The degree of differentiability of the regularization of simultaneous binary collisions in some N–body problems

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Abstract

Simultaneous binary collisions in the four–body problem are studied in cases which reduce to 1D problems: the collinear, biisosceles, trapezoidal cases and tetrahedron problem, some of them requiring certain symmetries of the masses. It is shown that in all these cases the regularization is differentiable but the map passing from initial to final conditions (in some suitable transversal sections) is C^r for any $r < 8/3$ but not $C^{8/3}$. Then the result is extended to other symmetric N–body problems.

1 Introduction

It is well known that binary collisions are regularizable but general triple collisions are non regularizable [2]. Triple collision can be regularizable in some special, highly symmetrical case. Here regularization should be understood as continuity with respect initial conditions (i.c.) across the singularity. See [3] for further motivations and details as well as for additional references. The remaining non fully elucidated case corresponds to simultaneous binary collisions (SBC). They are known to be regularizable but, in contrast with the case of a single binary collision, which is analytically regularizable, the question of the regularity of the regularization of the SBC is not fully understood. In [3] numerical evidence is given that the order of regularity is finite. The main goal of this paper is to give a proof of this fact in some cases of the four body problem and some highly symmetric N–body problems.

For the four–body problem the cases we shall consider are:

(a) The collinear problem: The bodies have arbitrary masses and coordinates of the form $x_1 \le x_2 \le x_3 \le x_4$ for all time. SBC is produced if $x_1 = x_2 < x_3 = x_4$.

(b) The biisosceles problem: The bodies move on the plane with coordinates of the form $(\mp x, y_1)$, $(0, y_3)$, $(0, y_4)$ for all time, with $y_3 \le y_4$. Furthermore $m_1 = m_2$ is required, m_3 and m_4 arbitrary. SBC is produced for $x = 0, y_1 \ne y_3, y_3 = y_4$.

(c) The trapezoidal problem: Again a planar case with coordinates of the masses of the form $(\mp x_1, y_1), (\mp x_3, y_3)$ and masses $m_1 = m_2, m_3 = m_4$. SBC appears for $x_1 = 0, x_3 = 0, y_1 \ne y_3$.

(d) The tetrahedron problem: Now the bodies move in $\mathbb{R}^3$ with masses $m_1 = m_2, m_3 = m_4$, as in the trapezoidal problem, with coordinates of the form $(\mp x_1, 0, z_1), (0, \mp y_3, z_3)$ for all time. SBC are obtained for $x_1 = 0, y_3 = 0, z_1 \ne z_3$.

All four cases require initial velocities sharing the symmetry properties of the positions for all time. They are subproblems of the four–body problem.

In what follows $|\cdot|$ will denote the euclidean norm and $(\,,\,)$ the euclidean scalar product.

In general, let $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4 \in \mathbb{R}^d$ ($d = 1, 2$ or 3 is enough for most of the physical applications) the positions of four punctual bodies of masses $m_j > 0, j = 1, \ldots, 4$. Let be

$$\mu_1 = m_1 + m_2, \quad \mu_2 = m_3 + m_4, \quad M = \mu_1 + \mu_2, \quad \alpha_1 = \frac{m_1}{\mu_1}, \quad \alpha_2 = \frac{m_2}{\mu_1}, \quad \alpha_3 = \frac{m_3}{\mu_2}, \quad \alpha_4 = \frac{m_4}{\mu_2},$$
$$\mathbf{q}_{12} = \alpha_1\mathbf{q}_1 + \alpha_2\mathbf{q}_2, \quad \mathbf{q}_{34} = \alpha_3\mathbf{q}_3 + \alpha_4\mathbf{q}_4, \quad \mathbf{Q}_1 = \mathbf{q}_2 - \mathbf{q}_1, \quad \mathbf{Q}_2 = \mathbf{q}_4 - \mathbf{q}_3 \ \text{ and } \ \mathbf{Q} = \mathbf{q}_{34} - \mathbf{q}_{12}.$$

The equations of motion follow from Newton law

$$\ddot{\mathbf{q}}_i = -\sum_{j\neq i} m_j r_{ij}^{-3}(\mathbf{q}_i - \mathbf{q}_j), \qquad \text{where} \qquad r_{ij} = |\mathbf{q}_i - \mathbf{q}_j|,$$

and can be written in Hamiltonian form with Hamiltonian function given by

$$H = H(\mathbf{Q}_1, \mathbf{Q}_2, \mathbf{Q}, \mathbf{P}_1, \mathbf{P}_2, \mathbf{P}) = T - U, \tag{1}$$

being

$$U = \frac{M}{\mu_1\mu_2}\left(\frac{m_1 m_2}{|\mathbf{Q}_1|} + \frac{m_3 m_4}{|\mathbf{Q}_2|} + \frac{\mu_1\mu_2}{|\mathbf{Q}|} + \mathcal{F}\right), \tag{2}$$

where

$$\begin{aligned}\mathcal{F} \;\; = \;\; \mu_1\mu_2 \Bigg(& \frac{\alpha_2\alpha_3}{|\mathbf{Q} - \alpha_1\mathbf{Q}_1 - \alpha_4\mathbf{Q}_2|} + \frac{\alpha_2\alpha_4}{|\mathbf{Q} - \alpha_1\mathbf{Q}_1 + \alpha_3\mathbf{Q}_2|} + \frac{\alpha_1\alpha_3}{|\mathbf{Q} + \alpha_2\mathbf{Q}_1 - \alpha_4\mathbf{Q}_2|} + \\ & \frac{\alpha_1\alpha_4}{|\mathbf{Q} + \alpha_2\mathbf{Q}_1 + \alpha_3\mathbf{Q}_2|} - \frac{1}{|\mathbf{Q}|}\Bigg)\end{aligned} \tag{3}$$

and

$$T = \frac{1}{2}\left(|\mathbf{P}|^2 + \frac{\mu_1^2\mu_2}{M}\frac{1}{m_1 m_2}|\mathbf{P}_1|^2 + \frac{\mu_1\mu_2^2}{M}\frac{1}{m_3 m_4}|\mathbf{P}_2|^2\right). \tag{4}$$

It is clear that H depends only on the relative positions $\mathbf{Q}_1, \mathbf{Q}_2, \mathbf{Q}$, and the related momenta $\mathbf{P}_1, \mathbf{P}_2$ and $\mathbf{P}$.

Definition 1 *A four–body problem as given by (1)–(4) is said reducible to 1D if the vectors* $\mathbf{Q}_1, \mathbf{Q}_2, \mathbf{Q}$ *remain with constant direction along the motion.*

The main result is as follows

Theorem 1 *In any one of the cases (a) to (d) before, the SBC is regularizable, the regularization being of order* $8/3 - \epsilon$ *for* ϵ *arbitrarily small, but it is never* $C^{8/3}$.

This theorem proves the conjecture stated in [3] for the four–body problems reducible to 1D.

2 Expanding the equations of motion

From (1), (2) and (4) we have that the equations of motion can be written as

$$\begin{aligned}\ddot{\mathbf{Q}}_1 &= -\frac{\mu_1}{|\mathbf{Q}_1|^3}\mathbf{Q}_1 + \frac{\mu_1}{m_1 m_2}\mathcal{F}_{\mathbf{Q}_1}, \qquad \ddot{\mathbf{Q}}_2 = -\frac{\mu_2}{|\mathbf{Q}_2|^3}\mathbf{Q}_2 + \frac{\mu_2}{m_3 m_4}\mathcal{F}_{\mathbf{Q}_2}, \\ \ddot{\mathbf{Q}} &= -\frac{M}{|\mathbf{Q}|^3}\mathbf{Q} + \frac{M}{\mu_1\mu_2}\mathcal{F}_{\mathbf{Q}},\end{aligned} \tag{5}$$

where $\mathcal{F}_{\mathbf{X}}$ stands for the partial derivative of $\mathcal{F}$ with respect to $\mathbf{X}$. Each one of the terms in $\mathcal{F}$, as given by (3), can be expanded as follows: Let be

$$f_1 = (\mathbf{Q}, \mathbf{Q}_1), \;\; f_2 = (\mathbf{Q}, \mathbf{Q}_2), \;\; f_{11} = (\mathbf{Q}, \mathbf{Q})(\mathbf{Q}_1, \mathbf{Q}_1), \;\; f_{12} = (\mathbf{Q}, \mathbf{Q})(\mathbf{Q}_1, \mathbf{Q}_2), \;\; f_{22} = (\mathbf{Q}, \mathbf{Q})(\mathbf{Q}_2, \mathbf{Q}_2).$$

Then, a part of $\mathcal{F}$ as

$$\mathcal{F}^{\alpha_1,\alpha_4} = |\mathbf{Q} - \alpha_1\mathbf{Q}_1 - \alpha_4\mathbf{Q}_2|^{-1} = |\mathbf{Q}|^{-1}\left[1 - 2(\alpha_1 f_1 + \alpha_4 f_2) + \alpha_1^2 f_{11} + 2\alpha_1\alpha_4 f_{12} + \alpha_4^2 f_{22}\right]^{-1/2},$$

can be expanded by using the Legendre polynomials. The final expression for $\mathcal{F}$ is

$$
\begin{aligned}
\mathcal{F} &= \mu_1\mu_2 \sum_{n\geq 1} |\mathbf{Q}|^{-1-2n} \sum_{k=0}^{\left[\frac{n}{2}\right]} \sum_{j+j'=n-2k} \sum_{k_1+k_2+k_3=k} (-1)^k \frac{(2n-2k)!}{2^{n-k_2}(n-k)!j!j'!k_1!k_2!k_3!} f_1^j f_2^{j'} f_{11}^{k_1} f_{12}^{k_2} f_{22}^{k_3} \\
& \left[\alpha_1^m\alpha_2\alpha_3\alpha_4^{m'} + (-1)^{m'}\alpha_1^m\alpha_2\alpha_3^{m'}\alpha_4 + (-1)^m\alpha_1\alpha_2^m\alpha_3\alpha_4^{m'} + (-1)^n\alpha_1\alpha_2^m\alpha_3^{m'}\alpha_4\right], \qquad (6)
\end{aligned}
$$

where $j, j', k_1, k_2, k_3 \geq 0$, m, m' stand for $j+2k_1+k_2, j'+k_2+2k_3$, respectively, and $m+m'=n$. $[\ \ ]$ denotes the integer part, as usual. Of course, (6) is suitable to derive some properties of $\mathcal{F}$, but for an efficient computation the use of recurrence relations is better. Note that the terms of a given value of n are all of them homogeneous of degree $-(n+1)$ with respect to $\mathbf{Q}$, and homogeneous of total degree n with respect to $\mathbf{Q}_1$ and $\mathbf{Q}_2$. Furthermore, the following lemma holds.

