The primitive function of an exact symplectomorphism

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Abstract. Let $\mathcal{O}$ be the zero-section of the cotangent bundle $T^*\mathcal{M}$ of a real analytic manifold $\mathcal{M}$. Let $F : (T^*\mathcal{M}, \mathcal{O}) \to (T^*\mathcal{M}, \mathcal{O})$ be a real analytic local diffeomorphism preserving the canonical symplectic form $\omega = \mathrm{d}\alpha$ of $T^*\mathcal{M}$, where α denotes the Liouville form. Suppose that $F^*\alpha - \alpha$ is an exact form $\mathrm{d}S$. We prove that:

- We can reconstruct F from S and $f = F_{|\mathcal{O}}$.
- If f is included into a flow, then F can be included into a Hamiltonian flow.

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1. Introduction

We recall that a symplectic structure on a manifold $\mathcal{N}$ is a 2-form ω which is closed and non-degenerate. Symplectic structures appear in Physics. For example, they appear naturally in Mechanics and Optics. In these cases, the symplectic structure has the extra feature of being exact, that is $\omega = \mathrm{d}\alpha$ for some 1-form α. In Mathematics, the natural symplectic form for a cotangent bundle is also exact.

For exact symplectic manifolds, we can consider exact symplectic maps F, that is, maps which satisfy

$$F^*\alpha - \alpha = \mathrm{d}S,$$

where S is a function we will call *primitive function*, and it is unique up to addition of a constant.

These primitive functions are interesting objects since, for instance, in Optics they have the interpretation of the optical length between the points [13] and in Mechanics they are the actions [1]. Moreover, they play an important role in variational calculations. Notably in [14, 15] we have shown how they can be used to provide useful variational principles in cases when there are no twist conditions and the usual variational principles based on (Lagrangian) generating functions do not work. In fact, the variational principles based on primitive functions can be defined in any cotangent bundle. In [14, 15] we have also shown how these variational principles lead to effective algorithms and to other consequences such as converse KAM theory.

The main goal of this paper is to explore the question of to what extent primitive functions determine the maps (Note that, of course, since there are maps F different from identity for which $F^*\alpha = \alpha$, we cannot expect full determination). This is a natural question, not only theoretically but also practically. For example in Optics, the primitive function (the optical length) is directly measurable.

This problem turns out to be intimately related to the problem of interpolation of exact symplectic maps. As it is well known, a time dependent (*Hamiltonian*) function H_t on $\mathcal{N}$ gives rise to a Hamiltonian vector field X_{H_t} by $\mathbf{i}_{X_{H_t}}\omega = -\mathrm{d}H_t$. This vector field can be integrated to a deformation h_t determined by $\frac{\partial h_t}{\partial t} = X_{H_t} \circ h_t; h_0 = \mathrm{id}$. These h_t are exact symplectomorphisms. The interpolation problem is: Given an exact symplectomorphism F to find a Hamiltonian H_t such that the family generated by it satisfies $h_1 = F$. In such a case, we say that F is *Hamiltonian*. The point is that we do not have F, but only its primitive function and some additional information. Appendix B is a digression on the general set up of the interpolation problem.

In both problems, determination and interpolation, given the applications that we have in mind (these primitive functions provide variational principles of great generality) we are interested in them also as a computational tool, so we have developed constructive proofs, and described algorithms that implement them. Indeed, we have implemented most of the algorithms described and used them. The numerical analysis issues associated to the implementation of these methods will appear in [16].

The main result of this paper is:

Theorem 1 *Let $\mathcal{M}$ be a real analytic manifold, and $\mathcal{N} \subset T^*\mathcal{M}$ a tubular neighborhood of its zero-section. Let $S : \mathcal{N} \to \mathbb{R}$ be a real analytic function such that $\mathrm{d}S(0_x) = 0$ at any point 0_x of the zero-section $\mathcal{O}$ of $T^*\mathcal{M}$. Let $f : \mathcal{M} \to \mathcal{M}$ a real analytic diffeomorphism. Then:*

> *There exists a unique real analytic exact symplectomorphism $F : \mathcal{N}_0 \to T^*\mathcal{M}$ defined in a neighborhood of the zero-section $\mathcal{N}_0 \subset \mathcal{N}$ such that:*
>
> - *The primitive function of F is S: $F^*\alpha - \alpha = \mathrm{d}S$;*
> - *The zero-section $\mathcal{O}$ is invariant by F, and $f = F_{|\mathcal{O}}$.*
>
> *Moreover, if f is interpolated by the flow $f_t = f_{t,0}$ of a real analytic time-dependent vector field $X_t \in \mathcal{X}(\mathcal{M})$ (i.e. $f = f_1$), then:*
>
> - *F is Hamiltonian;*
> - *A time-dependent Hamiltonian H_t generating F can be computed constructively from S and X_t.*

Remark: The last statement means that if we write $H_t(z) = \sum_{k\leq 0} H_k(z)t^k$ ($z \in \mathcal{N}$) and $X_t(x) = \sum_{k\leq 0} X_k(x)t^k$ ($x \in \mathcal{M}$), then the coefficients H_k can be computed from S and the coefficients X_k. $\triangleleft$

Remark: The flow $F_t = F_{t,0}$ of the Hamiltonian H_t restricted to the zero-section is given by f_t. $\triangleleft$

Since the zero-section of a cotangent bundle is the universal model of Lagrangian manifold, thanks to Weinstein's Theorem [28], the assumptions of the theorem are not so restrictive. Weinstein's Theorem states that a neighborhood of a Lagrangian manifold is symplectically equivalent to a neighborhood of the zero-section of its cotangent bundle. (Moreover, using a generalized Poincaré's lemma, if the Lagrangian manifold is exact then the symplectomorphism is also exact). Although the result belongs to the smooth category, there are in the literature analytic versions of this theorem [25], where only tori are considered.

Lagrangian manifolds are essential to Hamiltonian/symplectic dynamics. For instance, KAM tori of an exact map (or a Hamiltonian flow) are Lagrangian, and so are the stable and unstable manifolds of a hyperbolic fixed point or a whiskered torus. In these and other cases, one often works in the analytic category.

In summary, the exact symplectic dynamics around an exact Lagrangian manifold can be described by a function defined on a neighborhood of that submanifold (the primitive function of the symplectomorphism) and the dynamics on the submanifold. Moreover, the passage from the representation (primitive function, restricted dynamics) to the dynamics in a neighborhood can be effectively implemented as algorithms.

So, for instance, an important application we have in mind is the description of the dynamics around KAM tori. All the dynamics near a KAM torus can be described by just one function, since the dynamics on the KAM torus is prescribed. This allows us to describe normal forms in a very efficient way (see Appendix A). We note that this description does not need any non-degeneracy conditions of the KAM torus. This is an advantage of the description by primitive functions over the description in terms of more classical generating functions. We recall that degenerate tori play a role in several applications to diffusion [12, 8, 15] and hydrodynamics [7, 9].

2. Definitions

First of all, we are going to introduce the elementary definitions and results about exact symplectomorphisms and Hamiltonian flows. For the sake of simplicity, for the time being all the objects (manifolds, functions, diffeomorphisms, ...) will be C^∞. Later we will work in the analytic category.

