_1 F^j(x, \varphi(x, \varepsilon), \varepsilon)|^\alpha.$$

By Lemma 5.2.8, we have that

$$0 < \pi_1 F^j(x, \varphi(x,\varepsilon), \varepsilon) < \frac{1}{(j_x + j)^r} \tag{5.2.19}$$
$$|\pi_2 F^j(x, \varphi(x,\varepsilon), \varepsilon)| < c_1 \frac{1}{(j_x + j)^r} + c_2 \frac{1}{(j_x + j)^{r\alpha}}$$

with $c_1 = \max_{\varepsilon \in V_\varepsilon} c_1(\varepsilon) < +\infty$ and $c_2 = \max_{\varepsilon \in V_\varepsilon} c_1(\varepsilon) < +\infty$. Let $\varepsilon_0 \in V_\varepsilon$ and $\nu > 0$. Therefore, there exists $j_0 = j_0(\nu, \varepsilon_0)$ which does not depend on ε such that

$$\|F^{j_0}(x, \varphi(x,\varepsilon), \varepsilon)\| < \frac{\nu}{2}.$$

Therefore for all $\varepsilon \in V_\varepsilon$ we have that

$$\|F^{j_0}(x, \varphi(x,\varepsilon), \varepsilon) - F^{j_0}(x, \varphi(x,\varepsilon_0), \varepsilon_0)\| < \nu. \tag{5.2.20}$$

In view of Lemma 5.2.6 we define

$$f_1(x,\varepsilon) = -\sqrt{\mu}x - d\mu x - (c_2/2)x^\alpha$$
$$f_2(x,\varepsilon) = -2\sqrt{\mu}x - 2c_2 x^\alpha.$$

We write explicitly the dependence on ε of functions f_1 and f_2, which comes from the dependence on ε of the coefficient c_2 and, eventually, of the parameter μ. These functions depend continuously on ε. We define

$$A(\delta,\varepsilon) = \{(x,y) \in \mathbb{R}^2 : \ 0 < x < \delta, \ f_2(x,\varepsilon) < y < f_1(x,\varepsilon)\}$$

as in (5.2.7). From Lemma 5.2.6 we know that, for all ε,

$$f_2(x,\varepsilon) < \varphi(x,\varepsilon) < f_1(x,\varepsilon).$$

Let $f(x,\varepsilon_0)$ be a positive function such that

$$f_2(x,\varepsilon_0) + f(x,\varepsilon_0) \leq \varphi(x,\varepsilon_0) \leq f_1(x,\varepsilon_0) - f(x,\varepsilon_0).$$

Since the functions f_1 and f_2 depend continuously on ε, there exists $\delta_0^1 > 0$, which depend on x and ε_0, such that, if $\|\varepsilon - \varepsilon_0\| \leq \delta_0^1$, we have that

$$|f_i(x,\varepsilon) - f_i(x,\varepsilon_0)| \leq f(x,\varepsilon_0)$$

(we recall that x and ε_0 are fixed). Hence

$$f_2(x,\varepsilon) < \varphi(x,\varepsilon_0) < f_1(x,\varepsilon).$$

In the same way, we can prove that $(x, \varphi(x, \varepsilon_0), \varepsilon)$ belongs to the domain of F^{j_0}. Indeed, since $F^i(x, \varphi(x, \varepsilon_0), \varepsilon_0) \in A(\delta, \varepsilon_0)$ for all i and by the continuity of F with respect to ε, there exists $\delta_0^2 = \delta_0^2(x, \varepsilon_0, j_0(\nu, \varepsilon_0))$ such that, if $\|\varepsilon - \varepsilon_0\| < \delta_0^2$,

$$f_2(\pi_1 F^i(x, \varphi(x, \varepsilon_0), \varepsilon), \varepsilon) < \pi_2 F^i(x, \varphi(x, \varepsilon_0), \varepsilon) < f_1(\pi_1 F^i(x, \varphi(x, \varepsilon_0), \varepsilon), \varepsilon)$$

for all $0 \leq i \leq j_0(\nu)$. Thus, $(x, \varphi(x, \varepsilon_0), \varepsilon)$ belongs to the domain of F^{j_0}. Then, by the continuity of F^{j_0} with respect to ε, there exists δ_0^3 which depends on x, ε_0 and ν such that, for all ε satisfying $\|\varepsilon - \varepsilon_0\| \leq \delta_0^3$,

$$\|F^{j_0}(x, \varphi(x, \varepsilon_0), \varepsilon) - F^{j_0}(x, \varphi(x, \varepsilon_0), \varepsilon_0)\| < \nu. \tag{5.2.21}$$

On the other hand we observe that, by Lemma 5.2.8 and since $F^2(x, \varphi(x, \varepsilon_0), \varepsilon) \in A(\delta, \varepsilon)$ we have that,

$$\begin{aligned} 0 < \pi_1 F^2(x, \varphi(x, \varepsilon_0), \varepsilon) &< \frac{1}{(2 + j_x)^r} \\ 0 < \pi_1 F^2(x, \varphi(x, \varepsilon), \varepsilon) &< \frac{1}{(2 + j_x)^r}. \end{aligned}$$

Remark 5.2.13 *We observe that the infinite product $\prod_{i=2}^{+\infty}(1 + M(i + j_x)^{-r})$ is bounded if $r > 1$. Indeed*

$$\log\Big(\prod_{i=2}^{+\infty}(1 + M(i + j_x)^{-r})\Big) = \sum_{i=2}^{+\infty} \log(1 + M(i + j_x)^{-r})$$

and the series is convergent since $\sum_{i=2}^{+\infty} 1/(i + j_x)^{-r}$ is convergent.

Remark 5.2.14 *Using the expression (5.2.17) for points z_1, z_2 of the form $z_i = (x, y_i)$ (i.e. with the same first component), it is not difficult to check that*

$$F^2(x, y_1, \varepsilon) - F^2(x, y_2, \varepsilon) \in S'.$$

We define

$$C = \prod_{i=2}^{+\infty}(1 + M(i + j_x)^{-r}) < +\infty. \tag{5.2.22}$$

Therefore, by Remark 5.2.14 and Lemma 5.2.11 applied iteratively with

$$\begin{aligned} z_1 &= F^2(x, \varphi(x, \varepsilon_0), \varepsilon) \\ z_2 &= F^2(x, \varphi(x, \varepsilon), \varepsilon) \end{aligned}$$

we obtain that

$$\begin{aligned} |\pi_2(z_1 - z_2)| &\le \prod_{i=2}^{j_0} \Big(1 + \frac{M}{(i + j_x)^r}\Big) \|F^{j_0}(x, \varphi(x, \varepsilon_0), \varepsilon) - F^{j_0}(x, \varphi(x, \varepsilon), \varepsilon)\| \\ &\le C\|F^{j_0}(x, \varphi(x, \varepsilon_0), \varepsilon) - F^{j_0}(x, \varphi(x, \varepsilon), \varepsilon)\| \end{aligned} \tag{5.2.23}$$

and using Lemma 5.2.10, (in fact, using (5.2.14) in the proof of Lemma 5.2.10) we have that

$$\begin{aligned} |\pi_1(z_1 - z_2)| &\le \prod_{i=2}^{j_0} \Big(\frac{1}{c + \sqrt{\mu}} + \frac{M}{(i + j_x)^r}\Big) \|F^{j_0}(x, \varphi(x, \varepsilon_0), \varepsilon) - F^{j_0}(x, \varphi(x, \varepsilon), \varepsilon)\| \\ &\le \prod_{i=2}^{+\infty} \Big(1 + \frac{M}{(i + j_x)^r}\Big) \|F^{j_0}(x, \varphi(x, \varepsilon_0), \varepsilon) - F^{j_0}(x, \varphi(x, \varepsilon), \varepsilon)\| \\ &= \|F^{j_0}(x, \varphi(x, \varepsilon_0), \varepsilon) - F^{j_0}(x, \varphi(x, \varepsilon), \varepsilon)\|. \end{aligned} \tag{5.2.24}$$

with C defined in (5.2.22).

Moreover, by the mean value theorem applied to $F^{-2} \circ F^2$, there exists a constant K (independent of ε) such that

$$|\varphi(x, \varepsilon_0) - \varphi(x, \varepsilon)| \le K\|F^2(x, \varphi(x, \varepsilon_0), \varepsilon) - F^2(x, \varphi(x, \varepsilon), \varepsilon)\|. \tag{5.2.25}$$

By (5.2.23), (5.2.24) and (5.2.25) we obtain

$$|\varphi(x, \varepsilon_0) - \varphi(x, \varepsilon)| \le KC\|F^{j_0}(x, \varphi(x, \varepsilon_0), \varepsilon) - F^{j_0}(x, \varphi(x, \varepsilon), \varepsilon)\|.$$

Using the triangular inequality as well as (5.2.20) and (5.2.21) we have that

$$|\varphi(x, \varepsilon_0) - \varphi(x, \varepsilon)| \le 2KC\nu$$

if $\|\varepsilon - \varepsilon_0\| < \delta_0 = \min\{\delta_0^1, \delta_0^2, \delta_0^3\}$.

5.2.7 Analyticity of the stable curve in the weak hyperbolic case

Now we deal with the analytic case. We suppose that F_η (with $\eta = (\mu, \varepsilon)$ as before) is in the normal form, that is

$$F_\eta(x,y) = \begin{pmatrix} (c+\sqrt{\mu})x + y \\ (c-\sqrt{\mu})y \end{pmatrix} + \begin{pmatrix} 0 \\ a_{20}x^2 + a_{11}xy \end{pmatrix} + \begin{pmatrix} r_3^1(x,y) \\ r_3^2(x,y) \end{pmatrix}. \qquad (5.2.26)$$

Then we have the following result:

Theorem 5.2.15 *Let $F_\eta : U \subset \mathbb{C}^2 \to \mathbb{C}^2$, $\eta = (\mu, \varepsilon) \in V_\eta = V_\mu \times V_\varepsilon \subset \mathbb{C} \times \mathbb{R}^n$ with $\|\eta\|$ small enough, be a family of analytic map, depending analytically on μ and continuously on ε, of the form (5.2.26), satisfying that $a_{20}(0,\varepsilon) > 0$. Then the stable curve is the graph of a function φ_η analytic on U, depending analytically on μ and continuously on ε for any $(\mu, \varepsilon) \in V_\eta$.*

Preliminaries

We perform a change of variables to move the stable curve closer to the x-axis. Concretely we define the change $T(x,y) = (x, y - c_2x^\alpha - 2\sqrt{\mu}x)$. If

$$H = T^{-1} \circ F_\eta \circ T, \qquad (5.2.27)$$

it is not difficult to check that

$$\begin{aligned} H_1(x,y) &= (c-\sqrt{\mu})x + y - c_2x^\alpha + r_3^1(x, y - c_2x^\alpha - 2\sqrt{\mu}x) \\ H_2(x,y) &= (c+\sqrt{\mu})y + \tilde{T}_0(x) + \tilde{T}_1(x,y) + \tilde{T}_2(x,y) + \tilde{T}_3(x,y) \end{aligned}$$

where

$$\begin{aligned} \tilde{T}_0 &= -c_2x^\alpha(c+\sqrt{\mu} - (c-\sqrt{\mu})^\alpha), \\ \tilde{T}_1 &= x^2(a_{20} - 2\sqrt{\mu}a_{11}) + a_{11}x(y - c_2x^\alpha) + c_2\alpha x^{\alpha-1}(c-\sqrt{\mu})^{\alpha-1}(y - c_2x^\alpha), \\ \tilde{T}_2 &= c_2[(c-\sqrt{\mu})x + y - c_2x^\alpha + r_3^1(x, y - c_2x^\alpha - 2\sqrt{\mu}x)]^\alpha - c_2(c-\sqrt{\mu})^\alpha x^\alpha \\ &\quad -c_2\alpha x^{\alpha-1}(c-\sqrt{\mu})^{\alpha-1}[y - c_2x^\alpha + r_3^1(x, y - c_2x^\alpha - 2\sqrt{\mu}x)], \\ \tilde{T}_3 &= r_3^2(x, y - c_2x^\alpha - 2\sqrt{\mu}x) + 2\sqrt{\mu}r_3^1(x, y - c_2x^\alpha - 2\sqrt{\mu}x) \\ &\quad +c_2\alpha x^{\alpha-1}(c-\sqrt{\mu})^{\alpha-1}r_3^1(x, y - c_2x^\alpha - 2\sqrt{\mu}x). \end{aligned} \qquad (5.2.28)$$

Now we perform the change of coordinates given by $C(x,y) = (x + ax^\beta, y)$ with $a < 0$, a parameter to be determined below and

$$\beta = \frac{5}{4}.$$

We observe that β is such that

$$2\beta - 1 = \alpha.$$

Then the system (5.2.27) becomes

$$G = C^{-1} \circ H \circ C \tag{5.2.29}$$

with

$$\begin{aligned}
G_1(x,y) &= (c - \sqrt{\mu})x + y + R_0(x) + R_1(x) + R_2(x,y) \\
G_2(x,y) &= (c + \sqrt{\mu})y + T_0(x) + T_1(x,y) + T_2(x,y) + T_3(x,y)
\end{aligned}$$

where $T_0(x) = \tilde{T}_0(x + ax^\beta)$, $T_i(x,y) = \tilde{T}_i(x + ax^\beta, y)$ for $i = 1, 2, 3$ and

$$\begin{aligned}
R_0(x) &= a(\beta - 1)\sqrt{\mu}x^\beta - x^\alpha[c_2 + a^2\beta] \\
R_1(x) &= -ax^\beta\{(c - \sqrt{\mu})^\beta(1 + ax^{\beta-1})^\beta - (c - \sqrt{\mu}) + (\beta - 1)\sqrt{\mu}\} + a^2x^\alpha\beta \\
&\quad -c_2x^\alpha[(1 + ax^{\beta-1})^\alpha - 1] \\
R_2(x,y) &= -a\{[H_1(x + ax^\alpha, y)]^\beta - (c - \sqrt{\mu})^\beta x^\beta(1 + ax^{\beta-1})^\beta\} + O_{5/2}.
\end{aligned}$$

We define the set

$$\Omega(\delta) = \left\{ x \in \mathbb{C};\ |x| < \delta,\ |\arg(x)| < \frac{\pi}{4(\alpha - 1)} = \frac{\pi}{2} \right\}.$$

Lemma 5.2.16 *If δ is small enough there exists γ_0 small enough such that, if $x \in \Omega(\delta) - \{0\}$,*

1) if $|y| \le \gamma_0|x|^\alpha$ then $G_1(x,y) \in \Omega(\delta)$,

2) if $|y| = \gamma_0|x|^\alpha$ then $|G_2(x,y) - y| < |y|$,

3) if $|y| = \gamma_0|x|^\alpha$ then $\gamma_0|G_1(x,y)|^\alpha < |G_2(x,y)|$.

Proof. We begin by proving 1). First we bound $R_1(x)$ and $R_2(x,y)$. We observe that

$$\alpha-\beta=\beta-1=\frac{1}{4}.$$

Therefore there exists a positive constant C such that

$$\begin{aligned}|R_1(x)| &\le |a||x^\beta||(c-\sqrt{\mu})^\beta(1+ax^{\beta-1})^\beta-(c-\sqrt{\mu})+(\beta-1)\sqrt{\mu}-ax^{\alpha-\beta}\beta| \\ &\quad +c_2\alpha|a||x|^{\alpha+\beta-1} \\ &= |a||x|^\beta\left|(c-\beta\sqrt{\mu}+O(\mu))(1+\beta ax^{\beta-1}+O(x^{2\beta-2}))-(c-\sqrt{\mu})\right. \\ &\quad \left.+(\beta-1)\sqrt{\mu}-ax^{\alpha-\beta}\beta\right|+c_2|x|^{\alpha+\beta-1} \\ &\le C|x|^\beta\left\{\mu+|x|^{\alpha-\beta}\sqrt{\mu}+|x|^{2\beta-2}\right\}. \end{aligned} \tag{5.2.30}$$

To deal with R_2 we consider the function $\varphi_u(v)=(u+v)^\alpha-u^\alpha$ for $u\in\Omega(\delta)$ and $|v|\le\gamma|u|^\alpha$. Then, since $\alpha=3/2$, we have that

$$\begin{aligned}|\varphi_u(v)| &\le |v|\sup\{\alpha|u+\xi|^{\alpha-1},\ |\xi|\le\gamma|u|^\alpha\} \\ &\le \alpha|v||u|(1+\gamma|u|^{\alpha-1}) \\ &\le \alpha\gamma|u|^{5/2}(1+\gamma|u|^{\alpha-1})\end{aligned}$$

if δ and is small enough. This bound is true for all u, v, and γ such that $u\in\Omega(\delta)$, $|v|\le\gamma|u|^\alpha$. We take

$$\begin{aligned} &u=(c-\sqrt{\mu})(x+ax^\beta), \\ &v=y-c_2x^\alpha(1+ax^{\beta-1})^\alpha+r_3^1(x+ax^\beta,y-c_2(x+ax^\beta)^\alpha-2\sqrt{\mu}(x+ax^\beta)), \\ &\gamma=2(\gamma_0+c_2).\end{aligned}$$

With this choice, since $a<0$, $u\in\Omega(\delta)$ and

$$|v|\le|y|+c_2|x+ax^\beta|^\alpha+O_2\le 2\gamma_0|x|^\alpha+2c_2|x|^\alpha\le\gamma|x|^\alpha$$

we have that, there exists a constant K such that

$$|R_2(x,y)|=|\varphi_u(v)|+O_{5/2}\le K|u|^{5/2} \tag{5.2.31}$$

if δ is small enough. We denote $\theta=\arg(x)$ and we recall that

$$\begin{aligned}\mathrm{Re}\left(c-\sqrt{\mu}+\frac{R_0(x)}{x}\right) &= c-\sqrt{\mu}+\sqrt{\mu}a(\beta-1)\cos(\theta(\beta-1))|x|^{\beta-1} \\ &\quad -[c_2+\beta a^2]\cos(\theta(\alpha-1))|x|^{\alpha-1} \\ &\le c-\sqrt{\mu}+\sqrt{\mu}a(\beta-1)\frac{\sqrt{2}}{2}|x|^{\beta-1}-[c_2+\beta a^2]\frac{\sqrt{2}}{2}|x|^{\alpha-1} \\ \mathrm{Im}\left(c-\sqrt{\mu}+\frac{R_0(x)}{x}\right) &= \sqrt{\mu}a(\beta-1)\sin(\theta(\beta-1))|x|^{\beta-1} \\ &\quad -[c_2+\beta a^2]\sin(\theta(\alpha-1))|x|^{\alpha-1}\end{aligned}$$

Therefore, since $3\beta - 2 = 7/4$ and by (5.2.30) and (5.2.31), we have that

$$\begin{aligned} |G_1(x,y)| &\leq |x||c - \sqrt{\mu} + a(\beta-1)\sqrt{\mu}x^{\beta-1} - [c_2 + a^2\beta]x^{\alpha-1}| \\ &\quad + \gamma_0|x|^\alpha + C|x|^\beta\left\{\mu + |x|^{\alpha-\beta}\sqrt{\mu}\right\} + O_{7/4} \\ &\leq |x|\Big[c - \sqrt{\mu} + \sqrt{\mu}|x|^{\beta-1}\Big\{a(\beta-1)\frac{\sqrt{2}}{2} + O(\sqrt{\mu}) + O_{1/4}\Big\} \\ &\quad + |x|^{\alpha-1}\Big\{-[c_2 + \beta a^2]\frac{\sqrt{2}}{2} + \gamma_0\Big\} + O_{2\alpha-2}\Big] \\ &< |x| \end{aligned} \tag{5.2.32}$$

if we choose $a < 0$ and γ_0 such that

$$-[c_2 + \beta a^2]\frac{\sqrt{2}}{2} + \gamma_0 < 0. \tag{5.2.33}$$

Now we must see that $|\arg(G_1(x,y))| < \frac{\pi}{4(\alpha-1)}$. It is sufficient to consider $x \in \partial\Omega(\delta)$. We consider the case that $\arg(x) = \frac{\pi}{4(\alpha-1)}$, the other cases are analogous. Since

$$\sin(\theta(\beta-1)) = \sin\frac{\pi}{8} > 0$$

we have that

$$\begin{aligned} \mathrm{Im}\,(G_1(x,y)/x) &\leq \sqrt{\mu}a(\beta-1)\sin(\theta(\beta-1))|x|^{\beta-1} - [c_2 + \beta a^2]\frac{\sqrt{2}}{2}|x|^{\alpha-1} \\ &\quad + \gamma_0|x|^{\alpha-1} + C|x|^{\beta-1}\left\{\mu + |x|^{\alpha-\beta}\sqrt{\mu}\right\} + O_{3/4} \\ &< 0 \end{aligned}$$

if δ and μ are small enough, $a < 0$ and γ_0 satisfies (5.2.33). Hence

$$\arg(G_1(x,y)) = \arg(x) + \arg(G_1(x,y)/x) < \arg(x)$$

We note that we can choose γ_0 is independent of μ.

For the second property we estimate the expressions of $\tilde{T}_0$, $\tilde{T}_1$, $\tilde{T}_2$ and $\tilde{T}_3$ given in (5.2.28). For $\tilde{T}_0$ we have that

$$\begin{aligned} |\tilde{T}_0| &\leq c_2|x|^\alpha|c - \sqrt{\mu} - (c - \sqrt{\mu})^\alpha| \\ &= c_2|x|^\alpha|1 + \sqrt{\mu} + O(\mu) - (1 - \alpha\sqrt{\mu} + O(\mu))| \\ &= c_2\sqrt{\mu}|x|^\alpha(1 + \alpha + O(\mu)). \end{aligned} \tag{5.2.34}$$

We recall that $\alpha = 3/2$ and that $c_2^2 = \frac{a_{20}-2\sqrt{\mu}a_{11}}{\alpha(c-\sqrt{\mu})^{\alpha-1}}$. Then for T_1 we have that

$$\begin{aligned}
\tilde{T}_1 &= (c-1+\sqrt{\mu})y + x^2[a_{20} - 2\sqrt{\mu}a_{11} - c_2^2\alpha(c-\sqrt{\mu})^{\alpha-1}] \\
&\quad + yx^{\alpha-1}[c_2\alpha(c-\sqrt{\mu})^{\alpha-1} + a_{11}x^{2-\alpha}] - a_{11}x^{\alpha+1} \\
&= (c-1+\sqrt{\mu})y + yx^{\alpha-1}[c_2\alpha(c-\sqrt{\mu})^{\alpha-1} + a_{11}x^{2-\alpha}] - a_{11}x^{\alpha+1}. \quad (5.2.35)
\end{aligned}$$

We recall that $c = 1 + O(\mu)$. Now we bound the expression for $\tilde{T}_1$ and we obtain:

$$\begin{aligned}
|\tilde{T}_1| &\leq (c-1+\sqrt{\mu})\gamma_0|x|^\alpha + \gamma_0|x|^{2\alpha-1}[c_2\alpha(c-\sqrt{\mu})^{\alpha-1} + |a_{11}||x|^{2-\alpha}] + |a_{11}||x|^{\alpha+1} \\
&= (c-1+\sqrt{\mu})\gamma_0|x|^\alpha + \gamma_0|x|^2[c_2\alpha(1+O(\sqrt{\mu})) + O_{1/2}] \\
&= \gamma_0|x|^\alpha[O(\sqrt{\mu}) + O_{1/2}]. \quad (5.2.36)
\end{aligned}$$

To evaluate $\tilde{T}_2$ we consider the function $\varphi_u(v) = (u+v)^\alpha - u^\alpha - \alpha u^{\alpha-1}v$ for $u \in \Omega(\delta)$ and $|v| \leq \gamma|u|^\alpha$. Then, since $\alpha = 3/2$, we have that

$$\begin{aligned}
|\varphi_u(v)| &\leq \frac{1}{2}|v|^2 \sup\{\alpha(\alpha-1)|u+\xi|^{\alpha-2},\ |\xi| \leq \gamma|u|^\alpha\} \\
&\leq \frac{1}{2}|v|^2\alpha(\alpha-1)\frac{1}{|u|^{1/2}(1-\gamma|u|^{1/2})^{1/2}} \\
&\leq \frac{1}{2}\alpha(\alpha-1)|u|^{5/2}\frac{\gamma^2}{(1-\gamma|u|^{1/2})^{1/2}} \\
&\leq \alpha(\alpha-1)\gamma^2|u|^{5/2} \quad (5.2.37)
\end{aligned}$$

if δ and is small enough. This bound is true for all u, v, and γ such that $u \in \Omega(\delta)$, $|v| \leq \gamma|u|^\alpha$. We take

$$\begin{aligned}
u &= (c-\sqrt{\mu})x, \\
v &= y - c_2x^\alpha + r_3^1(x, y - c_2x^\alpha - 2\sqrt{\mu}x), \\
\gamma &= 2(\gamma_0 + c_2).
\end{aligned}$$

With this choice, $u \in \Omega(\delta)$,

$$|v| \leq |y| + c_2|x|^\alpha + O_2 \leq 2\gamma_0|x|^\alpha + c_2|x|^\alpha + O_2 \leq \gamma|x|^\alpha$$

Using estimate (5.2.37), we obtain

$$|\tilde{T}_2| \leq c_2\alpha(\alpha-1)(c-\sqrt{\mu})^{5/2}\gamma^2|x|^{5/2}. \quad (5.2.38)$$

Finally, it is clear that

$$|\tilde{T}_3| \leq \varepsilon|x|^2.$$

Collecting the bounds (5.2.34), (5.2.36) and (5.2.38), and using that $T_0(x) = \tilde{T}_0(x+ax^\beta)$ and $T_i(x,y) = T_i(x+ax^\beta, t)$ for $i = 1,2,3$, we obtain

$$\begin{aligned}
|G_2(x,y) - y| &\leq |T_0| + |T_1| + |T_2| + |T_3| \\
&\leq c_2\sqrt{\mu}|x+ax^\beta|^\alpha((1+\alpha) + O(\mu)) \\
&\quad +\gamma_0|x+ax^\beta|^\alpha[O(\sqrt{\mu}) + O_{1/2}] \\
&\quad +c_2\alpha(\alpha-1)\gamma^2(c-\sqrt{\mu})^{5/2}|x+ax^\beta|^{5/2} + O_2 \\
&\leq \gamma_0|x|^\alpha[O(\sqrt{\mu}) + O_{1/2}] \leq |y|
\end{aligned}$$

if we take μ, δ small enough .

Finally we prove the third bound. Let $x \in \Omega(\delta)$ and let y be such that $|y| = \gamma_0|x|^\alpha$. We must see that

$$\gamma_0|G_1(x,y)|^\alpha < |G_2(x,y)|.$$

We write $G_2 = (c+\sqrt{\mu})y + T_0 + T_1 + T_2 + T_3$ as before. We observe that by bound (5.2.38) and since $|y| = \gamma_0|x|^\alpha$

$$\left|\frac{T_2}{y} + \frac{T_3}{y}\right| \leq \frac{c_2\alpha(\alpha-1)(c-\sqrt{\mu})^{5/2}\gamma^2|x+ax^\beta|^{5/2}}{\gamma_0|x|^\alpha} + o_{\alpha-1} = o_{1/2}.$$

Since

$$\begin{aligned}
&\mathrm{Im}(c+\sqrt{\mu}+x^{\alpha-1}[c_2\alpha(1+O(\sqrt{\mu})) + O_{1/2}]) = O_{\alpha-1} \\
&\mathrm{Re}(c+\sqrt{\mu}+x^{\alpha-1}[c_2\alpha(1+O(\sqrt{\mu})) + O_{1/2}]) = \\
&\qquad c+\sqrt{\mu}+c_2\alpha(1+O(\sqrt{\mu})+O_{1/2})|x|^{\alpha-1}\cos\theta(\alpha-1),
\end{aligned}$$

we have that

$$\begin{aligned}
&|c+\sqrt{\mu}+x^{\alpha-1}[c_2\alpha(1+O(\sqrt{\mu})) + O_{1/2}]| \\
&\qquad\qquad \geq c+\sqrt{\mu}+c_2\alpha|x|^{\alpha-1}\sqrt{2}/2(1+O(\sqrt{\mu})) + O_1.
\end{aligned}$$

Now we bound $|1+T_1/y|$, from (5.2.35) we have that

$$\begin{aligned}
\left|1+\frac{T_1}{y}\right| &\geq |c+\sqrt{\mu}+(x+ax^\beta)^{\alpha-1}[c_2\alpha(1+O(\sqrt{\mu})) + O_{1/2}]| \\
&\quad -\frac{1}{|y|}a^\alpha|a_{11}(x+ax^\beta)^{\alpha+1}| \\
&\geq 1+\sqrt{\mu}+O(\mu)+c_2\alpha|x|^{\alpha-1}(\sqrt{2}/2+O(\sqrt{\mu})+O_{1/4}).
\end{aligned}$$

We note that, if $|a|$ is big enough, we can choose

$$\gamma_0 > 2c_2(\alpha+1), \tag{5.2.39}$$

therefore, collecting all the bounds that we do, we obtain,

$$\begin{aligned}
|G_2(x,y)| &\geq |y|\left(\left|1+\frac{T_1}{y}\right| - \left|\frac{T_2}{y}+\frac{T_3}{y}\right|\right) - |T_0| \\
&\geq |y|\Big[1+\sqrt{\mu}+O(\mu) \\
&\quad +c_2\alpha|x|^{\alpha-1}(\sqrt{2}/2+O(\sqrt{\mu})+O_{1/4})\Big] - |T_0| \\
&= \gamma_0|x|^\alpha[1+\sqrt{\mu}+O(\mu)+O_{1/4}] - c_2\sqrt{\mu}|x|^\alpha((1+\alpha)+O(\mu)) \\
&\geq \gamma_0|x|^\alpha\big(1+\sqrt{\mu}-\frac{c_2}{\gamma_0}\sqrt{\mu}((1+\alpha)+O(\mu)+O_{1/4})\big) \\
&> \gamma_0|x|^\alpha
\end{aligned}$$

if we choose $|a|$ big enough (see (5.2.39)) and μ, δ small enough. Therefore, since for all $x \in \Omega(\delta)$ and $|y| \leq \gamma_0|x|^a$

$$|G_1(x,y)| < |x|$$

(see (5.2.32)), we have that, if $|y| = \gamma_0|x|^\alpha$ and $x \in \Omega(\delta)$,

$$\gamma_0|G_1(x,y)|^\alpha < \gamma_0|x|^\alpha = |y| < |G_2(x,y)|.$$

■

Proof of the Theorem 5.2.15

We already have all that we need to prove the Theorem 5.2.15. The proof is as [30]. We work with the function G^η defined by (5.2.29). For convenience we write $G^\eta(x,y) = G(x,y,\mu,\varepsilon)$. We consider the set the functions H such that, $h : \Omega(\delta)\times V_\eta \to \mathbb{C}$ belongs to H if and only if

(a) h is real and analytic with respect to $x \in \Omega(\delta)$.

(b) For all $(\mu,\varepsilon) \in V_\eta$, h is real analytic with respect to μ and continuous with respect to ε.

(c) For all $x \in \Omega(\delta)$ and $\eta \in V_\eta$, $|h(x,\eta)| \leq \gamma_0|x|^\alpha$. We recall that γ_0 was introduced in the preliminaries.

For any $\varepsilon \in V_\varepsilon$ fixed, we define $\Gamma_\varepsilon : H \to H$ implicitly by

$$G_2(x, (\Gamma_\varepsilon h)(x), \mu, \varepsilon) - h(G_1(x, (\Gamma_\varepsilon h)(x), \mu, \varepsilon), \mu) = 0$$

for $h \in H$. We introduce the function

$$\Delta_\varepsilon(x, w, \mu) = G_2(x, w, \mu, \varepsilon) - h(G_1^\eta(x, w, \mu, \varepsilon), \mu, \varepsilon)$$

and the set

$$\Omega_1 = \{(x, w, \mu) \in \mathbb{C}^2 \times W_\mu : x \in \Omega(\delta),\ |w| < \gamma_0 |x|^\alpha,\ \mu \in W_\mu\}.$$

Then, by Lemma 5.2.16, Δ is analytic on Ω_1. The second estimate of the Lemma 5.2.16 implies, by Rouche's Theorem, that the functions $\tilde{G}_2(w) \equiv w$ and $G_2(x_0, w, \mu_0, \varepsilon)$ have the same number of zeros in the disc

$$D(x_0) = \{w \in \mathbb{C},\ |w| < \gamma_0 |x|^\alpha\}$$

if we fix any $x_0 \in \Omega(\delta)$ and $\mu_0 \in W_\mu$. Another application of Rouche's Theorem gives that the functions $\Delta_\varepsilon(x_0, w, \mu_0)$ and $G_2(x_0, w, \mu_0, \varepsilon)$ also have the same number of zeros in $D(x_0)$. Thus, we can solve uniquely the equation $\Delta_\varepsilon(x_0, w, \mu_0) = 0$ for w, for any $x_0 \in \Omega(\delta)$ and $\mu_0 \in W_\mu$ fixed. Moreover, the implicit function theorem and the uniqueness guarantees the analyticity of $\Gamma_\varepsilon(h)$ with respect to (x, μ). We observe that this function is real for real values of (x, μ).

From the estimates in Lemma 5.2.16 one can prove that $\Gamma_\varepsilon(h) \in H$ and using Montel's Theorem, if we fix any initial condition h_0 the iterates $h^k = \Gamma_\varepsilon^k h_0$ must have a subsequence converging to some $\bar{h} \in H$. The points on the graph of this function converge to the origin by G, and then by uniqueness, it must coincides with φ_η.

Now we prove the continuity with respect to ε. Since the function φ_η is real and analytic in x and since $\Omega(\delta)$ is compact, we can restrict ourselves to the real part of the domain and in the real case the continuity is proved in Subsection 5.2.6.

5.3 Local invariant manifolds in the parabolic case

In this section we want to prove that the local stable invariant curve of the system (5.1.1) can be expressed as the graph of a function. For this we deal with its Poincaré map defined by

$$P_\mu^{t_0/\varepsilon}(x, y) = \psi_{\mu,\varepsilon}(t_0 + 2\pi\varepsilon, t_0, x, y)$$

whhere $\psi_{\mu,\varepsilon}$ is the solution of (5.1.1). We observe that, in fact, the stable curve of $P_\mu^{t_0/\varepsilon}$ is the intersection with the plane $t = t_0$ of the local stable curve of the system (5.1.1).

We recall that the Poincaré maps given in (3.4.7) and (3.5.3) have the form

$$\begin{aligned} P_\mu^\theta(x,y) &= \begin{pmatrix} x + 2\pi\varepsilon y \\ y \end{pmatrix} + 2\pi\varepsilon \begin{pmatrix} 2\pi\varepsilon q_1(x,y,\varepsilon) \\ -V'(x) + 2\pi\varepsilon q_2(x,y,\varepsilon) \end{pmatrix} \\ &\quad + \mu\varepsilon^{p+2n+3}\psi_{\mu,\varepsilon}(x,y,\theta,\mu,\varepsilon) + \mu^2\varepsilon^{p+3}R_{2k-3}(x,y,\theta,\mu,\varepsilon) \end{aligned} \tag{5.3.1}$$

where $\theta = t/\varepsilon$, $V'(x)$, q_1, $q_2 \in P_{n-1}$(independent of μ), $\psi_{\mu,\varepsilon} \in P_{2n-3}$ and $R_{2k-3} \in P_{2k-3} \times P_{2k-3}$.

In [30] maps of the form

$$F_\eta(x,y) = \begin{pmatrix} x + cy + f(x,y,\eta) \\ y + g(x,y,\eta) \end{pmatrix}$$

with $c \neq 0$, are considered. The existence of local stable curves as graphs of analytic functions in x, and analytically dependent of parameter η is proved. After some changes of variables, the following normal form for $F_\eta(x,y)$ is obtained:

$$N_\eta(x,y) = \begin{pmatrix} x+y \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ x^k p(x,\eta) + x^{l-1} y q(x,\eta) \end{pmatrix} + \begin{pmatrix} r_n^1(x,y,\eta) \\ r_n^2(x,y,\eta) \end{pmatrix} \tag{5.3.2}$$

with $2 \le k, l \le n$, and $p(x,\eta) = a_k + a_{k+1}x + \cdots + a_n x^{n-k}$and $q(x,\eta) = b_l + b_{l+1}x + \cdots + b_n x^{n-l}$. Let

$$\Omega(\delta) = \left\{ x \in \mathbb{C} \,:\, 0 < |x| < \delta \;,\; |\arg(x)| < \frac{\pi}{2(k-1)} \right\}.$$

The following theorem is proven in [30].

Theorem 5.3.1 *Let $N_\eta : U \subset \mathbb{C}^2 \to \mathbb{C}^2$, $0 \in U$, $\eta \in V_\eta \subset \mathbb{C}$, $0 \in V_\eta$, be a family of analytic maps, depending analytically on η, of the form (5.3.2) satisfying the condition $l > \frac{k+1}{2}$ and $a_k > 0$. Then*

1) *if δ is small enough the right hand branch of the stable curve is the graph of a function φ analytic on $\Omega(\delta)$, depending analytically on η.*

2) *φ has the following asymptotic expression when $|x| \to 0$:*

$$\varphi(x,\eta) = \sqrt{2a_k/(k+1)}x^\beta + h.o.t., \qquad \beta = (k+1)/2.$$

Remark 5.3.2 *This is only one of the cases considered in [30], but it is sufficient for our purposes.*

If we apply Theorem 5.3.1 to P_μ^θ we will obtain that the domain of φ, $\Omega(\delta)$, will depend on ε, because the linear part of P_μ^θ do depend on ε. We can not exclude the possibility that $\delta \to 0$ when $\varepsilon \to 0$.

We will state a suitable modification of Theorem 5.3.1 which gives the existence of local invariants manifolds of the Poincaré maps given in (3.4.7) and (3.5.3) as the graph of a function in a uniform domain with respect to parameters. We are led to consider maps of the form

$$F_{\mu,\varepsilon}(x,y) = \begin{pmatrix} x + \varepsilon y + \varepsilon p_1(x,y,\mu,\varepsilon) \\ y + \varepsilon p_2(x,y,\mu,\varepsilon) \end{pmatrix} \tag{5.3.3}$$

with p_1, $p_2 \in P_k$, continuous in ε and analytic in x, y and μ. From now on, if there is not danger of confusion, we omit the dependence on μ. We perform the linear change of variables $C_1(x,y) = (\varepsilon x, y)$ in order to put the linear part in the form

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Then in the new variables, which we rename by (x,y), the map $F_{\mu,\varepsilon}$ reads as

$$F_{\mu,\varepsilon}(x,y) = \begin{pmatrix} x + y + p_1(\varepsilon x, y) \\ y + \varepsilon p_2(\varepsilon x, y) \end{pmatrix}.$$

We write

$$\begin{aligned} p_1(\varepsilon x, y) &= \sum_{i+j=k} f_{i,j} \varepsilon^i x^i y^j + R_k^1(\varepsilon x, y) \\ \varepsilon p_2(\varepsilon x, y) &= \sum_{i+j=k} g_{i,j} \varepsilon^{i+1} x^i y^j + \varepsilon R_k^2(\varepsilon x, y) \end{aligned}$$

where $R_k^l(x,y) = o(\|(x,y)\|^k)$.

Lemma 5.3.3 *There exists a change of variables C_2 of the form*

$$C_2(\xi,\eta) = \begin{pmatrix} \xi + \Phi(\xi,\eta) \\ \eta + \Psi(\xi,\eta) \end{pmatrix} = \begin{pmatrix} \xi + \sum_{i+j=k} \Phi_{ij} \xi^i \eta^j \\ \eta + \sum_{i+j=k} \Psi_{ij} \xi^i \eta^j \end{pmatrix}$$

such that

$$C_2 \circ N_{\mu,\varepsilon} = F_{\mu,\varepsilon} \circ C_2 \tag{5.3.4}$$

with

$$N_{\mu,\varepsilon}(\xi,\eta) = \begin{pmatrix} \xi + \eta + r_k^1(\varepsilon\xi,\eta) \\ \eta + \varepsilon^{k+1} g_{k,0}\xi^k + \varepsilon^k(\varepsilon g_{k-1,1} + k f_{k,0}) x^{k-1} y + \varepsilon r_k^2(\varepsilon\xi,\eta) \end{pmatrix}. \tag{5.3.5}$$

Moreover the coefficients $\Phi_{i,k-i} = O(\varepsilon^{i-1})$ *and* $\Psi_{i,k-i} = O(\varepsilon^i)$ *for* $0 < i < k$ *and* $\Phi_{0,k} = \Psi_{0,k} = 0$.

Proof. As in [30], we write

$$N_{\mu,\varepsilon}(\xi,\eta) = \begin{pmatrix} \xi + \eta + h^1(\xi,\eta) \\ \eta + h^2(\xi,\eta) \end{pmatrix} + \begin{pmatrix} r_k^1(\xi,\eta) \\ r_k^2(\xi,\eta) \end{pmatrix}$$

with

$$h^l(\xi,\eta) = \sum_{i+j=k} h_{ij}^l \xi^i \eta^j \qquad \text{for } l = 1,2$$

and $r_k^l = o(\|(\xi,\eta)\|^k)$. Collecting the terms of order k from the equality (5.3.4), we obtain that the coefficients Φ_{ij}, Ψ_{ij} and h_{ij}^l must be satisfy the following systems of linear equations:

$$\begin{pmatrix} 0 & & & & 0 \\ \binom{k}{1} & 0 & & & 0 \\ \binom{k}{2} & \binom{k-1}{1} & 0 & & \\ \vdots & & & & \\ \binom{k}{k} & \binom{k-1}{k-1} & \binom{k-2}{k-2} & \dots & 0 \end{pmatrix} \begin{pmatrix} \Psi_{k,0} \\ \Psi_{k-1,1} \\ \Psi_{k-2,2} \\ \vdots \\ \Psi_{0,k} \end{pmatrix} = \begin{pmatrix} \varepsilon^{k+1} g_{k,0} - h_{k,0}^2 \\ \varepsilon^k g_{k-1,1} - h_{k-1,1}^2 \\ \varepsilon^{k-1} g_{k-2,2} - h_{k-2,2}^2 \\ \vdots \\ \varepsilon g_{0,k} - h_{0,k}^2 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & & & & 0 \\ \binom{k}{1} & 0 & & & 0 \\ \binom{k}{2} & \binom{k-1}{1} & 0 & & \\ \vdots & & & & \\ \binom{k}{k} & \binom{k-1}{k-1} & \binom{k-2}{k-2} & \dots & 0 \end{pmatrix} \begin{pmatrix} \Phi_{k,0} \\ \Phi_{k-1,1} \\ \Phi_{k-2,2} \\ \vdots \\ \Phi_{0,k} \end{pmatrix} = \begin{pmatrix} \varepsilon^k f_{k,0} - h_{k,0}^1 + \Psi_{k,0} \\ \varepsilon^{k-1} f_{k-1,1} - h_{k-1,1}^1 + \Psi_{k-1,1} \\ \varepsilon^{k-2} f_{k-2,2} - h_{k-2,2}^1 + \Psi_{k-2,2} \\ \vdots \\ f_{0,k} - h_{0,k}^1 + \Psi_{0,k} \end{pmatrix}.$$

We impose $h^2_{k,0} = \varepsilon^{k+1} g_{k,0}$, $h^1_{i,j} = 0$, $h^2_{i,j} = 0$ if $j \neq 0, 1$ and $\Psi_{k,0} = -\varepsilon^k f_{k,0}$. Therefore, solving the first linear system we obtain that $h^2_{k-1,1} = \varepsilon^{k+1} g_{k-1,1} + k\varepsilon^k f_{k,0}$, and for all $j \neq 0$, $\Psi_{n-j,j} = O(\varepsilon^{n-j})$. Moreover, solving the second system we obtain that $\Phi_{n-j,j} = O(\varepsilon^{n-j-1})$. We observe that we can take $\Psi_{0,k} = \Phi_{0,k} = 0$.

Now we deal with the remainders r^1_k and r^2_k. We write $\Phi(\xi, \eta) = \sum_{i+j=k} \Phi_{ij} \xi^i \eta^j$ and $\Psi(\xi, \eta) = \sum_{i+j=k} \Psi_{ij} \xi^i \eta^j$ and we note that

$$\begin{aligned} C_2^{-1} \circ F_{\mu,\varepsilon} \circ C_2(\xi, \eta) &= C_2^{-1} \begin{pmatrix} \xi + \eta + \Phi + \Psi + p_1(\varepsilon(\xi + \Phi), \eta + \Psi) \\ \eta + \Psi + \varepsilon p_2(\varepsilon(\xi + \Phi), \eta + \Psi) \end{pmatrix} \\ &= \begin{pmatrix} \xi + \eta + h^1(\xi, \eta) + r^1_k(\varepsilon\xi, \eta) \\ \eta + h^2(\xi, \eta) + \varepsilon r^1_k(\varepsilon\xi, \eta) \end{pmatrix}. \end{aligned}$$

This proves the lemma. ■

We perform the change of variables $C_1^{-1}(\xi, \eta) = (\xi/\varepsilon, \eta)$. Then the map (5.3.5) takes the form

$$\begin{aligned} \tilde{N}_{\mu,\varepsilon}(u, v) &= C_1^{-1} \circ N_{\mu,\varepsilon} \circ C_1(u, v) \\ &= \begin{pmatrix} u + \varepsilon v + \varepsilon r^1_k(u, v) \\ v + \varepsilon g_{k,0} u^k + \varepsilon(\varepsilon g_{k-1,1} + k f_{k,0}) u^{k-1} v + \varepsilon r^2_k(u, v) \end{pmatrix} \end{aligned} \tag{5.3.6}$$

with $g_{k,0}$, $g_{k-1,1}$, $f_{k,0}$ and $r^l_k(u, v) = o(\|(u, v)\|)^k$ depending on ε.

Proposition 5.3.4 *Let $\tilde{N}_{\mu,\varepsilon} : U \subset \mathbb{C}^2 \to \mathbb{C}^2$, such that $|\mu| < \mu_0$ and $0 \leq \varepsilon < \varepsilon_0$ be a family of analytic maps, depending continuously on ε and analytically with respect to μ of the form (5.3.6) with the condition that the coefficient $g_{k,0}(\varepsilon, \mu)$ satisfies $g_{k,0}(0, \mu) > 0$. Then there exists $\delta > 0$ independent of ε and μ such that*

1) *the stable curve is the graph of a function φ analytic on $\Omega(\delta)$, depending continuously with respect to ε and analytically with respect to μ.*

2) *The function φ has the form $\varphi(u, \varepsilon, \mu) = f(u, \varepsilon) + \mu h(u, \varepsilon, \mu)$ with*

$$f(u, \varepsilon) = \sqrt{2 g^*_{k,0}/(k+1)}\, u^{(k+1)/2} + h.o.t..$$

where $g^*_{k,0} = g_{k,0}(\varepsilon, 0)$.

