Kovalevskaya, Liapounov, Painlevé, Ziglin and the Differential Galois Theory

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1 Introduction

At the light of several recent applications of the differential Galois theory, in this review I will try make some remarks of some important concepts connected with integrability of complex analytical dynamical systems started in the seminal works of Kowalevskaya, Liapounov, Painlevé, Picard and Vessiot at the end of the XIX century. More precisely, our objective is to study the deep connection between "integrability" and "singularity theory" of complex analytical dynamical systems. The revival of interest in these problems in the last two decades was apparently motivated by the Ziglin and Adler - Van Moerbeke works in the eighties on integrability of Hamiltonian systems, and by the discovery in the sixties of the inverse spectral method for solving "integrable" non-linear partial differential equations by Gardner, Green Kruskal and Miura and its connection with the singularity theory: Painlevé property, isomodromy deformations, etc... An idea of the amount of papers published in this field is given by the 52 pages of references quoted in the monograph [1].

Although along of this paper essentially there are no new results, in section 4 I state a new result. It is a generalization of the so-called Morales-Ramis theorem about the non-integrability of an analytical Hamiltonian system by means of the variational equations. This was conjectured by the author in [54].

As we will see it is very important to precise carefully the definition of integrability and also of the Painlevé property. The problem here is involved by the fact that a given differential equation can "live" in several different complex analytical manifolds and, of course, the concept of integrability depends on this manifold in a very sensible way. Our guiding idea will be that, as happens with other fields in mathematics, integrability should be defined in a intrinsic geometrical invariant form. The last section is devoted to some conclusions about that, based on previous sections, as well as to some related conjectures.

2 Kovalevskaya and Liapounov

The equations of Euler-Poisson governing the dynamics of a rigid body with a fixed point within a constant gravitational field, in the moving frame of coordinates attached to the principal axes of inertia are given by

$$\dot{M} + \Omega \times M = \mu k \times l, \quad \dot{k} = k \times \Omega, \tag{1}$$

being $M = (p, q, r)$ the angular momentum, $\Omega = (p/A, q/B, r/C)$ the angular velocity, (A, B, C) the principal momenta of inertia, $k = (\alpha, \beta, \gamma)$ the unitary vertical vector, $l = (x_0, y_0, z_0)$ the unitary vector with origin at the fixed point and directed towards the center of gravity and μ the weight of the body multiplied by the distance from the fixed point to the center of masses.

The system (1) is a Hamiltonian system

$$\dot{z} = X_H(z),$$

defined over $\mathbf{C}^6$ with Hamilton function

$$H = \frac{1}{2} M \cdot \Omega + \mu k \cdot l.$$

It has five free *real* parameters (l, A, B, C), $l^2 = 1$ and three classical first integrals:

i) The geometrical integral: $k^2 = 1$.

ii) The energy integral: H.

iii) The vertical angular momentum: $M \cdot k$.

From these integrals it is possible to reduce the system (1) to a two degrees of freedom Hamiltonian system, see [39], pp. 31-32. From a modern point of view this reduction is studied in the context of the Poisson actions of Lie groups on symplectic manifolds and the associated momentum application, see [5], appendice 5.

During a century the only cases of integrability of the Euler-Poisson equations were the isotropic case with $A = B = C$ and the cases of Euler (1758) with $\mu = 0$ and Lagrange (1788) with $A = B$, $x_0 = y_0 = 0$, but S. Kovalevskaya in 1888 obtained a new and highly non-trivial case of integrability for $A = B = 2C$, $z_0 = 0$, [37]. Her method was completely revolutionary: she check the possible cases where the general solution was meromorphic in the finite complex time plane and they are precisely the above four cases of integrability. Furthermore, by using the relatively new theory of abelian functions she shoved that for the new case of integrability, the system linearized on an abelian variety by means of the Jacobi-Abel theorem about the inversion of a system of abelian integrals.

This abelian variety is roughly spiking the complexified of the corresponding Liouville real torus.

The paper of Kovalevskaya was the starting point of the modern theory of algebraically integrable systems, see [70]. Also the very popular "Painlevé test" to identify integrable systems is, in fact, used for the first time in the above work of Kovalevskaya. For this reason I prefer to call it the Kovalevskaya-Painlevé test. We will come back on this test later. In fact Painlevé was critical with the Kovalevskaya result because her method does not detect the possible existence of uniform transcendental singular points, ie, the existence of essential singularities [59]. In other words, perhaps for others values of the parameters the general solution of the system (1) is uniform in the whole complex plane time but not necessarily meromorphic.

About five years later the publication of the Kovalevskaya paper, Liapounov solved the problem pointed out by Painlevé and he proved that *the only cases of the family of systems (1) with a general solution uniform over the whole complex time plane are the four integrable cases above: isotropic, Euler, Lagrange and Kovalevskaya* [41, 40]. In this way Liapounov improved the Kovalevskaya result and furthermore his method was different of the Kovalevskaya one: he used for the first time the Poincaré's variational equation (VE) of the system (1), with respect to the initial conditions, *in complex time* along a suitable particular solution. We recall that if M is a *complex* analytic manifold a holomorphic field X defined over it and Γ the Riemann surface corresponding to an integral curve $z = z(t)$ of the vector field X, then the variational equation along Γ is given by

$$\dot{\eta} = \frac{\partial X}{\partial x}(z(t))\eta.$$

This equation express the linear part of the "flow" of X along any path over Γ. Here we do not enter in the geometrical meaning of the Riemann surface Γ, but this will be important later.

The particular solution $\Gamma : z = z(t)$, $z = (M, k)$, considered by Liapounov was of the type

$$M = \frac{m}{t},\ k = \frac{n}{t^2}, \tag{2}$$

where, of course, the parameters $(m, n) \in \mathbf{C}^6$ related to the initial conditions are not free: by substitution into (1) we get some algebraic constraints between them. Then Liapounov proved that such solutions always exist in two cases: either the parameters (A, B, C) are all different, else among the other parameters (x_0, y_0, z_0) one of them is zero. Notice that in these cases the completed phase curve $\overline{\Gamma}$ is the Riemann sphere, because it is parametrized by rational functions (see the next two sections for the geometrical meaning of Γ or of $\overline{\Gamma}$). Then the variational equation along $\Gamma = \mathbf{P^1} - \{\mathbf{0}, \infty\}$ is a meromorphic linear differential equation over the Riemann sphere $\mathbf{P^1}$ with two singularities $z = 0, \infty$, corresponding to $t = \infty, 0$ (poles of the coefficients), and it is Fuchsian, ie, both singularities are regular singularities. Also it is important for the Liapounov analysis the fact that the solution above (2) is uniform in $t \in \mathbf{C}$.

The next step in Liapounov work was the study of the indicial equation of the variational equation at the singular point $t = 0$, ie, $z = \infty$. This indicial equation is given by an algebraic equation of degree six in the exponents of the solutions of (2). Then it is possible to apply the following lemma.

Lemma 1 *Assume that the general solution of the non-linear system $\dot{z} = X_H(z)$ is uniform in a neighbourhood of the particular integral curve $\Gamma : z = z(t)$. Then also the general solution of the variational equation along Γ, (2) must be uniform.*

Of course, the lemma is valid for any holomorphic field X, but we only consider Hamiltonian fields here. Also, the conclusion of the lemma is true for the higher order variational equations VE_k, not only for the first order variational equation $VE_1 := VE$ (see section 4 for notation and terminology).

For the more general case of higher order variational equations, we can translate the above lemma in modern abstract geometrical terms as the fact that, given a particular integral phase curve Γ of X_H, then the linearization morphism from the holonomy group $Hol(\Gamma)$ of X_H along Γ to the monodromy group $M_k := Mon_k(\Gamma)$ of the variational equation along Γ

$$Lin_k : Hol(\Gamma) \to Mon_k(\Gamma)$$

is exhaustive. This morphism is nothing else that the restriction of the order k jet

$$Diff \to J^k$$

given by the Taylor series up to order k of the flow of X_H in a neighbourhood of Γ along closed loops of $\pi(\Gamma)$. We write J^k for the jet group of order k over $\mathbf{C}^2 n$, being $2n$ the dimension of the symplectic variety M where is defined the Hamiltonian field X_H. We use the notation Lin_k for the morphism above, because the jet group J^k is in fact a linear group (see for instance, [35], section 13).