2.1. Exact symplectomorphisms

We recall some standard definitions: A *symplectic structure* on a manifold $\mathcal{N}$ is given by a closed 2-form ω ($\mathrm{d}\omega = 0$) which is non-degenerate on each point. We say that $(\mathcal{N}, \omega)$ is a *symplectic manifold* and that ω is its symplectic form. This implies that the dimension of $\mathcal{N}$ is even, $2d$.

If $\omega = \mathrm{d}\alpha$, for some 1-form α on $\mathcal{N}$, we say that ω is exact, α is an *action form* and $(\mathcal{N}, \omega = \mathrm{d}\alpha)$ is an *exact symplectic manifold.* Notice that given an exact ω, we can construct many action forms by adding to the original α any closed 1-form. In what follows, we will assume that one particular action form α is chosen. Indeed in many cases, such as cotangent bundles, there are natural choices.

Example: The *phase space* of a mechanical system is the cotangent bundle of its *configuration space* $\mathcal{M}$ [1]. The dimension of the manifold $\mathcal{M}$ is referred as the number of degrees of freedom, d.

The *Liouville form* is the 1-form defined on each $\rho_x \in T^*\mathcal{M}$ by

$$\alpha_{\rho_x}(X_{\rho_x}) = \rho_x(q_*(\rho_x)X_{\rho_x}),$$

for any $X_{\rho_x} \in T_{\rho_x}T^*\mathcal{M}$. Moreover, it is the unique 1-form on $T^*\mathcal{M}$ that satisfies $\rho^*\alpha = \rho$ for any 1-form ρ on $\mathcal{M}$. Then, $\omega = \mathrm{d}\alpha$ is the *canonical symplectic structure* on $T^*\mathcal{M}$ (and α is an *action form* for ω). In cotangent coordinates in $T^*\mathcal{M}$, $(x, y) = (x_1, \ldots, x_d, y_1, \ldots, y_d) \in \mathcal{U} \times \mathbb{R}^d$, these forms are $\alpha = y\ dx = \sum_{i=1}^d y_i\ dx_i$, $\omega = dy \wedge dx = \sum_{i=1}^d dy_i \wedge dx_i$. ◁

Let $F : \mathcal{N} \to \mathcal{N}$ be a diffeomorphism of $\mathcal{N}$ onto its image. The *exactness equation* associated to F is

$$F^*\alpha - \alpha = \mathrm{d}S. \tag{1}$$

Definition: If the exactness equation is solvable, we say that S is a *primitive function* of F (notice that S is unique up to an additive constant), and hence F preserves the symplectic form ($F^*\omega = \omega$). We say that F is an *exact symplectomorphism* (because the 1-form $F^*\alpha - \alpha$ is not only closed, as it should be for any symplectic map, but also exact). ◁

Remark: We use the name primitive function (cf. [2]) to avoid confusion with the classical generating functions, which are different objects (albeit related). In particular, the primitive function *does not determine the symplectomorphism.* That is, there are many exact symplectomorphisms with the same primitive function. ◁

Definition: If $S = 0$ is a solution of the exactness equation, we say that F is an *actionmorphism*, because it preserves the action form ($F^*\alpha = \alpha$). ◁

Remark: If $\phi : \mathcal{M} \to \mathcal{M}$ is a diffeomorphism then its *lift* $\hat{\phi} \stackrel{\text{def}}{=} (\phi^{-1})^* : T^*\mathcal{M} \to T^*\mathcal{M}$ preserves not only the symplectic form $\omega = \mathrm{d}\alpha$ but also the Liouville form. All the *actionmorphisms* on the whole cotangent bundle can be obtained in this way (see, for instance, [18]). In fact, it is enough for it to be defined in a tubular neighborhood of the zero-section (Proposition 3). ◁

Remark: Given an exact symplectomorphism F, it is determined by its primitive function S up to a composition with an actionmorphism L on the left. That is:

$$(L{\circ}F)^*\alpha - \alpha = F^*L^*\alpha - \alpha = F^*\alpha - \alpha = \mathrm{d}S,$$

and, conversely, if G is another exact symplectomorphism whose primitive function is also S, then

$$\begin{aligned}(G{\circ}F^{-1})^*\alpha - \alpha &= (F^{-1})^*{\circ}G^*\alpha - \alpha = (F^{-1})^*(\alpha + \mathrm{d}S) - \alpha \\ &= -\mathrm{d}(S{\circ}F^{-1}) + (F^{-1})^*\mathrm{d}S = 0.\end{aligned}$$

◁

2.2. Hamiltonian flows

It is well known that to a function $H : \mathcal{N} \to \mathbb{R}$ we associate a *Hamiltonian vector field* X_H, which is uniquely determined by $\mathbf{i}_{X_H}\omega = -\mathrm{d}H$. The *Poisson bracket* between two functions K,H is defined by $\{K,H\} = \omega(X_K, X_H) = -\mathrm{d}K(X_H) = \mathrm{d}H(X_K) = -\{H,K\}$. These operations are natural with respect to pull back by symplectomorphisms. That is, for any symplectomorphism F: $F^*X_H \overset{\text{def}}{=} (F^{-1})_*X_H{\circ}F = X_{H\circ F}$, $\{K,H\}{\circ}F = \{K{\circ}F, H{\circ}F\}$.

Recall that the space of smooth functions $\mathcal{F}(\mathcal{N})$ endowed with the Poisson bracket is a Lie algebra, and the relation between the Lie bracket and the Poisson bracket is given by $X_{\{K,H\}} = [X_K, X_H]$. Keep in mind that $\mathcal{F}(\mathcal{N})$ is infinite dimensional and the Lie algebra operation is discontinuous, so that several of the properties of finite dimensional Lie algebras do not necessarily hold.

In symplectic coordinates (x,y) (those such that $\omega = dy \wedge dx$), we write

$$X_H = \begin{pmatrix} \dfrac{\partial H}{\partial y} & -\dfrac{\partial H}{\partial x} \end{pmatrix}^\top \quad , \quad \{K,H\} = \frac{\partial K}{\partial y} \cdot \frac{\partial H}{\partial x} - \frac{\partial K}{\partial x} \cdot \frac{\partial H}{\partial y},$$

where $\cdot$ is the inner product.

2.3. A derivation associated to the action form

Let H_t be a time-dependent Hamiltonian function (where the subscript t represents the dependence on time), and X_{H_t} be the corresponding time-dependent Hamiltonian vector field. The time-(t,t_0) shift of its flow, φ_{t,t_0}, is an exact symplectomorphism (in the domain where it is defined): $\varphi^*_{t,t_0}\alpha - \alpha = \mathrm{d}S_{t,t_0}$, where

$$S_{t,t_0} = \int_{t_0}^{t} \mathbf{\Lambda}(H_s){\circ}\varphi_{s,t_0}\, ds \tag{2}$$

and $\mathbf{\Lambda}(H) = \alpha(X_H) - H$.

Remark: Applying (2), notice that the flow generated by a Hamiltonian H_t such that $\mathbf{\Lambda}(H_t) = 0$ is just a family of actionmorphisms. ◁

The operator $\mathbf{\Lambda}$ satisfies the following properties.

