Three dimensional dissipative diffeomorphisms with homoclinic tangencies

J. C. Tatjer

Dept. Matemàtica Aplicada i Anàlisi. Universitat de Barcelona
Gran Via 585. 08071 Barcelona. Spain.

Abstract

Given a two-parameter family of three-dimensional diffeomorphism $\{f_{a,b}\}_{a,b}$, with a dissipative (but not sectionally dissipative) saddle fixed point, assume that a special type of quadratic homoclinic tangency of the invariant manifolds exists. Then there is a return map $f^n_{a,b}$, near the homoclinic orbit, for values of the parameter near such tangency and for n large enough, such that, after a change of variables and reparametrization depending on n, this return map tends to a simple quadratic map. This implies the existence of strange attractors and infinitely many sinks as in other known cases. Moreover, there appear attracting invariant circles, implying the existence of quasiperiodic behaviour near the homoclinic tangency.

1 Introduction

1.1 Motivation

A main problem in Dynamical Systems is to describe the typical asymptotic behaviour of trajectories as time (continuous or discrete) goes to infinity. Two objects are crucial in this description: invariant manifolds and, in the dissipative case, attracting sets and, in particular, strange attractors.

Concerning the invariant manifolds, the existence of homoclinic orbits plays a fundamental role in the development of the chaotic behaviour observed in many systems. It is known that the presence of transversal intersections of the invariant manifolds of a periodic orbit produces hyperbolic behaviour of orbits via the creation of horseshoe maps. The situation changes dramatically when a homoclinic bifurcation appears in a family of dynamical systems. In this case nonhyperbolic objects like strange attractors arise.

In recent years, efforts has been devoted to prove the existence and persistence of strange attractors in several scenarios. In the case of the logistic family, Jakobson ([10]) was able to prove the existence of strange attractors for a subset of positive measure on the space of parameters. Later, Benedicks and Carleson [2] showed a similar theorem for the strongly dissipative Hénon family. An application of Benedicks-Carlesson's methods was done in Mora and Viana [13], where their results were generalized to the setting of generic one-parmeter families of surface diffeomorphisms, having a quadratic homoclinic tangency related to a dissipative saddle periodic point. A generalization to any dimension larger than two was performed by Viana [22], in the sectionally dissipative case (when the product of any pair of eigenvalues of the saddle is less than 1 in absolute value). All the proofs of

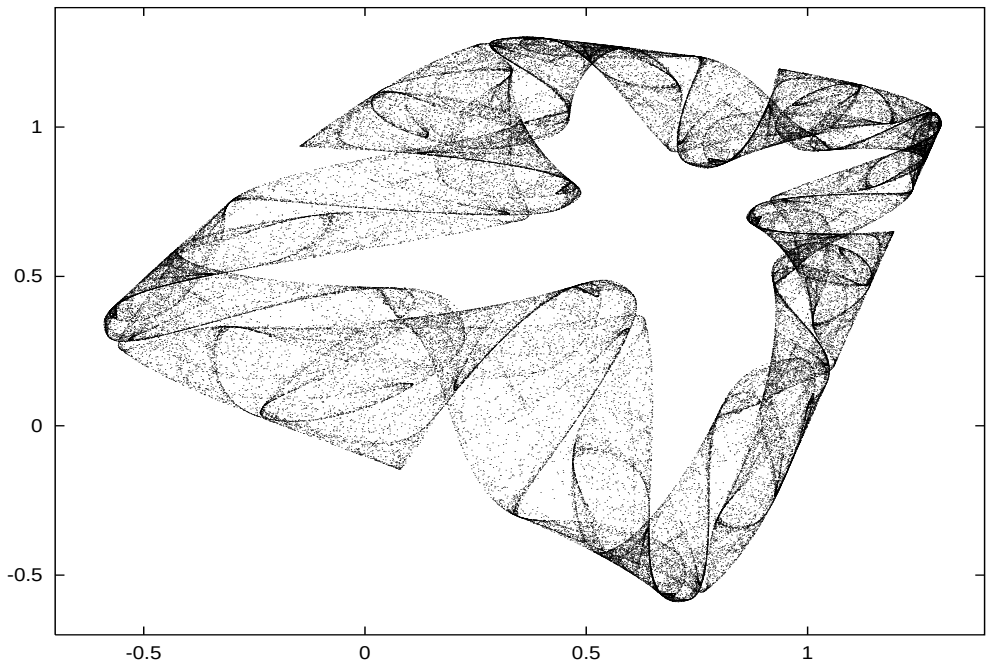


Figure 1: Two-dimensional strange attractor for a 2D non-invertible map.

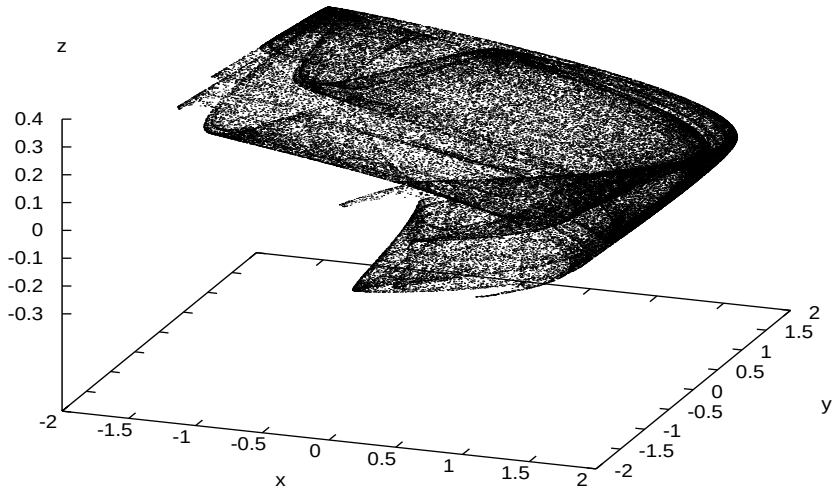
these theorems are based on the observation that the dynamics in a suitable subset of the phase space is very close to the dynamics of the logistic family. Moreover, all the strange attractors found, even in the multidimensional case, have topological dimension equal to one.

However, there are simple quadratic maps in dimension two (non-invertible) and in dimension three (invertible and with constant jacobian) for which strange attractors of topological dimension equal to two seem to appear. At least there is some numerical evidence of this fact. We give two examples, one in dimension two and the other in dimension three:

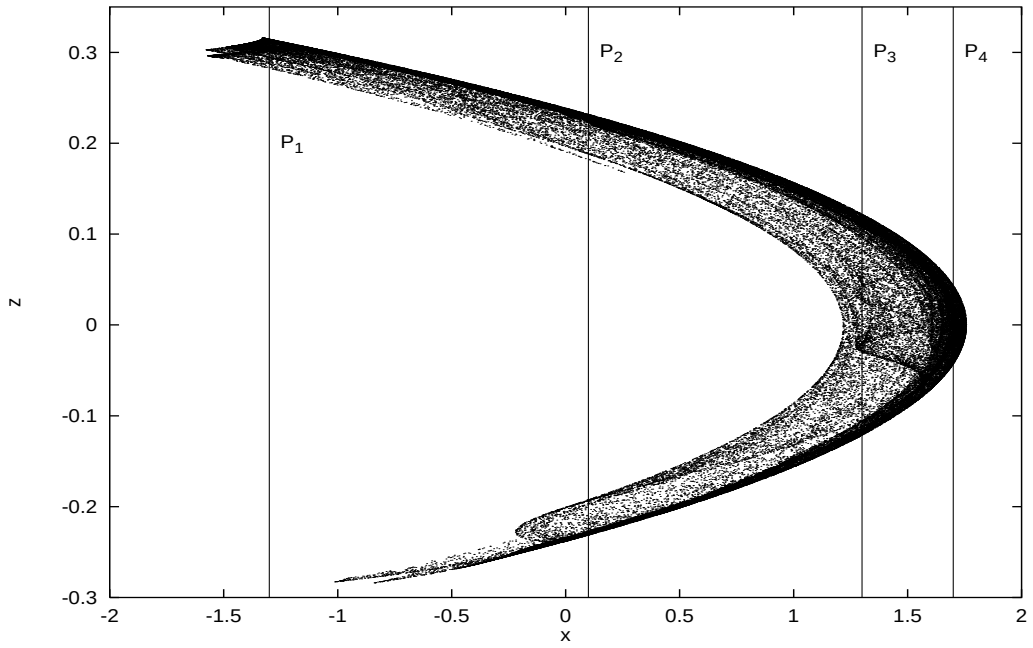
First consider the map $f_{a,b} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ given by $f_{a,b}(x,y) = (a + bx - y^2, x)$, with $a = 1$ and $b = 0.3$. We have taken as initial point $(x,y) = (0, 0.1)$, performed a transient of 10^6 iterates and drawn the next 10^5 consecutive iterates in figure 1. We note that $\det Df_{a,b}(x,y) = 2y$ and , therefore, the map is non-invertible.

In the three-dimensional case, we take $g_{a,b,c}(x,y,z) = (a + bx + z - y^2, x, cy)$, that has been obtained fattening the previous map. Now the jacobian determinant is constant and equal to c. Figure 2 displays two views of 10^5 iterates of a point in a two dimensional strange attractor, having the shape of a folded handkerchief. The corresponding values of the parameters are $a = 1.5$, $b = 0.1$ and $c = 0.18$. Intersections of the strange attractor with the planes $x = -1.3$ (P_1), $x = 0.1$ (P_2), $x = 1.3$ (P_3) and $x = 1.7$ (P_4), are drawn in figure 3.

As we will see later, the previous maps $f_{a,b}$ and $g_{a,b,c}$ are very important in two senses:



a) In the three-dimensional space.



b) Projected on the xz-plane.

Figure 2: Two-dimensional strange attractor for a 3D diffeomorphism

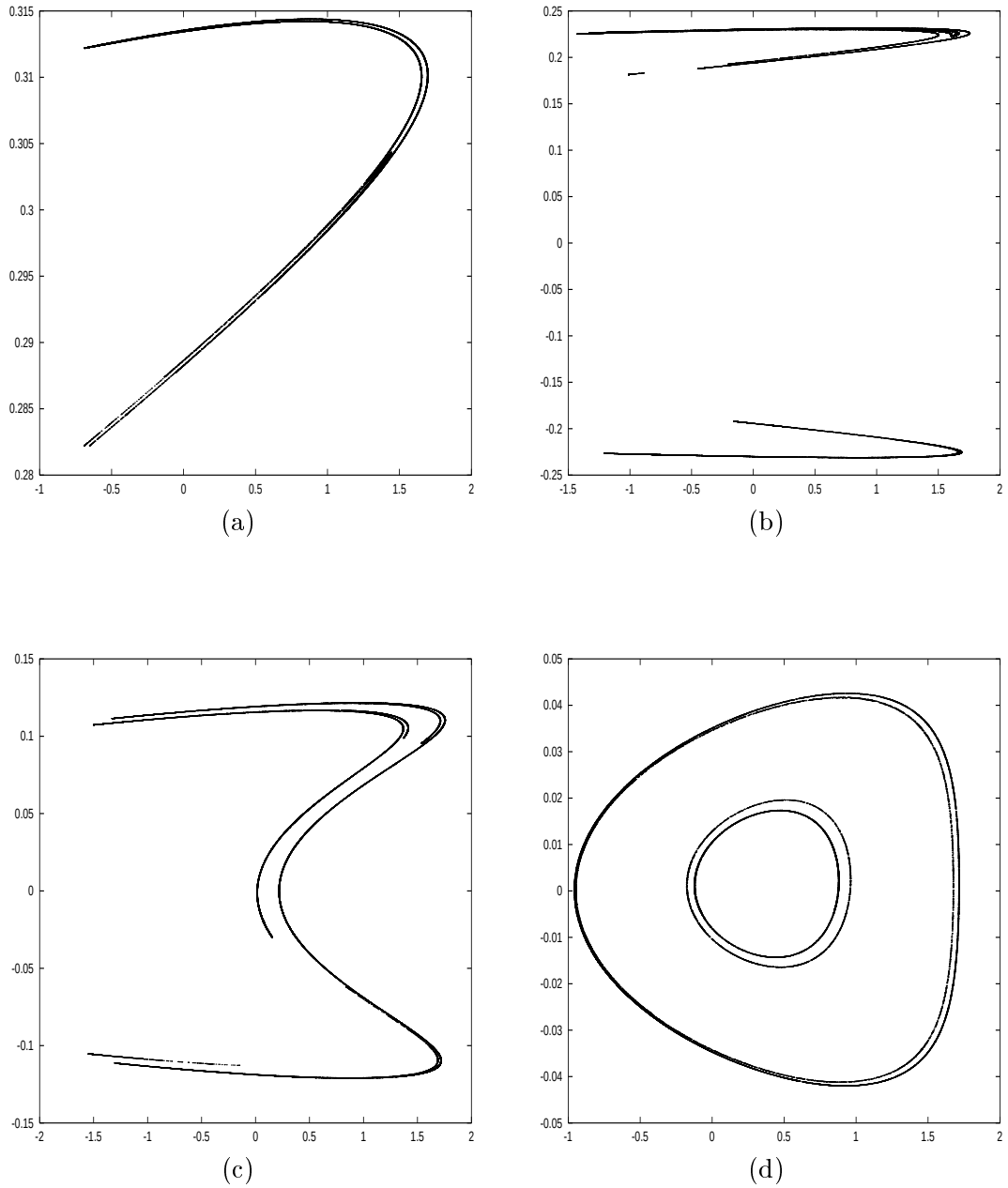


Figure 3: Intersections of the strange attractor of the previous figure with the planes P_1 (a), P_2 (b), P_3 (c) and P_4 (d). See the strong differences in the scales.

On the one hand, $f_{a,b}$ and $g_{a,b,c}$ have only one quadratic term and the others are linear. Therefore, they can be considered as the simplest families of maps in dimensions two and three, respectively, for which there is numerical evidence of existence of strange attractors of topological dimension equal to two.

On the other hand, the dynamical behaviour of the non-invertible map $f_{a,b}$ serves as a model of the behaviour of the orbits of a family of maps having a special type of codimension-two homoclinic tangency that we shall call *generalized homoclinic tangency*. Thus, we have a codimension-two global bifurcation that seems to force the existence of two-dimensional strange attractors for a set of values of the parameters of non-zero Lebesgue measure. Moreover, as the family $f_{a,b}$ exhibits quasiperiodic invariant circles, infinitely many sinks, and one-dimensional strange attractors, the same occurs for the initial family of diffeomorphisms. We shall prove the existence of all these objects except the two-dimensional strange attractor.

1.2 Statement of results

We begin with some definitions concerning fixed points of a diffeomorphism.

Let M be an m-dimensional smooth manifold and $f : M \longrightarrow M$ a diffeomorphism. Suppose that $p \in M$ is a fixed point of f ($f(p) = p$), and denote by $\lambda_1, \ldots, \lambda_m$ the eigenvalues of $Df(p)$.

Definition 1.1 *1. We say that p is* **dissipative** *if the product of the eigenvalues of $Df(p)$ is less than 1 in absolute value.*

2. If, in addition, $|\lambda_i \lambda_j| < 1$ for all $i, j \in \{1, \ldots, m\}$ such that $i \neq j$, we call p a **sectionally dissipative** *fixed point.*

3. The **index of stability** *of p is the number of eigenvalues of absolute value less than 1.*

4. The fixed point p is a (hyperbolic) **saddle** *if all the eigenvalues of $Df(p)$ have absolute value different from 1, and its index of stability is different from 0 and m.*

By the Stable Manifold Theorem (see, e.g., Palis and de Melo [15]), given a saddle fixed point p there exist the stable and unstable invariant manifolds that we denote by $W^s(f, p)$, and $W^u(f, p)$ respectively, and are defined by:

$$W^s(f,p) = \{q \in M \ : \ \lim_{n\to\infty} f^n(q) = p\}, \quad W^u(f,p) = W^s(f^{-1}, p).$$

As usual, if it is not necessary, we shall not write explicitely the dependence of the invariant manifolds on the map f.

Moreover, there are other invariant objects that we are interested in:

Suppose that $\dim M = 3$, p is a saddle fixed point and $|\lambda_1| < |\lambda_2| < 1 < |\lambda_3|$. Given a smooth submanifold $N \subset M$, we write

$$L_i(N) := \{(x, L) \,|\, x \in N,\ L \text{ is a } i\text{-dimensional linear subspace of } T_xM \text{ and } T_xN \cap L \neq \{0\}\},$$

and denote by E^{cu} (respectively E^{ss}) the invariant linear subspace of T_pM, associated to the eigenvalues λ_2 and λ_3 (resp. λ_1). Let $F_1 : L_1(W^s(p)) \longrightarrow L_1(W^s(p))$ and $F_2 : L_2(W^u(p)) \longrightarrow L_2(W^u(p))$ be maps induced by f on the respective spaces.

On the one hand, (p, E^{ss}) is a fixed point of F_1 with eigenvalues λ_1, λ_2 and $\lambda_2\lambda_1^{-1}$. Note that the corresponding stable manifold $W^s(p, E^{ss})$ has dimension two since $|\lambda_1| < |\lambda_2| < 1 < |\lambda_2\lambda_1^{-1}|$.

On the other hand, (p, E^{cu}) is a fixed point of F_2 with eigenvalues λ_3 and $\lambda_1\lambda_2^{-1}$. In this case, $|\lambda_1\lambda_2^{-1}| < 1 < |\lambda_3|$, and, therefore $W^u(p, E^{cu})$ is a one-dimensional manifold. Note, moreover, that the planes of this vector bundle are tangent to $W^u(p)$, by definition of $L_2(W^u(p))$.

Using these invariant manifolds, we can give the following definitions.

Definition 1.2 *Suppose that f is as before and p is a saddle fixed point.*

1. *If $|\lambda_1| < |\lambda_2| < 1 < |\lambda_2|$, the foliation $\mathcal{F}^{ss}(p)$ induced by $W^s(p, E^{ss})$, is called the* **strong stable foliation** *of p. Moreover, the* **unstable invariant tangent bundle** *of p is the vector bundle $W^u(p, E^{cu})$.*

2. *If $|\lambda_1| < 1 < |\lambda_2| < |\lambda_2|$, the* **strong unstable foliation** *of p, denoted by $\mathcal{F}^{uu}(p)$ is the strong stable foliation of p with respect f^{-1} and the* **stable invariant tangent bundle** *of p is the unstable invariant tangent bundle of p with respect to f^{-1}.*

Let $\{f_a\}_{a\in V}$ be a smooth family of diffeomorphisms in an n-dimensional smooth manifold M. This means that $a \in V \subset \mathbb{R}^m$ is a parameter, V is an open subset and that there exists a smooth map $f : M \times V \mapsto M$ such that $f_a = f(\cdot, a)$ and each $f_a : M \mapsto M$ is a diffeomorphism.

Let us suppose that for $a = a_0$, f_{a_0} has a dissipative saddle fixed point $p_0 \in M$. Taking V smaller if necessary, we can assume that for all $a \in V$, f_a has a dissipative saddle fixed point $p(a)$.

Assume that q is a nontransverse homoclinic point of the map f_{a_0}. Then, for values of the parameter near a_0, for points r near q, and $n \in \mathbb{N}$ large enough (depending on the distance between q and r and between a and a_0) $f_a^n(r)$ is near q. This allows us to define a return map near the homoclinic point. There are some known cases in which this return map has a simple limit (in a suitable way) when n tends to infinity:

- When $n = 2$, $m = 1$ and there exists a generically unfolding quadratic homoclinic tangency, the limit map is the logistic map. Several phenomena can be observed as: existence of a nonzero Lebesgue measure set A, in the parameter space, such that if $a \in A$ the map f_a exhibits a strange attractor (see Mora and Viana [13]); and coexistence of infinitely many attractors (see e. g. Newhouse [14] and Palis and Takens [16]). Moreover, the nontransverse homoclinic orbit is in the closure of the parameter values for which there are attracting periodic orbits, and it is possible to get some information about the shape of the basin of attraction of these orbits (see Tatjer and Simó [21]).

- When n is arbitrary, $m = 1$, $\dim W^u(p_0) = 1$, there exists a generically unfolding quadratic homoclinic tangency and p_0 is sectionally dissipative, then the limit return map is the logistic map. There are similar results concerning the coexistence of

infinitely many sinks (see Palis and Viana [17]) and strange attractors (see Viana [22]).

- When $n = 2$, $m = 2$ and the nontransverse homoclinic tangency is cubic, then the limit map is a one dimensional cubic map. In this case it is possible to prove the existence of cusp and codimension two flip bifurcations associated to this bifurcation (see Tatjer [20]).

- When n is arbitrary, $m = 1$ and there is a hyperbolic fixed point of index of stability $1 \leq s \leq n-2$, such that, for some value of the parameter, its stable and unstable invariant manifolds have a generically unfolding quadratic tangency of codimension greater or equal than 2 then, in general, infinitely many sinks or strange attractors do not appear. However, analogous results to the first and second items can be proven, if sinks are replaced by periodic orbits of index $s+1$ and strange attractors by strange saddle sets. See Romero [18].

Definition 1.3 *(see Viana [22]) We define an* **attractor** *of a transformation f to be a compact, f-invariant and transitive set A whose basin $W^s(A) = \{z \in M : d(f^n(z), A) \to 0$ as $n \to \infty\}$ has nonempty interior. We call the attractor* **strange** *if it contains a dense orbit $\{f^n(z_1) : n \geq 0\}$ displaying exponential growth of the derivative:*

$$\|Df^n(z_1)\| \geq e^{cn} \text{ for all } n \geq 0 \text{ and some } c > 0.$$

This definition does not specify how many expanding directions the map f has at the point z_1. In any case, all the strange attractors found in the cited cases have only one expanding direction. Another definition of strange attractor with two expanding directions is given in Viana [23].

The main goal of this paper is to obtain other cases in dimension $n = 3$ for which there exist limit return maps. This means to delete the condition of sectional dissipativeness. As, in general, a one-parameter family of diffeomorphisms without this condition, does not have such a limit map, we must add one more parameter ($m = 2$) and a new condition over the homoclinic tangency. This new type of homoclinic tangency will be called *generalized homoclinic tangency*, and will be defined in section 2. The existence of these limit return maps allow us to find some phenomena associated to this new type of tangencies.

The role that the saddle-node bifurcation plays in the two dimensional case ($n = 2$ and $m = 1$) will be played in the three dimensional case by the Bogdanov-Takens (BT) bifurcation. We recall that for such a bifurcation of periodic points, the corresponding spectrum has one double eigenvalue equal to 1, with associated eigenspace of dimension 1. See section 2 for more details. In particular, we shall find invariant circles, and therefore, quasiperiodic behaviour.

Before giving our main results we will introduce a linearization assumption, that simplify the study of families of diffeomorphisms, and the definition of families of *limit return maps* associated to a homoclinic tangency.

From now on, we suppose that the family $\{f_a\}_{a \in V}$ is defined on a smooth manifold M of dimension three, $p(a)$ is a saddle for all $a \in V$, and the eigenvalues of $p(a)$ satisfy $|\lambda_1| < |\lambda_2| < |\lambda_3|$.

Definition 1.4 *We say that* $\{f_a\}_{a\in V}$ *satisfies the* **linearization assumption** *with respect to the map of fixed points* $p(a)$ *if there exist an open (unbounded) set* $U \subset \mathbb{R}^{3+k}$ *and a* C^s *(*$s \geq 3$*) map* $\mathbf{x} : U \mapsto M$ *such that:*

1. $\Lambda(U) = U$, *where* $\Lambda(t_1, t_2, t_3, a) = (\lambda_1 t_1, \lambda_2 t_2, \lambda_3 t_3, a)$, *for* $(t_1, t_2, t_3, a) \in \mathbb{R}^{3+k}$.
2. U *is either a neighbourhood of* $\{(t_1, t_2, t_3) \ : \ (t_1^2 + t_2^2)t_3 = 0\} \times V$, *if* $|\lambda_1| < |\lambda_2| < 1$ *and* $|\lambda_3| > 1$, *or a neighbourhood of* $\{(t_1, t_2, t_3) \ : \ (t_2^2 + t_3^2)t_1 = 0\} \times V$, *if* $|\lambda_1| < 1 < |\lambda_2| < |\lambda_3|$.
3. $\mathbf{x}(0, 0, 0, a) = p(a)$, *for all* $a \in V$.
4. $\mathbf{x}(\lambda_1 t_1, \lambda_2 t_2, \lambda_3 t_3, a) = f_a(\mathbf{x}(t_1, t_2, t_3, a))$, *for* $(t_1, t_2, t_3, a) \in U$.

We note that if the family is linearizable the invariant manifolds have natural parametrizations. For example, if $|\lambda_1| < |\lambda_2| < 1 < |\lambda_3|$ then:

$$W^s(p(a)) = \{q \in M \ : \ q = \mathbf{x}(t_1, t_2, 0, a) \text{ for some } (t_1, t_2) \in \mathbb{R}^2\},$$

$$W^u(p(a)) = \{q \in M \ : \ q = \mathbf{x}(0, 0, t_3, a) \text{ for some } t_3 \in \mathbb{R}\}.$$

On the other hand, the leaves of the strong stable foliation of $p(a)$ are parametrized by the maps $\mathbf{x}(t, \bar{t}_2, 0, a)$ with t a real parameter small enough and $\bar{t}_2$ a small constant. Moreover, the fiber of the unstable tangent vector bundle at a point $\mathbf{x}(0, 0, t_3, a)$ of $W^u(p(a))$ is generated by the vectors $D_2\mathbf{x}(0, 0, t_3, a)$ and $D_3\mathbf{x}(0, 0, t_3, a)$ (tangent to $W^u(p(a))$).

In the case $|\lambda_1| < 1 < |\lambda_2| < |\lambda_3|$, the leaves of $\mathcal{F}^{uu}(p(a))$ are parametrized by $\mathbf{x}(0, \bar{t}_2, t, a)$ and the fiber of the stable vector bundle at $\mathbf{x}(t_1, 0, 0, a)$ is generated by the vectors $D_1\mathbf{x}(t_1, 0, 0, a)$ and $D_2\mathbf{x}(t_1, 0, 0, a)$.

The linearization assumption is a generic condition (open and dense), for families of diffeomorphisms having saddle fixed points. Indeed, we only need to have a C^s conjugacy of the map near the fixed point to its linear part. See, e.g. Banyaga et al. [1]. Then we use the maps f_a and f_a^{-1} to build up the map $\mathbf{x}$. Moreover, we can use, for all $a \in V$, the inverse of $\mathbf{x}(\cdot, a)$ in some neighbourhood of the fixed point $p(a)$, as a local chart of M. In these coordinates we have that $g_a^n(t_1, t_2, t_3) = \mathbf{x}_a^{-1}\mathbf{x}(\lambda_1^n t_1, \lambda_2^n t_2, \lambda_3^n t_3, a)$, where $\mathbf{x}_a(t_1, t_2, t_3) = \mathbf{x}(t_1, t_2, t_3, a)$, $g_a^n = \mathbf{x}_a^{-1} \circ f_a^n \circ \mathbf{x}_a$, and (t_1, t_2, t_3) and $\mathbf{x}(t_1, t_2, t_3, a)$ are in a small neighbourhood of $(0, 0, 0)$.

The following remark will be used repeatedly.

Remark 1.5 *Without loss of generality, when we take a family* $\{f_a\}_{a\in V}$ *of diffeomorphisms, satisfying the linearization assumption, we can suppose that* $M = \mathbb{R}^3$ *and* $p(a) = 0$, *for all* $a \in V$. *Moreover, consider the sets* $A_1 = \{(t_1, t_2, t_3) \in \mathbb{R}^3 \ : \ (t_1^2 + t_2^2)t_3 = 0\} \times V$, $A_2 = \{(t_1, t_2, t_3) \in \mathbb{R}^3 \ : \ (t_2^2 + t_3^2)t_1 = 0\} \times V$, $B_1 = \{(t_1, t_2, 0) \ : \ t_1^2 + t_2^2 \leq 3\}$ *and* $B_2 = \{(t_1, 0, 0) \ : \ |t_1| \leq 3\}$. *The linearization assumption allow us to suppose (via a change of coordinates and reparametrization if necessary) that if* $|\lambda_1| < |\lambda_2| < 1 < |\lambda_3|$ *(resp.* $|\lambda_1| < 1 < |\lambda_2| < |\lambda_3|$*) there exists a* C^k*-map (*$k \geq 3$*)* $\mathbf{x}_f$ *defined on a neighbourhood* U_f *of* A_1 *(resp.* A_2*) such that:*

1. *If* $(t_1, t_2, t_3, a) \in U_f$ *then* $(\lambda_1 t_1, \lambda_2 t_2, \lambda_3 t_3) \in U_f$, *and* $f_a(\mathbf{x}_f(t_1, t_2, t_3)) = \mathbf{x}_f(\lambda_1 t_1, \lambda_2 t_2, \lambda_3 t_3, a)$.

