A numerical method for computing unstable quasi-periodic solutions for the 2-D Poiseuille flow †

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We study the dynamics of two-dimensional Poiseuille flow. Firstly we obtain the family of periodic solutions which bifurcates from the laminar flow, together with its stability for several values of the wave number α. The curve of periodic flows presents several Hopf bifurcations. For $\alpha = 1.02056$ we follow the branches of quasi-periodic orbits that are born at one of the bifurcation points.

Poiseuille flow

We consider the flow of a viscous incompressible fluid, in a channel between two parallel walls at $y = \pm 1$, driven by a constant stream-wise pressure gradient and governed by the dimensionless Navier–Stokes equations in primitive variable formulation

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla p + \frac{1}{Re}\Delta \mathbf{u}, \tag{1}$$

where $\mathbf{u} = (u, v)$ are the components of the velocity, p is the pressure and Re is the Reynolds number defined in terms of the constant pressure gradient. As boundary conditions we suppose $u(x, \pm 1, t) = v(x, \pm 1, t) = 0$ and $(u, v, p)(x + L, y, t) = (u, v, p)(x, y, t)$ for $L = 2\pi/\alpha, x \in \mathbb{R}, y \in [-1, 1], t \geq 0$. The initial profile of velocities is simply subjected to the incompressibility condition, $\operatorname{div} \mathbf{u} = 0$. We are concerned with the dynamics of Poiseuille flow in varying the parameters Re and α. The stability of the laminar solution has been studied extensively through the literature. For instance, Orszag [1] obtained the critical Reynolds number, $Re_{\text{cr}} = 5772.22$ for $\alpha = 1.02056$, so that if $Re < Re_{\text{cr}}$ the laminar solution is stable for any value of α.

We construct a numerical integrator of system (1) by means of a spectral method. Since u, v, p are supposed periodic in x, we replace them by their truncated Fourier series:

$$(u, v, p)(x, y, t) = \sum_{k=-N}^{N} (\hat{u}_k, \hat{v}_k, \hat{p}_k)(y, t)e^{ik\alpha x}, \qquad x \in \mathbb{R}, \quad y \in [-1, 1], \quad t \geq 0.$$

We resolve the Fourier coefficients evaluating them at the Chebyshev abscissas. Next we eliminate v and p from the discretized system to obtain a set of ODE just in terms of u, $\partial u/\partial t = f(u)$. This system has dimension $(2N + 1)(M - 2) + 1$, being M the number of Chebyshev modes in y. For the temporal discretization we employ a semi-implicit finite difference method: Adams–Bashforth for advection and Crank–Nicolson for pressure and viscosity terms.

Periodic solutions

Due to the translational symmetry of the channel in x, it is showed in [3] that any time-periodic function $w(x, y, t)$, of period T, is a rotating wave, i.e. $w(x, y, t) = w(x - ct, y, 0)$, for $c = L/T$. In this way we may find periodic solutions in time as stationary flows in a system

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of reference moving with velocity c in the stream-wise direction, which can either correspond to stable or unstable time-periodic orbits. Given a profile of velocities (u, v) we define its amplitude A, as the distance to the laminar solution $\bar{\mathbf{u}} = (\bar{u}, 0), \bar{u} = 1 - y^2$, in the L_2–norm.

The bifurcating diagram for the periodic flows in the Re–A plane, together with their stability is represented in figure 1. We decide the stability of a time-periodic solution u upon the eigenvalues of the Jacobian matrix $Df(u)$. In the case of the laminar flow, we obtain the classical results of Orszag [1] about the critical Reynolds being at $Re_{\text{cr}} = 5772.22$ for $\alpha = 1.02056$. First, the bifurcation curve of periodic flows reaches the laminar solution at Re_{cr} and in addition, the laminar solution is checked to be stable when $Re < Re_{\text{cr}}$ and unstable if $Re > Re_{\text{cr}}$. Soibelman & Meiron [4] found analogous bifurcations for $\alpha = 1.1$, and the critical Reynolds number for which there are time-periodic solutions: $Re \approx 2900$ for $\alpha \approx 1.3$. In figure 1 this minimum Reynolds number is verified. We also notice that there exists an attractor periodic solution for Re as low as 3000.

