Numerical study of bifurcations for the 2–D Poiseuille problem

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Abstract

We study the dynamics of two-dimensional Poiseuille flow. Firstly we check our calculations with previous results concerning the laminar solution and the minimum Reynolds number such that it becomes unstable. Next we studied time periodic solutions which, because of the imposed periodicity in the stream-wise direction, are rotating waves, what allow us to treat them as stationary flows in a moving system of reference. We use this fact to obtain also unstable time periodic flows and bifurcation branches from the laminar solution for different values of the wave number. Finally we introduce the case of bifurcation to quasi-periodic solutions.

Poiseuille flow

We consider the flow of a viscous incompressible fluid, in a channel between two parallel walls at $y = \pm 1$, driven by a stream-wise pressure gradient and governed by the dimensionless Navier–Stokes equations

$$\begin{aligned} \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} &= -\frac{\partial p}{\partial x} + \frac{1}{Re}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + G \\ \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} &= -\frac{\partial p}{\partial y} + \frac{1}{Re}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0, \end{aligned} \tag{1}$$

where (u, v) are the components of the velocity, p is the pressure, Re is the Reynolds number and $G = 2/Re$ is the constant pressure gradient. The Reynolds number is defined in terms of the channel half-length and the center velocity of a stationary solution, known as laminar flow. In dimensionless coordinates this laminar solution is written as

$$\bar{u}(y) = 1 - y^2, \quad \bar{v} = 0, \quad \nabla \bar{p} = 0.$$

As boundary conditions we suppose no-slip on the channel walls and period $L = 2\pi/\alpha$ (α is the wave number) in the stream direction x, i.e.

$$\left.\begin{aligned} u(x, \pm 1, t) = v(x, \pm 1, t) = 0 \\ (u, v, p)(x + L, y, t) = (u, v, p)(x, y, t) \end{aligned}\right\} \qquad x \in \mathbb{R}, \quad y \in [-1, 1], \quad t \geq 0.$$

The initial profile of velocities is simply subjected to the incompressibility condition.

We are concerned with the dynamics of Poiseuille flow in varying the parameters Re and α. The stability of the laminar solution has been studied extensively through the literature. For instance, Orszag (see [3]), solved numerically the linearization around the laminar flow of Navier–Stokes equations expressed in terms of the vorticity, what is known as the Orr–Sommerfeld equation. This is an eigenvalue problem for each value of the parameters Re

and α, in such a way that if $c = c_r + ic_i$ is a complex eigenvalue with $c_i > 0$, then the laminar flow is unstable to small disturbances according to linear theory. The neutral stability curve is presented in figure 1.

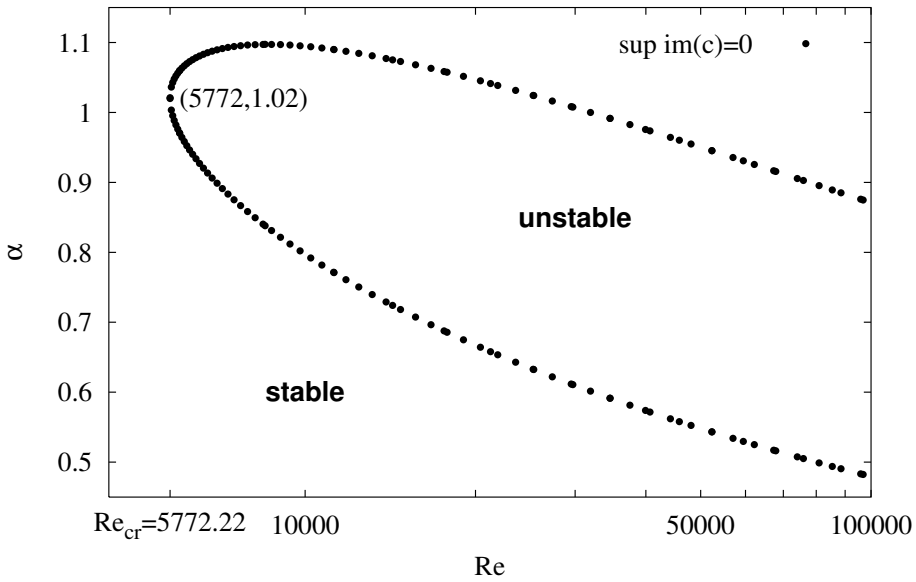

Fig. 1. Neutral stability curve for the laminar solution.