2. *For some neighbourhood $U_{1,f}$ of B_1 (resp. B_2) the map $\mathbf{x}_f(\cdot,\cdot,\cdot,a)$ restricted to $U_{1,f}$ is the identity, for all $a \in V$.*

Definition 1.6 *Let $\{f_a\}_{a\in V}$ be a smooth family of diffeomorphisms in some m-dimensional manifold M, depending on a parameter $a \in V \subset \mathbb{R}^k$, V being an open subset. Suppose that for $a = a_0$ there exists a homoclinic orbit O_0 of some dissipative fixed point. We say the the family $\{f_a\}_{a\in V}$ has a family of* **limit return map** *associated to the homoclinic orbit O_0, in the C^l topology, if there exist a point of the orbit q, and a natural number N, such that for any positive integer $n \geq N$ there exist reparametrizations $a = M_n(\tilde{a})$ of the a variable and $\tilde{a}$-dependent coordinate transformations $x = \Psi_{n,\tilde{a}}(\tilde{x})$ satisfying the following properties:*

1. *For each compact set K in the $(\tilde{a},\tilde{x})$ space the images of K under the maps*

$$(\tilde{a},\tilde{x}) \mapsto (M_n(\tilde{a}), \Psi_{n,\tilde{a}}(\tilde{x}))$$

   converge, for $n \to \infty$, in the (a,x) space, to (a_0,q).

2. *The domains of the maps*

$$(\tilde{a},\tilde{x}) \mapsto (\tilde{a}, (\Psi^{-1}_{n,\tilde{a}} \circ f^n_{M_n(\tilde{a})} \circ \Psi_{n,\tilde{a}}))$$

   converge, for $n \to \infty$, to all of $\mathbb{R}^{m+k}$, and the maps converge, in the C^l topology, to some map of the form

$$(\tilde{a},\tilde{x}) \mapsto (\tilde{a}, \tilde{f}_{\tilde{a}}(\tilde{x})).$$

In such a case, the map $\tilde{f}_{\tilde{a}}$ will be called a limit return map *and $\{\tilde{f}_{\tilde{a}}\}_{\tilde{a}\in\mathbb{R}^k}$ a family of limit return maps associated to the homoclinic orbit O_0.*

In the next theorem we study a special type of codimension-two homoclinic tangency, that we call generalized homoclinic tangency. Suppose, as before, that $p(a)$ is a saddle fixed point of f_a, $a \in V \subset \mathbb{R}^2$, with eigenvalues satisfying $|\lambda_1| < |\lambda_2| < |\lambda_3|$. We distinguish the cases $|\lambda_2| < 1$ and $|\lambda_2| > 1$.

- Let $|\lambda_2| < 1$. The following properties hold for a family $\{f_a\}_{a\in V}$ having a **generalized homoclinic tangency of type I** (resp. of **type II**) between $W^s(p(a))$ and $W^u(p(a))$ at $q \in W^s(p(a_0)) \cap W^u(p(a_0))$.

  1. $W^s(p(a_0))$ and $W^u(p(a_0))$ have a quadratic tangency at q, and q does not belong to the strong stable manifold of $p(a_0)$.
  2. $W^u(p(a_0))$ is tangent (resp. not tangent) at q to the leaf of $\mathcal{F}^{ss}(p(a_0))$ at q.
  3. The fiber of the unstable tangent bundle of $p(a_0)$ at q is not tangent (resp. tangent) to $W^s(p(a_0))$ at q.

- If $|\lambda_2| > 1$ we say that $\{f_a\}_{a\in V}$ has a generalized homoclinic tangency if $\{f_a^{-1}\}_{a\in V}$ has a generalized homoclinic tangency.

These types of tangencies will be studied in detail in section 2.

Our main theorem describes the dynamical behaviour of a two-parameter family of three-dimensional diffeomorphisms having a generalized homoclinic tangency, in several situations. We note that the corresponding fixed point is never sectionally dissipative.

Theorem 1 *Let $\{f_{a,b}\}_{a,b}$ be a two-parameter family of three dimensional diffeomorphisms, having a hyperbolic dissipative fixed point p_0 for $(a,b)=(0,0)$ with eigenvalues λ_1, λ_2 and λ_3. Assume that either $|\lambda_1|<|\lambda_2|<1$, $|\lambda_3|>1$, $|\lambda_1\lambda_3|<1$ and $|\lambda_2\lambda_3|>1$ (case A), or $|\lambda_1|<1$, $|\lambda_3|>|\lambda_2|>1$ (case B). Moreover, we suppose that the invariant manifolds of p_0 have a generalized homoclinic tangency which unfolds generically and the family $f_{a,b}$ satisfies the linearization assumption. Then:*

1. *For n large enough there are values of the parameter (a_n,b_n) for which the map $f_{a,b}$ undergoes a generic n-periodic Bogdanov-Takens bifurcation. Moreover $(a_n,b_n)\to(0,0)$ when $n\to\infty$.*

2. *There exists a family of limit return maps $\{\tilde{f}_{\tilde{a},\tilde{b}}\}_{(\tilde{a},\tilde{b})\in\mathbb{R}^2}$ associated to the generalized homoclinic tangency, such that*

   (a) *If $|\lambda_1|<|\lambda_2|<1$, $|\lambda_3|>1$, $|\lambda_1\lambda_3|<1$ and $|\lambda_2\lambda_3|>1$ then*

   $$\tilde{f}_{\tilde{a},\tilde{b}}(\tilde{x},\tilde{y},\tilde{z})=(\tilde{z},\tilde{b}\tilde{z},\tilde{a}+\tilde{y}+\tilde{z}^2),$$

   or

   $$\tilde{f}_{\tilde{a},\tilde{b}}(\tilde{x},\tilde{y},\tilde{z})=(0,\tilde{z},\tilde{a}+\tilde{b}\tilde{y}+\tilde{z}^2),$$

   depending on the type of generalized homoclinic tangency (I or II resp).

   (b) *If $|\lambda_1|<1$, $|\lambda_3|>|\lambda_2|>1$ then*

   $$\tilde{f}_{\tilde{a},\tilde{b}}(\tilde{x},\tilde{y},\tilde{z})=(\tilde{z},\tilde{a}+\tilde{b}\tilde{y}+\tilde{z}^2,\tilde{y}).$$

3. *$(a,b)=(0,0)$ is in the closure of the set of parameter values for which there exist differentiable invariant circles. In case A, the circles can be of attracting or saddle type (attracting or repelling inside the centre manifold), depending on the geometry of the tangency. In case B the circles are always attracting.*

4. *$(a,b)=(0,0)$ is in the closure of the set of parameter values for which there are attracting periodic orbits.*

5. *There is a set E of parameter values such that its intersection with any neighbourhood of $(0,0)$ has positive Lebesgue measure, and for $(a,b)\in E$ the diffeomorphism $f_{a,b}$ exhibits a strange attractor near the orbit of tangency.*

6. *There are open sets $U\subset\mathbb{R}^2$, arbitrarily near $(0,0)$ such that for a generic $(a,b)\in U$ the map $f_{a,b}$ has infinitely many sinks.*

Remark 1.7 *The two quadratic maps that appear in the previous theorem (case A) are conjugate if $\tilde{b}\neq 0$. When $\tilde{b}=0$ they are not conjugate but if we restrict the first map to the plane $\tilde{y}=0$ and the second to $\tilde{x}=0$, we obtain the same two-dimensional map. These restrictions are logistic maps in both cases.*

At first glace, one can see that the limit return maps in theorem 1 case A are essentially the Hénon map and in theorem 1 case B the non-invertible quadratic map that we called $f_{a,b}$ in the previous subsection. This means that the second situation (when the unstable invariant manifold has dimension two) is the natural setting in which topologically two dimensional strange attractors must show up. The main difference between theorem 1 and the results of Viana in the sectionally dissipative case is the existence of Bogdanov-Takens and Hopf bifurcations, and (attracting or saddle) invariant circles. As we shall see, the return maps in case A for which these bifurcations occur can be seen, restricted to a centre manifold of dimension two, as 'perturbations of the conservative Hénon map'.

We want to point out that the strange attractors of theorem 1 are topologically one-dimensional. We are not able to prove, for the moment, the existence of two-dimensional strange attractors. As far as we know, only Viana [23] has shown the existence of such attractors, but for a simpler case.

On the other hand, the shape of any strange attractor is very complicated. Indeed, it is the closure of an unstable invariant manifold. However, as we shall see, near a generalized homoclinic tangency there are one-dimensional strange attractors contained in a two-dimensional center manifold. However, this would not be the case for the topologically two-dimensional strange attractors. In this sense, the interesting dynamical phenomena is fully three-dimensional.

The last theorem involves the classification of certain quadratic maps in dimensions two and three. As it is known, all the nontrivial quadratic maps in $\mathbb{R}^2$ with constant jacobian are conjugate to a map of the Hénon family. We shall prove that the limit return map of theorem 1 (case B) is also representative of all the quadratic maps in $\mathbb{R}^2$ whose second iterate is also quadratic. Moreover, we find the simplest quadratic map in $\mathbb{R}^3$ that can be taken as a model to study chaotic behaviour. We note that the two model quadratic maps are precisely $f_{a,b}$ and $g_{a,b,c}$ which we used to give two examples of two-dimensional strange attractors, in the first section of the introduction.

Theorem 2 *1. Let $f : \mathbb{R}^2 \mapsto \mathbb{R}^2$ be a quadratic map such that $f \circ f$ is also quadratic. Suppose that f does not have any invariant linear foliation. Then it is linearly conjugate to the map $(x, y) \mapsto (a + bx - y^2, x)$ for some real numbers a and b.*

2. Let $f : \mathbb{R}^3 \mapsto \mathbb{R}^3$ be a quadratic map such that $f \circ f$ is also quadratic and having constant jacobian. Suppose that f does not have any invariant linear foliation. Then it is linearly conjugate to the map $(x, y, z) \mapsto (a + bx + z - y^2, x, cy)$, for some real numbers a, b and c.

It seems natural, in this moment, that to prove the existence of topologically 2D strange attractors, it is necessary, first, to prove that the maps of the non-invertible family of theorem 1 (case B) exhibit strange attractors for a set of non-zero Lebesgue measure in the plane of parameters. After this, one has to show that the maps of the three-dimensional quadratic family of theorem 2 also exhibit strange attractors for a very small c (constant of dissipativeness). Finally, a theorem à la Mora-Viana would be precise to prove that the return map of theorem 1 (case B) has also strange attractors of dimension two.

Other related topics that we are considering in a next future are: the effective computation of bifurcations and strange attractors in a family of diffeomorphisms displaying

generalized homoclinic tangencies and the study of more global behaviour near a generalized homoclinic tangency in the way of Broer et al. [5] for the two-dimensional case.

Finally, it would be interesting to look for semi-global examples of this new type of homoclinic bifurcation. In the case of codimension-two homoclinic bifurcations for singular points, in three-dimensional ordinary differential equations, some examples were found by Sandstede [19]. Our case is more involved, but one can expect that generalized homoclinic bifurcations can appear in suitable three-dimensional differential equations depending periodically on time.

Remark 1.8 *1. According to Gonchenko et al. [8, 9], the maps given in theorem 2 can be seen as approximations of return maps of special familes of diffeomorphisms having a (non-generalized) quadratic homoclinic tangency, when some of the eigenvalues of the initial fixed point are not real. For the second three-dimensional quadratic map the dimension of the phase space has to be at least four. However, to have these approximations it is necessary to impose very strong conditions to the eigenvalues of the fixed point (in the Lebesgue measure sense).*

2. As far as we know, the first study of the non-invertible family of maps of theorem 2, from the point of view of dynamic behaviour, was done by Whitley [24]. He gave properties of the non-wandering set, traced the genealogies of the periodic points of period ≤ 5, and described their stability types and bifurcations. Likewise, this map has been used as a discrete model of density-dependent population growth (see Maynard Smith [12]). Also Mira and co-workers have studied non-invertible maps (see e. g. [7] and references therein) from a phenomenological point of view.

The paper is organized as follows. In section 2 we introduce the definition of generalized homoclinic tangency and the Bogdanov-Takens bifurcations, giving the conditions of generic unfolding. In section 3 we obtain bifurcations associated to the existence of homoclinic tangencies in the one-parameter case, and to the generalized homoclinic tangency in the two-parameter case, and other related results. In section 4 we prove the existence of the limit return maps and theorem 1. Finally, theorem 2 is shown in section 5.

In what follows we always consider smooth maps. This could be replaced by differentiable enough maps.

2 Unfoldings of homoclinic tangencies and bifurcations

We have divided this section in four parts. In the first part, we study the generic unfolding of a quadratic tangency between a curve and a surface in a three-dimensional space. We use this to define a homoclinic quadratic tangency unfolding generically. In the second part, we define what we call generalized transversality between a curve and a surface. In this case it is supposed that there is a foliation in the surface. Then we give the definition of generalized homoclinic transversality. It is necessary in order to describe the behaviour of the generalized (quadratic) homoclinic tangency that is defined in the fourth section. We write out in all the cases the conditions of generic unfolding that are needed for the proof

of our main theorem. Finally, the conditions of existence of a generic Bogdanov-Takens bifurcation are obtained in the last part.

2.1 Unfolding of quadratic tangencies

The purpose of this section is to define a generic unfolding of the tangency of a curve and a surface in $\mathbb{R}^3$, when these objects depend on one real parameter.

First, we begin with the study of the behaviour of a quadratic tangency between a surface and a curve that depend on a parameter. For example, the plane $z = 0$ and the parabola $z = x^2 + a$, $y = 0$, have a quadratic tangency at $(x, y, z) = (0, 0, 0)$ for $a = 0$, that unfolds generically.

Definition 2.1 *Let $\{C_a\}_{a\in V}$, be a smooth family of smooth regular curves $C_a \subset \mathbb{R}^3$, and $\{S_a\}_{a\in V}$ be a smooth family of smooth regular surfaces $S_a \subset \mathbb{R}^3$ depending on a parameter $a \in V \subset \mathbb{R}$, V being an open subset, such that, for $a = a_0$, C_a and S_a intersect at a point $p_0 \in \mathbb{R}^3$. We say that C_a and S_a have a* **quadratic tangency** *at p_0 (or a contact of order 1) which unfolds generically with a at $a = a_0$ iff there exists some smooth change of variables, such that in the new variables $p_0 = (0, 0, 0)$, $S_a = \{(x, y, z) \in U : z = 0\}$, where U is a neighbourhood of $0 \in \mathbb{R}^3$, and C_a is represented by the parametrized curve $\gamma(t, a) = (x(t, a), y(t, a), z(t, a))$, and the following properties hold:*

1. $z(0, a_0) = 0$,
2. $D_1 z(0, a_0) = 0$,
3. $D_{11} z(0, a_0) \neq 0$ *and*
4. $D_2 z(0, a_0) \neq 0$.

It is easy to see that this definition does not depend on the change of variables used to set S_a as the plane of equation $z = 0$.

The manifolds C_a and S_a can be represented in the following way: $S_a = \{(x, y, z) \in U : f(x, y, z, a) = 0\}$, where U is an open neighbourhood of $p_0 = (x_0, y_0, z_0)$, and $C_a = \{\gamma(t, a) : t \in I \subset \mathbb{R}\}$, where I is an open interval, and $f(p_0, a_0) = 0$, and $\gamma(0, a_0) = p_0$. Then, it is easy to prove the following proposition:

Proposition 2.2 *Suppose that C_a and S_a have a quadratic tangency at p_0 unfolding generically at $a = a_0$, and f and γ are as before. Then:*

1. $Df_0(p_0)D_1\gamma(0, a_0) = 0$,
2. $D^2 f_0(p_0)(D_1\gamma(0, a_0), D_1\gamma(0, a_0)) + Df_0(p_0)D_{11}\gamma(0, a_0) \neq 0$,
3. $Df_0(p_0)D_2\gamma(0, a_0) + D_2 f(p_0, a_0) \neq 0$,

where $f_0 = f(\cdot, \cdot, \cdot, a_0)$.

The next proposition gives the behaviour of quadratic tangencies which unfold generically.

Proposition 2.3 *Let C_a and S_a be as in the beginning of the section, and $p_0 \in C_{a_0} \cap S_{a_0}$. If C_a and S_a has a quadratic tangency at p_0 which unfolds generically with a at $a = a_0$ then there are an open neighbourhood I_0 of $0 \in \mathbb{R}$, and a smooth map $a = a(t) \in \mathbb{R}$ such that:*

1. *$a(0) = a_0$, $\dot{a}(0) = 0$ and $\ddot{a}(0) \neq 0$.*
2. *$\gamma(t, a(t)) \in S_{a(t)}$ for all $t \in I_0$.*

For future use, we want to obtain simple necessary conditions of quadratic tangencies when the curve does not change with the parameter and is one of the coordinate axes. More precisely, suppose that $\gamma(t, a) = (t, 0, 0)$, and the surface S_a has the following parametrization:

$$\mathbf{x}(u, v, a) = (X(u, v, a), Y(u, v, a), Z(u, v, a)).$$

As $p_0 = (x_0, 0, 0) \in S_{a_0}$, there exists $(u_0, v_0) \in \mathbb{R}^2$ such that $p_0 = \mathbf{x}(u_0, v_0, a_0)$. If we define the map $g(u, v, a) = D_1Y(u, v, a)D_2Z(u, v, a) - D_2Y(u, v, a)D_1Z(u, v, a)$, we get the following result.

Proposition 2.4 *If the curve γ and the family of surfaces S_a have a quadratic tangency at $(x_0, 0, 0)$ which unfolds generically with a at a_0 then:*

1. *$Y(u_0, v_0, a_0) = Z(u_0, v_0, a_0) = 0$.*
2. *$g(u_0, v_0, a_0) = 0$.*
3. *If $g_0 = g(\cdot, \cdot, a_0)$, $Y_0 = Y(\cdot, \cdot, a_0)$ and $Z_0 = Z(\cdot, \cdot, a_0)$ then*

$$rank\,(Dg_0(u_0, v_0), DY_0(u_0, v_0), DZ_0(u_0, v_0)) = 2.$$

4. *$rank\,(DY(u_0, v_0, a_0), DZ(u_0, v_0, a_0)) = 2$.*

Proof:

We choose a map $G : U \times U_1 \subset \mathbb{R}^4 \longrightarrow \mathbb{R}^3$ such that:

1. U is an open neighbourhood of $(u_0, v_0, 0)$ and U_1 is an open neighbourhood of a_0.
2. $G_a : U \subset \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ is a diffeomorphism for all $a \in U_1$, where $G_a(u, v, w) = G(u, v, w, a)$.
3. If $(u, v, 0) \in U$ then $G(u, v, 0, a) = \mathbf{x}(u, v, a)$, for all $a \in U_1$.

This map can be easily obtained, taking into account that the surfaces S_a are regular. For example, if $D_1XD_2Y - D_1YD_2X \neq 0$, where the derivatives are evaluated at (u_0, v_0, a_0), we can take $G(u, v, w, a) = (X(u, v, a), Y(u, v, a), Z(u, v, a) + w)$.

Using G_a as a change of variables and applying definition 2.1, it is easy to see that if there exists a quadratic tangency in p_0 then $Y(u_0, v_0, a_0) = Z(u_0, v_0, a_0) = 0$ and $g(u_0, v_0, a_0) = 0$. This proves items 1 and 2 of the proposition.

On the other hand, as S_a is a regular surface, the previous result allow us to take the map G of the beginning of the proof as $G(u, v, w, a) = (X(u, v, a), Y(u, v, a) +$

$g_2(w), Z(u,v,a) + g_3(w))$, where either $g_2(w) = w$ and $g_3(w) = 0$ or $g_2(w) = 0$ and $g_3(w) = w$. Taking again the new map G_a as a change of variables we obtain the rest of the proposition from items 3 and 4 of definition 2.1. □

In order to define quadratic tangencies bewteen invariant manifolds (quadratic homoclinic tangencies), we will add a generic condition. It concerns the position of an orbit inside a two dimensional invariant manifold.

Definition 2.5 *Let f be a three-dimensional diffeomorphism, defined on a neighbourhood of a hyperbolic saddle fixed point p, with eigenvalues λ_i, $i = 1,2,3$ which are supposed to be real and different in absolute value. We denote by Λ the linear map $\Lambda(t_1,t_2,t_3) = (\lambda_1 t_1, \lambda_2 t_2, \lambda_3 t_3)$. Let W and $\tilde{W} \subset W$ be, respectively, the two-dimensional (stable or unstable) invariant manifold of p, and the strong stable or unstable manifold contained in W. We say that $q \in W$ is in* **general position** *if $q \notin \tilde{W}$ and there exists some smooth conjugacy F between f and Λ ($\Lambda = F \circ f \circ F^{-1}$) such that the orbit of $F(q)$, corresponding to the map Λ, has at least one point out of the coordinate axes.*

In many cases the condition for an orbit to be in general position is equivalent to be in the two dimensional stable or unstable invariant manifold but not in the corresponding strong (stable or unstable) one-dimensional manifold.

Suppose, for example, that the fixed point of the map f of the preceding definition has eigenvalues satisfying $0 < |\lambda_1| < |\lambda_2| < 1 < |\lambda_3|$, and that f itself is the linear map Λ, that is $f(x,y,z) = (\lambda_1 x, \lambda_2 y, \lambda_3 z)$. Moreover, we assume that there exists a natural number n such that $|\lambda_1 \lambda_2^{-n}| < 1$, and we allow the conjugacy of the definition to be a C^n diffeomorphism. If we take a point (x_0, y_0, z_0) in $W^s(0)$ but not in $W^{ss}(0)$ (the strong stable invariant manifold), then $z_0 = 0$ and $y_0 \neq 0$. We claim that the orbit of this point is in general position.

In fact, according to the definition, if $x_0 \neq 0$ then the assertion is true. On the other hand, if $x_0 = 0$ we can find a nontrivial C^n diffeomorphism G which commutes with Λ (that is $G \circ \Lambda = \Lambda \circ G$), and does not preserve the line $x = 0$. In the case $\lambda_1 > 0$ we take $G(x,y,z) = (x + C|y|^\beta, y, z)$, where C is a constant different from zero and $\beta = \log|\lambda_1| / \log|\lambda_2|$. For $\lambda_1 < 0$ $G(x,y,z) = (x + C \sin[\pi(\log|\lambda_2|)^{-1} \log|y|]|y|^\beta, y, z)$, where C and β are as before. This proves the claim.

Next, we can give the definition of quadratic homoclinic tangency.

Let $\{f_a\}_{a \in V}$ be a family of three-dimensional diffeomorphisms, having a saddle fixed point $p = p(a)$, for all $a \in V$. Suppose that the eigenvalues of $Df_a(p(a))$ that we denote by λ_i, $i = 1,2,3$, are all real and different in absolute value: $|\lambda_1| < |\lambda_2| < |\lambda_3|$. Moreover $|\lambda_1| < 1$ and $|\lambda_3| > 1$.

Definition 2.6 *We say that $\{f_a\}_{a \in V}$ has a* **quadratic homoclinic tangency** *at $q \in W^s(p(a_0)) \cap W^u(p(a_0))$ unfolding generically with a at $a = a_0$ if the manifolds $W^u(p(a))$ and $W^s(p(a))$ have a quadratic tangency at q which unfolds generically at $a = a_0$, and q is in general position.*

A typical quadratic homoclinic tangency is shown in figure 4.

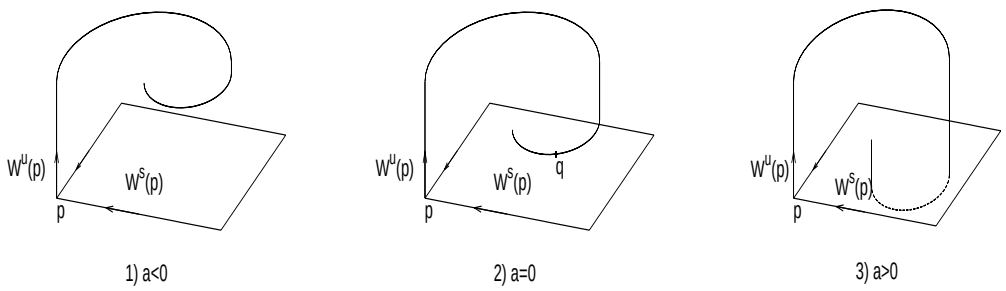


Figure 4: Quadratic homoclinic tangency. 1) No intersection of the invariant manifolds. 2) A tangencial intersection. 3) Two transversal intersections.

Now suppose that $\{f_a\}_{a\in V}$, has a quadratic homoclinic tangency at $q \in W^u(p(a_0)) \cap W^s(p(a_0))$ unfolding generically with a at $a_0 \in V$, and satisfies the linearization assumption. Without loss of generality, we can assume that $a_0 = 0$ and $V = (-\epsilon, \epsilon)$.

In the next two propositions we will give changes of coordinates and reparametrizations of $\{f_a\}_{a\in V}$, depending on the type of saddle, that simplify its study.

First, we consider the case in which the stable invariant manifold is two-dimensional. Using the next proposition, we can suppose that, near the point of quadratic homoclinic tangency, of coordinates $(x, y, z) = (1, 1, 0)$, $W^s(p(a))$ is the xy-coordinate plane and $W^u(p(a))$ has a local extremum with respect to the z coordinate at $(x, y, z) = (1, 1, a)$, for all $a \in (-\epsilon, \epsilon)$.

Proposition 2.7 *Suppose that $|\lambda_1| < |\lambda_2| < 1 < |\lambda_3|$. Then there exist an open neighbourhood $V_1 \subset V$ of a_0, a family $\{g_{\tilde{a}}\}_{\tilde{a}\in(-\epsilon,\epsilon)}$ conjugate to $\{f_a\}_{a\in V_1}$, and a C^k ($k \geq 3$) map $\mathbf{x} = \mathbf{x}(t_1, t_2, t_3, \tilde{a})$, defined on a neighbourhood U of the set $A_1 = \{(t_1, t_2, t_3) \in \mathbb{R}^3 : (t_1^2 + t_2^2)t_3 = 0\} \times (-\epsilon, \epsilon)$, such that:*

1. *The conjugacy transforms the fixed point $p(a)$ to $\tilde{p}(\tilde{a}) = (0, 0, 0)$, for all $\tilde{a} \in (-\epsilon, \epsilon)$.*
2. *The family $\{g_{\tilde{a}}\}_{\tilde{a}\in(-\epsilon,\epsilon)}$ has a quadratic homoclinic tangency at the point $\tilde{q} = (1, 1, 0) \in W^u((0, 0, 0)) \cap W^s((0, 0, 0))$ unfolding generically with $\tilde{a}$ at $\tilde{a} = 0$.*
3. *There exists a neighbourhood U_1 of the set $B_1 = \{(t_1, t_2, 0) : t_1^2 + t_2^2 \leq 3\}$, such that for all $\tilde{a} \in (-\epsilon, \epsilon)$, the map $\mathbf{x}(\cdot, \cdot, \cdot, \tilde{a})$ restricted to U_1 is the identity.*
4. *If $(t_1, t_2, t_3, \tilde{a}) \in U$ then $(\lambda_1 t_1, \lambda_2 t_2, \lambda_3 t_3, \tilde{a}) \in U$ and $g_{\tilde{a}}(t_1, t_2, t_3) = \mathbf{x}(\lambda_1 t_1, \lambda_2 t_2, \lambda_3 t_3, \tilde{a})$.*
5. *If $\mathbf{x} = (x, y, z)$ then*

$$x(0, 0, 1, \tilde{a}) = 1, \quad y(0, 0, 1, \tilde{a}) = 1, \quad z(0, 0, 1, \tilde{a}) = \tilde{a}, \quad D_3 z(0, 0, 1, \tilde{a}) = 0,$$

for all $\tilde{a} \in (-\epsilon, \epsilon)$, *and* $D_{33} z(0,0,1,0) \neq 0$.

Proof:

We can suppose that the family $\{f_a\}_{a \in V}$ satisfies the statements of Remark 1.5. The invariant manifolds of $p(a) = 0$ can be represented as

$$W^s(p(a)) = \{\, \mathbf{x}_f(t_1, t_2, 0, a) \;:\; (t_1, t_2) \in \mathbb{R}^2 \,\}$$

and

$$W^u(p(a)) = \{\, \mathbf{x}_f(0, 0, t_3, a) \;:\; t_3 \in \mathbb{R} \}.$$

Moreover, locally, the unstable invariant manifold is the z axis and the stable one is the xy-coordinate plane (if we take coordinates x, y, z in $\mathbb{R}^3$).

Taking a suitable point of the orbit of homoclinic tangency, we can always assume that the invariant manifolds are tangent on the plane $z = 0$. As $W^u(p(a))$ and $W^s(p(a))$ have a quadratic homoclinic tangency unfolding generically at $a = 0$, from definition 2.1 we see that there exists a real number $\bar{t}_3 \neq 0$ such that $z_f(0,0,\bar{t}_3,0) = D_3 z_f(0,0,\bar{t}_3,0) = 0$, $D_4 z_f(0,0,\bar{t}_3,0) \neq 0$ and $D_{33} z_f(0,0,\bar{t}_3,0) \neq 0$. Moreover, as the homoclinic point is in general position, $x_f(0,0,\bar{t}_3,0) \neq 0$ and $y_f(0,0,\bar{t}_3,0) \neq 0$. Note that the tangent homoclinic point is $\mathbf{x}_f(0,0,\bar{t}_3,0)$.