Proof. The steps of the proof of the Theorem 3.1 in [30] work in this case except by one technical lemma. We must substitute Lemma 3.4 of [30] (the equivalent lemma in the weak hyperbolic case is Lemma 5.2.8) by the following statement:

Let $0 < r < 2/(k-1)$. If δ is small enough, $(u, v) \in A(\delta)$ and $0 < u \leq 1/j^r$ then

$$0 < \pi_1 \tilde{N}_{\mu,\varepsilon}(u, v) < 1/(j+\varepsilon)^r.$$

The proof of this lemma follows immediately. Hence the stable manifold can be expressed as the graph of an analytic function φ in $\Omega(\delta)$, depending analytically with respect to μ. Moreover, we can prove the continuity with respect to ε in the same way as we did in Section 5.2.6.

In order to prove 2), we observe that the function φ is analytic with respect to μ and therefore

$$\varphi(u, \varepsilon, \mu) = \varphi(u, \varepsilon, 0) + \mu \partial_\mu \varphi(u, \varepsilon, 0) + O(\mu^2)$$

which gives the result. ∎

We state a useful corollary for our purposes.

Corollary 5.3.5 *There exists $\delta > 0$ independent of ε and μ such that the map (5.3.3) has a unique stable (an unstable) local invariant manifold which can be expressed as the graph of a function*

$$\varphi^s : \Omega(\delta) \times \{\mu \in \mathbb{C} : |\mu| \leq \mu_0\} \times [0, \varepsilon_0) \to \mathbb{C}$$

which is analytic in $x \in \Omega(\delta)$, analytic with respect to μ and continuous with respect to ε. Moreover, $\varphi^s(x, \varepsilon, \mu) = f(x, \varepsilon) + \mu g(x, \varepsilon, \mu)$.

Proof. We must go back to the original variables. It is clear that

$$F_{\mu,\varepsilon} = C_1^{-1} \circ C_2 \circ C_1 \circ \tilde{N}_{\mu,\varepsilon} \circ C_1^{-1} \circ C_2^{-1} \circ C_1.$$

Then if $(x, y) = C(u, v)$ where $C = C_1^{-1} \circ C_2 \circ C_1$, we have

$$\begin{aligned} \begin{pmatrix} x \\ y \end{pmatrix} &= C_1^{-1} \circ C_2 \begin{pmatrix} u/\varepsilon \\ v \end{pmatrix} = C_1^{-1} \begin{pmatrix} u/\varepsilon + \Phi(u/\varepsilon, v) \\ v + \Psi(u/\varepsilon, v) \end{pmatrix} \\ &= \begin{pmatrix} u + \varepsilon\Phi(u/\varepsilon, v) \\ v + \Psi(u/\varepsilon, v) \end{pmatrix} \end{aligned}$$

and by Lemma 5.3.3 (in particular by the conclusion on the orders of the coefficients $\phi_{i,k-i}$ and $\psi_{i,k-i}$ with respect to ε), the functions

$$\tilde{\Phi}(u, v, \varepsilon) \equiv \varepsilon\Phi(u/\varepsilon, v) \qquad \tilde{\Psi}(u, v, \varepsilon) \equiv \Psi(u/\varepsilon, v)$$

are of the form

$$\tilde{\Phi}(u,v,\varepsilon) = \sum_{i+j=k} \tilde{\Phi}_{i,j}(\varepsilon)u^i v^j$$
$$\tilde{\Psi}(u,v,\varepsilon) = \sum_{i+j=k} \tilde{\Psi}_{i,j}(\varepsilon)u^i v^j$$

where the coefficients $\tilde{\Phi}_{i,j}(\varepsilon)$ and $\tilde{\Psi}_{i,j}(\varepsilon)$ are continuous at $\varepsilon = 0$. Thus the change C is of the form

$$C(u,v) = (u + \tilde{\Phi}(u,v,\varepsilon), v + \tilde{\Psi}(u,v,\varepsilon)) \tag{5.3.7}$$

where $\tilde{\Phi}$ and $\tilde{\Psi}$ are of order k in (u,v) and continuous at $\varepsilon = 0$.

Since the local stable invariant manifold of $\tilde{N}_{\mu,\varepsilon}$ is the graph of φ and the change of variables C is of the form (5.3.7), the stable manifold of $F_{\mu,\varepsilon}$ can be expressed as the graph of a function φ^s. Moreover, expanding, using Taylor's theorem, with respect to μ at $\mu = 0$ we have

$$\varphi^s(x,\varepsilon,\mu) = f(x,\varepsilon) + \mu g(x,\varepsilon,\mu).$$

■

Proof of Proposition 5.1.1

We perform the change of variables $\tilde{C}_1(x,y) = (x/2\pi, y)$ and, since $2k-3 \geq n-1$, it is obvious that the Poincaré maps given in (3.4.7) and (3.5.3) have the form of (5.3.3). With slight changes in Corollary 5.3.5 we obtain that the local stable manifolds of the Poincaré map (5.3.1) and the Poincaré map of the auxiliary system given in (3.5.3) can be expressed as the graph of a function. Therefore, the stable manifold of systems (5.1.1) and (3.7.4) can be written as graphs from the graphs of their respective Poincaré maps.

We introduce the new parameter η. Later we will evaluate η at $\mu\varepsilon^{p+2}$, but for the moment it is useful to consider η as an independent parameter.We define the map

$$G(x,y,\theta,\eta,\mu,\varepsilon) = \tilde{C}_1^{-1} P_\mu^\theta(\tilde{C}_1(x,y))$$

where $\tilde{C}_1(x,y) = (x/2\pi, y)$. We observe that G has the form

$$G(x,y,\theta,\eta,\mu,\varepsilon) = \begin{pmatrix} x + \varepsilon y + \varepsilon p_1(x,y,\varepsilon) \\ y + \varepsilon p_2(x,y,\varepsilon) \end{pmatrix} + \eta\varepsilon r(x,y,\theta,\eta,\mu,\varepsilon)$$

where p_1, p_2, $r \in P_{n-1}$. This map is C^0, and analytic in x, y, μ and η. Applying Corollary 5.3.5 to G we obtain that the stable manifold of G can be expressed as the graph of a function ψ of the form

$$\bar{\varphi}(x,\theta,\eta,\mu,\varepsilon) = f(x,\varepsilon) + \eta g_1(x,\theta,\eta,\mu,\varepsilon) + \mu g_2(x,\theta,\eta,\mu,\varepsilon)$$

analytic with respect to x, η and μ. But, since $G(x,y,\theta,0,\mu,\varepsilon)$ does not depend on μ, by uniqueness of the stable manifold, $g_2(x,\theta,0,\mu,\varepsilon) = 0$, and, since ψ is analytic with respect to η, g_2 can be written as

$$g_2(x,\theta,\eta,\mu,\varepsilon) = \eta\tilde{g}_2(x,\theta,\eta,\mu,\varepsilon).$$

Hence, the function ψ is of the form

$$\bar{\varphi}(x,\theta,\eta,\mu,\varepsilon) = f(x,\varepsilon) + \eta\bar{g}(x,\theta,\eta,\mu,\varepsilon).$$

Therefore,

$$\begin{aligned}\bar{\varphi}(x,\theta,\mu\varepsilon^{p+2},\mu,\varepsilon) &= f(x,\varepsilon) + \mu\varepsilon^{p+2}\bar{g}(x,\theta,\mu\varepsilon^{p+2},\mu,\varepsilon)\\ &\equiv f(x,\varepsilon) + \mu\varepsilon^{p+2}g(x,\theta,\mu,\varepsilon)\end{aligned}$$

is the stable manifold of $\tilde{C}_1^{-1} \circ P_\mu^\theta \circ \tilde{C}_1$ which is C^0, analytic with respect to (x,μ) in $\Omega(\delta,\mu_0)$ and, by the fact that P_μ^θ is 2π-periodic with respect to θ and by the uniqueness of the local stable manifold it is 2π-periodic with respect to θ.

Going back to the original variables, the stable manifold of P_μ^θ is of the same form

$$\varphi_{\mu,\varepsilon}(x,\theta) = f(x,\varepsilon) + \mu\varepsilon^{p+2}g(x,\theta,\mu,\varepsilon).$$

Moreover we know that, if $\mu = 0$, the stable manifold of $P_0^\theta\circ$ does not depend on ε, therefore, by the uniqueness of the stable manifold, $f(x,\varepsilon) = -\sqrt{-2V(x)}$ which is the corresponding to the unperturbed system: f does not depend on ε.

In order to prove that g is C^1 in θ, we introduce

$$\psi_{\mu,\varepsilon}(t,t_0,x,y) = (\psi^1_{\mu,\varepsilon}(t,t_0,x,y), \psi^2_{\mu,\varepsilon}(t,t_0,x,y)),$$

the solution of the system (5.1.1). We observe that, for all θ,

$$\varphi_{\mu,\varepsilon}(\psi^1_{\mu,\varepsilon}(\varepsilon\theta,0,x,\varphi_{\mu,\varepsilon}(x,0)),\theta) = \psi^2_{\mu,\varepsilon}(\varepsilon\theta,0,x,\varphi_{\mu,\varepsilon}(x,0)).$$

We invert, with respect to x, the function

$$u = \psi^1_{\mu,\varepsilon}(\varepsilon\theta,0,x,\varphi_{\mu,\varepsilon}(x,0))$$

which depends C^1 with respect to θ and we obtain a function with the same kind of dependence with respect to θ:

$$x = h(u,\theta).$$

Therefore, we can obtain an explicit expression of $\varphi_{\mu,\varepsilon}(u,\theta)$ from $\varphi_{\mu,\varepsilon}(u,0)$:

$$\varphi_{\mu,\varepsilon}(u,\theta) = \psi^2_{\mu,\varepsilon}(\varepsilon\theta, 0, h(u,\theta), \varphi_{\mu,\varepsilon}(h(u,\theta),0))$$

which depends C^1 with respect to θ. Therefore, by the uniqueness of the local stable invariant manifold, the result holds.

5.4 Local invariant manifolds in the weak hyperbolic case

Now we prove that there exists an analytic function ψ_μ such that the local stable (unstable) manifold of the system (5.1.2) or equivalently of the map given by (4.4.2), can be expressed as the graph of this function. As in the previous section we use a suitable modification of Theorem 5.2.15 in order to prove the existence of invariant curves of the Poincaré map (4.4.2). We recall that the Poincaré map (4.4.2) has the form

$$\begin{aligned} P^\theta_\mu(x,y) \;=\; & \begin{pmatrix} 1+\mu^2\varepsilon^{2p+2}(c_{13}+\varepsilon c_1) & 2\pi\varepsilon + c_4\mu^2\varepsilon^{2p+2} \\ c_2\mu^2\varepsilon^{2p+2} & 1+\mu^2\varepsilon^{2p+2}(-c_{13}+\varepsilon c_3) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \\ & +2\pi\varepsilon \begin{pmatrix} 2\pi\varepsilon q_1(x,y,\varepsilon) \\ -V'(x)+2\pi\varepsilon q_2(x,y,\varepsilon) \end{pmatrix} + \mu\varepsilon^{p+9}\psi_{\mu,\varepsilon}(x,y,\theta,\mu,\varepsilon) \qquad (5.4.1) \\ & +\mu^2\varepsilon^{2p+2}R_2(x,y,\theta,\mu,\varepsilon) \end{aligned}$$

where c_{13}, c_i are constants and $q_1 \in P_2$, $q_2 \in P_2$, $\psi_{\mu,\varepsilon} \in P_3$ and $R_2 = (R^1_2, R^2_2) \in P_2$. We also recall that in Lemma 4.5.2 we prove that there exists a linear change of coordinates $C(\theta) = \mathrm{Id} + O(\mu^2\varepsilon^{2p+1})$ such that the Poincaré map has the form

$$\begin{aligned} G^{\theta_0}_\mu(u,v) \;=\; & \begin{pmatrix} \cosh(2\pi\varepsilon\rho) & \sinh(2\pi\varepsilon\rho)/\rho \\ \rho\sinh(2\pi\varepsilon\rho) & \cosh(2\pi\varepsilon\rho) \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \\ & +2\pi\varepsilon \begin{pmatrix} 2\pi\varepsilon q_1(u,v,\varepsilon) \\ -V'(x)+2\pi\varepsilon q_2(u,v,\varepsilon) \end{pmatrix} \qquad (5.4.2) \\ & +\mu\varepsilon^{p+9}\psi_{\mu,\varepsilon}(u,v,\theta_0) + \mu^2\varepsilon^{2p+2}\bar{R}_2(u,v,\theta_0) \end{aligned}$$

with $\rho = \rho(\mu, \varepsilon) = O(\mu\varepsilon^{p+1/2})$, q_1, q_2, $\psi_{\mu,\varepsilon}$ are the same that in (5.4.1) and, $\bar{R}_2 \in P_2$

This form suggests us to consider a model map, $F_{\mu,\varepsilon}$ given by

$$F_{\mu,\varepsilon}(x,y) = \begin{pmatrix} cx + \varepsilon y + \varepsilon p_1(x,y,\mu,\varepsilon) \\ \mu\varepsilon x + cy + \varepsilon p_2(x,y,\mu,\varepsilon) \end{pmatrix} \tag{5.4.3}$$

with $c = 1 + e\mu\varepsilon + O(\mu^2\varepsilon^2)$ and p_1, p_2 of order 2 in (x, y) variables. Moreover we suppose that it is continuous in ε and analytic in x, y and μ. From now on, if there are not danger of confusion, we omit the dependence on μ, ε. We perform the linear change of variables $C_1(x, y) = (\varepsilon x, y)$ in order to put the linear part in the form

$$\begin{pmatrix} c & 1 \\ \mu\varepsilon^2 & c \end{pmatrix}.$$

Then in the new variables, the map $F_{\mu,\varepsilon}$ reads as

$$F_{\mu,\varepsilon}(x,y) = \begin{pmatrix} cx + y + p_1(\varepsilon x, y) \\ \mu\varepsilon^2 x + cy + \varepsilon p_2(\varepsilon x, y) \end{pmatrix}.$$

We perform another linear change of variables $C_2(x, y) = (x, y - \varepsilon\sqrt{\mu}x)$ and then, the map $G = C_2 \circ F \circ C_2^{-1}$ is of the form

$$G_{\mu,\varepsilon}(x,y) = \begin{pmatrix} (c + \varepsilon\sqrt{\mu})x + y + q_1(\varepsilon x, y) \\ (c - \varepsilon\sqrt{\mu})y + \varepsilon q_2(\varepsilon x, y) \end{pmatrix}$$

with q_1, $q_2 \in P_2$. We observe that $G_{\mu,\varepsilon}$ is analytic with respect to $\sqrt{\mu}$ and continuous with respect to ε. We write

$$\begin{aligned} q_1(\varepsilon x, y) &= \sum_{i+j=2} f_{i,j}\varepsilon^i x^i y^j + R_2^1(\varepsilon x, y) \\ \varepsilon q_2(\varepsilon x, y) &= \sum_{i+j=2} g_{i,j}\varepsilon^{i+1} x^i y^j + \varepsilon R_2^2(\varepsilon x, y) \end{aligned}$$

where $R_2^l(x, y) = o(\|(x, y)\|^2)$. The map $G_{\mu,\varepsilon}$ has the form (5.2.4) considered in Section 5.2. Now we perform a change of variables in order to achieve a normal form.

Lemma 5.4.1 *There exists a change of variables C_3 such that*

$$C_3(\xi,\eta) = \begin{pmatrix} \xi + \Phi(\xi,\eta) \\ \eta + \Psi(\xi,\eta) \end{pmatrix} = \begin{pmatrix} \xi + \sum_{i+j=k} \Phi_{ij}\xi^i\eta^j \\ \eta + \sum_{i+j=k} \Psi_{ij}\xi^i\eta^j \end{pmatrix}$$

and

$$C_3 \circ N_{\mu,\varepsilon} = G_{\mu,\varepsilon} \circ C_3 \tag{5.4.4}$$

with

$$N_{\mu,\varepsilon}(\xi,\eta) = \begin{pmatrix} (c+\varepsilon\sqrt{\mu})\xi + \eta + r_2^1(\varepsilon\xi,\eta) \\ (c-\varepsilon\sqrt{\mu})\eta + \varepsilon^3 a_{20}\xi^2 + \varepsilon^2 a_{11}xy + \varepsilon r_2^2(\varepsilon\xi,\eta) \end{pmatrix}. \tag{5.4.5}$$

The coefficients $\Phi_{i,2-i} = O(\varepsilon^{i-1})$ *and* $\Psi_{i,2-i} = O(\varepsilon^i)$ *for* $i = 1,2$ *and* $\Phi_{0,2} = \Psi_{0,2} = 0$. *Moreover they are continuous with respect to* ε *and analytic with respect to* $\sqrt{\mu}$.

Proof. Analogously as we did in Section 5.2.2, we choose $h_{20}^1 = h_{11}^1 = h_{02}^1 = h_{02}^2 = 0$ and $\phi_{02}(\mu) = \psi_{02}(\mu) = 0$. Hence it remains to see that the solutions of the linear system

$$\begin{pmatrix} a_1(\mu,\varepsilon) & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & a_2(\mu,\varepsilon) & 0 & 1 & 0 \\ 2(c+\varepsilon\sqrt{\mu}) & a_3(\mu,\varepsilon) & 0 & -1 & 0 & 0 \\ 0 & 0 & 2(c+\varepsilon\sqrt{\mu}) & a_4(\mu,\varepsilon) & 0 & 1 \\ 1 & c-\varepsilon\sqrt{\mu} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & (c-\varepsilon\sqrt{\mu}) & 0 & 0 \end{pmatrix} \begin{pmatrix} \phi_{20} \\ \phi_{11} \\ \psi_{20} \\ \psi_{11} \\ h_{20}^2 \\ h_{11}^2 \end{pmatrix} = \begin{pmatrix} \varepsilon^2 f_{20} \\ \varepsilon^3 g_{20} \\ \varepsilon f_{11} \\ \varepsilon^2 g_{11} \\ f_{02} \\ \varepsilon g_{02} \end{pmatrix}$$

where

$$\begin{aligned} a_1(\mu,\varepsilon) &= (c+\varepsilon\sqrt{\mu})^2 - (c+\varepsilon\sqrt{\mu}) \\ a_2(\mu,\varepsilon) &= (c+\varepsilon\sqrt{\mu})^2 - (c-\varepsilon\sqrt{\mu}) \\ a_3(\mu,\varepsilon) &= c^2 - \varepsilon^2\mu - (c+\varepsilon\sqrt{\mu}) \\ a_4(\mu,\varepsilon) &= c^2 - \varepsilon^2\mu - (c-\varepsilon\sqrt{\mu}) \end{aligned}$$

give solutions of the form we have stated. We call A the matrix of the linear system and b its non-homogeneous term. We can write

$$A = A_0(\varepsilon) + \varepsilon\sqrt{\mu}B(\sqrt{\mu},\varepsilon)$$

and then the linear system can be written as

$$(A_0 + \varepsilon\sqrt{\mu}B)\zeta = b$$

with $\zeta = (\phi_{20}, \phi_{11}, \psi_{20}, \psi_{11}, h^2_{20}, h^1_{11})^T$, $b = (\varepsilon^2 f_{20}, \varepsilon^3 g_{20}, \varepsilon f_{11}, \varepsilon^2 g_{11}, f_{02}, \varepsilon g_{02})^T$. Therefore, the solutions of this linear system are

$$\zeta = A_0^{-1}(I + \varepsilon\sqrt{\mu}B)^{-1}b = A_0^{-1}b - \varepsilon\sqrt{\mu}A_0^{-1}Bb + O(\varepsilon^2\mu).$$

Therefore, as in Lemma 5.3.3, since $A_0^{-1}b$ satisfies the properties on coefficients ϕ_{ij}, ψ_{ij} and h^k_{ij} that we have stated, it is not difficult to see that $\Phi_{20} = O(\varepsilon)$, $\Phi_{11} = O(1)$, $\Psi_{20} = O(\varepsilon^2)$, $\Psi_{11} = O(\varepsilon)$, $h^2_{20} = O(\varepsilon^3)$ and $h^2_{11} = O(\varepsilon^2)$.

Now we deal with the remainders r^1_2 and r^2_2. We observe that

$$\begin{aligned} C_3^{-1} \circ G_{\mu,\varepsilon} \circ C_3(\xi,\eta) &= \begin{pmatrix} (c+\varepsilon\sqrt{\mu})(\xi+\Phi) + \eta + \Psi + q_1(\varepsilon(\xi+\Phi), \eta+\Psi) \\ (c-\varepsilon\sqrt{\mu})(\eta+\Psi) + \varepsilon q_2(\varepsilon(\xi+\Phi), \eta+\Psi) \end{pmatrix} \\ &= \begin{pmatrix} (c+\varepsilon\sqrt{\mu})\xi + \eta + h^1(\xi,\eta) + r^1_k(\varepsilon\xi,\eta) \\ (c-\varepsilon\sqrt{\mu})\eta + h^2(\xi,\eta) + \varepsilon r^1_k(\varepsilon\xi,\eta) \end{pmatrix}. \end{aligned}$$

This prove the result. ■

We perform the change of variables $(u,v) = C_1^{-1}(\xi,\eta) = (\xi/\varepsilon, \eta)$, then the map (5.4.5) takes the form

$$\begin{aligned} \tilde{N}_{\mu,\varepsilon}(u,v) &= C_1^{-1} \circ N_{\mu,\varepsilon} \circ C_1(u,v) \\ &= \begin{pmatrix} (c+\varepsilon\sqrt{\mu})u + \varepsilon v + \varepsilon r^1_2(u,v) \\ (c-\varepsilon\sqrt{\mu})v + \varepsilon a_{20}u^2 + \varepsilon a_{11}uv + \varepsilon r^2_k(u,v) \end{pmatrix} \end{aligned} \tag{5.4.6}$$

with $r^l_2(u,v) = o(\|(u,v)\|)^2$ depending on ε, $\varepsilon\sqrt{\mu}$ and μ. The coefficients a_{20} and a_{11} can depend on ε.

Proposition 5.4.2 *Let $\tilde{N}_{\mu,\varepsilon} : U \subset \mathbb{C}^2 \to \mathbb{C}^2$, such that $|\mu| < \mu_0$ and $0 \le \varepsilon < \varepsilon_0$ be a family of analytic maps, depending continuously on ε and analytically with respect to $\sqrt{\mu}$ of the form (5.4.6) with the condition that the coefficient $a_{20}(\varepsilon, \sqrt{\mu})$ satisfies $a_{20}(0, \sqrt{\mu}) > 0$. Then there exists $\delta > 0$ independent of ε and μ such that*

1) *the stable curve is the graph of a function φ analytic on $\Omega(\delta)$, depending continuously with respect to ε and analytically with respect to $\sqrt{\mu}$..*

2) *The function φ has the form $\varphi(u,\varepsilon,\mu) = f(u,\varepsilon) + \sqrt{\mu}g(u,\varepsilon,\sqrt{\mu})$ with*

$$f(u,\varepsilon) = \sqrt{2a^*_{20}/(k+1)}u^{(k+1)/2} + h.o.t.$$

where $a^*_{20} = a_{20}(\varepsilon, 0)$.

Proof. As in the parabolic case, the steps of the proof of Theorem 5.2.15 work in this case except one technical lemma. We must substitute Lemma 5.2.8 by the following:

Let $0 < r < 2$. If δ is small enough, $(u, v) \in A(\delta)$ and $0 < x \leq 1/j^r$ then

$$0 < \pi_1 \tilde{N}_{\mu,\varepsilon}(u, v) < 1/(j + \varepsilon)^r.$$

The proof of this lemma follows immediately. Hence the stable manifold can be expressed as the graph of an analytic function φ in $\Omega(\delta)$, depending analytically with respect to μ. Moreover, we can prove the continuity with respect to ε in the same way as we did in Section 5.2.6.

In order to prove the second property, we observe that the function φ is analytic with respect to $\sqrt{\mu}$ therefore

$$\varphi(u, \varepsilon, \mu) = \varphi(u, \varepsilon, 0) + \sqrt{\mu}\partial_\mu \varphi(u, \varepsilon, 0) + O(\mu).$$

■

We state a useful corollary for our purposes.

Corollary 5.4.3 *There exists $\delta > 0$ independent of ε and μ such that the map (5.4.3) has a unique stable local invariant manifold which can be represented as the graph of a function*

$$\varphi^s : \Omega(\delta) \times \{\mu \in \mathbb{C} : |\mu| \leq \mu_0\} \times [0, \varepsilon_0) \to \mathbb{C}$$

which is analytic in $x \in \Omega(\delta)$, analytic with respect to $\sqrt{\mu}$ and continuous with respect to ε. Moreover, $\varphi^s(x, \varepsilon, \mu) = f(x, \varepsilon) + \sqrt{\mu} g(x, \varepsilon, \mu)$.

Proof. We must go back to the original variables. It is clear that

$$F_{\mu,\varepsilon} = C_1^{-1} \circ C_2^{-1} \circ C_3 \circ C_1 \circ \tilde{N}_{\mu,\varepsilon} \circ C_1^{-1} \circ C_3^{-1} \circ C_2 \circ C_1.$$

Then the original variables $(x, y) = C(u, v)$ where $C = C_1^{-1} \circ C_2^{-1} \circ C_3 \circ C_1$, therefore,

$$\begin{aligned}
\begin{pmatrix} x \\ y \end{pmatrix} &= C_1^{-1} \circ C_2^{-1} \circ C_3 \begin{pmatrix} u/\varepsilon \\ v \end{pmatrix} = C_1^{-1} \circ C_2^{-1} \begin{pmatrix} u/\varepsilon + \Phi(u/\varepsilon, v) \\ v + \Psi(u/\varepsilon, v) \end{pmatrix} \\
&= C_1^{-1} \begin{pmatrix} u/\varepsilon + \Phi(u/\varepsilon, v) \\ v + \Psi(u/\varepsilon, v) + \sqrt{\mu} u + \sqrt{\mu}\varepsilon \Phi(u/\varepsilon, v) \end{pmatrix} \\
&= \begin{pmatrix} u + \varepsilon \Phi(u/\varepsilon, v) \\ v + \Psi(u/\varepsilon, v) + \sqrt{\mu} u + \sqrt{\mu}\varepsilon \Phi(u/\varepsilon, v) \end{pmatrix}
\end{aligned}$$

and by Lemma 5.4.1, the functions

$$\tilde{\Phi}(u,v,\varepsilon)=\varepsilon\Phi(u/\varepsilon,v) \qquad \tilde{\Psi}(u,v)=\Psi(u/\varepsilon,v)$$

are of order 2 in the (u,v) variables and continuous with respect to ε at $\varepsilon=0$. Therefore the change C has the form

$$C(u,v)=(u+\tilde{\Phi}(u,v,\varepsilon),v+\sqrt{\mu}u+\tilde{\Psi}(u,v,\varepsilon)+\sqrt{\mu}\tilde{\Phi}(u,v,\varepsilon)). \tag{5.4.7}$$

Since the local stable invariant manifold of $\tilde{N}_{\mu,\varepsilon}$ is the graph of φ and the change C has the form (5.4.7), the stable manifold of $F_{\mu,\varepsilon}$ can be expressed as the graph of the function φ^s. Moreover, expanding φ^s in $\sqrt{\mu}$, by using Taylor's theorem, we have

$$\varphi^s(x,\varepsilon,\mu)=f(x,\varepsilon)+\sqrt{\mu}g(x,\varepsilon,\mu).$$

■

Proof of Proposition 5.1.2

We recall that the Poincaré map P_μ^θ given in (5.4.1) is

$$P_\mu^\theta=C^\theta\circ G_\mu^\theta\circ(C^\theta)^{-1}$$

where $C^\theta=\mathrm{Id}+O(\mu^2\varepsilon^{2p+1})$ is linear and 2π-periodic in θ. Therefore it is sufficient to prove that the local stable invariant manifold of the map G_μ^θ can be expressed as the graph of a function having the properties we want. We write the map G_μ^θ given in (5.4.2) as

$$G_\mu^\theta(x,y)=\begin{pmatrix} cx+\frac{s}{\rho}y+\varepsilon^2p_1(x,y,\mu,\varepsilon)\\ \rho sx+cy+\varepsilon p_2(x,y,\mu,\varepsilon)\end{pmatrix}$$

and we recall that $c=\cosh(\rho 2\pi\varepsilon)$, $s=\sinh(\rho 2\pi\varepsilon)$, $\rho=O(\mu\varepsilon^{p+1/2})$ and ρ depends 2π periodically on θ. Moreover we recall that $p_i(x,y,\varepsilon,\mu)\in P_2$.

Let $\tilde{C}_1(x,y)=(x/2\pi,y)$. It is clear that the function $\tilde{G}_\mu^{\theta_0}=\tilde{C}_1\circ G_\mu^\theta\circ\tilde{C}_1^{-1}$ has the form given in (5.4.3) with the parameter $\mu=\rho^2$. Therefore, by Corollary 5.4.3 we obtain that the stable manifold of $\tilde{G}_\mu^\theta$ can be expressed as the graph of a function $\bar{\varphi}^s$, which is analytic with respect to $(x,\rho,\mu)\in\Omega(\delta_0)\times\{\eta\in\mathbb{C}^2:|\eta|\le\mu_0\}$ and continuous with respect to θ and ε, with δ_0 independent of ε, μ and θ. A similar argument to the one given in the proof of Proposition 5.1.1 gives the result.

Chapter 6

Flow box coordinates

6.1 Introduction

In this chapter we prove the existence of flow box coordinates of a system with generic hypotheses, in a neighborhood of the stable manifold which does not contain the origin and it is independent of the parameters μ and ε. A similar result is in [43]. There, the flow box coordinates are found implicitly using the variational equations in a neighborhood of the stable manifold. Our proof gives these coordinates in an explicit way and gives a careful estimate of the distance between the change of coordinates in the unperturbed case and the change in the perturbed case, using variational equations with respect to the parameter μ.

Delshams and M.T.Seara [20] [21] use flow-box coordinates defined near a hyperbolic fixed point. To construct such coordinates they use the Birkhoff Normal Form in an essential way. Also in [68] the authors construct similar flow box coordinates in the Arnold example. They use that the equation is analytic with respect to the time.

We begin by introducing notation and the hypotheses **H1**, **H2** and **H3** we will assume in this Chapter. With these hypotheses we will prove a result on the existence of flow box coordinates: Theorem 6.2.5. Then, the application of this theorem to equation (1.1.1) gives Corollary 6.2.6, in which the result is obtained applying Theorem 6.2.5, not directly to (1.1.1) but to the averaged equation.

To prove Theorem 6.2.5 first, in Section 6.3, we translate the stable manifold to the first axis of coordinates and in these coordinates, for the unperturbed system, we construct explicitely the flow box coordinates, just integrating the equation and using that the system is Hamiltonian.

To construct the flow box coordinates for the general system in a neighbourhood of (a part of) the stable manifold, we use a special parameterization of the solutions of the equation. We parameterize the solutions, $z(t, s, Y)$ with two parameters $(t, s) \in \mathbb{R} \times \mathbb{C}$ in such a way that $t \in \mathbb{R}$ is a time parameter, $Y \in \mathbb{C}$ and

$$z(t + 2\pi\varepsilon, s, Y) = z(t, s + 2\pi\varepsilon, Y)$$

in a suitable domain. To obtain this we use a technique designed by Lazutkin to do a controled analytic continuation. This is done in Sections 6.4.2 and 6.4.3. From this parameterization we easily obtain another parameterization of the form

$$w(t + s, t/\varepsilon, Y),$$

that is, we separate in some way the slow time $t + s$ and the fast time t/ε.

Next we find a first flow box coordinates $(\mathcal{T}, \mathcal{Y})$ from the condition

$$w(\mathcal{T}(x, v, \theta), \theta, \mathcal{Y}(x, v, \theta)) = (x, v)$$

using the scheme of the proof of the implicit function theorem. We obtain it close to the analogous ones we have calculated in the non perturbed case. Then easily we pass to new flow box coordinates $(\mathcal{T}, \mathcal{F})$ with $\mathcal{F}$ close to the energy variable (the Hamiltonian).

Finally, using the Hamiltonian character of the equations, we slightly modify these coordinates to make them canonical.

6.2 Definitions and main result

We consider Hamiltonian systems of the form

$$H(x, y, t/\varepsilon) = H_0(x, y) + \mu\varepsilon^q H_1(x, y, t/\varepsilon, \mu, \varepsilon)$$

where

$$H_0(x, y) = \frac{y^2}{2} + V(x).$$

Remark 6.2.1 *Since we will apply the result of existence of flow box coordinates to an averaged system, here q and H_1 mean a generic constant and a generic Hamiltonian respectively which (in general) do not coincide with p and h_1 introduced in Chapter 1.*

The associated equations are

$$
\begin{aligned}
\dot{x} &= y + \mu\varepsilon^q \partial_y H_1(x, y, t/\varepsilon, \mu, \varepsilon) \\
\dot{y} &= -V'(x) - \mu\varepsilon^q \partial_x H_1(x, y, t/\varepsilon, \mu, \varepsilon).
\end{aligned}
\tag{6.2.1}
$$

For $w = (w_1, w_2) \in \mathbb{C}^2$, we define

$$\|w\| = \max\{|w_1|, |w_2|\}.$$

We will assume the following hypotheses

H1 The potential V is an analytic function in $\{x \in \mathbb{C} : |x| < \delta_0\}$, $V(x) = -a_n x^n - \dots$ with $a_n > 0$, $n \in \mathbb{N}$ and $3 \leq n$.

H2 $H_1(x, y, \theta, \varepsilon, \mu)$ is C^0, 2π-periodic in θ and analytic in the x, y, μ variables. The variables (x, y, μ) belong to

$$B(\delta_0, \mu_0) \equiv \{(x, y) \in \mathbb{C}^2 : \|(x, y)\| < \delta_0\} \times \{\mu \in \mathbb{C} : |\mu| < \mu_0\},$$

$\theta \in \mathbb{R}$ and $0 < \varepsilon < \varepsilon_0$. Moreover $H_1(x, y, \theta, \varepsilon, \mu) = O(\|(x, y)\|^k)$ with $k \geq 2$.

In our applications, k wil be always greater of equal than $n - 1$.

Under hypotheses **H1** and **H2**, the origin is a fixed point. Next hypothesis deal with the stable manifold of the origin.

We define the open set

$$\Omega(\delta_0) = \left\{ x \in \mathbb{C} : 0 < \operatorname{Re} x < \delta_0,\ |\arg(x)| < \frac{\pi}{5(\beta - 1)} \right\}$$

where $\beta = n/2$.

H3 The origin has a stable manifold (in the sense of Chapter 5) which can be represented as a graph of a function $\varphi : \Omega(\delta_0) \times \mathbb{R} \to \mathbb{C}$ having the form

$$\varphi_{\mu,\varepsilon}(x, \theta) = \varphi(x, \theta, \varepsilon, \mu) = f(x) + \mu\varepsilon^q g(x, \theta, \varepsilon, \mu)$$

with $\beta = n/2$, $f(x) = -\sqrt{-2V(x)}$ and $g(x, \theta, \varepsilon, \mu)$ is C^0, C^1 and 2π-periodic in θ and analytic with respect to x in $\Omega(\delta_0)$ and with respect to μ in $\{\mu \in \mathbb{C} : |\mu| < \mu_0\}$. Moreover

$$g(x, \theta, \varepsilon, \mu) = O(x^l)$$

for some $l \geq k/2 \geq 1$.

Remark 6.2.2 *In Chapter 5 we have given sufficient conditions for the averaged systems we will consider, to satisfy **H3**.*

Remark 6.2.3 *By Chapter 2, the unperturbed system ($\mu = 0$) has a parameterization of the stable manifold defined in*

$$\{u \in \mathbb{C} : \operatorname{Re} u \geq T \quad , \quad |\operatorname{Im} u| \leq a\}$$

and denoted by $\gamma_0(u) = (\alpha_0(u), \beta_0(u))$.

In the next definition we fix some parameters.

Definition 6.2.4 *Let $\delta_0' > 0$ be such that $\delta_0' < \delta_0$ and let C_0 and C_1 be such that*

$$C_0 |x|^\beta \leq |f(x)| \leq C_1 |x|^\beta$$

for all $x \in \Omega(\delta_0)$ where $\beta = n/2$. We define

$$r_0 = \frac{1}{2}\Big(- C_1 \delta_0^\beta + \sqrt{C_1^2 \delta_0^{2\beta} + C_0^2 (\delta_0')^{2\beta}}\Big).$$

Consider μ and ε. For any δ, δ', r and η such that $0 < \delta_0' \leq \delta' < \delta \leq \delta_0$, $0 < r < r_0$ and $0 \leq \eta$ we define the open sets

$$\begin{aligned} &V(\delta', \delta, r, \eta) \\ &= \left\{ (x, y, \theta) \in \mathbb{C}^2 \times \mathbb{R} : \delta' < \operatorname{Re} x < \delta,\ |\arg(x)| < \frac{\pi}{4(\beta - 1)} - \eta,\ |y - \varphi_{\mu,\varepsilon}(x, \theta)| < r \right\} \end{aligned}$$

and

$$V_0(\delta', \delta, r, \eta) = \{(x, y) \in \mathbb{C}^2 : \exists \theta \in \mathbb{R} \text{ such that } (x, y, \theta) \in V(\delta', \delta, r, \eta)\}.$$

In fact, since $\varphi_{\mu,\varepsilon}$ depends on μ, ε, V and V_0 also depend on μ, ε,

We define $\eta_0 = \pi/(20(\beta - 1))$. The main result of this Chapter is:

Theorem 6.2.5 *(Flow-box coordinates). Let $0 < \delta < \delta_0/3$. If the hypotheses **H1-H3** hold, for any $0 < \delta' < \delta$ there exists $r > 0$ and a canonical change of variables*

$$(x, y, \theta = \frac{t}{\varepsilon}) \in V(\delta', \delta, r, \eta_0) \mapsto (T, I, \theta) = (\mathcal{T}^1(x, y, \theta), \mathcal{I}^1(x, y, \theta), \theta) \in \mathcal{V}$$

of class C^1, 2π-periodic in θ and analytic in the x, y variables, such that it transforms the system (6.2.1) to

$$\begin{aligned} \dot{T} &= 1 \\ \dot{I} &= 0 \end{aligned} \tag{6.2.2}$$

and satisfies

$$\mathcal{T}^1(x,y,\theta) = \mathcal{T}_0(x,y) + O(\mu\varepsilon^q), \;\; \mathcal{I}^1(x,y,\theta) = \mathcal{I}_0(x,y) + O(\mu\varepsilon^q),$$

where the change $(x,y) \mapsto (\mathcal{T}_0(x,y), \mathcal{I}_0(x,y))$ is the corresponding change for the unperturbed system. Moreover the change is continuous in $(x,y,\theta,\mu,\varepsilon)$.

To study the splitting of separatrices we will use the following corollary of Theorem 6.2.5 which is obtained when we apply it to a system of the form

$$\begin{aligned} x' &= \varepsilon y + \mu\varepsilon^{p+2n+3}\partial_y F + \mu^2\varepsilon^{p+3}\partial_y R_{2k-2} \\ y' &= -\varepsilon V'(x) - \mu\varepsilon^{p+2n+3}\partial_x F - \mu^2\varepsilon^{p+3}\partial_x R_{2k-2} \end{aligned}$$

which comes from the system (1.1.1) by the averaging procedure. The system (1.1.1) satisfies either the hypotheses **HP1-HP4** or **HP1-HP3**, **HP6** (according if we are considering the parabolic case or the weak hyperbolic case). In such cases the results of Chapters 3 and 4 apply and we have that the stable manifold can be parameterized by $\gamma^s_{\mu,\varepsilon}(t,s)$.

This corollary gives a new flow box coordinates and additional information over the values of these flow box coordinates on the stable manifold $\gamma^s_{\mu,\varepsilon}(t,s)$. Let $\mathcal{C}$ be the change which transforms the system (1.1.1) to the averaged system and is given in Lemma 3.3.4 and in Remark 4.3.4. (In the weak hyperbolic case, this Remark summarizes all changes designed in several previous lemmas).

Corollary 6.2.6 *Given δ_1' such that $0 < \delta_1' < \delta < \delta_0/3$, there exist $r_1 > 0$ and a canonical change of variables*

$$(x,y,\theta = \frac{t}{\varepsilon}) \in \mathcal{C}(V(\delta_1', \delta, r_1, \eta_0)) \mapsto (S,E,\theta) = (\mathcal{S}(x,y,\theta), \mathcal{E}(x,y,\theta), \theta) \in \mathcal{V}_1$$

of class C^1, 2π-periodic in θ and analytic in the x, y variables, such that it transforms the system (1.1.1) to

$$\begin{aligned} \dot{S} &= 1 \\ \dot{E} &= 0 \end{aligned}$$

and satisfies

$$\mathcal{S}(x,y,\theta)=\mathcal{S}_0(x,y)+O(\mu\varepsilon^{p+i_0}),\qquad \mathcal{E}(x,y,\theta)=\mathcal{E}_0(x,y)+O(\mu\varepsilon^{p+i_0})$$

where $i_0=1$ *in the parabolic case and* $i_0=1/2$ *in the weak hyperbolic case, and* $(x,y)\mapsto(\mathcal{S}_0(x,y),\mathcal{E}_0(x,y))$ *is the corresponding change when* $\mu=0$.

Moreover, there exists $T\geq 0$ *big enough such that for all* (t,s) *such that* $T\leq |t+\operatorname{Re}s|\leq 2T$*, the parameterization* $\gamma^s_{\mu,\varepsilon}(t,s)$ *of the local stable manifold given in Chapters 3 or in Chapter 4 satisfies*

$$\gamma^s_{\mu,\varepsilon}(t,s)\in\mathcal{C}(V(\delta_1',\delta,r_1,\eta_0))$$

and

$$\mathcal{S}(\gamma^s_{\mu,\varepsilon}(t,s),t/\varepsilon)=t+s+\mu\varepsilon^{p+i_0}\mathcal{X}(s)\qquad\text{and}\qquad\mathcal{E}(\gamma^s_{\mu,\varepsilon}(t,s),t/\varepsilon)=0$$

and $\mathcal{X}(s_0)$ *for some* s_0*, which we can choose freely, depending on initial conditions on the stable curve. Moreover* $\mathcal{X}(s)$ *is analytic and* $2\pi\varepsilon$*-periodic.*

In addition the change $(x,y,\theta)\mapsto(S,E,\theta)$ *is continuous in* $(x,y,\theta,\mu,\varepsilon)$ *and analytic in* (x,y,μ).

Remark 6.2.7 *To fix ideas we consider only the parabolic case and the weak hyperbolic case, which are the object of this memoir, but the proof also would work in the hyperbolic case, with some small changes.*

In this Chapter we omit the dependence on μ and ε which is assumed that it is analytical and continuous respectively.

6.3 Some preliminaries bounds

6.3.1 A preliminary change of variables

Since we have assumed that the stable manifold can be expressed as a graph of an analytic function, we can easily move the stable curve to the x-axis. We perform the change of variables $C:V(\delta_0',\delta_0,r_0,0)\to\mathbb{C}^2\times\mathbb{R}$ defined by

$$(x,y,\theta)\mapsto(x,v=y-f(x)-\mu\varepsilon^q g(x,\theta),\theta). \tag{6.3.1}$$

This change puts the local stable manifold at $v = 0$. The change is canonical. It maps $V(\delta_0', \delta_0, r_0, 0)$ onto $U(\delta_0', \delta_0, r_0, 0)$ where

$$U(\delta_0', \delta_0, r_0, 0) = \left\{ (x, v, \theta) \in \mathbb{C}^2 \times \mathbb{R} : \delta_0' < \operatorname{Re} x < \delta_0,\ |\arg(x)| < \frac{\pi}{4(\beta - 1)},\ |v| < r_0 \right\}.$$

We note that, in general, we can not extend C in such a way that it is analytic at $x = 0$ because f and g are not analytic at 0. The equations in these new variables are

$$\begin{aligned} \dot{x} &= v + f(x) + \mu\varepsilon^q g(x, \theta) + \mu\varepsilon^q \tilde{g}_1(x, v, \theta) \\ \dot{v} &= -v(f'(x) + \mu\varepsilon^q \partial_x g(x, \theta)) + \mu\varepsilon^q \tilde{g}_2(x, v, \theta) \\ \dot{\theta} &= 1/\varepsilon \end{aligned} \tag{6.3.2}$$

where

$$\begin{aligned} \tilde{g}_1(x, v, \theta) &= \partial_y H_1(x, v + f(x) + \mu\varepsilon^q g(x, \theta), \theta) \\ \tilde{g}_2(x, v, \theta) &= \partial_x H_1(x, v + f(x) + \mu\varepsilon^q g(x, \theta), \theta) - \partial_x H_1(x, f(x) + \mu\varepsilon^q g(x, \theta), \theta) \\ &\quad + [f'(x) + \mu\varepsilon^q \partial_x g(x, \theta)][\partial_y H_1(x, v + f(x) + \mu\varepsilon^q g(x, \theta), \theta) \\ &\quad - \partial_y H_1(x, f(x) + \mu\varepsilon^q g(x, \theta), \theta)]. \end{aligned}$$

To obtain the expression of $\tilde{g}_2$ we have used that $\dot{v} = 0$ when $v = 0$. This condition gives a relation which permits to simplify the form of $\tilde{g}_2$.

For any δ, δ', r and η such that $0 < \delta_0' \le \delta' < \delta \le \delta_0$, $r \le r_0$ and $0 \le \eta$ we define the sets

$$U(\delta', \delta, r, \eta) = \left\{ (x, v, \theta) \in \mathbb{C}^2 \times \mathbb{R} : \delta' < \operatorname{Re} x < \delta,\ |\arg(x)| < \frac{\pi}{4(\beta - 1)} - \eta,\ |v| < r \right\}$$

and

$$U_0(\delta', \delta, r, \eta) = \left\{ (x, v) \in \mathbb{C}^2 : \delta' < \operatorname{Re} x < \delta,\ |\arg(x)| < \frac{\pi}{4(\beta - 1)} - \eta,\ |v| < r \right\}.$$

We observe that these sets are convex. We denote by $X_\mu : U(\delta_0', \delta_0, r_0, 0) \to \mathbb{C}^2 \times \mathbb{R}$, the vector field $X_\mu = X_0 + \mu\varepsilon^q X_1$ with

$$X_0 = \begin{pmatrix} v + f(x) \\ -vf'(x) \\ 1/\varepsilon \end{pmatrix} \quad \text{and} \quad X_1 = \begin{pmatrix} g(x, \theta) + \tilde{g}_1(x, v, \theta) \\ -v\partial_x g(x, \theta) - \tilde{g}_2(x, v, \theta) \\ 0 \end{pmatrix}.$$

Along this section, K will denote a generic constant independent of $\mu, \varepsilon, \delta'$ and δ and $K(\delta', \delta)$ will denote a constant independent of μ and ε.