Notice that the monodromy and Galois group of the variational equation considered by Liapounov is abelian, because the fundamental group of $\pi(\Gamma)$ is abelian too, and the Galois group is the Zariski adherence of the monodromy group. From the abelian structure of the Galois group we get that the above variational equation will be integrable in the sense of the differential Galois theory, see section 4 later.

From the lemma a necessary condition for the general solution of (1) to be uniform is that the indicial equation of the variational equation at $t = 0$ along Γ must have only integer roots $m \in \mathbf{Z}$. Furthermore, for multiple roots $m \in \mathbf{Z}$ of the indicial equation logarithmic terms will not appear. By a careful analysis of this indicial equation, Liapounov was able to prove that this condition is only satisfied for the four already known integrable cases: isotropic, Euler, Lagrange and Kovalevskaya. As for these four cases the general solution was explicitly obtained and it is uniform, then they are the only cases of the Euler-Poisson equations with uniform general solution in the complex time plane.

As a conclusion we observe that the Liapounov result on the rigid body was obtained in the following way:
1. We start from a *uniform* particular solution representing the Riemann surface Γ.
2. The monodromy group of the variational equation along Γ is trivial.
3. The necessary conditions for uniformity obtained from the steps 1 and 2 above are also sufficient because the general solution for the systems which satisfies the conditions 1 and 2 are explicitly known and they are uniform for these cases.

Of course, it is possible to apply the Liapounov method to other systems and we can generalize his method including the consideration of the higher orders variational equations in the following way.

Proposition 1 *Let the analytical complex autonomous dynamical system $\dot{z} = X(z)$ be defined over the complex analytical manifold M. Assume that the general solution of the above dynamical system is single valued over the complex plane and let $\Gamma := \{z = z(t)\}$ be a single valued particular solution. Then the variational equation of order k, VE_k must be trivial for any $k \in N$.*

We will come back to the higher order variational equations in section 4, where we send to the reader for notation and terminology and then we will state a non-integrability theorem by means of the differential Galois theory. For the moment we observe that if we are able to prove the triviality of the monodromy group of the first variational equation, $VE := VE_1$, then the problem in order to prove the triviality of the monodromy group of the variational equation of order k, VE_k, is the non-existence of logarithmic terms when we solve, VE_2, VE_3, ..., by the method of variation of constants from the first order variational equation, ie, some suitable residues must be zero. A similar method will be used in section 5 for proving the non-integrability of a Hénon-Heiles system.

3 Ziglin's Monodromy Theorem

In 1982 Ziglin [81] proved a non-integrability theorem for complex analytical Hamiltonian systems. He used the constrains imposed by the existence of some first integrals on the monodromy group of the normal variational equation along some complex integral curve. This result concerns branching of solutions: the monodromy group expresses the ramifications of solutions of the normal variational equation in the *complex* domain. Then, it can be considered as an extension of the Liapounov method in the sense that as Liapounov, Ziglin also considers the ramification of the solutions given by the variational equations along particular solutions, but in fact as we known now it is a non-integrability criterion basically given by the non-integrability of the variational equation, whereas Liapounov considers an integrable VE, see next section. Furthermore, it is geometric as it does not depend on the choice of the parametrization on the Riemann surface defined by the particular solution, for instance, on the temporal parametrization. This last remark will be important later in connection with the Painlevé property.

Let M be a *complex* analytic symplectic manifold of complex dimension $2n$ with a holomorphic Hamiltonian system X_H defined over it. Let Γ be the Riemann surface corresponding to an integral curve $z = z(t)$ which is not an equilibrium point of the vector field X_H.

Over M a base of the topology is defined by the sets $\{\varphi(t,z) : t \in U, z \in W\}$, being $\varphi(t,z)$ the flow of X_H, U an open set of $\mathbf{C}$, W a neighbourhood of M. Then the Riemann surface Γ defined by the integral curve $z = z(t)$ with initial conditions (t_0, z_0) is, *by definition*, the connected component of z_0 for the above topology.

Here several remarks are in order. These remarks will be important later in the last section. The above definition of Γ is nothing else that the classical definition of the Riemann surface defined by an analytic curve, [34, 73]. We have a *local* temporal parametrization of Γ: $t \mapsto z(t)$, $t \in\subset U \subset \mathbf{C}$. In some cases it will be possible to extend this map globally to $\mathbf{C}$: $\mathbf{C} \to \Gamma$. For instance, if $\Gamma := \{x^2 + y^2 = 1\}=$ the Riemann sphere $\mathbf{P}^1$ (as an abstract Riemann surface), an obvious global parametrization is given by trigonometric functions defined over $\mathbf{C}$, but the object we are interested in is the geometrical "phase curve" $\Gamma = \mathbf{P}^1$ not the temporal parametrization defined over the Riemann surface $\mathbf{C}$. Another example is given by an elliptic curve $\Gamma = \{y^2 = P_3(x) : P_3$ a polynomial of degree 3 or 4 with simple roots$\}=$ a genus one Riemann surface Π , a complex torus; it is essentially parametrized by the elliptic functions $x = \wp(t)$, $y = \wp'(t)$ (really Π is the completed Riemann surface $\overline{\Gamma}$, defined in the next section). As above, the important geometrical object in the dynamics is Π, not the complex time plane. This Riemann surface will be precisely the phase curve for Hamiltonian systems with Lamé's equation as normal variational equation (NVE), see next two sections. In more abstract terms this two examples are particular cases of the fact that in some cases we have an infinite sheet covering $\mathbf{C} \to \Gamma$ (or $\overline{\Gamma}$), given by transcendental functions, but in most of the cases Γ corresponds to an algebraic curve, ie, it is a compact Riemann surface, despite of the fact that the covering is given by transcendental functions: it is a problem of the temporal parametrization. Then *we can not substitute the Riemann surface* Γ *(or* $\overline{\Gamma}$*) by the complex time plane* $\mathbf{C}$.

Another possible situation is when the phase curve Γ has not a global temporal parametrization. This happens typically when the phase curve Γ is a hyperelliptic curve. There is nothing strange or problematic in the fact that no global temporal parametrization exists over $\mathbf{C}$ for this kind of Riemann surfaces and we can study their associated variational equations without problems.

For instance, that happens in Yoshida's work about the non-integrability of the homogeneous potentials and their generalizations [75, 76, 77, 78, 47, 54]. Of course, we can uniformize this kind of Riemann surfaces over the whole unit disk by automorphic functions (see [66]), but it is not necessary to do that in this context. We come back to this problem in the last section of this review.

For more information about the Riemann surface of integral curves of holomorphic fields, see [46, 54].

Now as in the last section we can write the variational equation (VE) along Γ,

$$\dot{\eta} = \frac{\partial X_H}{\partial x}(z(t))\eta.$$

Using the linear first integral $dH(z(t))$ of the VE it is possible to reduce this variational equation and to obtain the so-called normal variational equation (NVE) which, in suitable coordinates, can be written as a linear Hamiltonian system

$$\dot{\xi} = JS(t)\xi,$$

where, as usual,

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

is the standard symplectic form of dimension $2(n-1)$.

More general, if, including the Hamiltonian, there are k meromorphic first integrals *independent over* Γ and in *involution*, we can reduce the number of degrees of freedom of the VE by k and get the normal variational equation (NVE) that, in suitable coordinates, can be written as above

$$\dot{\xi} = JS(t)\xi,$$

but now being J the matrix of the symplectic form of dimension $2(n-k)$.