Lemma 1 *In the expression of $\mathcal{F}$ there are no terms of degree of homogeneity equal to 1 with respect to $\mathbf{Q}_1$ or to $\mathbf{Q}_2$.*

Proof Terms in $|\mathbf{Q}_2|^1$ require $k_3=0$ and $j'+k_2=1$. In each case the coefficient in square brackets, depending on the masses, cancels. For $\mathbf{Q}_1$ is similar. □

The homogeneity properties of the terms appearing for a given n are preserved, decreasing the corresponding order by 1, in (5).

From (6) it follows that some terms can disappear in the cases reducible to 1D. Then, writing for instance $\mathbf{Q} = \hat{Q}\mathbf{u}$, where u is a unitary vector, we can assume $\hat{Q} \geq 0$, with an eventual relabelling of the masses. Equations (5) turn out to be scalar equations. We shall keep the same name for the variables Q_1, Q_2, Q as scalar variables, skipping the $\hat{}$. In case (b) $f_1 = f_{12} = 0$, in case (c) $f_1 = f_2 = 0$ and in case (d) $f_1 = f_2 = f_{12} = 0$, so less terms remain.

A key role is played by the terms in (6) which contain both Q_1 and Q_2. According to lemma 1 the minimum value of n to find these terms is $n=4$. It is interesting to consider which kind of terms will appear. For a more complete discussion we consider a general planar four–body problem. Let $\varphi_1 \in \mathbb{S}^1$ be the angle from $\mathbf{Q}$ to $\mathbf{Q}_1$ and $\varphi_2 \in \mathbb{S}^1$ from $\mathbf{Q}$ to $\mathbf{Q}_2$. A simple computation shows

Lemma 2 *For $n=4$ there appear for the first time "coupled terms" (i.e., containing both $\mathbf{Q}_1$ and $\mathbf{Q}_2$). They are given by*

$$
\begin{aligned}
\mathcal{F}_{4,c} &= \frac{3}{4}\alpha_1\alpha_2\alpha_3\alpha_4\mu_1\mu_2 \left(35f_1^2f_2^2 - 5f_1^2f_{22} - 20f_1f_2f_{12} - 5f_2^2f_{11} + f_{11}f_{22} + 2f_{12}^2\right)|\mathbf{Q}|^{-9} = \\
&= \frac{3}{16}\alpha_1\alpha_2\alpha_3\alpha_4\mu_1\mu_2|\mathbf{Q}_1|^2|\mathbf{Q}_2|^2|\mathbf{Q}|^{-5}g_0(\varphi_1,\varphi_2), \qquad (7)
\end{aligned}
$$

where

$$
g_0(\varphi_1,\varphi_2) = 3 + 5c_1 + 5c_2 + 19c_1c_2 - 16s_1s_2, \qquad (8)
$$

being $c_1 = \cos(2\varphi_1), c_2 = \cos(2\varphi_2), s_1 = \sin(2\varphi_1), s_2 = \sin(2\varphi_2)$.

Remark 1 *In cases (a), (b) and (c), we have $\varphi_1 = \varphi_2 = 0$; $|\varphi_1| = \pi/2, \varphi_2 = 0$; $\varphi_1 = \varphi_2 = \pm\pi/2$, respectively. Then the related values of g_0 are $32, -16$ and 12 respectively. The corresponding value in the 3D case considered in (d) is 4.*

From (8) in lemma 2 it follows that, in the general case, g_0 is zero only along 2 closed curves for $(\varphi_1,\varphi_2) \in \mathbb{T}^2$. For each value of φ_1 there are exactly two values of φ_2 such that $g_0(\varphi_1,\varphi_2) = 0$, and vice–versa.

Due to the fact that (7) is homogeneous with respect to $\mathbf{Q}_1, \mathbf{Q}_2$ and $\mathbf{Q}$ individually, a multiple of g_0 appears also in $\mathcal{F}_{\mathbf{Q}_1}$, $\mathcal{F}_{\mathbf{Q}_2}$ and $\mathcal{F}_{\mathbf{Q}}$. The other factors in (7), depending only on the masses, are never zero.

3 New variables and equations of motion

In [3] equations of motion have been described by using blow up coordinates, both for the biisosceles problem and for the general planar four–body problem. Last ones in [3] are based on doing, first, a Levi–Civita transformation, and then a blow up. In all cases the use of some *energies of the colliding binaries*, $m_1 - m_2$ and $m_3 - m_4$, is essential. However, the blown up system seems difficult to analyze to obtain the *in–out map*, that is, the map which tells us how some i.c. close to i.c. leading to a SBC are transformed when leaving from the vicinity of the SBC. This fact has also been pointed by ElBialy [1] who has produced several papers describing in a beautiful way the underlying geometry in the blown up variables. The expansions to obtain the order of regularity seem rather cumbersome in these variables.

Instead, we shall follow a more direct approach, using also the fact that, according to the results of [3] the passage close to a SBC is well defined and we have uniqueness of the solution.

In the cases reducible to 1D, to which we restrict from now on, an important remark is that for any i.c. $Q_1, \dot{Q}_1, Q_2, \dot{Q}_2, Q, \dot{Q}$ with Q_1, Q_2 small, $\dot{Q}_1, \dot{Q}_2 < 0$, $\dot{Q}$ bounded and Q bounded away from zero, there are nearby values of time for which $Q_1 = 0$ and $Q_2 = 0$. It is not restrictive to assume that $Q_1 = 0$ at $t = 0$ while $Q_2 = 0$ at $t = t_2$, small. We are precisely interested in the different behaviour when t_2 approaches 0 from the positive and negative sides. Note that any i.c. lead to binary collisions both between m_1 and m_2 and between m_3 and m_4 . The set of i.c. leading to SBC has codimension 1.

Let be

$$e_1 = \frac{1}{2}\dot{Q}_1^2 - \frac{\mu_1}{Q_1}, \qquad e_2 = \frac{1}{2}\dot{Q}_2^2 - \frac{\mu_2}{Q_2}, \tag{9}$$

the energies of the binaries. Then

$$\dot{e}_1 = \frac{\mu_1}{m_1 m_2}\dot{Q}_1 \mathcal{F}_{Q_1}, \qquad \dot{e}_2 = \frac{\mu_2}{m_3 m_4}\dot{Q}_2 \mathcal{F}_{Q_2}, \tag{10}$$

where $\dot{Q}_j$ can be obtained as $Q_j^{-1/2}\left[2(\mu_j + e_j Q_j)\right]^{1/2}, j = 1, 2$.

It is convenient now to do the (1D) Levi–Civita transformation. Let be $Q_j = B_j^2$. The equations of motion can be put in the form

$$\begin{aligned}
\dot{B}_1 &= B_1^{-2}\left(\frac{1}{2}\left(\mu_1 + e_1 B_1^2\right)\right)^{1/2}, \qquad \dot{B}_2 = B_2^{-2}\left(\frac{1}{2}\left(\mu_2 + e_2 B_2^2\right)\right)^{1/2}, \\
\dot{e}_1 &= \frac{2\mu_1}{m_1 m_2} B_1^{-1}\left(\frac{1}{2}\left(\mu_1 + e_1 B_1^2\right)\right)^{1/2} \mathcal{F}_{Q_1}|_{Q_1 = B_1^2, Q_2 = B_2^2}, \\
\dot{e}_2 &= \frac{2\mu_2}{m_3 m_4} B_2^{-1}\left(\frac{1}{2}\left(\mu_2 + e_2 B_2^2\right)\right)^{1/2} \mathcal{F}_{Q_2}|_{Q_1 = B_1^2, Q_2 = B_2^2}, \\
\dot{Q} &= P, \qquad \dot{P} = -\frac{M}{Q^2} + \frac{M}{\mu_1 \mu_2}\mathcal{F}_Q|_{Q_1 = B_1^2, Q_2 = B_2^2}.
\end{aligned} \tag{11}$$

Equations (11) have the seemingly difficult terms B_j^{-2} and B_j^{-1}. Last ones are fictitious because, by lemma 1, $\mathcal{F}_{Q_j}$ contains at least $Q_j = B_j^2$ as a factor. As usual in the case of the two–body problem one can introduce new time scales. The problem is that each one of the binaries requires a different time scaling. Let $\bar{s}_1$ be a new independent variable defined by

$$t' = B_1^2, \qquad t(\bar{s}_1 = 0) = 0, \tag{12}$$

where $'$ means $\frac{d}{d\bar{s}_1}$ in (12). Then it is obtained

$$B_1' = \left(\frac{1}{2}\left(\mu_1 + e_1 B_1^2\right)\right)^{1/2}, \quad e_1' = \frac{2\mu_1}{m_1 m_2} B_1 B_1' \mathcal{F}_{Q_1} \quad \text{and also} \quad B_1'' - \frac{e_1}{2} B_1 = \frac{\mu_1}{2 m_1 m_2} B_1^3 \mathcal{F}_{Q_1}, \tag{13}$$

the first equation in (13) being equivalent to the definition of e_1 in (9). In a similar way we can introduce a new variable $\bar{s}_2$ by

$$t' = B_2^2, \qquad t(\bar{s}_2 = 0) = t_2, \tag{14}$$

where now $'$ means $\frac{d}{d\bar{s}_2}$ and we obtain (with the same meaning of $'$)

$$B_2' = \left(\frac{1}{2}\left(\mu_2 + e_2 B_2^2\right)\right)^{1/2}, \quad e_2' = \frac{2\mu_2}{m_3 m_4} B_2 B_2' \mathcal{F}_{Q_2} \quad \text{and also} \quad B_2'' - \frac{e_2}{2} B_2 = \frac{\mu_2}{2m_3 m_4} B_2^3 \mathcal{F}_{Q_2}. \tag{15}$$

The problem with the system (13)–(15) is the use of different time scales, related through the common value of t and the fact that (13) and (15) are coupled through the terms in $\mathcal{F}$ containing Q_1 and Q_2. Furthermore we still have last two equations in (11). We shall introduce new dependent variables, keeping t as independent variable but splitting the equations in an uncoupled part and the remaining, coupled, part.