Some preliminary bounds of the vector field X_μ are necessary.

Lemma 6.3.1 *Let $\varkappa = \min\{k-1, l\} \geq 1$ (k and l have been introduced in **H2** and **H3** respectively). Then*

1) The vector field X_1 is bounded, more precisely, there exists a constant K such that for all $(x, v, \theta) \in \overline{U(\delta_0', \delta_0, r_0, 0)}$

$$\|X_1(x, v, \theta)\| \leq K\|(x, v)\|^{\varkappa}.$$

2) The vector field X_1 restricted to $\overline{U(\delta_0', \delta_0, r_0, 0)}$ is Lipschitz with $\mathrm{Lip}(X_1) \leq K\delta_0^{\varkappa-1}$.

3) For all $(x, v) \in U_0(\delta_0', \delta_0, r_0, 0)$ and $h \in \mathbb{R}^2$ such that $(x, v) + h \in U_0(\delta_0, \delta_0, r_0, 0)$,

$$\|X_0(x + h_1, v + h_2) - X_0(x, v) - DX_0(x, v)h\| \leq K(\delta_0', \delta_0)\|h\|^2.$$

Proof. We note that $f(x) = -\sqrt{-2V(x)} = O(x^{n/2})$. To prove the first bound we recall that $g(x, \theta)$ is $O(x^l)$ and that $H_1(x, y, \theta)$ is a function of order k. For $(x, v, \theta) \in \overline{U(\delta_0', \delta_0, r_0, 0)}$ fixed, we define the auxiliary functions:

$$\triangle_1 H_1(s) = \partial_x H_1(x, sv + f(x) + \mu\varepsilon^q g(x, \theta), \theta)$$

and

$$\triangle_2 H_1(s) = \partial_y H_1(x, sv + f(x) + \mu\varepsilon^q g(x, \theta), \theta).$$

We have

$$\begin{aligned} |\triangle_1 H_1(1) - \triangle_1 H_1(0)| &\leq \int_0^1 |\partial_{xy} H_1(x, sv + f(x) + \mu\varepsilon^q g(x, \theta), \theta)|\ |v|\ ds \\ &\leq K\|(x, v)\|^{k-1} \end{aligned}$$

and

$$\begin{aligned} |\triangle_2 H_1(1) - \triangle_2 H_1(0)| &\leq \int_0^1 |\partial_{yy} H_1(x, sv + f(x) + \mu\varepsilon^q g(x, \theta), \theta)|\ |v|\ ds \\ &\leq K\|(x, v)\|^{k-1}. \end{aligned}$$

Then, if we bound $\tilde{g}_2$, we obtain

$$\begin{aligned}|\tilde{g}_2(x,v,\theta)| &\le |\triangle_1 H_1(1)-\triangle_1 H_1(0)| + |f'(x)+\mu\varepsilon^q\partial_x g(x,\theta)|\ |\triangle_2 H_1(1)-\triangle_2 H_1(0)|\\ &\le K\|(x,v)\|^{k-1} + K|x|^{\min\{\beta-1,l-1\}}\|(x,v)\|^{k-1}\\ &\le K\|(x,v)\|^{k-1}.\end{aligned}$$

Hence

$$\begin{aligned}|X_1^2(x,v,\theta)| &\le |\tilde{g}_2(x,v,\theta)| + |v|\ |\partial_x g(x,\theta)|\\ &\le K\|(x,v)\|^{k-1} + K|v||x|^{l-1}\\ &\le K\|(x,v)\|^{\min\{k-1,l\}} = K\|(x,v)\|^{\varkappa}.\end{aligned}$$

Clearly, the first component satisfies

$$\begin{aligned}|X_1^1(x,v,\theta)| &\le |g(x,\theta)| + |\tilde{g}_1(x,v,\theta)| \le K|x|^l + K\|(x,v)\|^{k-1}\\ &\le K\|(x,v)\|^{\min\{k-1,l\}} = K\|(x,v)\|^{\varkappa}.\end{aligned}$$

Now we prove that the field X_1 is Lipschitz with respect to the (x,v) variables. We will apply the mean value theorem. Let $(\xi,\eta,\theta)\in U(\delta_0',\delta_0,r_0,0)$. Then, it is clear that

$$\begin{aligned}|\partial_\xi X_1^1(\xi,\eta,\theta)| &= |\partial_\xi g(\xi,\theta) + \partial_\xi\tilde{g}_1(\xi,\eta,\theta)|\\ &\le K|\xi|^{l-1} + K\|(\xi,\eta)\|^{k-2}\\ &= K\|(\xi,\eta)\|^{\min\{l-1,k-2\}} = K\|(\xi,\eta)\|^{\varkappa-1}\end{aligned}$$

and

$$|\partial_\eta X_1^1(\xi,\eta,\theta)| = |\partial_\eta\tilde{g}_1(\xi,\eta,\theta)| \le K\|(\xi,\eta)\|^{k-2}.$$

Concerning X_2^1, using the above notation, we have

$$\begin{aligned}|\partial_\xi X_1^2(\xi,\eta,\theta)| &= |\eta\partial_{\xi\xi}g(\xi,\theta) + \partial_\xi\tilde{g}_2(\xi,\eta,\theta)|\\ &\le |\eta\partial_{\xi\xi}g(\xi,\theta)| + |\partial_\xi\triangle_1 H_1(1) - \partial_\xi\triangle_1 H_1(0)|\\ &\quad +|f'(\xi)+\mu\varepsilon^q\partial_\xi g(\xi,\theta)|\ |\partial_\xi\triangle_2 H_1(1) - \partial_\xi\triangle_2 H_1(0)|\\ &\quad +|f''(\xi)+\mu\varepsilon^q\partial_{\xi\xi} g(\xi,\theta)|\ |\triangle_2 H_1(1) - \triangle_2 H_1(0)|\\ &\le K|\eta|\ |\xi|^{l-2} + K\|(\xi,\eta)\|^{k-2}(1+|\xi|^{\beta-1}+\mu\varepsilon^q|\xi|^{l-1})\\ &\quad +K(|\xi|^{\beta-1}+\mu\varepsilon^q|\xi|^{l-1})(1+|\xi|^{\beta-1}+\mu\varepsilon^q|\xi|^{l-1})\|(\xi,\eta)\|^{k-2}\\ &\quad +K(|\xi|^{\beta-2}+\mu\varepsilon^q|\xi|^{l-2})\|(\xi,\eta)\|^{k-1}\\ &\le K\|(\xi,\eta)\|^{\min\{l-1,k-2\}} = K\|(\xi,\eta)\|^{\varkappa-1}\end{aligned}$$

and

$$\begin{aligned}
|\partial_\eta X_1^2(\xi,\eta,\theta)| &= |\partial_\xi g(\xi,\theta) + \partial_\eta \tilde{g}_2(\xi,\eta,\theta)| \\
&\leq |\partial_\xi g(\xi,\theta)| + |\partial_{\eta\xi} H_1(\xi,\eta + f(\xi) + \mu\varepsilon^q g(\xi,\theta))| \\
&\quad + |f'(\xi) + \partial_\xi g(\xi,\theta)| \, |\partial_{\eta\eta} H_1(\xi,\eta + f(\xi) + \mu\varepsilon^q g(\xi,\theta))| \\
&\leq K[|\xi|^{l-1} + \|(\xi,\eta)\|^{k-2} + (|\xi|^{\beta-1} + \mu\varepsilon^q |\xi|^{l-1})\|(\xi,\eta)\|^{k-2}] \\
&\leq K\|(\xi,\eta)\|^{\min\{l-1,k-2\}} = K\|(\xi,\eta)\|^{\varkappa-1}.
\end{aligned}$$

Let (x,v) and $(\bar{x},\bar{v})$ be two points of $U_0(\delta_0',\delta_0,r_0,0)$ and $\theta \in \mathbb{R}$. By the mean value theorem, as well as the previous bounds, we obtain

$$\begin{aligned}
\|X_1(x,v,\theta) - X_1(\bar{x},\bar{v},\theta)\| &\leq \|DX_1\|\|(x,v) - (\bar{x},\bar{v})\| \\
&\leq K\delta_0^{\varkappa-1}\|(x,v) - (\bar{x},\bar{v})\|
\end{aligned}$$

as we wanted.

The third bound in the statement is a consequence of Taylor's theorem and the mean value theorem. We observe that, since X_0 is analytic,

$$\sup_{(x,v)\in U_0} \|DX_0^2(x,v)\| \leq K(\delta_0',\delta_0) = K[(\delta_0')^{\beta-2} + r_0(\delta_0')^{\beta-3}].$$

By Taylor's theorem we have

$$\begin{aligned}
X_0(x+h_1, v+h_2) &= X_0(x,v) + DX_0(x,v)h \\
&\quad + \int_0^1 (DX_0(x+sh_1, v+sh_2) - DX_0(x,v))h \; ds
\end{aligned}$$

and, by the mean value theorem

$$\|X_0(x+h_1,v+h_2) - X_0(x,v) - DX_0(x,v)h\| \leq \frac{1}{2}K(\delta',\delta)\|h\|^2$$

as we want. ∎

6.3.2 The unperturbed case

When $\mu = 0$, the system (6.3.2) is Hamiltonian with Hamiltonian

$$\mathcal{F}_0(x,v) = \frac{v^2}{2} + vf(x). \tag{6.3.3}$$

Then for any $z^0 = (x^0, v^0, \theta^0)$ the solution with initial condition z^0 is contained in the curve

$$v = -f(x) \pm \sqrt{f^2(x) + 2\mathcal{F}_0(x^0, v^0)}$$

when one has to choose the sign in such a way that the relation is satisfied by the initial condition. From system (6.3.2) and hypothesis **H3**, it is clear that

$$\dot{x} = \pm\sqrt{(v^0)^2 + 2v^0 f(x^0) - V(x)}. \tag{6.3.4}$$

Let $\psi_0(t, x, v)$ be the flow of the unperturbed Hamiltonian system. Integrating equation (6.3.4) we find the time (in general complex time) to arrive from (x, v) to (x^*, v^*) where $x^* = \delta$ and v^* is determined by the energy conservation. In this way we get that the functions $\mathcal{T}_0(x, v)$, $\mathcal{Y}_0(x, v)$ defined in $U_0(\delta_0', \delta_0, r_0, 0)$ by

$$\mathcal{T}_0(x, v) = -\int_x^\delta \frac{ds}{\sqrt{2\mathcal{F}_0(x, v) - V(s)}} \tag{6.3.5}$$

$$\mathcal{Y}_0(x, v) = -f(\delta) - \sqrt{f^2(\delta) + 2\mathcal{F}_0(x, v)} \tag{6.3.6}$$

are such that $\psi_0(\mathcal{T}_0(x, v), \delta, \mathcal{Y}_0(x, v)) = (x, v)$. We choose the sign minus in (6.3.4), because it is obvious that, in the real case and over the stable manifold, $x(t)$ must decrease as t goes to $+\infty$. In the coordinates

$$(T, Y) = (\mathcal{T}_0(x, v), \mathcal{Y}_0(x, v))$$

the equations of the unperturbed system become:

$$\dot{T} = 1$$
$$\dot{Y} = 0.$$

Also we can consider the change

$$(x, v) \in U_0(\delta_0', \delta_0, r_0, 0) \mapsto (\mathcal{T}_0(x, v), \mathcal{F}_0(x, v)) \in \mathcal{V},$$

where $\mathcal{F}_0$ is the Hamiltonian. The equations in the coordinates

$$(T, F) = (\mathcal{T}_0(x, v), \mathcal{F}_0(x, v))$$

also are

$$\dot{T} = 1$$
$$\dot{F} = 0.$$

This second change is canonical, i.e. $\partial_x \mathcal{T}_0 \partial_v \mathcal{F}_0 - \partial_v \mathcal{T}_0 \partial_x \mathcal{F}_0 = 1$.

6.4 Flow box coordinates in a complex domain

6.4.1 Introduction and definitions

Now, we fix δ, δ', r and η such that $0 < 3\delta_0' \leq \delta' < \delta \leq \delta_0/3$, $r \leq r_0/3$ and $\eta_0 = \pi/(20(\beta-1))$. Obviously $\delta_0' < \delta_0/9$. Let $z_0(u,Y)$ be the solution of the unperturbed system (the associated system to the Hamiltonian $\mathcal{F}_0$, (6.3.3)) with $z_0(0,Y) = (\delta, Y)$.

From the expressions (6.3.5) and (6.3.6) we deduce that there exist some functions $\kappa_1^{\pm}(Y)$ and $\kappa_2(Y)$, depending on the choice of δ, δ', r and η, such that $z_0(t+s,Y)$ belongs to $U_0(\delta',\delta,r,\eta_0)$ if $|Y| \leq r$, $\kappa_1^-(Y) \leq t + \operatorname{Re} s \leq \kappa_1^+(Y)$ and $|\operatorname{Im} s| \leq \kappa_2(Y)$. We remark that if u is such that $z_0(u,Y) \in U_0(\delta',\delta,r,\eta_0)$ then $\bar{u}$ (the complex conjugate of u) also has the same property. To see this we just recall that $U_0(\delta_0',\delta_0,r_0,0)$ is symmetric with respect to real axis and that $\mathcal{T}_0(x,v)$ is a real analytic function, in particular we have that

$$\operatorname{Im}\mathcal{T}_0(x,v) = -\operatorname{Im}\mathcal{T}_0(\bar{x},\bar{v}).$$

Let $\kappa_0 > 0$ be small. We define

$$\kappa_1^+ = \max_{|Y|\leq r} \kappa_1^+(Y)$$
$$\kappa_1^- = \min_{|Y|\leq r} \kappa_1^-(Y)$$
$$\kappa_2 = \max_{|Y|\leq r} \kappa_2(Y)$$

and the sets

$$\begin{aligned} D_0(\kappa_1^{\pm},\kappa_2,\kappa_0) &= \{s \in \mathbb{C} : \kappa_1^- - \kappa_0 < \operatorname{Re} s < \kappa_1^+ + \kappa_0 \text{ and } |\operatorname{Im} s| < \kappa_2 + \kappa_0\} \\ D(\kappa_1^{\pm},\kappa_2,\kappa_0) &= \{(t,s) \in \mathbb{R}\times\mathbb{C} : t+s \in D_0(\kappa_1^{\pm},\kappa_2,\kappa_0) \text{ and } |t| \leq 4\pi\varepsilon\} \\ W(r,\kappa_0) &= \{Y \in \mathbb{C} : |Y| \leq r + \kappa_0\}. \end{aligned} \tag{6.4.1}$$

Remark 6.4.1 *Since $v = 0$ is a solution of the system (6.3.2), there exist r and κ_0 small enough, such that for any $(t,s) \in D(\kappa_1^{\pm},\kappa_2,\kappa_0)$ and $Y \in W(r,\kappa_0)$, the solution $z_0(t+s,Y)$ belongs to $U_0(\delta'/2, 2\delta, 2r, \eta_0/2)$.*

Our goal is to find flow box coordinates in $U(\delta',\delta,r,\eta_0)$. We observe that this open set is a neighborhood of a part of the stable manifold $v = 0$.

We will find the solutions of the equations (6.3.2) parameterizated in the form

$$(z(t,s,Y),t/\varepsilon)$$

with

$$z(t,s,Y) = z_0(t+s,Y) + \mu\varepsilon^q z_1(t,s,Y),$$

$z(0,0,Y) = (\delta, Y)$ and the additional property

$$z(t+2\pi\varepsilon, s, Y) = z(t, s+2\pi\varepsilon, Y). \tag{6.4.2}$$

The relation (6.4.2) permits to give a dynamical interpretation of the parameter s: the iterations of the Poincaré map simply consists in increasing the variable s by $2\pi\varepsilon$. To get the solutions in this form we will rewrite (6.3.2) in the form

$$\begin{aligned} \dot{z} &= A(t+s)z + b(z)(t,s) \\ \dot{\theta} &= 1/\varepsilon \end{aligned}$$

and we will apply the fixed point theorem to a suitable operator in a Banach space. To construct this operator we will need another operator which we call increment operator. This operator was introduced by Lazutkin in [53].

Next, we will prove that, as in the unperturbed case, the solutions with initial condition in $U(\delta', \delta, r, \eta_0)$ arrive at $x = \delta$. Then we will prove that the flow can be straightened in $U(\delta', \delta, r, \eta_0)$.

Finally, we will construct another change in order to get that the composition of changes is canonical.

6.4.2 Increment operator and analytic continuation

Let h, $\tau_1^{\pm}$, τ_2. We define

$$\begin{aligned} D &= D(h, \tau_1^{\pm}, \tau_2) \\ &= \{(t,s) \in \mathbb{R} \times \mathbb{C} : |t| < 2h,\ \tau_1^- < t + \operatorname{Re} s < \tau_1^+,\ |\operatorname{Im} s| < \tau_2\} \end{aligned}$$

and

$$W = W(r_1) = \{Y \in \mathbb{C} : |Y| < r_1\}.$$

We consider the equation

$$\dot{z} = A(t+s)z + b(t,s,Y) \tag{6.4.3}$$

(here $\cdot$ denotes derivative with respect to t), where $A(u)$ is a 2×2 matrix whose elements are analytic in

$$D_0 = D_0(\tau_1^{\pm}, \tau_2) = \{u \in \mathbb{C} : \ \tau_1^- < \operatorname{Re} u < \tau_1^+, \ |\operatorname{Im} u| < \tau_2\} \tag{6.4.4}$$

and continuous in $\overline{D_0}$. The function $b : D \times W \to \mathbb{C}^2$ is continuous for any t, $|t| \leq h$ and $b(t,.,.)$ is analytic. We assume that b verifies

$$b(t+h,s,Y) = b(t,s+h,Y) \tag{6.4.5}$$

and we look for solutions $z(t,s,Y)$ of (6.4.3) analytic with respect to s and Y, and satisfying

$$z(t+h,s,Y) = z(t,s+h,Y).$$

Let $M(u)$ be a fundamental matrix of the homogeneous equation

$$\frac{d}{du}\zeta = A(u)\zeta.$$

By the general theory of linear equations, M is analytic in D_0 and there exists a constant C_M such that

$$|M(u)| \leq C_M, \qquad |M^{-1}(u)| \leq C_M, \qquad u \in D_0. \tag{6.4.6}$$

By the variation of constants method, the solutions can be expressed as

$$z(t,s,Y) = M(t+s)\Big[M^{-1}(s)c(s,Y) + \int_0^t M^{-1}(\xi+s)b(\xi,s,Y)\ d\xi\Big] \tag{6.4.7}$$

where $c(s,Y)$ is a arbitrary function. Therefore, if the function $c(s,Y)$ is analytic in $D_0 \times W$, $z(t,s,Y)$ given in (6.4.7) is continuous and analytic with respect to (s,Y) in $D_0 \times W$.

We write

$$z(t+h,s,Y) = M(t+h+s)\Big[M^{-1}(s)c(s,Y) + \int_0^{t+h} M^{-1}(\xi+s)b(\xi,s,Y)\ d\xi\Big].$$

Also

$$\begin{aligned} z(t,s+h,Y) &= M(t+s+h)\Big[M^{-1}(s+h)c(s+h,Y) \\ &\quad + \int_0^t M^{-1}(\xi+s+h)b(\xi,s+h,Y)\ d\xi\Big] \\ &= M(t+s+h)\Big[M^{-1}(s+h)c(s+h,Y) \\ &\quad + \int_h^{t+h} M^{-1}(\xi+s)b(\xi,s,Y)\ d\xi\Big] \end{aligned}$$

where we have made the obvious change of variables in the integral, and we have used (6.4.5).

We introduce the auxiliary function $f(s,Y) = M^{-1}(s)c(s,Y)$. We have that

$$z(t+h,s,Y) = z(t,s+h,Y)$$

if and only if

$$f(s,Y) - f(s+h,Y) = -\int_0^h M^{-1}(\xi+s)b(\xi,s,Y)\ d\xi. \tag{6.4.8}$$

Therefore, it is natural to study the operator

$$\triangle_h f(s,Y) = f(s+h,Y) - f(s,Y).$$

In a precise way, we want to find analytic solutions of the equation

$$\triangle_h f(s,Y) = g(s,Y), \qquad s, s+h \in D_0,\ Y \in W \tag{6.4.9}$$

where g is analytic in $D_0 \times W$.

We define the auxiliary open sets

$$D_0^- = \{s \in \mathbb{C} : \operatorname{Re} s < \tau_1^+ \text{ and } |\operatorname{Im} s| < \tau_2\}$$

and

$$D_0^+ = \{s \in \mathbb{C} : \tau_1^- < \operatorname{Re} s \text{ and } |\operatorname{Im} s| < \tau_2\}.$$

It is clear that $D_0 = D_0^+ \cap D_0^-$. For any open set $\Omega \subset \mathbb{C}$, we define the function space

$$\mathcal{A}(\Omega, W) = \{H : \overline{\Omega} \times W \to \mathbb{C} : H \text{ is analytic in } \Omega \times W \text{ and continuous in } \overline{\Omega} \times W\}.$$

The main idea of what was developed by Lazutkin in [53] is the following. Construct two analytic functions $g^+ \in \mathcal{A}(D_0^+, W)$ and $g^- \in \mathcal{A}(D_0^-, W)$ such that

$$g = g^+ + g^- \quad \text{in } D_0 \times W. \tag{6.4.10}$$

Then, because of the linearity of equation (6.4.9), the problem of finding the function f can be reduced to two simpler problems: to find two functions f^+ and f^- of $\mathcal{A}(D_0^+, W)$ and $\mathcal{A}(D_0^-, W)$ respectively such that

$$\triangle_h f^\pm = g^\pm.$$

Therefore, since the operator $\triangle_h$ is linear, the function

$$f = f^+ + f^-,$$

which is defined in $(D_0^+ \times W) \cap (D_0^- \times W) = D_0 \times W$, satisfies the equation:

$$\triangle_h f(s,Y) = \triangle_h f^+(s,Y) + \triangle_h f^-(s,Y) = g^+(s,Y) + g^-(s,Y) = g(s,Y)$$

if $s, s+h \in D_0$, $Y \in W$.

To follow the previous program the first thing we must do is to construct functions $g^\pm$, defined in the corresponding extended domain and verifying (6.4.10). This is done by using the next lemma which also provide useful bounds of the norm of $g^\pm$ in terms of the norm of g.

Lemma 6.4.2 *Let $\chi : \mathbb{C} \longrightarrow \mathbb{C}$ be a Lipschitz bounded function such that*

$$\operatorname{supp} \chi = \{\xi \in \mathbb{C} : \operatorname{Re} \xi \le \sigma\}.$$

Let

$$\Omega = \{\xi \in \mathbb{C} : \; s_1 < \operatorname{Re} \xi < s_2, \; |\operatorname{Im} \xi| < \tau_2\},$$

with $s_1 \in \mathbb{R}$, $s_1 < \sigma$ and $s_2 \in [\sigma, \infty) \cup \{\infty\}$. Let $\Omega^* = \Omega \cap \overset{\circ}{\overline{\operatorname{supp} \chi}}$ (small circle denotes topological interior) and let $g \in \mathcal{A}(\Omega^*, W)$. We define

$$h(\xi,\eta) = \frac{1}{2\pi i} \int_{\partial\Omega \cap \operatorname{supp} \chi} \frac{\chi(\zeta)}{\zeta - \xi} g(\zeta,\eta)\, d\zeta = \frac{1}{2\pi i} \int_{\partial\Omega^*} \frac{\chi(\zeta)}{\zeta - \xi} g(\zeta,\eta)\, d\zeta. \tag{6.4.11}$$

Then

1) h is analytic on $\Omega \times W$,

2) h extends continuously to $\overline{\Omega} \times W$,

3) if $(\xi_0, \eta_0) \in (\partial\Omega \cap \operatorname{supp}\chi) \times W$

$$\lim_{(\xi_0,\eta_0)} h(\xi,\eta) = \chi(\xi_0)g(\xi_0,\eta_0) + \frac{1}{2\pi i}\int_{\partial\Omega^*} \frac{\chi(\zeta)-\chi(\xi_0)}{\zeta-\xi_0} g(\zeta,\eta_0)\, d\zeta,$$

and if $(\xi_0, \eta_0) \in (\partial\Omega \cap (\operatorname{supp}\chi)^c) \times W$

$$\lim_{(\xi_0,\eta_0)} h(\xi,\eta) = \frac{1}{2\pi i}\int_{\partial\Omega^*} \frac{\chi(\zeta)-\chi(\xi_0)}{\zeta-\xi_0} g(\zeta,\eta)\, d\zeta,$$

4) if $(\xi,\eta) \in \Omega \times K$, *where* K *is a compact subset of* W *we have*

$$|h(\xi,\eta)| \leq \left(\|\chi\| + \frac{1}{2\pi} \operatorname{Lip}\chi \ \operatorname{length}(\partial\Omega^*) \right) \|g\|_K$$

where $\|\chi\| = \sup\{|\chi(\xi)| : \xi \in \mathbb{C}\}$ *and* $\|g\|_K = \sup\{|g(\xi,\eta)| : (\xi,\eta) \in \partial\Omega^* \times K\}$.

The same results hold in the case $\operatorname{supp}\chi = \{\xi \in \mathbb{C};\ \operatorname{Re}\xi \geq \sigma\}$, $s_1 \in \{-\infty\} \cup (-\infty, \sigma]$, $s_2 \in \mathbb{R}$, $s_2 > \sigma$.

Remark 6.4.3 *We observe that, in order to apply this result, we only need that the function g to be analytic in a bounded complex rectangle.*

This Lemma is a parameter (with respect to η) version of a Lemma in [53].

The proof of the present version of the Lemma can be found in [35]. Using the previous technical lemma, we will construct a right inverse of the operator $\triangle_h$.

Lemma 6.4.4 *Let* D_0 *be the set defined in (6.4.4). There is a continuous operator*

$$\triangle_h^{-1} : \mathcal{A}(D_0, W) \to \mathcal{A}(D_0, W)$$

such that given $g \in \mathcal{A}(D_0, W)$, $f = \triangle_h^{-1} g$ *is a solution of the equation*

$$f(s+h, Y) - f(s, Y) = g(s, Y) \qquad \text{for } s, s+h \in D_0,\ Y \in W \tag{6.4.12}$$

and its operator norm verifies $\|\triangle_h^{-1}\| \leq C_{D_0} e^{h/\tau_2} h^{-1}$, *where the constant* C_{D_0} *only depends on the size of the domain* D_0.

Proof. Here C_{D_0} denotes any constant which only depends on D_0. Let $\chi : \mathbb{R} \to [0,1]$, be the Lipschitz function defined by

$$\chi(u) = \begin{cases} 1 & \text{if } u \leq \tau_1^- \\ 1 - \frac{u-\tau_1^-}{\tau_1^+ - \tau_1^-} & \text{if } \tau_1^- < u < \tau_1^+ \\ 0 & \text{if } u \geq \tau_1^+. \end{cases}$$

Let $\chi_+(s) = \chi(\operatorname{Re} s)$ and $\chi_-(s) = 1 - \chi(\operatorname{Re} s)$, defined in $\mathbb{C}$. We observe that

$$\text{supp } \chi_+ = \{s \in \mathbb{C} : \operatorname{Re} s \leq \tau_1^+\}$$

and

$$\text{supp } \chi_- = \{s \in \mathbb{C} : \operatorname{Re} s \geq \tau_1^-\}.$$

Moreover it is clear that $D_0 = D_0^+ \cap \text{supp } \chi_+$ and $D_0 = D_0^- \cap \text{supp } \chi_-$. Let $\rho = \tau_2^{-1}$ and $g \in \mathcal{A}(D_0, W)$. By Lemma 6.4.2, the functions

$$g_\pm(s,Y) = \frac{1}{\cosh(\rho s)} \int_{\partial D_0} \frac{\chi_\pm(\xi)\cosh(\rho\xi)}{\xi - s} g(\xi, Y) d\xi$$

belong to $\mathcal{A}(D_0^\pm, W)$ respectively. Moreover, by 4) of Lemma 6.4.2, we have

$$\begin{aligned} |g_\pm(s,Y)| &\leq (\|\chi_\pm\| + \frac{1}{2\pi} \text{Lip } \chi_\pm \text{ length } (\partial D_0)) \frac{\|g\|}{|\cosh(\rho s)|} \max_{\xi \in \partial D_0} |\cosh(\rho\xi)| \\ &\leq C_{D_0} \|g\| \frac{1}{|\cosh(\rho s)|} \qquad \text{for } (s,Y) \in D_0^\pm \times W \end{aligned} \tag{6.4.13}$$

where $\|g\|$ means $\sup_{D_0 \times W} |g(s,Y)|$.

Now we construct the inverse of $\triangle_h$. Given $(s,Y) \in D_0^+ \times W$, we define

$$f_+(s,Y) = -\sum_{k \geq 0} g_+(s + kh, Y).$$

A direct substitution shows that f_+ satisfies (6.4.12) in $D_0^+ \times W$. In the same way, if $(s,Y) \in D_0^- \times W$,

$$f_-(s,Y) = \sum_{k \geq 1} g_-(s - kh, Y)$$

satisfies (6.4.12) in $D_0^- \times W$.

These series are convergent. Indeed, from (6.4.13) we have, for $(s,Y) \in D_0^+ \times W$

$$
\begin{aligned}
|f_+(s,Y)| &\leq \sum_{k\geq 0} |g_+(s+kh,Y)| \leq C_{D_0}\|g\| \sum_{k\geq 0} \frac{1}{|\cosh(\rho(s+kh))|} \\
&\leq C_{D_0}\|g\| \sum_{k\geq 0} \frac{1}{\cosh(\rho(\operatorname{Re} s + kh))|\cos(\rho \operatorname{Im} s)|} \\
&\leq C_{D_0}\|g\| \sum_{k\geq 0} \frac{2}{\cos(\rho \operatorname{Im} s)} e^{-\rho \operatorname{Re} s} e^{-\rho h k} \\
&\leq C_{D_0}\|g\| \frac{2}{\cos(1)} e^{-\rho \operatorname{Re} s} \sum_{k\geq 0} e^{-\rho h k} \\
&\leq C_{D_0}\|g\| \frac{1}{1-e^{-\rho h}} e^{-\rho \operatorname{Re} s} \leq C_{D_0}\|g\| h^{-1} e^{\rho h} e^{-\rho \operatorname{Re} s}.
\end{aligned}
$$

In the same way we obtain, for $(s,Y) \in D_0^- \times W$

$$|f_-(s,Y)| \leq C_{D_0}\|g\| h^{-1} e^{\rho h} e^{\rho \operatorname{Re} s}.$$

Now we consider the function $f : D_0 \times W \to \mathbb{C}$, defined by $f = f_+ + f_-$. It is clear that

$$\triangle_h f(s,Y) = \triangle_h f_+(s,Y) + \triangle_h f_-(s,Y) = g_+(s,Y) + g_-(s,Y) = g(s,Y)$$

for $s, s+h \in D_0$, $Y \in W$. Moreover, since $\rho = \tau_2^{-1}$, on $D_0 \times W$,

$$|f(s,Y)| \leq C_{D_0}\|g\| h^{-1} e^{(\tau_1+h)/\tau_2} = C_{D_0}\|g\| h^{-1} e^{h/\tau_2},$$

where C_{D_0} is a generic constant which may take different values in different formulas but only depends on D_0 and $\tau_1 = \max\{|\tau_1^+|, |\tau_1^-|\}$. Then the f so constructed solves (6.4.12). ■

6.4.3 A useful parameterization of the solutions of the system (6.3.2)

In this section we give a good parameterization of the solutions of the system associated to the vector field X_μ, passing through $x = \delta$. We introduce an additional parameter, $s \in \mathbb{C}$, to be able to reach $\{x = \delta\}$ and to obtain useful properties of the parameterization.

We recall that we denote by $z_0(u, Y)$ the solution of the unperturbed system

$$\begin{aligned} \dot{x} &= v + f(x) \\ \dot{v} &= -vf'(x) \end{aligned}$$

such that $z_0(0, Y) = (\delta, Y)$, which is analytic in $(u, Y) \in D_0(\kappa_1^\pm, \kappa_2, \kappa_0) \times W(r, \kappa_0)$ and continuous in this boundary. Therefore, since X_0 is analytic in $U_0(\delta_0', \delta_0, r_0)$, a fundamental matrix $M(u)$ of the system

$$\frac{d}{du} z = DX_0(z_0(u, Y))z$$

is analytic in $D_0(\kappa_1^\pm, \kappa_2, \kappa_0)$. Moreover as we pointed out in (6.4.6), $M(u)$ and $M^{-1}(u)$ are bounded in $D_0(\kappa_1^\pm, \kappa_2, \kappa_0)$.

Now we present the annunciated parameterization of the solutions of the system (6.3.2).

Proposition 6.4.5 *If ε and μ are small enough then the solutions of the equation (6.3.2) can be expressed as parameterized curves*

$$(z(t, s, Y), t/\varepsilon) = (x(t, s, Y), v(t, s, Y), t/\varepsilon)$$

with $(t, s, Y) \in \tilde{U}$ defined by

$$\tilde{U} = D(\kappa_1^\pm, \kappa_2, \kappa_0) \times W(r, \kappa_0),$$

satisfying the following properties:

1) $t \mapsto z(t, s, Y)$ is a solution of system (6.3.2).

2) $z(t, s, Y)$ is C^1 and analytic on (s, Y).

3) $z(t + 2\pi\varepsilon, s, Y) = z(t, s + 2\pi\varepsilon, Y)$.

4) The solution of the system (6.3.2) is of the form

$$z(t, s, Y) = z_0(t + s, Y) + \mu\varepsilon^q z_1(t, s, Y)$$

with

$$\sup_{\tilde{U}} |z_1(t, s, Y)| \leq K(\delta', \delta).$$

5) For all $Y \in W(r, \kappa_0)$, $z(0,0,Y) = (\delta, Y)$.

Proof. If

$$(z(t,s,Y), t/\varepsilon) = (z_0(t+s,Y) + \mu\varepsilon^q z_1(t,s,Y), t/\varepsilon)$$

is a solution of the equation (6.3.2), where z_0 is a solution of the unperturbed equation, it is clear that

$$\dot{z}_1 = DX_0(z_0(t+s,Y))z_1 + b(z_1)(t,s,Y) \tag{6.4.14}$$

with

$$\begin{aligned} &b(z_1)(t,s,Y) = \\ &\quad \frac{1}{\mu\varepsilon^q}[X_0(z(t,s,Y)) - X_0(z_0(t+s,Y)) - \mu\varepsilon^q DX_0(z_0(t+s,Y))z_1(t,s,Y)] \\ &\quad + X_1(z(t,s,Y), t/\varepsilon). \end{aligned} \tag{6.4.15}$$

Thus, z_1 is a solution of (6.4.14) if and only if

$$z_1(t,s,Y) = M(t+s)\left[M^{-1}(s)c(s,Y) + \int_0^t M^{-1}(\sigma+s)b(z_1)(\sigma,s,Y)d\sigma\right] \tag{6.4.16}$$

where $M(u)$ is a fundamental matrix of the homogeneous system. At this point $c(s,Y)$ is an arbitrary function. We choose the function $c(s,Y)$ as follows. We consider

$$g(z_1)(s,Y) = -\int_0^{2\pi\varepsilon} M^{-1}(\sigma+s)b(z_1)(\sigma,s,Y)d\sigma \tag{6.4.17}$$

and we take

$$c(z_1)(s,Y) \equiv c(s,Y) = M(s)\triangle_{2\pi\varepsilon}^{-1}g(z_1)(s,Y) \tag{6.4.18}$$

where $\triangle_{2\pi\varepsilon}^{-1}$ is the operator defined in Lemma 6.4.4. This choice of $c(s,Y)$ is the one which will permit us to check that an operator to be defined below is well defined in its domain.

We define Σ to be the space of functions $z_1 : \tilde{U} \to \mathbb{C}^2$ such that $z_1 \in \Sigma$ if and only if z_1 satisfies

(a) $z_1(t,s,Y)$ is C^0 and analytic on (s,Y).

(b) For all $(t,s,Y)\in\tilde{U}$, we have that

$$z_1(t+2\pi\varepsilon,s,Y)=z_1(t,s+2\pi\varepsilon,Y).$$

(c) $\|z_1\|=\sup_{\tilde{U}}|z_1(t,s,Y)|<+\infty$.

We endow Σ with the supremum norm and it becomes a Banach space. For any $\rho>0$, we define $\Sigma(\rho)$ as the closed ball of radius ρ of Σ. We define the operator $\mathcal{G}:\Sigma(\rho)\to\Sigma(\rho)$ to be the right hand side of (6.4.16):

$$\mathcal{G}(z_1)(t,s,Y)=M(t+s)\left[M^{-1}(s)c(z_1)(s,Y)+\int_0^t M^{-1}(\sigma+s)b(z_1)(\sigma,s,Y)d\sigma\right]$$

with $c(s,Y)$ chosen as (6.4.18). Our goal is to prove that $\mathcal{G}$ has a fixed point in $\Sigma(\rho)$. For that we will see that $\mathcal{G}$ is well defined and that it is a contraction in $\Sigma(\rho)$.

First we prove that it is well defined. Let $z_1\in\Sigma(\rho)$, then, by Remark 6.4.15

$$z(t,s,Y)\in U(\delta'/3,3\delta,3r,0)\subset U(\delta_0',\delta_0,r_0,0)$$

and the function $b(z_1)$ given in (6.4.15) is well defined. Moreover, it is clear that, since $M(t+s)$, $M^{-1}(t+s)$ and $z_1(t,s,Y)$ are C^0 and analytic on (s,Y), the function g defined in (6.4.17) is analytic in $\tilde{U}_0$ with

$$\tilde{U}_0=D_0(\kappa_1^{\pm},\kappa_2,\kappa_0)\times W(r,\kappa_0).$$

Therefore, by Lemma 6.4.4, the function $c(z_1)(s,Y)$ is analytic in $\tilde{U}_0$. Thus $\mathcal{G}(z_1)(t,s,Y)$ is also C^0 and analytic on (s,Y).

Now we prove that the property (b) holds for $\mathcal{G}(z_1)$. It is clear that, since $z_1\in\Sigma$ and $X_1(x,v,\theta)$ is 2π-periodic in θ,

$$b(z_1)(t+2\pi\varepsilon,s,Y)=b(z_1)(t,s+2\pi\varepsilon,Y).$$

Then

$$\begin{aligned}\mathcal{G}(z_1)(t+2\pi\varepsilon,s,Y) &= M(t+2\pi\varepsilon+s)\Big[M^{-1}(s)c(s,Y)\\ &\quad+\int_0^{t+2\pi\varepsilon} M^{-1}(\sigma+s)b(z_1)(\sigma,s,Y)d\sigma\Big]\\ &= M(t+s+2\pi\varepsilon)\Big[M^{-1}(s)c(s,Y)\\ &\quad+\int_{-2\pi\varepsilon}^{t} M^{-1}(\sigma+2\pi\varepsilon+s)b(z_1)(\sigma,s+2\pi\varepsilon,Y)d\sigma\Big]\end{aligned}$$

and

$$\begin{aligned}\mathcal{G}(z_1)(t, s+2\pi\varepsilon, Y) &= M(t+s+2\pi\varepsilon)\Big[M^{-1}(s+2\pi\varepsilon)c(s+2\pi\varepsilon, Y)\\ &\quad + \int_0^t M^{-1}(\sigma+2\pi\varepsilon+s)b(z_1)(\sigma, s+2\pi\varepsilon, Y)d\sigma\Big].\end{aligned}$$

Thus, $\mathcal{G}(z_1)(t, s+2\pi\varepsilon, Y) = \mathcal{G}(z_1)(t+2\pi\varepsilon, s, Y)$ if and only if

$$M^{-1}(s)c(s,Y) - M^{-1}(s+2\pi\varepsilon)c(s+2\pi\varepsilon, Y) = -\int_0^{2\pi\varepsilon} M^{-1}(\sigma+s)b(z_1)(\sigma, s, Y)d\sigma.$$

This last equality holds by definition of c in (6.4.18).

Next we will see that if we chose ρ in a suitable way, $\mathcal{G}(\Sigma(\rho)) \subset \Sigma(\rho)$. Indeed, let C_M be a constant such that $\|M(u)\|$, $\|M^{-1}(u)\| \leq C_M$. We recall that,

$$f(s,Y) = M^{-1}(s)c(s,Y).$$

We recall that we have defined $\varkappa = \min\{k-1, l\} > 0$. By Lemmas 6.4.4 and 6.3.1 we obtain

$$\|f\| \leq C_{D_0}\frac{e^{2\pi\varepsilon/\tau_2}}{2\pi\varepsilon}\|g(z_1)\| \leq C_M C_{D_0}\|b(z_1)\| \leq C_M C_{D_0}[K(\delta,\delta')|\mu|\varepsilon^q\|z_1\|^2 + K\delta^{\varkappa}].$$

Thus

$$\begin{aligned}\|\mathcal{G}(z_1)\| &\leq C_M^2 C_{D_0}(K(\delta,\delta')|\mu|\varepsilon^q\|z_1\|^2 + K\delta^{\varkappa}) + C_M^2|t|(K(\delta,\delta')|\mu|\varepsilon^q\|z_1\|^2 + K\delta^{\varkappa})\\ &\leq C_M^2 C_{D_0}(K(\delta,\delta')|\mu|\varepsilon^q\rho^2 + K\delta^{\varkappa}) + C_M^2 2\pi\varepsilon(K(\delta,\delta')|\mu|\varepsilon^q\rho^2 + K\delta^{\varkappa})\\ &\leq \rho\end{aligned}$$

if $\rho = 2C_M^2 K(C_{D_0} + 2\pi\varepsilon)\delta^{\varkappa}$ and $|\mu|\varepsilon^q$ is small enough.

Therefore, $\mathcal{G}(\sigma) \in \Sigma(\rho)$, and the operator $\mathcal{G}$ is well defined.

Finally we prove that $\mathcal{G}$ is a contraction. Let z_1 and z_2 be two functions that belong to $\Sigma(\rho)$:

$$\begin{aligned}|(\mathcal{G}(z_1) - \mathcal{G}(z_2))(t,s,Y)| \leq \Big|M(t+s)\Big[M^{-1}(s)(c(z_1)(s,Y) - c(z_2)(s,Y))\\ + \int_0^t M^{-1}(\sigma+s)(b(z_1)(\sigma,s,Y) - b(z_2)(\sigma,s,Y))d\sigma\Big]\Big|.\end{aligned} \tag{6.4.19}$$

We observe that, since the operator $\triangle_{2\pi\varepsilon}^{-1}$ is linear

$$M^{-1}(s)c(z_1) - M^{-1}(s)c(z_2) = \triangle_{2\pi\varepsilon}^{-1}(g(z_1) - g(z_2)).$$

By Lemma 6.4.4 we have

$$\|M^{-1}(s)c(z_1) - M^{-1}(s)c(z_2)\| \le C_{D_0}\frac{e^{2\pi\varepsilon/\tau_2}}{2\pi\varepsilon}\|g(z_1) - g(z_2)\|. \tag{6.4.20}$$

Now we bound $\|b(z_1) - b(z_2)\|$. Until the end of the proof, z_0, z_1 and z_2 will stand for $z_0(t+s, Y)$, $z_1(t, s, Y)$ and $z_2(t, s, Y)$ respectively.

It is clear that we can write $b(z_1) - b(z_2)$ as

$$\begin{aligned}
b(z_1) - b(z_2) &= \frac{1}{\mu\varepsilon^q}[X_0(z_0 + \mu\varepsilon^q z_1) - X_0(z_0 + \mu\varepsilon^q z_2) - \mu\varepsilon^q DX_0(z_0)(z_1 - z_2)] \\
&\quad + X_1(z_0 + \mu\varepsilon^q z_1, t/\varepsilon) - X_1(z_0 + \mu\varepsilon^q z_2, t/\varepsilon) \\
&= \frac{1}{\mu\varepsilon^q}(X_0(z_0 + \mu\varepsilon^q z_1) - X_0(z_0 + \mu\varepsilon^q z_2)) \\
&\quad - DX_0(z_0 + \mu\varepsilon^q z_1)(z_1 - z_2) \\
&\quad + DX_0(z_0 + \mu\varepsilon^q z_1)(z_1 - z_2) - DX_0(z_0)(z_1 - z_2) \\
&\quad + X_1(z_0 + \mu\varepsilon^q z_1, t/\varepsilon) - X_1(z_0 + \mu\varepsilon^q z_2, t/\varepsilon).
\end{aligned}$$

Using the bounds of Lemma 6.3.1 we get

$$\begin{aligned}
\|b(z_1) - b(z_2)\| &\le |\mu|\varepsilon^q K(\|z_1 - z_2\|^2 + \|z_1\|\|z_1 - z_2\|) \\
&\quad + (\mathrm{Lip}\ X_1)|\mu|\varepsilon^q\|z_1 - z_2\| \\
&= K(\delta', \delta)|\mu|\varepsilon^q\|z_1 - z_2\|.
\end{aligned} \tag{6.4.21}$$

Moreover, it is clear that

$$\|g(z_1) - g(z_2)\| \le C_M 2\pi\varepsilon\|b(z_1) - b(z_2)\|.$$

Then, using (6.4.20) and (6.4.21) in (6.4.19), we obtain

$$\begin{aligned}
\|\mathcal{G}(z_1) - \mathcal{G}(z_2)\| &\le C_M C_{D_0}\frac{e^{2\pi\varepsilon/\tau_2}}{2\pi\varepsilon}\|g(z_1) - g(z_2)\| + C_M^2|t|\|b(z_1) - b(z_2)\| \\
&\le K(D_0, M, \delta', \delta)\|b(z_1) - b(z_2)\| \\
&\le K(D_0, M, \delta', \delta)|\mu|\varepsilon^q\|z_1 - z_2\| \\
&\le \frac{1}{2}\|z_1 - z_2\|
\end{aligned}$$

if $|\mu|\varepsilon^q$ is small enough.