Also the variational equation along Γ admits a geometrical intrinsic formulation in terms of linear connections over the tangent bundle restricted to Γ given by the Lie derivative along the field X_H restricted to the phase curve Γ, see [46, 54].

As the NVE is a Hamiltonian linear system, its monodromy group is contained in the symplectic group.

Theorem 1 (Ziglin) *Suppose that an n-degrees of freedom Hamiltonian system has $n-k$ additional meromorphic first integrals, independent of the above integrals over a neighbourhood of Γ, but not necessarily over Γ itself, and assume also that the monodromy group of the NVE contains a non-resonant transformation g. Then, any other element of the monodromy group of the NVE sends eigendirections of g into eigendirections of g.*

We recall that a linear transformation of $Sp(m, \mathbf{C})$ is resonant if there are integers $r_1, ..., r_m$, such that $\lambda_1^{r_1} \cdots \lambda^{r_m} = 1$, with λ_i being its eigenvalues.

We remark that Ziglin's theorem *does not assume complete integrability* of the Hamiltonian system X_H. The n independent first integrals are of two types: k of them that are in involution and *independent over* Γ are used to make the reduction to the NVE only; the other $n-k$ first integrals, not necessarily in involution, imposed strong restrictions to the possible structure of the monodromy group of the normal variational equation. However for two degrees of freedom the Hamiltonian system is completely integrable.

Ziglin in a second paper devoted to applications [82], proved that the Euler-Poisson system of equations has an additional meromorphic first integral -ie, different from the classical ones: the geometric integral, the Hamiltonian and the vertical component of the angular momentum- only in the already known four cases of integrability.

4 The Differential Galois approach

At approximately the same time that were published the works of Kovaleskaya and Liapounov on the rigid body and motivated by the classical Galois theory of algebraic equations, Picard ([61, 62], [63], Chapitre XVII) and, in a more clear way Vessiot in his Ph.D.Thesis ([71]), created and developed the differential Galois Theory of linear differential equations. Then this theory is also called the Picard-Vessiot Theory and was also worked by Kolchin from the forties to the sixties along this century, who introduced the modern algebraic abstract terminology and obtained new important results, see [36] and references therein. The small but nice monograph [32] was during some years the only available monograph on this theory and contributed in a essential way to its diffusion. Other modern references on this field are [10, 12, 14, 43, 67, 69].

In the last 15 years we observe a new revival of interest in the differential Galois theory. This is partially due to the connections and applications to others areas of the mathematics: number theory [11, 33], asymptotic theory [42, 13], Ziglin's non-integrability method [45, 16, 52, 9, 46, 47, 48, 54], etc... Here we are interested in the applications to the Ziglin method and we follow partially [54]. As we shall see, inside the differential Galois theory there is also a very nice concept of "integrability" i.e., solutions in closed form. Furthermore, all information about the integrability of the equation is coded in the identity component of the Galois group: a linear equation is integrable if, and only if, the identity component of its Galois group is solvable. We shall only review the necessary definitions and results of the Picard-Vessiot theory in order to understand the applications to non-integrability.

A differential field K is a field with a derivative (or derivation) $\delta = {}'$, i.e., an additive mapping that satisfies the Leibniz rule. The only case we shall interested here is for $K = \mathcal{M}(\overline{\Gamma})$, the meromorphic functions over a connected

Riemann surface $\overline{\Gamma}$. The reason for this notation will be clear below: $\overline{\Gamma} - \Gamma$ will be the set of singular points of the linear differential equation, ie, poles of the coefficients with $\frac{d}{dt}$ as derivation, being t a local coordinate over the Riemann surface Γ. A particular classical case is when $K = \mathbf{C}(t) = \mathcal{M}(\mathbf{P^1})$ is the field of rational functions, ie, the field of meromorphic functions over the Riemann sphere $\mathbf{P^1}$.

We can define differential subfields and differential extensions in a direct way by requiring that inclusions must commute with the derivation. Analogously, a differential automorphism in K is an automorphism that commutes with the derivative. The field of constants of K is the kernel of the derivative. In the above examples $\mathbf{C}$ is such a kernel. From now on we will suppose that this is the case.

Let

$$\xi' = A\xi, \quad A \in Mat(m, K) \tag{3}$$

be a linear differential equation. We proceed now to associate to (3) the so-called Picard-Vessiot extension of K. The Picard-Vessiot extension L of (3) is an extension of K, such that if $u_1, ..., u_m$ is a "fundamental" system of solutions of the equation (3) (i.e., linearly independent over $\mathbf{C}$), then $L = K(u_{ij})$ (rational functions in K in the coefficients of the "fundamental" matrix $(u_1 \cdots u_m)$). This is the extension of K generated by K together with u_{ij}. We observe that L is a differential field (by (3)). The existence and unicity of the Picard-Vessiot extensions was proved by Kolchin. In the analytical case $K = \mathcal{M}(\overline{\Gamma})$, $' = d/dt$, this result is essentially the existence and uniqueness theorem for linear differential equations and it is already in Vessiot [71].

As in classical Galois theory of algebraic equations, we define the Galois group of (3) $G := Gal_K(L) = Gal(L/K)$ as the group of all the (differential) automorphisms of L which leave fixed the elements of K. Then one of the main results of the theory is that the Galois group of (3) is isomorphic to an *algebraic* linear group over $\mathbf{C}$, ie, to a subgroup of $GL(n, \mathbf{C})$ such that their matrix elements satisfy polynomial equations. The other fundamental result is the Galois correspondence between algebraic subgroups and intermediate extensions.

Theorem 2 *Let L/K be the Picard-Vessiot extension associated to a linear differential equation. Then there is a $1-1$ correspondence between the intermediary differential fields $K \subset M \subset L$ and the algebraic subgroups $H \subset G := Gal_K(L)$, such that $H = Gal_M(L)$ (the extension L/M is a Picard-Vessiot extension). Furthermore, we have*

(i)To the normal subgroups $H \subset G$ correspond the Picard-Vessiot extensions M/K. Then the group G/H is a linear algebraic group and $G/H = Gal_K(M)$.

(ii) Let F be a subgroup of G and K_F the subfield of L given by the elements of L fixed by F. Then $H := Gal_{K_F}(L)$ is the Zariski closure (over the complex field $\mathbf{C}$) of F.

We recall here that the Zariski topology over the field $\mathbf{C}$ is the topology such that the closed sets are precisely the algebraic sets: zeros of polynomial with coefficients in $\mathbf{C}$.

Given an algebraic linear group, there is the maximal connected group which contains the identity, G^0, it is possible to prove that it is an algebraic group called the identity component of G. For people familiarized with the Lie theory, as a group it coincide with the identity component of G considered as a complex analytical Lie group (as G satisfy algebraic equations, in particular, it is a complex analytical Lie group).

As a corollary, when we consider the algebraic closure $\overline{K}$ (of K in L), we obtain $Gal_K(\overline{K}) = G/G^0$, where $G^0 = Gal_{\overline{K}}(L)$ is the identity component of the Galois group G which corresponds to the transcendental part of the Picard-Vessiot extension, i.e., by definition, the extension $L/\overline{K}$ is the maximal transcendental extension between the extensions L/L_1, with L_1 an extension of K. If $\overline{K} = K$ (i.e., if $G = G^0$), we say that L/K is a purely transcendental extension.

We say that *a linear differential equation is integrable* if we can obtain its Picard-Vessiot extension $K \subset L$ and, hence, the general solution, by adjunction to K of integrals, exponentiation of integrals or algebraic functions of elements of K. In other words, there exists a chain of differential extensions $K_1 := K \subset K_2 \subset \cdots \subset K_r := L$, where each extension is given by the adjunction of one element a, $K_i \subset K_{i+1} = K_i(a, a', a'', ...)$, such that a satisfies one of the following conditions:

(i) $a' \in K_i$,

(ii) $a' = ba$, $b \in K_i$,

(iii) a is algebraic over K_i.

