Simple Choreographic Motions of N Bodies: A Preliminary Study

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Abstract

A "simple choreography" for an N-body problem is a periodic solution in which all N masses trace the same curve without colliding. We shall require all masses to be equal and the phase shift between consecutive bodies to be constant. The first 3-body choreography for the Newtonian potential, after Lagrange's equilateral solution, was proved to exist by Chenciner and Montgomery in December 1999. In this paper we prove the existence of planar N-body simple choreographies with arbitrary complexity and/or symmetry, and any number N of masses, provided the potential is of strong force type (behaving like $1/r^a$, $a \geq 2$ as $r \to 0$). The existence of simple choreographies for the Newtonian potential is harder to prove, and we fall short of this goal. Instead, we present the results of a numerical study of the simple Newtonian choreographies, and of the evolution with respect to a of some simple choreographies generated by the potentials $1/r^a$, focusing on the fate of some simple choreographies guaranteed to exist for $a \geq 2$ which disappear as a tends to 1.

Contents

1 Introduction

We will prove the existence of new families of periodic solutions to the N-body problem. In these solutions all N masses travel along a fixed curve in the plane. These solutions are topologically interesting, and pleasing to the eye. See the Figures herein, although it is better to look at animations [1].

The N-body problem with N equal masses concerns the study of the differential equations

$$\frac{d^2x_i}{dt^2} = \nabla_i U(x_1, \ldots, x_N). \tag{1.1}$$

Here $U(x) = U(x_1, \ldots, x_N)$ is the negative of the potential energy. The vectors $x_i \in \mathbb{R}^d$, $i = 1, 2, \ldots, N$ represent the positions of N masses moving in $\mathbb{R}^d$. We will

[1] Some animations, to be run under `linux` or `unix` using `gnuplot`, are available at `http://www.maia.ub.es/dsg`.

only be concerned here with the planar case, $d = 2$. We take all the masses to be equal to 1. We assume that U has the form

$$U(x) = \Sigma_{1\leq i<j\leq N} f(r_{ij}), \tag{1.2}$$

where $r_{ij} = |x_i - x_j|$ is the distance between the ith and jth mass and where the *two-body potential* $f(r)$ is a smooth non-negative function of $r > 0$ which blows up as r tends to 0. The potential is said to be *Newtonian* when $f = c/r$ for some $c > 0$.

A collision-free solution for the N-body problem in which all masses move on the same planar curve with a constant phase shift will be called a *simple choreography.* Lagrange [1772] found a simple choreography in the case $N = 3$, for the Newtonian potential. The three masses form the vertices of an equilateral triangle which rotates rigidly within its circumscribing circle. More generally, place N equal masses on a circle of radius r, so as to form the vertices of a regular N-gon. Rotate this N-gon rigidly about the center of the circle with angular velocity ω. The resulting curve will be a solution to the N-body equations. For concreteness, assume the potential is $f(r) = c/r^a$, $c > o$, $a > 0$. Then the condition on the radius of the circle is

$$r\omega^2 = \frac{ac}{r^{a+1}}\sigma_{a,N} \quad \text{where} \quad \sigma_{a,N} = \Sigma_{j=1}^{N-1}\left(2\sin\frac{j\pi}{N}\right)^{-a}.$$

We will call this the *trivial circular* simple choreography.

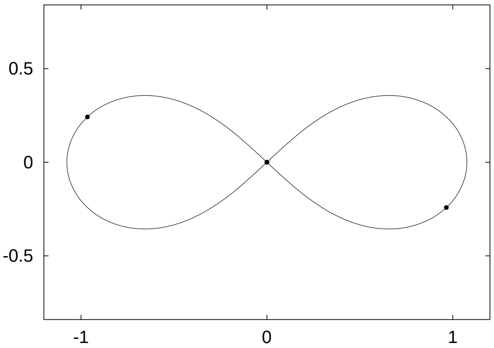


Figure 1: Three bodies on the eight.

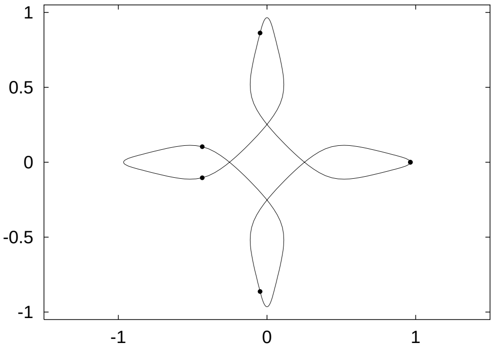


Figure 2: Fives bodies on a 4 petal flower.

In December of 1999 two of us (A. C. and R. M. [1999]) found another simple choreography for the Newtonian three-body problem. In this new solution three equal masses travel a fixed figure eight shaped curve in the plane (see Fig. 1). This figure eight started off a flurry of work. Soon afterwards another one of us (J. G.) wondered whether the circle and figure eight might be generalized to other Lissajous–like curves. He soon found initial conditions for $N = 4$ which led to a simple "chain" choreography in the Newtonian case (see Fig. 3 b). The four masses form a parallelogram at each time instant. Then C. S., the fourth member of our team, found a whole slew of numerical solutions in which all of the bodies move on a single curve, and with quite different shapes of curves (see Fig. 3 and 4). He coined the name "choreography" because of the dance-like movement of the bodies in animations. The qualification "simple" refers to the fact that all of the bodies lie on a single curve, "multiple" choreographies being reserved for solutions where the bodies move on different curves. As this paper deals only with simple choreographies we shall often skip the word "simple". Hundreds of simple Newtonian choreography solutions have now been found, the number of "distinct" choreographies growing quickly as a function of N. The largest N achieved so far is $N = 799$, with the bodies moving on a figure eight curve.

When we say "distinct", we are counting only what we call the "main" choreographies, that is those which are not derived from a given choreography either by travelling around it a multiple number of times (subharmonics) or by a continuation in which the angular momentum is varied, or even by a combination of these two constructions. The precise definition of "main" and "satellite" choreographies, together with examples and counting, will be given in section 5 .

Conjecture 1.1 *For every $N \geq 3$ there is a main simple choreography solution for the equal mass Newtonian N-body problem different from the trivial circular one. The number of such 'distinct' main simple choreographies grows rapidly with N.*

We will prove this conjecture, but only after replacing the Newtonian two-body potential $f = 1/r$ by a strong-force potential, a suggestion which goes back to Poincaré [1896]. For the precise statement, see theorem 2.2 below. We call a potential *strong-force* provided there exist positive constants c, δ such that its two-body potential f satisfies:

$$f(r) \geq c/r^2 \text{ whenever } r < \delta. \tag{1.3}$$

Imposing the strong force condition is a cheap way to get around the main obstruction to proving existence of simple choreographies, which is the existence of finite action collision solutions. Such collision solutions are present in the Newtonian case, and indeed for every $f(r) = 1/r^a$ with $a < 2$. Our real interest is establishing the existence of Newtonian choreographies and we will return to this in future papers.