In figure 1, each point in the Re–α plane represents a perturbation of the laminar solution whose stability is decided upon its position. On the neutral stability curve the corresponding perturbation has $c_i = 0$ for the most unstable eigenvalue $c = c_r + ic_i$. In figure 1 we also observe the critical Reynolds number, $Re_{\rm cr} = 5772.22$ for $\alpha = 1.02056$, so that if $Re < Re_{\rm cr}$ the laminar solution is stable for any value of α. We use these results to verify our numerical integrator of system (1). With it, we will follow the temporal evolution of an initial flow close to the laminar solution.

Numerical method

We use a spectral method to approximate velocities u, v and pressure p in (1). Since we suppose u, v, p periodic in x, they are replaced by their truncated Fourier series:

$$(u,v,p)(x,y,t) = \sum_{k=-N}^{N} (\hat{u}_k, \hat{v}_k, \hat{p}_k)(y,t) e^{ik\alpha x}, \qquad x \in \mathbb{R}, \quad y \in [-1,1], \quad t \geq 0,$$

which substituted in system (1) gives a system of PDE for the Fourier coefficients $(\hat{u}_k, \hat{v}_k, \hat{p}_k)$,

$$\begin{aligned}
\frac{\partial \hat{u}_k}{\partial t} + \widehat{\left[u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right]}_k &= -ik\alpha\hat{p}_k + \frac{1}{Re}\left(-k^2\alpha^2\hat{u}_k + \frac{\partial^2 \hat{u}_k}{\partial y^2}\right) + G\delta_{k0} \\
\frac{\partial \hat{v}_k}{\partial t} + \widehat{\left[u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right]}_k &= -\frac{\partial \hat{p}_k}{\partial y} + \frac{1}{Re}\left(-k^2\alpha^2\hat{v}_k + \frac{\partial^2 \hat{v}_k}{\partial y^2}\right) \qquad (2) \\
ik\alpha\hat{u}_k + \frac{\partial \hat{v}_k}{\partial y} &= 0,
\end{aligned}$$

where $\widehat{[\cdot]}_k$ stands for the kth Fourier coefficient. The boundary conditions result in

$$(\hat{u}_k, \hat{v}_k)(\pm 1, t) = 0, \text{for } t \geq 0 \text{ and } -N \leq k \leq N.$$

Because u, v, p are real functions, we just consider modes $0 \leq k \leq N$. To discretize the transversal variable y we make use of the collocation method, in which every equation is imposed at selected collocation points: the Chebyshev abscissas in this case. To this end we suppose $\hat{u}_k(y,t)$ and $\hat{v}_k(y,t)$ expanded as Chebyshev polynomials in y and consider their value on Chebyshev abscissas

$$y_j = \cos(\pi j/M), \quad 1 \leq j \leq M-1,$$

whereas for $\hat{p}_k(y,t)$ we take its value at the staggered grid

$$y_{j+\frac{1}{2}} = \cos(\pi(j+1/2)/M), \quad 0 \leq j \leq M-1.$$

Momentum equations in (2) are enforced on the first grid, while for the continuity equation the second is taken. If we used the same collocation points for the pressure and the continuity equation as for the velocity and momentum equations, then we would obtain an undetermined linear system for the discrete dependent variables (see [2]).

For the temporal discretization, the approximation is made with finite differences using Adams–Bashforth (explicit) for advection (nonlinear terms) and Crank–Nicolson (implicit) for diffusion (pressure and viscosity terms). Once the y and t derivatives in (2) are completely discretized we obtain a iterative scheme, in which it is solved a linear system of dimension $(2N+1)(3M-2)$ per time step.