Then, it can be proved that *a linear differential equation is integrable if, and only if, the identity component of the Galois group, G^0, is a solvable group.* In particular, if the identity component is abelian, the equation is integrable.

The usual terminology for integrable linear equations is that the associated Picard-Vessiot extension is a Liouville extension [32]. We prefer to use a terminology in agreement with our dynamical approach and with the creators of the theory ([71]).

Furthermore, by a classical theorem attributed to Schlesinger, the relation between the monodromy and the Galois group is as follows.

Let $\overline{\Gamma} - \Gamma$ be the set of singular points of the equation i.e., the poles of the coefficients on $\overline{\Gamma}$. We recall that the monodromy group of the equation is a subgroup of the linear group, given by the image of a representation of the fundamental group $\pi_1(\Gamma)$ into the linear group $GL(m, \mathbf{C})$. This representation is obtained by analytical continuation of the solutions along the elements of $\pi_1(\Gamma)$, see for instance [24]. The monodromy group is contained in the Galois group and if the equation is of Fuchsian class (i.e., it has regular singular points only), then the Galois group is dense in the monodromy group in the Zariski topology, ie, the closed sets are algebraic sets: zeros of polynomials, see [42]. In the general case, Ramis found a generalization of the above and, for example, he proved that the Stokes matrices associated to an irregular singularity belong to the Galois group. So, Ramis generalize the monodromy group to a group M_R which is Zariski-dense in the Galois group, this group contains the usual monodromy, but also the Stokes matrices (see[42, 13] and also [54, 46]).

As an example we study now some simple properties of the Lamé equation that we will need later. For a more complete analysis, see [54] and references therein.

The Weierstrass form of the Lamé equation is

$$\frac{d^2\eta}{dz^2} - (n(n+1)\wp(z) + B)\eta = 0, \tag{4}$$

where $\wp$ is the elliptic Weierstrass function with invariants g_2 and g_3. The equation has four parameters: n, B, g_2, g_3. It is a Fuchsian differential equation on the torus $\overline{\Gamma} = \Pi$ -a genus one Riemann surface- with only one singular point at the origin. Let $2w_1$, $2w_2$ be the two periods of the Weierstrass function $\wp$ and $\mathbf{g}_1$, $\mathbf{g}_2$ their corresponding monodromies in the above equation. If $\mathbf{g}_*$ represents the monodromy around the singular point, then $\mathbf{g}_* = [\mathbf{g}_1, \mathbf{g}_2]$. Of course, the differential field of coefficients is here the field of elliptic functions $K =$ meromorphic functions of periods $2w_1$, $2w_2$.

The Lamé equation historically arise in the context of the potential theory when we try to solve the Laplace equation by separation of variables in a system of confocal coordinates, see [64, 74].

Now it is easy to see that a necessary and sufficient condition for the *total* Galois group of (4) to be abelian is that $n \in \mathbf{Z}$. We sketch the steps of the proof. Indeed, this is a classical well-known necessary and sufficient condition for the monodromy group M of the equation(4) to be abelian. It is clear that, as M is generated by $\mathbf{g}_1$ and $\mathbf{g}_2$, an equivalent condition for the abelianess of M is $\mathbf{g}_* = \mathbf{1}(identity)$, and the indicial equation at the singularity is $\rho^2 - \rho - n(n+1) = 0$, and there is no logarithmic term for n integer. Therefore, as M is Zariski-dense in the Galois group G, this last group must also be abelian.

We observe that for n integer the Lamé equation is integrable, because the Galois group is abelian and its identity component is abelian too. In fact, Lamé and Hermite obtained explicitly the closed form solutions for this case. Here we only are interested in the case of the Lamé solutions. For a complete analysis of the integrability of the this equation see [54] and references therein.

The case considered by Lamé corresponds to n integer with some algebraic restrictions on the other three parameters of (4). Then one of the fundamental solutions of (4) is an elliptic function with the periods $2w_1, 2w_2$ and with a pole at $z = 0$, modulo periods. In other words one of the fundamental solutions η_1 is contained in the coefficient field K and the Picard-Vessiot extension of (4) splits in two steps:

$$K \subset K(\eta_1, \eta_1') \subset K(\eta_1, \eta_1', \eta_2, \eta_2'),$$

being η_2 the other fundamental solution. Now $K(\eta_1, \eta_1') = K$ and the Picard-Vessiot extension is simply

$$K \subset K(\eta_2, \eta_2'),$$

this is nothing else that the classical reduction method of D'Alambert when one particular solution η_1 is known, and η_2 is obtained by a quadrature, see for instance, [27].

From the above it is easy to see that the Galois group is contained in the unipotent triangular group in $SL(2, \mathbf{C})$. Effectively, the monodromy group is contained in the unipotent group, because the action (given by the analytical prolongation around $z = 0$) on η_1 is the identity and the action on η_2 is $\eta_2 + \alpha\eta_1$, with α different from zero if there are logarithmic term in η_2. As the Galois Group is obtained as the Zariski-adherence of the monodromy, it must be either trivial or the unipotent group. Hence, as an abstract algebraic group the Galois group is either trivial else the additive group $(\mathbf{C}, +)$. In any case the Galois group is connected and by the Galois correspondence the Picard-Vessiot extension is a purely transcendental extension, because G/G^0 is trivial. These are all the properties of the Lamé equation we need later.

The differential Galois approach to Ziglin's theory appeared for the first time, independently, in [16] and [45]. The papers [52, 9, 17, 53] followed. A common limitation of all these works is the restriction to Fuchsian variational equations (their singularities must be regular singular). Another problem inherent in Ziglin's original approach is that, except for 2-degrees of freedom, he only assume partial integrability but not complete integrability of the initial Hamiltonian system X_H. In the joint work of the author with Ramis [46, 47, 48](see also [54]) we overcame all these difficulties and we proved the following result.

Let us now consider, as in the Ziglin method, a complex analytic symplectic manifold of dimension $2n$ and X_H a holomorphic Hamiltonian system defined on it. Let Γ the Riemann surface corresponding to an integral curve $z = z(t)$ (which is not an equilibrium point) of X_H and we get the VE and the NVE associated to it. Now, we shall complete the Riemann surface Γ with some equilibrium points and (possibly) points at infinity, in such a way, that the coefficients of the matrix of the VE and of the NVE are meromorphic on this extended Riemann surface $\overline{\Gamma} \supset \Gamma$. So, the field of coefficients K of the VE (and of the NVE) is the field of meromorphic functions on $\overline{\Gamma}$. To be more precise, $\overline{\Gamma}$ is contained in the Riemann surface defined by the *desingularization* of the analytical, but in general singular, curve C in the phase space given by the integral curve $z = z(t)$ with their adherent equilibrium points, the singularities of the Hamiltonian system and the points at infinity, for more details see [46] or [54]. Then by an analysis of the Lie algebra of the Galois group of the VE along a particular solution we obtained the desired result.

Theorem 3 *[46, 54] (i)Assume there are n meromorphic first integrals of X_H in involution and independent in a neighbourhood of the analytical curve C, not necessarily on C itself, and the points at infinity of the VE are regular* *singularities. Then the identity component of the Galois group of the VE and of the NVE are abelian symplectic groups.*

(ii)Ziglin's theorem is valid mutatis mutandis if we replace in it monodromy group by Galois group.

We thank to Churchill who was pointed out [15] that in our results it is implicit the statement (ii) of the above theorem. In particular, Ziglin's theorem is a corollary of that because the monodromy group is contained in the Galois group.

The heuristic idea of the author for the search of (i) of the above theorem -usually called the Morales-Ramis theorem- when he started to thing about the application of the differential Galois theory to the Ziglin method 13 years ago, was that *a necessary condition for integrability of a non-linear dynamical system must be the Picard-Vessiot integrability of the VE along any particular solution* (see [45], pp. 18, 19). Furthermore, this theorem is a typical variant of several theorems, for instance, if the point at infinity gives rise to irregular singularities of the VE, then we only get obstructions to integrability by *rational* first integrals.