Let $\mathcal{F}_u$ be the uncoupled part of $\mathcal{F}$, i.e., the terms which do not contain simultaneously Q_1 and Q_2. These terms can be obtained from (6) by keeping only the indices with $j' = k_2 = k_3 = 0$ or $j = k_1 = k_2 = 0$. Let be $\mathcal{F}_c = \mathcal{F} - \mathcal{F}_u$. The system (13), (15) with the last two equations in (11) can be written as

$$\begin{aligned}
&\frac{d^2 B_1}{d\bar{s}_1^2} - \frac{e_1}{2} B_1 = \frac{\mu_1}{2m_1 m_2} B_1^3 \left(\mathcal{F}_{u,Q_1} + \mathcal{F}_{c,Q_1}\right), \\
&\frac{de_1}{d\bar{s}_1} = \frac{2\mu_1}{m_1 m_2} B_1 \left(\frac{1}{2}\left(\mu_1 + e_1 B_1^2\right)\right)^{1/2} \left(\mathcal{F}_{u,Q_1} + \mathcal{F}_{c,Q_1}\right), \qquad \frac{dt}{d\bar{s}_1} = B_1^2, \\
&\frac{d^2 B_2}{d\bar{s}_2^2} - \frac{e_2}{2} B_2 = \frac{\mu_2}{2m_3 m_4} B_2^3 \left(\mathcal{F}_{u,Q_2} + \mathcal{F}_{c,Q_2}\right), \\
&\frac{de_2}{d\bar{s}_2} = \frac{2\mu_2}{m_3 m_4} B_2 \left(\frac{1}{2}\left(\mu_2 + e_2 B_2^2\right)\right)^{1/2} \left(\mathcal{F}_{u,Q_2} + \mathcal{F}_{c,Q_2}\right), \qquad \frac{dt}{d\bar{s}_2} = B_2^2, \\
&\frac{dQ}{dt} = P, \qquad\qquad \frac{dP}{dt} = -\frac{M}{Q^2} + \frac{M}{\mu_1 \mu_2} \mathcal{F}_Q.
\end{aligned} \tag{16}$$

As i.c. we should take $B_1(\bar{s}_1) = 0, e_1(\bar{s}_1) = e_{10}$, $\frac{dB_1}{d\bar{s}_1}(\bar{s}_1) = \left(\frac{\mu_1}{2}\right)^{1/2}$ and $t(\bar{s}_1) = 0$ for $\bar{s}_1 = 0$, $B_2(\bar{s}_2) = 0, e_2(\bar{s}_2) = e_{20}$, $\frac{dB_2}{d\bar{s}_2}(\bar{s}_2) = \left(\frac{\mu_2}{2}\right)^{1/2}$ and $t(\bar{s}_2) = 0$ for $\bar{s}_2 = 0$, and $Q(t) = Q_0 > 0, P(t) = P_0$, for $t = 0$.

Now we consider the "uncoupled problem", obtained from (16) by skipping the terms containing $\mathcal{F}_c$ in the first 6 equations and the full term $\mathcal{F}_Q$ in the last one. Let $B_1^u, \ldots, P^u$ be the solutions. It is immediate (see [4]) that B_1^u, e_1^u, t^u are analytic in $\bar{s}_1$ and B_2^u, e_2^u, t^u are analytic in $\bar{s}_2$, while Q^u, P^u are analytic in t. Furthermore $B_1^u(\bar{s}_1) = \left(\frac{\mu_1}{2}\right)^{1/2} \bar{s}_1 + O(\bar{s}_1^3)$ and hence $t^u = \frac{\mu_1}{6}\bar{s}_1^3 + O(\bar{s}_1^5)$. This implies that B_1^u, e_1^u are analytic in $t^{1/3}$ if we skip the $\mathcal{F}_c$ terms. In a similar way B_2^u, e_2^u are analytic in $(t - t_2)^{1/3}$ and Q^u, P^u are analytic in either $t^{1/3}$ or in $(t - t_2)^{1/3}$.

Finally we introduce the variables to be used in the proof of the main theorem. Let $R_1 = B_1^3, R_2 = B_2^3, Q = x^{-2}, P = y$. Using again t as independent variable and keeping the splitting as in (16) we have

$$\begin{aligned}
\dot{R}_1 &= 3\left(\frac{1}{2}\left(\mu_1 + e_1 R_1^{2/3}\right)\right)^{1/2}, \\
\dot{e}_1 &= \frac{2\mu_1}{m_1 m_2} R_1^{-1/3} \left(\frac{1}{2}\left(\mu_1 + e_1 R_1^{2/3}\right)\right)^{1/2} \left(\mathcal{F}_{u,Q_1}|_{Q_1 = R_1^{2/3}} + \mathcal{F}_{c,Q_1}|_{Q_1 = R_1^{2/3}, Q_2 = R_2^{2/3}}\right), \\
\dot{R}_2 &= 3\left(\frac{1}{2}\left(\mu_2 + e_2 R_2^{2/3}\right)\right)^{1/2}, \\
\dot{e}_2 &= \frac{2\mu_2}{m_3 m_4} R_2^{-1/3} \left(\frac{1}{2}\left(\mu_2 + e_2 R_2^{2/3}\right)\right)^{1/2} \left(\mathcal{F}_{u,Q_2}|_{Q_2 = R_2^{2/3}} + \mathcal{F}_{c,Q_2}|_{Q_1 = R_1^{2/3}, Q_2 = R_2^{2/3}}\right), \\
\dot{x} &= -\frac{1}{2} x^3 y, \qquad \dot{y} = -M x^4 + \frac{M}{\mu_1 \mu_2} \mathcal{F}_Q|_{Q_1 = R_1^{2/3}, Q_2 = R_2^{2/3}},
\end{aligned} \tag{17}$$

where Q has been replaced by x^{-2} in all the apparitions of $\mathcal{F}$. As before, the uncoupled system is obtained by deleting the $\mathcal{F}_c$ terms in the first four equations and $\mathcal{F}_Q$ in the last one. By analyticity we have

Proposition 1 *Consider the uncoupled system associated to (17) with initial data $x(0) = x_0, y(0) = y_0, R_1(0) = 0, e_1(0) = e_{10}, R_2(t_2) = 0, e_2(t_2) = e_{20}$. Then there exists $\beta > 0$ such that if $|t_2| < \beta/3$ the solutions x^u, y^u are analytic in t, R_1^u, e_1^u are analytic in $t^{1/3}$ and R_2^u, e_2^u are analytic in $(t-t_2)^{1/3}$ if $|t| < \beta$. Furthermore $R_1^u = 3\left(\frac{\mu_1}{2}\right)^{1/2} t + O(|t|^{5/3}), R_2^u = 3\left(\frac{\mu_2}{2}\right)^{1/2}(t-t_2) + O(|t-t_2|^{5/3})$.*

For further use we recall that in all cases (a) to (d) $\mathcal{F}$ starts with terms containing $Q^{-3}Q_1^2$ and $Q^{-3}Q_2^2$, with nonzero coefficients, and $\mathcal{F}_c$ with terms containing $Q^{-5}Q_1^2Q_2^2$ also with non zero coefficient. This follows from (6) and lemma 2.

4 Sketch of the proof of theorem 1

In this section we give the idea of the proof of theorem 1. In what follows the set of variables will be denoted by $w = (R_1, e_1, R_2, e_2, x, y)$.

We recall that for problems reducible to 1D, orbits passing near a SBC have two single binary collisions which take place at two different but very close times. So, we can assume that these orbits have a binary collision between m_1 and m_2 at $t = 0$ and another one between m_3 and m_4 at $t = t_2$ for some t_2 near to zero.

Let us consider a SBC that we assume takes place at $t = 0$ having $e_1 = \hat{e}_{10}, e_2 = \hat{e}_{20}, x = \hat{x}_0$ and $y = \hat{y}_0$ as data at the collision. We know that the solution with these initial data is analytic in $t^{1/3}$ and the initial data (see [4]). Let $\hat{w}(t) = (\hat{R}_1(t), \hat{e}_1(t), \hat{R}_2(t), \hat{e}_2(t), \hat{x}(t), \hat{y}(t))$ be this solution.

Moreover, proposition 1 suggests us to consider two sections of the flow, Σ_1 and Σ_2, defined by $R_1 = \hat{R}_1(t_*) \approx a_3 t_*$ and $R_1 = \hat{R}_1(t_{**}) \approx a_3 t_{**}$, respectively, for some fixed values $t_* < 0$ and $t_{**} > 0$ near zero and $a_3 = 3\sqrt{\mu_1/2}$. We want to compute the Poincaré map T_P from Σ_1 to Σ_2 in a neighbourhood of the point $P_c = \hat{w}(t_*) \in \Sigma_1$. The orbit $\phi(t; P_c)$ goes to SBC forward in time.

We consider an arc $\bar{\gamma}(l) \subset \Sigma_1$ of i.c. defined by

$$w_i(l) = P_c + \Delta P_c = P_c + (0, \nu_2(l), \nu_3(l), \nu_4(l), \nu_5(l), \nu_6(l)), \quad l \in [-l_0, l_0], \tag{18}$$

where $\nu_j(l) = \sum_{k\geq 1} \nu_{jk} l^k, j = 2,3,4,5,6$, are analytic functions of l, l_0 a small positive number, and we assume that the following transversality condition

$$\frac{d\nu_3}{dl}(0) \neq 0 \tag{19}$$

is satisfied.

We remark that for $l = 0$, $w_i(0)$, corresponds to a SBC orbit. So, we are interested in $T_P(\bar{\gamma}(l))$ as a function of l in a neighbourhood of zero.

We note that each value of $l \in [-l_0, l_0]$ determines a unique orbit, $w(t; l)$, passing through $w_i(l)$. For this orbit we fix the origin of time at the moment that the single binary collision between m_1 and m_2 takes place. Then, we assume that $R_1(0) = 0$ for $w(t; l)$. Let be $\epsilon_-(l)$ and $\epsilon_+(l)$ such that $w(t_* + \epsilon_-(l); l) = w_i(l)$ and $t_{**} + \epsilon_+(l)$ the smallest positive time satisfying $w(t_{**} + \epsilon_+(l); l) \in \Sigma_2$. We call $t_* + \epsilon_-(l)$ and $t_{**} + \epsilon_+(l)$ the initial and final times respectively. Our purpose is to get $w(t_{**} + \epsilon_+(l); l)$ as a function of l, P_c, t_* and t_{**}. To do that we need to compute some approximation for a solution, $w(t)$, which passes near SBC. This approximation will depend on the i.c. In particular, it will depend on l. Then we compute the final time $t_{**} + \epsilon_-(l)$ and finally we get $w(t_{**} + \epsilon_+(l); l)$ up to a certain order. In fact, we are interested in the non analytic terms in l which appear in $w(t_{**} + \epsilon_+(l); l)$ at the lowest order. We shall see that these terms will not appear until order $8/3$ in l.