As $D_3 z_f(0,0,\bar{t}_3,0) = 0$, $D_{33} z_f(0,0,\bar{t}_3,0) \neq 0$, there exists a map $\tilde{t}_3(a)$, defined on a neighbourhood of zero, such that for all a

$$D_3 z_f(0,0,\tilde{t}_3(a),a) = 0 \text{ and } \tilde{t}_3(0) = \bar{t}_3.$$

Let $g_{\tilde{a}} = \Lambda^{-1}_{H(\tilde{a})} \circ f_{H(\tilde{a})} \circ \Lambda_{H(\tilde{a})}$ and $H = G^{-1}$, where

$$\Lambda_a(t_1, t_2, t_3) = (x(0,0,\tilde{t}_3(a),a)t_1, y(0,0,\tilde{t}_3(a),a)t_2, \tilde{t}_3(a)t_3),$$

$G(a) = z(0,0,\tilde{t}_3(a),a)/\tilde{t}_3(a)$, and H is defined on a neighbourhood of $0 \in \mathbb{R}$. The existence of H follows from the the Inverse Function theorem.

If we set $\mathbf{x}(\cdot,\cdot,\cdot,\tilde{a}) = \Lambda^{-1}_{H(\tilde{a})} \circ \mathbf{x}_f(\cdot,\cdot,\cdot,H(\tilde{a})) \circ \Lambda_{H(\tilde{a})}$, it is easy to show that $g_{\tilde{a}}$ and $\mathbf{x}$ satisfy the required properties. □

In the second proposition, we suppose that the stable invariant manifold is one-dimensional and the unstable one is two-dimensional.

Proposition 2.8 *Suppose that* $|\lambda_1| < 1 < |\lambda_2| < |\lambda_3|$. *Then there exist an open neighbourhood* $V_1 \subset V$ *of* a_0, *a family* $\{g_{\tilde{a}}\}_{\tilde{a} \in (-\epsilon,\epsilon)}$ *conjugate to* $\{f_a\}_{a \in V_1}$, *and a* C^k $(k \geq 3)$ *map* $\mathbf{x} = \mathbf{x}(t_1, t_2, t_3, \tilde{a})$, *defined on a neighbourhood* U *of* $A_2 = \{(t_1,t_2,t_3) \in \mathbb{R}^3 \;:\; (t_2^2 + t_3^2)t_1 = 0\} \times (-\epsilon, \epsilon)$, *such that:*

1. *The conjugacy transforms the fixed point* $p(a)$ *to* $\tilde{p}(\tilde{a}) = (0,0,0)$,
2. *The family* $\{g_{\tilde{a}}\}_{\tilde{a} \in (-\epsilon,\epsilon)}$ *has a quadratic homoclinic tangency at* $\tilde{q} = (1,0,0) \in W^u((0,0,0)) \cap W^s((0,0,0))$ *unfolding generically with* $\tilde{a}$ *at* $\tilde{a} = 0$.
3. *There exists a neighbourhood* U_1 *of* $B_2 = \{(t_1,0,0) \;:\; |t_1| \leq 3\}$, *such that for all* $\tilde{a} \in (-\epsilon, \epsilon)$, *the map* $\mathbf{x}(\cdot,\cdot,\cdot,\tilde{a})$ *restricted to* U_1 *is the identity.*
4. *If* $(t_1,t_2,t_3,\tilde{a}) \in U$ *then* $(\lambda_1 t_1, \lambda_2 t_2, \lambda_3 t_3, \tilde{a}) \in U$ *and* $g_{\tilde{a}}(t_1,t_2,t_3) = \mathbf{x}(\lambda_1 t_1, \lambda_2 t_2, \lambda_3 t_3, \tilde{a})$.

5. *If* $\mathbf{x} = (x, y, z)$ *and*

$$g(t_2, t_3, \tilde{a}) = D_2 y(0, t_2, t_3, \tilde{a}) D_3 z(0, t_2, t_3, \tilde{a}) - D_3 y(0, t_2, t_3, \tilde{a}) D_2 z(0, t_2, t_3, \tilde{a}),$$

then one of the following possibilities holds:

(a) $x(0, 1, 1, \tilde{a}) = 1$, $y(0, 1, 1, \tilde{a}) = 0$, $z(0, 1, 1, \tilde{a}) = \tilde{a}$, $g(1, 1, \tilde{a}) = 0$, *for all* $a \in (-\epsilon, \epsilon)$ *and*

$$D_1 g(1, 1, 0) D_3 y(0, 1, 1, 0) - D_2 g(1, 1, 0) D_2 y(0, 1, 1, 0) \neq 0.$$

(b) $x(0, 1, 1, \tilde{a}) = 1$, $y(0, 1, 1, \tilde{a}) = \tilde{a}$, $z(0, 1, 1, \tilde{a}) = 0$, $g(0, 1, 1, a) = 0$, *for all* $a \in (-\epsilon, \epsilon)$ *and*

$$D_1 g(1, 1, 0) D_3 z(0, 1, 1, 0) - D_2 g(1, 1, 0) D_2 z(0, 1, 1, 0) \neq 0.$$

Proof:
In this case the invariant manifolds have the following representation:

$$W^s(p(a)) = \{ \mathbf{x}_f(t_1, 0, 0, a) \ : \ t_1 \in \mathbb{R} \}$$

and

$$W^u(p(a)) = \{ \mathbf{x}_f(0, t_2, t_3, a) \ : \ (t_2, t_3) \in \mathbb{R}^2 \}.$$

Moreover, locally, the stable invariant manifold is the x axis and the unstable one is the yz-coordinate plane (if we take coordinates x, y, z in $\mathbb{R}^3$).

We can always take the tangent homoclinic point q to be on the straight line $y = 0$, $z = 0$. Therefore, from definition 2.6 and proposition 2.4, we see that there exists $(\bar{t}_1, \bar{t}_2) \in \mathbb{R}^2$ such that $q = \mathbf{x}_f(0, \bar{t}_2, \bar{t}_3, 0)$, $\bar{t}_2 \neq 0$, $\bar{t}_3 \neq 0$, $x_f(0, \bar{t}_2, \bar{t}_3, 0) \neq 0$ (because q is in general position) and $y_f(0, \bar{t}_2, \bar{t}_3, 0) = z_f(0, \bar{t}_2, \bar{t}_3, 0) = 0$, $g_f(\bar{t}_2, \bar{t}_3, 0) = 0$, and for $t_1 = 0$, $t_2 = \bar{t}_2$, $t_3 = \bar{t}_3$ and $a = 0$,

$$\operatorname{rank} \begin{pmatrix} D_1 g_f & D_2 g_f \\ D_2 y_f & D_3 y_f \\ D_2 z_f & D_3 z_f \end{pmatrix} = 2,$$

and

$$\operatorname{rank} \begin{pmatrix} D_2 y_f & D_3 y_f & D_4 y_f \\ D_2 z_f & D_3 z_f & D_4 z_f \end{pmatrix} = 2,$$

where $g_f(t_2, t_3, a) = D_2 y_f(0, t_2, t_3, a) D_3 z_f(0, t_2, t_3, a) - D_3 y_f(0, t_2, t_3, a) D_2 z_f(0, t_2, t_3, a)$.

We distinguish two cases:

a) $D_1 g_f D_3 y_f - D_2 g_f D_2 y_f \neq 0$.

As $g(\bar{t}_2, \bar{t}_3, 0) = 0$, $y(0, \bar{t}_2, \bar{t}_3, 0) = 0$ by the Implicit Function Theorem, the system

$$\begin{aligned} g(t_2, t_3, a) &= 0 \\ y(0, t_2, t_3, a) &= 0 \end{aligned}$$

has a solution $\tilde{t}_2(a)$, $\tilde{t}_3(a)$, for a small enough, such that $\tilde{t}_i(0) = \bar{t}_i$, for $i = 1, 2$.

We claim that $D_2 z_f D\tilde{t}_2(0) + D_3 z_f D\tilde{t}_3(0) + D_4 z_f \neq 0$. Suppose, indeed, that it is equal to zero. Differentiation of $y_f(0, \tilde{t}_2(a), \tilde{t}_3(a), a) = 0$ with respect to a, gives the system

$$\begin{aligned} D_2 y_f D\tilde{t}_2 + D_3 y_f D\tilde{t}_3 &= -D_4 y_f \\ D_2 z_f D\tilde{t}_2 + D_3 z_f D\tilde{t}_3 &= -D_4 z_f \end{aligned}$$

But this system is incompatible by definition of quadratic homoclinic tangency which unfolds generically. Therefore, the claim is true.

Let $\tilde{a}$ be a new parameter such that $\tilde{a} = G(a) = z_f(0, \tilde{t}_2(a), \tilde{t}_3(a), a)/\tilde{t}_3(a)$. Denote by H the inverse of G (that exists by the claim) and $g_{\tilde{a}} = \Lambda^{-1}_{H(\tilde{a})} \circ f_{H(\tilde{a})} \circ \Lambda_{H(\tilde{a})}$, where

$$\Lambda_a(t_1, t_2, t_3) = (x(0, \tilde{t}_2(a), \tilde{t}_3(a), a)t_1, \tilde{t}_2(a)t_2, \tilde{t}_3(a)t_3).$$

Taking $\mathbf{x}(\cdot, \cdot, \cdot, \tilde{a}) = \Lambda^{-1}_{H(\tilde{a})} \circ \mathbf{x}_f(\cdot, \cdot, \cdot, H(\tilde{a})) \circ \Lambda_{H(\tilde{a})}$, properties 1., 2., 3. and 4.(a) follows immediately.

b) $D_1 g D_3 z - D_2 g D_2 z \neq 0$.

We can proceed as before. The main differences are that the maps $\tilde{t}_2$ and $\tilde{t}_3$ satisfy the equations:

$$\begin{aligned} g(t_2, t_3, a) &= 0 \\ z(0, t_2, t_3, a) &= 0 \end{aligned}$$

and in the final step we take $\tilde{a} = y_f(0, \tilde{t}_2(a), \tilde{t}_3(a), a)/\tilde{t}_2(a)$, as the new parameter. Then, we obtain a map $\mathbf{x}$ which satisfies 1, 2, 3(b). □

The latter proposition has also a geometric interpretation. It means that we can obtain suitable variables and parameters for which the invariant manifolds near the point of quadratic homoclinic tangency behaves as follows: $W^s(p(a))$ is the straight line $y = z = 0$ and either (a) $(x, y, z) = (1, 0, a)$ is a local extremum of the z-coordinate of the curve in $W^u(p(a))$ for which the y-coordinate is always zero, or (b) $(x, y, z) = (1, a, 0)$ is a local extremum of the y-coordinate of the curve in $W^u(p(a))$ for which the z-coordinate is always zero.

2.2 Generalized homoclinic transversality

The goal of this section is to introduce a new type of transversality between invariant manifolds of a fixed point.

The first step is to define generalized transversality between a one-parameter family of foliated surfaces and a one-parameter family of vector bundles with a curve as zero section and planes as fibers. We begin with the local definition of foliation:

Definition 2.9 *Let $S \subset \mathbb{R}^3$ a surface and $\Phi = \{\mathcal{L}_\alpha\}_{\alpha \in A}$, A being a set of indices, a partition of S into disjoint sets called* **leaves**. *The partition Φ is called a* **foliation** *if there is a parametrization $\mathbf{x} : I \times J \subset \mathbb{R}^2 \mapsto \mathbb{R}^3$ of S such that $\mathcal{L}_\alpha = \mathbf{x}(I \times \{c_\alpha\})$, where I and J are open intervals, $S = \mathbf{x}(I \times J)$, and c_α is a constant for each $\alpha \in A$. We call the pair (S, Φ) a* foliated surface.

In order to define generalized transversality, we take a family of foliated surfaces $\{(S_a, \Phi_a)\}_{a\in V}$, $V \subset \mathbb{R}$ being an open set, and a family of vector bundles $\{\mathcal{V}_a\}_{a\in V}$ whose zero section is a curve $\gamma = \gamma(t, a)$ and with fibers that are tangent planes to the curve. We can associate to each fiber of the bundle, a vector $v_a(t) = v(t, a) \in T_{\gamma_a(t)}\mathbb{R}^3 \setminus T_{\gamma_a(t)}\gamma_a$, such that v is a smooth map, in such a way $v_a(t)$ and $\dot{\gamma}_a(t)$ generate the fiber. In this situation we say that the family of vector fields $\{v_a\}_{a\in V}$ along the family of curves $\{\gamma_a\}_{a\in V}$, is associated to the family of tangent vector bundles $\{\mathcal{V}_a\}_{a\in V}$.

Definition 2.10 *We say that $\{(S_a, \Phi_a)\}_{a\in V}$ and $\{\mathcal{V}_a\}_{a\in V}$ have a* **generalized transversal intersection** *at $p \in \mathbb{R}^3 \in S_{a_0}$ which unfolds generically with a at $a = a_0 \in V$, iff in some coordinate system, for which the leaves of the foliation are the straight lines $z = 0$, $y = c$, the following properties are satisfied:*

1. $p = \gamma(0, a_0)$,
2. $z(0, a_0) = 0$,
3. $D_1 z(0, a_0) \neq 0$,
4. *If* $T(t, a) = v_2(t, a)D_1 z(t, a) - v_3(t, a)D_1 y(t, a) = 0$ *then* $T(0, a_0) = 0$.
5. $D_1 z(0, a_0)D_2 T(0, a_0) - D_2 z(0, a_0)D_1 T(0, a_0) \neq 0$.

Here $v = (v_1, v_2, v_3)$ is a family of vector fields associated to the family of vector bundles and we write $\gamma(t, a) = (x(t, a), y(t, a), z(t, a))$.

This definition means that, at the point p of generalized transversality, the corresponding surface and curve have a transversal intersection and that the fiber of the vector bundle and the leaf of the foliation at p are tangent. This latter property is lost when we change the value of the parameter.

The next proposition confirms our geometric interpretation of the generalized transversality:

Proposition 2.11 *Suppose that $\{(S_a, \Phi_a)\}_{a\in V}$ and $\{\mathcal{V}_a\}_{a\in V}$ have a generalized transversal intersection at $p = \gamma(t_0, a_0) \in \mathbb{R}^3$ for $a = a_0 \in V$ which unfolds generically with a. Then there exists a map $t = t(a)$ defined on a neighbourhood of a_0 such that $t(a_0) = t_0$, and a neighbourhood $V_1 \subset V$ of a_0 satisfying:*

1. *S_a has a transversal intersection with γ_a at $\gamma(t(a), a)$, for all $a \in V_1$.*
2. *The fiber of $\mathcal{V}_a$ at $\gamma(t(a), a)$ is transversal to the leaf of (S_a, Φ_a) at $\gamma(t(a), a)$, if $a \in V_1 \setminus \{a_0\}$.*

Proof:

We take the coordinates of the previous definition. The map $t = t(a)$ exists because $D_1 z(0, a_0) \neq 0$ and $z(0, a_0) = 0$. So, we can apply the implicit function theorem to the equation $z(t, a) = 0$, in order to get $t(a)$. This proves item 1.

On the other hand, from the identity $z(t(a), a) = 0$ we obtain $\dot{t}(a_0) = -D_2 z(0, a_0)/D_1 z(0, a_0)$. To show the second part of the proposition, it is enough to see that the map $f(a) = T(t(a), a)$ is strictly monotone and $f(a_0) = 0$. Indeed,

$$D_1 z(0, a_0)f'(a_0) = D_1 z(0, a_0)D_2 T(0, a_0) - D_2 z(0, a_0)D_1 T(0, a_0) \neq 0,$$

and $f(a_0) = T(0, a_0) = 0$ by definition. $\square$

Let $\{f_a\}_{a\in V}$, $V \subset \mathbb{R}$ being an open set, be a smooth 1-parameter family of three dimensional diffeomorphisms having a saddle fixed point $p = p(a)$, for $a \in V$, with real eigenvalues λ_i, $i = 1, 2, 3$, such that $|\lambda_1| < |\lambda_2| < |\lambda_3|$. We will now define generalized homoclinic transversality. We distinguish two cases: $|\lambda_2| < 1$ and $|\lambda_2| > 1$.

Definition 2.12 *We say that $\{f_a\}_{a\in V}$ has a* **generalized homoclinic transversality** *at $q \in W^s(p(a_0)) \cap W^u(p(a_0))$, unfolding generically with a at $a = a_0$ if:*

1. *When $|\lambda_2| < 1$, the family of foliated surfaces $\{(W^s(p(a)), \mathcal{F}^{ss}(p(a)))\}_{a\in V}$ and the family of invariant tangent vector bundles $\{W^u(p(a), E^{cu}(p(a)))\}_{a\in V}$ have a generalized transversal intersection at q, which unfolds generically with a at $a = a_0$. Moreover $q \in W^s(p(a_0))$ is in general position.*

2. *When $|\lambda_2| > 1$, the invariant manifolds of $p(a)$ with respect to the family $\{f_a^{-1}\}_{a\in V}$ have a generalized homoclinic transversality at q, unfolding generically with a at $a = a_0$.*

In the next proposition we give adapted coordinates for such type of transversality.

Proposition 2.13 *Let $\{f_a\}_{a\in V}$ as before, and $|\lambda_1| < |\lambda_2| < 1 < |\lambda_3|$. We assume that this family has a generalized transversal homoclinic point q unfolding generically with a at $a = a_0$, and satisfies the linearization assumption. Then, there exist an open neighbourhood $V_1 \subset V$ of a_0, a family $\{g_{\tilde{a}}\}_{\tilde{a}\in(-\epsilon,\epsilon)}$ conjugate to $\{f_a\}_{a\in V_1}$, and a C^k ($k \geq 3$) map $\mathbf{x}(t_1, t_2, t_3, \tilde{a})$, defined on a neighbourhood U of $\{(t_1, t_2, t_3) \in \mathbb{R}^3 : (t_1^2 + t_2^2)t_3 = 0\} \times (-\epsilon, \epsilon)$ such that:*

1. *The conjugacy transforms the fixed point $p(a)$ to $\tilde{p}(\tilde{a}) = (0, 0, 0)$, for all $\tilde{a} \in (-\epsilon, \epsilon)$.*

2. *The family $\{g_{\tilde{a}}\}_{\tilde{a}\in(-\epsilon,\epsilon)}$ has a generalized homoclinic transversality at $\tilde{q} = (1, 1, 0) \in W^u((0, 0, 0)) \cap W^s((0, 0, 0))$ unfolding generically with $\tilde{a}$ at $\tilde{a} = 0$.*

3. *There exists a neighbourhood U_1 of $\{(t_1, t_2, 0) : t_1^2 + t_2^2 \leq 3\}$, such that for all $\tilde{a} \in (-\epsilon, \epsilon)$, the map $\mathbf{x}(\cdot, \cdot, \cdot, \tilde{a})$ restricted to U_1 is the identity.*

4. *If $(t_1, t_2, t_3, \tilde{a}) \in U$ then $(\lambda_1 t_1, \lambda_2 t_2, \lambda_3 t_3, \tilde{a}) \in U$ and $g_{\tilde{a}}(t_1, t_2, t_3) = \mathbf{x}(\lambda_1 t_1, \lambda_2 t_2, \lambda_3 t_3, \tilde{a})$.*

5. *If $\mathbf{x} = (x, y, z)$ then*

$$x(0, 0, 1, a) = 1, \quad y(0, 0, 1, \tilde{a}) = 1, \quad z(0, 0, 1, \tilde{a}) = 0,$$

and

$$D_2 y(0, 0, 1, \tilde{a}) D_3 z(0, 0, 1, \tilde{a}) - D_2 z(0, 0, 1, \tilde{a}) D_3 y(0, 0, 1, \tilde{a}) = \tilde{a},$$

for all $\tilde{a} \in (-\epsilon, \epsilon)$ and $D_3 z(0, 0, 1, 0) \neq 0$.

The proof is similar to that used in proposition 2.7. Note that this proposition allows us to suppose that, in a neighbourhood of $(1, 1, 0)$: a) $W^s(p(a))$ is the xy-coordinate plane, b) $(1, 1, 0) \in W^u(p(a)) \cap W^s(p(a))$ is the point of transversal intersection for all $a \in (-\epsilon, \epsilon)$, c) the leaves of the strong stable foliations are the straight lines $y = c$, $z = 0$, for constants c

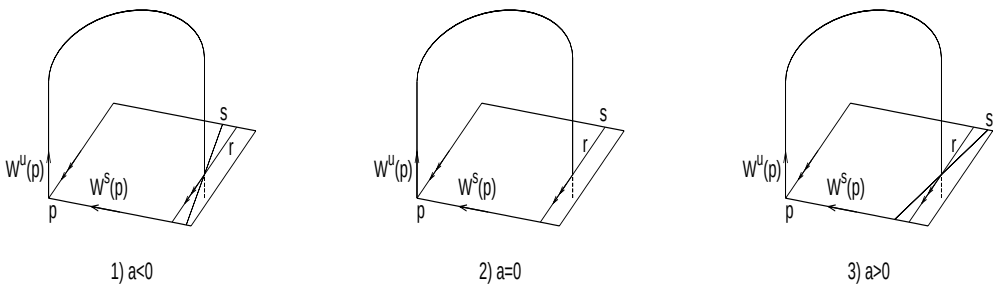


Figure 5: Generalized transversality. The line from p with two arrows represents the strong stable manifold $W^{ss}(p)$, the line r is a leaf of the strong stable foliation of $W^s(p)$ and the line s represents the intersection of $W^s(p)$ with the fiber of the unstable invariant vector bundle at the point of intersection of the invariant manifolds. In 1) and 3) r and s are different and in 2) (corresponding to the generalized homoclinic transversality) they coincide. Observe that in all the cases the invariant manifolds have transversal intersection.

small enough, d) the planes of the unstable tangent vector bundle at $(1,1,0)$ are generated by the vectors $D_2\mathbf{x}(0,0,1,a)$ and $D_3\mathbf{x}(0,0,1,a)$, and e) the parameter $\tilde{a}$ gives the value of the first component of the normal vector $D_2\mathbf{x}(0,0,1,a) \wedge D_3\mathbf{x}(0,0,1,a)$ to the fiber of the tangent vector bundle at $(1,1,0)$.

In figure 5 we have a typical example of a generalized homoclinic transversality.

2.3 Generalized homoclinic tangencies

Now we want to introduce a new type codimension-two homoclinic bifurcation, that we shall call generalized homoclinic tangency. It can be seen as a 'collision' of a quadratic homoclinic tangency and a generalized homoclinic transversality.

As before, we first study a new type of tangency between a two-parameter family of foliated surfaces $\{(S_{a,b}, \Phi_{a,b})\}_{(a,b)\in V}$, and a two-parameter family of vector bundles $\{\mathcal{V}_{a,b}\}_{(a,b)\in V}$, having a regular parametrized curve $\gamma_{a,b} = \gamma_{a,b}(t) = (x(t,a,b), y(t,a,b), z(t,a,b))$ as a zero section and tangent planes to $\gamma_{a,b}(t)$ as fibers. Here, $V \subset \mathbb{R}^2$ is an open neighbourhood of $(a,b) = (a_0,b_0)$. Recall that a smooth family of vector fields $\{v_{a,b}\}_{a,b\in V}$ associated to $\{\mathcal{V}_{a,b}\}_{(a,b)\in V}$ satisfies that for all t, $v_{a,b}(t) = (v_1(t,a,b), v_2(t,a,b), v_3(t,a,b))$ and $\dot{\gamma}_{a,b}(t)$ generate the corresponding fiber of $\mathcal{V}_{a,b}$.

Definition 2.14 *We say that $\{(S_{a,b}, \Phi_{a,b})\}_{(a,b)\in V}$ and $\{\mathcal{V}_{a,b}\}_{(a,b)\in V}$ have a* **generalized tangency of type I** *at $p_0 = \gamma_{a_0,b_0}(t_0) \in S_{a_0,b_0}$, which unfolds generically with (a,b) at (a_0,b_0), if there exist: a) a coordinate system (x,y,z) for which $S_{a,b}$ is the plane $z=0$ and the leaves of the foliation are the straight lines $z=0$, $y=c$, and b) a 2-parameter family of vector fields $\{v_{a,b}\}_{(a,b)\in V}$, associated to $\{\mathcal{V}_{a,b}\}_{(a,b)\in V}$, such that*

1. $z(t_0,a_0,b_0)=0,\ D_1z(t_0,a_0,b_0)=0,\ D_1y(t_0,a_0,b_0)=0,$

2. $D_{11}z(t_0,a_0,b_0)\neq 0,\ v_3(t_0,a_0,b_0)\neq 0,$

3. $$\begin{vmatrix} 0 & D_2z(t_0,a_0,b_0) & D_3z(t_0,a_0,b_0) \\ D_{11}z(t_0,a_0,b_0) & D_{12}z(t_0,a_0,b_0) & D_{13}z(t_0,a_0,b_0) \\ D_{11}y(t_0,a_0,b_0) & D_{12}y(t_0,a_0,b_0) & D_{13}y(t_0,a_0,b_0) \end{vmatrix} \neq 0,$$

   and

4. $D_{11}y(t_0,a_0,b_0)v_3(t_0,a_0,b_0)-D_{11}z(t_0,a_0,b_0)v_2(t_0,a_0,b_0)\neq 0.$

Before going to the next proposition, let us precise the geometric interpretation of the previous definition: The first condition means that the leaf of Φ_{a_0,b_0} at p_0 and the curve γ_{a_0,b_0} are tangent at $p_0=\gamma_{a_0,b_0}(t_0)$. The second says that the tangency is quadratic and the fiber of $\mathcal{V}_{a_0,b_0}$ at p_0 is not tangent to the leaf of Φ_{a_0,b_0} at p_0. In order to interpret the third and fourth condions, first we have to perform a reparametrization of (a,b) and t. Without loss of generality, we can suppose that $t_0\neq 0$.

Consider the following system of equations:

$$\begin{aligned} z(t,a,b) &= \tilde{a} \\ D_1z(t,a,b) &= 0 \\ D_1y(t,a,b) &= \tilde{b} \end{aligned}$$

From the first and third conditions, we see that there exist smooth maps $t=t(\tilde{a},\tilde{b})$, $a=a(\tilde{a},\tilde{b})$ and $b=b(\tilde{a},\tilde{b})$, defined on a neighbourhood of $(\tilde{a},\tilde{b})=(0,0)$, such that $t(0,0)=t_0$, $a(0,0)=a_0$ and $b(0,0)=0$. Moreover, a simple computation shows that the map

$$(\tilde{a},\tilde{b})=(z(t(a,b),a,b),D_1y(t(a,b),a,b)),$$

is invertible. Therefore, we can take $\tilde{a},\tilde{b}$ and $\tilde{t}=t/t(a(\tilde{a},\tilde{b}),b(\tilde{a},\tilde{b}))$ as new parameters. If we write

$$\tilde{\gamma}_{\tilde{a},\tilde{b}}(\tilde{t})=\gamma_{a,b}(t(a,b)\tilde{t})=(\tilde{x}(\tilde{t},\tilde{a},\tilde{b}),\tilde{y}(\tilde{t},\tilde{a},\tilde{b}),\tilde{z}(\tilde{t},\tilde{a},\tilde{b})),$$

where $(a,b)=(a(\tilde{a},\tilde{b}),b(\tilde{a},\tilde{b}))$, then:

$$\tilde{z}(1,\tilde{a},\tilde{b})=\tilde{a},\quad D_1\tilde{z}(1,\tilde{a},\tilde{b})=0\ \text{ and }\ D_1\tilde{y}(1,\tilde{a},\tilde{b})=\tilde{b},$$

for all $\tilde{a}$ and $\tilde{b}$ small enough.

With this parametrization, the third condition is superfluous. This means that it gives a generic unfolding of the generalized tangency.

It remains to explain the fourth condition of the definition. For the sake of simplifying notation, we remove the tildes of the new parametrization. Suppose that the point $\gamma_{a,b}(t)$ belongs to the surface $S_{a,b}$ and that the fiber of $\mathcal{V}_{a,b}$ at $\gamma_{a,b}(t)$ is tangent to the leaf of $\Phi_{a,b}$ at the same point. Then t, a and b satisfy the system of equations:

$$\begin{aligned} z(t,a,b) &= 0 \\ v_2(t,a,b)D_1z(t,a,b)-v_3(t,a,b)D_1y(t,a,b) &= 0 \end{aligned}$$

From condition three, we see that this system has a solution $t = t(b)$, $a = a(b)$, such that $t(0) = 1$ and $a(0) = 0$. In other words, this means that the curve $(a_2(s), b_2(s))$ in the parameter plane, such that $f_{a_2(s),b_2(s)}$ has a generalized transversality, is a regular curve. Here we can take the parameter $s = b$.

Note, finally, that the curve $(a_1(s), b_1(s))$, corresponding to the existence of a quadratic homoclinic tangency satisfies $(a_1(s), b_1(s)) = (0, s)$. This implies that the two curves $(a_1(s), b_1(s))$ and $(a_2(s), b_2(s))$ have a quadratic tangency for $s = 0$.