In the papers [47, 48] (see also [54]) the above theorem was applied to prove the non-integrability of several systems: homogeneous potentials, Bianchi IX cosmological model and Sitnikov three body problem. In [47] we generalize Yoshida's theorem [75, 76, 77, 78] on the non-integrability of homogeneous potentials in two ways: essentially we get a discrete set of possible integrable cases and furthermore, the result is valid for an arbitrary number of degrees of freedom, whereas Yoshida result was only valid for two degrees of freedom and with a continuum set of values as a necessary condition for integrability. In [48] it was proved the non-integrability of the Bianchi IX problem by *rational* first integrals and we obtain another proof of the non-integrability of the Sitnikov three body problem by *meromorphic* first integrals. In fact, we generalize a little the Moser result on the non-integrability of the Sitnikov system by *holomorphic* first integrals [56].

Some recent works with applications and reviews connected with the above results are [8, 15, 21, 22, 23, 30, 31, 55, 68, 72, 80].

The Ziglin method and the above results are based on a first order obstruction to integrability: the variational equation, *VE*, gives us the linear part of the flow of X_H along the integral curve Γ, but we can also consider the quadratic, cubic, etc... contributions to the flow along Γ, ie, the rest of the jet along Γ. These higher order terms are given by the solutions of the higher order variational equations along Γ and it is well-known that they can be solved by the method of variation of constants in a recurrent way, once we know the solution of the order one variational equation.

C. Simó proposed to the author some years ago the problem of study the possible extension of the first order non-integrability criterion to higher order. The difficulty was that at that time I have a weaker result and then I though that the integrability of initial Hamiltonian system implies *only* the integrability of the variational equations at any order, ie, that the identity component of the Galois group must be solvable. In particular, if the first variational equation is integrable, all the higher variational equations are also integrable, because they are obtained by the method of variation of constants from the first variational equation. Now with (i) of the above theorem the situation is different: the identity component of the Galois group is not only solvable but abelian and I proposed in [54], p.146 the following conjecture.

For a given k, to the variational equation up to order k, $(VE)_k$, we associate the chain of extensions of differential fields

$$K := L_0 \subset L_1 \subset L_2 \subset \cdots \subset L_{k-1} \subset L_k,$$

with K the field of meromorphic functions over $\overline{\Gamma}$ and $L_0 \subset L_k$ is the extension given by the solutions of $(VE)_k$, ie, the jet up to order k of the solutions of the Hamiltonian system over Γ. We know that the first extension of the above chain, $K \subset L_1$, is the Picard-Vessiot extension given by the solutions of the first order variational equation $(VE)_1$, written up to now as VE. It is not difficult to see that the other single extensions $L_s \subset L_{s+1}$ are also Picard-Vessiot extensions because they are obtained by the method of variation of constants, ie, by quadratures , and that the Galois group $Gal_{L_{s-1}}(L_s)$ is an abelian connected algebraic group, see [54],section 2.2 and references therein. From this it is not evident at all that the total extension $K \subset L_k$ is a Picard-Vessiot extension, because a composition of Picard-Vessiot extensions is not, in general, a Picard-Vessiot extension, ie, associated to a linear differential equation. Indeed this is the case.

Proposition 2 *([54] For each $k \geq 1$, the extension $K \subset L_k$ is a Picard-Vessiot extension.*

The key point in this proposition is that the variational equation up to order k, $(VE)_k$, is a linear differential equation *with the same coefficient field K* of meromorphic functions over $\overline{\Gamma}$ as the first order variational equation $(VE)_1$. Also, this proposition is very related with the remarks made in section 2 about the fact that the jet group is a linear group.

Now we can talk about the Galois group G_k : of the Picard-Vessiot extensions of the variational equation of order k,VE_k, and I conjectured that a necessary condition for the integrability of the initial Hamiltonian system is that the identity component of the Galois group $G_k := Gal_K L$ must be abelian for any order k. In others words, the abelian structure given by the Liouville theorem is "projected" onto the identity component of the variational equations at any order. This is clearly a natural generalization of the Morales-Ramis theorem and Ramis found recently how to apply basically the same arguments of the original proof of our theorem to this conjecture and it is now a well-established theorem.

Theorem 4 *Assume there are n meromorphic first integrals of X_H in involution and independent in a neighbourhood of the analytical curve C, not necessarily on C itself and the points at infinity of the first variational equation VE_1 are regular singularities. Then the identity component of the Galois group of the higher order variational equations VE_k are abelian symplectic groups.*

A proof of this theorem will be given in a forthcoming paper of Ramis, Simó and the author [49].

5 The Non-integrability of a Hénon-Heiles system

In 1985 Ito published a paper with applications of Ziglin's theorem to a family of generalized Hénon-Heiles potentials [28]. This family of potentials is given by

$$V(x_1, x_2) = \frac{1}{2}x_1^2 + \frac{1}{3}dx_1^3 + (\frac{1}{2} + cx_1)x_2^2, \tag{5}$$

with $c \neq 0$, d parameters (for $c = 0$ the above potential is separable). From Ito's paper until today this family of potentials was intensively studied for several people.

Ito itself showed that except for $c/d = 0, 1, 1/6, 1/2$, the Hamiltonian systems defined by the above potentials are non-integrable with meromorphic first integrals.For $c/d = 0, 1, 1/6$, it is well-known that the systems are integrable. For $c/d = 1/2$ numerical simulations suggest the non-integrability of the system, but a rigorous proof was up to now missing.

By applying the last theorem of the above section to this system we can prove that effectively for $c/d = 1/2$ it is non-integrable. We give the main ideas underlying the proof.

The Hamiltonian system can be written as

$$H = \frac{1}{2}(y_1^2 + y_2^2) + \frac{1}{2}x_1^2 + \frac{1}{3}x_1^3 + \frac{1}{2}x_1x_2^2. \tag{6}$$

The proof is based on the following two lemmas. Let G_k be the Galois group of the variational equation up to order k, VE_k, along one of the family of elliptic integral curves Γ, parametrized by the energy, and contained in the invariant plane $x_2 = y_2 = 0$. The Riemann surface Γ is a punctured torus and that the first order variational equation VE_1 splits into a direct sum of the tangential variational equation and the normal variational equation, because Γ is contained in an invariant plane. Both of them are of Lamé type. Then it is possible to verify that the Galois group G_1 of the first order variational equation is represented by unipotent triangular matrices of the type

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ \alpha & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \beta & 1 \end{pmatrix};$$

because for both -the tangential and the normal first order variational equations- we fall in the case of integrability of the Lamé equation given by Lamé, where one non-trivial solution of the Lamé equation is an elliptic function, i.e., it belong to the coefficient field K of elliptic functions (see [54],Subsection 2.8.4, or [64, 74]).

Notice that the Galois group G_1 is abelian, for this reason it is not possible to obtain a non-integrability criterion by an analysis up to first order.

As the general solution of VE_1 is obtained by two quadratures, then the extension L_1/K of the first order variational equation is transcendental (this is also easily proved by observing that G_1 is isomorphic to the vector group $(\mathbf{C}^2, +)$: it is connected and for the fundamental correspondence theorem of the differential Galois theory, the extension must be transcendent), and as the extensions L_k/L_1 are also transcendental, we have the following lemma.

Lemma 2 *The Galois group G_k of the variational equation up to order k of the Hamiltonian system (6), along the elliptic integral curves contained in the invariant plane $x_2 = y_2 = 0$, is connected, i.e., $G_k = (G_k)^0$.*

Now the coefficients of the order k variational equations are elliptic functions holomorphic over a punctured torus, and this singularity is a regular singular one of this variational equation (because their solutions are obtained by the method of variations of constants from the solution of the first order variational equation). Then we arrive at the same necessary and sufficient condition for abelianess of the monodromy (and Galois) group for the Lamé equation: the local monodromy $\mathbf{g}_* := [\mathbf{g_1}, \mathbf{g}_2]$ around the singular point must be the identity, where $\mathbf{g_i}$ are the monodromies along the periods of the coefficients.