1.1 Literature

The search for periodic solutions of the N-body problem is more tractable in the case of equal masses than in the general case due to the symmetries of mass interchange. To our knowledge, this observation first appears explicitly (but only in the spatial case) in the paper by Davies, Truman and Williams [1983]. Among other things, these authors were looking for periodic solutions of the equal mass Newtonian N-body problem in $\mathbb{R}^3$ whose configurations were invariant under an orientation-reversing isometry at each instant. After reducing by rotations, they get a two-degrees of freedom system, parameterized by the angular momentum (the symplectic reduction). They look for periodic orbits whose projection to the reduced phase space are "brake orbits". A brake orbit is a periodic solution which traces out the image of an interval, going back and forth, 'braking' to zero (reduced) velocity and changing direction at the endpoints of this interval. In order to give rise to a periodic solution of the unreduced system, the period of the reduced (brake) orbit must be in resonance with the period of rotation of the system. The existence of such resonant brake orbits was only established numerically. Periodic solutions of this kind were rediscovered recently at least twice: the "pelotes", found numerically by Hoynant [1999], which include an a priori infinite set of examples where at least 4 bodies travel on one and the same spatial curve, and the "Hip-Hop", whose existence is proved by Chenciner and Venturelli [1999] as a collisionless minimizer of the action under the appropriate symmetry conditions. The paper by Davies, Truman and Williams was the incentive for the systematic study by Stewart [1996] of symmetry methods in N-body problems with a non-singular potential.

Another important paper, of which we became aware only after our paper was nearly completed, is C. Moore [1993]. Moore investigates the possibility of realizing pure braids on N strands by periodic solutions to planar N-body problems. Simple choreographies correspond to certain special types of braids, hence Moore's paper has close relations to ours. His tool is the gradient flow for the action functional. He obtains the result, rediscovered by R.M. [1998], that for strong-force potentials any "tangled" braid type can be realized. He asserts the existence of the figure eight solution in the Newtonian case, based on a numerical investigation of the convergence of the gradient flow, and he discusses its dynamical stability. He also discusses the dependence of choreography solutions (and their existence or disappearance) on the exponent a of $f = 1/r^a$, thus presaging the discussion of our section 6.

Applications of the figure eight choreography start to appear. Heggie [2000] has numerical evidence that it can appear as an output of the interaction of two couples of binaries.

2 Simple choreographies. The theorem.

We are interested in periodic solutions in which all N masses travel the same curve $q(t)$. The period of these solutions is not important to us. Our proofs work for any period. Furthermore, in the homogeneous potential case, any periodic solution can be scaled to get any desired period. At this point it is convenient for us to take this period to be N. Thus we are searching for solutions to the N-body problem which have the form

$$x_j(t) = q(t + j), j = 0, \dots, N - 1 \tag{2.1}$$

with $q(t) = q(t + N)$ (see Chenciner-Montgomery [2000]).

We will say that a curve has a *collision* if $q(t) = q(t + i)$ for some time t, and some integer i, $i = 1, \dots N - 1$. We want solutions without collisions. With this in mind, let $\mathcal{C} = C^0(\mathbb{S}^1, \mathbb{C})$ be the set of all continuous curves $q : \mathbb{S}^1 = \mathbb{R}/N\mathbb{Z} \to \mathbb{C}$ endowed with the usual C^0-topology. Define the *discriminant locus* $\mathcal{D} \subset \mathcal{C}$ to be the set of all those curves along which there is some collision.

Definition 2.1 *A simple choreography class is a component of* $\mathcal{C} \setminus \mathcal{D}$.

The main theoretical result of this paper is :

Theorem 2.2 *Given any simple choreography class, there is a periodic solution of any planar strong-force N-body problem (see equation (1.3)) which realizes this class.*

Compare with Moore [1993] and Montgomery [1998] in which an analogous result is established for any braid class.

Examples of simple choreographies are given by the Figures in this paper. Most of these are for Newton's (non-strong force) potential. In Simó [2000] several other families are displayed. In section 5 it is proved that the number of "main" simple choreographies increases at least exponentially with N.

2.1 An alternative description

It is illuminating to have another description of simple choreographies (see Chenciner [2000]). The configuration space for the planar N-body problem is $\mathbb{C}^N$. Think of $\mathbb{S}^1 = \mathbb{R}/N\mathbb{Z}$ as a circle of circumference N, drawn in the plane. Inscribe within this $\mathbb{S}^1$ a regular N-gon, with vertices labelled in cyclic order and vertex 0 on the positive x-axis. The image of vertex j under a map $x : \mathbb{S}^1 \to \mathbb{C}^N \setminus \Delta$ is to represent the initial position $x_j(0)$ of mass j, $j = 0, 1, \dots N - 1$. As the N-gon rotates rigidly within the circle, these image points move, thus sweeping out a curve in $\mathbb{C}^N$. Now the group $\mathbb{Z}_N$ acts on $\mathbb{S}^1$ by rotations, taking our standard N-gon to itself, with the standard generator γ of the group acting on a point $t \in \mathbb{S}^1$ by $t \mapsto \gamma \circ \ t = t + 1$. This same generator acts on $\mathbb{C}^N$ by permuting the masses: $x = (x_0, \dots, x_{N-1}) \mapsto \gamma \circ x = (x_1, \dots, x_{N-1}, x_0)$.

Now we make a crucial observation. A map $x : \mathbb{S}^1 \to \mathbb{C}^N$ is equivariant with respect to this $\mathbb{Z}_N$ action, i.e. $x(\gamma \circ t) = \gamma \circ x(t)$, if and only if $x_j(t) = x_0(t + j)$. In other words, the $\mathbb{Z}_N$-equivariant maps into $\mathbb{C}^N$ correspond precisely to closed curves $q : \mathbb{S}^1 \to \mathbb{C}$ in the plane, with the correspondence being given by the equation (2.1) above. This defines a natural correspondence $\mathcal{C} := C^0(\mathbb{S}^1, \mathbb{C}) \leftrightarrow C^0(\mathbb{S}^1, \mathbb{C}^N)_{\mathbb{Z}_N}$, where the subscript $\mathbb{Z}_N$ denotes equivariance with respect to that group. Moreover, a curve is collision-free in our original sense if and only if its corresponding curve in $C^0(\mathbb{S}^1, \mathbb{C}^N)_{\mathbb{Z}_N}$ has no collisions, i.e., no points with $x_i = x_j$ for $i \neq j$, $i, j = 0, \dots N-1$. This establishes a natural correspondence between the space of loops $\mathcal{C} \setminus \mathcal{D}$ of the beginning and the space $C^0(\mathbb{S}^1, \mathbb{C}^N \setminus \Delta)_{\mathbb{Z}_N}$ where $\Delta \subset \mathbb{C}^N$ is the set of all possible collisions between any distinct masses. (Δ is sometimes called the "fat diagonal".) Thus a simple choreography class is the same as a component of the space $C^0(\mathbb{S}^1, \mathbb{C}^N \setminus \Delta)_{\mathbb{Z}_N}$ of collision-free equivariant loops in configuration space.