Therefore, since $\mathcal{G}$ is a contraction, by the fixed point theorem, there exists a unique $z_1 \in \Sigma(\rho)$ such that $z_0(t+s, Y) + \mu\varepsilon^q z_1(t, s, Y)$ satisfies the conclusions of the proposition, except that z is C^0. We recall that the function satisfies the equation

$$z(t, s, Y) = z(0, s, Y) + \int_0^t X_\mu(z(\sigma, s, Y), \sigma/\varepsilon)d\sigma.$$

Therefore z is C^1, and the proposition holds. ■

6.4.4 Proof of the Theorem 6.2.5

The proof of the Theorem 6.2.5 has two parts. The first one consists on constructing flow box coordinates in $D_0 \times W$ by using the previous proposition. The change of coordinates so obtained may be non canonical. In the second step we modify these flow box coordinates in such way they become canonical.

We begin by defining

$$w(u, \theta, Y) = z(\varepsilon\theta, u - \varepsilon\theta, Y).$$

Note that w is C^1 and analytic with respect to its first and third variables for $(u, Y) \in D_0(\kappa_1^{\pm}, \kappa_2, \kappa_0) \times W(r, \kappa_0)$. Moreover, since the solutions of the system (6.3.2) satisfy that $z(t + 2\pi\varepsilon, s, Y) = z(t, s + 2\pi\varepsilon, Y)$, we have that w is 2π-periodic respect to its second variable. This is a very important property because let us to extend the domain of w with respect to the θ variable, that is, we can consider w in the domain

$$D_0(\kappa_1^{\pm}, \kappa_2, \kappa_0) \times \mathbb{R} \times W(r, \kappa_0).$$

We have that

$$(w(t + s, t/\varepsilon, Y), t/\varepsilon) \tag{6.4.22}$$

is a new parameterization of the solutions of (6.3.2). Indeed,

$$\begin{aligned}
\partial_t[w(t + s, t/\varepsilon, Y)] &= \partial_u w(t + s, t/\varepsilon, Y) + \frac{1}{\varepsilon}\partial_\theta w(t + s, t/\varepsilon, Y) \\
&= \partial_s z(t, s, Y) + [\partial_t z(t, s, Y)\varepsilon + \partial_s z(t, s, Y)(-\varepsilon)](1/\varepsilon) \\
&= \partial_t z(t, s, Y).
\end{aligned}$$

We observe that, if $\mu = 0$, $w(u, \theta, Y) \equiv w_0(u, Y) = z_0(u, Y)$. We denote $z_1(\varepsilon\theta, u - \varepsilon\theta, Y)$ by $w_1(u, \theta, Y)$ and hence

$$w(u, \theta, Y) = w_0(u, Y) + \mu\varepsilon^q w_1(u, \theta, Y).$$

In the previous arguments we have not mentioned explicitely the dependence on the parameters, but it is clear that the continuity on ε and the analyticity on μ is manteined, and in particular w is C^0 in ε and C^w in μ.

Lemma 6.4.6 *Let $\delta \leq \delta_0/3$. Under the hypotheses **H1-H3**, for any $\delta' < \delta$ there exists $r > 0$ small enough and two unique functions $\mathcal{T}$ and $\mathcal{Y}$ defined in $U(\delta', \delta, r, \eta_0)$ such that*

$$w(\mathcal{T}(x, v, \theta), \theta, \mathcal{Y}(x, v, \theta)) = (x, v). \tag{6.4.23}$$

These functions are C^1, analytical in the (x, v) variables and 2π-periodic in θ. Moreover

$$\mathcal{T}(x, y, \theta) = \mathcal{T}_0(x, y) + O(\mu\varepsilon^q), \qquad \mathcal{Y}(x, y, \theta) = \mathcal{Y}_0(x, y) + O(\mu\varepsilon^q)$$

where $\mathcal{T}_0$ and $\mathcal{Y}_0$ are defined in (6.3.5) and (6.3.6).

Proof. We define the function

$$G(S, Y, x, v, \theta, \mu, \varepsilon) = w(S, \theta, Y, \mu, \varepsilon) - (x, v).$$

on the set

$$D_0(\kappa_1^{\pm}, \kappa_2, \kappa_0) \times W(r, \kappa_0) \times U_0(\delta', \delta, r) \times \mathbb{R} \times P$$

where

$$P = \{(\mu, \varepsilon) \in \mathbb{C} \times \mathbb{R} : |\mu| < \mu_0 \text{ and } 0 < \varepsilon < \varepsilon_0\}$$

with μ_0 and ε_0 small enough. Here we put explicitly the dependence on μ and ε of the solutions.

By the definitions of $\mathcal{T}_0$ and $\mathcal{Y}_0$ in (6.3.5) and (6.3.6) we have that, when $\mu = 0$,

$$z_0(\mathcal{T}_0(x, v), \mathcal{Y}_0(x, v)) = \psi_0(\mathcal{T}_0(x, v), \delta, \mathcal{Y}_0(x, v)) = (x, v) \tag{6.4.24}$$

where ψ_0 is introduced in Section 6.3.2. Then

$$\begin{aligned} G(\mathcal{T}_0(x, v), \mathcal{Y}_0(x, v), x, v, \theta, 0, \varepsilon) &= w(\mathcal{T}_0(x, v), \theta, \mathcal{Y}_0(x, v), 0, \varepsilon) - (x, v) \\ &= z_0(\mathcal{T}_0(x, v), \mathcal{Y}_0(x, v)) - (x, v) \\ &= 0. \end{aligned}$$

We observe that, by Proposition 6.4.5, G is analytic on $(S,Y) \in D_0(\kappa_1^{\pm}, \kappa_2, \kappa_0) \times W(r, \kappa_0)$.

Next we study the matrix $D_{S,Y}G$. Since the unperturbed system is Hamiltonian with Hamiltonian

$$\mathcal{F}_0(x,v) = \frac{v^2}{2} + vf(x)$$

the solution $z_0(u,Y) = (z_0^1(u,Y), z_0^2(u,Y))$ satisfies

$$\frac{(z_0^2(u,Y))^2}{2} + z_0^2(u,Y)f(z_0^1(u,Y)) = \frac{Y^2}{2} + Yf(\delta). \tag{6.4.25}$$

Differentiating with respect to Y in (6.4.25) we obtain

$$\begin{aligned}(z_0^2(u,Y) + f(z_0^1(u,Y)))\partial_Y z_0^2(u,Y) + z_0^2(u,Y)f'(z_0^1(u,Y))\partial_Y z_0^1(u,Y)& \\ = Y + f(\delta).&\end{aligned} \tag{6.4.26}$$

Evaluating this expression at $(u,Y) = (\mathcal{T}_0(x,v), \mathcal{Y}_0(x,v))$, we get

$$(v + f(x))\partial_Y z_0^2(\mathcal{T}_0, \mathcal{Y}_0) + vf'(x)\partial_Y z_0^1(\mathcal{T}_0, \mathcal{Y}_0) = \mathcal{Y}_0 + f(\delta).$$

Now we prove that the derivative $\partial_{SY}G$ at $\mu = 0$ is invertible. Of course,

$$D_{S,Y}G(S,Y,\theta,0,\varepsilon) = \begin{pmatrix} \partial_S z_0^1(S,Y) & \partial_Y z_0^1(S,Y) \\ \partial_S z_0^2(S,Y) & \partial_Y z_0^2(S,Y) \end{pmatrix}.$$

Using (6.4.24) and (6.4.26), its determinant evaluated at $(S,Y) = (\mathcal{T}_0(x,v), \mathcal{Y}_0(x,v))$ is

$$\begin{aligned}\det(D_{SY}G) &= \partial_S z_0^1(\mathcal{T}_0, \mathcal{Y}_0)\partial_Y z_0^2(\mathcal{T}_0, \mathcal{Y}_0) - \partial_Y z_0^1(\mathcal{T}_0, \mathcal{Y}_0)\partial_S z_0^2(\mathcal{T}_0, \mathcal{Y}_0) \\ &= X_0^1(x,v)\partial_Y z_0^2(\mathcal{T}_0, \mathcal{Y}_0) - X_0^2(x,v)\partial_Y z_0^1(\mathcal{T}_0, \mathcal{Y}_0) \\ &= \mathcal{Y}_0 + f(\delta)\end{aligned}$$

and by definition (6.3.6) of $\mathcal{Y}_0$, we obtain

$$\det(D_{SY}G) = -\sqrt{f^2(\delta) + v^2 + 2vf(x)}.$$

We recall that $|v| < r < r_0$, and that $C_0\delta^\beta \le |f(\delta)| \le C_1\delta^\beta$, hence, by Definition 6.2.4,

$$\begin{aligned}|\det(D_{SY}G)|^2 &\ge f(\delta)^2 - |v|^2 - 2|vf(x)| \\ &\ge C_0^2\delta^{2\beta} - r_0^2 - 2rC_1\delta^\beta \\ &> 0.\end{aligned}$$

At this point it would be natural to apply the implicit function theorem to the equation

$$G(S, Y, x, v, \theta, \mu, \varepsilon) = 0.$$

However to have a good control on the domains in which we will find the solution $\mathcal{T}$, $\mathcal{Y}$ in terms of $(x, v, \theta, \mu, \varepsilon)$ we follow the proof of the implicit function theorem using the special structure of the equation we have to deal with. We will work in a space of function

$$(S, Y) = h(x, v, \theta, \mu, \varepsilon).$$

In the rest of this proof we take the norm

$$\|(\xi, \eta)\| = \max\{|\xi|, |\eta|\}$$

for $(\xi, \eta) \in \mathbb{C}^2$. We define the space Γ of functions $h : U_0 \times \mathbb{R} \times P \to \mathbb{C}^2$ which satisfy (we call $(x, v, \theta, \mu, \varepsilon)$ the variables of h)

(a) h is C^0.

(b) h is analytic in $(x, v, \mu) \in U_0(\delta', \delta, r, \eta_0) \times \{\mu \in \mathbb{C} : |\mu| \leq \mu_0\}$.

(c) h is C^1 with respect to θ and 2π-periodic in θ.

(d) The norm

$$\begin{aligned}\|h\|_\Gamma &= \sup_{U_0\times\mathbb{R}\times P} \|h(x, v, \theta, \mu, \varepsilon)\| + \sup_{U_0\times\mathbb{R}\times P} \|\partial_\theta h(x, v, \theta, \mu, \varepsilon)\| \\ &\equiv \|h\|_\infty + \|\partial_\theta h\|_\infty\end{aligned}$$

is bounded.

We endow Γ with the norm $\|.\|_\Gamma$ and it becomes a Banach space. We call $\Gamma(\rho)$ the closed ball of radius ρ of Γ, centered at $(\mathcal{T}_0(x, v), \mathcal{Y}_0(x, v)) \in \Gamma$. We observe that, since $(\mathcal{T}_0(x, v), \mathcal{Y}_0(x, v))$ do not depend on θ, for any $h \in \Gamma(\rho)$,

$$\|h - (\mathcal{T}_0, \mathcal{Y}_0)\|_\Gamma = \|h - (\mathcal{T}_0, \mathcal{Y}_0)\|_\infty + \|\partial_\theta h\|_\infty$$

We define the operator $\mathcal{G} : \Gamma(\rho) \to \Gamma(\rho)$ by

$$\mathcal{G}(h)(x, v, \theta, \mu, \varepsilon) = h - (D_{S,Y} G(\mathcal{T}_0, \mathcal{Y}_0, \theta, 0, \varepsilon))^{-1} G(h, x, v, \theta, \mu, \varepsilon),$$

where in the right hand side, $h = h(x, v, \theta, \mu, \varepsilon)$, $\mathcal{T}_0 = \mathcal{T}_0(x, v)$ and $\mathcal{Y}_0 = \mathcal{Y}_0(x, v)$. $\mathcal{G}$ is well defined. Indeed, let $h \in \Gamma(\rho)$ with ρ small enough. By Section 6.4.1

$$(\mathcal{T}_0, \mathcal{Y}_0) \in D_0(\kappa_1^{\pm}, \kappa_2, 0) \times W(r, 0),$$

thus , if ρ is small enough, $h \in D_0(\kappa_1^{\pm}, \kappa_2, \kappa_0) \times W(r, \kappa_0)$ and then $\mathcal{G}(h) \in \Gamma$. Next we will check that $\mathcal{G}(h) \in \Gamma(\rho)$.

To shorten the notation we will not write the dependence on the variables $(x, v, \theta, \varepsilon)$, and we will denote $(\mathcal{T}_0, \mathcal{Y}_0)$ by h_0. By Taylor's theorem,

$$\begin{aligned}
\mathcal{G}(h)(\mu) &= h - (D_{S,Y}G(h_0, 0))^{-1}G(h, \mu) \\
&= h - (D_{S,Y}G(h_0, 0))^{-1}\Big(G(h_0, 0) + DG(h_0, 0)(h - h_0, \mu)^T \\
&\quad + \int_0^1 (DG(\mathcal{Z}(\zeta)) - DG(h_0, 0))(h - h_0, \mu)^T d\zeta\Big)
\end{aligned}$$

where $DG \equiv (\partial_S G, \partial_Y G, \partial_\mu G)$ and $\mathcal{Z}(\zeta) = (h_0 + \zeta(h - h_0), \zeta\mu)$. We observe that G is well defined in $\mathcal{Z}(\zeta)$ for all $\zeta \in [0, 1]$. Then, using that

$$G(h_0, 0) = G(\mathcal{T}_0, \mathcal{Y}_0, 0) = 0,$$

we obtain

$$\begin{aligned}
\mathcal{G}(h)(\mu) &= h_0 - (D_{S,Y}G(h_0, 0))^{-1}\Big(\mu\partial_\mu G(h_0, 0) \\
&\quad - \int_0^1 (DG(\mathcal{Z}(\zeta)) - DG(h_0, 0))(h - h_0, \mu)^T d\zeta\Big).
\end{aligned}$$

We observe that

$$\partial_\mu G(h_0, \mu) = \varepsilon^q w_1(\mathcal{T}_0, \theta, \mathcal{Y}_0, \mu, \varepsilon) + \mu\varepsilon^q \partial_\mu w_1(\mathcal{T}_0, \theta, \mathcal{Y}_0, \mu, \varepsilon). \tag{6.4.27}$$

Since G is analytical in h, G has its second derivative with respect to h bounded in $D_0(\kappa_1^{\pm}, \kappa_2, \kappa_0) \times W(r, \kappa_0)$. Therefore

$$\|\mathcal{G}(h) - h_0\|_\infty \le K(\delta', \delta)(|\mu|\varepsilon^q + \rho^2 + \rho|\mu|) \le \rho/2$$

if ρ and $|\mu|\varepsilon^q$ are small enough. Here we have used that $\|\partial_\mu G(h_0, 0)\|_\infty = O(\varepsilon^q)$ is bounded and that, by the mean value theorem,

$$\|D_{S,Y}G(\mathcal{Z}(\zeta)) - D_{S,Y}G(h_0, 0))(h - h_0)\|_\infty \le K(\delta', \delta)\rho(\rho + |\mu|).$$

Using (6.4.27), we have that

$$\partial_{\theta\mu} G = O(\varepsilon^q).$$

Moreover, since h_0 does not depends on θ, and that

$$\partial_\theta w(S, \theta, Y) = O(\mu\varepsilon^q)$$

and consequently $\partial_\theta D_{SY} G = O(\mu\varepsilon^q)$, we have that

$$\begin{aligned}
\|\partial_\theta \mathcal{G}(h)(\mu)\| &\leq |\mu||K(\delta', \delta) \int_0^1 |\partial_\theta (DG(\mathcal{Z}(\zeta)) - DG(h_0, 0))(h - h_0, \mu)^T| d\zeta \\
&\leq |\mu| K(\delta', \delta)| \Big(\int_0^1 \zeta |\partial_\theta DG(\mathcal{Z}(\zeta))(\partial_\theta h, 0)^T (h - h_0, \mu)^T| d\zeta \\
&\quad + \int_0^1 |(DG(\mathcal{Z}(\zeta)) - DG(h_0, 0))(\partial_\theta h, 0)^T d\zeta \Big) \\
&\leq |\mu| K(\delta', \delta)| (|\mu| \varepsilon^q \rho^2 + |\mu| \varepsilon^q + \rho^3) \leq \rho/2
\end{aligned}$$

if $|\mu|\varepsilon^q$ is small enough. In fact we can take $\rho = O(\mu\varepsilon^q)$.

This operator is a contraction, thus the fixed point theorem can be applied and we find functions $\mathcal{T}$ and $\mathcal{Y}$ such that for any $(x, v, \theta, \mu, \varepsilon) \in U(\delta', \delta, r) \times P$

$$w(\mathcal{T}, \theta, \mathcal{Y}, \mu, \varepsilon) = (x, v). \tag{6.4.28}$$

■

Now we prove that the flow can be straightened in $U(\delta', \delta, r, \eta_0)$.

Proposition 6.4.7 *Let $\delta \leq \delta_0/3$. If the hypotheses **H1-H3** hold, for any $\delta' < \delta$ there exists $r > 0$ and a change of variables*

$$(x, v, \theta = \frac{t}{\varepsilon}) \in U(\delta', \delta, r) \mapsto (T, F, \theta) = (\mathcal{T}(x, v, \theta), \mathcal{F}(x, v, \theta), \theta) \in \tilde{V}$$

analytic in the x, v variables, C^1 and 2π-periodic in θ, such that it transforms the system (6.3.2) to

$$\begin{aligned}
\dot{T} &= 1 \\
\dot{F} &= 0 \\
\dot{\theta} &= 1/\varepsilon
\end{aligned}$$

and satisfies $\mathcal{T}(x, v, \theta) = \mathcal{T}_0(x, v) + O(\mu\varepsilon^q)$, $\mathcal{F}(x, v, \theta) = \mathcal{F}_0(x, v) + O(\mu\varepsilon^q)$ where $(x, v) \mapsto (\mathcal{T}_0(x, v), \mathcal{F}_0(x, v))$ is the corresponding change for the unperturbed system and is given in (6.3.5) and (6.3.3).

Proof. We fix $(x, v) \in U_0(\delta', \delta, r, \eta_0)$ and we consider the solution $\psi(t)$ of the system (6.3.2) such that $\psi(0) = (x, v, 0)$. By Lemma 6.4.6, there exist $\mathcal{T}(x, v, 0)$ and $\mathcal{Y}(x, v, 0)$ such that

$$w(\mathcal{T}(x, v, 0), 0, \mathcal{Y}(x, v, 0)) = (x, v).$$

Moreover since the solutions of (6.3.2) can be parameterized as (6.4.22), taking $s = \mathcal{T}$ and $Y = \mathcal{Y}$ in (6.4.22) we obtain that

$$\tilde{\psi}(t) \equiv (w(\mathcal{T}(x, v, 0) + t, t/\varepsilon, \mathcal{Y}(x, v, 0)), t/\varepsilon), \tag{6.4.29}$$

also is a solution of (6.3.2) such that $\tilde{\psi}(0) = (x, v, 0) = \psi(0)$. By uniqueness, $\tilde{\psi} = \psi$. On the other hand, if t is such that, $\psi(t) \in U(\delta', \delta, r, \eta_0)$, by Lemma 6.4.6, applying (6.4.23) with $(x, v, \theta) = \psi(t)$ we obtain

$$\psi(t) = (w(\mathcal{T}(\psi(t)), t/\varepsilon, \mathcal{Y}(\psi(t))), t/\varepsilon). \tag{6.4.30}$$

Therefore (6.4.29) and (6.4.30) give us two expressions for the same solution $\psi(t)$. We observe that,

$$\mathcal{T}_0(x, v) + t = \mathcal{T}_0(\psi_0(t))$$

therefore, by the uniqueness of the functions $\mathcal{T}$ and $\mathcal{Y}$ given in Lemma 6.4.6, we have

$$\begin{aligned} \mathcal{T}(\psi(t)) &= \mathcal{T}(x, v, 0) + t \\ \mathcal{Y}(\psi(t)) &= \mathcal{Y}(x, v, 0) \end{aligned} \tag{6.4.31}$$

and then

$$\begin{aligned} \frac{d}{dt}\mathcal{T}(\psi(t)) &= 1 \\ \frac{d}{dt}\mathcal{Y}(\psi(t)) &= 0. \end{aligned}$$

We define a new function

$$\mathcal{F}(x, v, \theta) = \mathcal{F}_0(\delta, \mathcal{Y}(x, v, \theta))$$

where $\mathcal{F}_0$ is the Hamiltonian of the unperturbed system given in (6.3.3). We recall that

$$\|\mathcal{Y} - \mathcal{Y}_0\|_\infty = O(\mu\varepsilon^q),$$

then, since $\mathcal{F}_0$ is constant along the trajectories of the unperturbed system,

$$\begin{aligned}\mathcal{F}(x,v,\theta) &= \mathcal{F}_0(\delta, \mathcal{Y}_0(x,v)) + O(\mu\varepsilon^q)\\ &= \mathcal{F}_0(x,v) + O(\mu\varepsilon^q).\end{aligned}$$

Therefore, from (6.4.31), it is easily seen that

$$(T,F,\theta) = (\mathcal{T}(x,v,\theta), \mathcal{F}(x,v,\theta), \theta)$$

transforms (6.3.2) in $U(\delta', \delta, r)$ to

$$\begin{aligned}\dot{T} &= 1\\ \dot{F} &= 0\\ \dot{\theta} &= 1/\varepsilon\end{aligned}$$

and the statement holds. ■

Now we turn to modify the change of variables to get a canonical one. Before starting the result we need some preliminar calculations.

We denote by $\psi(t,x,v)$ the solution of the system (6.3.2) such that $(x,v,0) = \psi(0)$. In the proof of the Proposition 6.4.7, concretely in (6.4.31), we have seen that

$$\begin{aligned}\mathcal{T}(\psi(t)) &= t + \mathcal{T}(x,v,0)\\ \mathcal{F}(\psi(t)) &= \mathcal{F}(x,v,0).\end{aligned} \tag{6.4.32}$$

We introduce the matrix

$$\Phi(t) = \begin{pmatrix} \partial_x \mathcal{T}(\psi(t)) & \partial_v \mathcal{T}(\psi(t)) \\ \partial_x \mathcal{F}(\psi(t)) & \partial_v \mathcal{F}(\psi(t)) \end{pmatrix}.$$

Differentiating with respect to (x,v) in both sides of (6.4.32), we obtain

$$\Phi(t) \begin{pmatrix} \partial_x \psi_1(t) & \partial_v \psi_1(t) \\ \partial_x \psi_2(t) & \partial_v \psi_2(t) \end{pmatrix} = \Phi(0).$$

Since system (6.3.2) is Hamiltonian,

$$\det \begin{pmatrix} \partial_x \psi_1(t) & \partial_v \psi_1(t) \\ \partial_x \psi_2(t) & \partial_v \psi_2(t) \end{pmatrix} = 1$$

and we have that

$$\det \Phi(t) = \det \Phi(0) \tag{6.4.33}$$

for all t for which the solution is defined. Moreover, we know that for $\mu = 0$, $\det \Phi(t) = \det \Phi(0) = 1$, thus

$$\det \Phi(t) = 1 + \mu\varepsilon^q \tilde{g}(\psi(t)) \tag{6.4.34}$$

where $\tilde{g} = \tilde{g}(x, v, \theta)$ is some C^1 function, analytic in (x, v) and 2π-periodic in θ. Moreover from (6.4.33) it is clear that

$$\frac{d}{dt}\tilde{g}(\psi(t)) = 0. \tag{6.4.35}$$

We define the function $g : \tilde{U} \to \mathbb{C}$, by

$$g(T, F, \theta) = \tilde{g}(w(T, \theta, f(\delta) - \sqrt{f^2(\delta) + 2F}), \theta).$$

The function g is C^1, analytic in (T, F) and 2π-periodic in θ. If we differentiate with respect to the time, t, in g evaluated on the solutions of $\dot{T} = 1$, $\dot{F} = 0$, $\dot{\theta} = 1/\varepsilon$, by (6.4.35), we have the following equality:

$$0 = \partial_T g + \frac{1}{\varepsilon}\partial_\theta g. \tag{6.4.36}$$

To deal with equation (6.4.36), we define the change $(\xi, \eta) = (T + \varepsilon\theta, T - \varepsilon\theta)$ and the function

$$h(\xi, \eta, F) = g((\xi + \eta)/2, F, (\xi - \eta)/2\varepsilon)$$

defined in

$$\{(\xi, \eta, F) \in \mathbb{C}^3 : ((\xi + \eta)/2, F, (\xi - \eta)/2\varepsilon) \in \tilde{U}\}.$$

Then, by (6.4.36)

$$\partial_\xi h = 0$$

therefore,

$$g(T, F, \theta) = h(T + \varepsilon\theta, T - \varepsilon\theta, F) = h(0, T - \varepsilon\theta, F)$$

is a function that only depends on first integrals of the system (6.3.2). We define the function $\rho(F, S)$ by the condition

$$1 + \partial_F \rho(F, S) = \frac{1}{1 + \mu\varepsilon^q h(0, S, F)}.$$

We remark that, since h is analytic in F and S, the function ρ is $O(\mu\varepsilon^q)$.

Proposition 6.4.8 *The change of variables defined by*

$$(x, v, \theta = t/\varepsilon) \in U(\delta', \delta, r) \to (T, I, \theta) = (\mathcal{T}(x, v, \theta), \mathcal{I}(x, v, \theta), \theta) \in V$$

with $\mathcal{I}(x, v, \theta) = \mathcal{F}(x, v, \theta) + \rho(\mathcal{F}(x, v, \theta), \mathcal{T}(x, v, \theta) - \theta\varepsilon)$ *is canonical and is such that transforms the system (6.3.2) to*

$$\begin{aligned}\dot{T} &= 1\\ \dot{I} &= 0\\ \dot{\theta} &= 1/\varepsilon.\end{aligned}$$

Moreover, $\mathcal{T}(x, v, \theta) = \mathcal{T}_0(x, v) + O(\mu\varepsilon^q)$ *and* $\mathcal{I}(x, v, \theta) = \mathcal{F}_0(x, v) + O(\mu\varepsilon^q)$ *where* $\mathcal{T}_0$ *and* $\mathcal{F}_0$ *is the corresponding change in the unperturbed case.*

Proof. Let $\mathcal{I} = \mathcal{I}(x, v, \theta)$ as in the statement. Along the solutions of (6.3.2),

$$\dot{I} = \dot{F} + D_F\rho(F, S)\dot{F} + D_S\rho(F, S)(\dot{T} - \varepsilon\dot{\theta}) = 0.$$

Thus, this change transforms the system (6.3.2) to $\dot{T} = 1$, $\dot{I} = 1$, $\dot{\theta} = 1/\varepsilon$.

To see that the change is canonical we only have to calculate the determinant of

$$C(x, v, \theta) = \begin{pmatrix} \partial_x\mathcal{T} & \partial_v\mathcal{T} \\ \partial_x\mathcal{I} & \partial_v\mathcal{I} \end{pmatrix}.$$

We have

$$\begin{aligned}
\det C(x, v, \theta) &= \partial_x\mathcal{T}\partial_v\mathcal{I} - \partial_v\mathcal{T}\partial_x\mathcal{I}\\
&= [\partial_v\mathcal{F} + \partial_F\rho(\mathcal{F}, \mathcal{T} - \theta\varepsilon)\partial_v\mathcal{F} + \partial_S\rho(\mathcal{F}, \mathcal{T} - \theta\varepsilon)\partial_v\mathcal{T}]\,\partial_x\mathcal{T}\\
&\quad - [\partial_x\mathcal{F} + \partial_F\rho(\mathcal{F}, \mathcal{T} - \theta\varepsilon)\partial_x\mathcal{F} + \partial_S\rho(\mathcal{F}, \mathcal{T} - \theta\varepsilon)\partial_x\mathcal{T}]\,\partial_v\mathcal{T}\\
&= (\partial_v\mathcal{F}\partial_x\mathcal{T} - \partial_x\mathcal{F}\partial_v\mathcal{T})(1 + \partial_F\rho(\mathcal{F}, \mathcal{T} - \theta\varepsilon)\\
&= (1 + \mu\varepsilon^q h(0, \mathcal{T} - \theta\varepsilon, \mathcal{F}))(1 + \partial_F\rho(\mathcal{F}, \mathcal{T} - \theta\varepsilon))\\
&= 1.
\end{aligned}$$

■

Now we turn to the system in the original variables (x, y, θ). We define

$$\begin{aligned}\mathcal{T}^1(x, y, \theta) &= \mathcal{T}(C(x, y, \theta))\\ \mathcal{I}^1(x, y, \theta) &= \mathcal{I}(C(x, y, \theta))\end{aligned}$$

where C defined in (6.3.1). It is clear that the change

$$(x, y, \theta) \in V(\delta', \delta, r, \eta_0) \mapsto (T, I, \theta) = (\mathcal{T}^1(x, y, \theta), \mathcal{I}^1(x, y, \theta), \theta) \in \mathcal{V}$$

is canonical, since it is the composition of two canonical changes. Moreover

$$\begin{aligned} \dot{T} &= 1 \\ \dot{I} &= 0 \end{aligned}$$

and

$$\begin{aligned} \mathcal{T}^1(x, y, \theta) &= \mathcal{T}_0(x, y - f(x)) + O(\mu\varepsilon^q) \\ \mathcal{I}^1(x, v, \theta) &= \mathcal{I}_0(x, y - f(x)) + O(\mu\varepsilon^q) \\ &= \mathcal{F}_0(x, y - f(x)) + O(\mu\varepsilon^q) \end{aligned}$$

where $\mathcal{T}_0(x, y - f(x))$ and $\mathcal{F}_0(x, y - f(x))$ is the change when $\mu = 0$ in the original variables.

6.5 Proof of the Corollary 6.2.6

We consider the systems obtained scaling time by ε in the averaged systems (3.3.10) in Chapter 3 given by:

$$\begin{aligned} \dot{x} &= y + \mu\varepsilon^{p+2n+2}\partial_y F(x, y, \theta) + \mu^2\varepsilon^{p+2}\partial_y R_{2k-2}(x, y, \theta) \\ \dot{y} &= -V'(x) - \mu\varepsilon^{p+2n+2}\partial_x F(x, y, \theta) - \mu^2\varepsilon^{p+2}\partial_x R_{2k-2}(x, y, \theta) \\ \dot{\theta} &= 1/\varepsilon, \end{aligned} \tag{6.5.1}$$

and the system (4.3.6) given in Chapter 4 that is:

$$\begin{aligned} \dot{x} &= y + \mu\varepsilon^{p+8}\partial_y F(x, y, \theta) + \mu^2\varepsilon^{2p+1}(\partial_y f_3 + \partial_y R_2)(x, y, \theta) \\ \dot{y} &= -V'(x) - \mu\varepsilon^{p+8}\partial_x F(x, y, \theta) - \mu^2\varepsilon^{2p+1}(\partial_x f_3 + \partial_x R_2)(x, y, \theta) \\ \dot{\theta} &= 1/\varepsilon \end{aligned} \tag{6.5.2}$$

We denote by

$$\tilde{\gamma}^s_{\mu,\varepsilon}(t, s) = (\tilde{\alpha}(t, s), \tilde{\beta}(t, s))$$

the stable curve of any of those systems and by

$$\gamma_0(t + s) = (\alpha(t + s), \beta(t + s))$$

homoclinic orbit of the unperturbed system.

Definition 6.5.1 *We fix $0 < \delta \le \delta_0/3$ small enough. Let T be big enough such that*

$$\operatorname{Re}\tilde{\alpha}(t,s) \le \delta$$

for $t + \operatorname{Re} s \ge T/2$ and let δ_1' be such that

$$\delta_1' \le \operatorname{Re}\tilde{\alpha}(t,s) \le \delta$$

for $T/2 \le t + \operatorname{Re} s \le 3T$.

We observe that the systems (6.5.1) and (6.5.2) satisfy the hypotheses **H1-H3,** of Theorem 6.2.5 with $q = p+2$ and $q = p + 1/2$ respectively, therefore there exists a canonical change of variables, defined in the set $V(\delta_1', \delta, r, \eta_0)$, which we denote with the same notation $(T, I) = (\mathcal{T}^1(\bar{x}, \bar{y}, \theta), \mathcal{I}^1(\bar{x}, \bar{y}, \theta))$ in the two cases we consider, such that it transforms the systems (6.5.1) and (6.5.2) to

$$\begin{aligned} \dot{T} &= 1 \\ \dot{I} &= 0. \end{aligned}$$

Moreover, $\mathcal{T}^1(\bar{x}, \bar{y}, \theta) = \mathcal{T}_0(\bar{x}, \bar{y}) + O(\mu\varepsilon^{p+i_0})$ and $\mathcal{I}^1(\bar{x}, \bar{y}, \theta) = \mathcal{I}_0(\bar{x}, \bar{y}) + O(\mu\varepsilon^{p+i_0})$ with $i_0 = 1$ in the parabolic case and $i_0 = 1/2$ in the weak hyperbolic case.

We must see that the parametric representation of the stable manifold enters the domain of analyticity of $(\mathcal{T}^1(x, y, \theta), \mathcal{I}^1(x, y, \theta))$. This is done in the next lemma.

Lemma 6.5.2 *For any $0 < \eta < \frac{\pi}{4(\beta-1)}$ there exists $T > 0$ big enough such that, if $T/2 \le t + \operatorname{Re} s \le 3T$ and $|\operatorname{Im} s| \le a$, then $|\arg(\tilde{\alpha}(t,s))| < \frac{\pi}{4(\beta-1)} - \eta$.*

Proof. By, Chapters 2, 3 and 4, there exists the ocal stable manifold of (6.5.1) and (6.5.2) and, if $T/2 \le t + \operatorname{Re} s \le 3T$ and $|\operatorname{Im} s| \le a$ then

$$\begin{aligned} \tilde{\gamma}^s_{\mu,\varepsilon}(t,s) = {} & \left(\frac{c}{(t+s)^{2/(n-2)}}, -\frac{2c}{(n-2)(t+s)^{n/(n-2)}} \right) + \mu\varepsilon^{p+1+\lambda}\sigma(t,s) \\ & + o\Big(\frac{1}{(t+s)^{2/(n-2)}} \Big) \end{aligned}$$

with $c^{n-2} = -2/(a_n(n-2)^2)$, $\lambda = 1$ and $\sigma \in \mathcal{X}_{(2n-2)/(n-2)} \times \mathcal{X}_{(3n-4)/(n-2)}$ in the parabolic case and $\lambda = 1/4$ and $\sigma \in \mathcal{Y}^1_4 \times \mathcal{Y}^1_5$ in the weak hyperbolic case. We deal with the parabolic case, the other case is analogous. We denote $m = 2/(n-2)$, $u = u_1 + iu_2 = t + s$ and $z = u_2/u_1$. Note the $z = O(1/T)$ and that

$$|\sigma_1(t,s)| \le \|\sigma_1\|_{n/(n-2)}(t + \operatorname{Re} s)^{n/(n-2)}.$$

Then

$$\begin{aligned}
\tilde{\alpha}(t,s) &= c\frac{(u_1 - iu_2)^m}{(u_1^2 + u_2^2)^m} + \mu\varepsilon^{p+2}\sigma_1(t,s) + o\Big(\frac{1}{(t+s)^{2/(n-2)}}\Big)\\
&= c\frac{(1-iz)^m}{u_1^m(1+z^2)^m} + \mu\varepsilon^{p+2}\sigma_1(t,s) + o\Big(\frac{1}{(t+s)^{2/(n-2)}}\Big)\\
&= c\frac{1}{u_1^m}\Big(1 - imz + O(iz^2) + \mu\varepsilon^{p+2}u_1^m\sigma_1(t,s) + O\Big(\frac{1}{u}\Big)\Big). \qquad (6.5.3)
\end{aligned}$$

It is clear that $u_1^m\sigma_1(t,s)$ is bounded if $T/2 \le t + \operatorname{Re} s \le 3T$ and $|\operatorname{Im} s| \le a$. In fact it goes to zero when $T \to +\infty$. Therefore if T is big enough, using the estimate (6.5.3) we obtain

$$|\arg\tilde{\alpha}(t,s)| \le \frac{|\operatorname{Im}\tilde{\alpha}_\mu(t,s)|}{\operatorname{Re}\tilde{\alpha}_\mu(t+s)} = \frac{|O(z) + O(\mu\varepsilon^{p+2})|}{1 + O(z) + O(\mu\varepsilon^{p+2})} \le \frac{\pi}{4(\beta-1)} - \eta.$$

■

We call $\mathcal{C}$ the change from the initial to the averaged systems, defined in Chapter 3, Lemma 3.3.4 in the parabolic case, or in Chapter 4, Remark 4.3.4 in the weak hyperbolic case. We write $(x,y,\theta) = \mathcal{C}(\bar{x},\bar{y},\theta)$. Where $(\bar{x},\bar{y})$ denote the variables of the averaged systems. Moreover, we know that

$$(x,y) = (\bar{x},\bar{y}) + O(\mu\varepsilon^{p+1}).$$

We define new flow box coordinates

$$(\mathcal{T}^2(x,y,\theta), \mathcal{I}^2(x,y,\theta)) = (\mathcal{T}^1(\mathcal{C}^{-1}(x,y,\theta)), \mathcal{I}^1(\mathcal{C}^{-1}(x,y,\theta))).$$

Since the change $\mathcal{C}$ is $O(\mu\varepsilon^{p+1})$ close to the identity and canonical, the new change is also canonical and satisfies

$$(\mathcal{T}^2(x,y,\theta), \mathcal{I}^2(x,y,\theta)) = (\mathcal{T}_0^2(x,y), \mathcal{I}_0^2(x,y)) + O(\mu\varepsilon^{p+i_0})$$

with $i_0 = 1$ in the parabolic case and $i_0 = 1/2$ in the weak hyperbolic case.

The change $\mathcal{T}_0^2(x,y), \mathcal{I}_0^2(x,y) = h_0(x,y)$ is the corresponding change for $\mu = 0$. The domain of definition of the new change is $\mathcal{C}(V(\delta_1', \delta, r, \eta_0))$. Moreover it is clear that the change $(T,I) = (\mathcal{T}^2(x,y,\theta), \mathcal{I}^2(x,y,\theta))$ transforms the system (1.1.1) to

$$\begin{aligned}
\dot{T} &= 1\\
\dot{I} &= 0.
\end{aligned}$$

Let $\gamma^s_{\mu,\varepsilon}(t,s) = (\alpha(t,s), \beta(t,s))$ be the parameterization of the stable manifold of system (1.1.1) given in Theorems 3.2.1 and 4.2.1.

We define a new change of variables. Let s_0 be such that $|\operatorname{Im} s_0| \leq a$. We define

$$(x^*, y^*) = \gamma^s_{\mu,\varepsilon}(T - \operatorname{Re} s_0, s_0)$$

and the parameter

$$\tau = s_0 - \mathcal{T}^2(x^*, y^*, (T - \operatorname{Re} s_0)/\varepsilon).$$

We observe that, if $\mu = 0$, the constant τ is

$$\tau = \int_{x_0}^{\mathcal{C}^1(\delta, f(\delta), 0)} \frac{ds}{\sqrt{2h_0(x^*, y^*) - 2V(s)}}$$

where the superscript denotes the first and second component of the change and x_0 is the initial condition of γ_0 as it is defined in hypothesis **HP1**. τ does not depend on (x^*, y^*) while (x^*, y^*) belongs to the stable manifold of the unperturbed system.

Let $\delta_2' < \delta_2$ be such that $\delta_1' < \delta_2' < \delta_2 < \delta$ and

$$\delta_2' \leq \operatorname{Re} \alpha(t,s) \leq \delta_2$$

for $T \leq t + \operatorname{Re} s \leq 2T$ and $|\operatorname{Im} s| \leq a$. By Definition 6.5.1, if $|\mu|\varepsilon^{p+i_0}$ is small enough, it is immediate that there exist δ_2' and δ_2 with the above properties.

Remark 6.5.3 *Let T_1 and T_2 be such that $T < T_1 < T_2 < 2T$. We observe that, by Lemma 6.5.2, and by the fact that the change $\mathcal{C}$ is $O(\mu\varepsilon^{p+1})$ close to the identity, there exist $\delta_3' < \delta_3$ such that $\delta_2' < \delta_3' < \delta_3 < \delta_2$ and such that the stable manifold of the system (1.1.1) $(\gamma^s_{\mu,\varepsilon}(t,s), t/\varepsilon) = \mathcal{C}\ (\tilde{\gamma}^s_{\mu,\varepsilon}(t,s), t/\varepsilon)$ belongs to $V(\delta_3', \delta_3, r/2, 3\eta_0) \subset \mathcal{C}(V(\delta_1', \delta, r, \eta_0))$, if $|\mu|\varepsilon^{p+1}$ is small enough, for $T_1 \leq t + \operatorname{Re} s \leq T_2$ and $|\operatorname{Im} s| \leq a$. Therefore $(\gamma^s_{\mu,\varepsilon}(t,s), t/\varepsilon)$ belongs to the domain of the new flow box coordinates $(\mathcal{T}^2(x,y,\theta), \mathcal{I}^2(x,y,\theta))$.*

Moreover, since $\mathcal{C}$ is $O(\mu\varepsilon^{p+1})$ close to the identity, if $|\mu|\varepsilon^{p+1}$ is small enough, we have that

$$V(\delta_2', \delta_2, r/2, 2\eta_0) \subset \mathcal{C}(V(\delta_1', \delta, r, \eta_0)).$$

We define the functions

$$\begin{aligned} \mathcal{S}(x,y,\theta) &= \mathcal{T}^2(x,y,\theta) + \tau \\ \mathcal{E}(x,y,\theta) &= \mathcal{I}^2(x,y,\theta) - \mathcal{I}^2(x^*,y^*,\theta). \end{aligned} \tag{6.5.4}$$

Then, since $\dot{T}^2 = 1$ and $\dot{I}^2 = 0$, we have that for $s = s_0$,

$$\mathcal{S}(\gamma^s_{\mu,\varepsilon}(t, s_0), t/\varepsilon) = t + s_0. \tag{6.5.5}$$

Indeed,

$$\begin{aligned}
\mathcal{S}(\gamma^s_{\mu,\varepsilon}(t, s_0), t/\varepsilon) &= \mathcal{T}^2(\gamma^s_{\mu,\varepsilon}(t, s_0), t/\varepsilon) + \tau \\
&= t + \mathcal{T}^2(\gamma^s_{\mu,\varepsilon}(T - \operatorname{Re} s_0, s_0), (T - \operatorname{Re} s_0)/\varepsilon) + \tau \\
&= t + \mathcal{T}^2(x^*, y^*, (T - \operatorname{Re} s_0)/\varepsilon) + \tau \\
&= t + s_0.
\end{aligned}$$

Let $s \in \mathbb{C}$, $s \neq s_0$, then

$$\begin{aligned}
\mathcal{S}(\gamma^s_{\mu,\varepsilon}(t, s), t/\varepsilon) &= \mathcal{T}^2(\gamma^s_{\mu,\varepsilon}(t, s), t/\varepsilon) + \tau \\
&= t + s + \mu\varepsilon^{p+i_0}\mathcal{X}(s).
\end{aligned}$$

Here we have used that, for the unperturbed system we have that

$$\mathcal{S}_0(\gamma_0(t + s)) = t + s$$

and that

$$\begin{aligned}
\mathcal{S}(x, y, \theta) &= \mathcal{S}_0(x, v) + O(\mu\varepsilon^{p+i_0}) \\
\gamma^s_{\mu,\varepsilon}(t, s) &= \gamma_0(t + s) + O(\mu\varepsilon^{p+1}).
\end{aligned}$$

We observe that we can choose freely s_0 such that $\mathcal{X}(s_0) = 0$ and that for any s_0 we have a different definition of $\mathcal{S}$.

We also note that $\mathcal{X}(s)$ is $2\pi\varepsilon$-periodic in s. Indeed, we have that

$$\begin{aligned}
\mathcal{S}(\gamma^s_{\mu,\varepsilon}(t, s + 2\pi\varepsilon), t/\varepsilon) &= t + s + 2\pi\varepsilon + \mu\varepsilon^{p+i_0}\mathcal{X}(s + 2\pi\varepsilon) \\
\mathcal{S}(\gamma^s_{\mu,\varepsilon}(t + 2\pi\varepsilon, s), t/\varepsilon) &= t + 2\pi\varepsilon + s + \mu\varepsilon^{p+i_0}\mathcal{X}(s)
\end{aligned}$$

but, since $\gamma^s_{\mu,\varepsilon}(t, s + 2\pi\varepsilon) = \gamma^s_{\mu,\varepsilon}(t + 2\pi\varepsilon, s)$, from the previous equations we obtain

$$\mathcal{X}(s + 2\pi\varepsilon) = \mathcal{X}(s).$$

Finally,

$$\begin{aligned}
\mathcal{E}(\gamma^s_{\mu,\varepsilon}(t, s), t/\varepsilon) &= \mathcal{I}^2(\gamma^s_{\mu,\varepsilon}(t, s), t/\varepsilon) - \mathcal{I}^2(x^*, y^*, 0) \\
&= 0.
\end{aligned}$$

This ends the proof.

Chapter 7

The Extension Theorem

7.1 Introduction and main result

This short Chapter is devoted to recall the statement of the extension theorem which is given in [20]. This theorem is stated for systems of the form

$$\begin{aligned}\dot{x} &= y + \mu\varepsilon^p \partial_y h_1(x, y, t/\varepsilon) \\ \dot{y} &= -V'(x) - \mu\varepsilon^p \partial_x h_1(x, y, t/\varepsilon)\end{aligned}$$

such that the unperturbed system has a homoclinic orbit, $\gamma_0(u) = (\alpha_0(u), \beta_0(u))$ and β_0 is an analytic function in $|\operatorname{Im} u| < a$ and has singularities at $u = \pm ia$ which are poles.

Following the proof in [20] one can see that one can replace the condition of $u = \pm ia$ being poles by $u = \pm ia$ being branching points in the sense we have introduced in **HP1** in Chapter 1. For this reason here we do not reproduce the proof of the extension theorem and we will simply comment a small difference that appears in the weak hyperbolic case.

The goal of this theorem is to extend the domain of the parameterization $\gamma^u_{\mu,\varepsilon}(t, s)$ (in our case produced in Chapters 3 and 4) of the unstable manifold until $\gamma^u_{\mu,\varepsilon}$ enters the domain of the flow box coordinates. To do this, the parameterizations $\gamma^u_{\mu,\varepsilon}(t, s)$ and $\gamma_0(t + s)$ are compared in the complex domain $D^{\text{ext}}_\varepsilon$:

$$D^{\text{ext}}_\varepsilon \equiv \{(t, s) \in \mathbb{R} \times \mathbb{C} : |t + \operatorname{Re} s| \leq 2T \ , \ |\operatorname{Im} s| \leq a - \varepsilon\}.$$

The extension theorem gives an useful bound for the distance between the unstable manifold $\gamma^u_{\mu,\varepsilon}$ and the homoclinic orbit γ_0 of the unperturbed system for $(t, s) \in D^{\text{ext}}_\varepsilon$.