Proposition 1 *The linear operator* $\mathbf{\Lambda} : \mathcal{F}(\mathcal{N}) \to \mathcal{F}(\mathcal{N})$ *associated to* α, $\mathbf{\Lambda}(H) = \alpha(X_H) - H$, *is a derivation in* $\mathcal{F}(\mathcal{N})$. *Moreover, it is natural with respect to pull back by actionmorphisms.*

Proof: The proof of the product rule follows from

$$\begin{aligned}\alpha([X_{H_1}, X_{H_2}]) &= \mathrm{d}(\alpha(X_{H_2}))X_{H_1} - \mathbf{L}_{X_{H_1}}\alpha\ X_{H_2} \\ &= \{H_1, \alpha(X_{H_2})\} + \{\mathbf{\Lambda}(H_1), H_2\}.\end{aligned}$$

If $L^*\alpha = \alpha$ and $H \in \mathcal{F}(\mathcal{N})$: $\alpha(X_H)\circ L = L^*(\mathbf{i}_{X_H}\alpha) = \mathbf{i}_{L^*X_H}L^*\alpha = \mathbf{i}_{X_{H\circ L}}\alpha = \alpha(X_{H\circ L})$. $\square$

Example: If $\mathcal{N} = T^*\mathcal{M}$, we name the derivation associated to the Liouville form as the *Liouville derivative*. We can write $\mathbf{\Lambda}(H) = \mathrm{d}H(Z^*) - H$, where Z^* is the *Liouville vector field* on $T^*\mathcal{M}$, that is, the only vector field that satisfies the relation $\mathbf{i}_{Z^*}\mathrm{d}\alpha = \alpha$. If H is a Hamiltonian, then $\mathbf{\Lambda}(H)$ is also known as its *elementary action*, because it is used in order to define a variational principle for the orbits of the corresponding Hamiltonian vector field (see, for instance, [1, 18]). It is closely related to the *Legendre transformation.*

In cotangent coordinates $(x, y) \in \mathcal{U} \times \mathbb{R}^d$, the Liouville derivative is written

$$\mathbf{\Lambda}(H)(x, y) = y \cdot \nabla_y H(x, y) - H(x, y).$$

$\mathbf{\Lambda}$ is a vertical operator, because the value of $\mathbf{\Lambda}(H)$ on a fiber only depends on the value of H on such fiber. $\triangleleft$

3. Determination of an exact symplectomorphism

Since an exact symplectomorphism is determined by its primitive function up to an actionmorphism, it will be useful to characterize the actionmorphisms. From this point on, our exact symplectic manifold will be $T^*\mathcal{M}$. An actionmorphism defined in a tubular neighborhood of the zero-section does not move the zero-section, and it is the lift of the diffeomorphism induced in the zero-section. Finally, we can compute all the exact symplectomorphisms that fix the zero-section.

3.1. A condition on the primitive function

Since the zero-section is fixed, certain restrictions apply to the primitive function.

Proposition 2 *Let* $F : \mathcal{N} \to T^*\mathcal{M}$ *be an exact symplectomorphism and* $S : \mathcal{N} \to \mathbb{R}$ *be its primitive function. Then,*

$$F \text{ fixes the zero-section } \Leftrightarrow \forall x \in \mathcal{M}\ \mathrm{d}S(0_x) = 0.$$

Proof: Let $x \in \mathcal{M}$ be any point on the zero-section. Since $\alpha(0_x) = 0$ and

$$(F^*\alpha)(0_x) = \alpha(F(0_x))\circ F_*(0_x) = F(0_x)\circ(q\circ F)_*(0_x),$$

then $\mathrm{d}S(0_x) = F(0_x)\circ(q\circ F)_*(0_x)$. Finally, as $(q\circ F)_*$ is an epimorphism at all points, we obtain the result. $\square$

Remark: Similar results can be obtained for an exact Lagrangian invariant graph for an exact symplectomorphism defined on a cotangent bundle [14, 15]. In particular, we associate a conserved quantity to such a graph, and we can define variational principles for these invariant graphs [14, 15] $\triangleleft$

3.2. The actionmorphisms are lifts

Proposition 3 *Let $L : \mathcal{N} \to T^*\mathcal{M}$ be an actionmorphism. Let $\phi : \mathcal{M} \to \mathcal{M}$ be the dynamics on the zero-section ($\phi = L_{|\mathcal{O}}$). Then, $L = \hat{\phi}$.*

Proof: Consider $I = \hat{\phi}^{-1} \circ L$. It is also an actionmorphism $I : \mathcal{N} \to T^*\mathcal{M}$, and it fixes all the points on the zero-section. We are going to prove that I is the identity.

Take cotangent coordinates $(x, y) \in \mathcal{U} \times \mathbb{R}^d$ around each point of the zero-section $\{y = 0\}$. Hence, I is given by

$$\bar{x} = f(x, y), \bar{y} = g(x, y),$$

with $f(x, 0) = x, g(x, 0) = 0$. The condition $I^*\alpha = \alpha$ is

$$0 = g(x, y)^\top A(x, y) - y^\top, 0 = g(x, y)^\top B(x, y),$$

where the matrices

$$A(x, y) = \frac{\partial f}{\partial x}(x, y), B(x, y) = \frac{\partial f}{\partial y}(x, y),$$

satisfy $AB^\top = BA^\top$. Note that we recover $g(x, 0) = 0$, because the matrix $(A\ B)$ has maximum rank. Hence, we have

$$0 = (g(x, y)^\top A(x, y) - y^\top)B(x, y)^\top = -y^\top B(x, y)^\top,$$

and then each x-component f_i is a homogeneous function of degree 0 in the y-variables. By regularity on the zero-section, these functions f_i are constant with respect to the y-variables: $f(x, y) = f(x, 0) = x$. Finally, $g(x, y) = y$, because A is the identity matrix. $\square$

3.3. Effective computation of exact symplectomorphisms given the primitive function and a restricted dynamics

Let $S : \mathcal{N} \to \mathbb{R}$ be a function whose differential vanishes on the zero-section. We look for the symplectomorphisms F whose primitive function is S. All of them fix the zero-section, and each of them is determined by the dynamics on the zero-section. It is enough to compute the F that fixes all the points on the zero-section, and the we can obtain the whole family by composing that F with the lifts of the diffeomorphism on the zero-section.

From a formal/computational point of view we assume that:

- our manifold $\mathcal{M}$ is $\mathbb{R}^d$ (or we consider cotangent coordinates around each point on the zero-section);

- the primitive function S is a series in y:
$$S(x,y) = \sum_n s_n(x) y^n$$
where the s_n are x-functions (we use multi-indices $n = (n_1, \ldots, n_d) \in \mathbb{N}^d$ and $|n| = \sum_{i=1}^d n_1$),
- $F = (f, g)$ is given by
$$f(x,y) = \sum_n f_n(x) y^n \ , \ g(x,y) = \sum_n g_n(x) y^n,$$
where $f_n = (f_n^1, \ldots, f_n^d)^\top$, $g_n = (g_n^1, \ldots, g_n^d)^\top$ are vector functions, and $f_0 = x$, $g_0 = 0$.

From the exactness equation (1) in cotangent coordinates
$$\frac{\partial S}{\partial x_j}(x,y) = \sum_i g^i(x,y) \frac{\partial f^i}{\partial x_j}(x,y) - y_j,$$
$$\frac{\partial S}{\partial y_j}(x,y) = \sum_i g^i(x,y) \frac{\partial f^i}{\partial y_j}(x,y),$$
we can obtain the relationship between the sequences of f_n,g_n and s_n. In the following, $\sum_i$ means $\sum_{i=1}^d$ and $u, v \in \mathbb{N}^d$ are multi-indices. Moreover, $e_i = (0, \ldots, 1_i, \ldots, 0)$.