We summarize the properties of the generalized tangency in the following proposition:

Proposition 2.15 *Let $\{(S_{a,b}, \Phi_{a,b})\}_{(a,b)\in V}$ be a family of foliated surfaces, and $\{\mathcal{V}_{a,b}\}_{(a,b)\in V}$ a tangent vector bundle having a generalized homoclinic tangency of type I, at a point $p = \gamma(t_0, a_0, b_0)$ unfolding generically with (a, b) at (a_0, b_0). Then there exist two maps $(t_1(s), a_1(s), b_1(s))$ and $(t_2(s), a_2(s), b_2(s))$ such that:*

1. *$t_i(0) = t_0$, $(a_i(0), b_i(0)) = (a_0, b_0)$, for $i = 1, 2$.*
2. *$(a_i(s), b_i(s))$ are regular curves for $i = 1, 2$.*
3. *$S_{a_1(s),b_1(s)}$ has a quadratic tangency with the curve $\gamma(t, a_1(s), b_1(s))$ at $p_1(s) = \gamma(t_1(s), a_1(s), b_1(s))$, for s small enough.*
4. *$S_{a_2(s),b_2(s)}$ has a generalized transversal intersection with $\gamma(t, a_2(s), b_2(s))$ at $p_2(s) = \gamma(t_2(s), a_2(s), b_2(s))$, for $s \neq 0$ small enough. If $s = 0$ only the transversality of the curve and the surface does not remain.*
5. *Let $\mathcal{L}_{\alpha(s),a_1(s),b_1(s)}$ be the leaf of $\Phi_{a_1(s),b_1(s)}$ at $p_1(s)$. Then the curves $\mathcal{L}_{\alpha(s),a_1(s),b_1(s)}$ and $\gamma(t, a_2(s), b_2(s))$ are tangent for $s = 0$ and non-tangent for $s \neq 0$ small enough.*
6. *There is a quadratic tangency between $(a_1(s), b_1(s))$ and $(a_2(s), b_2(s))$ at (a_0, b_0).*

Definition 2.16 *We say that $\{(S_{a,b}, \Phi_{a,b})\}_{(a,b)\in V}$ and $\{\mathcal{V}_{a,b}\}_{(a,b)\in V}$ have a* **generalized tangency of type II** *at $p_0 = \gamma_{a_0,b_0}(t_0) \in S_{a_0,b_0}$, which unfolds generically with (a, b) at (a_0, b_0), if there exist: a) a coordinate system (x, y, z) for which $S_{a,b}$ is the plane $z = 0$ and the leaves of the foliation are the straight lines $z = 0$, $y = c$, and b) a 2-parameter family of vector fields $\{v_{a,b}\}_{(a,b)\in V}$, associated to $\{\mathcal{V}_{a,b}\}_{(a,b)\in V}$, such that*

1. *$z(t_0, a_0, b_0) = 0$, $D_1 z(t_0, a_0, b_0) = 0$, $v_3(t_0, a_0, b_0) = 0$,*
2. *$D_{11} z(t_0, a_0, b_0) \neq 0$, $D_1 y(t_0, a_0, b_0) \neq 0$,*
3. $$\begin{vmatrix} 0 & D_2 z(t_0, a_0, b_0) & D_3 z(t_0, a_0, b_0) \\ D_{11} z(t_0, a_0, b_0) & D_{12} z(t_0, a_0, b_0) & D_{13} z(t_0, a_0, b_0) \\ D_1 v_3(t_0, a_0, b_0) & D_2 v_3(t_0, a_0, b_0) & D_3 v_3(t_0, a_0, b_0) \end{vmatrix} \neq 0$$

   and

4. *$D_1 y(t_0, a_0, b_0) D_1 v_3(t_0, a_0, b_0) - D_{11} z(t_0, a_0, b_0) v_2(t_0, a_0, b_0) \neq 0$.*

Both definitions do not depend on changes of coordinates and parameters, and on the associated vector field $v_{a,b}$.

As in the previous definition, we can give a geometric explanation of the definition: Condition 1 means that the surface S_{a_0,b_0} and the curve γ_{a_0,b_0} are tangent at $p_0 = \gamma_{a_0,b_0}(t_0)$. Moreover, the fiber of $\mathcal{V}_{a_0,b_0}$ at p_0 is tangent to the leaf of Φ_{a_0,b_0} at p_0. From condition 2, the tangency is quadratic and the foliation Φ_{a_0,b_0} and the curve γ_{a_0,b_0} are not tangent at p_0. As before, performing a reparametrization of t, a, and b, we can suppose that $t_0 = 1$, $(a_0, b_0) = (0, 0)$, $D_1 z(1, a, b) = 0$, $z(1, a, b) = a$ and $v_3(1, a, b) = b$, for all (a, b) close to $(0, 0)$. Using this parametrization, condition 3 is not necessary, meannig that it yields a generic unfolding of the tangency.

Finally, condition 4 tell us that the curves $(a_i(s), b_i(s))$, $i = 1, 2$, defined as before, are regular. So we have:

Proposition 2.17 *Let $(S_{a,b}, \Phi_{a,b})$ be a family of foliated surfaces, and $\{\mathcal{V}_{a,b}\}_{(a,b)\in V}$ a tangent vector bundle having a generalized homoclinic tangency of type II, at a point $p = \gamma(t_0, a_0, b_0)$ unfolding generically with (a, b) at (a_0, b_0). Then there exist two maps $(t_1(s), a_1(s), b_1(s))$ and $(t_2(s), a_2(s), b_2(s))$ such that:*

1. *$t_i(0) = t_0$, $(a_i(0), b_i(0)) = (a_0, b_0)$, for $i = 1, 2$.*
2. *$(a_i(s), b_i(s))$ are regular curves for $i = 1, 2$.*
3. *$S_{a_1(s),b_1(s)}$ has a quadratic tangency with the curve $\gamma(t, a_1(s), b_1(s))$ at $p_1(s) = \gamma(t_1(s), a_1(s), b_1(s))$, for s small enough.*
4. *$S_{a_2(s),b_2(s)}$ has a generalized transversal intersection with $\gamma(t, a_2(s), b_2(s))$ in $p_2(s) = \gamma(t_2(s), a_2(s), b_2(s))$, for $s \neq 0$ small enough. If $s = 0$, neither the transversality of the curve and the surface nor the transversality of the fiber and the surface at $p_2(s)$ remain.*
5. *Let $\mathcal{L}_{\alpha(s),a_1(s),b_1(s)}$ be the leaf of $\Phi_{a_1(s),b_1(s)}$ at $p_1(s)$. Then the curves $\mathcal{L}_{\alpha(s),a_1(s),b_1(s)}$ and $\gamma(t, a_2(s), b_2(s))$ are not tangent for all s small enough.*
6. *There is a quadratic tangency between $(a_1(s), b_1(s))$ and $(a_2(s), b_2(s))$ at (a_0, b_0).*

The next step is to define generalized homoclinic tangencies for 2-parameter families of diffeomorphisms:

Let $\{f_{a,b}\}_{(a,b)\in V}$, $V \subset \mathbb{R}^2$ being an open set, be a smooth 2-parameter family of three dimensional diffeomorphisms having a saddle fixed point $p = p(a, b)$, for $(a, b) \in V$, with real eigenvalues λ_i, $i = 1, 2, 3$, such that $|\lambda_1| < |\lambda_2| < |\lambda_3|$.

Definition 2.18 *We say that $\{f_{a,b}\}_{(a,b)\in V}$ has a* **generalized quadratic homoclinic tangency** *of type I (respectively type II) at $q \in W^s(p(a_0, b_0)) \cap W^u(p(a_0, b_0))$ unfolding generically with (a, b) at $(a, b) = (a_0, b_0)$ if:*

1. *When $|\lambda_2| < 1$, the family of foliated surfaces $\{(W^s(p(a, b)), \mathcal{F}^{ss}(p(a, b)))\}_{(a,b)\in V}$ and the family of invariant tangent vector bundles $\{W^u(p(a, b), E^{cu}(p(a, b))\}_{(a,b)\in V}$ have a generalized tangency of type I (resp. type II) at q, which unfolds generically with (a, b) at $(a, b) = (a_0, b_0)$. Moreover, q is in general position.*

2. *When* $|\lambda_2| > 1$, *the invariant manifolds of* $p(a,b)$ *with respect to the family* $\{f_{a,b}^{-1}\}_{(a,b)\in V}$ *have a generalized quadratic homoclinic tangency of type I (resp. type II) at* q, *unfolding generically with* (a,b) *at* $(a,b)=(a_0,b_0)$.

In the last part of this section, we give adapted coordinates and parameters for families having a generalized homoclinic tangency.

Let $\{f_{a,b}\}_{(a,b)\in V}$ be as before, and assume that this family has a generalized tangencial homoclinic point q unfolding generically with (a,b) at $(a,b)=(a_0,b_0)$, and satisfies the linearization assumption. In the next two propositions we shall obtain adapted coordinates for this type of homoclinic bifurcation.

Proposition 2.19 *Suppose that* $|\lambda_1|<|\lambda_2|<1<|\lambda_3|$. *Then, there exists an open neighbourhood* $V_1\subset V$ *of* (a_0,b_0), *a family* $\{g_{\tilde a,\tilde b}\}_{(\tilde a,\tilde b)\in(-\epsilon,\epsilon)\times(-\epsilon,\epsilon)}$ *conjugate to* $\{f_{a,b}\}_{(a,b)\in V_1}$ *and a* C^k *(*$k\geq 3$*) map* $\mathbf{x}(t_1,t_2,t_3,\tilde a,\tilde b)$, *defined on a neighbourhood* U *of* $\{(t_1,t_2,t_3)\in\mathbb{R}^3 : (t_1^2+t_2^2)t_3=0\}\times(-\epsilon,\epsilon)\times(-\epsilon,\epsilon)$ *such that:*

1. *The conjugacy transforms the fixed point* $p(a,b)$ *to* $\tilde p(\tilde a,\tilde b)=(0,0,0)$, *for all* $(\tilde a,\tilde b)\in(-\epsilon,\epsilon)\times(-\epsilon,\epsilon)$.

2. *The family* $\{g_{\tilde a,\tilde b}\}_{(\tilde a,\tilde b)\in(-\epsilon,\epsilon)\times(-\epsilon,\epsilon)}$ *has a generalized quadratic homoclinic tangency at* $\tilde q=(1,1,0)\in W^u((0,0,0))\cap W^s((0,0,0))$ *unfolding generically with* $(\tilde a,\tilde b)$ *at* $(\tilde a,\tilde b)=(0,0)$.

3. *There exists a neighbourhood* U_1 *of* $\{(t_1,t_2,0) : t_1^2+t_2^2\leq 3\}$, *such that for all* $(\tilde a,\tilde b)\in(-\epsilon,\epsilon)\times(-\epsilon,\epsilon)$, *the map* $\mathbf{x}(\cdot,\cdot,\cdot,\tilde a,\tilde b)$ *restricted to* U_1 *is the identity.*

4. *If* $(t_1,t_2,t_3,\tilde a,\tilde b)\in U$ *then* $(\lambda_1t_1,\lambda_2t_2,\lambda_3t_3,\tilde a,\tilde b)\in U$ *and* $g_{\tilde a,\tilde b}(t_1,t_2,t_3)=\mathbf{x}(\lambda_1t_1,\lambda_2t_2,\lambda_3t_3,\tilde a,\tilde b)$.

5. *If we write* $\mathbf{x}=(x,y,z)$ *then*

   (a) *When the generalized homoclinic tangency is of type I:*

   $$x(0,0,1,\tilde a,\tilde b)=1,\quad y(0,0,1,\tilde a,\tilde b)=1,\quad z(0,0,1,\tilde a,\tilde b)=\tilde a,$$
   $$D_3z(0,0,1,\tilde a,\tilde b)=0,\quad D_3y(0,0,1,\tilde a,\tilde b)=\tilde b,$$

   for all $(\tilde a,\tilde b)\in(-\epsilon,\epsilon)\times(-\epsilon,\epsilon)$ *and* $D_{33}z(0,0,1,0,0)\neq 0$, $D_2z(0,0,1,0,0)\neq 0$, *and*

   $$\begin{vmatrix} D_{33}y(0,0,1,0,0) & D_{33}z(0,0,1,0,0)\\ D_2y(0,0,1,0,0) & D_2z(0,0,1,0,0)\end{vmatrix}\neq 0.$$

   (b) *When the generalized homoclinic tangency is of type II:*

   $$x(0,0,1,\tilde a,\tilde b)=1,\quad y(0,0,1,\tilde a,\tilde b)=1,\quad z(0,0,1,\tilde a,\tilde b)=\tilde a,$$
   $$D_3z(0,0,1,\tilde a,\tilde b)=0,\quad D_2z(0,0,1,\tilde a,\tilde b)=\tilde b,$$

   for all $(\tilde a,\tilde b)\in(-\epsilon,\epsilon)\times(-\epsilon,\epsilon)$ *and* $D_{33}z(0,0,1,0,0)\neq 0$, $D_3y(0,0,1,0,0)\neq 0$, *and*

   $$\begin{vmatrix} D_{23}z(0,0,1,0,0) & D_{33}z(0,0,1,0,0)\\ D_2y(0,0,1,0,0) & D_3y(0,0,1,0,0)\end{vmatrix}\neq 0.$$

Proof:

Let $\{f_{a,b}\}_{(a,b)\in V}$ and $\mathbf{x}_f$ be as in remark 1.5. In particular, we have that $p(a,b)=0$, for all $(a,b)\in V$ and $(a_0,b_0)=(0,0)$. Recall that in this situation:

a) The invariant manifolds of $p(a,b)=0$ can be represented as

$$W^s(p(a,b)) = \{\, \mathbf{x}_f(t_1,t_2,0,a,b) \;:\; (t_1,t_2)\in\mathbb{R}^2\,\}$$

and

$$W^u(p(a,b)) = \{\, \mathbf{x}_f(0,0,t_3,a,b) \;:\; t_3\in\mathbb{R}\}.$$

Moreover, locally, the unstable invariant manifold is the z axis and the stable one is the xy-coordinate plane (if we take coordinates x,y,z in $\mathbb{R}^3$).

b) The leaves of the strong stable foliation $\mathcal{F}^{ss}(p(a,b))$ are the curves $\mathbf{x}_f(t,\bar{t}_2,0,a,b)$, parametrized by t, where $\bar{t}_2$ is a constant.

c) The fibers of the unstable tangent vector bundle $W^u(p(a,b),E^{cu})$ are generated by the vectors $D_2\mathbf{x}_f(0,0,t,a,b)$ and $D_3\mathbf{x}_f(0,0,t,a,b)$, for all $t\in\mathbb{R}$. Following the previous notation, we can take the family of vector fields associated to the vector bundle as $v_{a,b}(t) = D_2\mathbf{x}_f(0,0,t,a,b)$.

We distinguish between the generalized tangencies of type I and II:

a) Type I:

We can always assume that the tangent homoclinic point q belongs to the plane $z=0$. Moreover, there exists a real number $\bar{t}_3\neq 0$ such that $q=\mathbf{x}_f(0,0,\bar{t}_3,0,0)$.

From definitions 2.18 and 2.14 we see that $z_f(0,0,\bar{t}_3,0,0)=D_3z_f(0,0,\bar{t}_3,0,0)=0$ and $D_{33}z_f(0,0,\bar{t}_3,0,0)\neq 0$, $x_f(0,0,\bar{t}_3,0,0)\neq 0$ and $y_f(0,0,\bar{t}_3,0,0)\neq 0$.

Using these conditions, we can prove the existence of a map $\tilde{t}_3(a,b)$ defined on a neighbourhood of $(0,0)$ such that $\tilde{t}_3(0,0)=\bar{t}_3$ and $D_3z_f(0,0,\tilde{t}_3(a,b),a,b)=0$.

Note that $W^u(p(a,b))$ has a local extremum with respect to the z coordinate at $(x,y,z)=\mathbf{x}_f(0,0,\tilde{t}_3(a,b),a,b)$.

Without loss of generality (via a change of scale if necessary), we can assume that $\tilde{t}_3(a,b)=1$, for all $(a,b)\in V$ and: $x_f(0,0,1,a,b)=y_f(0,0,1,a,b)=1$, $D_3z_f(0,0,1,a,b)=0$. This simply means that we can move the point $\mathbf{x}_f(0,0,\tilde{t}_3(a,b),a,b)$ to a point with the x and y coordinates equal to 1. In particular, the point of homoclinic tangency is $\tilde{q}=(1,1,0)$. Moreover, $D_2y_f(0,0,1,0,0)=0$, $D_{33}z_f(0,0,1,0,0)\neq 0$, $D_2z_f(0,0,1,0,0)\neq 0$ and

$$\left|\begin{array}{cc} D_4z_f(0,0,1,0,0) & D_5z_f(0,0,1,0,0) \\ D_{34}y_f(0,0,1,0,0) & D_{35}y_f(0,0,1,0,0)\end{array}\right| \neq 0.$$

These conditions are obtained from definition 2.14.

Finally, by the inverse function theorem, we can take an $\epsilon>0$, an open subset $V_1\subset V$ and new parameters

$$\begin{aligned}\bar{a} &= z(0,0,1,a,b)\\ \bar{b} &= D_3y(0,0,1,a,b),\end{aligned}$$

such that $(\tilde{a},\tilde{b}) \in (-\epsilon,\epsilon)\times(-\epsilon,\epsilon)$ and $(a,b)\in V_1$. It is a simple check to show that the family $\{g_{\tilde{a},\tilde{b}}\}_{(\tilde{a},\tilde{b})\in(-\epsilon,\epsilon)\times(-\epsilon,\epsilon)}$, given by $g_{\tilde{a},\tilde{b}} = f_{a,b}$, satisfies the properties 1, 2, 3(a) of the proposition.

b) Type II

As before, we can suppose that $W^u(p(a,b))$ has a local extremum, with respect to the z-coordinate, at a point with x and y coordinates equal to 1, and corresponding parameter $t_3 = 1$. This means that:

$$x_f(0,0,1,a,b) = y_f(0,0,1,a,b) = 1, \quad D_3 z_f(0,0,1,a,b) = 0,$$

for all $(a,b)\in V$, and the point of homoclinic tangency is $q = (1,1,0) = \mathbf{x}_f(0,0,1,0,0)$. From definitions 2.16 and 2.18 one has:

$D_2 z(0,0,1,0,0) = 0$, $D_{33} z(0,0,1,0,0) \neq 0$, $D_3 y(0,0,1,0,0) \neq 0$ and

$$\begin{vmatrix} D_4 z(0,0,1,0,0) & D_5 z(0,0,1,0,0) \\ D_{24} z(0,0,1,0,0) & D_{25} z(0,0,1,0,0) \end{vmatrix} \neq 0.$$

Finally we perform the change of parameters:

$$\begin{aligned} \bar{a} &= z(0,0,1,a,b) \\ \bar{b} &= D_2 z(0,0,1,a,b) \end{aligned}$$

obtaining the desired result. □

Remark 2.20 *The following properties hold for the family $\{g_{\tilde{a},\tilde{b}}\}$: a) $W^s(\tilde{p}(\tilde{a},\tilde{b}))$ is the xy-coordinate plane in a neighbourhood of $(1,1,0)$. b) $W^u(\tilde{p}(\tilde{a},\tilde{b}))$ has a local extremum with respect to the third variable at $(x,y,z) = \mathbf{x}(0,0,1,\tilde{a},\tilde{b}) = (1,1,\tilde{a})$. c) The leaves of $\mathcal{F}^{ss}(\tilde{p}(\tilde{a},\tilde{b}))$ are the straight lines $y = c$, $z = 0$, for constants c small enough, in a neighbourhood of $(1,1,0)$. d) The fibers of the unstable tangent vector bundle at $(1,1,\tilde{a})$ are generated by the vectors $D_2\mathbf{x}(0,0,1,\tilde{a},\tilde{b})$ and $D_3\mathbf{x}(0,0,1,\tilde{a},\tilde{b})$. e) For type I generalized homoclinic tangencies, the parameter $\tilde{b}$ gives the value of the second component of the tangent vector $D_3\mathbf{x}(0,0,1,\tilde{a},\tilde{b})$ to $W^u(\tilde{p}(\tilde{a},\tilde{b}))$ at $(1,1,\tilde{a})$. f) For type II, let $W^{cu}(\tilde{p}(\tilde{a},\tilde{b}))$ be a center-unstable invariant manifold such that:*

$$W^{cu}(\tilde{p}(\tilde{a},\tilde{b})) = \{\mathbf{x}(0,t_2,t_3,\tilde{a},\tilde{b}), \ \ (t_2,t_3)\in W\},$$

W being a suitable open neighbourhood of $\{(t_2,t_3), \quad t_2 = 0\}$. The parameter $\tilde{b}$ is the third component of the first tangent vector $D_2\mathbf{x}(0,0,1,\tilde{a},\tilde{b})$ of the parametrization of $W^{cu}(\tilde{p}(\tilde{a},\tilde{b}))$ at $(1,1,\tilde{a})$.

Proposition 2.21 *Suppose that $|\lambda_1| < 1 < |\lambda_2| < |\lambda_3|$. Then, there exists an open neighbourhood $V_1 \subset V$ of (a_0,b_0), a family $\{g_{\tilde{a},\tilde{b}}\}_{(\tilde{a},\tilde{b})\in(-\epsilon,\epsilon)\times(-\epsilon,\epsilon)}$ conjugate to $\{f_{a,b}\}_{(a,b)\in V_1}$ and a C^k ($k\geq 3$) map $\mathbf{x}(t_1,t_2,t_3,\tilde{a},\tilde{b})$, defined on a neighbourhood U of $\{(t_1,t_2,t_3)\in\mathbb{R}^3 : (t_2^2+t_3^2)t_1 = 0\}\times(-\epsilon,\epsilon)\times(-\epsilon,\epsilon)$ such that:*

1. *The conjugacy transforms the fixed point $p(a,b)$ to $\tilde{p}(\tilde{a},\tilde{b}) = (0,0,0)$, for all $(\tilde{a},\tilde{b})\in(-\epsilon,\epsilon)\times(-\epsilon,\epsilon)$.*

2. *The family* $\{g_{\tilde{a},\tilde{b}}\}_{(\tilde{a},\tilde{b})\in(-\epsilon,\epsilon)\times(-\epsilon,\epsilon)}$ *has a generalized quadratic homoclinic tangency at* $\tilde{q} = (1,0,0) \in W^u((0,0,0)) \cap W^s((0,0,0))$ *unfolding generically with* $(\tilde{a},\tilde{b})$ *at* $(\tilde{a},\tilde{b}) = (0,0)$.

3. *There exists a neighbourhood* U_1 *of* $\{(t_1,0,0) \ : \ |t_1| \leq 3\}$, *such that for all* $(\tilde{a},\tilde{b}) \in (-\epsilon,\epsilon)\times(-\epsilon,\epsilon)$, *the map* $\mathbf{x}(\cdot,\cdot,\cdot,\tilde{a},\tilde{b})$ *restricted to* U_1 *is the identity.*

4. *If* $(t_1,t_2,t_3,\tilde{a},\tilde{b}) \in U$ *then* $(\lambda_1 t_1, \lambda_2 t_2, \lambda_3 t_3, \tilde{a}, \tilde{b}) \in U$ *and* $g_{\tilde{a},\tilde{b}}(t_1,t_2,t_3) = \mathbf{x}(\lambda_1 t_1, \lambda_2 t_2, \lambda_3 t_3, \tilde{a}, \tilde{b})$.

5. *If we write* $\mathbf{x} = (x,y,z)$ *then*

   (a) *If the generalized homoclinic tangency is of type I:*

$$x(0,1,1,\tilde{a},\tilde{b}) = 1, \quad y(0,1,1,\tilde{a},\tilde{b}) = \tilde{a},$$

$$z(0,1,1,\tilde{a},\tilde{b}) = 0, \quad D_3 z(0,1,1,\tilde{a},\tilde{b}) = \tilde{b},$$

$$D_2 y(0,1,1,\tilde{a},\tilde{b}) D_3 z(0,1,1,\tilde{a},\tilde{b}) - D_3 y(0,1,1,\tilde{a},\tilde{b}) D_2 z(0,1,1,\tilde{a},\tilde{b}) = 0,$$

   for all $(\tilde{a},\tilde{b}) \in (-\epsilon,\epsilon)\times(-\epsilon,\epsilon)$, $D_3 y(0,1,1,0,0) = 0$, $D_2 z(0,1,1,0,0) \neq 0$, *and for* $(t_1,t_2,t_3,\tilde{a},\tilde{b}) = (0,1,1,0,0)$:

$$D_2 y D_{33} z - D_{33} y D_2 z \neq 0.$$

   (b) *If the generalized homoclinic tangency is of type II:*

$$x(0,1,1,\tilde{a},\tilde{b}) = 1, \quad y(0,1,1,\tilde{a},\tilde{b}) = 1,$$

$$z(0,1,1,\tilde{a},\tilde{b}) = \tilde{a}, \quad D_3 z(0,1,1,\tilde{a},\tilde{b}) = \tilde{b},$$

$$D_2 y(0,1,1,\tilde{a},\tilde{b}) D_3 z(0,1,1,\tilde{a},\tilde{b}) - D_3 y(0,1,1,\tilde{a},\tilde{b}) D_2 z(0,1,1,\tilde{a},\tilde{b}) = 0,$$

   for all $(\tilde{a},\tilde{b}) \in (-\epsilon,\epsilon)\times(-\epsilon,\epsilon)$,

$$D_2 z(0,1,1,0,0) = D_3 z(0,1,1,0,0) = 0, \quad D_3 y(0,1,1,0,0) \neq 0,$$

   and, for $(t_1,t_2,t_3,\tilde{a},\tilde{b}) = (0,1,1,0,0)$:

$$(D_2 y)^2 D_{33} z - 2 D_2 y D_3 y D_{23} z + (D_3 y)^2 D_{22} z \neq 0.$$

Proof:

As in the previous proposition, we can suppose that $f_{a,b}(t_1,t_2,t_3)$ and $\mathbf{x}_f$ satisfy the properties given in remark 1.5. Then:

a) The invariant manifolds of $p(a,b) = 0$ are

$$W^s(p(a,b)) = \{\mathbf{x}_f(t_1,0,0,a,b) \ : \ t_1 \in \mathbb{R}\}$$

and

$$W^u(p(a,b)) = \{\mathbf{x}_f(0,t_2,t_3,a,b) \ : \ (t_2,t_3) \in \mathbb{R}^2\}.$$

Moreover, locally the unstable invariant manifold is the yz-plane and the stable one is the x axis.

b) The leaves of the strong unstable foliation $\mathcal{F}^{uu}(p(a,b))$ are the curves $\mathbf{x}_f(0,\bar{t}_2,t,a,b)$, parametrized by t, where $\bar{t}_2$ is a constant.

c) The fibers of the stable tangent vector bundle $W^s(p(a,b),E^{cs})$ are generated by the vectors $D_1\mathbf{x}(t,0,0,a,b)$ and $D_2\mathbf{x}(t,0,0,a,b)$, for all $t\in\mathbb{R}$. We take $v_{a,b}(t) = D_2\mathbf{x}(t,0,0,a,b)$, as associated family of vector fields to the bundle.

We distinguish between types I and II:

a) Type I:

As, by definition 2.18, the tangent homoclinic point q is in general position, we can assume that q belongs to the straight line $y=0$, $z=0$ and that there exist $\bar{t}_2\neq 0$ and $\bar{t}_3\neq 0$ such that $q=\mathbf{x}_f(0,\bar{t}_2,\bar{t}_3,0,0)$. By using propositions 2.15 and 2.4, we easily prove that:

$$x_f(0,\bar{t}_2,\bar{t}_3,0,0)\neq 0,\quad y_f(0,\bar{t}_2,\bar{t}_3,0,0)=0,\quad z_f(0,\bar{t}_2,\bar{t}_3,0,0)=0.$$

Moreover, for $(t_1,t_2,t_3,a,b)=(0,\bar{t}_2,\bar{t}_3,0,0)$: $D_3y = D_3z = 0$, $D_2z\neq 0$, $D_2yD_{33}z - D_2zD_{33}y\neq 0$.

These conditios allow us to define (by the implicit function theorem) maps $\tilde{t}_2=\tilde{t}_2(a,b)$, $\tilde{t}_3=\tilde{t}_3(a,b)$, such that $\tilde{t}_2(0,0)=\bar{t}_2$, $\tilde{t}_3(0,0)=\bar{t}_3$, and satisfy, for (a,b) near $(0,0)$, the following system of equations:

$$\begin{aligned} D_2y(0,\tilde{t}_2,\tilde{t}_3,a,b)D_3z(0,\tilde{t}_2,\tilde{t}_3,a,b) - D_2z(0,\tilde{t}_2,\tilde{t}_3,a,b)D_3y(0,\tilde{t}_2,\tilde{t}_3,a,b) &= 0\\ z(0,\tilde{t}_2,\tilde{t}_3,a,b) &= 0\end{aligned}$$

Note that the point $\mathbf{x}_f(0,\tilde{t}_2,\tilde{t}_3,a,b)$ has the following property: It is a local extremum of the y coordinate of the curve in $W^u(p(a,b))$ for which the z-coordinate is always zero. Performing a change of scale if necessary, we can suppose that $\tilde{t}_2\equiv 1$, $\tilde{t}_3\equiv 1$, and

$$x(0,1,1,a,b)=1,\quad z(0,1,1,a,b)=0,$$

$$D_2y(0,1,1,a,b)D_3z(0,1,1,a,b) - D_2z(0,1,1,a,b)D_3y(0,1,1,a,b)=0,$$

for all $(a,b)\in V$. In particular $\bar{t}_1=1$, $\bar{t}_2=1$ and $q=(1,0,0)=\mathbf{x}_f(0,1,1,0,0)$.