To perform the proof of the theorem as pointed above, it is necessary to have a good representation of the solutions passing near SBC. To do that we characterize these solutions by t_2, e_{10}, e_{20}, x_0 and y_0, defined as

$$R_2(t_2) = 0, \quad e_{10} = e_1(0), \quad e_{20} = e_2(t_2), \quad x_0 = x(0), \quad y_0 = y(0). \tag{20}$$

We call $(t_2, e_{10}, e_{20}, x_0, y_0)$ the parameters of the orbit. The solutions $w(t)$ will be obtained in two steps.

First for a given set of values of the parameters we compute, up to a certain order, the solution $w(t)$ satisfying (20). (The meaning of the order for $w(t)$ shall be made precise below and we shall prove that some finite order is enough to our purposes.) To do that we split $w(t)$ in two parts, $w(t) = w^u(t) + \varphi(t)$, where $w^u(t)$ is the solution of the uncoupled equations and $\varphi(t)$ contains the terms due to the coupling. After proposition 1, $w^u(t)$ is analytic, with respect to suitable time variables, and can be easily obtained at lower order. Moreover, $\varphi(t)$ is approximated by the first iteration of a convergent iterative procedure to be introduced in the next section.

Once $w(t)$ is known up to a certain order as a function of $(t_2, e_{10}, e_{20}, x_0, y_0)$ we compute these parameters in terms of the i.c. $w_i(l)$, that is, as functions of l which have analytic and non analytic terms in l. By substituting the parameters into the expression of $w(t)$ obtained in the first step we get a convenient expression for the solution depending on the i.c.

5 A representation for the solution

In this section we shall give a representation of the solutions passing near SBC which will be convenient to our purposes.

Let be $s_1 = t^{1/3}$ and $s_2 = (t - t_2)^{1/3}$. From now on we shall use s_1 and s_2 instead of the uncoupled times $\bar{s}_1, \bar{s}_2$, introduced in section 3, and we write the equations (17) in the following way

$$
\begin{aligned}
\frac{dR_1}{ds_1} &= f_1(w), & \frac{dR_2}{ds_2} &= f_3(w), & \frac{dx}{ds_1} &= f_5(w), \\
\frac{de_1}{ds_1} &= f_2(w) + g_2(w), & \frac{de_2}{ds_2} &= f_4(w) + g_4(w), & \frac{dy}{ds_1} &= f_6(w) + g_6(w),
\end{aligned} \tag{21}
$$

where

$$
\begin{aligned}
f_1(w) &= 9s_1^2 \left(\frac{1}{2}\left(\mu_1 + e_1 R_1^{2/3}\right)\right)^{1/2}, & f_3(w) &= 9s_2^2 \left(\frac{1}{2}\left(\mu_2 + e_2 R_2^{2/3}\right)\right)^{1/2}, \\
f_2(w) &= \frac{2\mu_1}{3m_1m_2} f_1(w) R_1^{-1/3} \mathcal{F}_{u,Q_1}|_{Q_1 = R_1^{2/3}}, & f_4(w) &= \frac{2\mu_2}{3m_3m_4} f_3(w) R_2^{-1/3} \mathcal{F}_{u,Q_2}|_{Q_2 = R_2^{2/3}}, \\
f_5(w) &= -\frac{3}{2} s_1^2 x^3 y, & f_6(w) &= -3M s_1^2 x^4, \\
g_2(w) &= \frac{2\mu_1}{3m_1m_2} f_1(w) R_1^{-1/3} \mathcal{F}_{c,Q_1}|_{Q_1 = R_1^{2/3}, Q_2 = R_2^{2/3}}, \\
g_4(w) &= \frac{2\mu_2}{3m_3m_4} f_3(w) R_2^{-1/3} \mathcal{F}_{c,Q_2}|_{Q_1 = R_1^{2/3}, Q_2 = R_2^{2/3}}, & g_6(w) &= 3s_1^2 \frac{M}{\mu_1\mu_2} \mathcal{F}_Q|_{Q_1 = R_1^{2/3}, Q_2 = R_2^{2/3}}.
\end{aligned} \tag{22}
$$

Notice that the uncoupled equations are defined by the vector field $(f_1, f_2, f_3, f_4, f_5, f_6)^T$. Moreover g_2, g_4 and g_6 contain the coupling terms.

We fix arbitrary values e_{10}, e_{20}, x_0, y_0 and t_2, and look for the solution of the uncoupled equations such that

$$
R_1(0) = 0, \quad R_2(t_2) = 0, \quad x(0) = x_0, \quad e_1(0) = e_{10}, \quad e_2(t_2) = e_{20}, \quad y(0) = y_0. \tag{23}
$$

Let be $w^u(t) = (R_1^u(t), e_1^u(t), R_2^u(t), e_2^u(t), x^u(t), y^u(t))$ this solution. Usually we will denote w^u as a function of t but also we will write $R_1^u(s_1), e_1^u(s_1), R_2^u(s_2), e_2^u(s_2), x^u(s_1)$ and $y^u(s_1)$ to remark the dependence with respect the new times s_1 and s_2 .

Let be $I_\beta = \{t \in \mathbb{R} \,|\, |t| \leq \beta\}$ where $\beta > 0$ is the value used in proposition 1. As a consequence of that proposition and after some computations we get the following lemma

Lemma 3 *Let t_2 be such that $|t_2| \leq \beta/3$. Then $w^u(t)$ is analytic on s_1 and s_2 and there exist some positive constants $c_1 < c_2$, such that for all $t \in I_\beta$*

$$
\begin{aligned}
c_1|s_j|^3 \leq |R_j^u(s_j)| &\leq c_2|s_j|^3, & |e_j^u(s_j)| &\leq c_2, \quad j = 1, 2, \\
|x^u(s_1)| &\leq c_2, & y^u(s_1) &\leq c_2.
\end{aligned}
$$

Furthermore

$$\begin{aligned}
R_1^u(s_1) &= a_3 s_1^3 + a_5 s_1^5 + \dots, \qquad a_3 = 3\left(\frac{\mu_1}{2}\right)^{1/2}, \qquad a_5 = k_1 e_{10} a_3^{5/3},\\
R_2^u(s_2) &= c_3 s_2^3 + c_5 s_2^5 + \dots, \qquad c_3 = 3\left(\frac{\mu_2}{2}\right)^{1/2}, \qquad c_5 = k_2 e_{20} c_3^{5/3},\\
e_1^u(s_1) &= e_{10} + b_4 s_1^4 + \dots, \qquad b_4 = k_3 x_0^6 a_3^{4/3},\\
e_2^u(s_2) &= e_{20} + d_4 s_2^4 + \dots, \qquad d_4 = k_4 [x^u(t_2)]^6 c_3^{4/3},\\
x^u(t) &= x_0 + x_1 t + \dots, \qquad x_1 = -\frac{1}{2} x_0^3 y_0,\\
y^u(t) &= y_0 + y_1 t + \dots, \qquad y_1 = -M x_0^4,
\end{aligned}$$

for all $t \in I_\beta$. The coefficients k_m are constants depending on the masses.

Now we consider the complete equations (21). Let be $w(t) = (R_1(t), e_1(t), R_2(t), e_2(t), x(t), y(t))$ the solution of (21) such that (23) are satisfied. We write $w(t)$ as

$$w(t) = w^u(t) + \varphi(t), \tag{24}$$

where $\varphi(t) = (\varphi_1(t), \varphi_2(t), \varphi_3(t), \varphi_4(t), \varphi_5(t), \varphi_6(t))$ is the solution of

$$\begin{aligned}
\frac{d\varphi_j}{ds} &= F_j(\varphi; w^u) = f_j(w^u + \varphi) - f_j(w^u), \qquad j = 1,3,5,\\
\frac{d\varphi_j}{ds} &= F_j(\varphi; w^u) = f_j(w^u + \varphi) - f_j(w^u) + g_j(w^u + \varphi), \qquad j = 2,4,6,
\end{aligned} \tag{25}$$

where $s = s_1$ for $j = 1,2,5,6$, and $s = s_2$ for $j = 3,4$. It is also clear that

$$\varphi_j(0) = 0 \quad \text{for} \quad j = 1,2,5,6 \quad \text{and} \quad \varphi_j(t_2) = 0 \quad \text{for} \quad j = 3,4. \tag{26}$$

In order to get $\varphi(t)$ we shall apply an iterative Picard–like procedure to the equations (25).

Let α be a positive number, $\alpha < 1/2$ with $\alpha^3 \le \beta$, β being defined in proposition 1, and $I_\alpha = \{t \in \mathbb{R} \,|\, |t| \le \alpha^3/3\}$. So, $|s_1| \le \alpha$ and $|s_2| \le (4/3)^{1/3}\alpha$ for all $t, t_2 \in I_\alpha$.

For a continuous function $\varphi : I_\alpha \mapsto \mathbb{R}^6$ we define the following norm

$$\begin{aligned}
\|\varphi\| &= \max\{\|\varphi_j\|, j = 1,\dots,6\}, \quad \text{where}\\
&\|\varphi_j\| = \sup_{t\in I_\alpha}\left|\frac{\varphi_j(t)}{s^9}\right|, \quad j = 1,3, \qquad \|\varphi_j\| = \sup_{t\in I_\alpha}\left|\frac{\varphi_j(t)}{s^4}\right|, \quad j = 2,4,\\
&\|\varphi_5\| = \sup_{t\in I_\alpha}\left|\frac{\varphi_5 t)}{s_1^6}\right|, \qquad \|\varphi_6\| = \sup_{t\in I_\alpha}\left|\frac{\varphi_6(t)}{s_1^3}\right|,
\end{aligned} \tag{27}$$

being $s = s_1$ if $j = 1,2$ and $s = s_2$ is $j = 3,4$.

Let $C_b(I_\alpha)$ the vector space of continuous functions $\varphi : I_\alpha \mapsto \mathbb{R}^6$ such that $\|\varphi\|$ is bounded where $\|\cdot\|$ is the norm defined in (27). We note that if $\varphi \in C_b(I_\alpha)$ with $\|\varphi\| \le 1/2$, then $|\varphi_1| \le \frac{1}{2}|s_1|^9$ and $|\varphi_3| \le \frac{1}{2}|s_2|^9$ for all $t \in I_\alpha$. Therefore, using lemma 3, we can take α small enough such that for all $t \in I_\alpha$

$$\left|\frac{\varphi_j(t)}{R_j^u(t)}\right| \le \frac{1}{2}, \quad \text{for} \quad j = 1,3. \tag{28}$$

Let be $\bar{C}_b(I_\alpha) = \{\varphi \in C_b(I_\alpha) \,|\, \|\varphi\| \le 1/2,\}$. In the next we take α such that for any $\varphi \in \bar{C}_b(I_\alpha)$ (28) holds.