Proposition 4 *With the notation above:*

- *The function s_0 is constant and the functions s_{e_i} vanish:* $\mathrm{D}S(x, 0) = 0$.
- *We can recover the x-functions f_n and g_n from s_n by means of the recurrence:*
  - *$(k = 1)$ $\forall i, j = 1 \div d$:* $g_{e_i}^j = \delta_{ij}$, $f_{e_i}^j = (1 + \delta_{ij}) \ s_{e_i + e_j}$;
  - *$(k > 1)$ $\forall |n| = k, j = 1 \div d$:*
$$g_n^j = \frac{\partial s_n}{\partial x_j}(x) - \sum_i \sum_{\substack{u+v=n \\ u \neq 0, n}} \frac{\partial f_u^i}{\partial x_j}(x) g_v^i(x),$$
$$f_n^j = \frac{n_j + 1}{k} s_{n+e_j}(x) + \frac{n_j + 1}{k} \sum_l F_{n+e_j-e_l}^l(x) - F_n^j(x),$$
where
$$F_n^j(x) = -\sum_i \sum_{\substack{u+v=n \\ |v|>1}} (u_j + 1) f_{u+e_j}^i(x) g_v^i(x).$$

The proof of this proposition is straightforward.

Remark: This method does not use the implicit function theorem, usually used when obtaining a symplectomorphism from a generating function. Moreover, if the primitive function is real analytic, the expansions of f and g also converge. In fact, we can obtain the symplectomorphism as the time-1 flow of a real-analytic Hamiltonian (this is our main theorem). ◁

Remark: The calculations also lead very quickly to normal forms around exact Lagrangian invariant manifolds (see Appendix A). $\triangleleft$

Remark: The numerical analysis issues of this method will appear in [16]. $\triangleleft$

4. Proof of Theorem 1

To prove Theorem 1, we first introduce a general construction based on the homotopy method, in which the manifold $\mathcal{N}$ can be any exact symplectic manifold. The evolution equation that we shall obtain by this method can have *many* solutions. This is connected with the existence of *many* exact symplectomorphisms with the same primitive function, and with the fact that the $\mathbf{\Lambda}$-derivative vanishes for *many* functions. Then, we focus on the case with $\mathcal{N} = T^*\mathcal{M}$. The properties of the (now) Liouville derivative, which are geometrically connected with the foliation of our (new) phase space, are essential to obtain a (new) evolution equation with an unique solution. This solution can be efficiently computed (4.2) and its analyticity is proved (4.3). We shall use a majorant method. Finally, the Hamiltonian will be obtained from the solution of the evolution equation in a straightforward way.

4.1. The homotopy method

Let $H : \mathcal{N} \times \mathbb{R} \to \mathbb{R}$ be a time-dependent Hamiltonian function (denoted by H_t), X_{H_t} be the corresponding vector field, and $\varphi_t = \varphi_{t,0}$ be the corresponding flow from $t_0 = 0$. We want

$$\varphi_1^*\alpha - \alpha = \mathrm{d}S.$$

In fact, we impose $\forall t\ \varphi_t^*\alpha - \alpha = t\ \mathrm{d}S$ (which is the basic construction of a *homotopy method*). That is to say, we want $S_{t,0} = t \cdot S$ (with the notation of 2.2), which is equivalent, applying (2), to $S = \mathbf{\Lambda}(H_t)\circ\varphi_t$. Therefore, if H_0 satisfies $S = \mathbf{\Lambda}(H_0)$ and

$$\begin{aligned} 0 &= \frac{d}{dt}(\mathbf{\Lambda}(H_t)\circ\varphi_t)\circ\varphi_t^{-1} \\ &= \frac{\partial}{\partial t}(\mathbf{\Lambda}(H_t)) + \mathrm{d}(\mathbf{\Lambda}(H_t))\frac{\partial\varphi_t}{\partial t}\circ\varphi_t^{-1} \\ &= \frac{\partial}{\partial t}(\mathbf{\Lambda}(H_t)) + \{H_t, \mathbf{\Lambda}(H_t)\}, \end{aligned}$$

then H_t is a time-dependent Hamiltonian whose time-1 map is an exact symplectomorphism whose primitive function is S. We shall refer to this problem as the *homotopy problem.*

If we find a solution to the homotopy problem, we can only assure that the primitive function of its time-1 map is S, but this does not determine φ_1. This is related to the existence of functions whose $\mathbf{\Lambda}$-derivative vanishes, as the next proposition shows.

Proposition 5 *Let $S : \mathcal{N} \to \mathbb{R}$ be a function. Then, the time-t map $\varphi_t = \varphi_{t,0}$ of a time-dependent Hamiltonian $H_t : \mathcal{N} \to \mathbb{R}$ that satisfies $S = \mathbf{\Lambda}(H_0)$ and*

$$\frac{d}{dt}(\mathbf{\Lambda}(H_t)) = -\{H_t, \mathbf{\Lambda}(H_t)\},$$

is exact symplectic and its primitive function is $t \cdot S$.

Furthermore, let $\bar{H}_t$ be a solution of the homotopy problem and $\psi_t = \psi_{t,0}$ be the flow of the corresponding Hamiltonian vector field. Let $h_t \in \ker \mathbf{\Lambda}$ be a time dependent Hamiltonian, whose flow $L_t = L_{t,0}$ is a family of actionmorphisms. Consider the new family of deformations $\varphi_t = L_t \circ \psi_t$, which is also connected to the identity. Then:

- *φ_t is the flow of the Hamiltonian $H_t = h_t + \bar{H}_t \circ L_t^{-1}$;*
- *H_t is also a solution of the homotopy problem.*

Proof: The first point is proved (cf. [19]) from

$$\begin{aligned}
\frac{d\varphi_t}{dt} &= \frac{\partial L_t}{\partial t} \circ \psi_t + (L_t)_*(\psi_t)\, \frac{d\psi_t}{dt} = X_{h_t} \circ L_t \circ \psi_t + ((L_t)_*\, X_{\bar{H}_t}) \circ \psi_t \\
&= X_{h_t} \circ \varphi_t + (L_t)_*\, X_{\bar{H}_t} \circ L_t^{-1} \circ \varphi_t = X_{h_t} \circ \varphi_t + X_{\bar{H}_t \circ L_t^{-1}} \circ \varphi_t \\
&= X_{h_t + \bar{H}_t \circ L_t^{-1}} \circ \varphi_t.
\end{aligned}$$

The second one follows from

$$\begin{aligned}
\frac{d}{dt}(\mathbf{\Lambda}(H_t)) &= \frac{\partial}{\partial t}(\mathbf{\Lambda}(\bar{H}_t)) \circ L_t^{-1} + \mathrm{d}(\mathbf{\Lambda}(\bar{H}_t))(L_t^{-1})\, \frac{\partial L_t^{-1}}{\partial t} \\
&= -\{\bar{H}_t, \mathbf{\Lambda}(\bar{H}_t)\} \circ L_t^{-1} - \mathrm{d}(\mathbf{\Lambda}(\bar{H}_t))(L_t^{-1})\ (L_t^{-1})_*\, X_{h_t} \\
&= -\{\bar{H}_t \circ L_t^{-1}, \mathbf{\Lambda}(\bar{H}_t \circ L_t^{-1})\} - \{h_t, \mathbf{\Lambda}(\bar{H}_t \circ L_t^{-1})\} \\
&= -\{H_t, \mathbf{\Lambda}(H_t)\}
\end{aligned}$$

and $\mathbf{\Lambda}(H_0) = \mathbf{\Lambda}(h_0 + \bar{H}_0 \circ L_0^{-1}) = \mathbf{\Lambda}(\bar{H}_0) = S$. □

4.2. An evolution problem

In this section we make make use of the special features of $T^*\mathcal{M}$. In the following, $\mathcal{N}$ is a tubular neighborhood of the zero-section. The Liouville derivative satisfies the following property.