2.2 Remark on imposing additional symmetries

Various other groups act on $\mathbb{S}^1$ and on $\mathbb{C}^N$. By imposing these as additional symmetries we can obtain beautiful symmetric patterns for our N-body choreography solutions. Fix a finite group Γ containing $\mathbb{Z}_N$, and acting on both $\mathbb{S}^1$ and on $\mathbb{C}^N$ in such a way that it preserves the Lagrangian, and such that the restriction of the action to $\mathbb{Z}_N$ agrees with the previously defined action of $\mathbb{Z}_N$. Replace $C^0(\mathbb{S}^1, \mathbb{C}^N \setminus \Delta)_{\mathbb{Z}_N}$ by $C^0(\mathbb{S}^1, \mathbb{C}^N \setminus \Delta)_\Gamma \subset C^0(\mathbb{S}^1, \mathbb{C}^N \setminus \Delta)_{\mathbb{Z}_N}$, the space of Γ-equivariant loops. Then the definition of choreography classes extends to yield that of equivariant choreography classes, and our main theorem still holds in the equivariant case.

The examples we have in mind are of the form $\Gamma = \mathbb{Z}_N \times F$ where F is either the dihedral group $D_2 = \mathbb{Z}_2 \times \mathbb{Z}_2$ generated by reflections about the x and y axes, or the dihedral group $D_k, k \geq 2$, defined as the full symmetry group of the regular k-gon, or simply the cyclic group $\mathbb{Z}_k$ of order k generated by a rotation by $2\pi/k$ radians. We need to define their actions. For example, $D_2 = \{1, \sigma, \tau, \sigma\tau\}$ is generated by two commuting elements σ, τ whose square is 1. We may define two different actions of D_2 on the space $C^0(\mathbb{S}^1, \mathbb{C}^N \setminus \Delta)$. In both cases, we let D_2 act on the circle as $\sigma(t) = -t + N/2$, $\tau(t) = t + N/2$. The action on $\mathbb{C}^N$ is defined diagonally from an action on $\mathbb{C}$. In the first case (the bar denotes complex conjugation) we use $\sigma(z) = -\bar{z}$, $\tau(z) = -z$, and in the second $\sigma(z) = \bar{z}$, $\tau(z) = -\bar{z}$.

An example of an equivariant loop for the first action of $F = \mathbb{Z}_3 \times D_2$ is the Lagrange equilateral solution where the three bodies chase each other around a circle. An example of an equivariant loop for the second action is the eight. Note that, on the contrary, the supereight with four bodies (Fig. 3 b) shares the same equivariance class for the group $\mathbb{Z}_4 \times D_2$ as that of the relative equilibrium solution where the four bodies form a rigid square and chase each other around a circle. (These two represent different topological, or choreography classes, however).

Curves with $\mathbb{Z}_N \times \mathbb{Z}_k$ or $\mathbb{Z}_N \times D_k$ equivariance correspond to planar choreographies which enjoy k-fold rotational symmetry. They have the pattern of flowers with k petals (see Fig. 2, 3 c and 4 e), or, when the petals overlap tightly, they look like pictures drawn by the children's drawing toy, the spirograph. We leave the definitions of the corresponding representations of the group to the reader.

3 Proof

Except for the (fundamental) symmetry considerations, the following proof is essentially due to Poincaré [1896], with the following unimportant differences: Poincaré was working with homology instead of homotopy and looked for periodic orbits in a rotating frame.

We use the direct method of the calculus of variations (see Montgomery [2000] for more details). The action for our N-body problem is given by

$$A(x) = \int_0^T [\frac{1}{2}K(\dot{x}(t)) + U(x(t))]dt, \tag{3.1}$$

where $T = N$, $K(\dot{x}) = \Sigma_{i=0}^{N-1}|\dot{x}_i|^2$ and $U(x) = \Sigma_{1\leq i<j\leq N} f(|x_i - x_j|)$, with f as in (1.3). If $U(x) \geq 0$, which we henceforth assume, and if the action of the curve x is finite, then the derivative $\dot{x}$ is square integrable, which is to say that it lies in the Sobolev space $x \in H^1(\mathbb{S}^1, \mathbb{C}^N)$. If x is a critical point of A, and if x has no collisions, then x is a N-periodic solution to (1.1). This is a basic, well-known result in mechanics and in the calculus of variations. Collisions have to be excluded because (1.1) breaks down at collisions, and because the action is not differentiable at paths with collisions, despite some potentials (e.g., the Newtonian one) being regularizable.

According to "the principle of symmetric criticality" (see for example Palais [1979]) this same statement holds for Γ-equivariant paths. More precisely, let Γ be any finite group acting on both $\mathbb{S}^1$ and on $\mathbb{C}^N$ by isometries in such a way that it preserves the potential U. Then Γ preserves the Lagrangian and hence leaves the action unchanged: $A(x) = A(g \circ x)$, for $g \in \Gamma$. Let $H^1(\mathbb{S}^1, \mathbb{C}^N)_\Gamma \subset H^1(\mathbb{S}^1, \mathbb{C}^N)$ be the set of all equivariant paths with square-integrable derivative. Suppose that x is collision-free, and that $dA(x)(v) = 0$ for all $v \in H^1(\mathbb{S}^1, \mathbb{C}^N)_\Gamma$. Then x is a solution to (1.1). The proof proceeds by using reducibility of Γ-representations to show that $dA(x)(v) = 0$ for all $v \in H^1(\mathbb{S}^1, \mathbb{C}^N)_\Gamma$ implies that $dA(x)(v) = 0$ for all $v \in H^1(\mathbb{S}^1, \mathbb{C}^N)$, and that x is a critical point within the bigger loop space $H^1(\mathbb{S}^1, \mathbb{C}^N)$ (a direct proof is given in Chenciner [2000]).

Recall $H^1(\mathbb{S}^1, \mathbb{C}^N) \subset C^0(\mathbb{S}^1, \mathbb{C}^N)$. This is one of the simplest instances of the Sobolev inequalities. The direct method proceeds by fixing a choreography α, that is to say a component of $C^0(\mathbb{S}^1, \mathbb{C}^N)_\Gamma$, intersecting α with the subspace of H^1-paths, and then taking the infimum of the action $A(x)$ over all paths x realizing this choreography. By slight abuse of notation we will use the same symbol α to denote the intersection of the class α with the space of H^1-paths. Set:

$$a(\alpha) = \inf_{x \in \alpha} A(x). \tag{3.2}$$

Then, by definition of infimum, there is a sequence $x_n \in \alpha \subset H^1(\mathbb{S}^1, \mathbb{C}^N \setminus \Delta)_\Gamma$ with $A(x_n) \to a(\alpha)$. The idea is to show that x_n converges to a solution to (1.1), and this solution lies in the interior of α.

The Sobolev inequality $|x(t) - x(s)| \leq \sqrt{\int |\dot{x}|^2 dt}\sqrt{|t-s|}$ shows that the set of all H^1 paths with action A bounded by a fixed constant forms an equicontinuous family. This same argument shows that the length ℓ of any path $x \in \mathcal{C}$ is less than $\sqrt{2A(x)}$. Without loss of generality we may take the center of mass of each of our paths x_n to be identically zero:

$$\Sigma_j x(t+j) = 0. \tag{3.3}$$

An easy argument now shows that the set of all paths in $\mathcal{C}$ with center of mass at the origin, and with bounded length, is a pointwise bounded family. The Arzelà-Ascoli theorem asserts that any bounded, equicontinuous family of paths in $\mathbb{C}$ contains a convergent subsequence. So without loss of generality, we have the existence of a curve x_* such that $x_n \to x_*$ in the C^0-norm.