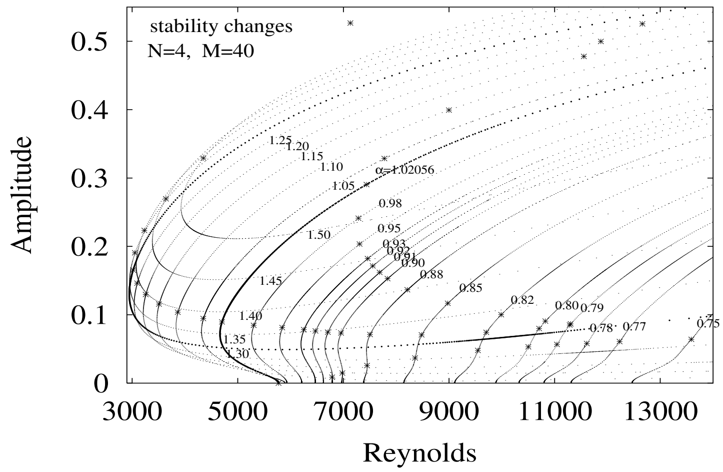

Fig. 1. Bifurcating curves for several values of α.

Quasi-periodic solutions

In this section we fix $\alpha = 1.02056$. Taking $Re \approx 7500$, the corresponding periodic flow is unstable so, as a test, we use it as initial condition. After some steps of numerical integration, the solution seems to fall in a regular régime. Then, in figure 2, we plot the projection of the solution vector $u(t) = (u_1, \ldots, u_K)(t),\ K = (2N+1)(M-2)+1$, over the plane of two arbitrary coordinates (the 120^{th} and 232^{th} to be precise). Each point in figure 2(a) corresponds to the value of the specified coordinates at a time instant. As we can observe, the trajectory seems to fill densely the projection of a torus. Taking an *ad hoc* Poincaré section Σ, of the space of solutions $u(t)$ and plotting the same coordinates as above, but only when crossing the Poincaré section, we see in figure 2(b), a closed curve, which confirms us that our flow lives in a 2-torus.

In [3] it is proved that, with the translational symmetry of the channel, every solution $w(x, y, t)$ that lies on an isolated invariant 2-torus, not asymptotic to a rotating wave, is a modulated wave, that is, exists $\tau > 0$ and $\phi \in \mathbb{R}$ such that for every integer n we have $w(x, y, n\tau) = w(x + n\phi, y, 0)$. As a consequence, this kind of wave has the property that, may be viewed as a periodic wave of period τ in a frame of reference moving with speed $c = (pL - \phi)/\tau$, for any integer p, as is easily proved. For each value of c we construct a Poincaré map as above $P_c : \Sigma \longrightarrow \Sigma$. Starting from an initial condition $u = u(0), u \in \Sigma$ we integrate (1) for an observer that moves with speed c, until a time τ such that $u(\tau) \in \Sigma$ and crosses it in the same sense as it did initially. We set $P_c(u) = u(\tau)$ and we try to find u, c such that $P_c(u) = u$. With this c the quasi-periodic flow u will be viewed as a periodic one.

To find the branches of quasi-periodic solutions at the bifurcation point $Re \approx 7400$, we search for zeros of the map $F(c, u) = P_c(u) - u$, using as a starting guess, points on a mesh around the bifurcating periodic solution u^*. This mesh is selected from the directions in the kernel of $DF(u^*)$. Taking Re as a continuation parameter, we can trace a curve of quasi-periodic flows in the Re–A plane as above. We emphasize that zeros of $F(c, u)$ can either correspond to stable or unstable quasi-periodic solutions. Unlike the periodic flow, in this case the amplitude A of the solution does depend on time so we evaluate it when $u \in \Sigma$. As we observe in figure 3(a), there is just one branch of quasi-periodic solutions which bifurcates from the curve of periodic flows. If we plot, instead of the amplitude, the value of a selected coordinate of the solution vector, we obtain the graph in figure 3(b), which confirms us the existence of two different branches of quasi-periodic orbits on the Hopf bifurcation at $Re \approx 7400$. As the periodic orbits go from stable to unstable when crossing the bifurcation point, the two branches of quasi-periodic solutions are locally stable to two-dimensional disturbances. By means of the eigenvalues of the Jacobian matrix $\partial P_c/\partial u(u)$ we can also estimate the stability of the obtained quasi-periodic flows. The continuation of those bifurcating branches, their changes of stability and other bifurcation curves is actually a work in progress.