Lemma 1 *The space of functions $\mathcal{F} \stackrel{\text{def}}{=} \mathcal{F}(\mathcal{N})$ splits as*

$$\mathcal{F} = \ker \mathbf{\Lambda} \oplus \mathbf{\Lambda}(\mathcal{F}).$$

Moreover, $\mathbf{\Lambda}(\mathcal{F})$ is the subspace of functions whose vertical derivatives vanish on the zero-section and $\ker \mathbf{\Lambda}$ *is the subspace of fiberwise homogeneous functions of degree* 1.

Proof: We have to solve the equation $\mathbf{\Lambda}(H) = S$. Since $\mathbf{\Lambda}$ is a vertical operator, we can restrict our attention to each fiber. On each fiber (fixed $x \in \mathcal{M}$) we have a linear operator transforming y-valued functions. Its properties are inherited by $\mathbf{\Lambda}$.

Hence, let $\mathcal{U} \subset \mathbb{R}^d$ be an open star-shaped neighborhood of the origin in $\mathbb{R}^d$, with coordinates $y = (y_1, \ldots, y_d)$, and $S : \mathcal{U} \to \mathbb{R}$ be a function. If $\nabla_y S(0) = 0$ the solutions of the partial differential equation $y \cdot \nabla_y H(y) - H(y) = S(y)$ depend on $a \in \mathbb{R}^d$ and are

$$H(y) = a \cdot y + \int_0^1 \frac{1}{t^2}(S(ty) - S(0))\, dt.$$

□

The homotopy problem is a family of evolution problems and, thanks to the previous lemma, we can specify a particular one. Notice that the lemma above states that $\mathbf{\Lambda}_| \stackrel{\text{def}}{=} \mathbf{\Lambda}_{|\mathbf{\Lambda}(\mathcal{F})}$ is an isomorphism in $\mathbf{\Lambda}(\mathcal{F})$. Thus, if we look for $E_t \in \mathbf{\Lambda}(\mathcal{F})$ as a solution of the *evolution problem*

$$\begin{cases} \dfrac{dE_t}{dt} = -\{\mathbf{\Lambda}_|^{-1}(E_t), E_t\}, \\ \\ E_0 = S. \end{cases}$$

then $\bar{H}_t = \mathbf{\Lambda}_|^{-1}(E_t)$ is a solution of the original problem. Of course, $\mathbf{\Lambda}(\mathcal{F})$ must be invariant under these operations to work inside this subspace. In fact, not only do the vertical derivatives of E_0 vanish on the zero-section, but *all* the derivatives vanish (recall that $\mathrm{d}S{\circ}z = 0$). The following lemma can be easily proved using cotangent coordinates.

Lemma 2 *Let $S, T \in \mathcal{F}(\mathcal{N})$ be two functions such that $\mathrm{d}S{\circ}z = 0$ and $\mathrm{d}T{\circ}z = 0$. Then,*

$$\mathrm{d}(\mathbf{\Lambda}_|^{-1}(S)){\circ}z = 0, \mathrm{d}\{S, T\}{\circ}z = 0.$$

Constructive method to solve the evolution problem To obtain the solution we use expansions in powers of t. If $E_t = \sum_{k\geq 0} E_k t^k$ is the expansion of E_t (where $E_0 = S$), then we can compute all the terms by the recurrence:

$$E_{k+1} = \frac{-1}{k+1} \sum_{u+v=k} \{\mathbf{\Lambda}_|^{-1}(E_u), E_v\}. \tag{3}$$

Hence, all the terms of the expansion satisfy $\mathrm{d}E_k{\circ}z = 0$. In fact, since the function $E_0 = S$ has y-order 2, the y-orders of the E_k increase: the y-order of E_k is $k+2$. This point is crucial to prove the convergence of the expansions.

4.3. Proof of the convergence of the expansions

Since we have used the majorant method to study the convergence of the expansions, will start by recalling some definitions and results on analytic functions. Given two real analytic functions in $z = (z_1, \ldots, z_m)$

$$f(z) = \sum_n f_n z^n, g(z) = \sum_n g_n z^n,$$

(using multi-index notation $n \in \mathbb{N}^m$), we can say that g is a majorant for f ($f \ll g$) if $\forall n\ |f_n| \leq g_n$.

Besides the standard estimates involving sums, products, derivatives, etc., which we will assume are well known, we shall use some more specialized estimates. Now, $m = 2d$ and $z = (x_1, \ldots, x_d, y_1, \ldots, y_d)$ are cotangent coordinates.

Lemma 3 *Given $b > 0$, we define $w_b(z) = \prod_{i=1}^d (b - x_i)(b - y_i)$. Let f, g be two real analytic functions such that $f \ll \prod_{t=1}^{u} y_{i_t} w_b^{-k}$, $g \ll \prod_{t=1}^{v} y_{j_t} w_b^{-l}$ where $u, v > 1$. Then:*

$$\{f, g\} \ll b^{2d-1} \begin{pmatrix} c_{uvkl} \prod_{t=1}^{u} y_{i_t} \cdot \sum_{s=1}^{v} \prod_{t=1, t\neq s}^{v} y_{j_t} \\ + \\ c_{uvlk} \prod_{t=1}^{v} y_{j_t} \cdot \sum_{s=1}^{u} \prod_{t=1, t\neq s}^{u} y_{i_t} \end{pmatrix} w_b^{-(1+k+l)}$$

and

$$\mathbf{\Lambda}_{|}^{-1}(f) \ll \frac{1}{u-1} \prod_{t=1}^{u} y_{i_t} w_b^{-k},$$

where $c_{uvkl} = k + \frac{2dkl}{u+v}$.

Proof: Since

$$\sum_{j=1}^{d} \frac{\partial f}{\partial x_j} \frac{\partial g}{\partial y_j} \ll \sum_{j=1}^{d} \left(\prod_{t=1}^{u} y_{i_t} \frac{k w_b^{-k}}{b - x_j} \left(\frac{\partial}{\partial y_j} \left(\prod_{t=1}^{v} y_{j_t} \right) w_b^{-l} + \prod_{t=1}^{v} y_{j_t} \frac{l w_b^{-l}}{b - y_j} \right) \right)$$

then

$$\begin{aligned} \{f, g\} &\ll b^{2d-1} \left(\begin{pmatrix} l \prod_{t=1}^{v} y_{j_t} \sum_{s=1}^{u} \prod_{t=1, t\neq s}^{u} y_{i_t} \\ + \\ k \prod_{t=1}^{u} y_{i_t} \sum_{s=1}^{v} \prod_{t=1, t\neq s}^{v} y_{j_t} \end{pmatrix} + \frac{2dkl}{b} \prod_{t=1}^{u} y_{i_t} \prod_{t=1}^{v} y_{j_t} \right) w_b^{-(1+k+l)} \\ &\ll b^{2d-1} \begin{pmatrix} c_{uvkl} \prod_{t=1}^{u} y_{i_t} \sum_{s=1}^{v} \prod_{t=1, t\neq s}^{v} y_{j_t} \\ + \\ c_{uvlk} \prod_{t=1}^{v} y_{j_t} \sum_{s=1}^{u} \prod_{t=1, t\neq s}^{u} y_{i_t} \end{pmatrix} w_b^{-(1+k+l)}. \end{aligned}$$

The second estimate is straightforward. □

Finally, we can prove the convergence of the expansions 3 for a sufficiently large t.