That is,

$$\gamma^u_{\mu,\varepsilon}(t,s) - \gamma_0(t+s) = O(\mu\varepsilon^\nu),$$

where ν is a parameter which depends on the system.

Therefore, if $\nu \geq 0$ and μ and ε are small enough $\gamma^u_{\mu,\varepsilon}(t,s)$, for some values of (t,s), belongs to the domain of the flow box coordinates.

We recall that ℓ is defined in Chapter 1.

The extension theorem is

Theorem 7.1.1 *Let $z(t,s) = (x(t,s), y(t,s))$ be a family of solutions of*

$$\begin{aligned} \dot{x} &= y + \mu\varepsilon^p \partial_y h_1(x,y,t/\varepsilon,\mu,\varepsilon) \\ \dot{y} &= -V'(x) - \mu\varepsilon^p \partial_x h_1(x,y,t/\varepsilon,\mu,\varepsilon) \end{aligned} \tag{7.1.1}$$

defined for $t_0 + \mathrm{Re}\, s = -2T$, for some $T > 0$, such that

$$z(t_0,s) - \gamma_0(t_0+s) - \mu\varepsilon^{p+1} G(\gamma_0(t_0+s), t_0/\varepsilon, \mu, \varepsilon) = O(\mu\varepsilon^{p+1+\lambda}), \tag{7.1.2}$$

where G is the function such that

$$\partial_\theta G(x,y,\theta,\mu,\varepsilon) = (\partial_y h_1(x,y,\theta,\mu,\varepsilon), -\partial_x h_1(x,y,\theta,\mu,\varepsilon))$$

and has zero mean with respect to θ, and $(t_0,s) \in D^{\mathrm{ext}}_\varepsilon$ verifies $t_0 + \mathrm{Re}\, s = -2T$. Also λ is an index which have the value 1 in the parabolic case and $1/4$ in the weak hyperbolic case.

We assume that

$$\nu \equiv p - \ell \geq 0. \tag{7.1.3}$$

Then, there exist ε_0, μ_0 and K such that the solution $z(t,s)$ can be extended to values of $t \in [t_0, 2T - \mathrm{Re}\, s]$, with the bound

$$|z(t,s) - \gamma_0(t+s)| \leq K\mu\varepsilon^{p-\ell} \tag{7.1.4}$$

for $(t,s) \in D^{\mathrm{ext}}_\varepsilon$, $0 < \varepsilon \leq \varepsilon_0$ and $|\mu| \leq \mu_0$.

Remark 7.1.2 *Theorem 7.1.1 is also valid if V is a trigonometric polynomial and we suppose that h_1 is also a trigonometric polynomial in x and a polynomial in y. We observe that in this case $\alpha_0(u) \sim iC \log(u \mp ia)$ near of singularity $u = \pm ia$ with C a constant which depends on the degree of V. (See [20] for more details about this case).*

Remark 7.1.3 *Although in the extension theorem given in [20], the initial condition of the solution* $z(t,s)$ *is*

$$z(t_0,s)-\gamma_0(t_0+s)-\mu\varepsilon^{p+1}G(\gamma_0(t_0+s),t_0/\varepsilon,\mu,\varepsilon)=O(\mu\varepsilon^{p+2})$$

and thus, this hypothesis is not satisfied in the weak hyperbolic case, it is not difficult to check that the proof also works in this case.

Remark 7.1.4 *We note that, if* $s\in\mathbb{R}$*, the estimate (7.1.4) can be improved. Concretely, for* $(t,s)\in\mathbb{R}^2$ *such that* $-2T\le t+s\le 2T$ *we have that*

$$z(t_0,s)-\gamma_0(t_0+s)=O(\mu\varepsilon^{p+i_0}).$$

Chapter 8

Splitting of separatrices

8.1 Introduction

This chapter is devoted to prove Theorem 1.2.1 and Corollary 1.2.3 which we reproduce below for the convenience of the reader. Let

$$\begin{aligned} \dot{x} &= y + \mu\varepsilon^p \partial_y h_1(x, y, t/\varepsilon) \\ \dot{y} &= -V'(x) - \mu\varepsilon^p \partial_x h_1(x, y, t/\varepsilon). \end{aligned} \tag{8.1.1}$$

We denote by A the area of a lobe generated by the stable and unstable manifold of system (8.1.1) associated to two homoclinic points and by ϑ the angle between them at one of these homoclinic points.

Theorem 8.1.1 *Under hypotheses **HP1-HP6**, for $\varepsilon \to 0^+$, $\mu \to 0$, the following formulae hold:*

$$A = \mu\varepsilon^p \int_{s_0}^{\bar{s}_0} M(\upsilon, \varepsilon)\ d\upsilon + O(\mu^2\varepsilon^{2\nu+r}, \mu^2\varepsilon^{\nu+p+i_0}, \mu\varepsilon^{p+1+i_0})e^{-a/\varepsilon},$$

$$\sin\vartheta = \mu\varepsilon^p \frac{M'(s_0, \varepsilon)}{||\dot{\gamma}_0(s_0)||^2} + O(\mu^2\varepsilon^{2\nu+r-2}, \mu^2\varepsilon^{\nu+p+i_0-2}, \mu\varepsilon^{p-1+i_0})e^{-a/\varepsilon},$$

where $s_0 < \bar{s}_0$ are the two consecutive zeros (associated to two consecutive homoclinic points) of the Melnikov function

$$M(s, \varepsilon) = \int_{-\infty}^{\infty} \{h_0, h_1\}(\gamma_0(t+s), t/\varepsilon)\ dt,$$

closest to zero and

$$i_0 = \begin{cases} 1 & \textit{in the parabolic case} \\ 1/2 & \textit{in the weak hyperbolic case.} \end{cases}$$

Corollary 8.1.2 *If **HP1-HP7** holds, then for* $\varepsilon \to 0^+$, $\mu \to 0$

$$A \sim \mu\varepsilon^{\nu+1} 8\pi |J_{1,0}^+| \frac{1}{\Gamma(\ell+1)} e^{-a/\varepsilon}$$
$$|\sin\vartheta| \sim \mu\varepsilon^{\nu-1} 4\pi |J_{1,0}^+| \frac{1}{\Gamma(\ell+1)} e^{-a/\varepsilon} \frac{1}{\|\dot\gamma_0(s_0)\|^2}$$

where Γ *is the Gamma function.*

Remark 8.1.3 *The constants* ν*,* a *and* $J_{1,0}^+$ *are introduced in the hypotheses **HP1-HP7**.*

To prove this theorem we will use many of the results established in the previous chapters. In particular the parameterizations of the stable manifolds, in Chapter 3 and 4 an the flow box coordinates developed in Chapter 6 will play an important role.

We recall that in Chapter 6, we have constructed flow box coordinates $(S, E) = (\mathcal{S}(x, y, t/\varepsilon), \mathcal{E}(x, y, t/\varepsilon))$ defined in a complex neighbourhood of a piece of the stable manifold of the system (8.1.1). In these coordinates the original system becomes the simple equation:

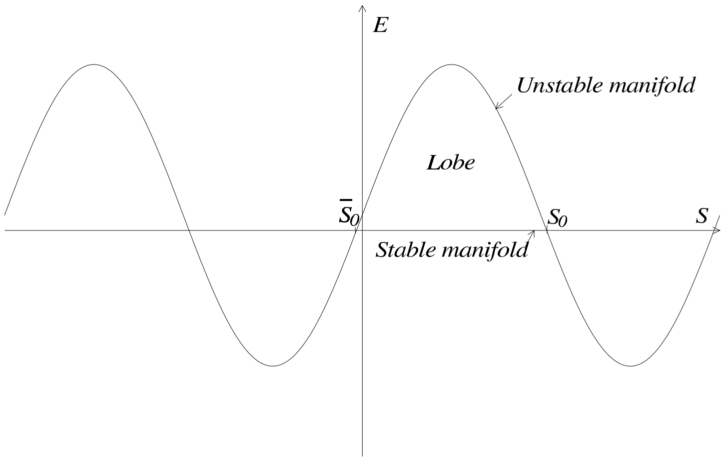
$$\dot S = 1$$
$$\dot E = 0.$$

Moreover, we had proved that on the parameterization of the stable manifold of the system (8.1.1), $\gamma^s_{\mu,\varepsilon}(t, s)$ (with s depending on the initial condition)

$$\mathcal{S}(\gamma^s_{\mu,\varepsilon}(t, s), t/\varepsilon) = t + s + \mu\varepsilon^{p+i_0}\mathcal{X}(s), \qquad \mathcal{E}(\gamma^s_{\mu,\varepsilon}(t, s), t/\varepsilon) = 0.$$

We can prove the existence of primary homoclinic points by using that the Poincaré map is area preserving and that the system (8.1.1) is a perturbation of one which has a homoclinic orbit. Moreover, by the extension theorem, we get that the parameterization of the unstable manifold $\gamma^u_{\mu,\varepsilon}(t, s)$ enters, for some values of (t, s) in the domain of the flow box coordinates.

In flow box coordinates the stable manifold corresponds to $E = 0$. If, in these variables, the unstable manifold can be written as $E = \phi(S)$, with ϕ a suitable function, in (S, E) coordinates we have the following situation:



Consequently the area of the lobe generated by two consecutive intersections between the stable and the unstable manifold, expressed in flow box coordinates, is

$$A = \left| \int_{S_0}^{\bar{S}_0} \phi(S)\ dS \right| . \tag{8.1.2}$$

The function ϕ is called the splitting function. Since the change from the original variables to the flow box variables is canonical, the area given by (8.1.2) is the same as the area of the corresponding lobe in the original variables (x, y).

The scheme of the proof is the same as the one given in [20]. For the convenience of the reader we present the main points of it. In Section 8.2 we construct the splitting function and we stablish its properties. In Section 8.3 we prove the main results. The flow box coordinates we use and the definition of the splitting function permit to prove that the Melnikov function is a good approximation of the splitting function.

Finally we will prove Corollary 1.2.3. We recall that we consider the case that the singularities of the homoclinic orbit may be branching points in the sense we have indicated in Chapter 1. This case is not considered in [20]. This fact forces several technicalities in the computation of the Melnikov function.

The asymptotic computations of the Melnikov function are deferred to Subsection 8.3.1.

8.2 The splitting function

To define the splitting function and to establish the properties we shall need we begin by recalling some notation and some previous results.

We recall the definitions of the sets

$$
\begin{aligned}
D_\varepsilon^{\mathrm{ext}} &= \{(t,s)\in\mathbb{R}\times\mathbb{C} : |t+\operatorname{Re}s|\le 2T,\ \ |\operatorname{Im}s|\le a-\varepsilon\},\\
D^s &= \{(t,s)\in\mathbb{R}\times\mathbb{C} : t+\operatorname{Re}s\ge T,\ \ |\operatorname{Im}s|\le a\},\\
D^u &= \{(t,s)\in\mathbb{R}\times\mathbb{C} : t+\operatorname{Re}s\le -T,\ \ |\operatorname{Im}s|\le a\},
\end{aligned}
$$

The homoclinic orbit of the unperturbed system is

$$\gamma_0(u) = (\alpha_0(u), \beta_0(u))$$

which is defined in (at least in) $\{u\in\mathbb{C} : |\operatorname{Im}u| < a\}$.

We denote $\gamma_{\mu,\varepsilon}^{s,u}(t,s)$ the parameterizations of the stable and unstable manifolds of the perturbed system and we recall that, by Theorem 3.2.1 (in the parabolic case) and Theorem 4.2.1 (in the weak hyperbolic case), the invariant curves are solutions with respect to t and satisfy

$$\gamma_{\mu,\varepsilon}^{s,u}(t+2\pi\varepsilon, s) = \gamma_{\mu,\varepsilon}^{s,u}(t, s+2\pi\varepsilon). \tag{8.2.1}$$

Moreover,

$$
\begin{aligned}
\gamma_{\mu,\varepsilon}^{s}(t,s) &= \gamma_0(t+s) + O(\mu\varepsilon^{p+1}) \quad \text{for } (t,s)\in D^s\\
\gamma_{\mu,\varepsilon}^{u}(t,s) &= \gamma_0(t+s) + O(\mu\varepsilon^{p+1}) \quad \text{for } (t,s)\in D^u.
\end{aligned}
$$

We call $U \equiv \mathcal{C}(V(\delta_1', \delta, r_1, \eta_0))$ the domain of the flow box coordinates (S,E), constructed in Chapter 6 (see in particular Corollary 6.2.6), which is a neighbourhood of

$$\{\gamma_{\mu,\varepsilon}^{s}(t,s) : T\le t+\operatorname{Re}s\le 2T,\ \ |\operatorname{Im}s|\le a\}$$

and it is independent of μ, ε.

By the extension theorem (Chapter 7) the domain of the parameterization of the unstable manifold $\gamma_{\mu,\varepsilon}^{u}(t,s)$ can be extended to values of $(t,s)\in D_\varepsilon^{\mathrm{ext}}$, and in this extended set verifies

$$\gamma_{\mu,\varepsilon}^{u}(t,s) = \gamma_0(t+s) + O(\mu\varepsilon^{\nu})$$

where $\nu = p - \ell$.

Since $\nu \geq 0$, for any $\varkappa_0 > 0$, there exist ε_0 and μ_0 such that, for all $0 < \varepsilon < \varepsilon_0$, $|\mu| < \mu_0$ and (t,s) such that $T + \varkappa_0 \leq t + \operatorname{Re} s \leq 2T - \varkappa_0$ and $|\operatorname{Im} s| \leq a - \varepsilon$, the unstable manifold, $\gamma^u_{\mu,\varepsilon}(t,s)$, is so close to $\gamma_0(t+s)$ and $\gamma_0(t+s)$ is so close to $\gamma^s_{\mu,\varepsilon}(t,s)$ that $\gamma^u_{\mu,\varepsilon}(t,s) \in U$ for those values of (t,s).

Hence the functions

$$\mathcal{S}^u(s) = \mathcal{S}(\gamma^u_{\mu,\varepsilon}(t,s), t/\varepsilon) - t, \qquad \mathcal{E}^u(s) = \mathcal{E}(\gamma^u_{\mu,\varepsilon}(t,s), t/\varepsilon), \tag{8.2.2}$$

are well defined for $s \in \mathbb{C}$ such that $T + \varkappa_0 \leq t + \operatorname{Re} s \leq 2T - \varkappa_0$ and $|\operatorname{Im} s| \leq a - \varepsilon$. We write some immediate properties of $\mathcal{S}^u$ and $\mathcal{E}^u$:

Remark 8.2.1 *By Theorem 6.2.6, $\mathcal{S}^u$ and $\mathcal{E}^u$ do not depend on time, then, in the definition of $\mathcal{S}^u$ and $\mathcal{E}^u$ the time t can be chosen arbitrarily. We choose it in such a way that $T + \varkappa_0 \leq t + \operatorname{Re} s \leq 2T - \varkappa_0$.*

Hence $\mathcal{S}^u$ and $\mathcal{E}^u$ can be analytically extended for all $s \in \mathbb{C}$ and $|\operatorname{Im} s| \leq a - \varepsilon$.

Remark 8.2.2 *The functions $\mathcal{S}^u(s) - s$ and $\mathcal{E}^u(s)$ are $2\pi\varepsilon$-periodic with respect to s. Indeed, we prove it for $\mathcal{S}^u(s) - s$. By property (8.2.1) and since $\mathcal{S}^u(s) - s$ does not depend on t, we have that*

$$\begin{aligned}
\mathcal{S}^u(s+2\pi\varepsilon) - (s+2\pi\varepsilon) &= \mathcal{S}(\gamma^u_{\mu,\varepsilon}(t, s+2\pi\varepsilon), t/\varepsilon) - t - (s+2\pi\varepsilon) \\
&= \mathcal{S}(\gamma^u_{\mu,\varepsilon}(t+2\pi\varepsilon, s), (t+2\pi\varepsilon)/\varepsilon) - (t+2\pi\varepsilon) - s \\
&= \mathcal{S}^u(s) - s.
\end{aligned}$$

Analogously, $\mathcal{E}^u(s+2\pi\varepsilon) = \mathcal{E}^u(s)$.

The next proposition asserts that the Melnikov function is a good approximation of the function $\mathcal{E}^u(s)$ for $|\operatorname{Im} s| \leq a - \varepsilon$ and it gives that, in particular when $s \in \mathbb{R}$, the approximation is exponentially small. We denote

$$\mathcal{E}^u_0(\varepsilon) = \frac{1}{2\pi\varepsilon} \int_0^{2\pi\varepsilon} \mathcal{E}^u(s)\,ds.$$

To treat simultaneously the parabolic case and the weak hyperbolic case we introduce the quantity i_0, which takes the values $i_0 = 1$ in the parabolic case and $i_0 = 1/2$ in the weak hyperbolic case.

We introduce the Melnikov function

$$M(s,\varepsilon) \equiv \int_{-\infty}^{+\infty} \{h_0, h_1\}(\gamma_0(t+s), t/\varepsilon)\ dt.$$

Since $h_1(x,y,t/\varepsilon)$ is $2\pi\varepsilon$-periodic in t, the Melnikov function has the same periodicity with respect to s. We denote by $M_k(\varepsilon)$ its Fourier's coefficients, i.e.,

$$M(s,\varepsilon) = \sum_{k\in\mathbb{Z}} M_k(\varepsilon) e^{iks/\varepsilon}.$$

Given $(t,s) \in D_\varepsilon^{\mathrm{ext}}$ and $\xi : D_\varepsilon^{\mathrm{ext}} \to \mathbb{C}^2$, we introduce $\tau = |t+s-ia|$ and

$$|\xi(t,s)|_\tau = |\xi_1(t,s)| + \tau|\xi_2(t,s)|.$$

Now we enunciate two lemmas which we will need. They can be found in [20].

Lemma 8.2.3 *For t, t_0, l real and s complex, such that $0 \le \operatorname{Im} s < a$ and*

$$-2T \le t_0 + \operatorname{Re} s \le t + \operatorname{Re} s \le 2T, \qquad t_0 + \operatorname{Re} s < 0$$

we denote

$$\rho_{[t_0,t]}^{-l}(s) \equiv \begin{cases} \sup \dfrac{1}{|\sigma+s-ia|^l}, & \text{if } l \neq 0 \\ \sup |\ln(|\sigma+s-ia|)|, & \text{if } l = 0 \end{cases}$$

where the supremum is taken for $\sigma \in [t_0,t]$.

Then there exists a constant K which only depends on l such that

$$\int_{t_0}^{t} \frac{d\sigma}{|\sigma+s-ia|^l} \le K\rho_{[t_0,t]}^{-(l-1)}(s), \tag{8.2.3}$$

Lemma 8.2.4 *Let $\delta_0 \in (0,1)$ and let $\delta : [0,+\infty) \to \mathbb{R}$ be a function such that $\delta(\tau) \le \delta_0/\tau^{r-1}$. Suppose that $\xi(t,s)$ and $\bar{\xi}(t,s)$ are two functions defined in $D_\varepsilon^{\mathrm{ext}}$. We will write $\xi(t,s) = \xi = (\xi_1,\xi_2)$ and $\bar{\xi}(t,s) = \bar{\xi} = (\bar{\xi}_1,\bar{\xi}_2)$. Assume that*

$$|\xi|_\tau, |\bar{\xi}|_\tau \le \delta(\tau).$$

Then we have that

$$|f(\alpha_0(t+s)+\xi_1) - f(\alpha_0(t+s)+\bar{\xi}_1)| \le K\frac{|\xi_1-\bar{\xi}_1|}{\tau^2}, \qquad (t,s)\in D_\varepsilon^{\mathrm{ext}}$$

$$|g(\gamma_0(t+s)+\xi, t/\varepsilon) - g(\gamma_0(t+s)+\bar{\xi}, t/\varepsilon)|_\tau \le K\frac{|\xi-\bar{\xi}|_\tau}{\tau^{\ell-2r+1}}, \qquad (t,s)\in D_\varepsilon^{\mathrm{ext}}$$

where $f(x) = -\sqrt{-V'(x)}$ and $g(x,y,t/\varepsilon) = (\partial_y h_1(x,y,t/\varepsilon), -\partial_x h_1(x,y,t/\varepsilon))$.

Remark 8.2.5 *Since*

$$f(\alpha_0(u)) = \dot{\beta}_0(u) = \ddot{\alpha}_0(u)$$

has a singularity of order $r+1$ *at* $u = ia$*, we have that, for* $(t,s) \in D_\varepsilon^{\mathrm{ext}}$ *such that* $0 \le \operatorname{Im} s < a$*:*

$$|f^{(j)}(\alpha_0(t+s))| \le K \frac{1}{\tau^{2-(j-1)(r-1)}}, \qquad \text{for } j \ge 0. \tag{8.2.4}$$

By hypothesis ***HP3****,* $h_1(x,y,\theta)$ *is a polynomial in* (x,y)*. When we evaluate* h_1 *at* $(x,y) = \gamma_0(u)$*, by the definition of* ℓ *in Chapter 1, the function has a singularity of order at most* ℓ *at* $u = ia$*, hence for* (t,s) *as before*

$$|\partial_x^{k_1}\partial_y^{k_2} h_1(\gamma_0(t+s), t/\varepsilon)| \le K \frac{1}{\tau^{\ell-k_1(r-1)-k_2 r}}, \qquad \text{for } k_1,\ k_2 \ge 0. \tag{8.2.5}$$

Next result establishes some important properties of the functions $\mathcal{S}^u$ and $\mathcal{E}^u$, in particular the closeness of $\mathcal{E}^u$ and the Melnikov function.

Proposition 8.2.6 *Under hypotheses* ***HP1****-****HP6****,* $\mathcal{S}^u$ *and* $\mathcal{E}^u$ *satisfy the following estimates:*

a) For $s \in \mathbb{C}$ *such that* $|\operatorname{Im} s| \le a - \varepsilon$*,*

$$\mathcal{E}^u(s) = \mu\varepsilon^p M(s,\varepsilon) + O(\mu^2\varepsilon^{2\nu+r-1}, \mu\varepsilon^{p+i_0}). \tag{8.2.6}$$

b) For $s \in \mathbb{R}$*, and* $\mathcal{E}_0^u(\varepsilon) = \frac{1}{2\pi\varepsilon}\int_0^{2\pi\varepsilon} \mathcal{E}^u(s)\, ds$*,*

$$\mathcal{E}^u(s) - \mathcal{E}_0^u(\varepsilon) = \mu\varepsilon^p M(s,\varepsilon) + O(\mu^2\varepsilon^{2\nu+r-1}, \mu\varepsilon^{p+i_0})e^{-a/\varepsilon}. \tag{8.2.7}$$

c) For $s \in \mathbb{R}$*,* $S = \mathcal{S}^u(s)$ *is real analytic and invertible, and its inverse* $s = s^u(S)$ *satisfies that* $s^u(S) - S$ *is* $O(\mu\varepsilon^{p+i_0})$ *and* $2\pi\varepsilon$*-periodic in* S*.*

Proof. In Corollary 6.2.6 we have proved that

$$\mathcal{E}(\gamma^s_{\mu,\varepsilon}(t,s), t/\varepsilon) = 0 \tag{8.2.8}$$

and

$$\mathcal{E}(x,y,\theta) = h_0(x,y) + O(\mu\varepsilon^{p+i_0}). \tag{8.2.9}$$

Since $\mathcal{E}^u(s)$ does not depend on t, for any s we choose $t = T_s$ with $T_s = T + \varkappa_0 - \operatorname{Re} s$ and therefore, for $(t, s) = (T_s, s)$, $\gamma^u_{\mu,\varepsilon}(t, s)$ and $\gamma^s_{\mu,\varepsilon}(t, s)$ belong to the domain of the flow box coordinates U. Then, from the definition (8.2.2) of $\mathcal{E}^u$and properties (8.2.8) and (8.2.9):

$$\begin{aligned}\mathcal{E}^u(s) &= \mathcal{E}(\gamma^u_{\mu,\varepsilon}(t,s), t/\varepsilon) - \mathcal{E}(\gamma^s_{\mu,\varepsilon}(t,s), t/\varepsilon)\\ &= h_0(\gamma^u_{\mu,\varepsilon}(t,s)) - h_0(\gamma^s_{\mu,\varepsilon}(t,s)) + O(\mu\varepsilon^{p+i_0}),\end{aligned} \tag{8.2.10}$$

if $|\operatorname{Im} s| \le a - \varepsilon$.

Since, for any s, such that $|\operatorname{Im} s| \le a - \varepsilon$ we have that

$$\begin{aligned}&\gamma^s_{\mu,\varepsilon}(t,s) \to 0 \qquad &\text{when } t \to +\infty\\ &\gamma^u_{\mu,\varepsilon}(t,s) \to 0 \qquad &\text{when } t \to -\infty\end{aligned}$$

we deduce

$$\lim_{t\to+\infty} h_0(\gamma^s_{\mu,\varepsilon}(t,s)) = \lim_{t\to-\infty} h_0(\gamma^u_{\mu,\varepsilon}(t,s)) = 0.$$

Then

$$\begin{aligned}h_0(\gamma^u_{\mu,\varepsilon}(T_s,s)) - h_0(\gamma^s_{\mu,\varepsilon}(T_s,s)) &= \int_{-\infty}^{T_s} \partial_t \left[h_0(\gamma^u_{\mu,\varepsilon}(t,s))\right] dt - \int_{T_s}^{+\infty} \partial_t \left[h_0(\gamma^s_{\mu,\varepsilon}(t,s))\right] dt\\ = \mu\varepsilon^p &\left[\int_{-\infty}^{T_s} \{h_0, h_1\}(\gamma^u_{\mu,\varepsilon}(t,s), t/\varepsilon)\; dt + \int_{T_s}^{+\infty} \{h_0, h_1\}(\gamma^s_{\mu,\varepsilon}(t,s), t/\varepsilon)\; dt\right].\end{aligned}$$

Adding and subtracting the Melnikov function we obtain

$$\begin{aligned}h_0(\gamma^u_{\mu,\varepsilon}(T_s,s)) - h_0(\gamma^s_{\mu,\varepsilon}(T_s,s)) &= \mu\varepsilon^p \int_{-\infty}^{-T_s} \{h_0, h_1\}(\gamma^u_{\mu,\varepsilon}, t/\varepsilon) - \{h_0, h_1\}(\gamma_0, t/\varepsilon)\; dt\\ &+ \mu\varepsilon^p \int_{-T_s}^{T_s} \{h_0, h_1\}(\gamma^u_{\mu,\varepsilon}, t/\varepsilon) - \{h_0, h_1\}(\gamma_0, t/\varepsilon)\; dt\\ &+ \mu\varepsilon^p \int_{T_s}^{+\infty} \{h_0, h_1\}(\gamma^s_{\mu,\varepsilon}, t/\varepsilon) - \{h_0, h_1\}(\gamma_0, t/\varepsilon)\; dt\\ &+ \mu\varepsilon^p \int_{-\infty}^{+\infty} \{h_0, h_1\}(\gamma_0, t/\varepsilon)\; dt\end{aligned} \tag{8.2.11}$$

where $\gamma^u_{\mu,\varepsilon}$, $\gamma^s_{\mu,\varepsilon}$ and γ_0 denote $\gamma^u_{\mu,\varepsilon}(t,s)$, $\gamma^s_{\mu,\varepsilon}(t,s)$ and $\gamma_0(t+s)$ respectively. The last term in (8.2.11) is the Melnikov function. By the conclusion 4) of Theorem 3.2.1, the first and the third lines in (8.2.11) are $O(\mu^2\varepsilon^{2p+1})$. It remains to bound the second line.

It is not difficult to see that, if we write $\gamma^*_{\mu,\varepsilon}(t,s) = (\alpha^*(t,s), \beta^*(t,s))$ for $* = u, s$,

$$\begin{aligned}\{h_0, h_1\}(\gamma^*_{\mu,\varepsilon}, t/\varepsilon) - \{h_0, h_1\}(\gamma_0, t/\varepsilon) = \\ - f(\alpha^*)[\partial_y h_1(\gamma^*_{\mu,\varepsilon}, t/\varepsilon) - \partial_y h_1(\gamma_0, t/\varepsilon)] - [f(\alpha^*) - f(\alpha)]\partial_y h_1(\gamma_0, t/\varepsilon) \\ - \beta^*[\partial_x h_1(\gamma^*_{\mu,\varepsilon}, t/\varepsilon) - \partial_x h_1(\gamma_0, t/\varepsilon)] - (\beta^* - \beta)\partial_x h_1(\gamma_0, t/\varepsilon).\end{aligned}$$

Using bounds (8.2.4), (8.2.5) and Lemma 8.2.4 and taking into account that, by the extension theorem, $\gamma^u_{\mu,\varepsilon} - \gamma_0 = O(\mu\varepsilon^\nu)$, we get

$$|\{h_0, h_1\}(\gamma^u_{\mu,\varepsilon}, t/\varepsilon) - \{h_0, h_1\}(\gamma_0, t/\varepsilon)| \leq K\frac{\mu\varepsilon^\nu}{\tau^{\ell-r+2}}$$

(we recall that $\tau = |t + s - ia|$). Then applying the estimate (8.2.3) with $l = \ell - r + 2$ we obtain that the second line in (8.2.11) is $O(\mu\varepsilon^{p+\nu-\ell+r-1})$. Thus

$$h_0(\gamma^u_{\mu,\varepsilon}(T_s, s)) - h_0(\gamma^s_{\mu,\varepsilon}(T_s, s)) = \mu\varepsilon^p M(s, \varepsilon) + O(\mu^2\varepsilon^{2p+1}, \mu^2\varepsilon^{2\nu+r-1}).$$

Now a) follows from (8.2.10) and from the previous expression. Note that, since $\ell \geq r - 1$, one has $2p + 1 \geq 2\nu + r - 1$.

To prove b) we recall that the function $\mathcal{E}^u(s)$ is $2\pi\varepsilon$-periodic in s and analytic in the complex strip $|\operatorname{Im} s| \leq a - \varepsilon$, then, expanding in Fourier series

$$\mathcal{E}^u(s) = \sum_{k\in\mathbb{Z}} \mathcal{E}^u_k(\varepsilon)e^{iks/\varepsilon}$$

with

$$\mathcal{E}^u_k(\varepsilon) = \frac{1}{2\pi\varepsilon}\int_0^{2\pi\varepsilon} \mathcal{E}^u(s)e^{-iks/\varepsilon}ds.$$

For $s \in \mathbb{R}$, since $\mathcal{E}^u$ is analytic for $|\operatorname{Im} s| \leq a - \varepsilon$ and it is $2\pi\varepsilon$-periodic we can write

$$\mathcal{E}^u_k(\varepsilon) = \frac{1}{2\pi\varepsilon}\int_0^{2\pi\varepsilon} \mathcal{E}^u(s \pm i(a - \varepsilon))e^{-ik(s\pm i(a-\varepsilon))/\varepsilon}\, ds \tag{8.2.12}$$

Thus, by the conclusion a) of this proposition about the estimate (8.2.6) of $\mathcal{E}^u(s)$ in the complex domain, for $k \neq 0$ we obtain

$$\begin{aligned}\mathcal{E}^u_k(\varepsilon) &= \frac{e^{-|k|r_\varepsilon/\varepsilon}}{2\pi\varepsilon}\int_0^{2\pi\varepsilon} \mathcal{E}^u(s \pm i(a - \varepsilon))e^{-iks/\varepsilon}\, ds \\ &= \mu\varepsilon^p M_k(\varepsilon) + O(\mu^2\varepsilon^{2\nu+r-1}, \mu\varepsilon^{p+i_0})e^{-|k|a/\varepsilon},\end{aligned}$$

where we consider the sign $+$ for $k < 0$ and the sign $-$ for $k > 0$. Here $M_k(\varepsilon)$ are the Fourier coefficients of the Melnikov function. Now b) follows from the above equality.

Next we prove c). We recall that, by Corollary 6.2.6:

$$\mathcal{S}(\gamma^s_{\mu,\varepsilon}(t,s), t/\varepsilon) = t + s + \mu\varepsilon^{p+i_0}\mathcal{X}(s)$$

and, for $(x,y) \in U$

$$\mathcal{S}(x,y,\theta) = \mathcal{S}_0(x,y) + O(\mu\varepsilon^{p+i_0})$$

where $\mathcal{S}_0$ is a flow box coordinate when $\mu = 0$. Also, by the extension theorem,

$$\gamma^u_{\mu,\varepsilon}(t,s) - \gamma_0(t+s) = O(\mu\varepsilon^\nu) \tag{8.2.13}$$

for any $t \in \mathbb{R}$ and $s \in \mathbb{C}$ such that $T + \varkappa_0 \leq t + \operatorname{Re} s \leq 2T - \varkappa_0$ and $|\operatorname{Im} s| \leq a - \varepsilon$. Then we obtain that

$$\begin{aligned}\mathcal{S}^u(s) - s &= \mathcal{S}(\gamma^u_{\mu,\varepsilon}(t,s), t/\varepsilon) - t - s \\ &= \mathcal{S}_0(\gamma_0(t+s)) - t - s + O(\mu\varepsilon^\nu, \mu\varepsilon^{p+i_0}) \\ &= O(\mu\varepsilon^\nu).\end{aligned}$$

We expand $\mathcal{S}^u(s) - s$ in Fourier's series,

$$\mathcal{S}^u(s) - s = \sum_{k\in\mathbb{Z}} \mathcal{S}^u_k(\varepsilon) e^{iks/\varepsilon}.$$

Thus for $s \in \mathbb{R}$, estimating the Fourier coefficients of $\mathcal{S}^u(s) - s$ for $s \in \mathbb{R}$, in the same way as we did for $\mathcal{E}^u$ we have that for $k \neq 0$,

$$\mathcal{S}^u_k(\varepsilon) = \frac{e^{-|k|r_\varepsilon/\varepsilon}}{2\pi\varepsilon}\int_0^{2\pi\varepsilon} \mathcal{S}^u(s \pm i(a-\varepsilon))e^{-iks/\varepsilon}\, ds = O(\mu\varepsilon^\nu)e^{-|k|a/\varepsilon}.$$

From this we deduce

$$\begin{aligned}\mathcal{S}^u(s) - s &= \mathcal{S}^u_0(\varepsilon) + O(\mu\varepsilon^\nu)e^{-a/\varepsilon} \\ \frac{d\mathcal{S}^u}{ds}(s) - 1 &= O(\mu\varepsilon^\nu)e^{-a/\varepsilon},\end{aligned}$$

(taking into account that $\frac{d\mathcal{S}^u}{ds}(s) - 1$ has zero mean).

On the other hand, if $s \in \mathbb{R}$, by Remark 7.1.4 we have that

$$\gamma^u_{\mu,\varepsilon}(t,s) - \gamma_0(t+s) = O(\mu\varepsilon^{p+i_0}),$$

hence

$$\mathcal{S}^u(s) - s = O(\mu\varepsilon^{p+i_0}).$$

Therefore $\mathcal{S}_0^u(\varepsilon) = O(\mu\varepsilon^{p+i_0})$. This implies that $S = \mathcal{S}^u(s)$ (restricted to $\mathbb{R}$) is invertible. We denote $s = s^u(S)$ its inverse which is analytic. Moreover $s^u(S) - S = O(\mu\varepsilon^{p+i_0})$. To see that $s^u(S) - S$ is $2\pi\varepsilon$-periodic, we observe that, by Remark 8.2.2, $\mathcal{S}^u(s + 2\pi\varepsilon) = \mathcal{S}^u(s) + 2\pi\varepsilon$, thus

$$\begin{aligned} s^u(S + 2\pi\varepsilon) - (S + 2\pi\varepsilon) &= s^u(\mathcal{S}^u(s) + 2\pi\varepsilon) - (S + 2\pi\varepsilon) \\ &= s^u(\mathcal{S}^u(s + 2\pi\varepsilon)) - (S + 2\pi\varepsilon) \\ &= s + 2\pi\varepsilon - (S + 2\pi\varepsilon) \\ &= s^u(S) - S \end{aligned}$$

as we wanted. ■

Now we define the splitting function. From Corollary 6.2.6 it follows that the local stable manifold $\gamma_\mu^s(t,s)$ (for (t,s) such that $|\operatorname{Im} s| \le a - \varepsilon$ and $T + \varkappa_0 \le t + \operatorname{Re} s \le 2T - \varkappa_0$) can be written in the (S,E) coordinates:

$$(S,E) = (\mathcal{S}(\gamma_{\mu,\varepsilon}^s(t,s), t/\varepsilon), \mathcal{E}(\gamma_{\mu,\varepsilon}^s(t,s), t/\varepsilon)) = (t + s + \mu\varepsilon^{p+i_0}\mathcal{X}(s), 0) \qquad (8.2.14)$$

and the local unstable manifold $\gamma_{\mu,\varepsilon}^u(t,s)$ (for (t,s) such that $|\operatorname{Im} s| \le a - \varepsilon$ and $T + \varkappa_0 \le t + \operatorname{Re} s \le 2T - \varkappa_0$) can be expressed as

$$(S,E) = (\mathcal{S}(\gamma_{\mu,\varepsilon}^u(t,s), t/\varepsilon), \mathcal{E}(\gamma_{\mu,\varepsilon}^u(t,s), t/\varepsilon)) = (t + \mathcal{S}^u(s), \mathcal{E}^u(s)).$$

We define the Poincaré map

$$P_{\mu,\varepsilon}(x,y) = \varphi_{\mu,\varepsilon}(2\pi\varepsilon, 0, x, y),$$

where $\varphi_\mu(t, t_0, x, y)$ is the solution of system (8.1.1).

The restriction to U of the unstable curve C^u of $P_{\mu,\varepsilon}$, is given by $\gamma_{\mu,\varepsilon}^u(0,s)$ parametrizes for $s \in \mathbb{C}$ such that $T + \varkappa_0 \le \operatorname{Re} s \le 2T - \varkappa_0$ and $|\operatorname{Im} s| \le a - \varepsilon$. therefore in the (S,E) variables C^u is represented by

$$(S,E) = (\mathcal{S}(\gamma_{\mu,\varepsilon}^u(0,s)), \mathcal{E}(\gamma_{\mu,\varepsilon}^u(0,s))) = (\mathcal{S}^u(s), \mathcal{E}^u(s)).$$

Thus, it is very natural to put the unstable manifold in implicit form, i.e., the variable E as a function of S, because the measure of splitting can be computed by using this function. The function ϕ is defined implicitly by

$$\phi(\mathcal{S}^u(s)) = \mathcal{E}^u(s).$$

By property c) of Proposition 8.2.6, the equality $S = \mathcal{S}^u(s)$ can be inverted for real values of s, $s = s^u(S)$, thus the function ϕ is, in fact, defined explicitly by:

$$\phi(S) = \mathcal{E}^u(s^u(S)). \tag{8.2.15}$$

We observe that the splitting function is defined in $\mathbb{R}$.

The parameterization for the unstable manifold introduced in Theorem 3.2.1 and defined in (6.5.5) in Chapter 6 is not uniquely determinate. Indeed, if we define $s = S + \varrho(S)$ where ϱ is a $2\pi\varepsilon$-periodic function which is $O(\mu\varepsilon^{p+i_0})$, then $\gamma^*_{\mu,\varepsilon}(t,S) = \gamma^*_{\mu,\varepsilon}(t, S+\varrho(S))$ is another parameterization which all properties we have proved until now.

Since $s^u(S) - S$ is $O(\mu\varepsilon^{p+i_0})$ and $2\pi\varepsilon$-periodic in S a new parameterization for the unstable manifold can be defined as

$$\tilde{\gamma}^u_{\mu,\varepsilon}(t,S) = \gamma^u_{\mu,\varepsilon}(t, s^u(S))$$

and for the stable manifold

$$\tilde{\gamma}^s_{\mu,\varepsilon}(t,S) = \gamma^s_{\mu,\varepsilon}(t,S).$$

Finally, after this change of parameter, the splitting function defined in (8.2.15) can be also represented in the form

$$\begin{aligned} \phi(S) &= \mathcal{E}(\gamma^u_{\mu,\varepsilon}(t, s^u(S)), t/\varepsilon) \\ &= \mathcal{E}(\tilde{\gamma}^u_{\mu,\varepsilon}(t,S), t/\varepsilon). \end{aligned} \tag{8.2.16}$$

8.3 Proof of the Theorem 1.2.1 and its corollary

First we will show that the function ϕ given in (8.2.15) can be used to measure some magnitudes related to the splitting. Then we will prove the formulas in Theorem 1.2.1. In the next proposition we prove the existence of primary homoclinic points and we give a formula for the area of the lobes, as well as useful properties of the splitting function.

Proposition 8.3.1 *The function $\phi : \mathbb{R} \to \mathbb{R}$ is $2\pi\varepsilon$-periodic, real analytic and satisfies the following properties:*

a) There exists $h^u \in \mathbb{R}$ *such that* $\gamma^u_{\mu,\varepsilon}(t,h^u) = \gamma^s_{\mu,\varepsilon}(t,h^s)$ *(giving a homoclinic orbit), with* $h^s = \mathcal{S}^u(h^u)$*. For* $n \in \mathbb{N}$*, we define*

$$h^s_n = h^s + 2\pi\varepsilon n$$

which give homoclinic points. Clearly, for all n*,* $\phi(h^s_n) = 0$*. Moreover,* $\phi'(h^s_n)$ *is independent of* n*, and*

$$\begin{aligned}\phi'(h^s_n) &= \partial_S\tilde{\gamma}^s_{\mu,\varepsilon}(t,h^s_n) \wedge \partial_S\tilde{\gamma}^u_{\mu,\varepsilon}(t,h^s_n)(1+O(\mu\varepsilon^{p+i_0})) \\ &= \|\partial_S\tilde{\gamma}^s_{\mu,\varepsilon}(t,h^s_n)\| \; \|\partial_S\tilde{\gamma}^u_{\mu,\varepsilon}(t,h^s_n)\| \sin\vartheta(t,h^s_n)(1+O(\mu\varepsilon^{p+i_0})),\end{aligned}$$

where $\wedge$ *denotes the exterior product on* $\mathbb{R}^2$*, and* $\vartheta(t,h^s_n)$ *is the angle between* $\tilde{\gamma}^u(t,h^s_n)$ *and* $\gamma^s(t,h^s_n)$*.*

b) The area of the lobe between the invariant curves is given by

$$A = \left| \int_h^{\bar{h}} \phi(S)\; dS \right|,$$

where h *and* $\bar{h}$ *are two consecutive zeros of* $\phi(S)$*. We may choose the ones which are closest to zero.*

c) $\phi_0 = \int_{h_n}^{h_n+2\pi\varepsilon} \phi(S) dS = 0$*.*

Proof. We begin by proving the existence of homoclinic orbits. Let $P_{\mu,\varepsilon}$ be the Poincaré map

$$P_{\mu,\varepsilon}(x,y) = \varphi_{\mu,\varepsilon}(2\pi\varepsilon, 0, x, y).$$

Let $W^{*,+}(P_{\mu,\varepsilon},0)$ $(* = s,u)$ be the right hand side of the stable and the unstable invariant curves of the origin of the map $P_{\mu,\varepsilon}$. Since the parameterizations $\gamma^*_{\mu,\varepsilon}$, $* = s,u$, as functions of t are solutions of the system (8.1.1), we have that

$$\begin{aligned}C^s &\equiv \{\gamma^s_{\mu,\varepsilon}(0,s) : \operatorname{Re} s \geq T, \quad |\operatorname{Im} s| \leq a - \varepsilon\} \subset W^{s,+}(P_{\mu,\varepsilon},0) \\ C^u &\equiv \{\gamma^u_{\mu,\varepsilon}(0,s) : \operatorname{Re} s \leq -T, \quad |\operatorname{Im} s| \leq a - \varepsilon\} \subset W^{u,+}(P_{\mu,\varepsilon},0).\end{aligned}$$

Moreover, since

$$\gamma^*_{\mu,\varepsilon}(t+2\pi\varepsilon, s) = \gamma^*_{\mu,\varepsilon}(t, s+2\pi\varepsilon) \qquad * = s,u,$$

in their respective domains, we have that

$$P_{\mu,\varepsilon}(\gamma^*_{\mu,\varepsilon}(0,s)) = \gamma^*_{\mu,\varepsilon}(2\pi\varepsilon, s) = \gamma^*_{\mu,\varepsilon}(0, s+2\pi\varepsilon),$$

which means that if we consider s as the variable in $C^s \subset W^{s,+}(P,0)$ the dynamics of $P_{\mu,\varepsilon}$ on C^s is just

$$s \mapsto s + 2\pi\varepsilon.$$

Since $P_{\mu,\varepsilon}$ is area preserving and $P_{0,\varepsilon}$ has a homoclinic connexion (which coincides with the homoclinic orbit of the unperturbed differential equation), a well known geometric argument, applied to $P_{\mu,\varepsilon}$ restricted to the real numbers, gives that $P_{\mu,\varepsilon}$ has (real) primary homoclinic points. Since the iterates of the homoclinic points also are homoclinic points there will be such points in U.

Then there exist h^u, $h^s \in \mathbb{R}$, $T + \varkappa_0 \le h^u, h^s \le 2T - \varkappa_0$, such that

$$z^h = \gamma^u_{\mu.\varepsilon}(0, h^u) = \gamma^s_{\mu.\varepsilon}(0, h^s).$$

Hence

$$\gamma^u_{\mu.\varepsilon}(t, h^u) = \gamma^s_{\mu.\varepsilon}(t, h^s)$$

are defined for all $t \in \mathbb{R}$ and are a homoclinic solution of (8.1.1).

Moreover, taking t such that

$$T + \varkappa_0 \le t + h^u, t + h^s \le 2T - \varkappa_0.$$

We recall by Corollary (6.2.6), we can choose $s_0 = h^s$. Therefore, we can write

$$\mathcal{S}(\gamma^s_{\mu,\varepsilon}(t, h^s), t/\varepsilon) = t + h^s,$$

therefore

$$h^s = \mathcal{S}(\gamma^s_{\mu,\varepsilon}(t, h^s), t/\varepsilon) - t = \mathcal{S}(\gamma^u_{\mu,\varepsilon}(t, h^u), t/\varepsilon) - t = \mathcal{S}^u(h^u).$$

Moreover, by the definition of ϕ in (8.2.15) and the definition of $\mathcal{E}^u$ in (8.2.2) we have

$$\begin{aligned}
\phi(h^s) &= \phi(\mathcal{S}^u(h^u)) = \mathcal{E}^u(s^u(\mathcal{S}^u(h^u)) \\
&= \mathcal{E}^u(h^u) = \mathcal{E}(\gamma^u_{\mu,\varepsilon}(t, h^u), t/\varepsilon) \\
&= \mathcal{E}(\gamma^s_{\mu,\varepsilon}(t, h^s), t/\varepsilon) = 0.
\end{aligned}$$

By the $2\pi\varepsilon$-periodicity of ϕ, $\phi(h^s_n) = \phi(h^s + 2\pi\varepsilon n) = \phi(h^s) = 0$.