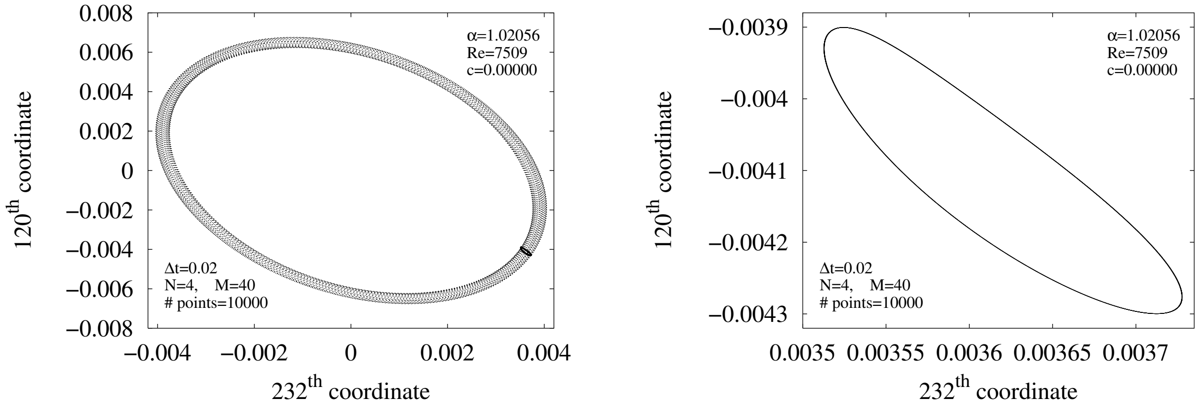

Fig. 2. Solution vector (a) each Δt units of time and (b) on the Poincaré section.

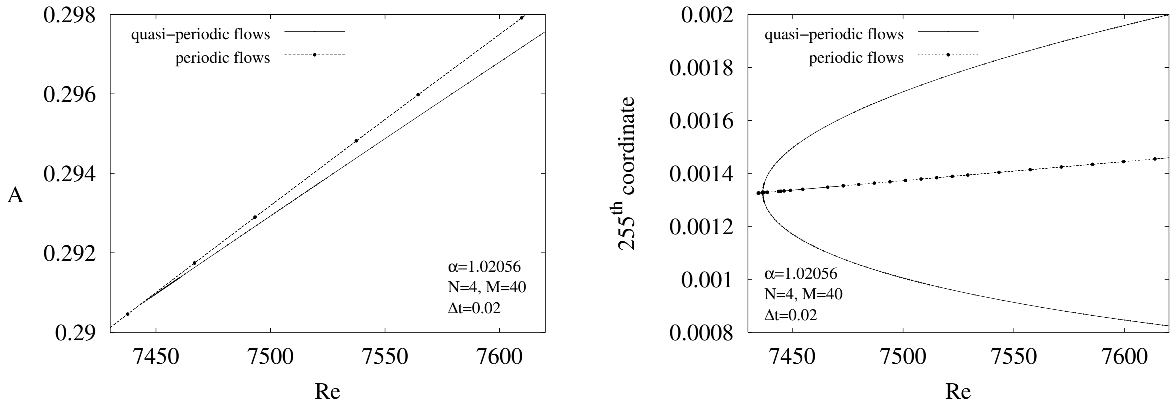

Fig. 3. Bifurcating curves of quasi-periodic flows.

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