Proposition 6 *Let $S : \mathcal{N} \to \mathbb{R}$ be a real analytic function, with $\mathrm{d}S \circ z = 0$. Then, there exists a tubular neighborhood of the zero-section where the solution of the evolution problem E_t is defined for all time $t \leq 1 + \delta$.*

Proof: We can use cotangent coordinates (x, y) in a neighborhood of each point of the zero-section. It is sufficient to prove that we can obtain a small neighborhood of zero where the series $\sum_{k\geq 0} E_k(x, y) t^k$ is defined for $t \leq 1 + \delta$.

$S(x,0)$ is constant, and we can suppose that it is 0. Moreover, since $\forall x\ \frac{\partial S}{\partial y}(x,0)=0$ we can write:

$$S(x,y)=\sum_{i,j} y_i y_j s_{ij}(x,y),$$

where the functions s_{ij} are real analytic (and $s_{ij}=s_{ji}$). Given a small enough radius $b>0$, let c be the maximum of the sup-norms of the functions s_{ij} on $\{\|(x,y)\|_\infty<b\}$. So then, $\forall i,j=1\div d\ s_{ij}\ll cb^{2d}w_b^{-1}$, and

$$E_0\ll cb^{2d}\sum_{i,j} y_i y_j\ w_b^{-1}.$$

Suppose that $\forall u\le n$

$$E_u\ll\gamma_u\sum_{i_1,\dots,i_{u+2}}\prod_{t=1}^{u+2} y_{i_t}\ w_b^{-(2u+1)},$$

where $\gamma_0=cb^{2d},\gamma_1,\dots,\gamma_n$ are constants. We want to estimate E_{n+1}. Applying the majorant estimates of Lemma 3, and taking $\hat c_{uv}=c_{(u+2),(v+2),(2u+1),(2v+1)}$, we obtain

$$\begin{aligned}
E_{n+1}&=\frac{-1}{n+1}\sum_{u+v=n}\{\boldsymbol{\Lambda}_|^{-1}(E_u),E_v\}\\
&\ll\frac{b^{2d-1}}{n+1}\sum_{\substack{u+v=n\\ i_1,\dots,i_{u+2}\\ j_1,\dots,j_{v+2}}}\frac{\gamma_u\gamma_v}{u+1}\left(\begin{array}{c}\hat c_{uv}\prod_{t=1}^{u+2}y_{i_t}\cdot\sum_{s=1}^{v+2}\prod_{t=1,t\ne s}^{v+2}y_{j_t}\\ +\\ \hat c_{vu}\prod_{t=1}^{v+2}y_{j_t}\cdot\sum_{s=1}^{u+2}\prod_{t=1,t\ne s}^{u+2}y_{i_t}\end{array}\right)w_b^{-(2n+3)}\\
&\ll\gamma_{n+1}\sum_{k_1,\dots,k_{n+3}}\prod_{t=1}^{n+3}y_{k_t}\ w_b^{-(2n+3)},
\end{aligned}$$

where

$$\begin{aligned}
\gamma_{n+1}&\stackrel{\text{def}}{=}4d(1+2d)b^{2d-1}\sum_{u+v=n}\gamma_u\gamma_v\\
&\ge\frac{db^{2d-1}}{n+1}\sum_{u+v=n}\frac{\gamma_u\gamma_v}{u+1}(\hat c_{uv}(v+2)+\hat c_{vu}(u+2))
\end{aligned}$$

Hence, the sequence $\{\gamma_n\}$ is the sequence of coefficients of the Taylor series of the function

$$f(t)=\frac{1-\sqrt{1-4K\gamma_0 t}}{2Kt},$$

where $K=4d(1+2d)b^{2d-1}$, and then $\lim_n\frac{\gamma_n}{\gamma_{n+1}}=\frac{1}{4K\gamma_0}$.

Finally, consider the ball $B_{\rho b}=\{\|x\|_\infty\le\rho b,\|y\|_\infty\le\rho b\}$, where $\rho\in[0,1[$. Then, we have

$$\sum_{n\ge0}|E_n(x,y)|t^n\le\sum_{n\ge0}\gamma_n(d\rho b)^{n+2}(b(1-\rho))^{-2d(2n+1)}t^n,$$

and this series converges in $B_{\rho b}$ for a fixed $t>1$ in that case where ρ is small enough. □

4.4. End of the proof

Let E_t be the solution of the evolution problem, which belongs to $\mathbf{\Lambda}(\mathcal{F})$. Then $\bar{H}_t = \mathbf{\Lambda}_|^{-1}(E_t)$ is a Hamiltonian whose $\varphi_t = \varphi_{t,0}$ has primitive function $t \cdot S$. Since $\mathrm{d}E_t \circ z = 0$ then $\mathrm{d}\bar{H}_t \circ z = 0$, and all the points of the zero-section are fixed. Hence, φ_1 is *the* exact symplectomorphism, whose primitive function is S, and fixes all the points of the zero-section.

Finally, if the dynamics *on* the zero-section can be interpolated by a flow, then the dynamics *around* the zero-section can be interpolated by a Hamiltonian flow. Notice that the Hamiltonian lift of X_t is given by a Hamiltonian $h_t(\rho_x) = \rho_x(X_t(x))$ which belongs to $\ker \mathbf{\Lambda}$, and its flow L_t is the lift of the flow on the zero-section: $L_t = \hat{f}_t$. Thus, it is enough to apply Proposition 5: $H_t = h_t + \bar{H}_t \circ L_t^{-1}$.

Example: For instance, let $\mathcal{M} = \mathbb{T} \stackrel{\text{def}}{=} \mathbb{R}/\mathbb{Z}$ be the 1-dimensional torus and $\mathcal{N} = T^*\mathbb{T} \simeq \mathbb{T} \times \mathbb{R}$ be the annulus. Suppose our exact symplectomorphism fixes the zero section $\mathbb{T}_0$ and its dynamics is given by an ergodic translation by $\omega \in \mathbb{R} \setminus \mathbb{Q}$ and its primitive function is $S(x, y) = \frac{1}{2}s(x)y^2$ (where s is 1-periodic, of course). Then

$$H(x, y, t) = \omega y + \frac{1}{2}s(x - \omega t)y^2.$$

In order to get a periodic Hamiltonian, we have to perform a time change of variables. A C^2 change of time is given by a strictly increasing C^3 function $t = t(s) = s + \bar{t}(s)$ such that:

- $\bar{t}$ is 1-periodic, and then $t'(s)$ is 1-periodic;
- $\bar{t}(0) = 0$, $\bar{t}'(0) = -1$, $\bar{t}''(0) = 0$, $\bar{t}'''(0) = 0$.