Obviously ϕ' is also $2\pi\varepsilon$-periodic, thus $\phi'(h_n)$ does not depend on n. Now we compute $\phi'(h^s)$. We recall that

$$\gamma^u_{\mu,\varepsilon}(t, h^u) = \gamma^u_{\mu,\varepsilon}(t, s^u(h^s)) = \tilde{\gamma}^u_{\mu,\varepsilon}(t, h^s)$$

and formula (8.2.16):

$$\phi(S) = \mathcal{E}(\tilde{\gamma}_{\mu,\varepsilon}(t, S), t/\varepsilon).$$

We differentiate the above equation at the point $S = h^s$ and then $\phi'(h_s)$ can be expressed in the following way:

$$\phi'(h^s) = \partial_x \mathcal{E}(\tilde{\gamma}^u_{\mu,\varepsilon}(t, h^s), t/\varepsilon)\partial_S \tilde{\alpha}^u(t, h^s) + \partial_y \mathcal{E}(\tilde{\gamma}^u_{\mu,\varepsilon}(t, h^s), t/\varepsilon)\partial_S \tilde{\beta}^u(t, h^s) \tag{8.3.1}$$

where $\tilde{\gamma}^u_{\mu,\varepsilon}(t, s) = (\tilde{\alpha}^u(t, s), \tilde{\beta}^u(t, s))$. Moreover, differentiating with respect to $s = S$ the equation

$$(\mathcal{S}(\gamma^s_{\mu,\varepsilon}(t, s), t/\varepsilon), \mathcal{E}(\gamma^s_{\mu,\varepsilon}(t, s), t/\varepsilon)) = (t + s + \mu\varepsilon^{p+i_0}\mathcal{X}(s), 0),$$

(given in (8.2.14)) we obtain

$$\begin{aligned}
1 + O(\mu\varepsilon^{p+i_0-1}) &= \partial_x \mathcal{S}(\gamma^s_{\mu,\varepsilon}(t, S), t/\varepsilon)\partial_S \alpha^s(t, S) + \partial_y \mathcal{S}(\gamma^s_{\mu,\varepsilon}(t, S), t/\varepsilon)\partial_S \beta^s(t, S)\\
0 &= \partial_x \mathcal{E}(\gamma^s_{\mu,\varepsilon}(t, S), t/\varepsilon)\partial_S \alpha^s(t, S) + \partial_y \mathcal{E}(\gamma^s_{\mu,\varepsilon}(t, S), t/\varepsilon)\partial_S \beta^s(t, S)
\end{aligned}$$

and from this, taking into account that the change $(x, y) \mapsto (S, E)$ is canonical, we get when $S = h^s$

$$\begin{aligned}
\partial_S \alpha^s(t, h^s)(1 + O(\mu\varepsilon^{p+i_0-1})) &= \partial_y \mathcal{E}(\gamma^s_{\mu,\varepsilon}(t, h^s), t/\varepsilon) = \partial_y \mathcal{E}(\gamma^u_{\mu,\varepsilon}(t, h^u), t/\varepsilon)\\
&= \partial_y \mathcal{E}(\tilde{\gamma}^u_{\mu,\varepsilon}(t, h^s), t/\varepsilon)\\
\partial_S \beta^s(t, h^s)(1 + O(\mu\varepsilon^{p+i_0-1})) &= -\partial_x \mathcal{E}(\gamma^s_{\mu,\varepsilon}(t, h^s), t/\varepsilon) = -\partial_x \mathcal{E}(\tilde{\gamma}^u_{\mu,\varepsilon}(t, h^s), t/\varepsilon).
\end{aligned}$$

Substituting the derivatives of $\mathcal{E}$ in (8.3.1) we obtain the formula stated in a).

In order to prove b) of Proposition 8.3.1 we recall that the change $\tilde{\mathcal{C}}$, given in Corollary 6.2.6, which transforms the initial coordinates (x, y) to the flow box coordinates (S, E), is canonical. Therefore.

$$A = \left| \int\int_{Lobe} dx dy \right| = \left| \int\int_{\tilde{\mathcal{C}}(Lobe)} dS dE \right|.$$

Moreover, since the Poincaré map $P_{\mu,\varepsilon}$ is orientation preserving, there exists at least one primary homoclinic point of $P_{\mu,\varepsilon}$ between $z^h = \gamma^s_{\mu,\varepsilon}(0,h^s)$ and $P_{\mu,\varepsilon}(z^h)$. We denote this homoclinic point by $\gamma^s_{\mu,\varepsilon}(0,\bar{h}^s)$. By definition of the splitting function, the area of a lobe, in (S,E) coordinates, is the area of the splitting function between two consecutive zeros of ϕ, hence

$$A = \left| \int_{h^s}^{\bar{h}^s} \phi(S)\ dS \right|$$

with h^s and $\bar{h}^s$ are two consecutive zeros of ϕ.

The conclusion c) asserts that the splitting function has zero mean. To prove it we note that since $P_{\mu,\varepsilon}$ is area preserving, a standard geometric argument gives that the area of two consecutive lobes one inner and the other outer, coincide. Therefore c) follows from b) and the fact that the change $\tilde{\mathcal{C}}$ is canonical. ■

Now we have a suitable expression, but not the final one, of the splitting function $\phi(S)$ and consequently of the area of the lobe. By the second estimate of Proposition 8.2.6 as well as by the definition of $\phi(S) = \mathcal{E}^u(s^u(S))$, we have that for real values of S

$$\phi(S) \equiv \mathcal{E}^u(s^u(S)) = \mathcal{E}^u_0(\varepsilon) + \mu\varepsilon^p M(s^u(S),\varepsilon) + O(\mu^2\varepsilon^{2\nu+r-1}, \mu\varepsilon^{p+i_0})e^{-a/\varepsilon},$$

where we recall $\mathcal{E}^u_0(\varepsilon)$ that is the 0-Fourier coefficient of $\mathcal{E}^u$. Therefore, in order to prove Theorem 1.2.1 we need to estimate $\mathcal{E}^u_0$. For this we enunciate a technical lemma which we will prove in Subsection 8.3.1.

Lemma 8.3.2 *Under the standing conditions we have*

$$\mu\varepsilon^p \frac{dM}{dS}(S,\varepsilon) = O(\mu\varepsilon^{\nu-1})e^{-a/\varepsilon}.$$

A direct consequence of this lemma is the following corollary which finishes the proof of Theorem 1.2.1.

Corollary 8.3.3 *For $S \in \mathbb{R}$, $\phi(S)$ satisfies the estimate*

$$\phi(S) = \mu\varepsilon^p M(S,\varepsilon) + O(\mu^2\varepsilon^{2\nu+r-1}, \mu^2\varepsilon^{\nu+p-1+i_0}, \mu\varepsilon^{p+i_0})e^{-a/\varepsilon}.$$

Proof. The proof of this corollary is a direct consequence of the Lemma 8.3.2 and the Taylor' Theorem. Indeed,

$$\begin{aligned}\mu\varepsilon^p M(s^u(S),\varepsilon) &= \mu\varepsilon^p M(S+O(\mu\varepsilon^{p+i_0}),\varepsilon)\\ &= \mu\varepsilon^p M(S,\varepsilon)+\mu\varepsilon^p\int_0^1 \frac{dM}{dS}(S+\zeta O(\mu\varepsilon^{p+i_0}),\varepsilon)O(\mu\varepsilon^{p+i_0})\ d\zeta\\ &= \mu\varepsilon^p M(S,\varepsilon)+O(\mu^2\varepsilon^{\nu+p-1+i_0})e^{-a/\varepsilon}\end{aligned}$$

therefore,

$$\phi(S)=\mathcal{E}_0^u+\mu\varepsilon^p M(S,\varepsilon)+O(\mu^2\varepsilon^{2\nu+r-1},\mu^2\varepsilon^{\nu+p-1+i_0},\mu\varepsilon^{p+i_0})e^{-a/\varepsilon}$$

and since by c) of Proposition 8.3.1 $\phi_0=0$, and the average of M is zero

$$\mathcal{E}_0^u=O(\mu^2\varepsilon^{2\nu+r-1},\mu^2\varepsilon^{\nu+p-1+i_0},\mu\varepsilon^{p+i_0})e^{-a/\varepsilon}$$

and the corollary holds. ∎

The proof of Theorem 1.2.1 is an immediate consequence of Proposition 8.3.1 and Corollary 8.3.3.

8.3.1 Proof of the Lemma 8.3.2

The proof of this lemma has big differences from the proof of the corresponding Lemma in [20]. As we pointed out before, we are considering the case such that the parameterization of the homoclinic orbit has a singularity which is a branching point. The proof of Lemma 8.3.2 is the place where this hypothesis has to be taken into account. Since $u=\pm ia$ are branching points, the homoclinic orbit is defined in a neighbourhood of the singularities except a segment starting at them, and therefore we can not use the residue theory in order to estimate the Melnikov integral.

The case such that the singularity is a pole also follows from this proof taking below $q=1$.

We recall that as we pointed out in Remark 1.1.1, near of singularities $\pm ia$, the homoclinic orbit $\gamma_0=(\alpha_0,\beta_0)$ can be written as

$$\alpha_0(u)=\frac{c_\pm}{(u\pm ia)^{\frac{p}{q}}}(1+O(u\pm ia)^{\frac{1}{q}}),\qquad \beta_0(u)=\frac{d_\pm}{(u\pm ia)^{\frac{p}{q}+1}}(1+O(u\pm ia)^{\frac{1}{q}}).$$

where we take

$$\arg(u+ia) \in \left(-\frac{\pi}{2}, \frac{3\pi}{2}\right), \qquad \text{near the singularity } -ia$$

and

$$\arg(u-ia) \in \left(-\frac{3\pi}{2}, \frac{\pi}{2}\right), \qquad \text{near the singularity } +ia.$$

Here we write

$$r = 1 + \frac{p}{q} \in \mathbb{Q}.$$

We recall the definition of J given in Chapter 1:

$$J(x,y,t/\varepsilon) \equiv \{h_0, h_1\}(x,y,t/\varepsilon) \sim \sum_{n\neq 0} J_n(x,y)e^{int/\varepsilon}$$

and that $J(\gamma_0(t+s), t/\varepsilon)$ has a singularity of order at most $\ell+1$. We also observe that the perturbation $h_1(x,y,\theta)$ can be written as

$$h_1(x,y,\theta) = \sum_{k\leq|l|\leq\kappa, l\in\mathbb{N}^2} a_l(\theta)x^{l_1}y^{l_2}.$$

We recall that, by its definition, ℓ, can be expressed in the form

$$\ell = j_1(r-1) + j_2 r = j_2 + (j_1+j_2)\frac{p}{q}, \tag{8.3.2}$$

where j_1 and j_2 are such that

$$j_1(r-1) + j_2 r = \max\{l_1(r-1) + l_2 r \ : \ l_1 + l_2 \geq k \ , \ l = (l_1, l_2) \ , \ a_l(\theta) \neq 0\}.$$

Now we write the Fourier's coefficients of $M(s,\varepsilon)$ in terms of the Fourier's coefficients of J evaluated at $\gamma_0(u)$: $J_n(\gamma_0(u))$. We note that, since J is only continuous with respect to t/ε, the Fourier's series may not converge, although their Fourier's coefficients are well defined. However, since $M(.,\varepsilon)$ is analytic and $2\pi\varepsilon$-periodic with respect to s, its Fourier series converges and thus, we have that

$$M(s,\varepsilon) = \sum_{n\in\mathbb{Z}} M_n(\varepsilon)e^{ins/\varepsilon}$$

with

$$M_n(\varepsilon) = \frac{1}{2\pi\varepsilon}\int_0^{2\pi\varepsilon} M(s,\varepsilon)e^{-ins/\varepsilon}\, ds.$$

We claim that we can relate the Fourier coefficients of M with the ones of J, concretely

$$M_n(\varepsilon) = \frac{1}{2\pi\varepsilon}\int_{-\infty}^{+\infty} e^{-iun/\varepsilon} J_{-n}(\gamma_0(u))\, du.$$

Indeed, by definition of $M(s,\varepsilon)$, we obtain

$$\begin{aligned}
M_n(\varepsilon) &= \frac{1}{2\pi\varepsilon}\int_0^{2\pi\varepsilon}\left(\int_{-\infty}^{+\infty} J(\gamma_0(t+s), t/\varepsilon)dt\right) e^{-ins/\varepsilon}\, ds\\
&= \frac{1}{2\pi\varepsilon}\int_0^{2\pi\varepsilon}\int_{-\infty}^{+\infty} e^{-ins/\varepsilon} J\Big(\gamma_0(u), \frac{u-s}{\varepsilon}\Big) du\, ds\\
&= \frac{1}{2\pi\varepsilon}\int_{-\infty}^{+\infty} e^{-iun/\varepsilon}\int_0^{2\pi\varepsilon} e^{in(u-s)/\varepsilon} J\Big(\gamma_0(u), \frac{u-s}{\varepsilon}\Big) ds\, du\\
&= \frac{1}{2\pi\varepsilon}\int_{-\infty}^{+\infty} e^{-iun/\varepsilon} J_{-n}(\gamma_0(u))\, du.
\end{aligned}$$

Here we have used that, since the integral

$$\int_{-\infty}^{+\infty} e^{-ins/\varepsilon} J\Big(\gamma_0(u), \frac{u-s}{\varepsilon}\Big) du$$

is absolutely convergent, we can change the order of integration.

Now, our goal is to estimate the integrals

$$\int_{-\infty}^{+\infty} e^{-iun/\varepsilon} J_{-n}(\gamma_0(u))\, du \tag{8.3.3}$$

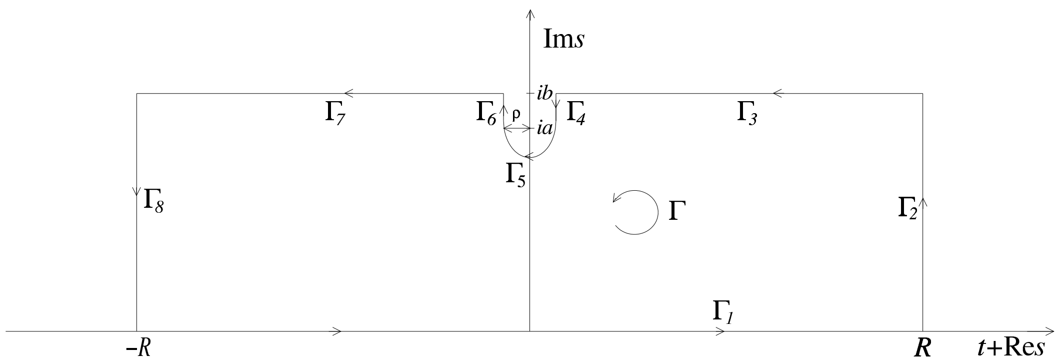
for $n \in \mathbb{Z}$.

We observe that, since J is a polynomial in x, y variables, near the singularities $u = \pm ia$, $J_n(\gamma_0(u))$ has the form:

$$\begin{aligned}
J_n(\gamma_0(u)) &= \frac{1}{(u\pm ia)^{\ell+1}}\Big(J^{\pm}_{n,0} + \sum_{m\geq 0} J^{\pm}_{n,m}(u\pm ia)^{\frac{m}{q}}\Big)\\
&= \sum_{-\infty<m\leq (j_1+j_2)p} \frac{J^{\pm}_{n,(j_1+j_2)p-m}}{(u\pm ia)^{\frac{m}{q}+j_2+1}}
\end{aligned} \tag{8.3.4}$$

where j_1 and j_2 are defined in (8.3.2) and $J^{\pm}_{n,(j_1+j_2)p-m}$ are coefficients which depend on ε and μ.

Now we proceed to evaluate the integrals (8.3.3). We consider first the case $n < 0$. We choose the path of integration $\Gamma = \Gamma_1 \vee \Gamma_2 \vee \ldots \vee \Gamma_8$ as indicated in the figure:



where $b > a$, ρ is small (obviously $\rho < a$) and R is big. Since we will play with the dependence of Γ on ρ, we will denote the path by $\Gamma(\rho)$.

Since the function $J_{-n}(\gamma_0(u))e^{-inu/\varepsilon}$ is analytic in the region enclosed by $\Gamma(\rho)$,

$$\int_{\Gamma(\rho)} J_{-n}(\gamma_0(u))e^{-inu/\varepsilon}\, du = 0, \qquad \forall \rho \in (0, \rho_0)$$

with ρ_0 small enough. The advantage of considering these curves is that the above integral does not depend on ρ. Therefore, in order to compute the dominant term of

$$\int_{\Gamma_1(\rho)} J_{-n}(\gamma_0(u))e^{-inu/\varepsilon}\, du = -\sum_{j=2}^{8} \int_{\Gamma_j(\rho)} J_{-n}(\gamma_0(u))e^{-inu/\varepsilon}\, du$$

the strategy consists on to expand the right hand side in terms of powers of ρ and then to take limit when ρ goes to zero. The terms with negative powers of ρ must cancel and the terms with positive powers of ρ tend to zero. Therefore we only have to take into account the coefficients of ρ^0 in such expansion.

We begin to look for the asymptotic expression of (8.3.3). First we observe that

$$\int_{\Gamma_2} J_{-n}(\gamma_0(u))e^{-inu/\varepsilon}\, du + \int_{\Gamma_8} J_{-n}(\gamma_0(u))e^{-inu/\varepsilon}\, du \to 0 \qquad \text{when } R \to +\infty$$

and that

$$\int_{\Gamma_3(\rho)} J_{-n}(\gamma_0(u))e^{-inu/\varepsilon}\, du,\ \int_{\Gamma_7(\rho)} J_{-n}(\gamma_0(u))e^{-inu/\varepsilon}\, du = O(e^{nb/\varepsilon}),$$

uniformly with respect to ρ. Next we will compute the integrals over the paths $\Gamma_5(\rho)$, $\Gamma_4(\rho)$ and $\Gamma_6(\rho)$. For these three integrals we stay near the singularity ia, thus we can use the expansion of $J_{-n}(\gamma_0(u))$ given in (8.3.4). For $j = 4, 5, 6$, we have that

$$\int_{\Gamma_j(\rho)} J_{-n}(\gamma_0(u))e^{-inu/\varepsilon}\, du = \sum_{m\le(j_1+j_2)p} J^{-}_{n,(j_1+j_2)p-m} \int_{\Gamma_j(\rho)} \frac{e^{-inu/\varepsilon}}{(u-ia)^{\frac{m}{q}+j_2+1}}\, du. \quad (8.3.5)$$

To evaluate the integrals in the right hand side of (8.3.5) we distinguish two cases: $m/q \notin \mathbb{N}$ and $m/q \in \mathbb{N}$.

First we deal with the case $m/q \notin \mathbb{N}$:

1. Integral over $\Gamma_5(\rho)$. This path can be parameterized by $g_5(\theta) = ia + \rho e^{-i\theta}$ where $\theta \in [0, \pi]$. Using the series expansion of the exponential we have that

$$\begin{aligned}\int_{\Gamma_5(\rho)} \frac{e^{-inu/\varepsilon}}{(u-ia)^{\frac{m}{q}+j_2+1}}\, du &= -ie^{na/\varepsilon}\rho \int_0^{\pi} \frac{e^{-in\rho e^{-i\theta}/\varepsilon}e^{-i\theta}}{(\rho e^{-i\theta})^{\frac{m}{q}+j_2+1}}\, d\theta \\ &= -ie^{na/\varepsilon}\rho^{-\frac{m}{q}-j_2} \int_0^{\pi} e^{-in\rho e^{-i\theta}/\varepsilon}e^{i\theta(\frac{m}{q}+j_2)}\, d\theta \\ &= -ie^{na/\varepsilon}\rho^{-\frac{m}{q}-j_2} \sum_{l\ge 0} \int_0^{\pi} \left(\frac{-in\rho}{\varepsilon}\right)^{l} e^{-i\theta(\frac{m}{q}+j_2-l)}\frac{1}{l!}\, d\theta.\end{aligned}$$

   Therefore, if $\frac{m}{q} \notin \mathbb{N}$, the integral over $\Gamma_5(\rho)$ has not constant term in ρ. Then the integral

$$\int_{\Gamma_5(\rho)} \frac{e^{-inu/\varepsilon}}{(u-ia)^{\frac{m}{q}+j_2+1}}\, du$$

   has not contribution.

2. Integral over $\Gamma_4(\rho)$. This path can be parameterized by

$$g_4(\theta) = \rho - i\theta, \qquad \theta \in [-b, -a].$$

Then

$$\int_{\Gamma_4(\rho)} \frac{e^{-inu/\varepsilon}}{(u-ia)^{\frac{m}{q}+j_2+1}}\,du = -i\int_{-b}^{-a} \frac{e^{-\rho ni/\varepsilon}e^{-n\theta/\varepsilon}}{(\rho-(a+\theta)i)^{\frac{m}{q}+j_2+1}}\,d\theta \tag{8.3.6}$$

$$= -i\int_{a}^{b} \frac{e^{-\rho ni/\varepsilon}e^{n\theta/\varepsilon}}{(\rho+(\theta-a)i)^{\frac{m}{q}+j_2+1}}\,d\theta.$$

We need some notation, we introduce

$$I_l(\rho) = \int_a^b \frac{e^{n\theta/\varepsilon}}{(\rho+(\theta-a)i)^{\eta+l}}\,d\theta$$

where $l = j_2 + 1 + \left[\frac{m}{q}\right]$ and $\eta = \frac{m}{q} - \left[\frac{m}{q}\right] \in (0,1)$. Here $[.]$ denotes the integer part function. Integrating by parts in $I_l(\rho)$ we obtain a recurrence formula for $I_l(\rho)$:

$$I_l(\rho) = \frac{1}{i(\eta+l-1)}\Big(f_{l-1}(\rho) + \frac{n}{\varepsilon}I_{l-1}\Big)$$

where

$$f_{l-1}(\rho) = \frac{e^{na/\varepsilon}}{\rho^{\eta+l-1}} - \frac{e^{nb/\varepsilon}}{(\rho+i(b-a))^{\eta+l-1}}.$$

From the recurrence relation it is not difficult to prove by induction that,

$$\begin{aligned} I_l(\rho) \;&=\; \sum_{j=1}^{l}\left(\frac{n}{\varepsilon}\right)^{j-1}\frac{1}{i^j}\frac{1}{(\eta+l-1)\cdots(\eta+l-j)}f_{l-j}(\rho) \\ &\quad+\left(\frac{n}{\varepsilon}\right)^{l}\frac{1}{i^l}\frac{1}{(\eta+l-1)\cdots\eta}I_0(\rho), \end{aligned} \tag{8.3.7}$$

(we observe that $\eta > 0$). We recall that we only have to look for the constant terms in ρ of $I_l(\rho)$. The contribution of the j-term in the sum (8.3.7) is

$$-\left(\frac{n}{\varepsilon}\right)^{j-1}\frac{1}{i^j}\frac{1}{(\eta+l-1)\cdots(\eta+l-j)}\frac{e^{nb/\varepsilon}}{(i(b-a))^{\eta+l-1}}. \tag{8.3.8}$$

Now we analyze $I_0(\rho)$. We observe that, since $\rho > 0$, for $z > a$, we have that

$$\arg(\rho + i(z-a)) \to \frac{\pi}{2} \qquad \text{when } \rho \to 0.$$

Therefore, using the dominate convergence theorem

$$I_0(\rho) = \int_a^b \frac{e^{n\theta/\varepsilon}}{(\rho + i(\theta - a))^\eta}\, d\theta \to e^{-\eta i\pi/2} \int_a^b \frac{e^{n\theta/\varepsilon}}{(\theta - a)^\eta}\, d\theta \qquad \text{when } \rho \to 0.$$

With elemental changes of variables we get

$$\int_a^b \frac{e^{n\theta/\varepsilon}}{(\theta - a)^\eta}\, d\theta = \left(\frac{\varepsilon}{|n|}\right)^{1-\eta} e^{na/\varepsilon} \int_0^{|n|(b-a)/\varepsilon} s^{-\eta} e^{-s}\, ds. \tag{8.3.9}$$

Moreover we have that

$$\int_0^{|n|(b-a)/\varepsilon} s^{-\eta} e^{-s}\, ds = \Gamma(1-\eta) + \psi(\varepsilon) \tag{8.3.10}$$

where Γ is the Gamma function and

$$\psi(\varepsilon) = \int_{|n|(b-a)/\varepsilon}^{+\infty} s^{-\eta} e^{-s}\, ds \le \left(\frac{\varepsilon}{|n|(b-a)}\right)^\eta e^{-|n|(b-a)/\varepsilon}$$

which is exponentially small. Using (8.3.8) and (8.3.9), we obtain that the constant term in ρ of $I_l(\rho)$ is

$$\begin{aligned} I_l \;&\equiv\; \sum_{j=1}^{l} \left(\frac{n}{\varepsilon}\right)^{j-1} \frac{1}{i^j} \frac{-e^{nb/\varepsilon}}{(\eta + l - 1)\cdots(\eta + l - j)(i(b-a))^{\eta + j - 1}} \\ &\quad + \left(\frac{n}{\varepsilon}\right)^{l} \frac{1}{i^l} \frac{1}{(\eta + l - 1)\cdots\eta} \left(\frac{\varepsilon}{|n|}\right)^{1-\eta} e^{na/\varepsilon} e^{-\eta i\pi/2} [\Gamma(1-\eta) + \psi(\varepsilon)]. \end{aligned}$$

Clearly, by (8.3.9), the dominant term of I_l is

$$\begin{aligned} I_l &= \frac{(-1)^l}{i^l} \left(\frac{|n|}{\varepsilon}\right)^{l-1+\eta} \frac{e^{-\eta i\pi/2}}{(\eta + l - 1)\cdots\eta} \Gamma(1-\eta) e^{-|n|a/\varepsilon}(1 + O(\varepsilon)) \\ &= \frac{(-1)^l}{i^l} \left(\frac{|n|}{\varepsilon}\right)^{l-1+\eta} e^{-\eta i\pi/2} \frac{\pi}{\sin \pi\eta} \frac{1}{\Gamma(l + \eta)} e^{-|n|a/\varepsilon}(1 + O(\varepsilon)). \end{aligned}$$

where we have used the formula $\Gamma(1-\eta)\Gamma(\eta) = \pi/\sin(\eta\pi)$. Thus the constant term in ρ of

$$\int_{\Gamma_4(\rho)} \frac{e^{-inu/\varepsilon}}{(u - ia)^{\frac{m}{q} + j_2 + 1}}$$

is

$$-i\frac{(-1)^l}{i^l}\left(\frac{|n|}{\varepsilon}\right)^{l-1+\eta} e^{-\eta i\pi/2}\frac{\pi}{\sin\pi\eta}\frac{1}{\Gamma(l+\eta)}e^{-|n|a/\varepsilon}(1+O(\varepsilon)), \tag{8.3.11}$$

where we recall that

$$l = j_2 + 1 + \left[\frac{m}{q}\right] \qquad \eta = \frac{m}{q} - \left[\frac{m}{q}\right].$$

3. Integral over Γ_6. This path can be parameterized by

$$g_6(z) = -\rho + i\theta, \qquad \theta \in [a, b].$$

Therefore,

$$\int_{\Gamma_6(\rho)} \frac{e^{-inu/\varepsilon}}{(u-ia)^{\frac{m}{q}+j_2+1}}\,du = i\int_a^b \frac{e^{\rho ni/\varepsilon}e^{n\theta/\varepsilon}}{(-\rho+(\theta-a)i)^{\frac{m}{q}+j_2+1}}\,d\theta.$$

Thus, if we define

$$J_l(\rho) = \int_a^b \frac{e^{\rho ni/\varepsilon}}{(-\rho+(\theta-a)i)^{l+\eta}}\,d\theta$$

we have that $J_l(\rho) = I_l(-\rho)$ and for $\eta > 0$, by using the previous computations, we obtain

$$\begin{aligned} J_l(\rho) \quad &= \quad \sum_{j=1}^{l}\left(\frac{n}{\varepsilon}\right)^{j-1}\frac{1}{i^j}\frac{1}{(\eta+l-1)\cdots(\eta+l-j)}f_{l-j}(-\rho) \\ &\quad + \left(\frac{n}{\varepsilon}\right)^l\frac{1}{i^l}\frac{1}{(\eta+l-1)\cdots\eta}J_0. \end{aligned}$$

As before we calculate the constant term J_0 of $J_0(\rho)$. In this case, the argument of $-\rho+(z-a)i$ belongs to $(-\frac{3\pi}{2}, -\pi)$ and therefore,

$$J_0(\rho) = \int_a^b \frac{e^{n\theta/\varepsilon}}{(-\rho+i(\theta-a))^\eta}\,d\theta \to e^{\eta i3\pi/2}\int_a^b \frac{e^{n\theta/\varepsilon}}{(\theta-a)^\eta}\,d\theta \qquad \text{when } \rho\to 0.$$

Consequently the constant term in ρ of

$$\int_{\Gamma_6(\rho)} \frac{e^{-inu/\varepsilon}}{(u-ia)^{\frac{m}{q}+j_2+1}}\,du$$

is

$$i\frac{(-1)^l}{i^l}\left(\frac{|n|}{\varepsilon}\right)^{l-1+\eta} e^{\eta i3\pi/2}\frac{\pi}{\sin\pi\eta}\frac{1}{\Gamma(l+\eta)}e^{-|n|a/\varepsilon}(1+O(\varepsilon)). \tag{8.3.12}$$

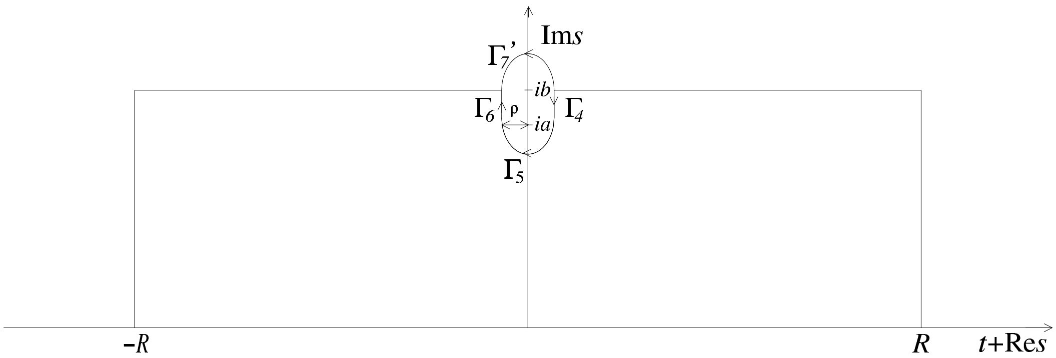
Now we consider the case such that $\frac{m}{q} \in \mathbb{N}$. Taking into account that, $\frac{m}{q} + j_2 + 1 \in \mathbb{N}$, the functions

$$\frac{e^{-inu/\varepsilon}}{(u-ia)^{\frac{m}{q}+j_2+1}}$$

have a pole of order $\frac{m}{q} + j + 1$. Therefore we can apply the residue theory. We consider the integration path $\Gamma'(\rho)$ given by

$$\Gamma'(\rho) = \Gamma_4(\rho) \vee \Gamma_5(\rho) \vee \Gamma_6(\rho) \vee \Gamma_7'(\rho)$$

where $\Gamma_7'(\rho)$ is



Then

$$\int_{\Gamma'(\rho)} \frac{e^{-inu/\varepsilon}}{(u-ia)^{\frac{m}{q}+j_2+1}}\, du = 2\pi i \mathrm{Res}(f, ia)$$

where $f(u) = \dfrac{e^{-inu/\varepsilon}}{(u-ia)^{\frac{m}{q}+j_2+1}}$. It is not difficult to calculate the following residue:

$$\mathrm{Res}(f, ia) = \frac{1}{(\frac{m}{q}+j_2)!}\left(\frac{-in}{\varepsilon}\right)^{\frac{m}{q}+j_2} e^{na/\varepsilon}(1+O(\varepsilon)).$$

Then,

$$\int_{\Gamma_4(\rho)\vee\Gamma_5(\rho)\vee\Gamma_6(\rho)} f = 2\pi i \mathrm{Res}(f, ia) - \int_{\Gamma_7'(\rho)} f.$$

It remains to estimate the integral over the path $\Gamma_7'(\rho)$. This curve can be parameterized by $g_7'(\theta) = ib - \rho e^{i\theta}$, $\theta \in [0, \pi]$, thus

$$\int_{\Gamma_7'(\rho)} \frac{e^{-inu/\varepsilon}}{(u-ia)^{\frac{m}{q}+j_2+1}}\, du = -\rho i e^{nb/\varepsilon} \int_0^\pi \frac{e^{in\rho e^{i\theta}/\varepsilon}}{(i(b-a)-\rho e^{i\theta})^{\frac{m}{q}+j_2+1}}\, d\theta.$$

This integral goes to zero when $\rho \to 0$, therefore the constant term in ρ of

$$\int_{\Gamma_4(\rho)\vee\Gamma_5(\rho)\vee\Gamma_6(\rho)} \frac{e^{-inu/\varepsilon}}{(u-ia)^{\frac{m}{q}+j_2+1}}\, du$$

is

$$2\pi i \mathrm{Res}(f, ia) = 2\pi i \frac{1}{(\frac{m}{q}+j_2)!} \left(\frac{-in}{\varepsilon}\right)^{\frac{m}{q}+j_2} e^{-|n|a/\varepsilon}(1+O(\varepsilon)).$$

Now we compute the dominant term of $M_n(\varepsilon)$ for $n<0$. We recall that

$$\ell = j_2 + \frac{(j_1+j_2)p}{q}$$

and we denote

$$\eta_\ell = \frac{(j_1+j_2)p}{q} - \left[\frac{(j_1+j_2)p}{q}\right] \in (0,1).$$

Finally, if $\ell \notin \mathbb{N}$ and $n<0$, by (8.3.11) and (8.3.12) we obtain that the dominant term of $M_n(\varepsilon)$ is

$$\begin{aligned} M_n(\varepsilon) &= \sum_{m\le(j_1+j_2)p} J^+_{-n,(j_1+j_2)p-m} \int_{-\infty}^{\infty} \frac{e^{-inu/\varepsilon}}{(u-ia)^{\frac{m}{q}+j_2+1}}\, du \\ &= -\sum_{m\le(j_1+j_2)p} J^+_{-n,(j_1+j_2)p-m}(1+O(\varepsilon)) \int_{\Gamma_4\vee\Gamma_6} \frac{e^{-inu/\varepsilon}}{(u-ia)^{\frac{m}{q}+j_2+1}}\, du \\ &= -i^{[\ell]} \left(\frac{|n|}{\varepsilon}\right)^{\ell} \frac{J^+_{-n,0}\pi}{\sin\pi\eta_\ell} \frac{e^{-|n|a/\varepsilon}}{\Gamma(\ell+1)} e^{-\eta_\ell i\pi/2}(1-e^{\eta_\ell i2\pi})(1+O(\varepsilon)). \end{aligned} \tag{8.3.13}$$

And if $\ell \in \mathbb{N}$, the dominant term of $M_n(\varepsilon)$ is

$$\begin{aligned} M_n(\varepsilon) &= \sum_{m\le(k+j)p} J^+_{-n,(j_1+j_2)p-m} \int_{-\infty}^{\infty} \frac{e^{-inu/\varepsilon}}{(u-ia)^{\frac{m}{q}+j_2+1}} du \\ &= -\sum_{m\le(k+j)p} J^+_{-n,(j_1+j_2)p-m}(1+O(\varepsilon)) \int_{\Gamma_4\vee\Gamma_5\vee\gamma_6} \frac{e^{-inu/\varepsilon}}{(u-ia)^{\frac{m}{q}+j_2+1}} \\ &= -i^{\ell} \left(\frac{|n|}{\varepsilon}\right)^{\ell} \frac{1}{\ell!} e^{na/\varepsilon} 2\pi i J^+_{-n,0}(1+O(\varepsilon)). \end{aligned} \tag{8.3.14}$$

Remark 8.3.4 *We observe that the expression of $M_n(\varepsilon)$ in (8.3.13) goes to (8.3.14) when η goes to zero.*

For the case $n > 0$, it is sufficient to observe that $M_n(\varepsilon) = \overline{M_{-n}(\varepsilon)}$ and $J^-_{-n,0} = \overline{J^+_{n,0}}$. Thus, if $\ell \notin \mathbb{N}$ and $n > 0$

$$M_n(\varepsilon) = (-i)^{[\ell]} \left(\frac{|n|}{\varepsilon}\right)^\ell \frac{\pi}{\sin \pi\eta_\ell} \frac{e^{-|n|a/\varepsilon}}{\Gamma(\ell+1)} e^{-\eta_\ell i\pi/2}(1 - e^{\eta_\ell i 2\pi}) J^-_{-n,0}(1 + O(\varepsilon))$$

and if $\ell \in \mathbb{N}$ and $n > 0$

$$M_n(\varepsilon) = (-i)^\ell \left(\frac{|n|}{\varepsilon}\right)^\ell \frac{1}{\ell!} e^{-|n1a/\varepsilon} 2\pi i J^-_{-n,0}(1 + O(\varepsilon)).$$

Consequently, for all n,

$$\mu\varepsilon^p M_n(\varepsilon) = \mu\varepsilon^{p-\ell} e^{-|n|a/\varepsilon} \tilde{M}_n = \mu\varepsilon^\nu e^{-|n|a/\varepsilon} \tilde{M}_n \tag{8.3.15}$$

where, choosing the sign $+$ for $n < 0$ and the sign $-$ for $n > 0$

$$\begin{aligned}
\tilde{M}_n &= (-i)^{[\ell]} |n|^\ell \frac{\pi}{\sin \pi\eta_\ell} \frac{1}{\Gamma(\ell+1)} e^{-\eta_\ell i\pi/2}(1 - e^{\eta_\ell i2\pi}) J^\pm_{-n,0}(1 + O(\varepsilon)), \quad \text{if } \ell \notin \mathbb{N} \\
\tilde{M}_n &= (-i)^\ell |n|^\ell \frac{1}{\ell!} 2\pi i J^\pm_{-n,0}(1 + O(\varepsilon)), \quad \text{if } \ell \in \mathbb{N}.
\end{aligned}$$

8.3.2 Proof of the Corollary 1.2.3

Assume the same hypotheses as Theorem 1.2.1 and the further hypothesis **HP7**. We note that, if **HP7** is satisfied, $J^+_{1,0} = \overline{J^-_{-1,0}}$ are different to zero, and then by (8.3.15)

$$\begin{aligned}
\mu\varepsilon^p M(s,\varepsilon) &= \mu\varepsilon^\nu \sum_{n>0} e^{-na/\varepsilon}(\tilde{M}_n e^{ins/\varepsilon} + \tilde{M}_{-n} e^{-ins/\varepsilon}) \\
&= \mu\varepsilon^\nu e^{-a/\varepsilon}(\tilde{M}_1 e^{is/\varepsilon} + \tilde{M}_{-1} e^{-is/\varepsilon}) + \mu\varepsilon^\nu O(e^{-2a/\varepsilon}).
\end{aligned}$$

Next we compute $\tilde{M}_1 e^{is/\varepsilon} + \tilde{M}_{-1} e^{-is/\varepsilon}$. If $\ell \notin \mathbb{N}$

$$\begin{aligned}
\tilde{M}_1 e^{is/\varepsilon} + \tilde{M}_{-1} e^{-is/\varepsilon} &= -\frac{\pi}{\sin \pi\eta} \frac{1}{\Gamma(\ell+1)} i^{[\ell]} (e^{\eta i\pi/2}(1 - e^{-\eta i2\pi}) J^-_{-1,0}) e^{is/\varepsilon} \\
&= -\frac{\pi}{\sin \pi\eta} \frac{1}{\Gamma(\ell+1)} (-i)^{[\ell]} (e^{-\eta i\pi/2}(1 - e^{\eta i2\pi}) J^+_{1,0}) e^{-is/\varepsilon}
\end{aligned}$$

where $\eta_\ell = \ell - [\ell]$ and, if we write $J^+_{1,0} = |J^+_{1,0}|e^{i\theta}$, and since $J^-_{-1,0} = |J^+_{1,0}|e^{-i\theta}$, with $|J^+_{1,0}| > 0$, we obtain

$$\begin{aligned}\mu\varepsilon^p M(s,\varepsilon) &= -\mu\varepsilon^\nu \frac{2\pi}{\sin\pi\eta}\frac{e^{-a/\varepsilon}}{\Gamma(\ell+1)}|J^+_{1,0}|\,\mathrm{Re}((-i)^{[\ell]}e^{-\eta i\pi/2}(1-e^{\eta i2\pi})e^{i(\theta-\frac{s}{\varepsilon})})\\ &\quad +\mu\varepsilon^\nu O(e^{-2a/\varepsilon}).\end{aligned}$$

We can calculate more explicitly this formula. We consider three cases. If $\ell \in \mathbb{Q}$ and $[\ell]$ is odd,

$$\begin{aligned}\mathrm{Re}(i^{[\ell]}e^{-\eta i\pi/2}(1-e^{\eta i2\pi})e^{i(\theta-s/\varepsilon)}) &= (-1)^{([\ell]-1)/2}\left(\mathrm{Im}\,e^{i(\theta-s/\varepsilon-\eta\pi/2)}) - \mathrm{Im}(e^{i(\theta-s/\varepsilon-\eta3\pi/2)})\right)\\ &= (-1)^{([\ell]-1)/2}\left(\sin\left(\theta-\frac{s}{\varepsilon}-\frac{\eta\pi}{2}\right)-\sin\left(\theta-\frac{s}{\varepsilon}+\frac{\eta3\pi}{2}\right)\right)\\ &= -(-1)^{([\ell]-1)/2}2\sin(\pi\eta)\left(\cos\left(\theta-\frac{s}{\varepsilon}-\frac{\eta\pi}{2}\right)\right)\end{aligned}$$

and then we get

$$\begin{aligned}\mu\varepsilon^p\left|\int_{s_0}^{\bar s_0} M(s,\varepsilon)\right| &= \mu\varepsilon^{\nu+1}\frac{\pi}{\sin\pi\eta}\frac{1}{\Gamma(\ell+1)}|J_{1,0}|8\sin(\pi\eta)e^{-a/\varepsilon}+\mu\varepsilon^{\nu+1}O(e^{-2a/\varepsilon})\\ &= \mu\varepsilon^{\nu+1}8\pi|J^+_{1,0}|\frac{1}{\Gamma(\ell+1)}e^{-a/\varepsilon}+\mu\varepsilon^{\nu+1}O(e^{-2a/\varepsilon})\end{aligned}$$

where s_0 and $\bar s_0$ are two consecutive zeros of $M(s,\varepsilon)$. Moreover,

$$|M'(s_0,\varepsilon)| = \mu\varepsilon^{\nu-1}4\pi|J^+_{1,0}|\frac{1}{\Gamma(\ell+1)}e^{-a/\varepsilon}+\mu\varepsilon^{\nu-1}O(e^{-2a/\varepsilon})$$

If $\ell \in \mathbb{Q}$ and $[\ell]$ is even,

$$\begin{aligned}\mathrm{Re}(i^{[\ell]}e^{-\eta\pi/2}(1-e^{\eta i2\pi})e^{i(\theta-s/\varepsilon)}) &= (-1)^{[\ell]/2}\left(\mathrm{Re}(e^{i(\theta-s/\varepsilon-\eta\pi/2)}) - \mathrm{Re}(e^{i(\theta-s/\varepsilon-\eta3\pi/2)})\right)\\ &= (-1)^{[\ell]/2}\left(\cos\left(\theta-\frac{s}{\varepsilon}-\frac{\eta\pi}{2}\right)-\cos\left(\theta-\frac{s}{\varepsilon}-\frac{\eta3\pi}{2}\right)\right)\\ &= (-1)^{[\ell]/2}2\sin(\pi\eta)\left(\sin\left(\theta-\frac{s}{\varepsilon}-\eta\pi\right)\right)\end{aligned}$$

and therefore

$$\begin{aligned}\mu\varepsilon^p\left|\int_{s_0}^{\bar s_0} M(s,\varepsilon)\right| &= \mu\varepsilon^{\nu+1}8\pi|J^+_{1,0}|\frac{1}{\Gamma(\ell+1)}e^{-a/\varepsilon}+\mu\varepsilon^{\nu+1}O(e^{-2a/\varepsilon})\\ |M'(s_0,\varepsilon)| &= \mu\varepsilon^{\nu-1}4\pi|J^+_{1,0}|\frac{1}{\Gamma(\ell+1)}e^{-a/\varepsilon}+\mu\varepsilon^{\nu-1}O(e^{-2a/\varepsilon})\end{aligned}$$

Therefore, in the two cases, applying the formula of Theorem 1.2.1 we obtain that

$$\begin{aligned} A &= \mu\varepsilon^{\nu+1} 8\pi |J^+_{1,0}| \frac{1}{\Gamma(\ell+1)} e^{-a/\varepsilon} + \mu\varepsilon^{\nu+1} O(e^{-2a/\varepsilon}) \\ &\quad + O(\mu^2\varepsilon^{2\nu+r}, \mu^2\varepsilon^{\nu+p+i_0}, \mu\varepsilon^{p+1+i_0}) e^{-a/\varepsilon} \\ |\sin\vartheta| &= \mu\varepsilon^{\nu-1} 4\pi |J^+_{1,0}| \frac{1}{\Gamma(\ell+1)} e^{-a/\varepsilon} \frac{1}{\|\dot\gamma_0(s_0)\|^2} + \mu\varepsilon^{\nu-1} O(e^{-2a/\varepsilon}) \\ &\quad + O(\mu^2\varepsilon^{2\nu+r-2}, \mu^2\varepsilon^{\nu+p+i_0-2}, \mu\varepsilon^{p-1+i_0}) e^{-a/\varepsilon}. \end{aligned} \tag{8.3.16}$$

Finally, if $\ell \in \mathbb{N}$,

$$\tilde M_1 e^{is/\varepsilon} + \tilde M_{-1} e^{-is/\varepsilon} = \frac{2\pi}{\ell!}\left((-i)^{\ell+1} J^-_{-1,0} e^{is/\varepsilon} + i^{\ell+1} J^+_{1,0} e^{-is/\varepsilon}\right)$$

and thus

$$\mu\varepsilon^p M(s,\varepsilon) = \mu\varepsilon^\nu 2\pi |J^+_{1,0}| \frac{1}{\ell!} \operatorname{Re}(i^{\ell+1} e^{i(\theta - s/\varepsilon)}) e^{-a/\varepsilon} + \mu\varepsilon^{\nu+1} O(e^{-2a/\varepsilon}).$$

And, with a similar argument to the previous ones, we deduce the formulas:

$$\begin{aligned} A &= \mu\varepsilon^{\nu+1} 8\pi |J^+_{1,0}| \frac{1}{\ell!} e^{-a/\varepsilon} + \mu\varepsilon^{\nu+1} O(e^{-2a/\varepsilon}) \\ &\quad + O(\mu^2\varepsilon^{2\nu+r}, \mu^2\varepsilon^{\nu+p+i_0}, \mu\varepsilon^{p+1+i_0}) e^{-a/\varepsilon} \\ |\sin\vartheta| &= \mu\varepsilon^{\nu-1} 4\pi |J^+_{1,0}| \frac{1}{\ell!} e^{-a/\varepsilon} \frac{1}{\|\dot\gamma_0(s_0)\|^2} + \mu\varepsilon^{\nu-1} O(e^{-2a/\varepsilon}) \\ &\quad + O(\mu^2\varepsilon^{2\nu+r-2}, \mu^2\varepsilon^{\nu+p+i_0-2}, \mu\varepsilon^{p-1+i_0}) e^{-a/\varepsilon}. \end{aligned}$$

Remark 8.3.5 *We note that if ℓ is an integer, the formulae of the area and the angle given in (8.3.16) also apply and they coincide with the last formulas explicitely computed for $\ell \in \mathbb{N}$.*

Part II

Invariant manifolds of maps

Chapter 9

Invariant manifolds of parabolic points in higher dimensions

9.1 Introduction

Invariant manifolds are very important objects in dynamics because they provide essential information for the analysis of the dynamical structure of a system.