Lemma 3 *The Galois group G_k of the variational equation of order k, VE_k, of the Hamiltonian system given by (6) is abelian, if and only if, the local monodromy $\mathbf{g}_*$ of the variational equation VE_k, around the singular point of the coefficients is the identity.*

Theorem 5 *The Hamiltonian system defined by the function (6) has no additional meromorphic integral independent of the Hamiltonian.*

Sketch of Proof. By Theorem 4 we need to find some k such that $(G_k)^0$ is not abelian. But by Lemma 2 this is equivalent to saying that G_k is not abelian, which means equivalent, by Lemma 3, to $\mathbf{g}_*$ different from the identity for some k. Then we start with the solution of the first order variational equation, and we solve the second order variational equation by the method of variation of constants, then the third order variational equation by the same method and so on ... If in some step of this process we get a logarithm around the singular point, i.e., a residue different from zero in the Laurent series of some integrand, then $\mathbf{g}_*$ is not the identity for the corresponding variational equation and G^0 is not abelian. A long but straightforward computation gives us a logarithm for $k = 3$.

Q.E.D.

In the above cited forthcoming paper by Jean-Pierre Ramis, Carles Simó and the author [49], the details of this computation will be explained.

6 Integrability and the Painlevé property

From the Hénon-Heiles system studied above we observe the similarity between the variational method to prove non-integrability used here and the Kovalevskaya-Painlevé test to detect "integrable systems": the existence of a logarithm is, in general, an obstruction to integrability, see [19, 65, 1] and references therein. Motivated by that, in this section I try to make some remarks about this connection and also to make some ideas on the meaning of the Painlevé property for the study of integrable systems. For this we need to precise the concept of integrability for complex analytical systems.

6.1 What is integrability?

As very often happens in mathematics, the key point is to state adequate definitions with invariant geometrical meaning. There are at least two classes of dynamical systems for which the integrability is well-defined: the Hamiltonian systems and the linear differential equations.

It seems that integrability is related to the existence of global invariants. These invariants are scalar ones (first integrals), vector invariants (symmetry fields) or, in general, tensor invariants. For finite dimensional Hamiltonian systems integrability means complete integrability in the Liouville sense, in these case we have a tensor invariant -the symplectic form- and a set of scalar invariants in involution. From the duality given by the symplectic form, instead of the scalar invariants, we can consider an abelian set of symmetry fields. Then the system can be integrated by quadratures.

In the definition of integrability of Hamiltonian systems is hidden the degree of regularity allowed to the first integrals. For complex analytical symplectic manifolds, M, it is natural to consider holomorphic or, more general, meromorphic first integrals. Notice that it is very important to precise what is M: if it is compact or not, if it is a neighbourhood of a equilibrium point, etc... For instance, a polynomial Hamiltonian field is defined over $\mathbf{C}^{2n}$, but it is also defined over the projective space $\mathbf{P}^{2n}$, then the meromorphic functions over $\mathbf{C}^{2n}$

are more general objects than the meromorphic functions over $\mathbf{P}^{2n}$: in the last case they are rational functions, for more details see [46, 54].

A widely studied class of integrable Hamiltonian systems are the algebraically integrable ones. For these systems the first integrals are polynomial and furthermore, roughly spiking they can be linearized on abelian varieties, corresponding to the Liouville real tori. It is well-known that *the general solution of an algebraically integrable Hamiltonian system is a meromorphic function over the whole time complex plane.* For instance, the Kovalevskaya top is an algebraically integrable Hamiltonian system. In fact the work of Kovalevskaya on the top was the motivation for the actual definition and works on these systems. A brief introduction to algebraically integrable Hamiltonian systems is found in [39]; for more details see [3, 4, 57, 70].

Despite of the fact that very important families of integrable Hamiltonian systems are algebraically integrable, there are integrable Hamiltonian systems that are not algebraically integrable. May be the most simple ones are given by 1-degree of freedom classical Hamiltonians, $H = 1/2y^2 + V(x)$, being the potential V a polynomial of degree less than 3 three or greater than 4. In the first case the general solution is given by transcendental elementary functions. For the second case we have to invert a *single* hyperelliptic integral, which have a infinite set of sheets. Anyway the geometry in the phase space of these systems is very clear: it is foliated by rational (ie, isomorphic to the Riemann sphere) curves or by hyperelliptic curves. The problem here is that, in the author opinion, the suitable definition of the phase curve, Γ or $\overline{\Gamma}$, for an analytical autonomous system is the one given in section 3, and not only based on the global behaviour of the temporal parametrization. For algebraically integrable Hamiltonian systems, the definition of the phase integral curve coincides with the given in section 3. We illustrate that with the above example of one degree of freedom. If the potential V has degree 3 or 4, the system is algebraically integrable and the general solution is given by elliptic functions in time. The phase curve $\overline{\Gamma}$ is a genus one compact Riemann surface, ie, an elliptic curve and it is isomorphic as Riemann surface to the complex plane *module periods* of the solution $z = z(t)$, but not to its universal covering $\mathbf{C}$.

Some authors conjectured that the fact that the general solution is meromorphic in time is not only a necessary condition, but also a sufficient condition for algebraic integrability. If we consider again the harmonic oscillator, the general solution is holomorphic over the whole complex time, but the system is not algebraically integrable. Here it is forbidden to replace the Riemann surface $\overline{\Gamma} = \mathbf{P}^1$, defined by the general solution, by the plane t module periods (which is isomorphic to the complex plane with a point deleted: $\mathbf{C} - \{0\}$), due to the existence of a essential singularity in the general solution at $t = \infty$.

For autonomous non-Hamiltonian systems I do not know a completely general suitable definition of integrability. Some authors try to define it by the existence of $m - 1$ global first integrals, where m is the dimension of the manifold where the system is defined. But for instance, if the system is Hamiltonian

- and, a priori, we do not know if this is the case- that seems not be the correct definition. Also, may be there are other tensor invariant objects, not only scalar ones, that reduce the system to quadratures; for instance, symmetry fields.

Although a general definition of integrability is today missing, we have some necessary or sufficient conditions in order to test if a definition of integrability is "good" or not:
1)If a system is "solvable by quadratures" then it must be considered as integrable, but now we need to define precisely what means solvable by quadratures. It is a sufficient condition for integrability.
2)If a system have a "chaotic behaviour" in some region of the phase space, then the system must be considered as non-integrable. Also here we have to define the concept of chaos, but it seems there exist some agreement in the mathematical community about this concept: ergodic behaviour, symbolic dynamics, positive entropy, etc... Clearly this dynamical test is a necessary condition for integrability.

I conjectured that in the complex case these two "test" should be equivalents: for complex analytical dynamical systems, solvability by quadratures is a necessary and sufficient condition in order to have regular non-chaotic behaviour over the whole phase space.

There is an agreement on the fact that the definition of integrability of Hamiltonian systems satisfies the sufficient condition 1) above, but it is necessary to be precise on the degree of regularity allowed to the first integrals. It is author's opinion that for complex analytical Hamiltonian systems the correct definition of integrability should be (complete) integrability by *meromorphic* first integrals.

For linear differential equations, integrability is made precise in the context of the differential Galois theory of linear differential equations as it was introduced in section 4. In the analytical case where the coefficients are meromorphic functions over a Riemann surface $\overline{\Gamma}$, it is a geometrical theory with an invariant meaning: the linear differential equation is defined by a meromorphic connection over a meromorphic vector bundle with base $\overline{\Gamma}$, see [54] and references therein. We recall that integrability for these linear differential equations was defined in section 4 as solvability by quadratures in a very precise way.