Then we define the new Hamiltonian by

$$H_p(x, y, s) = t'(s) \cdot H(x, y, t(s))$$

in the time interval $[0, 1]$ and extend the function by periodicity. It is easy also to perform a C^∞ change of time. However, it is more difficult to obtain a time-periodic Hamiltonian which is analytic (see [17, 26]). ◁

Conclusion

In this paper we propose a new way to describe an exact symplectomorphism, besides the classical methods using generating functions and Lie transforms. We use the so-called primitive function of an exact symplectomorphism.

Our method allows us to obtain *all* the exact symplectic dynamics around invariant exact Lagrangian manifolds. Accordingly, it can be used to study the symplectomorphism in a vicinity of such a manifold (for instance, to obtain normal forms).

To reconstruct the exact symplectomorphism from its primitive function, we propose two methods, both of them implementable on a computer:

- a "generating-function-like" method: but, unlike the classical one, our method does not involve the implicit function theorem (and non-degeneracy conditions are not needed) and compositions, and the coefficients of the symplectomorphism are computed by means of recurrences only;
- a "Hamiltonian like method": we demonstrate how to construct a time dependent Hamiltonian whose time-1 map is the symplectomorphism.

Appendix A. Normal forms

We want to compute normal forms for the dynamics around an invariant exact Lagrangian manifold for a given exact symplectomorphism, that is, to obtain the more simplified expression for this dynamics. We will show that our methods also let us to get normal forms very quickly.

The main formula that we need is that, for given $F, G : \mathcal{N} \to \mathcal{N}$ exact symplectomorphisms on the exact symplectic manifold $(\mathcal{N}, \omega = \mathrm{d}\alpha)$, whose primitive functions are S, T, respectively, then the primitive function of the conjugation $\hat{F} = G^{-1} \circ F \circ G$ is $\hat{S} = S \circ G + T - T \circ \hat{F}$.

Making use of Weinstein's Theorem [28], we assume without loss of generality that the Lagrangian manifold is the zero-section of a cotangent bundle.

Appendix A.1. General construction

For the sake of simplicity, assume that $\mathcal{M}$ is a parallelizable manifold, that is $T^*\mathcal{M} \simeq \mathcal{M} \times \mathbb{R}^d$. $\mathcal{N} \subset T^*\mathcal{M}$ is a tubular neighborhood of the zero-section and $F = (f, g) : \mathcal{N} \to \mathcal{M} \times \mathbb{R}^d$ is an exact symplectomorphism whose primitive function is S. The zero-section is fixed and its dynamics is given by $\phi(x) = f(x, 0)$.

We want to obtain the simplest expression of F in the suitable coordinates. As the dynamics on the zero-section is already given, we have to obtain the simplest expression of the primitive function (as a previous step, we could obtain the nicest expression for the map ϕ).

As is 3.3, we are going to expand the (x, y)-functions in y-Taylor series, whose coefficients are x-functions. For the capital letter (x, y)-function $S(x, y)$ we shall write

$$S(x, y) = \sum_n s_n(x) y^n$$

where the coefficients s_n are lower case x-functions ($n = (n_1, \ldots, n_d) \in \mathbb{N}^d$). Moreover, for a given $k \in \mathbb{N}$, we shall write

$$S_k(x, y) = \sum_{|n|=k} s_n(x) y^n \ , \ S_{\leq k}(x, y) = \sum_{|n| \leq k} s_n(x) y^n \ ,$$

etc.

In this case, as S is the primitive function of F, we know that s_0 is a constant (which we will take zero) and the functions s_{e_i} vanish. This is the first step of normal form (we have eliminated the terms of y-order 1) in S.

Suppose that we have already done $k-1$ steps of normal form, and we want to complete the step k. Then, the primitive function of F is given by

$$S(x,y) = N_{\le k-1} + R_{>k-1}$$

(N from normal form and R from remainder). N corresponds to the terms that we have not be able to eliminate in the previous steps. In this step, we want to eliminate $S_k = R_k$.

Consider then the change of variables G as given by the time-1 map associated to a Hamiltonian $H = H_k$, which is y-homogeneous of degree k. Notice that G does not move the points on the zero-section. Since $\mathbf{\Lambda}(H) = (k-1)H$ and H is preserved by the flow, the primitive function of G and G^{-1} are $T = (k-1)H$ and $-T$, respectively (and $T = T_k$). The computation of G and G^{-1} can be performed by means of the recurrence in Proposition 4, and it is not necessary the use of the Lie series method.

Then, the primitive function $\hat{S}$ of $\hat{F} = G^{-1}\circ F\circ G$ is given by

$$\begin{aligned}
\hat{S}(x,y) &= S\circ G(x,y) + T(x,y) - T\circ\hat{F}(x,y) \\
&= N_{\le k-1}(x,y) + R_k(x,y) + \ldots + \\
&\quad\, T_k(x,y) + \ldots - \\
&\quad\, T_k(\phi(x), \mathrm{D}\phi(x)^{-\top}y) + \ldots \\
&= N_{\le k-1}(x,y) + R_k(x,y) + T_k(x,y) - T_k(\phi(x), \mathrm{D}\phi(x)^{-\top}y) + \ldots,
\end{aligned}$$

where '$\ldots$' means terms with higher order (we have only take care of the terms to degree k). Then, we look for T_k (that is, H_k) such that

$$\mathrm{L}_\phi^k T_k(x,y) \stackrel{\text{def}}{=} T_k(\phi(x), \mathrm{D}\phi(x)^{-\top}y) - T_k(x,y) = R_k(x,y).$$

This is the *homological equation* of degree k associated to the diffeomorphism ϕ. The operator L_ϕ^k acts on the space of y-homogeneous functions of degree k, $\mathcal{F}_k$. We can also consider L_ϕ as acting on the graded algebra $\bigoplus_{k=0}^\infty \mathcal{F}_k$ (formally, the algebra of all the functions).

For the sake of simplicity, suppose that L_ϕ^k applies $\mathrm{L}_\phi^k(\mathcal{F}_k)$ onto its image, that is, $\mathcal{F}_k = \ker \mathrm{L}_\phi \oplus \mathrm{L}_\phi^k(\mathcal{F}_k)$. Then R_k decomposes uniquely as

$$R_k = N_k + L_k,$$

and we can only solve the linear equation $\mathrm{L}_\phi^k T_k = L_k$ and the terms in N_k will remain. Hence, the normal form to degree k is given by

$$\hat{S}(x,y) = N_{\le k} + \hat{R}_{>k}.$$

In summary, the normal form to degree k is given by a k degree y-polynomial $N_{\le k}$ that belongs to $\ker \mathrm{L}_\phi$.

Appendix A.2. An example: the normal form near a KAM torus

Assume $\mathcal{M} = \mathbb{T}^d$ and $\phi(x) = x + \omega$, where $\omega \in \mathbb{R}^d$ is a Diophantine vector (that is that there exists $C > 0$ and $\tau \ge 1$ such that $\forall n \in \mathbb{Z}$ $|q\cdot\omega - n| \ge C/\|q\|_1^\tau$). The motion on the zero-section is quasi-periodic.