The crux of the matter is to show that this C^0 limit x_* is collision-free, or what is the same thing, that minimizing sequences cannot tend to the boundary of a component α. If so, this limit is automatically in α. Fatou's lemma $A(x_*) \leq \lim_n A(x_n)$ then shows that x_* is a minimizer, and hence a critical point for the action restricted to Γ-equivariant loops. The principle of symmetric criticality applies, yielding that x_* is a solution realizing the given choreography. That x_* is collision-free follows directly from

Proposition 3.1 *If U is a strong-force potential, then any path with collision has infinite action.*

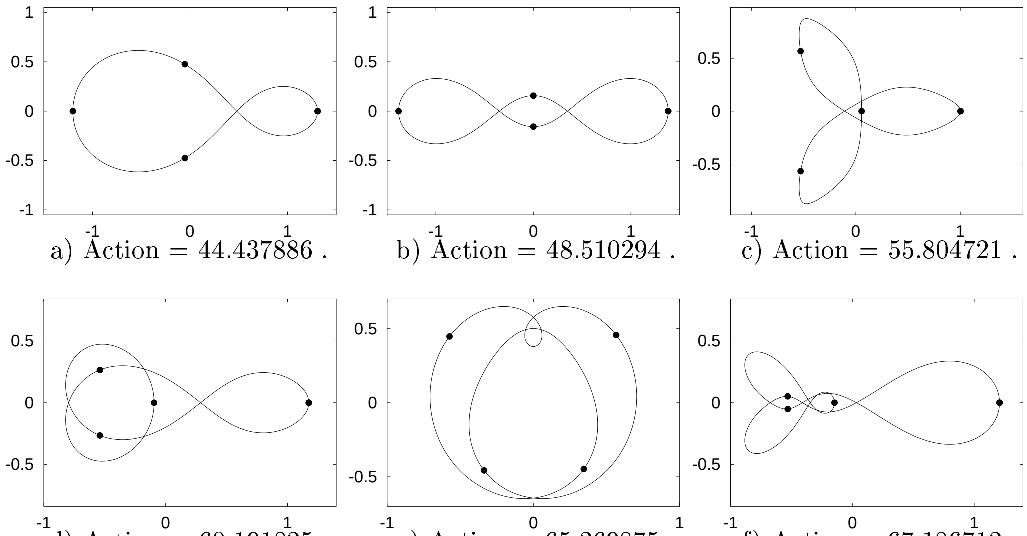
Proof. Suppose the path x suffers a collision, with masses i and j colliding at time t_c. Write r for r_{ij}. The kinetic term K in the action satisfies $K \geq \dot{r}^2$. Since x is continuous, we have $r < \delta$ for some time interval $|t - t_c| \leq \epsilon$ about the collision time. The strong force assumption yields $U \geq c/r^2$ over this interval. Thus the Lagrangian satisfies $L = \frac{1}{2}K + U \geq \frac{1}{2}\dot{r}^2 + c/r^2$. Using $a^2 + b^2 \geq 2ab$ we have that $L \geq \sqrt{2c}|\frac{\dot{r}}{r}|$ for $|t| \leq \epsilon$. But $|\int_{t_1}^{t_2} \frac{\dot{r}}{r} dt| = |\log r(t_2) - \log r(t_1)|$, and $r(t_c) = 0$. From this we conclude that the partial action $\int_t^{t_c+\epsilon} L dt$ diverges at least logarithmically as $t \to t_c$ from above, and so the action of the collision path is infinite. QED

Remark. This proposition is proved in Poincaré [1896] under the stronger assumption of (almost) conservation of energy. More precisely, Poincaré makes use of the fact that at collision, kinetic and potential energy are of the same order of magnitude.

4 Numerical investigations

We concentrate on the Newtonian potential. Easier cases (strong-force $1/r^a$, $a \geq 2$) and harder cases ($1/r^a, 0<a<1$) of homogeneous potentials have also been successfully searched for simple choreographies (see section 6). Except in that section all Figures presented here are of Newtonian solutions. Note, also, that a natural continuation to the logarithmic potential, $f(r) = -\log r$, is possible and it is better done by using $f(r) = 1/(ar^a)$ instead of $f(r) = 1/r^a$ for small positive a, but we shall not report on these results here. All the Figures in the paper represent solutions with period $T = 2\pi$. From now on we set $\mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$ and we shall use $\mathbb{S}^1_N$ for $\mathbb{R}/N\mathbb{Z}$.

For $N = 3$ only Lagrange's equilateral solution, the eight and some satellites of the eight (see section 5) are known. From now on we skip the trivial circular case in which the N masses form a regular N-gon which rotates within its circumscribing circle. Figure 3 presents some simple choreographies for four bodies. The positions of the bodies at some initial time are displayed. The values of the actions are shown. Compare with the action for the circle choreography, $A = 36.613230$. Several examples with $N = 5$ can be found in Simó [2000].

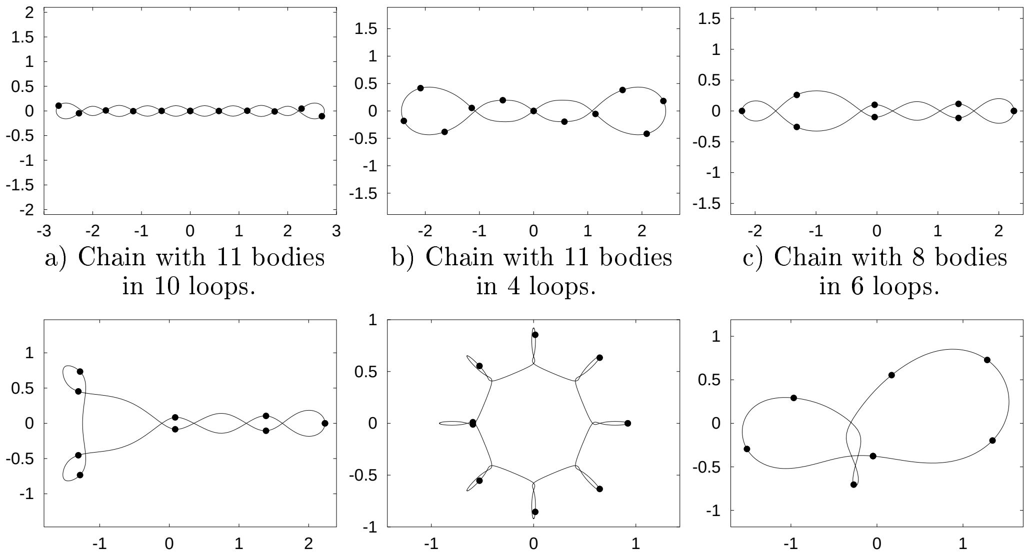


a) Action = 44.437886 . b) Action = 48.510294 . c) Action = 55.804721 .

d) Action = 60.191825 . e) Action = 65.269875 . f) Action = 67.186712 .

Figure 3: Simple choreographies for four bodies under the Newtonian potential.

In Fig. 4 we display some simple choreographies for several values of N. They show just a few of the types found. We refer to Simó [2000] for additional families.



a) Chain with 11 bodies in 10 loops.
b) Chain with 11 bodies in 4 loops.
c) Chain with 8 bodies in 6 loops.
d) Bifurcated chain with 9 bodies
e) 8 petal daisy with 9 bodies.
f) An asymmetric case with 7 bodies.