In order to get $\{g_{\tilde{a},\tilde{b}}\}$ and $\mathbf{x}$, we need to reparametrize the initial family. We will use the third condition of definition 2.14 (generic unfolding):

Denote by $G_{a,b}$ the inverse map of $\mathbf{x}_{a,b} = \mathbf{x}_f(\cdot,\cdot,\cdot,a,b)$, in a neighbourhood of $(1,0,0)$, for all (a,b) in a neighbourhood of $(0,0)$. Moreover, we write $G(s_1,s_2,s_3,a,b) = G_{a,b}(s_1,s_2,s_3)$. Let $h_{a,b} = G_{a,b}\circ f_{a,b}\circ \mathbf{x}_{a,b}$. As $h_{a,b}$ has a generalized quadratic homoclinic tangency of type I at $(0,1,1)$, this means that, the family of foliated surfaces $W^u(h_{a,b},(0,0,0))$ and the family of tangent vector bundles $W^s(((0,0,0),E^{cs}_{a,b}))$, with zero section $W^s(h_{a,b},(0,0,0))$, have a generalized quadratic tangency of type I at $(0,1,1)$ for $(a,b)=(0,0)$. Moreover, for (x,y,z) near $(0,0,1)$, $W^u(h_{a,b},(0,0,0))$ is the plane $x=0$, the leaves of the foliations are the straight lines $x=y=0$, and $W^s(h_{a,b},(0,0,0))$ is the curve $\gamma_{a,b}(t)=G_{a,b}(t,0,0)$.

From condition 3 of definition 2.14, we see that for $(s_1, s_2, s_3, a, b) = (0, 1, 0, 0, 0)$:

$$\begin{vmatrix} 0 & D_4G_1 & D_5G_1 \\ D_{11}G_1 & D_{14}G_1 & D_{15}G_1 \\ D_{11}G_2 & D_{14}G_2 & D_{15}G_2 \end{vmatrix} \neq 0,$$

where $G = (G_1, G_2, G_3)$. A simple computation shows that this condition is equivalent to:

$$D_{34}z_f(0,1,1,0,0)D_5y_f(0,1,1,0,0) - D_{35}z_f(0,1,1,0,0)D_4y_f(0,1,1,0,0) \neq 0.$$

Finally, we consider the following reparametrization:

$$\begin{aligned} \bar{a} &= y(0,1,1,a,b) \\ \bar{b} &= D_3z(0,1,1,a,b), \end{aligned}$$

that is well-defined by the previous condition. Then the family $\{g_{\tilde{a},\tilde{b}}\}_{(\tilde{a},\tilde{b})\in(-\epsilon,\epsilon)\times(-\epsilon,\epsilon)}$, for a suitable $\epsilon > 0$, defined by $g_{\tilde{a},\tilde{b}} = f_{a,b}$, and $\mathbf{x}(\cdot,\cdot,\cdot,\tilde{a},\tilde{b}) = \mathbf{x_f}(\cdot,\cdot,\cdot,a,b)$ satisfies all the properties that we wanted to prove.

b) Type II

Also in this case there exist $\bar{t}_2 \neq 0$ and $\bar{t}_3 \neq 0$ such that $x_f(0,\bar{t}_2,\bar{t}_3,0,0) \neq 0$, $y_f(0,\bar{t}_2,\bar{t}_3,0,0) = 0$, $z_f(0,\bar{t}_2,\bar{t}_3,0,0) = 0$. By using propositions 2.17 and 2.4, we get

$$D_2y_fD_3zx_f - D_2z_fD_3y_f = 0, \quad D_2z_f = D_3z_f = 0, \quad D_3y_f \neq 0,$$

and

$$D_{22}z_f(D_3y_f)^2 + D_{33}z_f(D_2y_f)^2 - 2D_2y_fD_3y_fD_{23}z_f \neq 0,$$

for $(t_1, t_2, t_3, a, b) = (0, \bar{t}_2, \bar{t}_3, 0, 0)$.

Then we consider maps $\tilde{t}_2 = \tilde{t}_2(a,b)$, $\tilde{t}_3 = \tilde{t}_3(a,b)$, such that $\tilde{t}_2(0,0) = \bar{t}_2$, $\tilde{t}_3(0,0) = \bar{t}_3$, and satisfy, for (a,b) near $(0,0)$, the following system of equations:

$$\begin{aligned} D_2y_f(0,\tilde{t}_2,\tilde{t}_3,a,b)D_3z_f(0,\tilde{t}_2,\tilde{t}_3,a,b) - D_2z_f(0,\tilde{t}_2,\tilde{t}_3,a,b)D_3y_f(0,\tilde{t}_2,\tilde{t}_3,a,b) &= 0 \\ y_f(0,\tilde{t}_2,\tilde{t}_3,a,b) &= 0 \end{aligned}$$

Note that the z-coordinate of the curve contained in $W^u(p(a,b))$ with y-coordinate equal to zero, has a local extremum at $\mathbf{x}_f(0,\tilde{t}_2,\tilde{t}_3,a,b)$, for all $(a,b) \in V$. As in the previous case, without loss of generality, we can suppose that $\tilde{t}_2 \equiv 1$, $\tilde{t}_3 \equiv 1$,

$$x_f(0,1,1,a,b) = 1, \quad y_f(0,1,1,a,b) = 0,$$

$$D_2y_f(0,1,1,a,b)D_3z_f(0,1,1,a,b) - D_2z_f(0,1,1,a,b)D_3y_f(0,1,1,a,b) = 0,$$

for all $(a,b) \in V$. Moreover, the point of homoclinic tangency is $q = (1,0,0) = \mathbf{x}_f(0,1,1,0,0)$ and, for $(t_1,t_2,t_3,a,b) = (0,1,1,0,0)$,

$$D_2z_f = D_3z_f = 0, \quad D_3y_f \neq 0 \text{ and } D_{22}z_f(D_3y)^2 + D_{33}z_f(D_2y_f)^2 - 2D_2y_fD_3y_fD_{23}z_f \neq 0.$$

Proceeding similarly to the first case, we can prove that the third condition of definition 2.16 implies that:

$$D_{34}z_f(0,1,1,0,0)D_5z_f(0,1,1,0,0) - D_{35}z_f(0,1,1,0,0)D_4z_f(0,1,1,0,0) \neq 0.$$

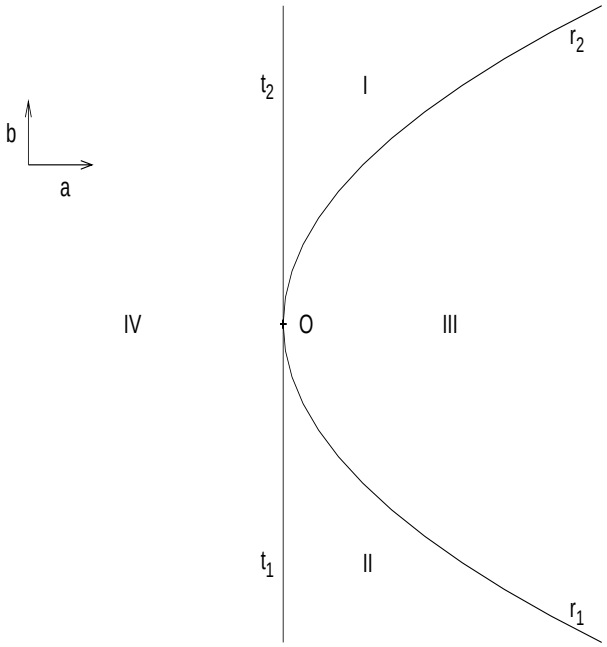


Figure 6: Generalized homoclinic tangency (type I). Parameter plane

Finally, we consider the following change of parameters:

$$\begin{aligned} \bar{a} &= z(0,1,1,a,b) \\ \bar{b} &= D_3 z(0,1,1,a,b) \end{aligned}$$

Then the family $\{g_{\tilde{a},\tilde{b}}\}_{(\tilde{a},\tilde{b})\in(-\epsilon,\epsilon)\times(-\epsilon,\epsilon)}$, for a suitable $\epsilon > 0$, defined by $g_{\tilde{a},\tilde{b}} = f_{a,b}$, and $\mathbf{x}(\cdot,\cdot,\cdot,\tilde{a},\tilde{b}) = x_f(\cdot,\cdot,\cdot,a,b)$ satisfy all the properties that we wanted to prove.

This finishes the proof of the proposition. □

In figures 6, 7 and 8 we have a typical example of a generalized homoclinic tangency.

In figure 6 we have drawn the plane of parameters, and marked several lines (t_1, t_2, r_1, r_2), a point (O) and regions (I, II, III, IV). The lines t_1 and t_2 correspond to a homoclinic tangency, and the lines r_1 and r_2 to a generalized homoclinic transversality. In region IV there is no intersection between the invariant manifolds.

In figures 7 and 8 several behaviours of the invariant manifolds are displayed, depending on the values of the parameters considered. The meaning of the letters on the pictures are the following: p is the fixed point, q is a point of homoclinic tangency, S, S_1 and S_2 are leaves of the strong stable foliation of $W^s(p)$, L, L_1, L_2 are the intersection of a fiber of the unstable vector bundle with $W^s(p)$ (that is the plane $z = 0$, in coordinates (x, y, z)) In the case of generalized homoclinic tangency, S and L coincide, and in the case of generalized transversality, S_i, L_i (for $i = 1$ or 2, depending on the case) coincide. More arrows means more attraction. Finally, all the dashed curves are below the plane $z = 0$.

Remark 2.22 *The following properties hold for* $\{g_{\tilde{a},\tilde{b}}\}_{(\tilde{a},\tilde{b})\in(-\epsilon,\epsilon)\times(-\epsilon,\epsilon)}$ *: a)* $W^s(\tilde{p}(\tilde{a},\tilde{b}))$ *is the* x *axis in a neighbourhood of* $(1,0,0)$*. b) For the type I, the* y*-coordinate of the*

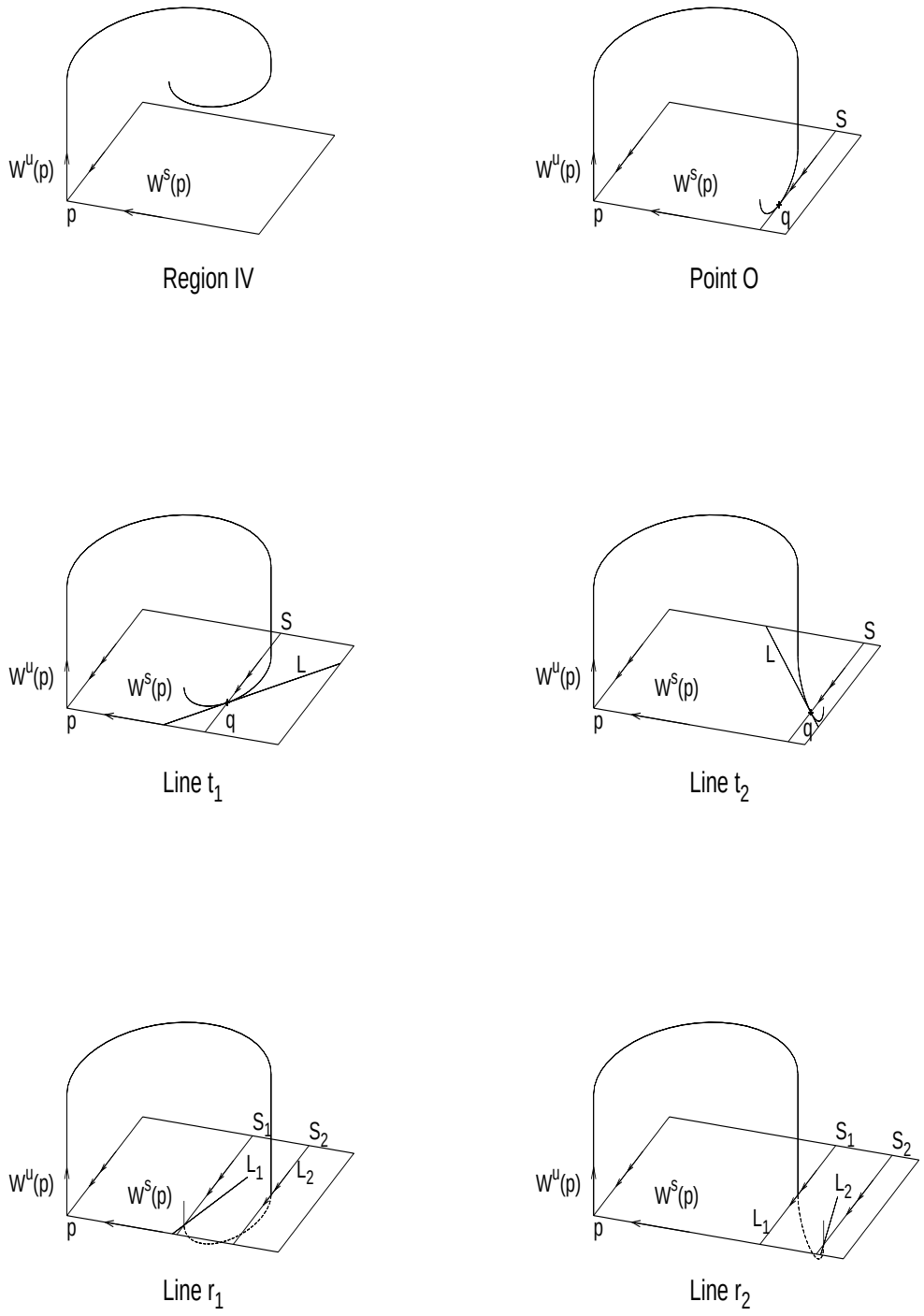


Figure 7: Generalized homoclinic tangency (type I). Phase space (see explanations in the text).

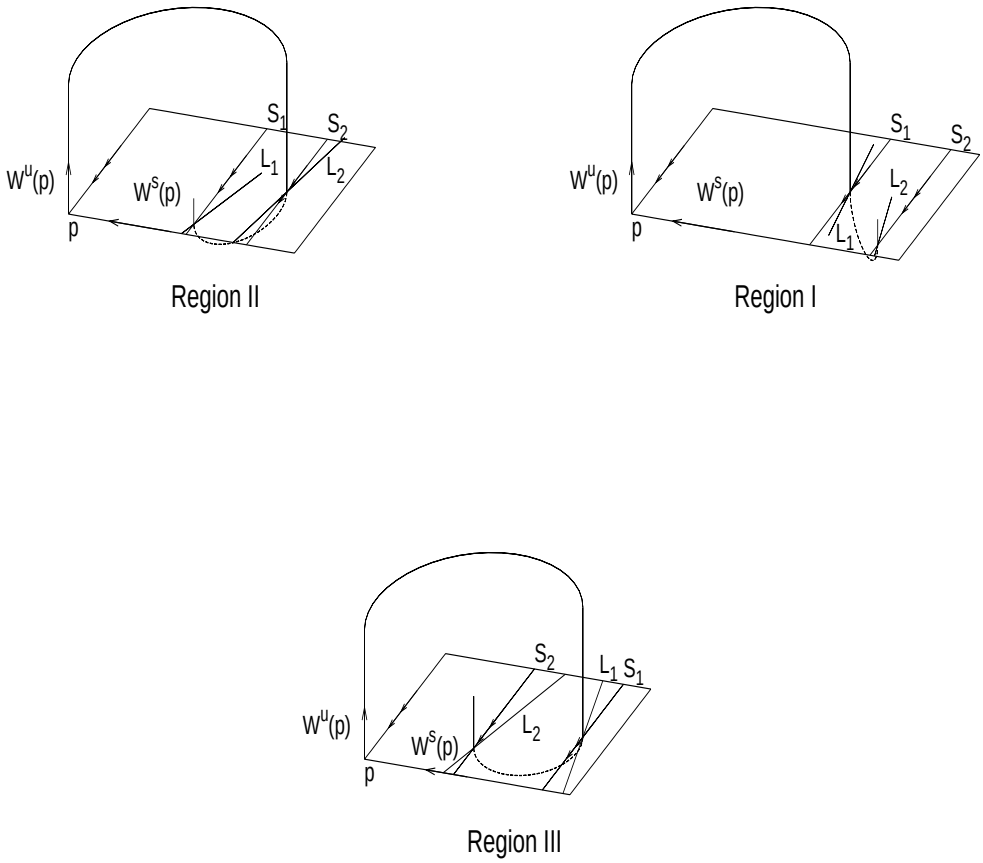


Figure 8: Generalized homoclinic tangency (type I). Continuation of the previous figure.

intersection of $W^u(\tilde{p}(\tilde{a},\tilde{b}))$ with $z=0$ has a local extremum at $(1,\tilde{a},0)$, for all $(\tilde{a},\tilde{b}) \in (-\epsilon,\epsilon)$. c) For the type II, the z-coordinate of the intersection of $W^u(\tilde{p}(\tilde{a},\tilde{b}))$ with $y=0$ has a local extremum at $(1,0,\tilde{a})$, for all $(\tilde{a},\tilde{b}) \in (-\epsilon,\epsilon)$. d) The fibers of the stable tangent vector bundle are the planes $z=0$, in a neighbourhood of $(1,0,0)$. e) Let $\mathbf{x}_f(0,1,t,\tilde{a},\tilde{b})$ the leaf of $\mathcal{F}^{uu}(\tilde{p}(\tilde{a},\tilde{b}))$ passing through $(1,\tilde{a},0)$ (type I) or $(1,0,\tilde{a})$ (type II), and parametrized by t. Then, $\tilde{b}$ is the z-coordinate of the tangent vector of this leaf at $(1,\tilde{a},0)$ (type I) or $(1,0,\tilde{a})$ (type II).

2.4 The Bogdanov-Takens bifurcation

Let $\{f_{a,b}\}_{a,b\in V}$, $V \subset \mathbb{R}^2$ being an open neighbourhood of (a_0,b_0), be a smooth 2-parameter 2-dimensional family of diffeomorphisms, having a fixed point p_0 for $(a,b)=(a_0,b_0)$, with two eigenvalues equal to 1 and $Df_{a_0,b_0}(p_0)$ different from the identity. The objective of this section is to give explicit conditions on the family to have a Bogdanov-Takens family of diffeomorphisms unfolding generically, as defined in Broer et al. [4].

Definition 2.23 *We say that $\{f_{a,b}\}_{(a,b)\in V}$ has a* **Bogdanov-Takens bifurcation** *at p_0 which unfolds generically with (a,b) at (a_0,b_0), if via an smooth change of variables and a reparametrization, we can obtain the family of maps*

$$\begin{pmatrix} u \\ v \end{pmatrix} \mapsto \begin{pmatrix} u+v \\ v+\mu+u^2+v(\nu+\tilde{\gamma}u)+\tilde{c}v^2+\tilde{r}_3(u,v) \end{pmatrix}, \tag{1}$$

where: p_0 is transformed in $(0,0)$, $\mu=\mu(a,b)$ and $\nu=\nu(a,b)$ are the new parameters such that $(\mu(a_0,b_0),\nu(a_0,b_0))=(0,0)$ and $|\frac{\partial(\mu,\nu)}{\partial(a,b)}|$ is different from zero when $(a,b)=(a_0,b_0)$, $\tilde{\gamma}$ and $\tilde{c}$ are functions of (a,b) such that $\tilde{\gamma}(a_0,b_0)\neq 2$. Moreover, $\tilde{r}_3=\tilde{r}_3(u,v,a,b)$ is a smooth function whose Taylor expansion around $(u,v)=(0,0)$, for each fixed (a,b), begins at degree 3.

In what follows, we include some properties related to the B-T bifurcation. For a proof and deeper properties see Broer et al. [4].

1. There exists a saddle-node bifurcation of (1) at $(0,0)$, for $\mu=0$ and all ν small.

2. The map (1) undergoes a Hopf bifurcation at $(u,0)$ for

$$(\mu,\nu)=(-u^2+O_3,(2-\tilde{\gamma})u+O_2),$$

   and all $u<0$ small enough. Here O_i means order i in u. Consequently, the curves of saddle-node and Hopf bifurcations have a quadratic contact at the origin.

3. Let $\mu_H=\mu_H(\nu)$ be the curve in the (μ,ν)-plane for which a Hopf bifurcation occurs. Note that $\nu\leq 0$ if $\tilde{\gamma}<2$ and $\nu\geq 0$ if $\tilde{\gamma}>2$. Then, there exist a function $\mu_1=\mu_1(\nu)$ and $\mu_0<0$ such that: a) If $\tilde{\gamma}<2$ then there exists a $\nu_0<0$ such that the map (1) has an attracting C^1 invariant curve for any (ν,μ) with $\mu_1(\nu)<\mu<\mu_H(\nu)$ and $0>\nu>\nu_0$. b) If $\tilde{\gamma}>2$ then there exists a $\nu_0>0$ such that the map (1) has a repelling C^1 invariant curve for any (ν,μ) with $\mu_1(\nu)<\mu<\mu_H(\nu)$ and $\nu_0>\nu>0$.

   Moreover the maps μ_1 and μ_H have a quadratic contact at the origin.

4. Suppose that $\{f_{a,b}\}_{a,b\in V}$ is as before and M is a a 2×2 invertible matrix such that

$$M^{-1}Df_{a_0,b_0}(p_0)M=\begin{pmatrix}1&1\\0&1\end{pmatrix}.$$

Let $v_i=Me_i$ and $w_i^t=e_i^tM^{-1}$, where $i=1,2$ and e_1, e_2 is the canonical basis of $\mathbb{R}^2$.

If

$$w_2^tD^2f_{a_0,b_0}(p_0)(v_1,v_1)\neq 0$$

and

$$w_2^tD^2f_{a_0,b_0}(p_0)(v_1,v_2)+w_1^tD^2f_{a_0,b_0}(p_0)(v_1,v_1)-w_2^tD^2f_{a_0,b_0}(p_0)(v_1,v_1)\neq 0,$$

then, by means of a smooth change of variables, it is possible to transform our family to one like (1). If after the transformation we have that $|\frac{\partial(\mu,\nu)}{\partial(a,b)}|$ is different from zero when $(a,b)=(a_0,b_0)$, then $\{f_{a,b}\}_{a,b\in V}$ has a Bogdanov-Takens bifurcation at p_0 which unfolds generically with (a,b) at (a_0,b_0).

3 Bifurcations of periodic points related to homoclinic tangencies

In this section, we will find bifurcations of periodic points near a homoclinic quadratic tangency, a generalized transversal homoclinic point (one-parameter case) and a generalized quadratic homoclinic tangency (two-parameter case). Let $m=1$ or $m=2$. We will consider an m-parameter family of diffeomorphisms $\{f_c\}_{c\in V}$, where V is an open neighbourhood of $c_0\in\mathbb{R}^m$ and $f_c:\mathbb{R}^3\longrightarrow\mathbb{R}^3$. We suppose, as before, that $\{f_c\}_c$ has a dissipative saddle fixed point $p=p(c)\in\mathbb{R}^3$, for c near c_0. We denote the eigenvalues of $Df_c(p)$ by λ_1, λ_2 and λ_3, satisfying $|\lambda_1|<|\lambda_2|<|\lambda_3|$, and $|\lambda_1\lambda_2\lambda_3|<1$. Moreover we introduce the following notation:

If $A=(a_{i,j})_{1\leq i,j\leq 3}$ is a 3×3 matrix then $d_1(A)=\operatorname{trace}A$, $d_2(A)=\det A_1+\det A_2+\det A_3$ (second trace), and $d_3(A)=\det A$, where A_i is the 2×2 matrix obtained by removing the row and the column i of the matrix A.

3.1 Quadratic tangency

In this case $c=a\in\mathbb{R}$. Assume that $\{f_a\}_{a\in V}$ has a quadratic homoclinic tangency at $q_0\in W^u(p(a_0))\cap W^s(p(a_0))$, unfolding generically with a at $a=a_0$, and that $\{f_a\}_{a\in V}$ satisfies the linearization assumption.

We shall distinguish between the cases $\dim W^u(p)=1$ or 2.

Proposition 3.1 *Suppose that $|\lambda_1|<|\lambda_2|<1<|\lambda_3|$ and, if $|\lambda_2\lambda_3|>1$, the homoclinic tangency is not generalized. Then, for n large enough, there exist values of the parameter a_n^+ and a_n^- such that*

1. *$f_{a_n^+}$ undergoes an n-periodic saddle-node bifurcation at p_n^+.*
2. *$f_{a_n^-}$ undergoes an n-periodic flip bifurcation at p_n^-.*

3. $\lim_{n\to\infty} a_n^+ = \lim_{n\to\infty} a_n^- = a_0$.

4. $\lim_{n\to\infty} p_n^+ = \lim_{n\to\infty} p_n^- = q_0$.

Proof:
We shall only consider the saddle-node case, the flip one being similar.

By using proposition 2.7, we can suppose that

$$f_a^n(t_1,t_2,t_3) = \mathbf{x}(\lambda_1^n t_1, \lambda_2^n t_2, \lambda_3^n t_3, a),$$

for all $(t_1,t_2,t_3,a) \in U$, where U is a suitable neighbourhood of $\{(t_1,t_2,t_3) \in \mathbb{R}^3 \ : \ (t_1^2+t_2^2)t_3 = 0\} \times (-\epsilon,\epsilon)$. The map $\mathbf{x}$ satisfies:

$$x(0,0,1,a) = 1,\ y(0,0,1,a) = 1,\ z(0,0,1,a) = a,\ D_3 z(0,0,1,a) = 0, \tag{2}$$

for $a \in (-\epsilon,\epsilon)$, and $D_{33}z(0,0,1,0) \neq 0$. Moreover, $\mathbf{x}(\cdot,a)$ is the identity in a neighbourhood of the disk $\{(t_1,t_2,0) \ : \ t_1^2+t_2^2 \leq 3\}$, and $q_0 = (1,1,0)$ can be taken as the tangential homoclinic point.

Now we want to find parameter values for which f_a^n undergoes a saddle-node bifurcation. Let $(t_1,t_2,\lambda_3^{-n}t_3)$ be a saddle-node fixed point of f_a^n. Then:

$$\mathbf{x}(\lambda_1^n t_1, \lambda_2^n t_2, t_3) = (t_1, t_2, \lambda_3^{-n} t_3),$$

and if we define $A = Df_a^n(t_1,t_2,\lambda_3^{-n}t_3) = D\mathbf{x}(\lambda_1^n t_1, \lambda_2^n t_2, t_3, a) \circ \Lambda_n$, where Λ_n is the diagonal matrix $\Lambda_n = \mathrm{diag}\,(\lambda_1^n, \lambda_2^n, \lambda_3^n)$, the condition of the existence of a saddle-node bifurcation is

$$1 - d_1(A) + d_2(A) - d_3(A) = 0. \tag{3}$$

Set $s_1 = \lambda_1^n$, $s_2 = \lambda_2^n$ and $s_3 = \lambda_3^{-n}$. To find an n-periodic saddle-node bifurcation is sufficient to solve the equation $\mathbf{F}(s_1,s_2,s_3,t_1,t_2,t_3,a) = 0$, where $\mathbf{F} = (F_1,F_2,F_3,F_4)$,

$$\begin{aligned}
F_1(s_1,s_2,s_3,t_1,t_2,t_3,a) &= x(s_1t_1, s_2t_2, t_3, a) - t_1,\\
F_2(s_1,s_2,s_3,t_1,t_2,t_3,a) &= y(s_1t_1, s_2t_2, t_3, a) - t_2,\\
F_3(s_1,s_2,s_3,t_1,t_2,t_3,a) &= z(s_1t_1, s_2t_2, t_3, a) - s_3t_3,\\
F_4(s_1,s_2,s_3,t_1,t_2,t_3,a) &= s_3 - s_3d_1(A) + s_3d_2(A) - s_3d_3(A),
\end{aligned}$$

and, taking the derivatives of x, y, z at (s_1t_1, s_2t_2, t_3, a),

$$s_3 d_1(A) = s_1s_3D_1x + s_2s_3D_2y + D_3z$$

$$s_3 d_2(A) = s_1s_2s_3 \begin{vmatrix} D_1x & D_2x \\ D_1y & D_2y \end{vmatrix} + s_1 \begin{vmatrix} D_1x & D_3x \\ D_1z & D_3z \end{vmatrix} + s_2 \begin{vmatrix} D_2y & D_3y \\ D_2z & D_3z \end{vmatrix}$$

$$s_3 d_3 = s_1s_2 \det D\mathbf{x}(s_1t_1, s_2t_2, t_3, a).$$

From (2), we see that $\mathbf{F}(0,0,0,1,1,1,0) = 0$. On the other hand, $\mathbf{F}$ is a differentiable map near $(0,0,0,1,1,1,0)$.