Invariant manifolds are associated to invariant objects, the simplest ones being fixed points. There are many results about invariant manifolds associated to objects having some kind of hyperbolicity [50] [51] [58].

The case of invariant objects without hyperbolic "directions" is more complicated. The full neighborhood of the object is a central manifold. If we consider dynamical systems generated by maps, the fact that a neighbourhood of the fixed point is a central manifold means that all the eigenvalues of the linear part of the map at the fixed point have modules one. The case that all eigenvalues are exactly equal to one is the most degenerate. For two dimensional maps, this case is considered in [63] [8] [30].

Some cases in higher dimension have been considered in [70] [26]. Such maps appear as Poincaré maps in some problems of Celestial Mechanics [63] [70] [62] [14].

In this case (all the eigenvalues are exactly equal to one) the sets of points whose positive iterates converge to the fixed point may be non-void. This set is invariant by the map. We can call it stable invariant set or stable invariant manifold in some generalized sense. In the analogous way we can define the unstable invariant set.

In this context, it may happen that both the stable and the unstable invariant sets are open sets. See an example of such case in [30].

Here we will consider multidimensional maps with a fixed point with linear part equal to the identity and we will give sufficient conditions to find stable invariants sets formed by points such that their iterates converge to the fixed point and some projection of them stay in a chosen set.

First we will obtain these sets as graphs of Lipschitz functions. Then we will add analyticity hypotheses and we will obtain these sets as graphs of analytic functions.

The methods we will use in this chapter are generalizations of the ones of McGehee in [63] where he studies a two dimensional case.

Finally we will present some simple examples to illustrate the application of the results.

9.2 Definitions and notation

We consider maps $F : U \subset \mathbb{R}^{n+m} \to \mathbb{R}^{n+m}$ of the form

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x + p(x,y) + f(x,y) \\ y + q(x,y) + g(x,y) \end{pmatrix} \tag{9.2.1}$$

where $p(x,y), q(x,y)$ are homogeneous polynomials of degree N_p, N_q respectively with $N_p,\ N_q \geq 2$, $f(x,y)$, $g(x,y)$ are differentiable functions of orders $o(\|(x,y)\|^{N_p})$ and $o(\|(x,y)\|^{N_q})$ respectively, and their derivatives $Df(x,y)$, $Dg(x,y)$ are $o(\|(x,y)\|^{N_p-1})$ and $o(\|(x,y)\|^{N_q-1})$ respectively.

Given a subset $V \subset \mathbb{R}^n$ we define

$$W_V^s = \{(x,y) \in U : \pi^1 F^k(x,y) \in V,\ k \geq 0,\ F^k(x,y) \to 0, \text{ as } k \to \infty\} \tag{9.2.2}$$

and its local version

$$\begin{aligned} &W_{V,r}^s \\ &\quad = \{(x,y) \in U : \pi^1 F^k(x,y) \in V \cap B(0,r),\ k \geq 0,\ F^k(x,y) \to 0, \text{ as } k \to \infty\} \end{aligned} \tag{9.2.3}$$

These definitions depend on the decomposition $\mathbb{R}^{n+m} = \mathbb{R}^n \times \mathbb{R}^m$. Particular cases are $n = 1$ or $m = 1$. In the two dimensional case, if $n = 1$ and $m = 1$, V can be taken as $(0,r)$ or $(-r,0)$. When $V = (0,r)$ the corresponding invariant manifold is denoted by W^{s+} in [30].

We shall use the following two norms: if $z \in \mathbb{R}^k$,

$$\|z\| = \max(|z_1|, \dots, |z_k|)$$

and therefore, if $(x, y) \in \mathbb{R}^{n+m}$, $\|(x, y)\| = \max(\|x\|, \|y\|)$, and

$$\|z\|_2 = \Big(\sum_{l=1}^{k} z_i^2\Big)^{1/2}.$$

Of course both norms are equivalent, but some sets defined through the norm will have different shape.

Given $V \subset \mathbb{R}^n$, we introduce the following notation:

$$V(r) = \{x \in V : \|x\|_2 < r\},$$

$$V^1(r) = \overline{\{x/\|x\|_2 : x \in V(r)\}}.$$

Notice that if $r_1 < r_2$ then $V^1(r_1) \subset V^1(r_2)$.

Also we introduce the projectors:

$$\begin{array}{lll} \pi^1(x, y) = x, & \pi_i^1(x, y) = x_i, & 1 \leq i \leq n, \\ \pi^2(x, y) = y, & \pi_j^2(x, y) = y_j, & 1 \leq j \leq m \end{array}$$

and the following sets,

$$\begin{aligned} V(r, \beta) &= \{(x, y) \in \mathbb{R}^{n+m} \;:\; x \in \overline{V(r)},\; \|y\|_2 \leq \beta\|x\|_2\}, \\ V^+(r, \beta) &= \{(x, y) \in \mathbb{R}^{n+m} \;:\; x \in \overline{V(r)},\; \|y\|_2 \geq \beta\|x\|_2\}, \\ v^+(r, \beta) &= \{(x, y) \in \mathbb{R}^{n+m} \;:\; (x, y) \in V(r, \beta), \|y\|_2 = \beta\|x\|_2\}, \\ S(\alpha) &= \{(\xi, \eta) \in \mathbb{R}^{n+m} \;:\; \|\eta\| \geq \alpha\|\xi\|\}, \\ S_j^+(\alpha) &= \{(\xi, \eta) \in \mathbb{R}^{n+m} \;:\; \eta_j \geq \alpha\|\xi\|\}, \\ S_j^-(\alpha) &= \{(\xi, \eta) \in \mathbb{R}^{n+m} \;:\; \eta_j \leq -\alpha\|\xi\|\}, \end{aligned}$$

for $j \in \{1, \dots, m\}$. Notice that

$$S(\alpha) = \bigcup_{j=1}^{m} (S_j^+(\alpha) \cup S_j^-(\alpha)). \tag{9.2.4}$$

9.3 The Lipschitz case

This section is devoted to prove, under suitable hypotheses, the existence of a Lipschitz stable invariant manifold in the sense of definitions (9.2.2) and (9.2.3). We consider maps F of the form (9.2.1).

We will assume that there exists $r > 0$ such that:

H1. The polynomial p satisfies

$$-D_{x_i}p_i(x,0) > \sum_{k=1,k\neq i}^{n} |D_{x_k}p_i(x,0)| + \sum_{k=1}^{m} |D_{y_k}p_i(x,0)|,$$

$\forall i \in \{1,\dots,n\}$ and $\forall x \in V^1(r)$.

H2. The polynomial q satisfies

$$D_x q(x,0) = 0 \qquad \text{and} \qquad D_{y_j}q_j(x,0) > \sum_{k=1,k\neq j}^{m} |D_{y_k}q_j(x,0)|,$$

$\forall j \in \{1,\dots,m\}$ and $\forall x \in V^1(r)$.

H3. $\exists A > 0$ such that $\forall x \in V(r)$, $\operatorname{dist}(x + p(x,0), V(r)^c) \geq A\|x\|^{N_p}$.

We remark that in **H3**, $\operatorname{dist}(x, A) = \inf_{z\in A} \operatorname{dist}(x,z)$ where $\operatorname{dist}(x,z)$ is measured with the norm $\|.\|$.

The main theorem of this section is:

Theorem 9.3.1 *Let $F : U \subset \mathbb{R}^{n+m} \longrightarrow \mathbb{R}^{n+m}$ be a map of class C^N, $N \geq 2$, of the form*

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x + p(x,y) + f(x,y) \\ y + q(x,y) + g(x,y) \end{pmatrix} \tag{9.3.1}$$

where $p(x,y), q(x,y)$ are homogeneous polynomials of degree N_p and N_q respectively ($N_p, N_q \geq 2$), $f(x,y)$ is of order $o(\|(x,y)\|^{N_p})$, $Df(x,y)$ is of order $o(\|(x,y)\|^{N_p-1})$, $g(x,y)$ is of order $o(\|(x,y)\|^{N_q})$ and $Dg(x,y)$ is of order $o(\|(x,y)\|^{N_q-1})$.

*Then, if there exists a convex open set $V \subset \mathbb{R}^n$, $0 \in \partial V$ and $r > 0$ such that the hypotheses **H1-H3** hold, $W^s_{V,r}$ is the graph of a Lipschitz function*

$$\varphi : V(r) \to \mathbb{R}^m.$$

Remark 9.3.2 *Hypotheses* ***H1*** *and* ***H2*** *provide a kind of weak hyperbolicity for the fixed points in a suitable domain,through the nonlinear terms.*

Remark 9.3.3 *Hypothesis* ***H2*** *implies that* $q(x,0)=0$.

Remark 9.3.4 *An unstable manifold theorem can be obtained by considering the inverse map.*

The rest of this section is devoted to prove Theorem 9.3.1. For that we need several lemmas. In all of them we will assume implicitly the hypotheses of Theorem 9.3.1.

We shall use several times the following simple lemma on homogeneous functions.

Lemma 9.3.5 *Let* $V \subset \mathbb{R}^n$ *be an open set with* $0 \in V$, *and let* $h : V \times \mathbb{R}^m \to \mathbb{R}$ *be a homogeneous continuous function of degree* N *and* $r_0 > 0$ *such that*

$$h(x,0) > 0, \qquad \forall x \in V^1(r_0).$$

Let $\tilde{h} : V \times \mathbb{R}^m \to \mathbb{R}$ *be such that* $\tilde{h}(x,y) = o(\|(x,y)\|_2^N)$. *Then* $\exists r, \beta, K > 0$, *such that*

$$h(x,y) + \tilde{h}(x,y) \geq K\|x\|_2^N, \qquad \forall x \in V(r), \ \ \|y\|_2 \leq \beta\|x\|_2.$$

As a consequence $\exists r, K > 0$, *such that*

$$h(x,0) + \tilde{h}(x,0) \geq K\|x\|_2^N, \qquad \forall x \in V(r).$$

Proof. Since $V^1(r_0)$ is compact, by continuity there exists $\beta > 0$ such that $h(x,y) > 0$ if $(x,y) \in B(r_0,\beta) = \{(x,y) : x \in V^1(r_0), \ \|y\|_2 \leq \beta\}$. Also since $B(r_0,\beta)$ is a compact set there exists $M > 0$ such that $h(x,y) \geq M$ in $B(r_0,\beta)$.

If $x \in V(r)$, $r \leq r_0$, $\|y\|_2 \leq \beta\|x\|_2$ and $x \neq 0$, by the homogeneity of h, we have that

$$h(x,y) = \|x\|_2^N h\Big(\frac{x}{\|x\|_2}, \frac{y}{\|x\|_2}\Big) \geq M\|x\|_2^N.$$

On the other hand, if r is small enough, we have

$$\|\tilde{h}(x,y)\|_2 \leq \frac{M}{2}\|x\|_2^N, \qquad \forall x \in V(r), \ \ \|y\|_2 \leq \beta||x||_2.$$

Hence $h(x,y) + \tilde{h}(x,y) \geq h(x,y) - |\tilde{h}(x,y)| \geq \frac{M}{2}\|x\|_2^N$ which proves the lemma. ∎

Lemma 9.3.6 *If $r > 0$ and $\beta > 0$ small enough, then $\forall (x,y) \in V(r,\beta)$ we have that*

$$\pi^1 F(x,y) \in V(r).$$

Proof. It is a consequence of hypothesis **H3**. Note that $\forall (x,y) \in V(r,\beta)$

$$\begin{aligned}\|\pi^1 F(x,y) - x - p(x,0)\| &\leq \|p(x,y) - p(x,0)\| + \|f(x,y)\| \\ &\leq \sup_{\|\xi\| \leq \|y\|} \|D_y p(x,\xi)\| \, \|y\| + \eta \|x\|^{N_p} \leq (C\beta + \eta)\|x\|^{N_p}\end{aligned}$$

with suitable C and η depending on r_0 and β small enough, and hence

$$\begin{aligned}\operatorname{dist}(\pi^1 F(x,y), V(r)^c) &\geq \operatorname{dist}(x + p(x,0), V(r)^c) - \|\pi^1 F(x,y) - x - p(x,0)\| \\ &\geq A\|x\|^{N_p} - (C\beta + \eta)\|x\|^{N_p} > 0\end{aligned}$$

if $(C\beta + \eta) < A$ which implies that $\pi^1 F(x,y) \in V(r)$. ∎

Lemma 9.3.7 *There exist $r, K, \beta > 0$ such that for all $(x,y) \in V(r,\beta)$ we have*

1) $\|\pi^1 DF(x,y)\| < 1 - K\|x\|^{N_p - 1}$,

2) $\|\pi^1 F(x,y)\| < \|x\|$,

Proof. 1) With the norm we are working with:

$$\|\pi^1 DF\| = \max_{i=1,\dots,n} \Big(\sum_{k=1}^{n} |D_{x_k} F_i| + \sum_{k=1}^{m} |D_{y_k} F_i| \Big).$$

For $i \in \{1, \dots, n\}$ we introduce

$$h(x,y) = D_{x_i} p_i(x,y) + \sum_{k=1, k\neq i}^{n} |D_{x_k} p_i(x,y)| + \sum_{k=1}^{m} |D_{y_k} p_i(x,y)|$$

and

$$\tilde{h}(x,y) = \sum_{k=1}^{n} |D_{x_k} f_i(x,y)| + \sum_{k=1}^{m} |D_{y_k} f_i(x,y)|.$$

By hypothesis **H1**, for $x \in V^1(r)$ we have that $-h(x,0) > 0$. By the hypotheses of Theorem 9.3.1 we also have that $-\tilde{h}(x,y) = o(\|(x,y)\|^{N_p - 1})$.

Then, by Lemma 9.3.5 there exist r, K, β such that $\forall (x,y) \in V(r,\beta)$,

$$-h(x,y) - \tilde{h}(x,y) \geq K\|x\|^{N_p-1},$$

and therefore

$$\begin{aligned}\sum_{k=1}^{n} |D_{x_k}F_i(x,y)| + \sum_{k=1}^{m} |D_{y_k}F_i(x,y)| &\leq 1 + h(x,y) + \tilde{h}(x,y)\\ &\leq 1 - K\|x\|^{N_p-1},\end{aligned}$$

since $|D_{x_i}F_i(x,y)| = 1 + D_{x_i}p_i(x,y) + D_{x_i}f_i(x,y) > 0$ if r is small.

2) It follows from

$$\|\pi^1 F(x,y)\| \leq \int_0^1 \|\pi^1 DF(tx,ty)(x,y)\|\,dt < \int_0^1 (1 - K\|tx\|^{N_p-1})\|(x,y)\|\,dt < \|x\|$$

if β and r are small enough. ∎

Lemma 9.3.8 *There exists a constant M_1 such that for $(x,y) \in V(r,\beta)$ and for any $t \in [0,1]$,*

$$\|x + tp(x,y) + tf(x,y))\|_2 \leq \|x\|_2(1 - tM_1\|x\|_2^{N_p-1}).$$

Proof. Let $(x,y) \in V(r,\beta)$ and $t \in [0,1]$. We define two auxiliary functions

$$\begin{aligned}h(x,y) &= \frac{2}{N_p}(x^T D_x p(x,y)x + x^T D_y p(x,y)y)\\ \tilde{h}_t(x,y) &= 2\langle x, f(x,y)\rangle + t\|p+f\|_2^2\end{aligned}$$

where $\langle .,. \rangle$ denotes the euclidean scalar product. Since hypothesis **H1**, implies that

$$D_{x_i}p_i(x,0) < -\sum_{k\neq i} |D_{x_k}p_i(x,0)|$$

it follows that

$$h(x,0) = \frac{2}{N_p}x^T D_x p(x,0)x < 0, \qquad x \neq 0.$$

Moreover, since $\tilde{h}_t(x,y) = o(\|(x,y)\|^{N_p+1})$ and $0 \le t \le 1$, by Lemma 9.3.5 there exists β small enough and a constant M such that, if $\|y\|_2 \le \beta \|x\|_2$ then

$$h(x,y) + \tilde{h}_t(x,y) \le -M\|x\|_2^{N_p+1}.$$

We observe that, by Euler's theorem, the homogeneous polynomial p can be written as

$$p(x,y) = \frac{1}{N_p} D_x p(x,y)x + \frac{1}{N_p} D_y p(x,y)y.$$

Therefore,

$$\begin{aligned}
\|x + tp(x,y) + tf(x,y)\|_2^2 &= \|x\|_2^2 + 2t\langle x, p(x,y) + f(x,y)\rangle + t^2\|p(x,y) + f(x,y)\|_2^2 \\
&= \|x\|_2^2 + 2t\langle x, p(x,y)\rangle + t\tilde{h}_t(x,y) \\
&= \|x\|_2^2 + th(x,y) + t\tilde{h}_t(x,y) \\
&< \|x\|_2^2 - Mt\|x\|_2^{N_p+1}
\end{aligned}$$

and the statement holds. ■

Lemma 9.3.9 *There exists a constant M_2 such that for any $(x,y) \in v^+(r,\beta)$ and for any $0 \le t \le 1$ we have*

$$\|y + tq(x,y) + tg(x,y)\|_2 \ge \|y\|_2(1 + tM_2\|x\|_2^{N_q-1})$$

Proof. Let $(x,y) \in v^+$. We study carefully the polynomial $q(x,y)$. By hypothesis **H2**, $D_x q(x,0) = 0$, therefore we have that $q(x,0) = 0$. Moreover for $j \in \{1,\dots,m\}$

$$q_j(x,y) = y_j q_j^{(j)}(x,y) + \sum_{k=1,k\neq j}^{m} y_k q_j^{(k)}(x,y)$$

where $q_j^{(k)}(x,y) = \int_0^1 D_{y_k} q_j(x,sy)ds$ are homogeneous polynomials of degree $N_q - 1$. Therefore if we define the matrix

$$Q(x,y) \equiv \begin{pmatrix} q_1^{(1)} & \dots & q_1^{(m)} \\ \vdots & \dots & \vdots \\ q_m^{(1)} & \dots & q_m^{(m)} \end{pmatrix}$$

we have that

$$q(x,y) = Q(x,y)y$$

On the other hand, since $q_j^{(k)}(x,0) = D_{y_k} q_j(x,0)$, hypothesis **H2** gives that for all $x \in V^1(r)$

$$q_j^{(j)}(x,0) > \sum_{k=1,k\neq j}^{m} |q_j^{(k)}(x,0)|.$$

By Lemma 9.3.5, there exists $\beta > 0$ and a constant K, such that, if $(x,y) \in V(r,\beta)$ and for any $j \in \{1,\dots,m\}$

$$q_j^{(j)}(x,y) - \sum_{k=1,k\neq j}^{m} |q_j^{(k)}(x,y)| \geq K\|x\|_2^{N_q-1}.$$

Consequently, the matrix $Q(x,y)$ verifies

$$v^T Q(x,y) v > 0 \quad \text{for} \quad v \neq 0 \;\text{ and }\; (x,y) \in V(r,\beta).$$

Let $M > 0$ be such that for all $(v,\xi,\eta) \in \mathbb{R}^m \times \mathbb{V}^1 \times \mathbb{R}^m$ such that $\|v\|_2 = \|(\xi,\eta)\|_2 = 1$

$$v^T Q(\xi,\eta) v \geq M.$$

Such M exists by compactness. We define the functions

$$\begin{aligned} h(x,y) &= 2y^T Q(x,y) y \\ \tilde{h}_t(x,y) &= 2y^T g(x,y) + t\|q(x,y) + g(x,y)\|_2^2. \end{aligned}$$

We deal with $h(x,y)$. Taking $(x,y) \in v^+(r,\beta)$, since $\|y\|_2 = \beta\|x\|_2$, we have that

$$\begin{aligned} h(x,y) &= 2\|y\|_2^2 \|(x,y)\|_2^{N_q-1} \frac{y^T}{\|y\|_2} Q\Big(\frac{1}{\|(x,y)\|_2}(x,y)\Big)\frac{y}{\|y\|_2} \\ &\geq 2\beta^2 \|x\|_2^{N_q+1} M. \end{aligned}$$

Therefore, since $\tilde{h}_t(x,y) = o(\|(x,y)\|_2^{Nq+1})$ and $0 \leq t \leq 1$, by Lemma 9.3.5 there exists C such that

$$h(x,y) + \tilde{h}_t(x,y) \geq C\beta^2 \|x\|_2^{N_q+1}$$

if β and r are small enough. Now we compute the norm $\|y + q(x,y) + g(x,y)\|_2^2$:

$$\begin{aligned} \|y + tq(x,y) + tg(x,y)\|_2^2 &= \|y\|_2^2 + 2t\langle y, q+g\rangle + t^2\|p(x,y) + g(x,y)\|_2^2 \\ &\geq \|y\|_2^2 + th(x,y) + t\tilde{h}_t(x,y) \\ &\geq \|y\|_2^2 (1 + tC\|x\|_2^{N_q-1}) \end{aligned}$$

and the statement holds. ■

Lemma 9.3.10 *There exist $r > 0$ and $\alpha \in (0,1]$ such that $\forall (x,y) \in V(r,\beta)$,*

$$DF(x,y) : S(\alpha) \to S(\alpha).$$

Proof. We write

$$DF = \begin{pmatrix} Id + D_x p + D_x f & D_y p + D_y f \\ D_x q + D_x g & Id + D_y q + D_y g \end{pmatrix}.$$

We define the auxiliary functions

$$P_1(x,y) = D_{x_i} p_i(x,y) + \sum_{k=1, k\neq i}^{n} |D_{x_k} p_i(x,y)| + \alpha \sum_{k=1}^{m} |D_{y_k} p_i(x,y)|,$$
$$P_2(x,y) = -D_{y_j} q_j(x,y) + \frac{1}{\alpha} \sum_{k=1}^{n} |D_{x_k} q_j(x,y)| + \sum_{k=1, k\neq j}^{m} |D_{y_k} q_j(x,y)|$$

and

$$Q_1(x,y) = \sum_{k=1}^{n} |D_{x_k} f_i(x,y)| + \alpha \sum_{k=1}^{m} |D_{y_k} f_i(x,y)|,$$
$$Q_2(x,y) = \frac{1}{\alpha} \sum_{k=1}^{n} |D_{x_k} g_j(x,y)| + \sum_{k=1}^{m} |D_{y_k} g_j(x,y)|.$$

By hypotheses **H1** and **H2**, since $\alpha \in (0,1]$, we have $P_1(x,0) < 0$ and $P_2(x,0) < 0$ for $x \in V^1(r)$. Moreover $Q_1(x,y) = o(\|(x,y)\|^{N_p - 1})$ and $Q_2(x,y) = o(\|(x,y)\|^{N_q - 1})$.

By Lemma 9.3.5 we have that $\sum_{j=1}^{2} (P_j(x,y) + Q_j(x,y)) < 0$ for $(x,y) \in V(r,\beta)$.

Let $(\xi, \eta) \in S(\alpha)$ and let $j \in \{1, \dots, m\}$ be such that $\|\eta\| = |\eta_j|$. We are going to check that $\forall i \in \{1, \dots, n\}$

$$|\alpha(\xi + D_x p\, \xi + D_x f\, \xi + D_y p\, \eta + D_y f\, \eta)_i| \le |(D_x q\, \xi + D_x g\, \xi + \eta + D_y q\, \eta + D_y g\, \eta)_j|.$$

By the choice of j

$$|\eta_k| \le |\eta_j| \quad \text{for all } k \in \{1, \dots, m\},$$

and by the definition of $S(\alpha)$

$$\alpha |\xi_l| \le |\eta_j| \quad \text{for all } l \in \{1, \dots, n\}.$$

Using all this we obtain

$$\begin{aligned}
&\alpha\left|\xi_i + D_{x_i}p_i\ \xi_i + \sum_{k=1,k\neq i}^{n} D_{x_k}p_i\ \xi_k + \sum_{k=1}^{n} D_{x_k}f_i\ \xi_k + \sum_{k=1}^{m}(D_{y_k}p_i + D_{y_k}f_i)\ \eta_k\right| \\
&-\left|\eta_j + D_{y_j}q_j\ \eta_j + \sum_{k=1}^{n}(D_{x_k}q_j + D_{x_k}g_j)\ \xi_k + \sum_{k=1,k\neq j}^{m} D_{y_k}q_j\ \eta_k + \sum_{k=1}^{m} D_{y_k}g_j\ \eta_k\right| \\
\leq\ & \alpha|\xi_i|(1 + D_{x_i}p_i) + \alpha \sum_{k=1,k\neq i}^{n} |D_{x_k}p_i||\xi_k| + \alpha\sum_{k=1}^{n} |D_{x_k}f_i|\,|\xi_k| \\
&+ \alpha\sum_{k=1}^{m}(|D_{y_k}p_i| + |D_{y_k}f_i|)|\eta_k| - (1 + D_{y_j}q_j)|\eta_j| \\
&+ \sum_{k=1}^{n}(|D_{x_k}q_j| + |D_{x_k}g_j|)|\xi_k| + \sum_{k=1,k\neq j}^{m} |D_{y_k}q_j||\eta_k| + \sum_{k=1}^{m} |D_{y_k}g_j|\,|\eta_k| \\
\leq\ & \Big[(1 + D_{x_i}p_i) + \sum_{k=1,k\neq i}^{n} |D_{x_k}p_i| + \sum_{k=1}^{n} |D_{x_k}f_i| + \alpha\sum_{k=1}^{m}(|D_{y_k}p_i| + |D_{y_k}f_i|) \\
&- (1 + D_{y_j}q_j) + \frac{1}{\alpha}\sum_{k=1}^{n}(|D_{x_k}q_j| + |D_{x_k}g_j|) + \sum_{k=1,k\neq j}^{m} |D_{y_k}q_j| + \sum_{k=1}^{m} |D_{y_k}g_j|\Big]|\eta_j| \\
=\ & \sum_{k=1}^{2}\left(P_k(x,y) + Q_k(x,y)\right)|\eta_j| \leq 0.
\end{aligned}$$

Then for all $i \in \{1, \dots, n\}$

$$\alpha|\pi_i^1[DF(x,y)(\xi,\eta)^T]| \leq |\pi_j^2[DF(x,y)(\xi,\eta)^T]| \leq \|\pi^2[DF(x,y)(\xi,\eta)^T]\|$$

∎

Lemma 9.3.11 *If $r, \beta > 0$ are small enough, $(x,y) \in V(r,\beta)$ and $\zeta \in S(\alpha)$ we have that*

$$\|\pi^2 DF^{-1}(x,y)\zeta\| \leq \|\pi^2\zeta\|.$$

Proof. It is clear that F is locally invertible in a neighbourhood of the origin and that F^{-1} is defined in a set of the form $V(r,\beta)$. Moreover F^{-1} and DF^{-1} can be written as

$$F^{-1} : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x - p(x,y) + \tilde{f}(x,y) \\ y - q(x,y) + \tilde{g}(x,y) \end{pmatrix}$$

and

$$DF^{-1} = \begin{pmatrix} Id - D_x p + D_x \tilde{f} & -D_y p + D_y \tilde{f} \\ -D_x q + D_x \tilde{g} & Id - D_y q + D_y \tilde{g} \end{pmatrix}$$

with $\tilde{f}(x,y) = o(\|(x,y)\|^{N_p})$, $\tilde{g}(x,y) = o(\|(x,y)\|^{N_q})$, $D\tilde{f}(x,y) = o(\|(x,y)\|^{N_p-1})$ and $D\tilde{g}(x,y) = o(\|(x,y)\|^{N_q-1})$. Given $j \in \{1,\dots,m\}$ we define the auxiliary functions

$$P_j(x,y) = -D_{y_j} q_j(x,y) + \frac{1}{\alpha}\sum_{k=1}^{n} |D_{x_k} q_j(x,y)| + \sum_{k=1,k\neq j}^{m} |D_{y_k} q_j(x,y)|$$

and

$$Q_j(x,y) = \frac{1}{\alpha}\sum_{k=1}^{n} |D_{x_k}\tilde{g}_j(x,y)| + \sum_{k=1}^{m} |D_{y_k}\tilde{g}_j(x,y)|.$$

By hypothesis **H2** we have

$$P_j(x,0) = -D_{y_j} q_j(x,0) + \sum_{k=1,k\neq j}^{m} |D_{y_k} q_j(x,0)| < 0$$

and hence, by Lemma 9.3.5, there exist r, β and K such that

$$P_j(x,y) + Q_j(x,y) < -K\|x\|^{N_q-1}$$

for $(x,y) \in V(r,\beta)$. Let $\zeta = (\xi,\eta) \in S(\alpha)$. We have

$$\begin{aligned}
&|\pi_j^2 DF^{-1}(x,y)\zeta| \\
&= \left|\sum_{k=1}^{n}(-D_{x_k} q_j + D_{x_k}\tilde{g}_j)\xi_k + \eta_j - D_{y_j} q_j\,\eta_j - \sum_{k=1,k\neq j}^{m} D_{y_k} q_j\,\eta_k + \sum_{k=1}^{m} D_{y_k}\tilde{g}_j\,\eta_k\right| \\
&\le \sum_{k=1}^{n}(|D_{x_k} q_j| + |D_{x_k}\tilde{g}_j|)|\xi_k| + |1 - D_{y_j} q_j|\,|\eta_j| + \sum_{k=1,k\neq j}^{m} |D_{y_k} q_j|\,|\eta_k| \\
&\quad + \sum_{k=1}^{m} |D_{y_k}\tilde{g}_j|\,|\eta_k| \\
&\le \frac{1}{\alpha}\sum_{k=1}^{n}(|D_{x_k} q_j| + |D_{x_k}\tilde{g}_j|)\|\eta\| + (1 - D_{y_j} q_j)\|\eta\| + \sum_{k=1,k\neq j}^{m} |D_{y_k} q_j|\,\|\eta\| \\
&\quad + \sum_{k=1}^{m} |D_{y_k}\tilde{g}_j|\,\|\eta\| \\
&\le (1 + P_j(x,y) + Q_j(x,y))\|\eta\| \\
&\le (1 - K\|x\|^{N_q-1})\|\eta\| \le \|\eta\|.
\end{aligned}$$

■

Lemma 9.3.12 *Let $j \in \{1, \dots, m\}$, $(\xi^i, \eta^i) \in S_j^+(\alpha)$ and $c_i \geq 0$, $1 \leq i \leq l$, with $l \geq 1$. Then*

$$\sum_{i=1}^{l} c_i(\xi^i, \eta^i) \in S_j^+(\alpha).$$

The same is true for $S_j^-(\alpha)$.

Proof. For $S_j^+(\alpha)$ it follows immediately from

$$\alpha|\sum_{i=1}^{l} c_i \xi_k^i| \leq \sum_{i=1}^{l} c_i \alpha |\xi_k^i| \leq \sum_{i=1}^{l} c_i \eta_j^i$$

and for $S_j^-(\alpha)$from

$$-\alpha|\sum_{i=1}^{l} c_i \xi_k^i| \geq -\sum_{i=1}^{l} c_i \alpha |\xi_k^i| \geq \sum_{i=1}^{l} c_i \eta_j^i.$$

■

Lemma 9.3.13 *Let r and β be small enough. Let $z^1, z^2 \in V(r, \beta)$ be two different points such that $z^2 - z^1 \in S(\alpha)$. Then, if $\alpha \geq \sqrt{n}\beta$,*

$$z^1 + t(z^2 - z^1) \in V(r, \beta) \qquad \textit{for all } t \in [0, 1].$$

Proof. We put $z^i = (x^i, y^i)$ for $i = 1, 2$. Since, by hypothesis, V is convex, we must see that for all $t \in [0, 1]$

$$\|y^1 + t(y^2 - y^1)\|_2 \leq \beta \|x^1 + t(x^2 - x^1)\|_2.$$

We note that, if $\alpha \geq \beta\sqrt{n}$,

$$\|y^1 - y^2\|_2 \geq \|y^1 - y^2\| \geq \alpha \|x^1 - x^2\| \geq \frac{\alpha}{\sqrt{n}} \|x^1 - x^2\|_2 \geq \beta \|x^1 - x^2\|_2. \tag{9.3.2}$$

We claim that

$$\beta^2 \langle x^1, x^2 \rangle - \langle y^1, y^2 \rangle \geq 0. \tag{9.3.3}$$

Indeed, by (9.3.2)

$$\begin{aligned}
\beta^2\langle x^1, x^2\rangle - \langle y^1, y^2\rangle &= \beta^2\frac{1}{2}\left[(\|x^1\|_2^2 + \|x^2\|_2^2 - \|x^1 - x^2\|_2^2)\right.\\
&\quad \left.- (\|y^1\|_2^2 + \|y^2\|_2^2 - \|y^1 - y^2\|_2^2)\right]\\
&\geq \frac{1}{2}\left[\|y^1 - y^2\|_2^2 - \beta^2\|x^1 - x^2\|_2^2\right]\\
&\geq 0.
\end{aligned}$$

Then, using (9.3.3) as well as that z^1, $z^2 \in V(r,\beta)$, we have that

$$\begin{aligned}
&\beta^2\|x^1 + t(x^2 - x^1)\|_2^2 - \|y^1 + t(y^2 - y^1)\|_2^2 = t^2(\beta^2\|x^2\|_2^2 - \|y^2\|_2^2)\\
&\qquad + (1-t)^2(\beta^2\|x^1\|_2^2 - \|y^1\|_2^2) + 2t(1-t)(\beta^2\langle x^1, x^2\rangle - \langle y^1, y^2\rangle)\\
&\qquad \geq 0
\end{aligned}$$

as we wanted. ■

Lemma 9.3.14 *Let r and β be small enough. Let $z^1, z^2 \in V(r,\beta)$ be different points such that $z^2 - z^1 \in S(\alpha)$ and $F(z^2), F(z^1) \in V(r,\beta)$. Then*

1) $F(z^2) - F(z^1) \in S(\alpha)$,

2) $\|\pi^2(F(z^2) - F(z^1))\| \geq \|\pi^2(z^2 - z^1)\|$.

Proof. By Lemma 9.3.13, $z^1 + t(z^2 - z^1) \in V(r,\beta)$, then since DF is continuous we write

$$\begin{aligned}
F(z^2) - F(z^1) &= \int_0^1 DF(z^1 + t(z^2 - z^1))(z^2 - z^1)\,dt\\
&= \lim_{n\to\infty}\sum_{k=0}^{n}\frac{1}{n}DF(z^1 + \frac{k}{n}(z^2 - z^1))(z^2 - z^1).
\end{aligned}$$

From (9.2.4) there exists j such that $z^2 - z^1 \in S_j^+(\alpha) \cup S_j^-(\alpha)$. We suppose that $z^2 - z^1 \in S_j^+(\alpha)$, the other case being analogous. By Lemma 9.3.12

$$\sum_{k=0}^{n}\frac{1}{n}DF\Big(z^1 + \frac{k}{n}(z^2 - z^1)\Big)(z^2 - z^1) \in S_j^+(\alpha) \subset S(\alpha)$$

for all $n \in \mathbb{N}$ and hence the limit when $n \to \infty$ has to belong to $\overline{S(\alpha)} = S(\alpha)$.

To deal with the second property, we observe that, since $F(z^1)$, $F(z^2) \in V(r, \beta)$ and as we have already seen in 1), $F(z^2) - F(z^1) \in S(\alpha)$, hence by Lemma 9.3.13 applied to $F(z^1)$, $F(z^2)$, the segment

$$\psi(t) = (1-t)F(z^1) + tF(z^2) \in V(r, \beta), \qquad t \in [0,1].$$

By the mean value theorem, Lemma 9.3.11 and the definition of ψ, we have that

$$\begin{aligned}
\|\pi^2(z^2 - z^1)\| &= \|\pi^2 \circ F^{-1} \circ \psi(1) - \pi^2 \circ F^{-1} \circ \psi(0)\| \\
&\leq \int_0^1 \|(\pi^2 \circ F^{-1} \circ \psi)'(t)\| \, dt \\
&= \int_0^1 \|\pi^2 DF^{-1}(\psi(t))\psi'(t)\| \, dt \\
&\leq \int_0^1 \|\pi^2(\psi'(t))\| \, dt \\
&= \int_0^1 \|\pi^2(F(z^2) - F(z^1))\| \, dt \\
&= \|\pi^2(F(z^2) - F(z^1))\|
\end{aligned}$$

and the statement holds. ∎

We will use a little bit of degree theory. See [59] for details. We denote by $d(f, D, p)$ the degree of f at p relative to D. We recall that if $d(f, D, p) \neq 0$, then $p \in f(D)$.

We recall that two functions $f, g \in C^0(\overline{D})$ are homotopic if there exists a continuous function $H : [0,1] \times \bar{D} \to \mathbb{R}^n$ such that $H(0,.) = f$ and $H(1,.) = g$. We say that H is a homotopy from f to g.

Proposition 9.3.15 *Let $f, g : D \subset \mathbb{R}^n \to \mathbb{R}^n$ be two continuous maps. If there exists a homotopy $H : [0,1] \times \bar{D} \to \mathbb{R}^n$ from f to g and*

$$p \notin H(t, \partial D) \quad \textit{for all } t \in [0,1]$$

then

$$d(f, D, p) = d(g, D, p).$$

We consider the set of m dimensional manifolds defined as follows: let $\mathcal{V}$ be an open neighbourhood of $V(r, \beta)\backslash\{(0,0)\}$ such that

$$\mathcal{V} \cap \{x = 0\} = \emptyset.$$

Below D_γ^m will denote an open set of $\mathbb{R}^m$, such that $0 \in D_\gamma^m$ and that $\overline{D_\gamma^m}$ is homeomorphic to a closed ball. Therefore ∂D_γ^m will be homeomorphic to a sphere. Given $\gamma : D_\gamma^m \to \mathcal{V}$ we will denote by Γ the image of γ, i.e. $\gamma(D_\gamma^m)$. At some places we will identify γ with Γ.

Let

$$\begin{aligned} &H(\alpha) \\ &\quad = \{\gamma : \overline{D_\gamma^m} \to \mathcal{V} : \gamma \in C^1,\, T_z\Gamma \subset S(\alpha) \ \ \forall z \in \Gamma \cap V(r,\beta),\, \gamma(\partial D_\gamma^m) \subset (V(r,\beta))^c\}. \end{aligned}$$

We note that the condition $T_z\Gamma \subset S(\alpha)$ implies that $\Gamma \cap V(r,\beta)$ can be expressed as a graph of a function $\psi : \pi^2(\Gamma \cap V(r,\beta)) \to \mathbb{R}^2$, in the form

$$\Gamma = \{(\psi(y), y) : y \in \Gamma \cap V(r,\beta)\}$$

and

$$\|D\psi(y)\| \le \frac{1}{\alpha}. \tag{9.3.4}$$

This is easily seen because if $v \in \mathbb{R}^m \setminus \{0\}$ we have that $t \mapsto (\psi(y+tv), y+tv)$ is a curve in Γ and hence its derivative at $t=0$, $(D\psi(y)v, v)$ belongs to $T_{(\psi(y),y)}\Gamma \subset S(\alpha)$ and then

$$\|v\| \ge \alpha \|D\psi(y)v\|$$

which proves (9.3.4).

Our goal is to iterate manifolds of $H(\alpha)$ by F. A subtle and delicate point is to check that the iterates remain non-void. When $m=1$ this is a simple consequence of Bolzano's theorem, but if $m>1$ we are forced to apply degree theory. This motivates in part the definition of $H(\alpha)$.

Lemma 9.3.16 *If $\beta < \alpha/\sqrt{n}$, we have that if $\Gamma \in H(\alpha)$ then*

$$F(\Gamma) \cap V(r,\beta) \in H(\alpha).$$

Proof. We perform the change of variables C defined by

$$(x, v) \mapsto (x, y = v\|x\|_2) \tag{9.3.5}$$

which transform the cone-like domain $V(r,\beta)$ to the cylinder-like domain

$$\tilde{V}(r,\beta) = \{(x,v) \in \mathbb{R}^{n+m} : x \in V(r),\ \|v\|_2 \leq \beta\}.$$

This change is a diffeomorphism when restricted to $\tilde{V}(r,\beta)\backslash\{x=0\}$. Indeed if $(x,y) \in V(r,\beta)$ then $x \neq 0$, and we can write the inverse explicitly as

$$(x,y) \mapsto \Big(x, v = \frac{y}{\|x\|_2}\Big).$$

Note that $x \mapsto \|x\|_2$ is C^1 except at $x=0$.

In these new variables F is expressed as $\tilde{F} = C^{-1} \circ F \circ C$ with

$$\begin{aligned} \pi^1 \tilde{F}(x,v) &= \pi^1 F(x, v\|x\|_2) = x + p(x, v\|x\|_2) + f(x, v\|x\|_2) \\ \pi^2 \tilde{F}(x,v) &= \pi^2 F(x, v\|x\|_2) = \frac{v\|x\|_2 + q(x, v\|x\|_2) + g(x, v\|x\|_2)}{\|x + p(x, v\|x\|_2) + f(x, v\|x\|_2)\|_2}. \end{aligned}$$

If $\Gamma \in H(\alpha)$, we denote by $\tilde{\Gamma}$ the image of Γ by this change of variables, i.e. $\tilde{\Gamma} = C^{-1}(\Gamma)$. We claim that $\tilde{\Gamma}$ can also be represented as a graph of a function $\tilde{\psi}$. Indeed, if

$$\Gamma = \{(\psi(y), y) : y \in D_\psi\},$$

then

$$\tilde{\Gamma} = \left\{ \Big(\psi(y), \frac{y}{\|\psi(y)\|_2}\Big) : y \in D_\psi \right\}.$$

Now, we are going to check that

$$\mathcal{X} : y \mapsto \frac{y}{\|\psi(y)\|_2}$$

is a diffeomorphism. First note that $\psi \neq 0$ and then it is well defined and C^1. Now we prove that $\mathcal{X}$ is one to one. If $y_1,\, y_2 \in D_\psi$ and $\mathcal{X}(y_1) = \mathcal{X}(y_2)$ we can write

$$y_1\,[\|\psi(y_2)\|_2 - \|\psi(y_1)\|_2] + (y_1 - y_2)\|\psi(y_1)\|_2 = 0$$

and then, if we assume that $y_1 \neq y_2$

$$\frac{\|\psi(y_2)\|_2 - \|\psi(y_1)\|_2}{\|y_1 - y_2\|_2} = \frac{\|\psi(y_1)\|_2}{\|y_1\|_2}. \tag{9.3.6}$$

By (9.3.4)

$$\frac{\|\psi(y_2)\|_2 - \|\psi(y_1)\|_2}{\|y_1 - y_2\|_2} \le \frac{\|\psi(y_2) - \psi(y_1)\|_2}{\|y_1 - y_2\|_2} \le \frac{1}{\alpha}$$

and by the fact that $(\psi(y_1), y_1) \in V(r, \beta)$

$$\frac{\|\psi(y_1)\|_2}{\|y_1\|_2} \ge \frac{1}{\beta}.$$

Putting these two last bounds in (9.3.6) we obtain

$$\frac{1}{\beta} \le \frac{1}{\alpha},$$

which gives a contradiction.

Next we prove that $\mathcal{X}$ is a C^1 diffeomorphism. Since we already know that it is one to one we only have to check that $D\mathcal{X}(y)$ is invertible. We compute

$$\begin{aligned} D\mathcal{X}(y) &= \frac{1}{\|\psi(y)\|_2} \operatorname{Id} - y \frac{1}{\|\psi(y)\|_2^3} \langle \psi(y), D\psi(y) \rangle \\ &= \frac{1}{\|\psi(y)\|_2} \left[\operatorname{Id} - \frac{y}{\|\psi(y)\|_2} \left\langle \frac{\psi(y)}{\|\psi(y)\|_2}, D\psi(y) \right\rangle \right]. \end{aligned}$$

Since $\|y\|_2/\|\psi(y)\|_2 \le \beta$, $\|D\psi(y)\| \le 1/\alpha$ and $\beta < \alpha$ we immediately see that $D\mathcal{X}(y)$ is invertible.

Therefore

$$\tilde{\Gamma} = \{(\psi(\mathcal{X}^{-1}(v)), v) : v \in \mathcal{X}(D_\psi)\}.$$

We call $\tilde{\psi} = \psi \circ \mathcal{X}^{-1}$.