Figure 4: A sample of different kinds of simple choreographies for the Newtonian potential.

For the numerical computation of simple choreographies, two methods have been used: minimization and Newton's method.

4.1 Minimization methods

These proceed by searching for local minima of A. In general we cannot ensure that the value of the minimum found is $a(\alpha)$, see (3.2). We represent a curve q whose components in $\mathbb{R}^2$ are denoted as (u, v), by an approximation $\hat{q} = (\hat{u}, \hat{v})$ with

$$\hat{u}(t) = \Sigma_{k=1}^{M} a_k \cos(kt) + b_k \sin(kt), \qquad \hat{v}(t) = \Sigma_{k=1}^{M} c_k \cos(kt) + d_k \sin(kt). \quad (4.1)$$

At time t the bodies are located in $q(t + 2\pi j/N)$, $j = 0, \ldots, N-1$, with velocities $\dot{q}(t+2\pi j/N)$. These values are substituted in (3.1). The integral $\int_{\mathbb{S}^1} L(\hat{q}(t), d\hat{q}/dt)dt$ is computed using a trapezoid rule with time step $2\pi/n$, where n is a multiple of the number of bodies, $n = pN$, $p \in \mathbb{N}$. Only the values for $t = t_j = 2\pi j/n$, $j = 0, \ldots, p-1$ are needed, because after $2\pi/N$ each body is shifted to the position of the next one. The approximate value of the action $\hat{A}$ depends on $P = \{a_k, b_k, c_k, d_k, k = 1, \ldots, M\}$ through (4.1). Because of (3.3), with j replaced by $2\pi/N$, all the coefficients with $N|k$ must be zero. By imposing symmetries on a choreography we can further decrease the cardinality of P. Collision-free solutions are analytic and the use of the trapezoid rule is suitable.

In this way we obtain a discretized functional $\hat{A}(P)$. It is minimized by using the gradient method and variants. Several practical problems appear: a) The action is quite "flat" and lots of local minima seem to exist. b) In case we look for a solution having a passage close to collision, the number of harmonics should be large, of the order of several thousands. Both problems slow down the minimization.

Typically the computations have been stopped when two consecutive estimates of $\hat{A}$, all of them being local minima along a search line, differ by less than 10^{-10}. We started with any arbitrary set P or with data obtained after smoothing and filtering a hand drawn curve. As a test of goodness we have used the conservation of the energy and the residual acceleration: the difference between the value given by (1.1) and by using $\frac{d^2}{dt^2}\hat{q}$ at the times $t = t_j$.

4.2 Newton's method

Let $\Phi_{2\pi/N}$ be the flow of (1.1) for a time interval $2\pi/N$. Starting with given values of positions and velocities at $t = 0$, $x_j, \dot{x}_j$, $j = 0, \ldots, N-1$, the transport by $\Phi_{2\pi/N}$ should give the same values, with the indices shifted cyclically by one unit. This gives a set of $4N$ scalar equations, which is solved by Newton method starting at an approximate solution found by minimization. The map $\Phi_{2\pi/N}$ is computed by numerical integration of (1.1). The simultaneous integration of the variational equations is also needed. In most of the cases parallel shooting (see, e.g., Stoer-Bulirsch [1983]) has been required, especially if passages close to collision occur. Typically Newton iterations are stopped when the "closing error" is below 10^{-12}. As a byproduct, linear stability properties have been obtained. Note that these $4N$ equations are not independent. Use has been made of (3.3) and the rotation and time shift invariance to decrease by 6 the dimension of the system to be solved.

The only choreography found to be linearly stable, up to now, for the Newtonian potential is the eight (again, see section 5). Furthermore, on the manifold of angular momentum zero, where the eight lives, the hypothesis of the KAM theorem has been checked to hold by a numerical computation of the torsion (see Simó (2) [2000]).

5 Main choreographies, satellites and linear chains

5.1 On main and satellite choreographies

As we announced in the Introduction, starting with one choreography, we show how to construct, under some hypotheses, a family of new choreographies, by either travelling around the initial choreography a multiple number of times, or by a continuation in which the angular momentum is varied, or by a combination of these two constructions.

This suggests distinguishing between *main* and *satellite* choreographies, as we have done preceding conjecture 1.1. in the first section: a *main* choreography is one which is not the satellite of another one.

5.1.1 Subharmonics

Let us start with a description of the Poincaré map in the neighborhood of an N-periodic choreography $x(t) = (q(t), q(t+1), \cdots, q(t+N-1))$. Recall (subsection 2.1) that the solution $x(t)$ is characterized by the fact that

$$\forall t,\ x(t+1) = Sx(t),$$

where $S : \mathbb{C}^N \to \mathbb{C}^N$ is the isometry of the configuration space defined by

$$S(x_0, x_1, \cdots, x_{N-1}) = (x_1, \cdots, x_{N-1}, x_0).$$

Note that this fact has been strongly used in the numerical methods of the previous section.

Let us fix the energy and the angular momentum to the value they have for our solution. After reduction of the translational and rotational symmetries, we get a $(4N-7)$-dimensional manifold (counting the dimension over $\mathbb{R}$) on which S operates naturally. Let us call Σ_0 a piece of $(4N-8)$-dimensional submanifold transverse to the periodic orbit at a point $(x(t_0), \dot{x}(t_0))$. Let $\Sigma_1, \cdots, \Sigma_{N-1}$ be the images of Σ_0 by $S, \cdots, S^{N-1}$. These submanifolds are transverse to the periodic orbit respectively at points

$$(x(t_0+1), \dot{x}(t_0+1)), \cdots, (x(t_0+N-1), \dot{x}(t_0+N-1)).$$

Let $P_i : \Sigma_i \to \Sigma_{i+1},\ i = 0, \cdots, N-1,$ be the Poincaré maps (of course, $\Sigma_N = \Sigma_0$). One verifies readily that $S \circ P_i = P_{i+1} \circ S$, where S is extended diagonally to $\mathbb{C}^N \times \mathbb{C}^N$. Let us define $P : \Sigma_0 \to \Sigma_0$ by the formula $P = S^{-1} \circ P_0$. One deduces from the above and from the fact that $S^N = Id$, that the first return map $\mathcal{P} = P_{N-1} \circ \cdots \circ P_1 \circ P_0$ to Σ_0 is equal to P^N. This is what characterizes the first return maps along choreographies: *they admit an N-th root* (which is nothing but the return map to the corresponding section in the quotient by S, which acts freely in the neighborhood of a choreography).

Now, for any choreography of N bodies, a subharmonic solution gives rise to a choreography each time it corresponds to a periodic point, say of order k, of $P = \sqrt[N]{\mathcal{P}}$, the Poincaré map of the quotient by S. This is because, lifted to the phase space, such a subharmonic will give rise to a choreography $\tilde{x}(t) \sim x(t),\ t \in [0, \tilde{T}]$, of period $\tilde{T} \sim kN$ (if we chose the period T of $x(t)$ to be equal to N), precisely $\tilde{x}(t + \tilde{T}/N) = S^k \tilde{x}(t)$, *as long as* $(k, N) = 1$. This is because

1) if $(k, N) = 1$, N is a generator of $\mathbb{Z}/k\mathbb{Z}$, so that all the inverse images of the periodic solution of P under the quotient map by S belong to the same periodic orbit;

2) the time spent by this orbit to go from Σ_i to Σ_{i+1} is independent of i because the vector field commutes with S.