As $D_{33}z(0,0,1,0) \neq 0$, we can use the Implicit Function Theorem to prove that, for s_1, s_2 and s_3 small enough, there exist $t_i = t_i(s_1, s_2, s_3)$, $i = 1, 2, 3$, and $a = a(s_1, s_2, s_3)$ solving the equation $\mathbf{F} = 0$, for fixed s_i, $i = 1, 2, 3$, and satisfying:

$$t_i(0,0,0) = 1, \quad i = 1, 2, 3, \quad a(0,0,0) = 0.$$

Taking into account the values of s_i, we get that

$$p_n^+ = (t_1(\lambda_1^n, \lambda_2^n, \lambda_3^{-n}), t_2(\lambda_1^n, \lambda_2^n, \lambda_3^{-n}), \lambda_3^{-n} t_3(\lambda_1^n, \lambda_2^n, \lambda_3^{-n}))$$

is a saddle-node n-periodic point for $a = a_n^+ = a(\lambda_1^n, \lambda_2^n, \lambda_3^{-n})$, if n is large enough, and $p_n^+ \to q_0$, $a_n^+ \to 0$ when $n \to \infty$.

In order to see that p_n^+ is really a saddle-node, let 1, α_n and β_n denote the eigenvalues of $Df_{a_n^+}^n(p_n^+)$. It is easy to show that $|\alpha_n \beta_n| < 1$ and

$$d_{2,n} = d_2(Df_{a_n^+}^n(p_n^+)) = \alpha_n + \beta_n + \alpha_n\beta_n = \lambda_2^n \lambda_3^n (D_2 y D_3 z - D_2 z D_3 y + o(1)),$$

where all the derivatives of y, z are evaluated at $(0, 0, 1, 0)$.

If $|\lambda_2 \lambda_3| < 1$ then $d_{2,n} \to 0$ when $n \to \infty$. If $|\lambda_2 \lambda_3| > 1$, to say that the homoclinc tangency is not generalized implies that $D_2 z(0,0,1,0) \neq 0$ and $D_3 y(0,0,1,0) \neq 0$ (see proposition 2.19). As $D_3 z(0,0,1,0) = 0$, we have that $|d_{2,n}| \to \infty$. The behaviour of $d_{2,n}$ in both cases allows to show that neither α_n nor β_n has absolute value equal to 1, if n is large enough. This ends the proof of the proposition. □

Remark 3.2 *If the saddle fixed point is sectionally dissipative, it is easy to see that, for n large enough, there are also n-periodic attractors near q_0.*

Proposition 3.3 *Suppose that $|\lambda_1| < 1 < |\lambda_2| < |\lambda_3|$ and the homoclinic tangency is not generalized. Then, for n large enough there exist values of the parameter a_n^+, a_n^- and a_n^c such that*

1. *$f_{a_n^+}$ has an n-periodic saddle-node bifurcation at p_n^+.*
2. *$f_{a_n^-}$ has an n-periodic flip bifurcation at p_n^-.*
3. *$f_{a_n^c}$ has an n-periodic point p_n^c with the product of two of its eigenvalues equal to 1.*
4. *$\lim_{n\to\infty} a_n^+ = \lim_{n\to\infty} a_n^- = \lim_{n\to\infty} a_n^c = a_0$.*
5. *$\lim_{n\to\infty} p_n^+ = \lim_{n\to\infty} p_n^- = \lim_{n\to\infty} p_n^c = q_0$.*

Moreover, the saddle-node and the flip have index of stability 1.

Proof:

In this case, using proposition 2.8, we can write, after a change of variables and reparametrization:

$$f_a^n(t_1, t_2, t_3) = \mathbf{x}(\lambda_1^n t_1, \lambda_2^n t_2, \lambda_3^n t_3, a),$$

where $\mathbf{x}$ is a smooth map, $\mathbf{x}(\cdot, a)$ is the identity in a neighbourhood of $\{(t_1, 0, 0) \ : \ |t_1| \leq 3\}$, and one of the two possibilities holds:

1. $x(0,1,1,a)=1,\ y(0,1,1,a)=0,\ z(0,1,1,a)=a,$

$$D_2y(0,1,1,a)D_3z(0,1,1,a)-D_2z(0,1,1,a)D_3y(0,1,1,a)=0$$

and

$$D_1g(1,1,0)D_3y(0,1,1,0)-D_2g(1,1,0)D_2y(0,1,1,0)\neq 0.$$

2. $x(0,1,1,a)=1,\ y(0,1,1,a)=a,\ z(0,1,1,a)=0,$

$$D_2y(0,1,1,a)D_3z(0,1,1,a)-D_2z(0,1,1,a)D_3y(0,1,1,a)=0$$

and

$$D_1g(1,1,0)D_3z(0,1,1,0)-D_2g(1,1,0)D_2z(0,1,1,0)\neq 0.$$

Here $g(t_2,t_3,a)=D_2yD_3z-D_2zD_3y$, where all the derivatives are evaluated at $(0,t_2,t_3,a)$. Moreover, the point of homoclinic tangency is $q_0=(1,0,0)$, and the condition of not being a generalized homoclinic tangency is $D_3z(0,1,1,0,0)\neq 0$ (see proposition 2.21).

We only consider the saddle-node case (the flip one being similar). Suppose that $(t_1,\lambda_2^{-n}t_2,\lambda_3^{-n}t_3)$ is a saddle-node n-periodic point. Then,

$$\mathbf{x}(\lambda_1^n t_1,t_2,t_3,a)=(t_1,\lambda_2^{-n}t_2,\lambda_3^{-n}t_3), \tag{4}$$

and

$$1-d_1(A)+d_2(A)-d_3(A)=0, \tag{5}$$

where $A=Df_a^n(t_1,\lambda_2^{-n}t_2,\lambda_3^{-n}t_3)$. If we define $s_1=\lambda_1^n$, $s_2=\lambda_2^{-n}$ and $s_3=\lambda_3^{-n}$, we can show that to find (t_1,t_2,t_3,a) satisfying (4) and (5) is equivalent to solve a system of equations $\mathbf{F}(s_1,s_2,s_3,t_1,t_2,t_3,a)=0\in\mathbb{R}^3$. As before, using the properties of the map $\mathbf{x}$, we see that there exist differentiable maps $t_i=t_i(s_1,s_2,s_3)$, $i=1,2,3$ and $a=a(s_1,s_2,s_3)$ such that, $a_n^+=a(\lambda_1^n,\lambda_2^{-n},\lambda_3^{-n})$, and

$$p_n^+=(t_1(\lambda_1^n,\lambda_2^{-n},\lambda_3^{-n}),\lambda_2^{-n}t_2(\lambda_1^n,\lambda_2^{-n},\lambda_3^{-n}),\lambda_3^{-n}t_3(\lambda_1^n,\lambda_2^{-n},\lambda_3^{-n})),$$

satisfy the properties that we wanted in the saddle-node case. We note that to see that only one eigenvalue of the saddle-node has absolute value equal to 1, we use the condition $D_3z(0,1,1,0,0)\neq 0$.

On the other hand, there are also periodic points with the product of two of their eigenvalues equal to 1. To see this, first it is easy to see that the equation (5) has to be replaced by

$$1+(d_1(A)-d_3(A))d_3(A)-d_2(A)=0.$$

By using variables $s_1=\lambda_1^n\lambda_3^n$, $s_2=\lambda_2^{-n}$, $s_3=\lambda_2^n\lambda_3^{-n}$, we can proceed as in the saddle-node case. □

3.2 Generalized transversality

For this type of transversality, we want to find hyperbolic periodic points having the product of two of their eigenvalues equal to 1. In order to get such situation we have to impose that the fixed point $p(a)$ is not sectionally dissipative. These periodic points are important in the context of the generalized homoclinic tangency, because they are the candidates to undergo a Hopf bifurcation, and a Bogdanov-Takens bifurcation, when we add a second parameter.

As before, we work with a one-parameter family $\{f_a\}_{a\in V}$, and assume that, for $a = a_0$, it has a generalized homoclinic transversality at $q_0 \in W^u(p(a_0)) \cap W^s(p(a_0))$, unfolding generically, and satisfies the linearization assumption.

Proposition 3.4 *Suppose that $|\lambda_1| < |\lambda_2| < 1 < |\lambda_3|$ and $|\lambda_2\lambda_3| > 1$. Then, for n large enough, there exist values of the parameter a_n^c such that*

1. *$f_{a_n^c}$ has an n-periodic point p_n^c having the product of two of its eigenvalues equal to 1.*
2. *$\lim_{n\to\infty} a_n^c = a_0$.*
3. *$\lim_{n\to\infty} p_n^c = q_0$.*

Proof:
From proposition 2.13, we can write

$$f_a^n(t_1, t_2, t_3) = \mathbf{x}(\lambda_1^n t_1, \lambda_2^n t_2, \lambda_3^n t_3),$$

where

$$x(0,0,1,a) = 1, \quad y(0,0,1,a) = 1, \quad z(0,0,1,a) = 0,$$

and

$$D_2y(0,0,1,a)D_3z(0,0,1,a) - D_2z(0,0,1,a)D_3y(0,0,1,a) = a,$$

for all $a \in V$ and $D_3z(0,0,1,0) \neq 0$. Moreover the point of generalized homoclinic transversality is $q_0 = (1,0,0) = \mathbf{x}(0,0,1,0)$.

Let $(t_1, t_2, \lambda_3^{-n}t_3)$ be an n-periodic point. and $A = Df_a^n(t_1, t_2, \lambda_3^{-n}t_3)$. If we suppose that the product of two eigenvalues of A is one, then:

$$1 + (d_1(A) - d_3(A))d_3(A) - d_2(A) = 0. \tag{6}$$

Define $s_1 = \lambda_2^{-n}\lambda_3^{-n}$, $s_2 = \lambda_2^n$ and $s_3 = \lambda_1^n\lambda_3^n$ and consider the equation $\mathbf{F}(s_1, s_2, s_3, t_1, t_2, t_3, a) = 0 \in \mathbb{R}^3$, where $\mathbf{F} = (F_1, F_2, F_3, F_4)$ and

$$\begin{aligned}
F_1(s_1, s_2, s_3, t_1, t_2, t_3, a) &= x(s_1s_2s_3t_1, s_2t_2, t_3, a) - t_1,\\
F_2(s_1, s_2, s_3, t_1, t_2, t_3, a) &= y(s_1s_2s_3t_1, s_2t_2, t_3, a) - t_2,\\
F_3(s_1, s_2, s_3, t_1, t_2, t_3, a) &= z(s_1s_2s_3t_1, s_2t_2, t_3, a) - s_1s_2t_3,\\
F_4(s_1, s_2, s_3, t_1, t_2, t_3, a) &= D_2yD_3z - D_3yD_2z + O(|s_1, s_2, s_3|),
\end{aligned}$$

In the last equation, the derivatives are evaluated at $(s_1s_2s_3t_1, s_2t_2, t_3, a)$. Note that the first three equations imply that $(t_1, t_2, \lambda_3^{-n}t_3)$ is an n-periodic point, and the last is obtained multiplying equation (6) by $s_1 = \lambda_2^{-n}\lambda_3^{-n}$. $O(|s_1, s_2, s_3|)$ means first order in s_1, s_2, s_3.

The properties of $\mathbf{x}$ imply that $\mathbf{F}(0,0,0,1,1,1,0) = 0$, and, by the implicit function theorem, there exist differentiable maps $t_i = t_i(s_1, s_2, s_3)$, $i = 1, 2, 3$, $a = a(s_1, s_2, s_3)$, such that $t_i(0,0,0) = 1$, for $i = 1, 2, 3$ and $a(0,0,0) = 0$. This means that

$$p_n^c = (t_1(\lambda_2^{-n}\lambda_3^{-n}, \lambda_2^n, \lambda_1^n\lambda_3^n), t_2(\lambda_2^{-n}\lambda_3^{-n}, \lambda_2^n, \lambda_1^n\lambda_3^n), \lambda_3^{-n}t_3(\lambda_2^{-n}\lambda_3^{-n}, \lambda_2^n, \lambda_1^n\lambda_3^n))$$

is an n-periodic point of $f_{a_n^c}$, where $a_n^c = a(\lambda_2^{-n}\lambda_3^{-n}, \lambda_2^n, \lambda_1^n\lambda_3^n)$, and has eigenvalues α_n, β_n, γ_n satisfying: $\alpha_n\beta_n = 1$, $|\alpha_n\beta_n\gamma_n| < 1$ and all are real and different from 1 and -1. Moreover, $\lim_{n\to\infty} a_n^c = 0$ and $\lim_{n\to\infty} p_n^c = q_0$. This finishes the proof of the proposition. □

3.3 Generalized quadratic tangency

Now, we deal with a family $\{f_{a,b}\}_{(a,b)\in V}$, having a generalized homoclinic tangency at $q_0 \in W^u(p(a_0,b_0)) \cap W^s(p(a_0,b_0))$, unfolding generically with (a,b) at (a_0,b_0). We use the same notation as before, and distinguish between two cases, depending on the dimension of the stable invariant manifold of the fixed point.

Proposition 3.5 *Suppose that $|\lambda_1| < |\lambda_2| < 1 < |\lambda_3|$ and $|\lambda_2\lambda_3| > 1$. Then, for n large enough there exist values of the parameters (a_n^{++}, b_n^{++}), (a_n^{+-}, b_n^{+-}) and (a_n^{--}, b_n^{--}), such that*

1. *$f_{a_n^{++},b_n^{++}}$ has an n-periodic point p_n^{++} with 1 as a double eigenvalue.*
2. *$f_{a_n^{+-},b_n^{+-}}$ has an n-periodic point p_n^{+-} with one eigenvalue equal to 1 and another equal to -1.*
3. *$f_{a_n^{--},b_n^{--}}$ has an n-periodic point p_n^{--} with two eigenvalues equal to -1.*
4. *$\lim_{n\to\infty}(a_n^{++}, b_n^{++}) = \lim_{n\to\infty}(a_n^{+-}, b_n^{+-}) = \lim_{n\to\infty}(a_n^{--}, b_n^{--}) = (a_0, b_0)$.*
5. *$\lim_{n\to\infty} p_n^{\pm\pm} = q_0$.*

Proof:
By proposition 2.19, we can assume that

$$f_{a,b}^n(t_1,t_2,t_3) = \mathbf{x}(\lambda_1^n t_1, \lambda_2^n t_2, \lambda_3^n t_3, a, b),$$

where $\mathbf{x}(\cdot, a, b)$ is the identity in a neighbourhood of $\{(t_1, t_2, 0) \ : \ t_1^2 + t_2^2 \leq 3\}$.

We only study the generalized homoclinic tangency of type I. We can proceed in a similar way for the other type. This means that the map $\mathbf{x}$ satisfies:

$$x(0,0,1,a,b) = 1, \quad y(0,0,1,a,b) = 1, \quad z(0,0,1,a,b) = a,$$

$$D_3 z(0,0,1,a,b) = 0, \quad D_3 y(0,0,1,a,b) = b,$$

for all $a \in V$ and $D_{33}z(0,0,1,0,0) \neq 0$, $D_2 z(0,0,1,0,0) \neq 0$, and

$$\begin{vmatrix} D_{33}y(0,0,1,0,0) & D_{33}z(0,0,1,0,0) \\ D_2 y(0,0,1,0,0) & D_2 z(0,0,1,0,0) \end{vmatrix} \neq 0.$$

Now, we look for periodic points with two eigenvalues equal to 1. Let $p_n = (t_1, t_2, \lambda_3^{-n} t_3)$ be an n-periodic point. Define $s_1 = \lambda_2^{-n}\lambda_3^{-n}$, $s_2 = \lambda_2^n$ and $s_3 = \lambda_1^n \lambda_2^{-n}$. Then $f_{a,b}^n(p_n) = p_n$ and, therefore,

$$\begin{aligned} x(s_2 s_3 t_1, s_2 t_2, t_3, a, b) - t_1 &= 0 \\ y(s_2 s_3 t_1, s_2 t_2, t_3, a, b) - t_2 &= 0 \\ z(s_2 s_3 t_1, s_2 t_2, t_3, a, b) - s_1 s_2 t_3 &= 0 \end{aligned}$$

In order to have two eigenvalues equal to 1 if $A = Df_{a,b}^n(p_n)$, one has to impose:

$$\begin{aligned} 1 - d_1(A) + d_2(A) - d_3(A) &= 0 \\ 1 - d_2(A) + 2d_3(A) &= 0. \end{aligned}$$

If we multiply the fourth equation by λ_3^{-n} and the fifth by $\lambda_2^{-n}\lambda_3^{-n}$, we easily see that the above system of five equations is equivalent to $\mathbf{F}(s_1, s_2, s_3, t_1, t_2, t_3, a, b) = 0 \in \mathbb{R}^5$, where $\mathbf{F} = (F_1, F_2, F_2, F_3, F_4, F_5)$ and

$$\begin{aligned} F_1(0,0,0,t_1,t_2,t_3,a,b) &= x(0,0,t_3,a,b) - t_1, \\ F_2(0,0,0,t_1,t_2,t_3,a,b) &= y(0,0,t_3,a,b) - t_2, \\ F_3(0,0,0,t_1,t_2,t_3,a,b) &= z(0,0,t_3,a,b), \\ F_4(0,0,0,t_1,t_2,t_3,a,b) &= -D_3 z(0,0,t_3,a,b), \\ F_5(0,0,0,t_1,t_2,t_3,a,b) &= D_2 y D_3 z - D_2 z D_3 y, \end{aligned}$$

where the derivatives of the last equation are evaluated at $(0,0,t_3,a,b)$. As $\mathbf{F}(0,0,0,1,1,1,0,0) = 0$ and

$$\det D\tilde{F}(0,0,0,1,1,1,0,0) = -D_{33} z(0,0,1,0,0) D_2 z(0,0,1,0,0) \neq 0,$$

by the conditions of existence of a generalized homoclinic tangency, there are differentiable maps $t_i = t_i(s_1, s_2, s_3)$, $i = 1, 2, 3$ and $(a, b) = (a(s_1, s_2, s_3), b(s_1, s_2, s_3))$ such that $t_i(0,0,0) = 1$, for $i = 1, 2, 3$ and $(a(0,0,0), b(0,0,0)) = (0,0)$, and satisfying $\mathbf{F}(s_1, s_2, s_3, t_1, t_2, t_3, a, b) = 0$, for all (s_1, s_2, s_3) small enough. Then

$$p_n^{++} = (t_1(\lambda_2^{-n}\lambda_3^{-n}, \lambda_2^n, \lambda_1^n\lambda_2^{-n}), t_2(\lambda_2^{-n}\lambda_3^{-n}, \lambda_2^n, \lambda_1^n\lambda_2^{-n}), \lambda_3^{-n} t_3(\lambda_2^{-n}\lambda_3^{-n}, \lambda_2^n, \lambda_1^n\lambda_2^{-n}))$$

and $a_n^{++} = a(\lambda_2^{-n}\lambda_3^{-n}, \lambda_2^n, \lambda_1^n\lambda_2^{-n})$, satisfy the statements of the proposition.

We can proceed similarly to obtain periodic points with eigenvalues $1, -1$ and with $-1, -1$. □

Remark 3.6 *We postpone the proof of the existence of Bogdanov-Takens bifurcations, unfolding generically, because we will use a limit return map to simplify the computations. It is also interesting to see if the other bifurcations unfold generically, but we only want to use the properties that can be deduced from the existence of a BT bifurcation, so we do not take in consideration this problem.*

Proposition 3.7 *Suppose that $|\lambda_1| < 1 < |\lambda_2| < |\lambda_3|$. Then, for n large enough, there exist values of the parameters (a_n^{++}, b_n^{++}), (a_n^{+-}, b_n^{+-}) and (a_n^{--}, b_n^{--}), such that*

1. *$f_{a_n^{++},b_n^{++}}$ has an n-periodic point p_n^{++} with 1 as a double eigenvalue.*

2. *$f_{a_n^{+-},b_n^{+-}}$ has an n-periodic point p_n^{+-} with one eigenvalue equal to 1 and another equal to -1.*

3. *$f_{a_n^{--},b_n^{--}}$ has an n-periodic point p_n^{--} with two eigenvalues equal to -1.*

4. $\lim_{n\to\infty}(a_n^{++},b_n^{++}) = \lim_{n\to\infty}(a_n^{+-},b_n^{+-}) = \lim_{n\to\infty}(a_n^{--},b_n^{--}) = (a_0,b_0)$.

5. $\lim_{n\to\infty} p_n^{\pm\pm} = q_0$.

Proof:
First, we can write, as before:

$$f_{a,b}^n(t_1,t_2,t_3) = \mathbf{x}(\lambda_1^n t_1, \lambda_2^n t_2, \lambda_3^n t_3, a, b).$$

We suppose that the generalized homoclinic tangency is of type I (the other case being similar). We can suppose, by proposition 2.21, that

$$x(0,1,1,a,b) = 1, \quad y(0,1,1,a,b) = a,$$

$$z(0,1,1,a,b) = 0, \quad D_3 z(0,1,1,a,b) = b,$$

$$D_2 y(0,1,1,a,b) D_3 z(0,1,1,a,b) - D_3 y(0,1,1,a,b) D_2 z(0,1,1,a,b) = 0,$$

for all $(a,b) \in V$, and $D_3 y(0,1,1,0,0) = 0$, $D_2 z(0,1,1,0,0) \neq 0$, and

$$D_2 y D_{33} z - D_{33} y D_2 z \neq 0,$$

where all the derivatives are evaluated at $(0,1,1,0,0)$.

Define $s_1 = \lambda_1^n \lambda_2^n$, $s_2 = \lambda_2^{-n}$ and $s_3 = \lambda_2^n \lambda_3^{-n}$, and suppose that $(t_1, \lambda_2^{-n} t_2, \lambda_3^{-n} t_3)$ is an n-periodic point with two eigenvalues equal to 1. Proceeding as in the previous proposition, we obtain a system of equations: $\mathbf{F}(s_1,s_2,s_3,t_1,t_2,t_3,a,b) = 0 \in \mathbb{R}^5$, such that $\mathbf{F} = (F_1,F_2,F_3,F_4,F_5)$ and

$$\begin{aligned}
F_1(0,0,0,t_1,t_2,t_3,a,b) &= x(0,t_2,t_3,a,b) - t_1,\\
F_2(0,0,0,t_1,t_2,t_3,a,b) &= y(0,t_2,t_3,a,b),\\
F_3(0,0,0,t_1,t_2,t_3,a,b) &= z(0,t_2,t_3,a,b),\\
F_4(0,0,0,t_1,t_2,t_3,a,b) &= D_2 y D_3 z - D_3 y D_2 z,\\
F_5(0,0,0,t_1,t_2,t_3,a,b) &= -D_3 z(0,t_2,t_3,a,b),
\end{aligned}$$

where the derivatives of the fourth equation are evaluated at $(0,t_2,t_3,a,b)$. Then, we have that $F(0,0,0,1,1,1,0,0) = 0$ and $\det DF(0,0,0,1,1,1,0,0) = -D_2 z(D_2 y D_{33} z - D_{33} y D_2 z) \neq 0$, where the derivatives are evaluated at $(0,1,1,0,0)$. This means that there exist differentiable maps $t_i = t_i(s_1,s_2,s_3)$, $i = 1,2,3$, $a = a(s_1,s_2,s_3)$ and $b = b(s_1,s_2,s_3)$ such that $t_i(0,0,0) = 1$, for $i = 1,2,3$ $(a,b)(0,0,0) = (0,0)$ and satisfy $\mathbf{F}(s_1,s_2,s_3,t_1,t_2,t_3,a,b) = 0$ for all (s_1,s_2,s_3) small enough. To finish the prove in the case of double eigenvalue equal to one, we can verify that $p_n^{++} = (t_1, \lambda_2^{-n} t_2, \lambda_3^{-n} t_3)$ and $(a_n^{++},b_n^{++}) = (a,b)(\lambda_1^n \lambda_2^n, \lambda_2^{-n}, \lambda_2^n \lambda_3^{-n})$ have the properties we wanted.

The other cases can be shown in the same way. □

4 Limit return maps

In this section we will prove the main theorem of the paper, concerning the existence of limit return maps and consequences. The needed changes of variables and reparametrizations to obtain these limit maps are suggested by the values of the parameters corresponding to the several bifurcations we have obtained. We use the same notation as in the previous section:

Let $\{f_{a,b}\}_{(a,b)\in V}$ be a two-parameter family of three dimensional maps having a dissipative saddle fixed point $p(a,b)$, for all $(a,b) \in V$. As before, we denote by λ_1, λ_2 and λ_3 the (real) eigenvalues of $Df_{a,b}p(a,b)$. Assume that this family has a generalized quadratic homoclinic tangency at $q \in W^u(p(a_0,b_0)) \cap W^s(p(a_0,b_0))$, unfolding generically with (a,b) at $(a_0,b_0) \in V$.

We first suppose that the fixed point is not sectionally dissipative, and the stable invariant manifold is two-dimensional. We have the following result:

Proposition 4.1 *Suppose that* $|\lambda_1| < |\lambda_2| < 1 < |\lambda_3|$, $|\lambda_1\lambda_3| < 1$ *and* $|\lambda_2\lambda_3| > 1$. *Then,* $\{f_{a,b}\}_{(a,b)\in V}$ *has a family of limit return maps* $\{H_{\bar{a},\bar{b}}\}_{(\bar{a},\bar{b})\in\mathbb{R}^2}$, *associated to the orbit of* q, *such that:*

1. *If the generalized homoclinic tangency is of type I then:*

$$H_{\bar{a},\bar{b}}(\bar{x},\bar{y},\bar{z}) = (\bar{z}, \bar{b}\bar{z}, \bar{a} + \bar{y} + \bar{z}^2). \tag{7}$$

2. *If the generalized homoclinic tangency is of type II then:*

$$H_{\bar{a},\bar{b}}(\bar{x},\bar{y},\bar{z}) = (0, \bar{z}, \bar{a} + \bar{b}\bar{y} + \bar{z}^2). \tag{8}$$

Proof:

We can suppose that the family $\{f_{a,b}\}_{(a,b)\in V}$ satisfies the same properties as $\{g_{\bar{a},\bar{b}}\}_{(\bar{a},\bar{b})\in(-\epsilon,\epsilon)\times(-\epsilon,\epsilon)}$, in Proposition 2.19. Thus, $V = (-\epsilon,\epsilon)\times(-\epsilon,\epsilon)$, the fixed point is $(0,0,0)$, for all $(a,b) \in V$, and the values of the parameters for which there exists a generalized homoclinic tangency are $a_0 = 0$ and $b_0 = 0$. Moreover, we can write $f^n_{a,b}(t_1,t_2,t_3) = \mathbf{x}(\lambda_1^n t_1, \lambda_2^n t_2, \lambda_3^n t_3, a, b)$, for (t_1,t_2,t_3,a,b) in a suitable neighbourhood of $\{(t_1,t_2,t_3) \in \mathbb{R}^3 \ : \ (t_1^2+t_2^2)t_3 = 0\} \times V$, where $\mathbf{x} = (x,y,z)$ is as in proposition 2.19.

By definition 1.6, in order to find a limit return map, we have to build up n-depending changes of variables and reparametrizations for the maps $f^n_{a,b}$ near the generalized homoclinic tangency.

We first need to obtain some information about the points and values of parameters corresponding to saddle-node and BT bifurcations. This is the content of the following lemma.

Lemma 4.2 *There exist smooth maps* $T_i = T_i(s_1,s_2,s_3,b)$, $u = u(s_1,s_2,s_3,b)$ *and* $v = v(s_2)$, *defined for* (s_1,s_2,s_3,b) *in a neighbourhood of* $(0,0,0,0) \in \mathbb{R}^4$, *such that:*

1. $T_i(0,0,0,b) = 1$, $i = 1,2,3$, $u(0,0,0,b) = 0$, *for* $|b|$ *small enough, and* $v(0) = 0$.

2. *If we take* $s_1 = \lambda_1^n$, $s_2 = \lambda_2^n$ *and* $s_3 = \lambda_3^{-n}$ *then the map* $f_{u,b}$ *has an* n*-periodic saddle-node bifurcation at* $(T_1, T_2, \lambda_3^{-n} T_3)$, *that is:*

$$\begin{aligned} x(\lambda_1^n T_1, \lambda_2^n T_2, T_3, u, b) - T_1 &= 0 \\ y(\lambda_1^n T_1, \lambda_2^n T_2, T_3, u, b) - T_2 &= 0 \\ z(\lambda_1^n T_1, \lambda_2^n T_2, T_3, u, b) - \lambda_3^{-n} T_3 &= 0 \\ \lambda_3^{-n}(1 - d_1 + d_2 - d_3) &= 0, \end{aligned} \tag{9}$$

where $d_i = d_i(Df_{u,b}(T_1, T_2, \lambda_3^{-n} T_3))$.