We define the following map

$$\begin{aligned}
T : \bar{C}_b(I_\alpha) &\mapsto \bar{C}_b(I_\alpha),\\
(T\varphi)_j(t) &= \int_0^{s_*} F_j(\varphi; w^u)\,ds, \qquad j = 1,\dots,6,
\end{aligned}$$

where $s_* = s_2$ if $j = 3,4$, $s_* = s_1$ otherwise. It is clear that a fixed point, φ, of T is a solution of (25) satisfying (26).

Next we state two lemmas whose proof will be given at the end of this section.

Lemma 4 *Let be $\psi, \varphi \in \bar{C}_b(I_\alpha)$ such that for some positive constant k_1, the following inequalities hold*

$$\|\psi_j - \varphi_j\| \leq k_1\alpha^2, \quad \text{if} \quad j = 1,3,5, \qquad \|\psi_j - \varphi_j\| \leq k_1\alpha^4, \quad \text{if} \quad j = 2,4,6. \tag{29}$$

There exists some constant c, such that if $C = c\alpha^2$, then

$$\|(T\psi)_j - (T\varphi)_j\| \leq k_1\alpha^2 C, \quad \text{if} \quad j = 1,3,5, \qquad \|(T\psi)_j - (T\varphi)_j\| \leq k_1\alpha^6 C, \quad \text{if} \quad j = 2,4,6. \tag{30}$$

We take as starting point $\varphi^{(0)} \in \bar{C}_b(I_\alpha)$, the vector $\varphi^{(0)} \equiv 0$, and define $\varphi^{(n)} = T\varphi^{(n-1)}$ for $n \geq 1$. Then it is easy to check that $\varphi^{(1)} = (0, \varphi_2^{(1)}, 0, \varphi_4^{(1)}, 0, \varphi_6^{(1)})$ where

$$\varphi_j^{(1)}(t) = \int_0^{s_1} g_j(w^u)ds_1, \quad j = 2,6, \qquad \varphi_4^{(1)}(t) = \int_0^{s_2} g_4(w^u)ds_2. \tag{31}$$

Lemma 5 *There exists some constant k such that $\|\varphi^{(1)}\| \leq k\alpha^4$.*

Proposition 2 *let α be small enough. Then $\varphi^{(n)} \in \bar{C}_b(I_\alpha)$ for all $n \geq 0$ and $\varphi^{(n)}$ tends to the solution φ, of (25) which satisfies (26), when n tends to ∞. Furthermore, there exist some constants k and c such that for all $n \geq 1$*

$$\|\varphi_j - \varphi_j^{(n)}\| \leq k\alpha^2 \frac{C^n}{1-C}, \quad j = 1,3,5, \quad \text{and} \quad \|\varphi_j - \varphi_j^{(n)}\| \leq k\alpha^6 \frac{C^n}{1-C}, \quad j = 2,4,6, \tag{32}$$

where $C = c\alpha^2$.

Proof

From lemma (5) $\varphi^{(1)} \in \bar{C}_b(I_\alpha)$ if α is sufficiently small. Furthermore $\varphi^{(1)}$ and $\varphi^{(0)}$ satisfy the hypothesis of lemma 4 with $k_1 = k$. Then there exists some constant c such that

$$\|\varphi_j^{(2)} - \varphi_j^{(1)}\| \leq k\alpha^2 C, \quad j = 1,3,5, \quad \text{and} \quad \|\varphi_j^{(2)} - \varphi_j^{(1)}\| \leq k\alpha^6 C, \quad j = 2,4,6,$$

where $C = c\alpha^2$ and so, $\|\varphi^{(2)} - \varphi^{(1)}\| \leq k\alpha^2 C$. This implies

$$\|\varphi^{(2)}\| \leq k\alpha^2 C + \|\varphi^{(1)}\| \leq k\alpha^2(\alpha^2 + C) \leq \frac{1}{2},$$

if α is small enough.

By using lemma 4 we obtain by induction that for all $n \geq 1$ the following assertions hold if α is small enough

(a) $\|\varphi_j^{(n+1)} - \varphi_j^{(n)}\| \leq k\alpha^2 C^n$, $j = 1,3,5$, and $\|\varphi_j^{(n+1)} - \varphi_j^{(n)}\| \leq k\alpha^6 C^n$, $j = 2,4,6$,

(b) $\|\varphi_j^{(n+1)}\| \leq k\alpha^2(\alpha^2 + C + C^2 + \ldots + C^n) \leq 1/2$, and so, $\varphi^{(n+1)} \in \bar{C}_b(I_\alpha)$.

Now we use a standard argument to show that $\varphi^{(n)}$ is a Cauchy sequence. More precisely, for $n, m \geq 1$, $m > n$ we get

$$\|\varphi_j^{(m)} - \varphi_j^{(n)}\| \leq k\alpha^2(C^{m-1} + C^{m-2} + \ldots + C^n) \leq k\alpha^2 \frac{C^n}{1-C}, \quad j = 1,3,5, \tag{33}$$

$$\|\varphi_j^{(m)} - \varphi_j^{(n)}\| \leq k\alpha^6(C^{m-1} + C^{m-2} + \ldots + C^n) \leq k\alpha^6 \frac{C^n}{1-C}, \quad j = 2,4,6. \tag{34}$$

For a given $\epsilon > 0$ we have $\|\varphi^{(m)} - \varphi^{(n)}\| \leq \epsilon$ if n is sufficiently large.

Now, for $\varphi = \lim_{n\to\infty} \varphi^{(n)}$, the inequalities (32) are easily obtained from (33) and (34) respectively. □

Remark 2 *In particular, for $n = 1$, we get from (32)*

$$\|\varphi_j - \varphi_j^{(1)}\| \leq \frac{kc}{1 - c\alpha^2}\alpha^4, \quad j = 1,3,5 \quad \text{and} \quad \|\varphi_j - \varphi_j^{(1)}\| \leq \frac{kc}{1 - c\alpha^2}\alpha^8, \quad j = 2,4,6. \tag{35}$$

Our purpose is to approximate $\varphi(t)$ by the dominant terms of $\varphi^{(1)}$ which is given in (31). In order to get a nice representation of $\varphi^{(1)}$ we introduce for $p,q\geq 0$

$$I_{p,q}(t;t_2)=\int_0^t z^{p/3}(z-t_2)^{q/3}dz, \qquad J_{p,q}(t;t_2)=\int_{t_2}^t z^{p/3}(z-t_2)^{q/3}dz. \tag{36}$$

Some useful properties of $I_{p,q}$ and $J_{p,q}$, that can be checked easily, are stated without proof in the following lemmas

Lemma 6 *The following properties hold*

(a) $I_{p,q}(t_2;t_2)=(-1)^{q/3}t_2^{\frac{p}{3}+\frac{q}{3}+1}\beta\left(\frac{p}{3}+1,\frac{q}{3}+1\right)$, *where* $\beta(a,b)=\int_0^1 x^{a-1}(1-x)^{b-1}dx$,

(b) $I_{p,q}(t;t_2)=(-1)^{(p+q+3)/3}I_{p,q}(-t;-t_2), \qquad J_{p,q}(t;t_2)=(-1)^{(p+q+3)/3}J_{p,q}(-t;-t_2)$,

(c) $J_{p,q}(t;t_2)=I_{p,q}(t;t_2)-I_{p,q}(t_2;t_2)$,

(d) $|I_{p,q}(t;t_2)|\leq 2^{q/3}\alpha^{p+q+3}, \quad |J_{p,q}(t;t_2)|\leq 2^{(p+q+3)/3}\alpha^{p+q+3}$, *for all* $t,t_2\in I_\alpha$.

In the next lemma we give explicitly the integrals which will be important to our purposes.

Lemma 7

$$I_{1,4}(t;t_2)=\Gamma_{1,1}(t;t_2)t_2^{8/3}+\Gamma_{1,a}(t;t_2), \qquad J_{4,1}(t;t_2)=\Gamma_{2,1}(t;t_2)t_2^{8/3}+\Gamma_{2,a}(t;t_2),$$

where

$$\begin{aligned}
\Gamma_{1,1}(t;t_2) &= \left\{\begin{array}{lll} \gamma_{1,1} & \text{if} & t>t_2,\quad t_2>0,\\ \gamma_{1,-1} & \text{if} & t<0,\quad t_2>0,\\ \gamma_{1,-1} & \text{if} & t>0,\quad t_2<0,\\ \gamma_{1,1} & \text{if} & t<t_2,\quad t_2<0,\end{array}\right. ,\\
\Gamma_{1,a}(t;t_2) &= \left\{\begin{array}{lll} \Gamma^+_{1,a}(t;t_2) & \text{if} & t>t_2,\quad t_2>0,\\ \Gamma^-_{1,a}(t;t_2) & \text{if} & t<0,\quad t_2>0,\\ \Gamma^-_{1,a}(-t;-t_2) & \text{if} & t>0,\quad t_2<0,\\ \Gamma^+_{1,a}(-t;-t_2) & \text{if} & t<t_2,\quad t_2<0,\end{array}\right. ,\\
\gamma_{1,1} &= \sum_{j\geq 0}\binom{4/3}{j}(-1)^j\frac{4/3-2j}{(4/3+j)(8/3-j)}, \qquad \gamma_{1,-1}=\sum_{j\geq 0}\binom{1/3}{j}\frac{(-1)^{j+1}}{8/3-j},\\
\Gamma^+_{1,a}(t;t_2) &= \sum_{j\geq 0}\binom{4/3}{j}\frac{(-1)^j}{8/3-j}t_2^j t^{(8-3j)/3},\\
\Gamma^-_{1,a}(t;t_2) &= \sum_{j\geq 0}\binom{1/3}{j}\frac{(-1)^j}{8/3-j}t_2^j(t_2-t)^{(8-3j)/3},\\
\Gamma_{2,1}(t;t_2) &= \beta(7/3,4/3)-\Gamma_{1,1}(t;t_2), \qquad \Gamma_{2,a}(t;t_2)=\frac{3}{4}t^{4/3}(t-t_2)^{4/3}-\Gamma_{1,a}(t;t_2).
\end{aligned}$$

Now we return to the computation of $\varphi^{(1)}$. Using (31) and the expressions of $g_2(w), g_4(w)$ and $g_6(w)$ given in (22) we get

$$\begin{aligned}
\varphi_2^{(1)}(t) &= c_{11}I_{1,4}(t;t_2)+\sum_{p+q\geq 7}c_{p,q}^{(2)}I_{p,q}(t;t_2)=\gamma_1(t)+O_{10},\\
\varphi_4^{(1)}(t) &= c_{21}J_{4,1}(t;t_2)+\sum_{p+q\geq 7}c_{p,q}^{(4)}J_{p,q}(t;t_2)=\gamma_2(t)+O_{10},\\
\varphi_6^{(1)}(t) &= c_{32}I_{4,0}(t;t_2)+c_{31}I_{0,4}(t;t_2)+\sum_{p+q\geq 6}c_{p,q}^{(6)}I_{p,q}(t;t_2)=\gamma_3(t)+O_9,
\end{aligned} \tag{37}$$

where

$$\gamma_1(t) = c_{11} I_{1,4}(t;t_2), \qquad \gamma_2(t) = c_{21} J_{4,1}(t;t_2), \qquad \gamma_3(t) = c_{32} I_{4,0}(t;t_2) + c_{31} I_{0,4}(t;t_2),$$

and $c_{11}, c_{21}, c_{31}, c_{32}$ and $c_{p,q}^{(j)}, j = 2,4,6$ are constants depending on e_{10}, e_{20}, x_0, y_0 and the masses. For instance

$$c_{11} = \frac{3^{8/3}}{4} g_0 \left(\frac{\mu_1\mu_2}{4}\right)^{2/3} \frac{m_3 m_4}{\mu_2} x_0^{10}, \qquad c_{21} = \frac{3^{8/3}}{4} g_0 \left(\frac{\mu_1\mu_2}{4}\right)^{2/3} \frac{m_1 m_2}{\mu_1} (x(t_2))^{10},$$

where g_0 is the value given at the remark following lemma 2, and depending on the cases (a) to (d). In (37) and in the next, O_r stands for a function of order r in α.