We remark that it is possible to define integrability in this theory by means of the existence of a suitable set of invariants which satisfies other linear differential equations obtained from the initial linear differential equation. For instance, it is a classical fact from Jacobi and Abel that the determinant of a fundamental matrix of solutions-the Wronskian-satisfies a scalar differential equation, then if it is constant we get a trivial solution of the above scalar equation. From this it is possible to prove that the Galois group of the original equation belong to $SL(n, \mathbf{C})$. This approach to the differential Galois theory by means of invariants is called the Tannakian approach and was based on an original idea that goes back to Grothendieck. An introduction to this with some applications and references is given in [46, 54].

For linear differential equations is very important to precise which is the base Riemann surface $\overline{\Gamma}$. This is the analogous object for linear differential equations to the symplectic manifold M and to the kind of regularity allowed to the first integrals, for Hamiltonian systems. One of the reasons of the importance of $\overline{\Gamma}$ is that the coefficient field is, by definition, the field of meromorphic functions over it and, by the Galois correspondence, to bigger coefficient fields (ie, smaller $\overline{\Gamma}$) correspond smaller Galois groups, see [54]. Hence, the integrability depends in a clear way on this Riemann surface.

If the linear differential equations are given by variational equations along a particular solution of non-linear autonomous systems, it was already defined the Riemann surface $\overline{\Gamma}$. If the linear differential equation is given in other way, the definition of the base Riemann surface depends on the context but, in general, it is important to guide our intuition by the geometrical point of view. For instance, it is possible to consider the Lamé equation (4) as defined over the complex plane $t \in \mathbf{C}$ or over the Riemann sphere $\mathbf{P}^1$, but it is more natural to consider it as defined over a genus one compact Riemann surface Π. The problem here is similar to the fact that classical Fourier analysis is an analysis over the circle S^1 and not over the real line, the circle is a compact Lie group, and in this way Fourier analysis can be generalized to other compact Lie groups.

As an illustration of the problems that can arise if the base Riemann surface of the Lamé equation is considered as $\mathbf{C}$ or $\mathbf{P}^1$, we take some remarks from the paper [18]. In their study of the Chazy equation the authors analyze some transformations of the particular two-parametric family of Lamé equations in Weiertrass's form

$$\frac{d^2\eta}{dz^2} - \frac{36-k^2}{4k^2}\wp(z+k)\eta = 0, \tag{7}$$

with invariants of the $\wp$ function given by

$$g_2 = 0,\ g_3 = -\frac{4ck^4}{(36-k^2)^2},$$

being c and k parameters, with $k \neq 0$. This $\wp$ function has an associated elliptic curve $y^2 = 8k^2/(36-k^2)x^3 + c$. Then by the change of the independent variable, $z \mapsto t$,

$$z = \int^{1-t)^{1/3}} \frac{d\tau}{\sqrt{1-\tau^3}},$$

the equation (7) is mapped onto the hypergeometric equation. Then the authors are very surprised from the fact that the Lamé equation (7) which have an irregular singularity at $z = \infty$, can be mapped to an equation with regular singularities. In other words, the authors consider the Lamé equation as defined over the Riemann sphere $z \in \mathbf{P}^1$, with an infinite number of regular singularities which accumulate to an irregular singularity at ∞.

From a geometrical point of view with the Lamé equation defined on the elliptic curve $y^2 = 8k^2/(36-k^2)x^3 + c$, ie, on the corresponding genus one Riemann surface Π as defined in section 4, the above transformation is given by the pull-back of a branched finite sheet covering between compact Riemann surfaces: $\phi : \Pi \to \mathbf{P}^1$, $z \mapsto t$ (we write z, but really we consider z, modulo periods). Of course, it is a pull-back between two Fuchsian differential equations. The existence of this pull-back that maps the equation (7) onto the hipergeometric equation is allowed by the symmetries of the elliptic curve $y^2 = 8k^2/(36-k^2)x^3 + c$, ie, by the fact that $g_2 = 0$. For a general Lamé equation, (4), we can write the equation over the Riemann sphere with four regular singular points (that is given by the pull-back of other related covering $\varphi : \Pi \to \mathbf{P}^1$), it is the so-called algebraic form of the Lamé equation. For $g_2 = 0$ the map $\phi : \Pi \to \mathbf{P}^1$,$z \mapsto t$, studied in [18] factorize in two maps $\Pi \to \mathbf{P}^1 \to \mathbf{P}^1$, being the first of them φ. Furthermore, the identity component of the Galois group of the corresponding differential equations is preserved by all these coverings. In particular, integrability is preserved by this change of variables, see [54, 46].

Of course, the definition of integrability given in section 4 for linear differential equations satisfy condition 1) above but by my joint research with Peris, under the assumption that there are a finite number of singular points, *it also satisfy condition 2)*, see [51]. I will not enter into the technical details of that, but I try to give heuristic reasons for the chaotic behaviour of the non-integrable linear ordinary differential systems, see the above reference for precise statements. The key point is that by a deep theorem proved by Tits, the Ramis monodromy group M_R, already mentioned in section 4, *contains a free non-abelian group* and a free group is a very rich object. The natural action of the monodromy group associated to non-integrable linear differential equation is the *linear* analogous here of the chaotic Poincaré map of a non-linear differential equation.

I take a well-known example from real dynamical systems in order to convince to the reader that natural actions of *linear* free groups can give rise to very complicated dynamics. The Anosov map on the *real* two dimensional torus $\mathbf{T}^2 = \mathbf{R}^2/\mathbf{Z}^2$ is defined by the action of the matrix

$$\Lambda = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix},$$

on $\mathbf{R}^2$, (mod 1), see [7, 6, 26]. As every body knows the discrete dynamical system defined by the action of the above map on $\mathbf{T}^2$ is chaotic: all the set $\mathbf{T}^2$ is an hyperbolic attractor, it has an infinite number of periodic orbits, etc...

It is possible to look at the Anosov map from another point of view. This map is equivalent to the natural action on the two dimensional real projective space $\mathbf{P}^2_{\mathbf{R}}$ of the linear group M generated by $A, B, C \subset PGL(3,\mathbf{R})$, being $A[x,y,z] = [\Lambda(x,y),z]$, $B[x,y,z] = [x+z,y,z]$, $C[x,y,z] = [x,y+z,z]$. So, we homogenize in order to include the affine action given by the periods. Then using Tit's theorem it is possible to prove that the group M contains a (non-abelian) free group. In fact, by an extension of Tit's theorem otained by Margulis, it contains a free *discrete group* of two generators, see [51] for references. It seems that the Anosov map-and others chaotic linear diffeomorphisms-has never been studied using this approach.

Hence the definition of integrability of linear differential equations with meromorphic coefficients over a Riemann surface, given in the context of the differential Galois theory is a very suitable definition, because it satisfies both conditions 1) and 2) given above. Furthermore, from concrete examples it is well-known that if a linear differential equation depends on parameters, for the majority of the values of the parameters the equation will be non-integrable, the reason is that the solvable connected algebraic groups are a very small class within the total class of connected algebraic groups.

However, in the last decades it was widely assumed that every differential equation reducible to a linear differential equation is integrable. In particular, the linear differential equations themselves are considered also as integrables, see, for instance [65]. The reason behind that seems related to the so-called singularity analysis of differential equations: the Kovalevskaya-Painlevé test and the Painlevé property. We turn now our attention to these concepts.

6.2 The Painlevé property

The Painlevé property, P, for a system of differential equations

$$\frac{dy}{dx} = f(x, y), \tag{8}$$

was defined by Painlevé at the end of the XIX century, as the fact that the critical points of its general solution are fixed singular points, see [59, 60]. By a critical point of a complex function Painlevé understood a point where the function is no single-valued in any neighbourhood of it and a fixed singular point is a singular point of the function f, while movable singular are points which change when the initial conditions are changed. We remark that Painlevé was mainly interested by systems -the algebraic differential systems (8)- which are meromorphic in the"base variable" x and rational in the "fibre variable" y, see the references quoted above. The phase space of an algebraic differential equation must be a fiber bundle with base $\overline{\Gamma}$, a *suitable* Riemann surface where are defined the coefficients as in the case of linear differential equations, and $\overline{\Gamma} - \Gamma$ the set of fixed singular points on the base. Modern references on the P property are in [19, 65, 1].