3. *If we define* $\bar{T}_i = T_i(0, \lambda_2^n, 0, v(\lambda_2^n))$, $i = 1, 2, 3$, *and* $\bar{u} = u(0, \lambda_2^n, 0, v(\lambda_2^n))$ *then:*

   (a) *If the generalized homoclinic tangency is of type I:*

$$\begin{aligned} x(0, \lambda_2^n \bar{T}_2, \bar{T}_3, \bar{u}, v(\lambda_2^n)) - \bar{T}_1 &= 0 \\ y(0, \lambda_2^n \bar{T}_2, \bar{T}_3, \bar{u}, v(\lambda_2^n)) - \bar{T}_2 &= 0 \\ z(0, \lambda_2^n \bar{T}_2, \bar{T}_3, \bar{u}, v(\lambda_2^n)) &= 0 \\ D_3 z(0, \lambda_2^n \bar{T}_2, \bar{T}_3, \bar{u}, v(\lambda_2^n)) &= 0 \\ D_3 y(0, \lambda_2^n \bar{T}_2, \bar{T}_3, \bar{u}, v(\lambda_2^n)) &= 0 \end{aligned} \tag{10}$$

   (b) *If the generalized homoclinic tangency is of type II:*

$$\begin{aligned} x(0, \lambda_2^n \bar{T}_2, \bar{T}_3, \bar{u}, v(\lambda_2^n)) - \bar{T}_1 &= 0 \\ y(0, \lambda_2^n \bar{T}_2, \bar{T}_3, \bar{u}, v(\lambda_2^n)) - \bar{T}_2 &= 0 \\ z(0, \lambda_2^n \bar{T}_2, \bar{T}_3, \bar{u}, v(\lambda_2^n)) &= 0 \\ D_3 z(0, \lambda_2^n \bar{T}_2, \bar{T}_3, \bar{u}, v(\lambda_2^n)) &= 0 \\ D_2 z(0, \lambda_2^n \bar{T}_2, \bar{T}_3, \bar{u}, v(\lambda_2^n)) &= 0, \end{aligned} \tag{11}$$

This lemma is an immediate consequence of the properties of the map $\mathbf{x}$ and the implicit function theorem.

Remark 4.3 *If* $(a_n^{++}, b_n^{++}) = (a, b)(\lambda_2^{-n}\lambda_3^{-n}, \lambda_2^n, \lambda_1^n \lambda_2^{-n})$ *are the values of the parameters for which there is an* n*-periodic Bogdanov-Takens bifurcations at* $p_n^{++} = P(\lambda_2^{-n}\lambda_3^{-n}, \lambda_2^n, \lambda_1^n \lambda_2^{-n})$, *in Proposition 3.5, it is not difficult to prove that* $a(0, \lambda_2^n, 0) = \bar{u}(\lambda_2^n)$, $b = (0, \lambda_2^n, 0) = v(\lambda_2^n)$ *and* $P(0, \lambda_2^n, 0) = (\bar{T}_1(\lambda_2^n), \bar{T}_2(\lambda_2^n), 0)$.

With the information of the previous lemma, we can perform a changes of variables and reparametrization for the map $f_{a,b}^n$, depending on the type of tangency. Denote by $(\bar{x}, \bar{y}, \bar{z})$ the new variables and by $(\bar{a}, \bar{b})$ the new parameters.

Type I. We consider the change:

$$\begin{aligned} t_1 &= T_1(\lambda_1^n, \lambda_2^n, \lambda_3^{-n}, b) + \lambda_3^{-n} \bar{x} \\ t_2 &= T_2(\lambda_1^n, \lambda_2^n, \lambda_3^{-n}, b) + \lambda_2^{-n} \lambda_3^{-2n} \bar{y} \\ t_3 &= \lambda_3^{-n} T_3(\lambda_1^n, \lambda_2^n, \lambda_3^{-n}, b) + \lambda_3^{-2n} \bar{z} \\ a &= u(\lambda_1^n, \lambda_2^n, \lambda_3^{-n}, b) + \lambda_3^{-2n} \bar{a} \\ b &= v(\lambda_2^n) + \bar{b} \lambda_2^{-n} \lambda_3^{-n} \end{aligned} \tag{12}$$

We want to express $f^n_{a,b}$ in terms of $\bar{x}, \bar{y}, \bar{z}, \bar{a}, \bar{b}$. Let $(\bar{x}, \bar{y}, \bar{z}, \bar{a}, \bar{b})$ denote a point. The (t_1, t_2, t_3, a, b) variables of this point are given by (12). After applying $f^n_{a,b}$, we get as (t_1, t_2, t_3) coordinates:

$$\begin{aligned}
t_1 &= x(\lambda_1^n T_1 + \lambda_1^n \lambda_3^{-n} \bar{x}, \lambda_2^n T_2 + \lambda_3^{-2n} \bar{y}, T_3 + \lambda_3^{-n} \bar{z}, u + \lambda_3^{-2n} \bar{a}, v + \bar{b} \lambda_2^{-n} \lambda_3^{-n}) \\
t_2 &= y(\lambda_1^n T_1 + \lambda_1^n \lambda_3^{-n} \bar{x}, \lambda_2^n T_2 + \lambda_3^{-2n} \bar{y}, T_3 + \lambda_3^{-n} \bar{z}, u + \lambda_3^{-2n} \bar{a}, v + \bar{b} \lambda_2^{-n} \lambda_3^{-n}) \\
t_3 &= z(\lambda_1^n T_1 + \lambda_1^n \lambda_3^{-n} \bar{x}, \lambda_2^n T_2 + \lambda_3^{-2n} \bar{y}, T_3 + \lambda_3^{-n} \bar{z}, u + \lambda_3^{-2n} \bar{a}, v + \bar{b} \lambda_2^{-n} \lambda_3^{-n})
\end{aligned}$$

where $T_i = T_i(\lambda_1^n, \lambda_2^n, \lambda_3^{-n}, b)$, $i = 1, 2, 3$, $u = u(\lambda_1^n, \lambda_2^n, \lambda_3^{-n}, b)$, and $v = v(\lambda_2^n)$. Transforming this back to the $(\bar{x}, \bar{y}, \bar{z})$ coordinates, and denoting the values of the coordinates of the new point by $\bar{x}'$, $\bar{y}'$, $\bar{z}'$,

$$\begin{aligned}
\bar{x}' &= (x - T_1(\lambda_1^n, \lambda_2^n, \lambda_3^{-n}))\lambda_3^n \\
\bar{y}' &= (y - T_2(\lambda_1^n, \lambda_2^n, \lambda_3^{-n}))\lambda_2^n \lambda_3^{2n} \\
\bar{z}' &= (z - \lambda_3^{-n} T_3(\lambda_1^n, \lambda_2^n, \lambda_3^{-n}))\lambda_3^{2n}
\end{aligned}$$

where the maps x, y, z are taken at the same point as before.

In order to obtain a limit map, we define $s_1 = \lambda_2^{-n}\lambda_3^{-n}$, $s_2 = \lambda_2^n$ and $s_3 = \lambda_1^n \lambda_3^n$, and use the Taylor expansion of the map $\mathbf{x}(\cdot, b)$, about the point

$$(\lambda_1^n T_1, \lambda_2^n T_2, T_3, u) = (s_1 s_2 s_3 T_1, s_2 T_2, T_3, u),$$

where now $T_i = T_i(s_1 s_2 s_3, s_2, s_1 s_2, b)$ and $u = u(s_1 s_2 s_3, s_2, s_1 s_2, b)$.

From equation (9), we see that:

$$\begin{aligned}
\bar{x}' &= D_3 x(0,0,1,0,0)\bar{z} + O_1(s_1, s_2, s_3) \\
\bar{y}' &= D_3 y(s_1 s_2 s_3 T_1, s_2 T_2, T_3, u, v(s_2) + \bar{b} s_1)\bar{z} s_1^{-1} + O_1(s_1, s_2, s_3) \\
\bar{z}' &= D_2 z(0,0,1,0,0)\bar{y} + D_3 z\, \bar{z} s_1^{-1} s_2^{-1} + \bar{a} + \frac{1}{2} D_{33} z(0,0,1,0,0)\bar{z}^2 + O_1(s_1, s_2, s_3)
\end{aligned}$$

where $D_3 z = D_3 z(s_1 s_2 s_3 T_1, s_2 T_2, T_3, u, v(s_2) + \bar{b} s_1)$ and $O_1(s_1, s_2, s_3)$ means first order in s_1, s_2 and s_3.

We know, by the fifth equation of (10), that $D_3 y(0, s_2 \bar{T}_2, \bar{T}_3, \bar{u}, v(s_2)) = 0$. Hence, if we expand the map $D_3 y(s_1 s_2 s_3 T_1, s_2 T_2, T_3, u, v + \bar{b} s_1)$ about $s_1 = 0$, and use that $D_3 y(0, 0, 1, a, b) = b$ (see Proposition 3.5), we get

$$\bar{y}' = \bar{b}\bar{z} + O_1(s_1, s_2, s_3).$$

Finally, from the last equation of (9) we have:

$$(1 - s_2 D_2 y) D_3 z(s_1 s_2 s_3 T_1, s_2 T_2, T_3, u, v + \bar{b} s_1) = s_1 s_2 (1 + O_1(s_1, s_2, s_3)) + D_3 y\, D_2 z\, s_2,$$

where $D_2 y$, $D_3 y$ and $D_2 z$ are taken at the same point as $D_3 z$.

As $D_3 y = (\bar{b} + O_1(s_1, s_2, s_3))s_1$, and $D_2 z(0,0,1,0,0) \neq 0$ (see again proposition 3.5), the component $\bar{z}'$ satisfies:

$$\bar{z}' = \bar{a} + D_2 z\, \bar{y} + (1 - D_2 z\, \bar{b})\bar{z} + \frac{1}{2} D_{33} z\, \bar{z}^2 + O_1(s_1, s_2, s_3),$$

where $D_2 z$, $D_3 z$ and $D_{33} z$ are evaluated at $(0,0,1,0,0)$.

This means that in the type I generalized homoclinic tangency we obtain, after a change of variables and a reparametrization, the following limit return map:

$$\begin{pmatrix} \bar{x}' \\ \bar{y}' \\ \bar{z}' \end{pmatrix} = \begin{pmatrix} \alpha \bar{z} \\ \bar{b} \bar{z} \\ \bar{a} + \beta \bar{y} + (1 - \beta \bar{b}) \bar{z} + \gamma \bar{z}^2 \end{pmatrix} \tag{13}$$

where α, β and γ are non-zero constants, and $\bar{a}$ and $\bar{b}$ are parameters. By means of a change of variables and parameters, it is easy to show that this return map is conjugate to (7).

Type II. We use the same notation as in the previous case. Now, we perform the change:

$$\begin{array}{rcl} t_1 & = & T_1(\lambda_1^n, \lambda_2^n, \lambda_3^{-n}, b) + \lambda_3^{-n} \bar{x} \\ t_2 & = & T_2(\lambda_1^n, \lambda_2^n, \lambda_3^{-n}, b) + \lambda_3^{-n} \bar{y} \\ t_3 & = & \lambda_3^{-n} T_3(\lambda_1^n, \lambda_2^n, \lambda_3^{-n}, b) + \lambda_3^{-2n} \bar{z} \\ a & = & u(\lambda_1^n, \lambda_2^n, \lambda_3^{-n}, b) + \lambda_3^{-2n} \bar{a} \\ b & = & v(\lambda_2^n) + \bar{b} \lambda_2^{-n} \lambda_3^{-n}. \end{array} \tag{14}$$

where T_i, $i = 1, 2, 3$, u and v satisfy equations (9) and (11).

As before, we want to express $f_{a,b}^n$ in the new coordinates and parameters. The (t_1, t_2, t_3, a, b) variables of a point $(\bar{x}, \bar{y}, \bar{z}, \bar{a}, \bar{b})$ are given by (14). After applying $f_{a,b}^n$ we get:

$$\begin{array}{rcl} t_1 & = & x(\lambda_1^n T_1 + \lambda_1^n \lambda_3^{-n} \bar{x}, \lambda_2^n T_2 + \lambda_2^n \lambda_3^{-n} \bar{y}, T_3 + \lambda_3^{-n} \bar{z}, u + \lambda_3^{-2n} \bar{a}, v + \bar{b} \lambda_2^{-n} \lambda_3^{-n}) \\ t_2 & = & y(\lambda_1^n T_1 + \lambda_1^n \lambda_3^{-n} \bar{x}, \lambda_2^n T_2 + \lambda_2^n \lambda_3^{-n} \bar{y}, T_3 + \lambda_3^{-n} \bar{z}, u + \lambda_3^{-2n} \bar{a}, v + \bar{b} \lambda_2^{-n} \lambda_3^{-n}) \\ t_3 & = & z(\lambda_1^n T_1 + \lambda_1^n \lambda_3^{-n} \bar{x}, \lambda_2^n T_2 + \lambda_2^n \lambda_3^{-n} \bar{y}, T_3 + \lambda_3^{-n} \bar{z}, u + \lambda_3^{-2n} \bar{a}, v + \bar{b} \lambda_2^{-n} \lambda_3^{-n}) \end{array}$$

We expand the maps $x(\cdot, b)$, $y(\cdot, b)$, $z(\cdot, b)$ about $(s_1 s_2 s_3 T_1, s_2 T_2, T_3, u)$ and obtain:

$$\begin{array}{rcl} \bar{x}' & = & D_3 x(0,0,1,0,0) \bar{z} + O_1(s_1, s_2, s_3) \\ \bar{y}' & = & D_3 y(0,0,1,0,0) \bar{z} + O_1(s_1, s_2, s_3) \\ \bar{z}' & = & D_2 z \, \bar{y} s_1^{-1} + D_3 z \, \bar{z} s_1^{-1} s_2^{-1} + \bar{a} + \dfrac{1}{2} D_{33} z(0,0,1,0,0) \bar{z}^2 + O_1(s_1, s_2, s_3) \end{array}$$

where $D_2 z$ and $D_3 z$ are taken at $(s_1 s_2 s_3 T_1, s_2 T_2, T_3, u, v + \bar{b} s_1)$.

In this case, $D_2 z(0, s_2 \bar{T}_2, \bar{T}_3, \bar{u}, v(s_2)) = 0$, by the fifth equation of (11). Expanding the map

$$D_2 z(s_1 s_2 s_3 T_1, s_2 T_2, T_3, u, v + \bar{b} s_1),$$

about $s_1 = 0$, and using that $D_2 z(0,0,1,a,b) = b$ (see proposition 3.5), we obtain

$$\bar{z}' = \bar{b} \bar{y} + + D_3 z \, \bar{z} s_1^{-1} s_2^{-1} + \bar{a} + \frac{1}{2} D_{33} z(0,0,1,0,0) \bar{z}^2 + f_3.$$

By the last equation of (9), we get:

$$\bar{z}' = \bar{a} + \bar{b} \bar{y} + (1 - \bar{b} D_3 y) \bar{z} + \frac{1}{2} (D_{33} z) \bar{z}^2 + O_1(s_1, s_2, s_3),$$

where $D_3 y$ and $D_{33} z$ are taken at $(0,0,1,0,0)$.

Therefore, a limit return map is:

$$\begin{pmatrix} \bar{x}' \\ \bar{y}' \\ \bar{z}' \end{pmatrix} = \begin{pmatrix} \alpha \bar{z} \\ \beta \bar{z} \\ \bar{a} + \bar{b}\bar{y} + (1 - \beta\bar{b})\bar{z} + \gamma \bar{z}^2 \end{pmatrix} \tag{15}$$

where β and γ are non-zero constants.

Again, by means of a change of variables and parameters, we obtain the map (8). This finishes the proof of the proposition. □

Now, we want to prove that the bifurcations of proposition 3.5 at $(a,b) = (a_n^{++}, b_n^{++})$ are generic Bogdanov-Takens bifurcations. This will imply the existence of invariant circles near the generalized homoclinic tangency.

There is a difficulty here. The limit map near the BT bifurcation is the conservative Hénon map. So we must include some terms of higher order in the return map. We have the following proposition:

Proposition 4.4 *Assume that we are in the hypotheses of proposition 4.1. Then:*

1. *For n large enough there are values of the parameter (a_n^{++}, b_n^{++}) for which the map $f_{a,b}^n$ undergoes a generic Bogdanov-Takens bifurcation. These values of the parameters are those of proposition 3.5.*

2. *$(a,b) = (a_0, b_0)$ is in the closure of parameter values for which there exist differentiable invariant circles. Depending on the geometry of the tangency and in the sign of λ_2, we can add attracting or saddle (repelling inside the centre manifold).*

Proof:

We shall distinguish between generalized tangencies of type I and II. It will be useful, as before, to define $s_1 = \lambda_2^{-n}\lambda_3^{-n}$, $s_2 = \lambda_2^n$ and $s_3 = \lambda_1^n \lambda_3^n$.

Type I:

Given the return map $f_{a,b}^n$, we perform the change of variables and parameters (12). We will prove that this map undergoes a generic Bogdanov-Takens bifurcation. In order to show this, we shall verify the conditions of item 4 in section 2.4. Then the statements of the proposition will be deduced from the properties of a generic BT bifurcation.

After the change, the return map has the following expression:

$$\begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{pmatrix} \mapsto \begin{pmatrix} D_3 x\, \bar{z} + f_1 \bar{z} + s_2 O_1 \\ D_2 y\, \bar{y} s_2 + \frac{1}{2} D_{33} y\, \bar{z}^2 s_2 + \bar{b}\bar{z} + f_2 \bar{z} + s_2 O_1 \\ \bar{a} + D_2 z\, \bar{y} + (1 - D_2 z\, \bar{b})\bar{z} + \frac{1}{2} D_{33} z\, \bar{z}^2 + f_3 \end{pmatrix}, \tag{16}$$

where all the derivatives are taken at $(0,0,1,0,0)$, f_i, $i = 1,2,3$, are smooth functions that are equal to zero when $s_1 = s_2 = s_3 = 0$ and depend on $s_1, s_2, s_3, \bar{a}, \bar{b}, \bar{x}, \bar{y}, \bar{z}$. Moreover, O_i means i-th order in (s_1, s_2, s_3), and f_3 satisfies $\partial^2 f_3/\partial \bar{x}^2 = s_2 O_1$, $\partial^2 f_3/\partial \bar{x}\partial \bar{y} = s_2 O_1$, $\partial^2 f_3/\partial \bar{x}\partial \bar{z} = s_2 O_1$, $\partial^2 f_3/\partial \bar{y}^2 = s_2 O_1$ and $\partial^2 f_3/\partial \bar{y}\partial \bar{z} = s_2 O_1$.

On the other hand, the map (16) has a fixed point $(\bar{x},\bar{y},\bar{z}) = O_1$, for $\bar{a} = \bar{a}_0 = O_1$, $\bar{b} = \bar{b}_0 = -1/D_2 z(0,0,1,0,0) + O_1$, with one eigenvalue, of multiplicity two, equal to 1, and the other less than one in absolute value. Moreover, a centre invariant manifold of the map (16), associated to the eigenvalue equal to 1 has the expression:

$$\bar{x} = \frac{D_3 x(0,0,1,0,0)}{\bar{b}}\bar{y} + O_1.$$

Recall that $D_3 x(0,0,1,0,0) \neq 0$ and $D_2 z(0,0,1,0,0) \neq 0$ (see proposition 2.19).

The restriction of the map to the centre manifold, which we denote by $F_{\bar{a},\bar{b}}$, has the form:

$$\begin{pmatrix} \bar{y} \\ \bar{z} \end{pmatrix} \overset{F_{\bar{a},\bar{b}}}{\mapsto} \begin{pmatrix} \bar{b}\bar{z} + \tilde{f} \\ \bar{a} + D_2 z\, \bar{y} + (1 - D_2 z\, \bar{b})\bar{z} + \frac{1}{2} D_{33} z\, \bar{z}^2 + \tilde{g} \end{pmatrix},$$

where the derivatives are taken at $(0,0,1,0,0)$, $\tilde{f} = O_1$, $\tilde{g} = O_1$, and satisfy:

$$\frac{\partial \tilde{f}}{\partial \bar{y}} = D_2 y\, s_2 + s_2 O_1, \quad \frac{\partial^2 \tilde{f}}{\partial \bar{y}^2} = s_2 O_1, \quad \frac{\partial^2 \tilde{f}}{\partial \bar{y}\partial \bar{z}} = s_2 O_1, \quad \frac{\partial^2 \tilde{f}}{\partial \bar{z}^2} = D_{33} y\, s_2 + s_2 O_1,$$

$$\frac{\partial^2 \tilde{g}}{\partial \bar{y}^2} = s_2 O_1, \quad \frac{\partial^2 \tilde{g}}{\partial \bar{y}\partial \bar{z}} = s_2 O_1,$$

where the derivatives of y are taken at $(0,0,1,0,0)$. Moreover, the fixed point $\bar{p}_0 \in \mathbb{R}^2$ of $F_{\bar{a}_0,\bar{b}_0}$ with a double eigenvalue 1 satisfies $\bar{p}_0 = O_1$.

In order to prove that the Bogdanov-Takens bifurcation is generic we have to see that $c_1 = w_2^t D^2 F_{\bar{a}_0,\bar{b}_0}(\bar{p}_0)(v_1,v_1) \neq 0$ and

$$c_2 = w_2^t D^2 F_{\bar{a}_0,\bar{b}_0}(\bar{p}_0)(v_1,v_2) + w_1^t D^2 F_{\bar{a}_0,\bar{b}_0}(\bar{p}_0)(v_1,v_1) - w_2^t D^2 F_{\bar{a}_0,\bar{b}_0}(\bar{p}_0)(v_1,v_1) \neq 0,$$

where $w_i^t = e_i^t M^{-1}$, $v_i = M e_i$, $i = 1,2$, being e_1, e_2 the canonical basis of $\mathbb{R}^2$ and M the linear map that conjugates $DF_{\bar{a}_0,\bar{b}_0}(\bar{p}_0)$ to

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

If $DF_{\bar{a}_0,\bar{b}_0}(\bar{p}_0) = (a_{ij})_{1\leq i,j\leq 2}$, it is easy to see that

$$v_1 = \begin{pmatrix} a_{11} - 1 \\ -(a_{11}-1)^2/a_{12} \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

$$w_1 = \begin{pmatrix} 0 \\ -a_{12}/(a_{11}-1)^2 \end{pmatrix}, \quad w_2 = \begin{pmatrix} 1 \\ a_{12}/(a_{11}-1) \end{pmatrix}.$$

Moreover $a_{11} = D_2 y(0,0,1,0,0) s_2 + s_2 O_1$, $a_{12} = -1/D_2 z(0,0,1,0,0) + O_1$.

Finally, we get $c_1 = D_{33} z(0,0,1,0,0) D_2 z(0,0,1,0,0) + O_1 \neq 0$ and $c_2 = ((D_2 y D_{33} z - D_{33} y D_2 z) D_2 z + O_1) s_2 \neq 0$, where all the derivatives are taken at $(0,0,1,0,0)$, as before. From proposition 2.19, we see that c_1 and c_2 are different from zero if s_1, s_2, s_3 are small enough, that is n is large enough.

To see that the BT bifurcation unfolds generically, we have to compute the values of the parameters μ and ν of the definition. These are $\mu = \frac{1}{2} D_{33} z \bar{a} + O_1$, and $\nu = -D_2 z \bar{b} - 1 + O_1$.

As $|\frac{\partial(\mu,\nu)}{\partial(\bar{a},\bar{b})}| \neq 0$ for $(\bar{a},\bar{b}) = (\bar{a}_0,\bar{b}_0)$ and n large enough, this means that the BT is generic, and by the properties of such bifurcation we obtain the properties that we wanted.

Type II:

We follow the same steps as before. In this case the return map is:

$$\begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{pmatrix} \mapsto \begin{pmatrix} D_3x\,\bar{z} + f_1\bar{z} + s_2O_1 \\ D_2y\,\bar{y}s_2 + D_3y\,\bar{z} + f_2\bar{z} + s_2O_1 \\ \bar{a} + \bar{b}\bar{y} + (1 - D_3y\bar{b})\bar{z} + \frac{1}{2}D_{33}z\,\bar{z}^2 + D_{23}z\,\bar{y}\bar{z}s_2 + f_3 \end{pmatrix}, \tag{17}$$

where all the derivatives are taken at $(0,0,1,0,0)$, f_i, $i = 1,2,3$, are smooth functions that are equal to zero when $s_1 = s_2 = s_3 = 0$ and depend on $s_1, s_2, s_3, \bar{a}, \bar{b}, \bar{x}, \bar{y}, \bar{z}$. Moreover, f_3 satisfies $\partial^2 f_3/\partial\bar{x}^2 = s_2O_1$, $\partial^2 f_3/\partial\bar{x}\partial\bar{y} = s_2O_1$, $\partial^2 f_3/\partial\bar{x}\partial\bar{z} = s_2O_1$, $\partial^2 f_3/\partial\bar{y}^2 = s_2O_1$, $\partial^2 f_3/\partial\bar{y}\partial\bar{z} = s_2O_1$.

The map (17) has a fixed point in $(\bar{x},\bar{y},\bar{z}) = O_1$ for $\bar{a} = \bar{a}_0 = O_1$ and $\bar{b} = \bar{b}_0 = -1/D_3y(0,0,1,0,0) + O_1$ with one eigenvalue, of multiplicity two, equal to 1, and the other less than one in absolute value. Moreover, the centre manifold associated to the eigenvalue equal to 1 has the expression:

$$\bar{x} = \frac{D_3x(0,0,1,0,0)}{D_3y(0,0,1,0,0)}\bar{y} + O_1.$$

Note that $D_3y(0,0,1,0,0) \neq 0$ (see proposition 2.19).

The restriction of the map to the centre manifold, that we denote again by $F_{\bar{a},\bar{b}}$, is

$$\begin{pmatrix} \bar{y} \\ \bar{z} \end{pmatrix} \overset{F_{\bar{a},\bar{b}}}{\mapsto} \begin{pmatrix} D_3y\,\bar{z} + \tilde{f} \\ \bar{a} + \bar{b}\bar{y} + (1 - D_3y\,\bar{b})\bar{z} + \frac{1}{2}D_{33}z\,\bar{z}^2 + \tilde{g} \end{pmatrix},$$

where the derivatives are taken at $(0,0,1,0,0)$, $\tilde{f} = O_1$ $\tilde{g} = O_1$, and satisfy:

$$\frac{\partial\tilde{f}}{\partial\bar{y}} = D_2y\,s_2 + s_2O_1, \quad \frac{\partial^2\tilde{f}}{\partial\bar{y}^2} = s_2O_1, \quad \frac{\partial^2\tilde{f}}{\partial\bar{y}\partial\bar{z}} = s_2O_1, \quad \frac{\partial^2\tilde{f}}{\partial\bar{z}^2} = s_2O_1,$$

$$\frac{\partial^2\tilde{g}}{\partial\bar{y}^2} = s_2O_1, \quad \frac{\partial^2\tilde{g}}{\partial\bar{y}\partial\bar{z}} = D_{23}z\,s_2 + s_2O_1,$$

where the derivatives of y are taken at $(0,0,1,0,0)$. Moreover, the fixed point $\bar{p}_0 \in \mathbb{R}^2$ of $F_{\bar{a}_0,\bar{b}_0}$ with a double eigenvalue 1 satisfies $\bar{p}_0 = O_1$.

Proceeding as in the previous case (and with the same notation) we obtain $c_1 = -D_{33}z/D_3y + O_1$ and $c_2 = s_2[(D_{23}zD_3y - D_2yD_{33}z)/D_3y + O_1]$. From proposition 2.19, we see that c_1 and c_2 are different from zero if n is large enough.

As before, in order to see that the BT bifurcation unfolds generically, we compute μ and ν and see that $|\frac{\partial(\mu,\nu)}{\partial(\bar{a},\bar{b})}| \neq 0$, but again this is an easy check. This finishes the proof of the proposition. □

In the next proposition, we assume that the stable invariant manifold of the fixed point, for which there is a generalized homoclinic bifurcation, has dimension 1. Recall that we denote by q a point in the orbit of generalized homoclinic tangency.

Proposition 4.5 *Suppose that* $|\lambda_1| < 1 < |\lambda_2| < |\lambda_3|$. *Then,* $\{f_{a,b}\}_{(a,b)\in V}$ *has the following family of limit return maps, associated to the orbit of* q :

$$(\bar{x}, \bar{y}, \bar{z}) \mapsto (\bar{z}, \bar{a} + \bar{b}\bar{y} + \bar{z}^2, \bar{y}), \tag{18}$$

where $\bar{a}$ *and* $\bar{b}$ *are parameters.*

Proof:

By means of a change of variables and a reparametrization, we can assume that $\{f_{a,b}\}_{(a,b)\in V}$ satisfies all the properties of the family $\{g_{\bar{a},\bar{b}}\}_{(\bar{a},\bar{b}\in(-\epsilon,\epsilon)\times(-\epsilon,\epsilon)}$, in proposition 2.21. In particular, the fixed point is always equal to $(0,0,0)$, and the values of the parameters for which there is a generalized homoclinic tangency are $a_0 = 0$ and $b_0 = 0$. Moreover, we have $f^n_{a,b}(t_1,t_2,t_3) = \mathbf{x}(\lambda_1^n t_1, \lambda_2^n t_2, \lambda_3^n t_3, a, b)$, where $\mathbf{x} = (x, y, z)$ satisfies all the properties given in proposition 2.21.