Finally, for the complete solution (24) we get the following representation

$$\begin{aligned}
R_1(t) &= R_1^u(t) + \varphi_1^{(1)}(t) + R_{1r}(t) = R_1^u(t) + R_{1r}(t), \\
R_2(t) &= R_2^u(t) + \varphi_3^{(1)}(t) + R_{2r}(t) = R_2^u(t) + R_{2r}(t), \\
e_1(t) &= e_1^u(t) + \varphi_2^{(1)}(t) + e_{1r}(t) = e_1^u(t) + \gamma_1(t) + E_1(t), \\
e_2(t) &= e_2^u(t) + \varphi_4^{(1)}(t) + e_{2r}(t) = e_2^u(t) + \gamma_2(t) + E_2(t), \\
x(t) &= x^u(t) + \varphi_5^{(1)}(t) + x_r(t) = x^u(t) + x_r(t), \\
y(t) &= y^u(t) + \varphi_6^{(1)}(t) + y_r(t) = y^u(t) + \gamma_3(t) + E_3(t),
\end{aligned} \tag{38}$$

where, after remark 2

$$\begin{aligned}
&|R_{jr}(t)| \le \frac{kc}{1-c\alpha^2}\alpha^4 |s_j|^9 = O_{13}, \qquad |e_{jr}(t)| \le \frac{kc}{1-c\alpha^2}\alpha^8 |s_j|^4 = O_{12}, \qquad j = 1,2, \\
&|x_r(t)| \le \frac{kc}{1-c\alpha^2}\alpha^4 |s_1|^6 = O_{10}, \qquad |y_r(t)| \le \frac{kc}{1-c\alpha^2}\alpha^8 |s_j|^3 = O_{11},
\end{aligned}$$

for all $t \in I_\alpha$. From (37) and using the property (d) of lemma 6 we get that

$$E_j(t) = O_{10}, \quad j = 1,2 \quad \text{and} \quad E_3(t) = O_9. \tag{39}$$

Next two lemmas are needed for the proofs of lemmas 4 and 5 and follow from (28) and inspection of (22).

Lemma 8 *Let α be small enough. There exist some constants b_2, b_4 and b_6 such that for any $\varphi \in \bar{C}_b(I_\alpha)$*

$$\begin{aligned}
& & |f_i(w^u + \varphi)| \le \tfrac{9\sqrt{3\mu_i}}{2} |s_i|^2, & \qquad i = 1,3, \\
&|g_2(w^u + \varphi)| \le b_2 \alpha^4 |s_1|^3, & |g_4(w^u + \varphi)| \le b_4 \alpha^4 |s_2|^3, & \qquad |g_6(w^u + \varphi)| \le b_6 \alpha^4 s_1^2
\end{aligned}$$

for all $t \in I_\alpha$.

Lemma 9 *Let α be small enough. There exist some constants $\bar{a}_j, j = 1, \ldots, 9$ such that for any $\varphi, \psi \in \bar{C}_b(I_\alpha)$ the following inequalities are satisfied*

$$\begin{aligned}
|f_1(w^u + \psi) &- f_1(w^u + \varphi)| \le \bar{a}_1 s_1^4 \max\left\{ \frac{1}{|s_1|^3} |\psi_1(t) - \varphi_1(t)|, |\psi_2(t) - \varphi_2(t)| \right\} \\
|f_2(w^u + \psi) &- f_2(w^u + \varphi)| \le \\
&\le \bar{a}_2 |s_1|^3 \max\left\{ \frac{1}{|s_1|^3} |\psi_1(t) - \varphi_1(t)|, |s_1|^2 |\psi_2(t) - \varphi_2(t)|, |\psi_5(t) - \varphi_5(t)| \right\} \\
|f_3(w^u + \psi) &- f_3(w^u + \varphi)| \le \bar{a}_3 s_2^4 \max\left\{ \frac{1}{|s_2|^3} |\psi_3(t) - \varphi_3(t)|, |\psi_4(t) - \varphi_4(t)| \right\} \\
|f_4(w^u + \psi) &- f_4(w^u + \varphi)| \le \\
&\le \bar{a}_4 |s_2|^3 \max\left\{ \frac{1}{|s_2|^3} |\psi_3(t) - \varphi_3(t)|, |s_2|^2 |\psi_4(t) - \varphi_4(t)|, |\psi_5(t) - \varphi_5(t)| \right\}
\end{aligned}$$

$$
\begin{aligned}
|f_5(w^u+\psi) &- f_5(w^u+\varphi)| \le \bar a_5 s_1^2 \max\{|\psi_5(t)-\varphi_5(t)|, |\psi_6(t)-\varphi_6(t)|\} \\
|f_6(w^u+\psi) &- f_6(w^u+\varphi)| \le \bar a_6 s_1^2 |\psi_5(t)-\varphi_5(t)|, \\
|g_2(w^u+\psi) &- g_2(w^u+\varphi)| \le \bar a_7 |s_1|^3 \\
& \max\{\frac{1}{|s_1|^3}|\psi_1(t)-\varphi_1(t)|, s_1^2|\psi_2(t)-\varphi_2(t)|, |\psi_3(t)-\varphi_3(t)|, |\psi_5(t)-\varphi_5(t)|\}, \\
|g_4(w^u+\psi) &- g_4(w^u+\varphi)| \le \bar a_8 |s_2|^3 \\
& \max\{|\psi_1(t)-\varphi_1(t)|, \frac{1}{|s_2|^3}|\psi_3(t)-\varphi_3(t)|, s_2^2|\psi_4(t)-\varphi_4(t)|, |\psi_5(t)-\varphi_5(t)|\}, \\
|g_6(w^u+\psi) &- g_6(w^u+\varphi)| \le \bar a_9 |s_1|^2 \\
& \max\{|\psi_1(t)-\varphi_1(t)|, |\psi_3(t)-\varphi_3(t)|, |\psi_5(t)-\varphi_5(t)|\},
\end{aligned}
$$

for all $t \in I_\alpha$.

Proof of lemma 4

From the definition of T and using lemma 9 we get

$$
\begin{aligned}
|(T\psi)_1(t) &- (T\varphi)_1(t)| \le \left| \int_0^{s_1} |f_1(w^u+\psi) - f_1(w^u+\varphi)| ds_1 \right| \le \\
&\le \bar a_1 \left| \int_0^{s_1} s_1^4 \max\{\frac{1}{|s_1|^3}|\psi_1(t)-\varphi_1(t)|, |\psi_2(t)-\varphi_2(t)|\} ds_1 \right| \le k_1 \alpha^2 |s_1|^9 \left(\frac{\bar a_1}{9}\alpha^2\right).
\end{aligned}
$$

Then $\|(T\psi)_1 - (T\varphi)_1\| \le k_1\alpha^2 \left(\frac{\bar a_1}{9}\alpha^2\right)$. In a similar way we get $\|(T\psi)_3 - (T\varphi)_3\| \le k_1\alpha^2 \left(\frac{\bar a_3}{9}\alpha^2\right)$.

$$
\begin{aligned}
|(T\psi)_5(t) &- (T\varphi)_5(t)| \le \left| \int_0^{s_1} |f_5(w^u+\psi) - f_5(w^u+\varphi)| ds_1 \right| \le \\
&\le \bar a_5 \left| \int_0^{s_1} s_1^2 \max\{|\psi_5(t)-\varphi_5(t)|, |\psi_6(t)-\varphi_6(t)|\} ds_1 \right| \le \bar a_5 k_1 \alpha^4 \frac{|s_1|^6}{6},
\end{aligned}
$$

and therefore $\|(T\psi)_5 - (T\varphi)_5\| \le k_1\alpha^2 \left(\frac{\bar a_5}{6}\alpha^2\right)$.

$$
\begin{aligned}
|(T\psi)_2(t) - (T\varphi)_2(t)| &\le \left| \int_0^{s_1} |f_2(w^u+\psi) - f_2(w^u+\varphi)| ds_1 \right| + \left| \int_0^{s_1} |g_2(w^u+\psi) - g_2(w^u+\varphi)| ds_1 \right| \\
&\le \bar a_2 \left| \int_0^{s_1} |s_1|^3 \max\{\frac{1}{|s_1|^3}|\psi_1(t)-\varphi_1(t)|, s_1^2|\psi_2(t)-\varphi_2(t)|, |\psi_5(t)-\varphi_5(t)|\} ds_1 \right| + \\
&\quad \bar a_7 \left| \int_0^{s_1} |s_1|^3 \max\{\frac{1}{|s_1|^3}|\psi_1(t)-\varphi_1(t)|, s_1^2|\psi_2(t)-\varphi_2(t)|, |\psi_3(t)-\varphi_3(t)|, |\psi_5(t)-\varphi_5(t)|\} ds_1 \right| \\
&\le k_1(\bar a_2 + \bar a_7)\alpha^8 \frac{s_1^4}{4},
\end{aligned}
$$

and $\|(T\psi)_2 - (T\varphi)_2\| \le k_1\alpha^4 \left(\frac{\bar a_2+\bar a_7}{4}\alpha^4\right)$. In a similar way we obtain $\|(T\psi)_4 - (T\varphi)_4\| \le k_1\alpha^4 \left(\frac{\bar a_4+\bar a_8}{4}\alpha^4\right)$ and $\|(T\psi)_6 - (T\varphi)_6\| \le k_1\alpha^4 \left(\frac{\bar a_6+\bar a_9}{3}\alpha^4\right)$. Now we only need to take $c = \max\{\frac{\bar a_1}{9}, \frac{\bar a_3}{9}, \frac{\bar a_5}{6}, \frac{\bar a_2+\bar a_7}{4}, \frac{\bar a_4+\bar a_8}{4}, \frac{\bar a_6+\bar a_9}{3}\}$. □

Proof of lemma 5

It follows immediately from (31) and lemma 8 for $\varphi \equiv 0$. □

6 The transition map. Proof of the theorem

As sketched in section 4 we consider now an orbit leading to SBC forward in time. Hence $t_2 = 0$. Let $\hat e_{10}, \hat e_{20}, \hat x_0$ and $\hat y_0$ the remaining parameters of the orbit, denoted by $\hat w(t) = (\hat R_1(t), \hat e_1(t), \hat R_2(t), \hat e_2(t), \hat x(t), \hat y(t))$.