5.1.2 Relative choreographies

Another possibility is to change the angular momentum level. Call a solution of (1.1) of the form $x(t) = (q(t), q(t+1), \ldots, q(t+N-1))$ a *relative simple choreography* of period N if there is a rotation R_β of fixed angle β, such that, for all t,

$$q(t+N) = R_\beta q(t).$$

In a rotating frame with angular velocity β/N, it becomes an honest choreography. If the angle of this rotation is a rational multiple m/d of 2π then x is periodic with period $T = dN$ and

$$X(t) = (q(t), q(t+d), \ldots, q(t+(N-1)d))$$

is a choreography (in the fixed frame) provided there are no collisions, that is if d and N are mutually prime.

Otherwise, the solution is quasiperiodic. If the Poincaré return map of an initial choreography q^0 is nondegenerate, and if that choreography has angular momentum C_0, then according to the implicit function theorem there will exist a family of relative simple choreographies near q^0 with angular momentum C taking any value within an interval about C_0. Most of these will be quasiperiodic, but a dense set will have rational rotational angle.

5.1.3 Satellites of the eight

Let us describe now some basic facts regarding the dynamics near the figure eight choreography (see Simó (2) [2000]). We shall show that it has many "satellite" choreographies. Indeed:

As a fixed point of the above Poincaré map $\mathcal{P}$ (with fixed energy and zero angular momentum), the figure eight orbit is totally elliptic with torsion: the frequencies do change locally when one gets away from the fixed point. Hence there exists a family of periodic points (subharmonics) parameterized by a rational rotation vector, whose components tend to limit values given by the eigenvalues of the Poincaré map $\mathcal{P}$, approximately 0.00842272 and 0.29809253, when we approach the fixed point.

As, for the eight solution, $P = \sqrt[3]{\mathcal{P}}$ is also totally elliptic, this implies the existence of a family of choreographies accumulating to the eight. In turn, some of these choreographies are totally elliptic and the same argument is likely to apply indefinitely, giving also rise to *choreographic solenoids*.

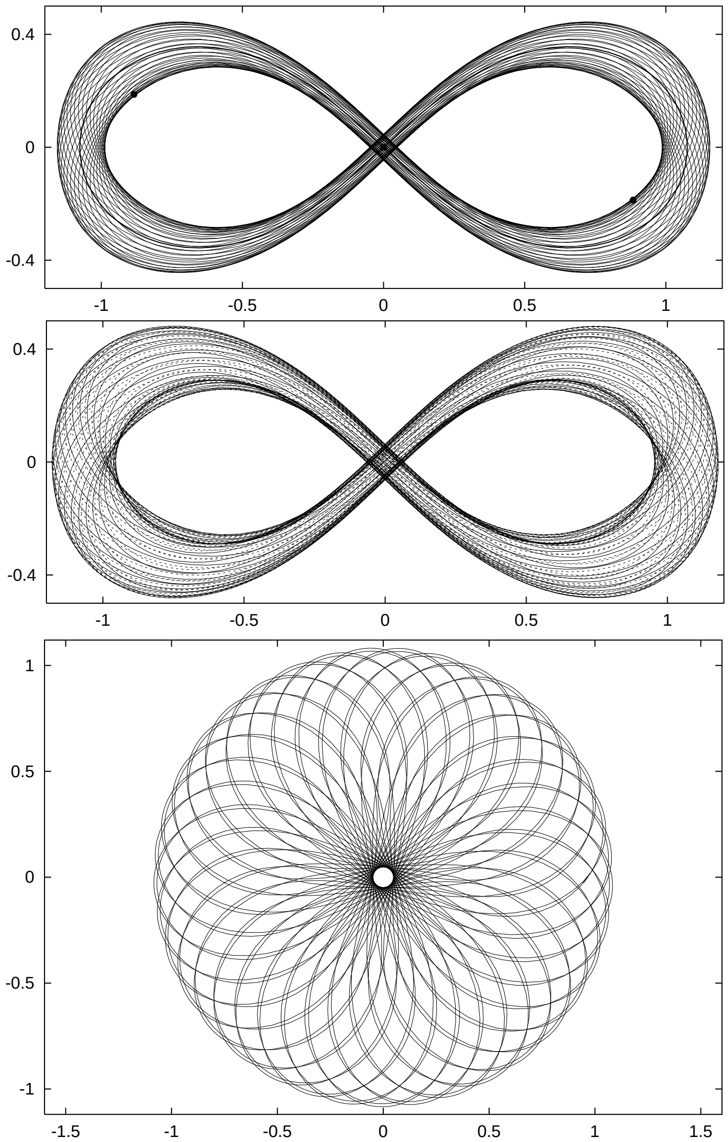


Figure 5: Several examples of satellite orbits, the top and bottom ones being choreographies, but not the middle one. See the text for explanation.

The figure eight periodic solution can also be continued to different angular momenta. As we saw, a possible way to proceed is to use a rotating frame with angular velocity ω. As first found by Hénon [2000], the periodic orbit becomes a distorted figure eight, with the three bodies travelling on the same path in rotating coordinates. For that purpose Hénon used the same program he had been using in Hénon [1976] to continue the collinear Schubart's orbit. According to the preceding discussion, this gives rise to new choreographies, satellites of the eight. If some of these are still totally elliptic – and this will happen for a small enough angular velocity – new satellites of them shall appear, and so on.

Figure 5 shows an illustration of these two possibilities. On the top we display a satellite choreography of the figure eight one, obtained from a periodic point under the Poincaré map around the fixed point and having only component along the fast frequency. Therefore, it lives on a "subcenter" manifold. The rotation number is $11/37 \approx 0.297297297$, quite close to the limit rotation number at the fixed point. Indeed, the variation of the rotation number is quite flat along that mode. All the bodies describe the same path and this seems to be a local minimum of the action. The dots on the Figure (one on the left, one on the right and the third one at the origin) show the initial position of the three bodies. For reference also the path of the figure eight solution is plotted. The middle subfigure shows a satellite orbit with rotation number 8/27. Note that the denominator is now a multiple of 3. The three bodies describe slightly different paths. On the Figure the path of one of the bodies is plotted in continuous lines, while the path of another body is plotted in broken lines. These paths look like curves with rational slope on a torus, slightly shifted the one from the other. To prevent to have too many lines the path of the third body is skipped, but can be clearly seen where it should be.

On the bottom we display a satellite choreography with non–zero angular momentum. This value, $C \approx 0.03125986$, has been selected to have a solution which precesses and closes also after 37 "revolutions" along the eight and 3 full revolutions around the center of mass. That is $m/d = 3/37$. The three bodies describe, again, the same path. Linear stability has been checked for this choreography. Figure 6 displays 1/37 of the period, both in fixed and rotating axes.