Now, The x-functions are 1-periodic in all their variables. The operator L_ϕ^k is given by

$$\mathrm{L}_\phi^k T_k(x,y) = T_k(x+\omega, y) - T_k(x,y)$$

$$= \sum_{|n|=k} (t_n(x+\omega) - t_n(x)) y^n.$$

Define the operator $\mathrm{l}_\omega t(x) = t(x+\omega) - t(x)$ on the space of functions on the d-torus: $\mathcal{T} \overset{\text{def}}{=} \mathcal{F}(\mathbb{T}^d)$. Then, formally speaking, $\mathcal{T} = \ker \mathrm{l}_\omega \oplus \mathrm{l}_\omega(\mathcal{T})$, and $\ker \mathrm{l}_\omega$ is the set of constants and $\mathrm{l}_\omega(\mathcal{T})$ is the set of periodic functions with zero average. This is not completely correct, and depends on the category of functions we are working in. For instance, in the C^k category, the homology equation decreases the degree of differentiability (and we can only do a finite number of steps of normal form), and in the C^ω category the width of the complex band around x = 0 is also decreased (see, for instance, [3]).

Taking into account all these remarks, we obtain that the normal form of order k is given by a primitive function S with

$$N_{\le k}(y) = \sum_{2\le |n| \le k} \nu_n y^n,$$

where the coefficients ν_n are constants. Applying Proposition 4, we get that the normal form is

$$\begin{aligned} f(x,y) &= \omega + x + \nabla_y H_{\le k}(y) + O(y^k),\\ g(x,y) &= y(I + O(y^k)), \end{aligned}$$

where

$$H_{\le k}(y) = \mathbf{\Lambda}_|^{-1}(N_{\le k}) = \sum_{2\le |n|\le k} \frac{\nu_n}{|n|-1} y^n.$$

This is the *Birkhoff's normal form* near a KAM torus.

Remark: The symplectomorphism corresponding to $N_{\le k}$ is the time-1 map of the Hamiltonian $\omega \cdot y + H_{\le k}(y)$. This Hamiltonian is integrable (as the corresponding map). Thus, we can construct d approximate first integrals around a neighborhood of an invariant torus (cf. [11, 27]). ◁

Remark: This method is useful to obtain partial justifications of the Greene's criterion of breakdown of an invariant torus [21, 11, 27], and also for estimates on the diffusion around a KAM torus [25, 22]. ◁

We can also obtain normal forms around a hyperbolic fixed point, that is, around its stable (or unstable) invariant manifold. Mixed situations can also be considered, that is, we can compute normal forms around the stable (or unstable) manifold of a hyperbolic quasi-periodic torus (although we need some reducibility hypothesis). See [14] for more information about this subject.

Appendix B. On the interpolation problem

The interpolation problem has been widely studied, and its many variants have been examined (for instance, [10, 6, 24, 5] for smooth diffeomorphisms and [23, 10, 17, 26] for analytic maps). Sometimes one uses the existence of generating functions, or some kind of closeness to an integrable map, or one uses that the symplectomorphism is already connected to the identity. Once we have obtained the Hamiltonian, we can then obtain the time dependence of period 1 (in the smooth category is straightforward, but in the analytic category is more difficult [26]).

Interpolation problems are not only interesting by themselves, but they also have applications. Many problems in Hamiltonian/symplectic dynamics are easier when considered in terms of maps, but it is often simpler to give a proof in terms of flows. Interpolation allows us to show that results proved for all periodic flows hold for all maps. See [17] for applications to Nekhoroshev estimates for almost integrable exact symplectic maps (in which the interpolation is important to be analytic), and also to KAM theorems for maps [10].

Since we are interested in a particular case, let $F : \mathcal{N} \to T^*\mathcal{M}$ be an exact symplectomorphism defined in a neighborhood of the zero-section and fixes all the points of the zero-section. Recall that $\mathrm{d}S(0_x) = 0$, and then S is constant on the zero-section. We are going to interpolate it by a Hamiltonian flow [20]. Notice that, in this case, we have *all* the information about the problem, not only the primitive function S. First of all, consider the fiberwise homotheties $h_\lambda : T^*\mathcal{M} \to T^*\mathcal{M}$, where $\lambda \in \mathbb{R}$, defined by $h_\lambda(\rho_x) = \lambda \cdot \rho_x$. Of course, ${h_\lambda}^*\alpha = \lambda \cdot \alpha$. Then, we can connect F with the identity by means of a 'symplectic path':

$$\begin{aligned} &F_0 = \mathrm{id}, \\ &F_t = h_{1/t}\circ F\circ h_t \quad (t \neq 0). \end{aligned}$$

Notice that the primitive function of F_t, for $t \neq 0$, is $S_t = \frac{1}{t} S\circ h_t$. This fits with the case $t = 0$, thanks to the conditions satisfied by S. It is an standard result about exact symplectic isotopies that F_t is generated by the flow of the vector field

$$X_t = \frac{dF_t}{dt}\circ F_t^{-1},$$

which is Hamiltonian. The Hamiltonian is $H_t = \alpha(X_t) - \frac{dS_t}{dt}\circ F_t^{-1}$. From a computational point of view, this method seems to be less efficient that the given in the proof of the Theorem 1, because now we have to inverse and compose some maps, while in the resolution of the evolution problem we just have to compute a recurrence of functions (where only sums, products and derivatives are needed).

Another approach can be the use of a 'parametric' Darboux's theorem [4]. That is, given a family of symplectic forms ω_t such that $\omega_0 = \omega_1 = \omega$ on a symplectic manifold $(\mathcal{N}, \omega)$ we can obtain a family of (local) diffeomorphisms $h_t : \mathcal{N} \to \mathcal{N}$ such that $h_t^* \omega_t = \omega$, with $h_0 = h_1 = \mathrm{id}$. Now, let $F = F_1$ be a symplectomorphism, connected with the identity $F_0 = \mathrm{id}$ by means of a path F_t of diffeomorphisms. Then, define $\omega_t = F_t^* \omega$ and construct h_t as before. Thus, define $\bar{F}_t = F_t \circ h_t$, that satisfies $\bar{F}_0 = F_0 \circ h_0 = \mathrm{id}$, $\bar{F}_1 = F_1 \circ h_1 = F$ and $\bar{F}_t^* \omega = \omega$. So, $\bar{F}_t$ is a symplectic deformation, that is generated by a (local) Hamiltonian vector field (the Hamiltonian can be a multi-valued function). In order to get a 1-valued function we need some additional exactness conditions: $\omega_t = \mathrm{d}\alpha_t$ $(\alpha_0 = \alpha)$ and $h_t^* \alpha_t = \alpha + \mathrm{d}S_t$ $(S_0 = 0)$, where we take $\alpha_t = F_t^* \alpha$. Notice that $\bar{F}_t^* \alpha = h_t^* F_t^* \alpha = h_t^* \alpha_t = \alpha + \mathrm{d}S_t$, and the deformation $\bar{F}_t$ is exact symplectic.

As a final comment, notice that when we interpolate we always loose one order of differentiability (if the symplectomorphism is C^k, the Hamiltonian vector field is C^{k-1}). This is not so strange. After all, the lift of a C^k diffeomorphism is a C^{k-1} symplectomorphism, and also the Hamiltonian lift of a C^k vector field is a C^{k-1} Hamiltonian vector field. We do not know whether this loss in the conclusion could be sidesteped using other methods.

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