We proceed as in proposition 4.5, considering a change of variables and parameters for the map $f^n_{a,b}$ near the generalized homoclinic tangency. Before giving these changes we need more information about the points and the values of the parameters for which there are saddle-node and BT bifurcations:

Lemma 4.6 *There exist smooth maps* $T_i = T_i(s_1,s_2,s_3,b)$, $i = 1,2,3$, $u = u(s_1,s_2,s_3,b)$ *defined on a neighbourhood of* $(0,0,0,0)$ *and* $v = v(s_1,s_2,s_3)$, *defined on a neighbourhood of* $(0,0,0)$ *such that:*

1. $T_i(0,0,0,b) = 1$, $i = 1,2,3$, $u(0,0,0,b) = 0$, *for* $|b|$ *small enough, and* $v(0,0,0) = 0$.
2. *If we define* $s_1 = \lambda_1^n$, $s_2 = \lambda_2^{-n}$ *and* $s_3 = \lambda_3^{-n}$ *then the map* $f_{u,b}$ *has an* n*-periodic saddle-node bifurcation at* (T_1,T_2,T_3), *that is:*

$$\begin{aligned} x(\lambda_1^n T_1, T_2, T_3, u, b) - T_1 &= 0 \\ y(\lambda_1^n T_1, T_2, T_3, u, b) - \lambda_2^{-n} T_2 &= 0 \\ z(\lambda_1^n T_1, T_2, T_3, u, b) - \lambda_3^{-n} T_3 &= 0 \\ \lambda_2^{-n}\lambda_3^{-n}(1 - d_1 + d_2 - d_3) &= 0, \end{aligned} \tag{19}$$

   where $d_i = d_i(Df_{u,b}(T_1, \lambda_2^{-n}T_2, \lambda_3^{-n}T_3))$.

3. *Define* $\bar{v} = v(\lambda_1^n\lambda_2^n\lambda_3^n, \lambda_2^{-n}, \lambda_2^n\lambda_3^{-n})$, $\bar{T}_i = T_i(\lambda_1^n, \lambda_2^{-n}, \lambda_3^{-n}, \bar{v})$ *and* $\bar{u} = u(\lambda_1^n, \lambda_2^{-n}, \lambda_3^{-n}, \bar{v})$. *Then,* $f_{\bar{u},\bar{v}}$ *has an* n*-periodic BT bifurcation at* $(\bar{T}_1, \bar{T}_2, \bar{T}_3)$. *This means that*

$$\begin{aligned} x(\lambda_1^n \bar{T}_1, \bar{T}_2, \bar{T}_3, \bar{u}, \bar{v}) - T_1 &= 0 \\ y(\lambda_1^n \bar{T}_1, \bar{T}_2, \bar{T}_3, \bar{u}, \bar{v}) - \lambda_2^{-n} \bar{T}_2 &= 0 \\ z(\lambda_1^n \bar{T}_1, \bar{T}_2, \bar{T}_3, \bar{u}, \bar{v}) - \lambda_3^{-n} \bar{T}_3 &= 0 \\ \lambda_2^{-n}\lambda_3^{-n}(1 - d_1 + d_2 - d_3) &= 0, \\ \lambda_3^{-n}(2 - d_1 + d_3) &= 0 \end{aligned} \tag{20}$$

To prove this lemma we only have to impose the conditions of saddle-node and BT bifurcation and apply the implicit function theorem, taking into account the properties of the map $\mathbf{x}$.

We will give different changes of variables and reparametrizations, depending on the type of generalized homoclinic tangency. Denote by $(\bar{x}, \bar{y}, \bar{z})$ the new variables and by

$(\bar{a}, \bar{b})$ the new parameters. From now on, we shall denote by α_i, $i \in \mathbb{N}$, suitable constants not depending on the variables s_1, s_2, s_3.

Type I

We consider the change:

$$\begin{array}{rcl} t_1 & = & T_1(\lambda_1^n, \lambda_2^{-n}, \lambda_3^{-n}, b) + \lambda_2^{-n}\lambda_3^{-n}\bar{x} \\ t_2 & = & \lambda_2^{-n} T_2(\lambda_1^n, \lambda_2^{-n}, \lambda_3^{-n}, b) + \lambda_2^{-2n}\lambda_3^{-2n}\bar{y} + \frac{D_2 y}{D_2 z}\lambda_2^{-n}\lambda_3^{-2n}\bar{z} \\ t_3 & = & \lambda_3^{-n} T_3(\lambda_1^n, \lambda_2^{-n}, \lambda_3^{-n}, b) + \lambda_2^{-n}\lambda_3^{-2n}\bar{z} \\ a & = & u(\lambda_1^n, \lambda_2^{-n}, \lambda_3^{-n}, b) + \lambda_2^{-2n}\lambda_3^{-2n}\bar{a} \\ b & = & v(\lambda_1^n\lambda_2^n\lambda_3^n, \lambda_2^{-n}, \lambda_2^n\lambda_3^{-n}) + \lambda_3^{-n}\bar{b}, \end{array} \tag{21}$$

where $D_2 y$ and $D_2 z$ are taken at $(\lambda_1^n T_1, T_2, T_3, u, b)$.

We want to express $f_{a,b}^n$ in terms of $\bar{x}, \bar{y}, \bar{z}, \bar{a}, \bar{b}$. Let $(\bar{x}, \bar{y}, \bar{z}, \bar{a}, \bar{b})$ denote a point. The (t_1, t_2, t_3, a, b) variables of this point are given by (21). After applying $f_{a,b}^n$, we get as (t_1, t_2, t_3) coordinates:

$$(t_1, t_2, t_3) = \mathbf{x}(\bar{t}_1, \bar{t}_2, \bar{t}_3, u + \lambda_2^{-2n}\lambda_3^{-2n}\bar{a}, v + \lambda_3^{-n}\bar{b}),$$

where

$$\begin{array}{rcl} \bar{t}_1 & = & \lambda_1^n T_1 + \lambda_1^n\lambda_2^{-n}\lambda_3^{-n}\bar{x}, \\ \bar{t}_2 & = & T_2 + \lambda_2^{-n}\lambda_3^{-2n}\bar{y} + \dfrac{D_2 y}{D_2 z}\lambda_3^{-2n}\bar{z}, \\ \bar{t}_3 & = & T_3 + \lambda_2^{-n}\lambda_3^{-n}\bar{z}, \end{array}$$

$T_i = T_i(\lambda_1^n, \lambda_2^{-n}, \lambda_3^{-n}, b)$, $i = 1, 2, 3$, $u = u(\lambda_1^n, \lambda_2^{-n}, \lambda_3^{-n}, b)$, and $v = v(\lambda_1^n\lambda_2^n\lambda_3^n, \lambda_2^{-n}, \lambda_2^n\lambda_3^{-n})$.

Transforming back to the $(\bar{x}, \bar{y}, \bar{z})$ coordinates, and denoting the values of the coordinates of the new point by $\bar{x}'$, $\bar{y}'$, $\bar{z}'$:

$$\begin{array}{rcl} \bar{x}' & = & (x - T_1(\lambda_1^n, \lambda_2^{-n}, \lambda_3^{-n}, b))\lambda_2^n\lambda_3^n, \\ \bar{y}' & = & \left(y - \lambda_2^{-n}T_2 - \dfrac{D_2 y}{D_2 z}(z - \lambda_3^{-n}T_3)\right)\lambda_2^{2n}\lambda_3^{2n}, \\ \bar{z}' & = & (z - \lambda_3^{-n}T_3)\lambda_2^n\lambda_3^{2n}, \end{array}$$

where x, y, z and their derivatives are taken at the same point as before.

Now, we define the following variables: $s_1 = \lambda_1^n\lambda_2^n\lambda_3^n$, $s_2 = \lambda_2^{-n}$ and $s_3 = \lambda_2^n\lambda_3^{-n}$. Performing the Taylor expansion of the map $\mathbf{x}(\cdot, b)$ about the point:

$$(\lambda_1^n T_1, T_2, T_3, u) = (s_1 s_2^2 s_3 T_1, T_2, T_3),$$

where $T_i = T_i(s_1 s_2^2, s_2, s_2 s_3, b)$, $u = u(s_1 s_2^2, s_2, s_2 s_3, b)$, we obtain, using the system of equations (19) and that $D_4 y = 1 + O_1$, $D_4 z = O_1$:

$$\begin{array}{rcl} \bar{x}' & = & D_3 x\,\bar{z} + O_1 \\ \bar{y}' & = & D_2 y\Big(s_2^{-1}\bar{y} + \frac{D_2 y}{D_2 z}s_2^{-2}\bar{z}\Big) + D_3 y s_2^{-2}s_3^{-1}\bar{z} + \bar{a} + \frac{1}{2}D_{33}y\,\bar{z}^2 - \frac{D_2 y}{D_2 z}\bar{z}' s_2^{-1} + O_1, \\ \bar{z}' & = & D_2 z\,\bar{y} + D_2 y s_2^{-1}\bar{z} + D_3 z\, s_2^{-1}s_3^{-1}\bar{z} + \frac{1}{2}D_{33}z s_2\bar{z}^2 + s_2 O_1, \end{array} \tag{22}$$

where the derivatives of x, y and z are taken at $(s_1 s_2^2 s_3 T_1, T_2, T_3, u, b)$, and O_1 means first order with respect to (s_1, s_2, s_3).

Replacing in the second equation of (22), $\bar{z}'$ by its value in the third equation, we obtain:

$$\bar{y}' = \left(D_3 y - \frac{D_2 y}{D_2 z} D_3 z \right) s_2^{-2} s_3^{-1} \bar{z} + \bar{a} + \frac{1}{2} \left(D_{33} y - \frac{D_2 y}{D_2 z} D_{33} z \right) \bar{z}^2.$$

We define

$$\tilde{T}_i = \tilde{T}_i(s_1, s_2, s_3) = T_i(s_1 s_2^2 s_3, s_2, s_2 s_3, v(s_1, s_2, s_3)), \quad i = 1, 2, 3,$$

$$\tilde{u} = \tilde{u}(s_1, s_2, s_3) = u(s_1 s_2^2 s_3, s_2, s_2 s_3, v(s_1, s_2, s_3)).$$

Note that when $s_1 = \lambda_1^n \lambda_2^n \lambda_3^n$, $s_2 = \lambda_2^{-n}$ and $s_3 = \lambda_2^n \lambda_3^{-n}$, the maps $\tilde{T}_i$ and $\tilde{u}$ are, respectively, equal to the maps $\bar{T}_i$ and $\bar{u}$ of the previous lemma.

If we take $b = v + \bar{b} s_2 s_3$, we can consider T_i and u as functions of s_1, s_2 and s_3. We claim that

$$T_i - \tilde{T}_i = \alpha_i \bar{b} s_2^2 s_3 + s_2^2 s_3 \, O_1, \quad i = 1, 2, 3,$$

$$u - \tilde{u} = \alpha_4 \bar{b} s_2^2 s_3 + s_2^2 s_3 \, O_1.$$

where

$$T_i = T_i(s_1 s_2^2 s_3, s_2, s_2 s_3, v + \bar{b} s_2 s_3),$$

$$u = u(s_1 s_2^2 s_3, s_2, s_2 s_3, v + \bar{b} s_2 s_3).$$

Here, and from now on, α_i, for some natural index i, is a constant that does not depend on s_j, $j = 1, 2, 3$.

Indeed, by the previous lemma we can write $T_i = 1 + s_2 h_i(b, s_1, s_2, s_3)$, $u = s_2 h_4(b, s_1, s_2, s_3)$, where h_i, $i = 1, 2, 3, 4$, are smooth maps, and this implies the claim.

On the other hand, taking into account that $D_3 z(0, 1, 1, a, b) = b$ (see proposition 2.21), we obtain the following expressions:

$$D_2 y(s_1 s_2^2 s_3 T_1, T_2, T_3, u, b) - D_2 y(s_1 s_2^2 s_3 \tilde{T}_1, \tilde{T}_2, \tilde{T}_3, \tilde{u}, v) = \alpha_5 s_2 s_3 + s_2 s_3 \, O_1, \tag{23}$$

$$D_3 z(s_1 s_2^2 s_3 T_1, T_2, T_3, u, b) - D_3 z(s_1 s_2^2 s_3 \tilde{T}_1, \tilde{T}_2, \tilde{T}_3, \tilde{u}, v) = \bar{b} s_2 s_3 + s_2 s_3 \, O_1 \tag{24}$$

and if we define $G = D_3 y D_2 z - D_2 y D_3 z$, then

$$G(s_1 s_2^2 s_3 T_1, T_2, T_3, u, b) - G(s_1 s_2^2 s_3 \tilde{T}_1, \tilde{T}_2, \tilde{T}_3, \tilde{u}, v) = D_5 G s_2 s_3 \bar{b} + \alpha_6 s_2^2 s_3 + s_2^2 s_3 \, O_1,$$

where $D_5 G = D_5 G(s_1 s_2^2 s_3 \tilde{T}_1, \tilde{T}_2, \tilde{T}_3, \tilde{u}, v)$.

We know, by proposition 2.21, that $G(0, 1, 1, a, b) = 0$ for all a and b small enough. This means that $D_5 G = s_2(\alpha_7 + O_1)$.

If we replace d_1 in the fourth equation of the system (20) by the value obtained isolating it in the fifth equation, we see that

$$s_2^2 s_3 (1 - d_2 + 2 d_3) = 0.$$

Setting d_i, $i = 1, 2, 3$, in terms of the derivatives of $\mathbf{x}$, we get

$$G(s_1 s_2^2 s_3 \tilde{T}_1, \tilde{T}_2, \tilde{T}_3, \tilde{u}, v) = s_2^2 s_3 (\alpha_8 + O_1)$$

and, therefore,

$$G(s_1 s_2^2 s_3 T_1, T_2, T_3, u, b) = s_2^2 s_3 (A + O_1),$$

where $A = A(\bar{b})$ does not depend on (s_1, s_2, s_3).

As $D_2 z(0,1,1,0,0) \neq 0$ we deduce that:

$$\bar{y}' = A\bar{z} + \bar{a} + \frac{1}{2}(D_{33} y - \frac{D_2 y}{D_2 z} D_{33} z)\bar{z}^2 + O_1.$$

On the other hand, computing explicitly the last equation of (20), we obtain:

$$D_3 z(s_1 s_2^2 s_3 \tilde{T}_1, \tilde{T}_2, \tilde{T}_3, \tilde{u}, v) = (2 + O_1) s_2 s_3 - s_3 D_2 y(s_1 s_2^2 s_3 \tilde{T}_1, \tilde{T}_2, \tilde{T}_3, \tilde{u}, v).$$

Combining this equation with (23) and (24) we get:

$$\bar{z}' = D_2 z(0,1,1,0,0)\, \bar{y} + (\bar{b} + 2)\bar{z} + O_1.$$

Summarizing, we have that the limit return map is

$$\begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{pmatrix} \mapsto \begin{pmatrix} D_3 x\, \bar{z} \\ A\bar{z} + \bar{a} + \frac{1}{2}(D_{33} y - \frac{D_2 y}{D_2 z} D_{33} z)\bar{z}^2 \\ D_2 z\, \bar{y} + (\bar{b} + 2)\bar{z} \end{pmatrix},$$

where $D_2 z = D_2 z(0,1,1,0,0) \neq 0$, $D_3 x = D_3 x(0,1,1,0,0) \neq 0$, and for $(t_1, t_2, t_3, a, b) = (0,1,1,0,0)$, $D_{33} y - \frac{D_2 y}{D_2 z} D_{33} z \neq 0$.

It is easy to prove that by means of a change of variables and reparametrization, this map is conjugate to (18).

Type II

We use the same notation as in the previous case. In this case, we consider the following change of variables and reparametrization for the map $f^n_{a,b}(t_1, t_2, t_3) = \mathbf{x}(\lambda_1^n t_1, \lambda_2^n t_2, \lambda_3^n t_3, a, b)$:

$$\begin{aligned}
t_1 &= T_1 + \lambda_2^{-n} \lambda_3^{-n} \bar{x} \\
t_2 &= \lambda_2^{-n} T_2 + \lambda_2^{-2n} \lambda_3^{-n} \bar{y} \\
t_3 &= \lambda_3^{-n} T_3 + \lambda_2^{-2n} \lambda_3^{-2n} \bar{z} - \frac{D_2 y}{D_3 y} \lambda_2^{-n} \lambda_3^{-2n} \bar{y} \\
a &= u + \lambda_2^{-2n} \lambda_3^{-2n} \bar{a} \\
b &= v + \lambda_3^{-n} \bar{b}
\end{aligned}$$

where $D_2 y$ and $D_3 y$ are taken at $(\lambda_1^n T_1, T_2, T_3, u, b)$, $T_i = T_i(\lambda_1^n, \lambda_2^{-n}, \lambda_3^{-n}, b)$, for $i = 1, 2, 3$, $u = u(\lambda_1^n, \lambda_2^{-n}, \lambda_3^{-n}, b)$, $v = v(\lambda_1^n \lambda_2^n \lambda_3^n, \lambda_2^{-n}, \lambda_2^n \lambda_3^{-n})$. Using similar estimates to the previous case, we can see that the limit return map is:

$$\begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{pmatrix} \mapsto \begin{pmatrix} (D_2 x - D_3 x \frac{D_2 y}{D_3 y})\bar{y} \\ D_3 y \bar{z} \\ \bar{a} + (\bar{b} + 2)\bar{z} + A\bar{y} + B\bar{y}^2 \end{pmatrix},$$

where A is a constant and $B = \frac{1}{2(D_3 y)^2}(D_{33} z (D_2 y)^2 - 2 D_{23} z D_2 y D_3 y + D_{22} z (D_3 y)^2) \neq 0$. It is also easy to see that this map is conjugate to the map (18).

This finishes the proof of the proposition. □

We end this section with the proof of Theorem 1, performed item by item.

Proof of Theorem 1

1. In case A, it is true by proposition 4.4. In case B, we can easily see, using the limit return map, that the values of the parameters $(a, b) = (a_n^{++}, b_n^{++})$, in proposition 3.7, give a generic BT bifurcation.

2. It has been proven in proposition 4.1 (case A) and 4.5 (case B).

3. This is a consequence of the properties of the BT bifurcation. Using the return maps, it is easy to see that in case A the invariant circles can be attracting or saddles (depending on the geometry of the tangency and the sign ofλ_2^n) and, in case B, they are always attracting (see section 2.4).

4. Again a consequence of the existence of BT bifurcations.

5. Working with the limit return map, it is possible to obtain values of the parameters for which there is a quadratic homoclinic tangency of a sectionally dissipative fixed point for the return map, if n is large enough. In case A, this can be made taking into account that the limit return maps are, essencially, Hénon maps with two parameters. In case B, one can use that, near a BT bifurcation, there exist generic homoclinic tangencies of dissipative periodic points (see again Broer et al. [4]). Then we can apply the result of Viana [22].

6. Proceed as before but now applying Palis and Viana[17].

Remark 4.7 *In case B, we can find strange attractors that are contained in a two-dimensional centre manifold associated to the BT bifurcation. However, in case A this is only possible if the invariant circles of the Theorem are attracting.*

5 Quadratic maps in dimensions 2 and 3

The objective of this section is to characterize the quadratic maps $f : \mathbb{R}^i \longrightarrow \mathbb{R}^i$, $i = 2$ or $i = 3$, such that $f \circ f$ is quadratic, and, if $i = 3$ f has also constant jacobian determinant. We will see that any map that does not have any invariant linear foliation is linearly conjugate to the map $(x, y) \longrightarrow (a + bx - y^2, x)$, in the first case, and to the map $(x, y, z) \longrightarrow (a + bx + z - y^2, x, cy)$, in the second case. In the two cases we also suppose that the jacobian of the map is not identically equal to zero. This will prove theorem 2.

Case i=2

We shall suppose that $\det Df$ is not identically equal to zero. We can write the map as $f(x, y) = v_0 + x v_1 + y v_2 + x^2 v_{11} + y^2 v_{22} + 2xy v_{12}$, where $v_i, v_{ij} \in \mathbb{R}^2$. As $f \circ f$ is a

quadratic map, then the map $P(x,y) = \det Df(f(x,y)) \det Df(x,y)$ has to be at most quadratic. We can write

$$P(x,y) = (Q(x,y) + P_1(x,y)) \det Df(f(x,y)),$$

where P_1 is at most linear and

$$Q(x,y) = 4(\det(v_{11}, v_{12})x^2 + \det(v_{12}, v_{22})y^2 + \det(v_{11}, v_{22})xy).$$

Then we have two possibilities: Either

$$\det(v_{11}, v_{12}) = \det(v_{12}, v_{22}) = \det(v_{11}, v_{22}) = 0$$

or there is a nonzero constant C such that $\det Df(f(x,y)) = C$ for all $(x,y) \in \mathbb{R}^2$. By the Inverse Function Theorem, $\det Df(x,y) = C$, for (x,y) belonging to some open set. This means that the jacobian is constant in the whole $\mathbb{R}$. But then we arrive to the same condition on the vectors v_{ij} as before.

It is easy to see that by means of a linear change of coordinates, we can suppose that the map f satisfies:

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_0 + a_1x + a_2y + a_{11}x^2 + 2a_{12}xy + a_{22}y^2 \\ b_0 + b_1x + b_2y \end{pmatrix}.$$

If we impose again that $f \circ f$ is quadratic, we get $a_{11} = a_{12} = 0$. As $b_1 \neq 0$ (because the map has not any linear foliation) we can perform the change $x = \bar{x} + \alpha\bar{y} + \beta$, $y = \bar{y} + \gamma$, with suitable constants α, β and γ, obtaining the map that we wanted.

Case i=3

In this case, it is possible to see (Bosch and Tatjer [3], Fornaess and Wu [6]) that any map with constant jacobian and without linear invariant foliations, is conjugate to one of the following maps:

$$\begin{aligned} x' &= a_0 + a_1x + a_2y + a_3z + a_{11}x^2 + a_{22}y^2 + 2a_{12}xy \\ y' &= b_0 + b_1x + b_2y + b_{11}x^2 \\ z' &= c_0 + c_1x \end{aligned} \tag{25}$$

$$\begin{aligned} x' &= a_0 + a_1x + a_2y + a_3z + a_{11}x^2 + a_{22}y^2 + 2a_{12}xy \\ y' &= b_0 + b_1x + b_2y + b_{22}y^2 \\ z' &= c_0 + c_2y \end{aligned} \tag{26}$$

$$\begin{aligned} x' &= a_0 + a_1x + a_2y + a_3z + a_{11}x^2 + a_{33}z^2 + 2a_{13}xz \\ y' &= b_0 + b_1x + b_3z + b_{11}x^2 \\ z' &= c_0 + c_1x \end{aligned} \tag{27}$$

$$\begin{aligned} x' &= a_0 + a_1x + a_2y + a_3z + a_{22}y^2 + a_{33}z^2 + 2a_{23}yz \\ y' &= b_0 + b_1x + b_3z + b_{22}y^2 \\ z' &= c_0 + c_2y \end{aligned} \tag{28}$$

If we impose that $f \circ f$ is quadratic, it is easy to see that the last case has a linear invariant foliation, and the other cases, after a suitable linear change, can be always be written as:

$$\begin{aligned} x' &= a_0 + a_1x + a_2y + a_3z + a_{11}x^2 + a_{22}y^2 + 2a_{12}xy \\ y' &= b_0 + b_1x + b_2y \\ z' &= c_0 + c_1x + c_2y \end{aligned} \tag{29}$$

Indeed, in the map (25) we have that $b_{11} = 0$ or $a_{11} = a_{12} = a_{22} = 0$. In the first case we have the result we want. In the second case, we perform the change of variables

$$\bar{x} = y - (a_3/a_2)z, \quad \bar{y} = x, \quad \bar{z} = z.$$

Note that, as there is no invariant linear foliation, $a_2 \neq 0$ in this case.

For (26), we have that $b_{22} = 0$.

In the map (27), $b_{11} = 0$ or $a_{11} = a_{33} = a_{13} = 0$. In the first case consider the change $(\bar{x}, \bar{y}, \bar{z}) = (x, z, y)$. The second case is the same as the second case of the map (25).

Coming back to equation (29), it is immediate that $a_{11} = a_{12} = 0$. If we perform a change of variables

$$\begin{aligned} x &= \bar{x} + \alpha_1 \bar{y} + \alpha_0 \\ y &= \bar{y} + \beta_0 \\ z &= \bar{z} + \gamma_1 \bar{y} + \gamma_0 \end{aligned}$$

and a change of scale, we obtain the desired map.

Remark 5.1 *As a corollary of this theorem (case $i = 2$), a quadratic map f without an invariant linear foliation, with constant jacobian and satisfying that $f \circ f$ is quadratic has a quadratic inverse. This means that these maps belong to the set of quadratic maps with constant jacobian and with quadratic inverse. Compare with Lomelí and Meiss [11].*

Acknowledgements.

The author is indebted to H. Broer, L. Gardini, C. Simó, F. Takens and M. Viana for helpful conversations and suggestions. He is thankful to IMPA (Rio de Janeiro, Brasil) for hospitality and partial support, during the preparation of this paper. A grant no. PR97-692 of the Ministerio de Educación (Spain) is also acknowledged. This work has been partially supported by the DGICYT grant PB 94-0215 (Spain), and by catalan grants CIRIT 1996SGR-00105 and 1998SGR-00042.

References

[1] A. Banyaga, R. de la Llave, C. E. Wayne: Cohomology equations near hyperbolic points and geometric versions of Sternberg linearization theorem. *J. Geom. Anal.* **6** (1996), no. 4, 613–649.

[2] M. Benedicks, L. Carlesson: The dynamics of the Hénon map. *Ann. Math.* **133** (1991), 73-169.

[3] M. Bosch, J. C. Tatjer: Difeomorfismos cuadráticos de $\mathbb{R}^n$ con jacobiano constante. Actas VIII CEDYA, Universidad de Santander (1985). To appear.

[4] H. Broer, R. Roussarie, C. Simó: Invariant circles in the Bogdanov-Takens bifurcation for diffeomorphisms. *Ergodic Theory and Dynam. Systems* (1996), **16**, 1147-1172.

[5] H. Broer, C. Simó, J. C. Tatjer: Towards global models near homoclinic tangencies of dissipative diffeomorphisms. *Nonlinearity* **11** (1998) 667-770.

[6] J. E. Fornaess, H. Wu: Classification of degree 2 polynomial automorphisms of $\mathbb{C}^3$. *Publicacions Matemàtiques,* **42**, 1998, pp. 195-210.

[7] C. E. Frouzakis, L. Gardini, I. G. Kevrekidis, G. Millerioux, C. Mira: On some properties of invariant sets of two-dimensional noninvertible maps. *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* **7** (1997), no. 6, 1167–1194.

[8] S. V. Gonchenko, D. Tuarev, L. P. Shil'nikov: Dynamical Phenomena in multidimensional systems with a structurally unstable homoclinic Poincaré curve. *Russian Acad. Sci. Dokl. Math.*, **47**(3) (1993), 410-415.

[9] S. V. Gonchenko, L. P. Shil'nikov, D. Tuarev: Quasiattractors and homoclinic tangencies. *Computers Math. Applic.* **34**(2-4) (1997), 195-227.

[10] M. Jakobson: Absolutely continuous invariant measures for one-parameter families of one-dimensional maps. *Commun. Math. Phys.* **81** (1981), 39-88.

[11] H. Lomelí, J. D. Meiss: Quadratic Volume Preserving Maps. *Nonlinearity* **11** (1998), 557-574.

[12] J. Maynard Smith: *Mathematical Ideas in Biology.* Cambridge Univ. Press, 1968.

[13] L. Mora, M. Viana: Abundance of strange attractors. *Acta Math.* **171**, 1-71 (1993).

[14] S. Newhouse: The abundance of wild hyperbolic sets and nonsmooth stable sets for diffeomorphisms. *Publ. Math. I. H. E. S.* **50** (1979), 101-151.

[15] J. Palis, W. de Melo: *Geometric Theory of Dynamical Systems. An Introduction.* Springer-Verlag, New-York (1982).

[16] J. Palis, F. Takens: *Hyperbolicity & sensitive chaotic dynamics at homoclinic bifurcations.* Cambridge University Press, Cambridge (1993).

[17] J. Palis, M. Viana: High dimension diffeomorphisms displaying infinitely many sinks. *Ann. Math.* **140** (1994), 91-136.

[18] N. Romero: Persistence of homoclinic tangencies in higher dimensions. *Ergodic Theory and Dynam. Systems* **15** (1995), 735-757.

[19] B. Sandstede: Constructing Dynamical Systems Having Homoclinic Bifurcation Points of Codimension Two. *J. Dynam. Differential Equations* **9** (1997), no. 2, 269-288.

[20] J. C. Tatjer: Codimension two bifurcations near a cubic homoclinic tangency. *ECIT 91* (Singapore: World Scientific) pp. 308-318.

[21] J. C. Tatjer, C. Simó: Basins of attraction near homoclinic tangencies. *Ergodic Theory Dynam. Systems* **14** (1994), no. 2, 351-390

[22] M. Viana: Strange attractors in higher dimensions. *Bol. Soc. Bras. Mat.* **24**, 13-62 (1993).

[23] M. Viana: Multidimensional nonhyperbolic attractors. *Inst. Hautes Études Sci. Publ. Math.* No. **85** (1997), 63–96.

[24] D. Whitley: On the Periodic Points of a Two-Parameter Family of Maps of the Plane. *Acta Applicandae Mathematicae* **5**, 279-311 (1986).