Periodic solutions

Following the evolution of an initial condition through the numerical scheme of system (2), we are able to test our results with the Re_{cr} computed by Orszag in [3]. Taking $\alpha = 1.02056, Re > Re_{\text{cr}}$ and the laminar solution as initial condition, we observe how this solution is no longer stable, as it evolves to other steady flow which turns out to be periodic in time. Due to the translational symmetry of the channel in x, it is showed in [5] that every periodic function on time $w(x,y,t)$, of period T, is a rotating wave, i.e.

$$w(x,y,t) = w(x-ct, y, 0), \quad c = \frac{L}{T}. \tag{3}$$

Considering a system of reference moving with velocity c in the stream-wise direction, i.e. making the change of variable $\tilde{x} = x - ct$ and writing again x instead of $\tilde{x}$, system (1) becomes

$$\begin{aligned}
\frac{\partial u}{\partial t} + (u-c)\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} &= -\frac{\partial p}{\partial x} + \frac{1}{Re}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + G \\
\frac{\partial v}{\partial t} + (u-c)\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} &= -\frac{\partial p}{\partial y} + \frac{1}{Re}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \\
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0.
\end{aligned} \tag{4}$$

Periodic solutions obtained integrating system (2) are checked (within a reasonable accuracy) to be rotating waves as theory states. A periodic flow on t in the fixed system of reference is viewed as a stationary one in the moving reference. This fact allows us to search for periodic solutions as the functions $(u, v, p)(x, y)$ which solve the stationary equations

$$\begin{aligned}(u-c)\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y} &= -\frac{\partial p}{\partial x}+\frac{1}{Re}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)+G\\ (u-c)\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y} &= -\frac{\partial p}{\partial y}+\frac{1}{Re}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)\\ \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y} &= 0,\end{aligned} \tag{5}$$

Discretizing derivatives as in (2), system (5) becomes a problem of finding zeros of a set of equations which is carried out with Newton's method. As a initial guess we take the approximate periodic solution obtained from the numerical integration of (2).

If we take Re as a continuation parameter, we can trace a curve of periodic flows by solving (5) for each Reynolds number. This is implemented numerically by pseudo-arclength continuation. We notice that zeros of system (5) can either correspond to stable or unstable time periodic solutions. A stable solution is the one for which any small disturbance ultimately decays to zero, whereas if some of those disturbances remain permanently different from zero, it is called unstable.

Given a profile of velocities (u, v) we define its amplitude A, as the distance to the laminar solution $(\bar{u}, 0)$ in the L_2–norm

$$A=\frac{1}{2L}\|(u-\bar{u},v)\|_2, \qquad \|(u,v)\|_2^2=\int_0^L\int_{-1}^1 \left[u(x,y)^2+v(x,y)^2\right]dy\,dx. \tag{6}$$

For a rotating wave as in (3), its amplitude does not depend on time as is easily verified from the definition. The bifurcating diagram for the periodic flows in the Re–A plane is represented in figure 2.