It is clear that linear differential equations have the P property. In fact, the equations with the P property are the natural generalization of the linear differential equations. There are also non-linear differential equations that satisfies the P, property. For instance, the Riccati equations belong to this class, but they are linearizable in a direct way, ie, by a change of variables which preserve the P property. As was already commented these equations -linear differential equations or reducible to linear ones- are, in general, non-integrables.

There are also genuine non-linear differential equations with the P property. Between them the more widely known families are the six Painlevé transcendents, $P_I - P_{VI}$ obtained by Painlevé himself. In fact the list of these equations was completed by a student of Painlevé, but the essential ideas are from Painlevé. All of them are second order scalar differential equations $y'' = f(x, y, y')$ with f rational, and except for P_I all the other five depends of parameters. It is widely assumed that, in general, these six equations are not reducible *in a direct way* to linear differential equations. Three important properties of these transcendents are the following:

a) They admit a Hamiltonian formulation, see [58].

b) They can be expressed as isomonodromic deformations of some linear differential equations with rational coefficients, see [1, 20, 29] and references therein.

c) Except for P_I all the other five equations admit one-parameter families of solutions by means of special functions, ie, solutions of second order linear equations with rational coefficients. Also for some very special values of the parameters, they have particular rational solutions. See [1].

As the Painlevé transcendents are Hamiltonian systems, then their integrability should be considered in the context of Hamiltonian systems. Most of the people that works in integrability consider the families of Painlevé transcendents as integrable. The opinion of the author is that *in general, they are not integrable.* Although we have not a completely general rigorous proof of this fact, we try to give some arguments in this direction.

In that what follows I *define* the base variable x of the Painlevé transcendents as defined over the Riemann sphere, ie, the base Riemann surface $\overline{\Gamma}$ is the Riemann sphere. In my opinion it is the natural base Riemann surface for the Painlevé transcendents. If we consider $\overline{\Gamma} = \mathbf{C}$, then perhaps some of then are integrable, for instance the general solution of P_I or P_{II} is meromorphic over $\mathbf{C}$ and all the interesting complicated behaviour of their solutions is concentrated at $x = \infty$. This is similar to the problem of integrability for the Airy equation

$$\frac{d^2\eta}{dx^2} = x\eta. \tag{9}$$

It is linear differential equation integrable over $\mathbf{C}$ but non-integrable over $\mathbf{P}^1$ (here integrability means in the sense of the differential Galois theory, as defined in section 4) because of the existence of a non-trivial Stokes phenomenon at $x = \infty$. For an algebraic proof of the non-integrability of (9), see the paper by Kovacic [37]. More general, any differential equation like (9) with an odd polynomial instead x is non-integrable.

By c) above there are particular families of the Painlevé transcendents which reduces to non-integrable linear differential equations, then it seems that in any reasonable sense we should consider these families as non-integrables. We illustrate that with the second transcendent P_{II}. This transcendent is defined by the 1-parameter family

$$\frac{d^2y}{dx^2} = 2y^3 + xy + \alpha, \tag{10}$$

being α a parameter. When $\alpha = 1/2 + n$, with n integer, (10) can be reduced to the Airy equation (9). For instance, for $\alpha = 1/2$, the general solution of the Riccati equation

$$\frac{dy}{dx} = \frac{1}{2}xy + y^2 \tag{11}$$

is a 1-parameter family of solutions of (10). The above Riccati equation is reduced to Airy's equation (9) by the standard change of a logarithmic derivative. In [51] we showed that the Riccati equation associated to a non-integrable linear differential equation with a non-compact Ramis monodromy group have a very complicated dynamics, because this group contains a free discrete group. Here the Ramis monodromy group is non-compact, because it contains the Stokes matrices.

From a) and c) above it will be an interesting exercise to apply Theorem 3 (or Theorem 4) to the Painlevé transcendents which have rational particular solutions in order to obtain non-integrability precise results for these Hamiltonian systems.

As a final remark based on b) above, I proposed a question about the integrability of these transcendents: Which is the relation between the integrability of the Painlevé transcendents -as Hamiltonian systems- and the integrability of the corresponding linear differential equation that define the transcendent by isomonodromic deformation? In my opinion both integrabilities are strongly connected, may be they are equivalent.

A more restrictive class of equations (8) which satisfies the P property are the equations with single-valued general solution. Still inside this class are the equations with meromorphic general solution.

We know that between the systems with meromorphic general solution are the algebraically integrable Hamiltonian systems. The singularity theory and the integrability of these systems is well-established and have a geometrical invariant meaning, by the works of Adler and van Moerbeke [3, 4, 57, 70]. The Kovalevskaya top belong to this class. We remark that, as was remarked by Buium in [12], the algebraically integrable systems are very related to Kolchin's strongly normal extensions. These extensions are differential extensions introduced by Kolchin in order to generalize the Picard-Vessiot extensions and have a nice differential Galois theory. The singularity theory of the strongly normal extensions was formulated in a geometrical way by Buium in the above quoted reference. However the author do not know a reference with a complete study of the relation between the strongly normal extension and the algebraically integrable Hamiltonian systems, although the particular example of the Euler top is studied by Buium in [12]. It is possible that the Kolchin-Buium theory of strongly normal extensions is the more general rigorous theory (ie, formulated in a geometrical way, not only by the local behaviour in time) on the equations which only movable poles singularities. As any strongly normal extension is reduced to a linear differential equation and to abelian functions. Then we can define the integrability of such system as the integrability of the associated linear differential equation. For the algebraically integrable Hamiltonian systems, the associated linear differential equation should be trivial.

We finish this section with some brief comments about the connection between the Kovalevskaya-Painlevé test and the approach to non-integrability by means of the higher order variational equations as was stated in sections 4 and 5. So, in this way we come back at the start of this review in section 2: the relationship between the Kovalevskaya method and the Liapounov method.

We observe that the Kowalevskaya-Painlevé method, as well as their recent generalizations [2, 65, 19], is a local method which is based on an analysis of the general solution around a singular point. The Liapounov method, as was generalized by Ziglin, is a global method based on the study of the first variational along a particular solution, and it gives a rigorous criterion of non-integrability of Hamiltonian systems. But there exist an intermediate criterion between the Kowalevskaya method and the Ziglin method it is the so-called Fuchsian perturbative method. This method is essentially based on the computation of the principal part of the Laurent series of a family of *particular solutions* and on the computation of the principal part of the solutions of the higher order variational equations VE_k along the above family, see [19], section 5.7. This last method was also the one used by Yoshida as a criterion to obtain necessary conditions for the existence of algebraic first integrals in similarity invariant systems by means of the Kovalevskaya exponents [79] and the Liapounov method sketched in section 2 also belong to this category, because the principal part of the solution (2) coincides with the complete solution. This is also the case in the particular solutions considered by Yoshida in [79].

We remark that by the application of Theorem 3 to the homogeneous potentials ([47, 54]), it is possible to prove that, in general, the Kovalevskaya exponents are very special rational numbers with the same denominator provided the system is integrable. Then *this justify the weak Painlevé criterion as a criterion to detect integrable system in the case of the homogeneous potentials for the first time* [80]. We recall that the weak Painlevé criterion considers as "integrable" systems which by a change of variable in time by a power with a rational exponent the new system satisfies the P property, see [65].

Now the method used in Theorem 4 by means of the higher order variational equations can be considered as a global geometrical version of the Fuchsian perturbative method, as well as of the Liapounov-Ziglin method. For instance, it includes the Stokes multipliers as possible non-integrability objects, that is not possible for one method based only on the ramification of the solutions, ie, based only on topological arguments. Furthermore, it gives a rigorous criterion of non-integrability for Hamiltonian systems. This achievement was possible by means of a very powerful theory: the differential Galois theory that started to be constructed at the same time as the research of Kovalevskaya and Liapounov on the top.

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