If the total energy is fixed, $H = h$, these parameters are not independent. Indeed from (1)–(4) we have

$$H = \frac{M}{\mu_1\mu_2}\left(\frac{m_1m_2}{\mu_1}e_1 + \frac{m_3m_4}{\mu_2}e_2\right) + \frac{1}{2}|P|^2 - \frac{M}{|Q|} + \frac{M}{\mu_1\mu_2}\mathcal{F}.$$

From this, and taking into account that $\mathcal{F} = 0$ if $Q_1 = Q_2 = 0$, we obtain

$$h = \frac{M}{\mu_1\mu_2}\left(\frac{m_1m_2}{\mu_1}\hat{e}_{10} + \frac{m_3m_4}{\mu_2}\hat{e}_{20}\right) + \frac{1}{2}\hat{y}_0^2 - M\hat{x}_0^2.$$

Let us take $t_* < 0$ and $t_{**} > 0$ with $|t_*|$ and $|t_{**}|$ small enough ($< \alpha^3$). We have to study the transition map $T_P : \Sigma_1 \mapsto \Sigma_2$ as defined in section 4.

Given $l \in [-l_0, l_0]$ and $\nu(l) = (0, \nu_2(l), \nu_3(l), \nu_4(l), \nu_5(l), \nu_6(l))$ as in (18) we want to compute the parameters and the initial time of the orbit $w(t_* + \epsilon_-; l)$ passing through $P_c + \Delta P_c(l)$ at $t = (t_* + \epsilon_-)$, that is, we want to compute $t_2(l), e_{10}(l), e_{20}(l), x_0(l), y_0(l)$ and $\epsilon_-(l)$. To this end we must solve the system $w(t_* + \epsilon_-; l) = \hat{w}(t_*) + \nu(l)$ that can be written as $G(\epsilon_-, t_2, e_{10}, e_{20}, x_0, y_0, l) = 0$, where the components of G are easily obtained from (38) using (37) and lemma 3, and are given by

$$\begin{array}{ll}
a_3(t_* + \epsilon_-) + k_1 e_{10} a_3^{5/3}(t_* + \epsilon_-)^{5/3} + \ldots & -\hat{R}_1(t_*), \\
c_3(t_* - t_2 + \epsilon_-) + k_2 e_{20} c_3^{5/3}(t_* - t_2 + \epsilon_-)^{5/3} + \ldots & -\hat{R}_2(t_*) - \nu_3(l), \\
e_{10} + k_3 x_0^6 a_3^{4/3}(t_* + \epsilon_-)^{4/3} + \ldots + c_{11} I_{1,4}(t_* + \epsilon_-; t_2) + \ldots & -\hat{e}_1(t_*) - \nu_2(l), \\
e_{20} + k_4(x_0 - \frac{1}{2}x_0^3 y_0 t_2 + \ldots)^6 c_3^{4/3}(t_* - t_2 + \epsilon_-)^{4/3} + \ldots + & \\
\qquad\qquad c_{21} J_{4,1}(t_* + \epsilon_-; t_2) + \ldots & -\hat{e}_2(t_*) - \nu_4(l), \\
x_0 - \frac{1}{2}x_0^3 y_0(t_* + \epsilon_-) + \ldots & -\hat{x}(t_*) - \nu_5(l), \\
y_0 - M x_0^4(t_* + \epsilon_-) + \ldots + c_{32} I_{4,0}(t_* + \epsilon_-; t_2) + c_{31} I_{0,4}(t_* + \epsilon_-; t_2) + \ldots & -\hat{y}(t_*) - \nu_6(l),
\end{array} \tag{40}$$

where the dots mean higher order terms and we have only included explicitly among them the terms coming from $\gamma_1(t), \gamma_2(t)$ and $\gamma_3(t)$ (see (37) and (38)). We remark that, like in section 5, the order here corresponds to order in α, taking into account that t_* and t_{**} are $O(\alpha^3)$. If we compute the differential of G with respect to $(\epsilon_-, t_2, e_{10}, e_{20}, x_0, y_0)$ we have that the only terms $O(\alpha^0)$ are $\mathrm{diag}(a_3, -c_3, 1, 1, 1, 1)$ and the term at the (2,1) place, which equals c_3. As $G = 0$ has the trivial solution $\epsilon_- = 0, t_2 = 0, e_1 = \hat{e}_{10}, e_2 = \hat{e}_{20}, x = \hat{x}_0, y = \hat{y}_0$ if $l = 0$, the implicit function theorem assures the existence and uniqueness of solution if l_0 is small enough. Furthermore the solution is given by

$$\begin{aligned}
t_2 &= -\frac{1}{c_3}\nu_3(l) + \ldots = \sigma l + O(l^2), \\
e_{10} &= \hat{e}_{10} + \ldots - c_{11} I_{1,4}(t_* + \epsilon_-; t_2) + \ldots = \hat{e}_{10} + \ldots - c_{11}\Gamma_{1,1}(t_* + \epsilon_-; t_2)(\sigma l)^{8/3} + \ldots, \\
e_{20} &= \hat{e}_{20} + \ldots - c_{21} J_{4,1}(t_* + \epsilon_-; t_2) + \ldots = \hat{e}_{20} + \ldots - c_{21}\Gamma_{2,1}(t_* + \epsilon_-; t_2)(\sigma l)^{8/3} + \ldots, \\
x_0 &= \hat{x}_0 + \ldots, \\
y_0 &= \hat{y}_0 + \ldots - c_{31} I_{0,4}(t_* + \epsilon_-; t_2) + \ldots = \hat{y}_0 + \ldots + \frac{3c_{31}}{7}(\sigma l)^{7/3} + \ldots, \\
\epsilon_- &= -\frac{1}{c_3}\Big(R_1(t_*, t_2, e_{10}, e_{20}, x_0, y_0) - \hat{R}_1(t_*)\Big) + \ldots,
\end{aligned} \tag{41}$$

where $\sigma = -\frac{1}{c_3}\frac{d\nu_3(0)}{dl}$ and the condition (19) and lemma 7 have been used. In (41) beyond the dominant terms we have only isolated the first terms which are non analytic in l. In particular, from $I_{0,4}(t_* + \epsilon_-; t_2) = \frac{3}{7}(t_* - t_2 + \epsilon_-)^{7/3} - \frac{3}{7}(-t_2)^{7/3}$, only the term $-\frac{3}{7}(-t_2)^{7/3}$ is kept. However, this term will cancel.

Having the parameters of the orbit at hand we can proceed to compute the image point under T_P, that is, $w(t_{**} + \epsilon_+; l)$ where ϵ_+ is provided by the implicit function theorem from the equation $R_1(t_{**} + \epsilon_+) = \hat{R}_1(t_{**})$. The terms containing a non analytic contribution in l of lower order appear in $e_1(t_{**} + \epsilon_+)$ and $e_2(t_{**} + \epsilon_+)$ at order 8/3. We remark that there seem to be non analytic terms of order 7/3 in $y(t_{**} + \epsilon_+)$. But these terms are cancelled by the corresponding ones of y_0 given in (41), that is, using (38), (37), lemma 3 and (41), we have

$$\begin{aligned}
y(t_{**} + \epsilon_+) &= y_0 + \ldots + c_{31} I_{0,4}(t_{**} + \epsilon_+; t_2) + \ldots = \\
&= \hat{y}_0 + \ldots - \frac{3c_{31}}{7}(-\sigma l)^{7/3} + \ldots + \frac{3c_{31}}{7}(-\sigma l)^{7/3} + \ldots = \hat{y}_0 + \ldots.
\end{aligned}$$

Now, using lemma 7, we write

$$\begin{aligned}
e_1(t_{**}+\epsilon_+) &= e_{10}+\ldots+c_{11}I_{1,4}(t_{**}+\epsilon_+;t_2)+\ldots= \\
&= \hat{e}_{10}+\ldots+c_{11}\left[-I_{1,4}(t_*+\epsilon_-;t_2)+I_{1,4}(t_{**}+\epsilon_+;t_2)\right]+\ldots= \\
&= \hat{e}_{10}+\ldots+c_{11}\left[-\Gamma_{1,1}(t_*+\epsilon_-;t_2)+\Gamma_{1,1}(t_{**}+\epsilon_+;t_2)\right](\sigma l)^{8/3}+\ldots, \\
e_2(t_{**}+\epsilon_+) &= e_{20}+\ldots+c_{21}J_{4,1}(t_{**}+\epsilon_+;t_2)+\ldots= \\
&= \hat{e}_{20}+\ldots+c_{21}\left[-J_{4,1}(t_*+\epsilon_-;t_2)+J_{4,1}(t_{**}+\epsilon_+;t_2)\right]+\ldots= \\
&= \hat{e}_{20}+\ldots+c_{21}\left[-\Gamma_{2,1}(t_*+\epsilon_-;t_2)+\Gamma_{2,1}(t_{**}+\epsilon_+;t_2)\right](\sigma l)^{8/3}+\ldots.
\end{aligned}$$

We note that according to the transversality condition in section 4, $\sigma \neq 0$. Moreover $t_*+\epsilon_- < 0$ and $t_{**}+\epsilon_+ > 0$. Using lemma 7 we have that if $t_2 > 0$ the coefficients of $l^{8/3}$ for $e_1(t_{**}+\epsilon_+)$ and $e_2(t_{**}+\epsilon_+)$ are

$$c_{11}\left(-\gamma_{1,-1}+\gamma_{1,1}\right)\sigma^{8/3}, \qquad c_{21}\left(\gamma_{1,-1}-\gamma_{1,1}\right)\sigma^{8/3},$$

respectively. So, one has to check that $\gamma_{1,1}-\gamma_{1,-1}\neq 0$. This will ensure that the terms of the form $Al^{8/3}$, with $A\neq 0$, appear as lower order non analytic terms if $t_2>0$ for $e_1(t_{**}+\epsilon_+)$ and $e_2(t_{**}+\epsilon_+)$. From lemma 7 it follows that the terms appearing if $t_2<0$ are $-Al^{8/3}$. Hence, T_P is C^r for any $r<8/3$ but not for $r=8/3$.