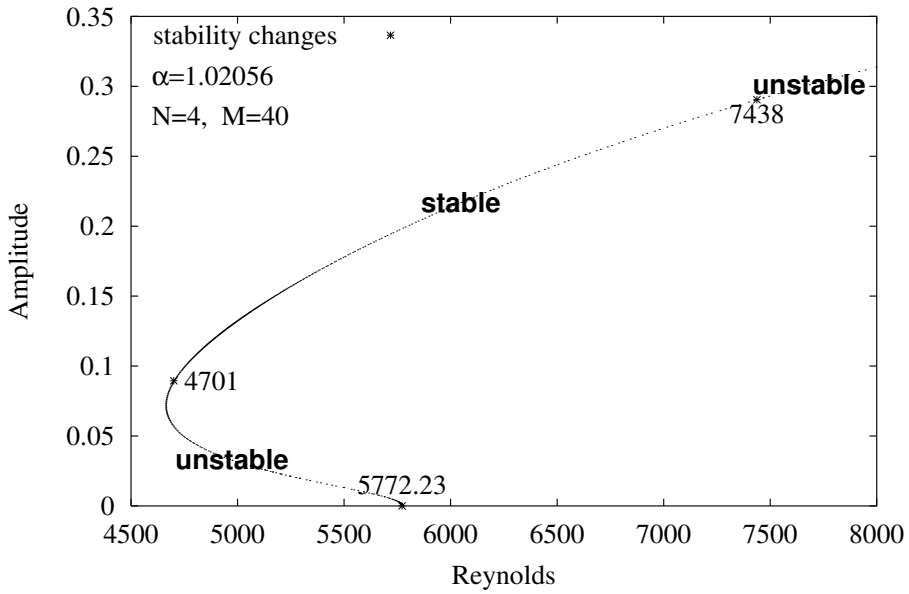

Fig. 2. Bifurcating curve of periodic flows.

To decide whether a time periodic flow (obtained by Newton's method) is stable or not we eliminate the divergence free condition from system (5), in order to get an ODE system from where we can linearize around a steady solution. In the discretized version of (4) (analogous to (2)), we resolve v in terms of u from the divergence equation which, substituted in the second momentum equation, allows us to write pressure simply as a function of u. Finally we may express the first momentum equation as only depending on u,

$$\frac{\partial u}{\partial t} = f(u), \tag{7}$$

being f a linear combination of the three discretized equations in (4). This system of ODE has dimension $(2N+1)(M-2)+1$, roughly one third of system (2).

By means of the real jacobian matrix $Df(u)$ (obtained analytically) around a time periodic solution, we calculate its eigenvalues to decide the stability of this solution, whose changes are represented in figure 2. In the case of the laminar flow, we obtain the classical results of Orszag in [3] about the critical Reynolds being at $Re_{\text{cr}} = 5772.22$. First, the bifurcation curve of periodic flows reaches the laminar solution at the above mentioned Re_{cr} and in addition, the laminar solution is checked to be stable when $Re < Re_{\text{cr}}$ and unstable if $Re > Re_{\text{cr}}$. In figure 2 we observe $Re_{\text{cr}} = 5772.23$ which was refined by increasing resolution up to $M = 64$ with $N = 4$, where we also obtained $Re_{\text{cr}} = 5772.22$.