In general, we do not need to start with a totally elliptic periodic solution. It is enough that it have some elliptic eigenvalues. For instance, for the choreographies for $N = 4$ shown in Fig. 3, we have that the dimension of the center manifold W^c equals 2 in all cases except d), where it is 4, and e), where it is 0, as it also is for the trivial circle case. (We always ignore the 4 couples of eigenvalues equal to 1 due to the first integrals.)

Remark. To decide if a given choreography is main is not easy. It is not excluded that some of the choreographies presented here as of main type could be related by a family of continuous solutions if we allow for periodic solutions in the complex phase space with the complex period.

By homotoping the potential we can sometimes connect two different main choreographies. This happens to Fig. 3 e where the homotopy parameter is the exponent

a in the potential $1/r^a$. In Simó [2000] another choreography, very similar to 3 e, but having $\dim W^c = 2$, is found by following choreography 3 e upon changing the potential. Both choreographies arise for the Newtonian potential and belong to the same class. See section 6 for further discussion. Note that although we do not allow such potential–varying homotopies in our definition of "satellite" choreography, they are nevertheless useful in understanding choreographies.

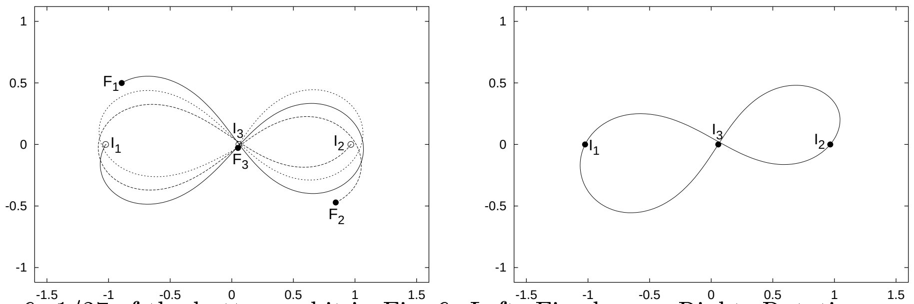


Figure 6: 1/37 of the bottom orbit in Fig. 6. Left: Fixed axes. Right: Rotating axes. The points marked I_j (resp. F_j) for $j = 1, 2, 3$ denote initial (resp. final) conditions.

5.2 The linear chains

Among the simplest choreographies are the "linear chains" formed by different "bubbles". Figures 1, 3 a, 3 b, 4 a, 4 b and 4 c show examples. All of them seem to be of main type. Working in $\mathbb{S}^1_N$ a double point z in a chain (or in a general choreography) has two values of t associated to it, say t_a and t_b. The (integer) length of the loop related to z is defined to be $[t_b - t_a]$ (in $\mathbb{S}^1_N$), where [ ] denotes the integer part. As $t_b - t_a \notin \mathbb{Z}$ the complementary length is $N - 1 - [t_b - t_a]$. A linear chain with J bubbles has $J - 1$ double points all lying on the x-axis, which we will label $z_1, \ldots z_{J-1}$ in order of increasing x-coordinate. Note that, if we try to produce a similar chain without having z_i on the x-axis, a kind of *principle of minimum interaction of the bubbles* leads, by minimization of the action, to a solution as described.

Upon reorienting the loop (reversing time) if necessary, we may assume that the corresponding lengths of the left hand loops defined by these double points yields an increasing sequence of integers: $1 \leq \ell_1 < \ell_2 < \ldots < \ell_{J-1}$. The values $\ell_1, \ell_2 - \ell_1, \ldots, \ell_{i+1} - \ell_i, \ldots, N - 1 - \ell_{J-1}$ are the lengths associated to the bubbles and characterize the choreography class of a linear chain. Note that if $\ell_{i+1} = \ell_i$ then the bubble between z_i and z_{i+1} can be destroyed without passing through a collision and hence represents the same choreography as a linear chain with one fewer bubble. So we assume that the sequence of lengths is strictly increasing. For completeness we also include the case $J = 1$, i.e., the trivial circular solution, as a linear chain.

Proposition 5.1 *The number of linear chains for N bodies is* $2^{N-3} + 2^{[(N-3)/2]}$

A proof can be found in Simó [2000]. In particular, the number of main choreographies increases exponentially with N.

6 Evolution of the choreographies with the potential

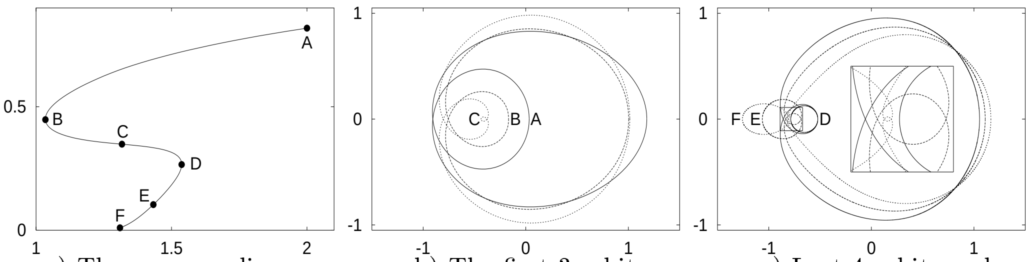
As anticipated, we can take a family of homogeneous potentials with $f(r) = 1/r^a$, $a > 0$ in (1.2). It is reasonable to ask several questions: what happens to a choreography which exists for $a = 1$ when a is decreased approaching zero? What is the fate of a choreography which exists for $a = 2$ but fails to exist for $a = 1$? Are the difficulties encountered in trying to prove the conjecture for the Newtonian potential just technical, or are they deeper? It has already been said that from a given choreography for $a = 1$ it is possible to find, by continuation with respect to a, another choreography also for $a = 1$. The main goal of this section is to present several numerical results in these directions.

The eight can be continued without difficulty to any value of $a > 0$. It is found to be stable in a short domain, roughly $a \in [0.86, 1.23]$. Concerning the cases $N = 4$ given in Fig. 3, decreasing a they reach a *saddle–node* bifurcation (*s-n* for short) and the continuation of the family is only possible by increasing a again. With the exception of the mentioned case e), all of them seem to approach a collision (either a single double collision, several double collisions or a triple collision) before reaching again $a = 1$. Case c) displays a short stability interval around $a = 0.63$.

For $N = 5$ similar things happen. Most of the cases go to a collision (a quadruple collision being now possible), after or before reaching a *s-n*. Some cases present several *s-n* before approaching a collision, the variation of a being monotone between two successive *s-n*. A couple of cases return to $a = 1$, after having a *s-n* for $a < 1$, before approaching a collision. On the other side the example with 5 bodies on a symmetrical eight can be continued to any value $a > 0$, while the linear chain with $J = 4$ (a super-super-eight) can be continued up to $a \simeq 0.0288854$, where it has a *s-n* and a starts to increase again.

Now let us consider a choreography with $N = 4$ which seems not to exist for the Newtonian potential. It should look like Fig. 3 a but with the small loop inside the larger one. It certainly exists for $a = 2$. Figure 7 shows what happens when a continuation for decreasing a is attempted. As a characteristic of a given choreography we have taken the minimum distance $r_{\min} = \min_{1\leq i<j\leq N,\, t\in[0,2\pi]} r_{i,j}(t)$.