Now we look at the image of $\tilde{\Gamma} = \operatorname{graph} \tilde{\psi}$ by $\tilde{F}$. First we prove that $\pi^2 \tilde{F} \circ (\tilde{\psi}(y), y)$ covers

$$B_\beta^m(0) = \{y \in \mathbb{R}^m : \|y\| \le \beta\}.$$

For this we will use degree theory. Let

$$H(t, v) = \frac{v\|\tilde{\psi}(v)\|_2 + tq(\tilde{\psi}(v), v\|x\|_2) + tg(\tilde{\psi}(v), v\|\tilde{\psi}(v)\|_2)}{\|\tilde{\psi}(v) + tp(\tilde{\psi}(v), v\|x\|_2) + tf(\tilde{\psi}(v), v\|\tilde{\psi}(v)\|_2)\|_2}$$

be a homotopy from the identity to $\pi^2\tilde{F}\circ(\tilde{\psi}(y),y)$. Let $v_0 \in B^m_\beta(0)$. If $v_0 \in \partial B^m_\beta(0)$ then $(\tilde{\psi}(y),y) \in \partial\tilde{V}(r,\beta)$ and by the conclusions of Lemmas 9.3.8 and 9.3.9 translated to $\tilde{F}$ we deduce that

$$v_0 \notin H(t, \partial B^m_\beta(0))$$

and hence from Proposition 9.3.15 we get that

$$d(\pi^2\tilde{F}\circ(\tilde{\psi},\mathrm{Id}), B^m_\beta(0), v_0) = d(\mathrm{Id}, B^m_\beta(0), v_0) = 1.$$

Going back to the variables (x,y) we obtain that $F(\Gamma)$ is the image of

$$\gamma_1 = F\circ\gamma = C\circ(C^{-1}\circ F\circ C)\circ(C^{-1}\circ\gamma) = C\circ\tilde{F}\circ(C^{-1}\circ\gamma).$$

We will need to restrict the domain D_γ to D_{γ_1} so that

$$\forall\zeta\in D_{\gamma_1}, \qquad F\gamma(\zeta)\in\mathcal{V}.$$

Therefore we also obtain that

$$F(\partial D_{\gamma_1}) \subset (V(r,\beta))^c.$$

Finally the fact that

$$T_z(F(\Gamma)) \subset S(\alpha)$$

for all $z \in F(\Gamma)\cap V(r,\beta)$ comes from Lemma 9.3.14. ■

With the previous lemmas we can prove Theorem 9.3.1, that is the existence of a Lipschitz manifold.

Proof of the Theorem 9.3.1. Given $\Gamma \in H(\alpha)$ we define the sequence

$$\begin{aligned}&\Gamma_0 = \Gamma,\\ &\Gamma_k = F(\Gamma_{k-1})\cap V(r,\beta), \qquad k\ge 1.\end{aligned}$$

By Lemma 9.3.16 all the elements of this sequence belong to $H(\alpha)$. We introduce $I_k = F^{-k}(\Gamma_k)$. We claim that $(I_k)_k$ is a nested sequence of nonempty compact sets. Indeed:

$$I_k = F^{-k}(F(\Gamma_{k-1})\cap V(r,\beta)) \subset F^{-k}(F(\Gamma_{k-1})) = F^{-(k-1)}(\Gamma_{k-1}) = I_{k-1}.$$

The fact that I_k are nonempty comes from Lemma 9.3.16. Hence $\bigcap_{k\geq 0} I_k \neq \emptyset$. Next we will prove that $\bigcap_{k\geq 0} I_k$ reduces to a point. For that we consider a particular sort of initial Γ_0. Given $x^0 \in V$ we define $\Gamma = \Gamma_0$ by

$$\Gamma : \begin{cases} x = x^0 \\ \|y\|_2 \leq \beta \|x\|_2. \end{cases}$$

It is clear that $\Gamma \in H(\alpha)$ and that $\forall z^1, z^2 \in \Gamma_0,\ z^2 - z^1 \in S(\alpha)$.

Assume that there exist $z^1, z^2 \in \bigcap_{k\geq 0} I_k$. Then $F^k(z^1), F^k(z^2) \in V(r,\beta),\ \forall k \geq 0$. By Lemma 9.3.7 we have that $\|\pi_1^1(F^k(z^1))\|$ is a strictly decreasing sequence of real numbers. Therefore it has a limit which must be 0. Moreover, for all k, $\|\pi^2(F^k(z^1))\| \leq \beta\|\pi^1(F^k(z^1))\|$, so that $\pi^2(F^k(z^1))$ also goes to 0. The same happens to $\pi^2(F^k(z^2))$. Applying Lemma 9.3.14 iteratively we get

$$\|\pi^2(F^k(z^2) - F^k(z^1))\| \geq \|\pi^2(z^1 - z^2)\|.$$

Taking the limit when $k \to \infty$ we obtain $\pi^2(z^2) = \pi^2(z^1)$. Also, since $z^2 - z^1 \in S(\alpha)$, we have that $\pi^1(z^2) = \pi^1(z^1)$ and $z^2 = z^1$. Therefore $\bigcap_{k\geq 0} I_k$ is a point and has the form (x^0, y^0). Furthermore

$$\bigcap_{k\geq 0} I_k = \Gamma \cap \{(x,y) \in \mathbb{R}^{n+m} :\ \lim_{k\to\infty} F^k(z) = 0,\ F^k(z) \in V(r,\beta),\ k \geq 0\}.$$

We define φ by $\varphi(x^0) = y^0$. The graph of φ is the invariant manifold we looked for. Now it remains to be proved that φ is Lipschitz. If we assume that $\operatorname{Lip}\varphi$ is not smaller than α, there would exist two different points $x^1, x^2 \in V(r,\beta)$ such that

$$\frac{\|\varphi(x^2) - \varphi(x^1)\|}{\|x^2 - x^1\|} \geq \alpha.$$

Applying Lemma 9.3.14 iteratively we have

$$\|\pi^2(F^k(x^2,\varphi(x^2)) - F^k(x^1,\varphi(x^1)))\| \geq \|\varphi(x^2) - \varphi(x^1)\| \geq \alpha\|x^2 - x^1\|.$$

Since $(x^1, \varphi(x^1))$ and $(x^2, \varphi(x^2))$ belong to the stable manifold

$$\lim_{k\to\infty} \pi^2\left(F^k(x^2,\varphi(x^2)) - F^k(x^1,\varphi(x^1))\right) = 0$$

and hence we deduce that $x^2 = x^1$, which is a contradiction. Therefore φ is Lipschitz with $\operatorname{Lip}\varphi < \alpha$. ∎

Remark 9.3.17 *From the fact that we can take α as small as we want if we take r small enough, we get that φ has an arbitrarily small Lipschitz constant in a sufficiently small neighbourhood of the origin. Therefore φ is differentiable at 0 and $D\varphi(0) = 0$.*

9.4 The analytic case

In this Section we shall prove that if F is analytic φ is also analytic in a suitable domain. We consider F defined in an open set of $\mathbb{C}^{n+m}$. We introduce the following notation: if $x \in \mathbb{C}^n$

$$\begin{aligned}\|\operatorname{Re} x\| &= \max(|\operatorname{Re} x_1|, \ldots, |\operatorname{Re} x_n|),\\ \|\operatorname{Im} x\| &= \max(|\operatorname{Im} x_1|, \ldots, |\operatorname{Im} x_n|),\\ \|x\| &= \max(|x_1|, \ldots, |x_n|).\end{aligned}$$

Given $\gamma, r > 0$ we define

$$\Omega(\gamma) = \{x \in \mathbb{C}^n : \operatorname{Re} x \in V(r),\ \|\operatorname{Im} x\| < \gamma \|\operatorname{Re} x\|\}.$$

We will need that the set $\Omega(\gamma)$ to be invariant. In fact, we only need that there exists a invariant set contained in $\Omega(\gamma)$. A sufficient condition for that is:

H4 For all $i, l \in \{1, \ldots, n\}$ and for all $x \in V^1(r)$,

$$N_p\Big(D_{x_i} p_i(x,0) + \sum_{k=1, k\neq i} |D_{x_k} p_i(x,0)|\Big) < D_{x_l} p_l(x,0) + \sum_{k=1, k\neq l} |D_{x_k} p_l(x,0)|.$$

Theorem 9.4.1 *Let F defined as in (9.3.1). Assume that the hypotheses **H1-H3** hold for $x \in \Omega(\gamma) \cap \mathbb{R}^n$ and that **H5** holds in $\Omega(\delta)$ (or more generally that $\Omega(\delta)$ is invariant by $\pi^1 F(.,0)$). Then, the map φ obtained in Theorem 9.3.1 is analytic in $V(r)$.*

The following estimate will be required.

Proposition 9.4.2 *There exist $r, \gamma, \beta > 0$ such that*

1) if $x \in \Omega(\gamma)$, and $\|y\| \leq \beta \|x\|$ then $\pi^1 F(x,y) \in \Omega(\gamma)$,

2) if $x \in \Omega(\gamma)$, and $\|y\| = \beta \|x\|$ then $\|\pi^2 F(x,y) - y\| < \|y\|$,

3) if $x \in \Omega(\gamma)$, and $\|y\| = \beta \|x\|$ then $\beta \|\pi^1 F(x,y)\| < \|\pi^2 F(x,y)\|$.

Proof. 1) Let $x \in \Omega(\gamma)$. We write $x = x_1 + ix_2$ with $x_1, x_2 \in \mathbb{R}^n$ and

$$p(x,0) = p^1(x_1,x_2) + ip^2(x_1,x_2), \qquad p^1(x_1,x_2), p^2(x_1,x_2) \in \mathbb{R}^n.$$

We want to see that there exists a constant B such that

$$\|x_2 + p^2(x_1,x_2)\| - \gamma\|x_1 + p^1(x_1,x_2)\| \le -B\|x\|^{N_p}.$$

Since $p^2(x_1,0) = 0$, we have that

$$p^2(x_1,x_2) = \int_0^1 D_{x_2}p^2(x_1,sx_2)x_2 \, ds$$

Moreover, since p^1 is a vector of homogeneous polynomials, by Euler's theorem

$$p^1(x_1,x_2) = \frac{1}{N}D_{x_1}p^1(x_1,x_2)x_1 + \frac{1}{N}D_{x_2}p^1(x_1,x_2)x_2. \tag{9.4.1}$$

Also, since $p(x,0)$ is analytic, by the Cauchy-Riemann equations, we have that

$$D_{x_1}p^1(x_1,x_2) = D_{x_2}p^2(x_1,x_2) \tag{9.4.2}$$
$$D_{x_2}p^1(x_1,x_2) = -D_{x_1}p^2(x_1,x_2). \tag{9.4.3}$$

We observe that, since $p^2(x_1,0) = 0$, by (9.4.3)

$$D_{x_2}p^1(x_1,0) = -D_{x_1}p^2(x_1,0) = 0. \tag{9.4.4}$$

We denote

$$C(x_1,x_2) = (c_{ij}(x_1,x_2))_{1\le i,j\le n} \equiv \int_0^1 D_{x_2}p^2(x_1,sx_2)ds \tag{9.4.5}$$
$$A(x_1,x_2) = (a_{ij}(x_1,x_2))_{1\le i,j\le n} \equiv \frac{1}{N_p}D_{x_1}p^1(x_1,x_2) \tag{9.4.6}$$
$$B(x_1,x_2) = (b_{ij}(x_1,x_2))_{1\le i,j\le n} \equiv \frac{1}{N_p}D_{x_2}p^1(x_1,x_2) \tag{9.4.7}$$

and we note that, by (9.4.2) and (9.4.4)

$$\begin{aligned} C(x_1,0) &= D_{x_2}p^2(x_1,0) = D_{x_1}p^1(x_1,0) = N_pA(x_1,0) \\ B(x_1,0) &= 0. \end{aligned} \tag{9.4.8}$$

Let $x \in \Omega(\gamma)$ be such that $\|x_2\| \le \gamma\|x_1\|$. Then

$$\begin{aligned} \|x_2 + p^2(x_1,x_2)\| &= \|(\mathrm{Id} + C(x_1,x_2))x_2\| \le \|x_2\|\|\mathrm{Id} + C(x_1,x_2)\| \\ &\le \gamma\|x_1\|\|\mathrm{Id} + C(x_1,x_2)\| \end{aligned}$$

and, by (9.4.1)

$$\begin{aligned}\|x_1 + p^1(x_1, x_2)\| &= \|(\mathrm{Id} + A(x_1, x_2))x_1 + B(x_1, x_2)x_2\| \\ &\geq \|x_1\|(1 - \|A(x_1, x_2)\| - \gamma\|B(x_1, x_2)\|).\end{aligned}$$

Since that in the norm we are working with, the operator norm is the maximum over the ℓ^1 norm of rows, there exist $i, l \in \{1, \dots, n\}$ such that

$$\begin{aligned}\|A(x_1, x_2)\| &= |a_{ii}(x_1, x_2)| + \sum_{k=1, k\neq i} |a_{ij}(x_1, x_2)| \\ \|\mathrm{Id} + C(x_1, x_2)\| &= 1 + c_{ll}(x_1, x_2) + \sum_{k=1, k\neq l} |c_{lj}(x_1, x_2)|.\end{aligned}$$

Then

$$\begin{aligned}\|x_2 + p^2(x_1, x_2)\| - &\gamma\|x_1 + p^1(x_1, x_2)\| \\ &\leq \gamma\|x_1\|(\|\mathrm{Id} + C(x_1, x_2)\| - 1 + \|A(x_1, x_2)\| + \gamma\|B(x_1, x_2)\|) \\ &= \gamma\|x_1\|\Big(c_{ll}(x_1, x_2) + \sum_{k=1, k\neq l} |c_{lj}(x_1, x_2)| - |a_{ii}(x_1, x_2)| \\ &\qquad - \sum_{k=1, k\neq i} |a_{ij}(x_1, x_2)| + \gamma\|B(x_1, x_2)\|\Big)\end{aligned}$$

and it is negative if and only if the functions

$$\begin{aligned}h_{li}(x_1, x_2) \equiv c_{ll}(x_1, x_2) &+ \sum_{k=1, k\neq l} |c_{lj}(x_1, x_2)| - |a_{ii}(x_1, x_2)| \\ &- \sum_{k=1, k\neq i} |a_{ij}(x_1, x_2)| + \gamma\|B(x_1, x_2)\|\end{aligned}$$

are negative. We observe that, by (9.4.8)

$$\begin{aligned}h_{li}(x_1, 0) &= N_p\Big(a_{ll}(x_1, 0) + \sum_{k=1, k\neq l} |a_{lj}(x_1, 0)|\Big) \\ &\quad - |a_{ii}(x_1, 0)| - \sum_{k=1, k\neq i} |a_{ij}(x_1, 0)|\end{aligned}$$

and by hypotheses **H4** $h_{li}(x_1, 0) < 0$. Thus, by Lemma 9.3.5 there exist $\gamma > 0$ small enough and a constant B such that

$$|h_{li}(x_1, x_2)| \leq -2B\|x\|^{N_p - 1}$$

if $\|x_2\| \le \gamma\|x_1\|$. Thus, for all $x \in \Omega(\gamma)$ with γ small enough, we have

$$\|x_2 + p^2(x_1,x_2)\| - \gamma\|x_1 + p^1(x_1,x_2)\| \le -B\gamma\|x\|^{N_p}.$$

Let $\beta > 0$ be small enough such that for all $x \in \Omega(\gamma)$ and $\|y\| \le \beta\|x\|$ we have that

$$\begin{aligned} \|\operatorname{Im} p(x,y) - p^2(x_1,x_2)\| &\le \|D_y \operatorname{Im} p\|\|y\| < \gamma\frac{B}{4}\|x\|^{N_p} \\ \|\operatorname{Re} p(x,y) - p^1(x_1,x_2)\| &\le \|D_y \operatorname{Re} p\|\|y\| < \gamma\frac{B}{4}\|x\|^{N_p}. \end{aligned} \tag{9.4.9}$$

On the other hand there exists $r > 0$ such that if $\|x\| < r$ and $\|y\| \le \beta\|x\|$,

$$\begin{aligned} \|\operatorname{Re} f(x,y)\| &< \gamma\frac{B}{8}\|x\|^{N_p} \\ \|\operatorname{Im} f(x,y)\| &< \gamma\frac{B}{8}\|x\|^{N_p}. \end{aligned} \tag{9.4.10}$$

We have that

$$\operatorname{Re}\pi^1 F(x,y) = \operatorname{Re}(x + p(x,0)) + \operatorname{Re}(p(x,y) - p(x,0)) + \operatorname{Re}(f(x,y))$$

by (9.4.9) and (9.4.10)

$$|\operatorname{Re}(p(x,y) - p(x,0)) + \operatorname{Re}(f(x,y))| \le \gamma\frac{30}{8}\|x\|^{N_p}$$

and then, if γ is small, by **H3**, $\operatorname{Re}\pi^1 F(x,y) \in V(r)$.

Moreover,

$$\begin{aligned} \|\operatorname{Im}(\pi^1 F(x,y))\| - \gamma\|\operatorname{Re}(\pi^1 F(x,y))\| &\le \|x_2 + p^2(x_1,x_2)\| - \gamma\|x_1 + p^1(x_1,x_2)\| \\ &\quad + \|p^2(x_1,x_2) - \operatorname{Im} p(x,y)\| + \gamma\|p^1(x_1,x_2) - \operatorname{Re} p(x,y)\| \\ &\quad + \|\operatorname{Im} f(x,y)\| + \gamma\|\operatorname{Re} f(x,y)\| \\ &\le -B\gamma\|x\|^{N_p} + B\left(\frac{1}{4} + \frac{1}{8}\right)(\gamma + \gamma^2)\|x\|^{N_p} \le -\gamma\frac{B}{2}\|x\|^{N_p} \end{aligned}$$

which implies that $\pi^1 F(x,y) \in \Omega(\gamma)$.

2) We let $x \in \Omega(\gamma)$ and $\|y\| = \beta\|x\|$, Let $j \in \{1,\dots,m\}$ be such that $|y_j| = \|y\|$. Then

$$\begin{aligned} \|\pi^2 F(x,y) - y\| &= \|q(x,y) + g(x,y)\| \\ &\le K\|x\|^{N_q} \\ &= K\|x\|^{N_q - 1}\frac{|y_j|}{\beta} \\ &< |y_j| = \|y\| \end{aligned}$$

if r is such that $Kr^{N-1} < \beta$.

3) We fix $j \in \{1, \dots, m\}$. Since $D_{x_k} q_j(x,0) = 0$, $1 \le k \le n$, we have that $q(x,0) = 0$ and

$$q_j(x,y) = y_j q_j^{(j)}(x,y) + \sum_{k=1, k\neq j}^{m} y_k q_j^{(k)}(x,y)$$

where $q_j^{(k)}(x,y) = \int_0^1 D_{y_k} q_j(x, sy)\, ds$, $1 \le k \le m$, are homogeneous polynomials of degree $N_q - 1$. Clearly

$$D_{y_k} q_j(x,0) = q_k^{(j)}(x,0) \qquad 1 \le k \le m.$$

If we restrict us to $\mathbb{R}^{n+m}$, by hypotheses **H2** we have that for $x \in \overline{V(r)}$,

$$D_{y_j} q_j(x,0) > \sum_{k=1, k\neq j}^{m} |D_{y_k} q_j(x,0)| \ge 0.$$

Hence

$$q_j^{(j)}(x,0) > \sum_{k=1, k\neq j}^{m} |q_j^{(k)}(x,0)| \ge 0. \tag{9.4.11}$$

Now let $(x,y) \in \mathbb{C}^{n+m}$ be such that $x \in \Omega(\gamma)$ and $\|y\| \le \beta \|x\|$. In order to apply Lemma 9.3.5, we consider $q_j^{(k)}$ as maps of the form $q_j^{(k)} : V(r) \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R} \times \mathbb{R}$, depending on the real and the imaginary parts of x and y. Then

$$\begin{aligned} \operatorname{Re}(q_j^{(j)}(\operatorname{Re} x, 0,0,0)) &= q_j^{(j)}(\operatorname{Re} x, 0,0,0) \\ &> \sum_{k=1, k\neq j}^{m} |q_j^{(k)}(\operatorname{Re} x, 0,0,0)|. \end{aligned}$$

There exist $\gamma, \beta, K > 0$ such that if, $\|\operatorname{Im} x\| < \gamma \|\operatorname{Re} x\|$ and $\|y\| \le \beta \|x\|$,

$$\operatorname{Re}(q_j(x,y)) - \sum_{k=1, k\neq j}^{m} |q_j^{(k)}(x,y)| \ge K_1 \|\operatorname{Re} x\|^{N_q - 1} \ge K_2 \|x\|^{N_q - 1}.$$

Now we bound $|\pi_j^2 F(x,y)|$, with $x \in \Omega(\gamma)$, and $|y_j| = \beta\|x\|$

$$\begin{aligned}
|\pi_j^2 F(x,y)| &= |y_j + q_j(x,y) + g_j(x,y)| \\
&= |y_j + y_j q_j^{(j)}(x,y) + \sum_{k=1,k\neq j}^{m} y_k q_j^{(k)}(x,y) + g_j(x,y)| \\
&\geq |y_j| \Big(|1 + q_j^{(j)}(x,y)| - \sum_{k=1,k\neq j}^{m} |q_j^{(k)}(x,y)| \Big) - |g_j(x,y)| \\
&\geq |y_j| \Big(1 + \mathrm{Re}(q_j^{(j)}(x,y)) - \sum_{k=1,k\neq j}^{m} |q_j^{(k)}(x,y)| \Big) - |g_j(x,y)| \\
&\geq |y_j|(1 + K\|x\|^{N_q-1}) - |g_j(x,y)| > |y_j|
\end{aligned}$$

if r is small enough since $g_j(x,y) = o(\|(x,y)\|^{N_q})$.

Now we claim that if β and γ are small then $\|\pi^1 F(x,y)\| \leq \|x\|$. Indeed, by Euler's theorem,

$$p(x,y) = \frac{1}{N_p} D_x p(x,y)x + \frac{1}{N_p} D_y p(x,y)y.$$

We denote

$$\begin{aligned}
\tilde{A}(x,y) = (\tilde{a}_{ij}(x,y))_{i,j} &\equiv \frac{1}{N_p} D_x p(x,y) \\
\tilde{B}(x,y) = (\tilde{b}_{ij}(x,y))_{i,j} &\equiv \frac{1}{N_p} D_y p(x,y).
\end{aligned}$$

By hypothesis **H1**, there exists a constant K such that, for $i \in \{1,\dots,n\}$ and for $x \in \Omega(\gamma) \cap \mathbb{R}^n$

$$\mathrm{Re}\,\tilde{a}_{ii}(x,0) + \sum_{j=1,j\neq i}^{n} |\mathrm{Re}\,\tilde{a}_{ij}(x,0)| \leq -K\|x\|^{N_p-1}.$$

Therefore, by Lemma 9.3.5, there exist γ and β small enough such that if $\|\mathrm{Im}\,x\| \leq \gamma\|\mathrm{Re}\,x\|$ and $\|y\| \leq \beta\|x\|$, then

$$\mathrm{Re}\,\tilde{a}_{ii}(x,y) + \sum_{j=1,j\neq i}^{n} |\mathrm{Re}\,\tilde{a}_{ij}(x,y)| \leq -K\|x\|^{N_p-1}. \tag{9.4.12}$$

We consider the functions $a_{ij} = (a^1_{ij}, a^2_{ij}) : V \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^2$ defined by

$$\begin{aligned} a^1_{ij}(\operatorname{Re} x, \operatorname{Im} x, \operatorname{Re} y, \operatorname{Im} y) &= \operatorname{Re} \tilde{a}_{ij}(x,y), \\ a^2_{ij}(\operatorname{Re} x, \operatorname{Im} x, \operatorname{Re} y, \operatorname{Im} y) &= \operatorname{Im} \tilde{a}_{ij}(x,y). \end{aligned}$$

Then

$$a_{ij}(\operatorname{Re} x, 0, 0, 0) = (\operatorname{Re} \tilde{a}_{ij}(\operatorname{Re} x, 0), 0).$$

We denote $v = (\operatorname{Im} x, \operatorname{Re} y, \operatorname{Im} y)$, and we recall that $\|v\| \leq \delta\|x\| = \max\{\beta,\gamma\}\|x\|$. There exists a constant K such that

$$|\operatorname{Im} \tilde{a}_{ij}(x,y)| = |a^2_{ij}(\operatorname{Re} x, v)| \leq \int_0^1 |D_v a^2_{ij}(\operatorname{Re} x, sv)|\|v\| ds \leq \delta K \|x\|^{N_p - 1}. \tag{9.4.13}$$

Let $i \in \{1, \dots, n\}$ be such that

$$\|\operatorname{Id} + \tilde{A}(x,y)\| = |1 + \tilde{a}_{ii}(x,y)| + \sum_{j=1, j\neq i}^{n} |\tilde{a}_{ij}(x,y)|$$

then, by (9.4.12) and (9.4.13)

$$\begin{aligned} \|\operatorname{Id} + \tilde{A}(x,y)\| &= |1 + \tilde{a}_{ii}(x,y)| + \sum_{j=1, j\neq i}^{n} |\tilde{a}_{ij}(x,y)| \\ &= [(1 + \operatorname{Re} \tilde{a}_{ii}(x,y))^2 + (\operatorname{Im} \tilde{a}_{ii}(x,y))^2]^{1/2} \\ &\quad + \sum_{j=1, j\neq i}^{n} [(\operatorname{Re} \tilde{a}_{ij}(x,y))^2 + (\operatorname{Im} \tilde{a}_{ij}(x,y))^2]^{1/2} \\ &\leq [(1 + \operatorname{Re} \tilde{a}_{ii}(x,y))^2 + \delta^2 C^2 \|x\|^{2N_p - 2}]^{1/2} \\ &\quad + \sum_{j=1, j\neq i}^{n} [(\operatorname{Re} \tilde{a}_{ij}(x,y))^2 + \delta^2 C^2 \|x\|^{2N_p - 2}]^{1/2} \\ &\leq 1 + \operatorname{Re} \tilde{a}_{ii}(x,y) + \delta C \|x\|^{N_p - 1} + \sum_{j=1, j\neq i}^{n} |\operatorname{Re} \tilde{a}_{ij}(x,y)| + \delta C \|x\|^{N_p - 1} \\ &\leq 1 - (K - \delta C)\|x\|^{N_p - 1}. \end{aligned}$$

Moreover, it is clear that $\|\tilde{B}(x,y)\|\|y\| \leq K_0\beta\|x\|^{N_p}$ and since, $f(x,y) = o(\|(x,y)\|^{N_p})$, there exists r small enough such that $\|f(x,y)\| \leq \frac{K}{2}\|x\|^{N_p}$. We bound $\|\pi^1 F(x,y)\|$,

$$\begin{aligned}
\|\pi^1 F(x,y)\| &= \|x + p(x,y) + f(x,y)\| \\
&\leq \|\operatorname{Id} + \tilde{A}(x,y)\|\|x\| + \|\tilde{B}(x,y)\|\|y\| + \|f(x,y)\| \\
&\leq (1 - (K - \delta C)\|x\|^{N_p - 1})\|x\| + \beta K_0\|x\|^{N_p} + \frac{K}{2}\|x\|^{N_p} \\
&\leq \|x\| - (K - \delta C - \beta K_0 - \frac{K}{2})\|x\|^{N_p} \\
&\leq \|x\|
\end{aligned}$$

if δ is small enough.

As before the take $x \in \Omega(\gamma)$, $x \neq 0$, and $|y_j| = \beta\|x\|$. We can bound the expression $|\pi_j^2 F(x,y)| - \beta\|\pi^1 F(x,y)\|$:

$$\begin{aligned}
|\pi_j^2 F(x,y)| - \beta\|\pi^1 F(x,y)\| &= |y_j + q_j(x,y) + g_j(x,y)| \\
&\quad - \frac{|y_j|}{\|x\|}\|x + p(x,y) + f(x,y)\| \\
&\geq |y_j| - |y_j|\frac{\|x\|}{\|x\|} = 0
\end{aligned}$$

which proves 3). ■

We also will need a multidimensional version of the classical Rouche's theorem. First we recall the definitions of index and multiplicity.

Definition 9.4.3 *Let $D \subset \mathbb{C}^n$ be an open set and let $f \in \mathcal{H}(\overline{D})$ be a holomorphic function. Let x_0 be an isolated solution of $f(x_0) = p$.*

(1) We define the index of x_0 as $i(f, x_0, p) = d(f, \mathcal{U}, p)$ where $\mathcal{U}$ is a neighborhood of x_0 which does not contain any solution of $f(x) = p$ different from x_0.

(2) We define the multiplicity of x_0 as a p-point of f as $i(f, x_0, p)$. We say that x_0 is simple if its multiplicity is 1.

The following theorem can be found in [59].

Theorem 9.4.4 *(Rouche's theorem) Let D be a bounded, open set in $\mathbb{C}^n$. Suppose that $f, g \in \mathcal{H}(\overline{D})$ are such that $\|g(z)\| < \|f(z)\|$ for all $z \in \partial D$. Then f has finitely many zeros in D, and, counting multiplicity, f and $f+g$ have the same number of zeros in D.*

In particular, if f has a unique zero in D of multiplicity one, $f+g$ also has a unique zero in D.

We define the set of functions

$$\mathcal{H} = \{h : \Omega(\gamma) \to \mathbb{C}^m : \text{real analytic in } \Omega,\ \|h(x)\| \le \beta\|x\|\}.$$

and also the sets

$$\begin{aligned}\Lambda^0 &= \{(x,y) \in \mathbb{C}^n \times \mathbb{C}^m : x \in \Omega(\gamma),\ \|y\| < \beta\|x\|\},\\ \Lambda &= \{(x,y) \in \mathbb{C}^n \times \mathbb{C}^m : x \in \Omega(\gamma),\ \|y\| \le \beta\|x\|\},\\ D(x_0) &= \{z \in \mathbb{C}^n : \|z\| < \beta\|x_0\|\}.\end{aligned}$$

where $x_0 \in \Omega(\gamma)$.

For $x_0 \in \Omega(\gamma)$, $y \in D(x_0)$ and $h \in \mathcal{H}$, we define

$$H(x_0, y) = \pi^2 F(x_0, y) - h(\pi^1 F(x_0, y))$$

and we want to solve $H(x_0, y) = 0$ with respect to y. The interpretation of $H(x_0, y) = 0$ is that, if we solve $y = y^*(x_0)$, graph y^* is the preimage by F of graph h. Notice that if $x_0 \in \Omega(\delta)$ and $y \in D(x_0)$, H is well defined and analytic in Λ^0. Let us see that $H(x_0, y) = 0$ has a unique solution in $D(x_0)$. Indeed, by Lemma 9.4.2, if $x_0 \in \Omega(\gamma)$ and $\|y\| = \beta\|x_0\|$ then

$$\|\pi^2 F(x_0, y) - y\| < \|y\|.$$

Therefore by Rouché's theorem, the functions y and $\pi^2 F(x_0, y)$ (as functions of y) have the same number of zeros in $D(x_0)$. Since the first function is the identity they have a unique zero.

On the other hand, if $\|y\| = \beta\|x_0\|$, by Lemma 9.4.2 we have that

$$\beta\|\pi^1 F(x_0, y)\| < \|\pi^2 F(x_0, y)\|$$

and hence

$$\begin{aligned}\|H(x_0, y) - \pi^2 F(x_0, y)\| &= \|h(\pi^1 F(x_0, y))\|\\ &\le \beta\|\pi^1 F(x_0, y)\|\\ &< \|\pi^2 F(x_0, y)\|\end{aligned}$$

and again by Rouché's theorem, H has a unique zero in $D(x_0)$ which we denote by $y^*(x_0)$. Clearly $\|y^*(x_0)\| \le \beta\|x_0\|$.

By the implicit function theorem, since this zero is unique, it depends analytically with respect to x_0. Hence we can define the map $\mathcal{F} : \mathcal{H} \to \mathcal{H}$ by

$$\mathcal{F}h(x) = y^*(x)$$

where $y^*(x)$ such that $H(x, y^*(x)) = 0$ for all $x \in \Omega(\delta)$.

Since H is real analytic and the solution $y(x)$ is unique it must be real analytic. Otherwise the conjugate would be another solution on $D(x)$ we have proved that $\mathcal{F}$ sends $\mathcal{H}$ into $\mathcal{H}$

Furthermore by construction we have

$$F(\mathrm{graph}(\mathcal{F}h)) \subset \mathrm{graph}(h)$$

and if $0 \le m \le n$

$$F^m(\mathrm{graph}(\mathcal{F}^n h)) \subset \mathrm{graph}(\mathcal{F}^{n-m}h) \in \Lambda.$$

Given $h_0 \in \mathcal{H}$ we define the sequence $h_n = \mathcal{F}^n h_0 \in \Lambda$. Since $h_n \in \mathcal{H}$ the sequence is uniformly bounded and, by Montel's theorem, it has a subsequence convergent to some function $h \in \mathcal{H}$. To check that $F^m(\mathrm{graph}(h)) \in \Lambda$, we shall assume the contrary, that is, that there exist $m \ge 0$ and $x \in \Omega$ such that $F^m(\mathrm{graph}(h)) \notin \Lambda$. Since F^m is continuous there exists $\varepsilon > 0$ such that if $\|y - h(x)\| < \varepsilon$ then $F^m(x, y) \notin \Lambda$, but for $n > m$ big enough $\|h_n(x) - h(x)\| < \varepsilon$, and this would imply $F^m(x, h_n(x)) \notin \Lambda$ which is a contradiction. Hence

$$F^m(\mathrm{graph}(h)) \in \Lambda, \quad \forall m \in \mathbb{N}.$$

If $x \in \Omega \cap \mathbb{R}^n = V$ we have, if β is small enough, that

$$\begin{aligned} \mathrm{graph}\, h_{|V} &\subset W^s_{V,r} \cap \{y \in \mathbb{R}^m : \|y\| \le \beta\|x\|\} \\ &\subset W^s_{V,r} \cap \{y \in \mathbb{R}^m : \|y\|_2 \le \sqrt{n}\beta\|x\|_2\} \\ &= \mathrm{graph}\, \varphi. \end{aligned}$$

Therefore, $h_{|V} = \varphi$ which implies that φ is a real analytic function.

9.5 Example 1

A simple example of application of the above theorem is the map $F : \mathbb{R}^{2+1} \to \mathbb{R}^{2+1}$ defined by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x - x^3 + 3xy^2 + f_1(x, y, z) \\ y + y^3 - 3x^2y + f_2(x, y, z) \\ z + q(x, y, z) + g(x, y, z) \end{pmatrix}$$

where $q(x, y, z)$ is an homogeneous polynomial of degree 3, f_1, f_2 and g have order 4 and their derivatives have order 3. Let

$$V(r) = \{(x, y) \in \mathbb{R}^2;\ x \in (0, r),\ 4|y| < |x|\}.$$

We assume that $q(x, y, z) = z\tilde{q}(x, y, z)$ and $\tilde{q}(x, y, 0) > 0$ on

$$V^1(r) = \overline{\{(x, y)/\|(x, y)\|_2 : (x, y) \in V(r)\}}.$$

Below we will check that F satisfies the hypotheses of Theorem 9.4.1. Therefore there exists a stable invariant manifold of the origin given by the graph of an analytic function $\varphi : V(r) \to \mathbb{R}$. Next we check the hypotheses of the theorem.

Let

$$\begin{pmatrix} p_1(x, y, z) \\ p_2(x, y, z) \end{pmatrix} = \begin{pmatrix} -x^3 + 3xy^2 \\ y^3 - 3yx^2 \end{pmatrix}.$$

We have

$$Dp(x, y, z) = \begin{pmatrix} -3x^2 + 3y^2 & 6xy & 0 \\ -6xy & 3y^2 - 3x^2 & 0 \end{pmatrix}.$$

The conditions of hypothesis **H1** read $3x^2 - 3y^2 > |6xy|$ for $(x, y) \in V^1(r)$ (the conditions for p_1 and p_2 coincide). The condition is equivalent to

$$1 - 2y^2 > |2y\sqrt{1 - y^2}| \qquad \text{for } y < 1/\sqrt{17}$$

which is easy to verify that it holds.

Since $D_xq(x, y, z) = zD_x\tilde{q}(x, y, z)$ we have that $D_xq(x, y, 0) = 0$ and $D_yq(x, y, 0) = 0$.

Also $D_zq(x, y, 0) = \tilde{q}(x, y, 0) > 0$ which implies the second condition of **H2**.

Now we deal with **H3**. We have to prove that $\exists A > 0$ such that if $(x,y) \in V(r)$ $\operatorname{dist}(\pi^1 F(x,y,0), V(r)^c) \geq A|(x,y)|^3 = Ax^3$. We estimate the distances of $\pi^1 F(x,y,0)$ to the three parts of $\partial V(r)$:

$$\begin{aligned}
&\{(x,y) : x - 4y = 0, \quad 0 \leq x \leq r\} \\
&\{(x,y) : x + 4y = 0, \quad 0 \leq x \leq r\} \qquad \text{and} \\
&\{(x,y) : x = r, \quad |y| \leq 1/4\}.
\end{aligned}$$

We have that for $x - 4y > 0$ and $x > 0$

$$\begin{aligned}
\operatorname{dist}(\pi^1 F(x,y,0), X - 4Y = 0) &= \frac{x - x^3 + 3xy^2 - 4(y + y^3 - 3yx^2)}{\sqrt{17}} \\
&= \frac{x(1 - x^2 + 3y^2) - 4y(1 + y^2 - 3x^2)}{\sqrt{17}} \\
&\geq \frac{x(2x^2 + 2y^2)}{\sqrt{17}} \\
&\geq \frac{2}{\sqrt{17}} x^3 > 0
\end{aligned}$$

which means that $\pi^1 F(x,y,0)$ stays at the same side of $X - 4Y = 0$ than (x,y) and that the distance is $O(\|(x,y)\|^3)$.

Also, if $x + 4y > 0$ and $x > 0$

$$\begin{aligned}
\operatorname{dist}(\pi^1 F(x,y,0), X + 4Y = 0) &= \frac{x - x^3 + 3xy^2 + 4(y + y^3 - 3yx^2)}{\sqrt{17}} \\
&= \frac{x(1 - x^2 + 3y^2) + 4y(1 + y^2 - 3x^2)}{\sqrt{17}} \\
&\geq \frac{x(2x^2 + 2y^2)}{\sqrt{17}} \\
&\geq \frac{2}{\sqrt{17}} x^3.
\end{aligned}$$

The third distance we must compute is, if $-x + r > 0$ and $x > 0$

$$\begin{aligned}
\operatorname{dist}(\pi^1 F(x,y,0), -X + r = 0) &= -x + x^3 - 3xy^2 + r \\
&\geq x(x^2 - 3y^2) \geq \frac{13}{16} x^3 > 0.
\end{aligned}$$

the conditions of the hypothesis **H4** is equivalent to

$$3[(-3x^2 + 3y^2 + 6|xy|] < 6|xy| + 3y^2 - 3x^2, \qquad (x,y) \in V^1(r)$$

which is satisfied since $6|xy| + 3y^2 - 3x^2$ is strictly negative.

9.6 Example 2

The second example is the elliptic three body problem. It consists in the study of the motion of three bodies of masses $1-\mu$, μ, 0, with $\mu \in (0,1)$. The first two bodies, called primaries, move on ellipses of eccentricity e and semimajor axis a in a plane. The third body moves in the plane of motion under the effect of the attraction of the two primaries. In a fixed system of coordinates X, Y we use complex variables notation $Z = X + iY$. The formulae $Z_1 = z_1 e^{if}$, $Z_2 = -z_2 e^{if}$ with

$$z_1 = \frac{\mu(1-e^2)}{1+e\cos f}, \quad z_2 = \frac{(1-\mu)(1-e^2)}{1+e\cos f}, \text{ and } \frac{df}{dt} = \frac{(1+e\cos f)^2}{(1-e^2)^{3/2}}$$

describe the position of the primaries. The motion of the third body is governed by the equation

$$\ddot{Z} = -(1-\mu)\frac{Z-Z_1}{R_1^3} - \mu\frac{Z-Z_2}{R_2^3}$$

where $R_1 = \|Z - Z_1\|$ and $R_2 = \|Z - Z_2\|$. McGehee introduced the set of coordinates x, y, ρ, α to study the behavior in a vicinity of infinity. These are defined by

$$Z = \frac{2}{x^2}e^{i\alpha} \qquad \dot{Z} = (y + i\frac{x^2\rho}{2})e^{i\alpha}.$$

The equations become

$$\begin{aligned}
\dot{x} &= -\frac{1}{4}x^3 y \\
\dot{y} &= -\frac{1}{4}x^4 + \frac{1}{8}x^6\rho^2 + F_1 \\
\dot{\alpha} &= \frac{1}{4}x^4\rho \\
\dot{\rho} &= F_2
\end{aligned}$$

where

$$
\begin{aligned}
F_1 &= \frac{1}{4}x^4\left(1-\left(\frac{1-\mu}{\sigma_1^3}+\frac{\mu}{\sigma_2^3}\right)\right)+\frac{1}{8}x^6\cos(\alpha-f)\frac{(1-\mu)\mu(1-e^2)}{1+e\cos(f)}\sigma \\
F_2 &= \frac{\mu(1-\mu)(1-e^2)}{4(1+e\cos(f))}x^4\sin(\alpha-f)(-\sigma) \\
\sigma &= \frac{1}{\sigma_1^3}-\frac{1}{\sigma_2^3} \\
\sigma_1^2 &= 1-z_1x^2\cos(\alpha-f)+\frac{1}{4}z_1^2x^4 \\
\sigma_2^2 &= 1+z_2x^2\cos(\alpha-f)+\frac{1}{4}z_2^2x^4.
\end{aligned}
$$

The set

$$I=\{(x,y,\alpha,\rho,f):x=0\}$$

is called the infinity manifold. The flow extends analytically to it and it is invariant by it. $I_0=I\cap\{y=0\}$ is called the parabolic infinity. It is foliated by periodic orbits which can be labeled by α and ρ. Our objective is to prove that they have an analytic stable invariant manifold.

We will compute the Poincaré map from $f=0$ to $f=2\pi$. For this we calculate $x(f)_{|f=2\pi}, y(f)_{|f=2\pi}, \alpha(f)_{|f=2\pi}$ and $\rho_{|f=2\pi}$. We need some preliminary computations. Evaluating at $f=0$:

$$
\begin{aligned}
\frac{1-\mu}{\sigma_1^3} &= \frac{1-\mu}{(1-x^2(z_1\cos\alpha-z_1^2\frac{1}{4}x^2))^{3/2}} \\
&= (1-\mu)(1+\frac{3}{2}x^2z_1\cos\alpha+O(x^4)) \\
&= 1-\mu+O(x^2)
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\mu}{\sigma_2^3} &= \mu\frac{1}{(1+x^2(z_2\cos\alpha+z_2^2\frac{1}{4}x^2))^{3/2}} \\
&= \mu(1-\frac{3}{2}x^2z_2\cos\alpha+O(x^4)) \\
&= \mu+O(x^2).
\end{aligned}
$$

Therefore

$$\frac{1-\mu}{\sigma_1^3}+\frac{\mu}{\sigma_2^3}=1-\mu+O(x^2)+\mu+O(x^2)=1+O(x^2).$$

We compute σ

$$\begin{aligned}
\sigma &= 1 - \frac{3}{2}x^2 z_2 \cos\alpha + O(x^4) - (1 + \frac{3}{2}x^2 z_1 \cos\alpha + O(x^4)) \\
&= -\frac{3}{2}x^2 \cos\alpha(z_1 + z_2) \\
&= -\frac{3}{2}x^2 \cos\alpha \frac{1-e^2}{1+e} = -\frac{3}{2}x^2 \cos\alpha(1-e).
\end{aligned}$$

We write $(x, y, \alpha, \rho) = (x(0), y(0), \alpha(0), \rho(0))$, evaluated at $f = 0$, and then the Poincaré map is:

$$\begin{aligned}
x(2\pi) &= x - Kx^3 y + h.o.t. \\
y(2\pi) &= y - Kx^4 + h.o.t. \\
\alpha(2\pi) &= \alpha + Kx^4\rho + h.o.t. \\
\rho(2\pi) &= \rho - Cx^6\alpha + h.o.t.
\end{aligned}$$

where

$$K = \frac{\pi}{2}\frac{(1-e)^3 2}{(1+e)^1 2} \qquad C = \frac{3}{2}\mu(1-\mu)(1-e)^2 K.$$

We observe that the fixed points of the Poincaré map are of the form $(0, 0, \alpha_\infty, \rho_\infty)$. This map is not yet in a suitable form. We perform the change of variables,

$$\begin{aligned}
u &= x + y \\
v &= x - y \\
t &= (\alpha - \alpha_\infty)v \\
z &= (\rho - \rho_\infty)v.
\end{aligned}$$

Then, the Poincaré map in this variables is

$$\begin{aligned}
u(2\pi) &= u - Ku\left(\frac{u+v}{2}\right)^3 + h.o.t. \\
v(2\pi) &= v + Kv\left(\frac{u+v}{2}\right)^3 + h.o.t. \\
t(2\pi) &= t + Kt\left(\frac{u+v}{2}\right)^3 + h.o.t. \\
z(2\pi) &= z + Kz\left(\frac{u+v}{2}\right)^3 + h.o.t..
\end{aligned}$$

This map satisfies the hypotheses **H1-H4** of Theorem 9.4.1, when we consider it defined in a neighbourhood of $\mathbb{R} \times \mathbb{R}^3$, and therefore there exists a one dimensional stable invariant manifold of the origin, which can be expressed as the graph of a Lipschitz function $\varphi : (0, r) \to \mathbb{R}^3$ where

$$\begin{aligned} v &= \varphi_1(u) \\ t &= \varphi_2(u) \\ z &= \varphi_3(u). \end{aligned}$$

We note that $v(2\pi) = 0$ if and only if $u = v = 0$. Therefore, $v = \varphi_1(u) \neq 0$. It remains to put the invariant manifold in the originals coordinates. On the invariant manifold,

$$x = \frac{u+v}{2} = \frac{u + \varphi_1(u)}{2} = h(u).$$

We observe that h is a Lipschitz function such that $\operatorname{Lip} h \leq \frac{1}{2} + \operatorname{Lip} \varphi_1 < 1$, therefore h is invertible. In other words, there exists ψ such that $u = \psi(x)$, thus the manifold can be represented as

$$\begin{aligned} y &= \frac{\psi(x) - \varphi_2(\psi(x))}{2} \\ \alpha &= \alpha_\infty + \frac{\varphi_2(\psi(x))}{\varphi_1(\psi(x))} \\ \rho &= \rho_\infty + \frac{\varphi_3(\psi(x))}{\varphi_1(\psi(x))}. \end{aligned}$$

With x belonging to a complex neighbourhood of and interval $(0, r)$.

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