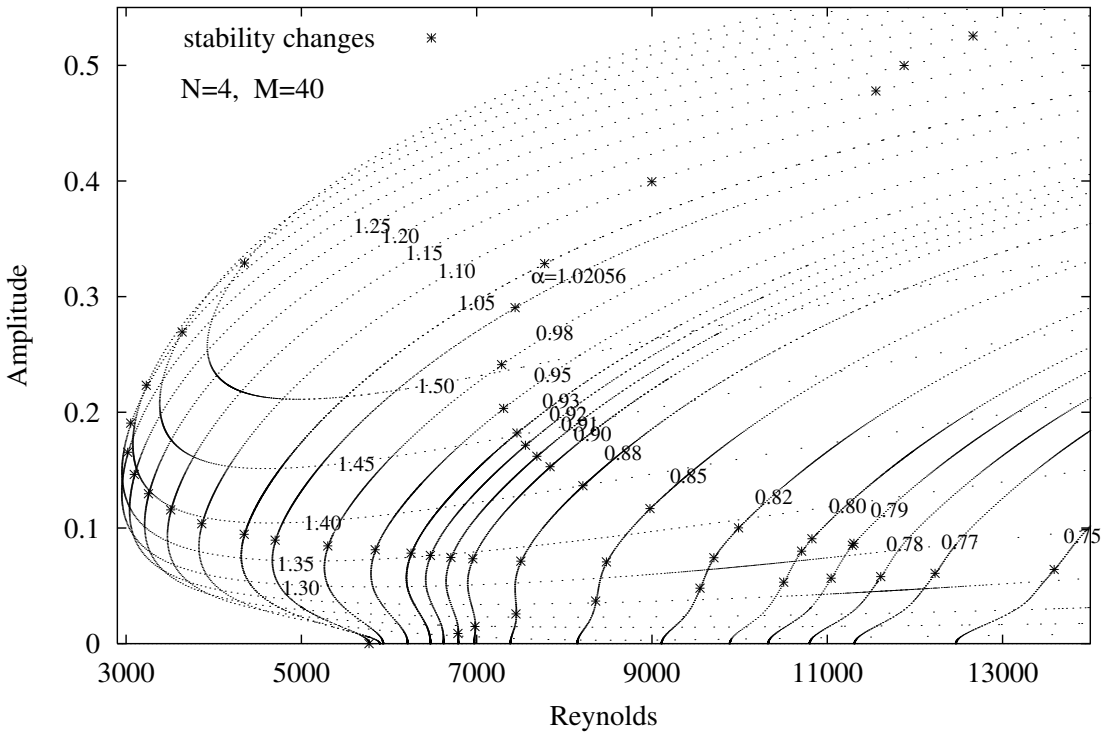

Fig. 3. Bifurcating curves for several values of α.

Now let us discuss the bifurcation diagram shown in figure 2. First, the laminar solution becomes unstable at $Re_{\text{cr}} = 5772.23$ due to a Hopf bifurcation, that gives rise to a unstable family of periodic orbits. This family continues backwards (with respect to the Reynolds number) until $Re \approx 4660$, when a turning point is reached. Before this turning point, there

is a single eigenvalue of $Df(u)$ on the real positive axis, while the remaining eigenvalues have negative real part. When going through the turning point, a real and negative eigenvalue becomes real positive, so the number of unstable eigenvalues is now two. Shortly after that, these two unstable eigenvalues collide and become a conjugate complex pair (still with positive real part), and then they cross the imaginary axis for $Re \approx 4700$ producing a new Hopf bifurcation. Between this value and $Re \approx 7400$, the family of periodic orbits is stable. For this last value, there is (again) a Hopf bifurcation produced by a conjugate pair of eigenvalues crossing the imaginary axis. These bifurcations persist when increasing the number of Chebyshev modes M, and no new bifurcations seems to appear.

Previously Soibelman & Meiron (see [6]) found analogous bifurcations for $\alpha = 1.1$, and the critical Reynolds number for which there are time-periodic solutions: $Re \approx 2900$ for $\alpha \approx 1.3$. In figure 3 this minimum Reynolds number is verified and different bifurcation curves are calculated for several values of α, together with its stability. Likewise, in this plot is observed that for $\alpha = 1.02056$, is attained the minimum Reynolds number for the laminar flow where takes place the transition from stable to unstable solution, what was formerly calculated in figure 1. We also notice in figure 3 that there exists an attractor periodic solution for $Re \approx 3000$.

Quasi-periodic solutions

As we did previously with an unstable laminar solution to find other class of attractor flows (the periodic solutions), we take now as an initial condition, an unstable periodic flow and follow its temporal evolution using the spectral discretization of (7). To start with, we fix $\alpha = 1.02056$ and take $Re \approx 7500$, for which the corresponding periodic flow is unstable. After some steps of numerical integration, the solution seems to fall in a regular régime. Then, in figure 4, we plot the projection of the solution vector

$$u(t) = (u_1, \ldots, u_K)(t), \quad K = (2N+1)(M-2)+1,$$

over the plane of two arbitrary coordinates (the 120^{th} and 232^{th} to be precise).

Each point in this figure represents the value of the specified coordinates at a time instant. As we can observe, the two represented coordinates fill densely the projection of a torus on the plane. Taking an *ad hoc* Poincaré section (when a coordinate crosses 0 from negative to positive) of the space of solutions $u(t)$ and plotting the same coordinates as above, but only when crossing the Poincaré section, we see in figure 5 a closed curve, which confirm us that our flow lives in a torus, that is, we have found a quasi-periodic solution.

In [5] it is proved that, with the translational symmetry of the channel, every solution $u(x, y, t)$ that lies on an isolated invariant 2-torus, not asymptotic to a rotating wave, is a modulated wave, that is, exists $\tau > 0$ and $\phi \in \mathbb{R}$ such that for every integer n we have

$$u(x, y, n\tau) = u(x + n\phi, y, 0).$$

As a consequence, this kind of wave has the property that, may be viewed as a periodic wave of period τ in a frame of reference moving with speed

$$c = \frac{pL - \phi}{\tau}, \quad \text{for any integer } p,$$

as is easily proved. The test of our quasi-periodic solution using this result and a more detailed analysis of the bifurcation curve of quasi-periodic solutions together with its stability, is actually a working progress.