On Fig. 7 a we display the evolution of $r_{\min}$ with a, starting at $a = 2$ (marked as A) and decreasing a. Two *s-n* are seen, marked as B and D. Proceeding along the family a collision is approached at the point marked as F. On Fig. 7 b the three orbits shown correspond to A, B and C in Fig. 7 a, the size of the inner loop decreasing when a does. Next orbits, D to F, are displayed on Fig. 7 c. The magnification shows a clear approximation of F to a binary collision. A very small loop appears for orbit C. It becomes as large as the inner one in D, then part of it moves outside the large loop in E and, finally, most of the small loop appears outside the large one in F. This scenario is frequently observed in the evolution of the action minimization procedure with the Newtonian potential, when it seems that no local minimum exists inside the chosen choreography class.



a) The $a - r_{\min}$ diagram   b) The first 3 orbits   c) Last 4 orbits and a magnification

Figure 7: Evolution of a choreography as a function of a for r^{-a} potentials. See the text.

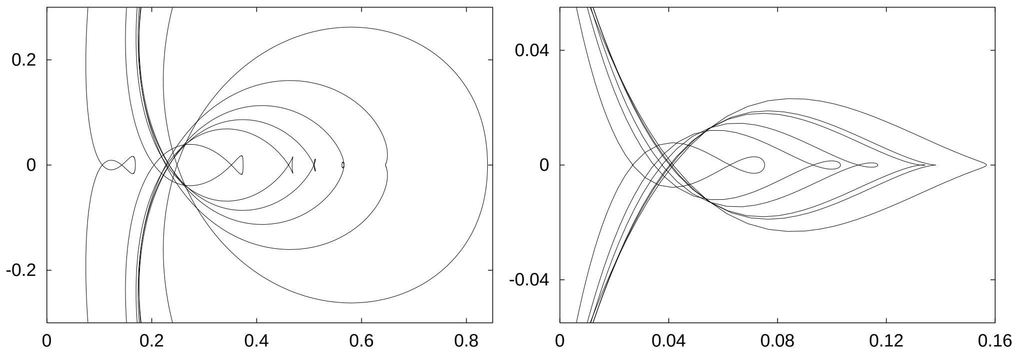


Figure 8: Details of small loops for different N. Left: Inner loops for r^{-a} potentials with values of a for which a s-n occurs. Right: Outer loops for the Newtonian potential. See the text for additional explanations.

The case $N = 4$ with a small loop inside is not an exception. Instead of $N = 4$ we can take $N > 4$ and ask for a small loop of integer length $[\ell] = 1$ inside a large loop. Only these two loops are requested for the choreography. In all cases the behavior seems to be the same one. Figure 8 a displays what happens for the equivalent of point B in Fig. 7 a, i.e., the first s-n encountered when we evolve from $a = 2$ downwards. In this Figure we show the small inner loops for several values of N: 4 to 8, 12 and 36, the loops going to the left for increasing N. To be able to put them on the same window we have added to each curve the coordinates of the rightmost point in the large loop. For $N = 4$ we have in Fig. 8 a the same loop shown in Fig. 7 b with label B. Note the evolution of the tiny loop which is created in Fig. 7 a for increasing N, like a swallowtail unfolding.

Let $a_{N,k}$ be the value of a in the continuation, started going down from $a = 2$ with N bodies, and when the k-th s-n is found. The data corresponding to $N = 4$, displayed in Fig. 7 a, are $a_{N,1} \simeq 1.0344$, $a_{N,2} \simeq 1.5374$. It is quite instructive to look at similar values for other values of N. They are given in the next table. In particular all the $a_{N,1}$ values are greater than 1. This gives an evidence of the lack of existence of this very simple choreography for all N. Furthermore, a tentative guess of the behavior of $a_{N,1}$ as a function of N for increasing N is $a_{N,1} \simeq 1 + c/N^2$, for some $c > 0$. For a value of a slightly larger than 1, the choreography with a small loop of $[\ell] = 1$ inside a large loop should exist for N large enough, while it seems not to exist for $a = 1$. It looks difficult to take into account this tiny difference in an analytical reasoning towards existence proofs.

$a_{5,1} \simeq 1.1720$	$a_{5,2} \simeq 1.3862$	$a_{12,1} \simeq 1.0449$	$a_{28,1} \simeq 1.0096$
$a_{6,1} \simeq 1.1401$	$a_{6,2} \simeq 1.3255$	$a_{16,1} \simeq 1.0273$	$a_{32,1} \simeq 1.0073$
$a_{7,1} \simeq 1.1103$	$a_{7,2} \simeq 1.2914$	$a_{20,1} \simeq 1.0183$	$a_{36,1} \simeq 1.0057$
$a_{8,1} \simeq 1.0887$	$a_{8,2} \simeq 1.2680$	$a_{24,1} \simeq 1.0129$	$a_{40,1} \simeq 1.0046$

On the other side, we can consider the small loop with $[\ell] = 1$, *outside* the large loop, generalizing to arbitrary N the case shown in Fig. 3 a. The shape of the small loop is shown in Fig. 8 b for different values of N: 40, 48, 50, 60, 70 and 100. As before the size of the loop decreases with increasing N, while to keep the loops on the same window we have added the coordinates of the leftmost point on the large loop. The value $N = 48$ has been selected because it is very close to having a cusp point. The evolution of the shape of the small loops is different from the one found in the preceding case.

We finish with a discussion of a different type. Consider an N-gon. It seems to be a global minimum for the action. Assume it is travelled $k > 1$ times. Is it still a local minimum? The simplest counterexample we have found appears for $N = 7$, $k = 2$. Taking small deviations from this solution (of the order of 10^{-3}) the minimization leads to an inner loop with $[\ell] = 3$ inside a larger loop. It looks similar to case A in Fig. 7 b, but the inner loop is closer to the outer one. It has an action $A \simeq 182.326$, while the 7-gon travelled twice has $A \simeq 182.729$. The continuation, starting at $a = 1$, has a s-n for $a \simeq 0.304557$. But it returns to $a = 1$ giving a new choreography in the same class. This one is a saddle of the action functional, with $A \simeq 186.705$. This is also in contrast with the case of Fig. 3 e, for which the choreography in the same class, obtained by continuation, is also a local minimum of the action.

7 Conclusions

Simple choreographies are N-body solutions in which all N masses chase each other around the same curve. We have proved the existence of simple planar choreographies of arbitrary complexity and symmetry for strong-force N-body problems. Most of these choreographies vanish as the strong force potential tends to the Newtonian potential, but still a large number persist. We have investigated this vanishing fact numerically, and have found a large number of individual Newtonian choreographies. An analytic existence proof for the Newtonian choreographies beyond $N = 3$ remains to be found. Which simple choreography classes survive in the Newtonian limit, and what determines whether or not they survive? Is the number of these classes finite? Are all the linear chains with k bubbles, $k < N$, represented? Can the figure eight solution with N bodies be continued for all $a > 0$ and even for the logarithmic potential if N is odd? It is also an open question whether simple choreographies can exist when the masses are not all equal and $N \geq 6$. For $N < 6$ it has been proved (Chenciner (2) [2000]) that all masses must be equal, using the fact that choreography for any set of masses implies choreography for equal masses (the arithmetic mean). The existence of choreographies with arbitrary time intervals (not necessarily equal) is completely open.

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