Using the representation of $\gamma_{1,1}$ and $\gamma_{1,-1}$ given in lemma 7 we have

$$\gamma_{1,1}-\gamma_{1,-1} = \frac{3}{4}+\frac{1}{35}-\frac{13}{30}+\sum_{j\geq 3}\frac{(-1)^j}{j!}\frac{1}{3}\left(-\frac{2}{3}\right)\cdots\left(\frac{7}{3}-j\right)\left[\frac{4\left(\frac{4}{3}-2j\right)}{4+3j}+\frac{4}{3}-j\right]\frac{1}{\frac{8}{3}-j}. \quad (42)$$

The first three terms in (42) add to $\frac{29}{84}$ and all the other terms are positive. This implies $A\neq 0$.

To conclude the proof of the theorem we have to show that really the terms in $l^{8/3}$ displayed above can not be cancelled by other higher order terms no matter how small is l. Indeed, terms $l^{8/3}$ can be, in principle cancelled by terms of the order t_*^r, t_{**}^r with $r\geq 9$ for a fixed $t_*, t_{**}=O(\alpha^3)$ and l (or t_2) much less that α^3. To show that this is not the case we proceed as follows. Given $l\neq 0$ we can determine t_2. Then we select some intermediate sections $\bar{\Sigma}_1, \bar{\Sigma}_2$ by choosing the related times $\bar{t}_* = -3|t_2|, \bar{t}_{**}=3|t_2|$. For the map $\bar{T}_P : \bar{\Sigma}_1 \mapsto \bar{\Sigma}_2$ the terms $O(l^{8/3})$ dominate any other term $O(l^{r/3}), r\geq 9$, if l is small enough. The maps $T_1 : \Sigma_1 \mapsto \bar{\Sigma}_1$, $T_2 : \bar{\Sigma}_2 \mapsto \Sigma_2$ are analytic diffeomorphisms. If T_P has a degree of regularity $\hat{r}\geq 8/3$, as $\bar{T}_P = T_2^{-2}T_P T_1^{-1}$, we would conclude that $\bar{T}_P$ is $C^{\hat{r}}$, which leads to an absurdity.

Finally the representation of $R_2(t)$ and the expression of t_2 as a function of l ensure that if the transversality condition is satisfied on Σ_1, then it is also satisfied on $\bar{\Sigma}_1$. This ends the proof of the theorem.

7 Several interpretations and extensions

From the proof of the main theorem it is clear that a key role, to prevent from higher differentiability, is played by t_2. That is, by the difference between the time of collision of m_1 and m_2, from one side, and m_3 and m_4 from the other. This fact can be interpreted in different ways.

First we consider again the computation of the functions $I_{p,q}(t;t_2)$. If $t>|t_2|$, it is enough to consider the value of $\hat{I}_{p,q}=\int_{-t}^{t} z^{p/3}(z-t_2)^{q/3}dz$. Other integrals, like $\int_{-t_1}^{t_3}$ with $t_1, t_3 > |t_2|$ differ from the previous one in terms which are analytic in t_2. It follows immediately that $\hat{I}_{p,q}(t;t_2)=(-1)^{p+q}\hat{I}_{p,q}(t;-t_2)$. As we are interested in $(q,p)=(4,1)$ this implies a change of sign if t_2 changes sign. Furthermore, for $t_2>0$,

$$\hat{I}_{4,1}(t;t_2)=\int_0^t\left[z^{4/3}(z-t_2)^{1/3}-z^{4/3}(z+t_2)^{1/3}\right]dz < -\int_0^{t_2} z^{4/3}\left[(t_2+z)^{1/3}-(t_2-z)^{1/3}\right]dz,$$

which is negative and $O(t_2^{8/3})$.

In terms of the blown up variables introduced in [3], the SBC manifold, $\mathcal{C}$, is four dimensional (see [3]) inside the 5D manifold of constant (total) energy. There are two critical points: one corresponding to simultaneous collision, (C), the other to simultaneous ejection, (E). At both points there are two neutral directions corresponding to the "outer variables" x and y. Inside $\mathcal{C}$, if we skip the neutral directions, both points are saddles. The center–unstable manifold W_C^{cu} coincides with W_E^{cs}. Along one of the branches of W_C^{cu} the binary collision (in $\mathcal{C}$) between m_1 and m_2 occurs first, and then the collision between m_3 and m_4 is produced. The order of these binary collisions is exchanged along the other branch. For orbits close to SBC this means $t_2 > 0$ ($t_2 < 0$ respectively). Given a point on the part of the center–stable manifold of C lying outside $\mathcal{C}$, nearby points to one side or another of this manifold pass close to C in the physical space (i.e., outside $\mathcal{C}$) which is 4D (codimension 1) due to the neutral character of (x, y). The orbits of these points leave $\mathcal{C}$ close to E tending to the same orbit on the center–unstable manifold of E. The theorem proved assures that this passage is only C^r for $r < 8/3$.

A simple physical interpretation proceeds as follows. For simplicity assume we consider the collinear four–body problem, such that all bodies have mass 1, the distance between both binaries is 1 at the SBC and let us denote as $2a$ the distance between m_1 and m_2 and $2b$ the distance between m_3 and m_4. From

$$2\ddot{b} = \ddot{q}_4 - \ddot{q}_3 = -\frac{1}{(1+a+b)^2} - \frac{1}{(1-a+b)^2} + \frac{1}{(1+a-b)^2} + \frac{1}{(1-a-b)^2},$$

a simple computation, skipping terms depending only on b, shows that the effect of $m_1 + m_2$ on the distance between m_3 and m_4 is an additional acceleration of the form $ka^2b(1+o(1)), k > 0$, constant. In fact, this is a particular case following from (6). This acceleration contributes to variations of the energy e_2 as $ka^2b\dot{b}$. Physically, if $\dot{b} < 0$ then the additional (positive) acceleration decreases the kinetic energy without essentially changing the potential one. Hence, the total energy decreases. On the other hand, $\dot{b} > 0$ contributes to increase the energy. We can think of the effect of the first binary on the second one as a series of kicks (always non negative). Each kick produces a jump in velocity, but has a much smaller effect on position. A kick produced when m_3 and m_4 approach (respectively separate) decreases (respectively increases) the energy. If $t_2 > 0$ the period of decrease of the energy is longer. The main contribution to the loss of energy is of the order of magnitude of $a^2b\dot{b} = O(t_2^{5/3})$ during a time $O(t_2)$. This gives a loss of energy $O(t_2^{8/3})$ if $t_2 > 0$ and a similar increase if $t_2 < 0$. This interpretation is nothing else than a restatement of the formula for $\hat{I}_{4,1}$ above.

Finally, from the point of view of complex variable, the integral $\hat{I}_{4,1}$ can be thought as the integral from $-t$ to t (t real positive) along a path turning by 3π around 0 and around t_2 to follow the suitable leaves of the Riemann surface at the branching points. Each one of the turns leaves "inside" only one of the collision times. In particular, the result is robust when the branching points move outside the real axis. This is what happens in planar problems or in problems reducible to 1D when they are considered inside $\mathbb{R}^2$ or $\mathbb{R}^3$ and orbits close to SBC do not pass through binary collisions.

Now we consider problems with a larger number of bodies, having SBC with m couples involved and the possible presence of other, non–colliding, k bodies. Hence $N = 2m + k$. We assume the j–th binary formed by masses m_{2j-1} at $\mathbf{q}_{2j-1}$ and m_{2j} at $\mathbf{q}_{2j}$. Let $\mathbf{Q}_j = \mathbf{q}_{2j} - \mathbf{q}_{2j-1},\ j = 1, \ldots, m$.

We can define again subproblems reducible to 1D if all the $\mathbf{Q}_j(t)$ are of the form $\mathbf{Q}_j(t) = u_j|\mathbf{Q}_j(t)|$, where u_j are fixed unitary vectors, $j = 1, \ldots, m$. This can require some symmetries on the masses and on the positions of the single bodies. As examples, beyond the general collinear problem, we have the multiisosceles problem, where m_1 is at $(-x_1, 0)$, m_2 at $(x_1, 0)$, with $m_2 = m_1$, and other couples are on the y axis. They collide simultaneously: the first couple at $x_1 = 0$ and the other at points of the form $(0, y_k),\ k = 2, \ldots, m,\ y_k \neq 0\ \forall k,\ y_{k_1} \neq y_{k_2}$ if $k_1 \neq k_2$. In that case other isolated bodies can be on the y axis. If they are outside the y axis, then they must appear in pairs with the same mass and should be located at points of the form $(-x_*, y_*), (x_*, y_*)$. The velocities must satisfy similar symmetries.

A 3D example, generalizing both the tetrahedron and the trapezoidal cases, can be obtained with couples of equal masses located at points of the form $(\mp x_k, 0, z_k)$ and of the form $(0, \mp y_j, w_j)$, with all the z_k and w_j taking different values when all the x_k and y_j become zero at the SBC. The reader can imagine easily other nice geometries. The natural extension of theorem 1 is as follows

Theorem 2 *Consider a subproblem of the N–body problem reducible to 1D. Then, any SBC is C^r–regularizable for $r < 8/3$, but not for $r = 8/3$.*

Proof

Initial conditions close to the ones leading to SBC will have binary collisions at times $0, t_2, \ldots, t_m$. Let $s_1 = t^{1/3},\ s_j = (t - t_j)^{1/3}$ for $j = 2, \ldots, m$. For the uncoupled problem one has $|\mathbf{Q}_j^u| = c_j s_j^2 + \ldots$ with c_j depending on the masses of the related binaries. Formulas like (9) allow to define the energies, e_j, for all the binaries, and expressions like (10), with the suitable summations, give the related variations. When we study, for instance, the variation of e_1, integrals of the form $\int t^{1/3}(t - t_j)^{4/3} dt$ appear. They give contributions with terms having a non zero coefficient in front of $\mathrm{sign}(t_j) t_j^{8/3}$. From this the theorem follows. □

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