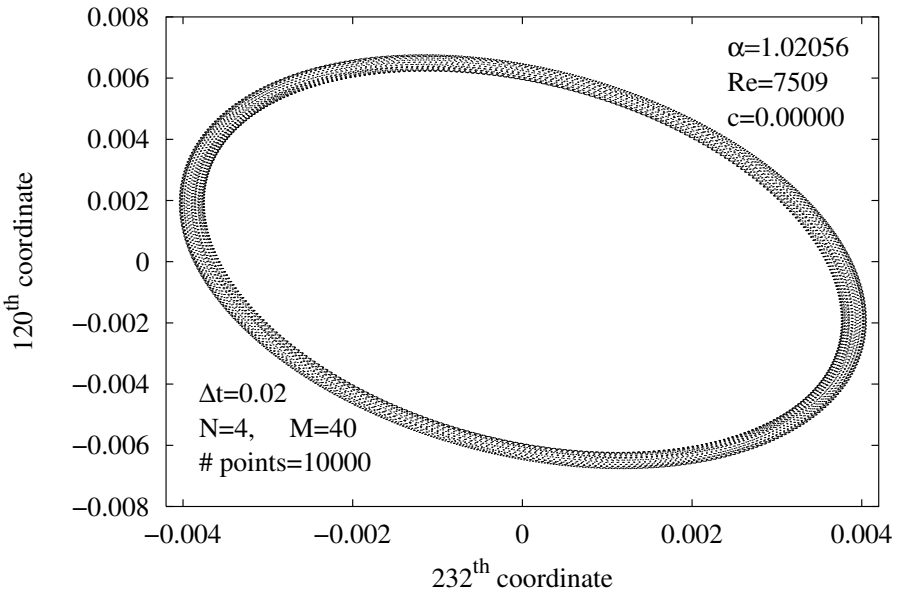

Fig. 4. Projection of the solution vector each Δt units of time.

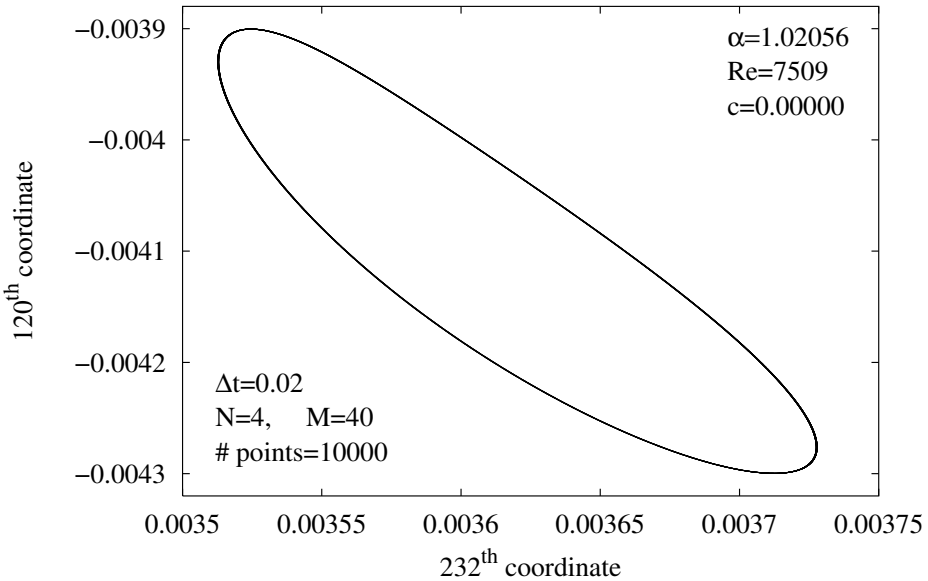

Fig. 5. Projection of the solution vector on the Poincaré section.

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