Reducible linear quasi–periodic systems with positive Lyapunov exponent and varying rotation number

H.W. Broer * and C. Simó †

Abstract

A linear system in two dimensions is studied. The coefficients are 2π–periodic on three angles, θ_j, $j = 1, 2, 3$, and these angles are linear with respect to time, with incommensurable frequencies. The system has positive Lyapunov coefficients and the rotation number changes in a continuous way when some parameter moves. A lift to $\mathbb{T}^3 \times \mathbb{R}^2$, however, is only of class L^p, for any $p < 2$.

Consider a class of linear differential equations

$$\dot{M} = A(t)M, \ t \in \mathbb{R}, \tag{1}$$

where $M \in Sl(2, \mathbb{R})$, $A(t) \in sl(2, \mathbb{R})$ and $\dot{} = d/dt$. Corresponding evolutions frequently occur as fundamental solutions of second order equations of Hill (or Schrödinger) type

$$\ddot{x} + q(t)x = 0. \tag{2}$$

Generically also the converse is true: the equation (1) can be associated to a second order equation of the form (2).

1 Preliminaries

Presently we consider an example of this where

$$M(t; t_0) = C(t) \begin{pmatrix} e^{\lambda t} & 0 \\ 0 & e^{-\lambda t} \end{pmatrix} C(t_0)^{-1}, \tag{3}$$

for $\lambda > 0$, where $C(t) \in SO(2, \mathbb{R})$. First note that this flow is reducible, in the sense that the transformation C renders it into a flow generated by the constant matrix

$$\begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}.$$

Let us further specify

$$C(t) = \begin{pmatrix} v_1(t) & -v_2(t) \\ v_2(t) & v_1(t) \end{pmatrix} \text{ with } v_1^2 + v_2^2 \equiv 1. \tag{4}$$

*Dept. of Mathematics and Computer Science, University of Groningen, PO Box 800, 9700 AV Groningen, The Netherlands

†Dept. de Matemàtica Aplicada i Anàlisi, Universitat de Barcelona, Gran Via 585, 08007 Barcelona, Spain

Lemma 1 *The flow (3) is generated by the linear differential equation (1) where*

$$A = \begin{pmatrix} \lambda(v_1^2 - v_2^2) & 2\lambda v_1 v_2 - w \\ 2\lambda v_1 v_2 + w & \lambda(v_2^2 - v_1^2) \end{pmatrix},$$

where $w = v_1 \dot v_2 - v_2 \dot v_1$ *and, hence,* $|w| = |\dot v| = ((\dot v_1)^2 + (\dot v_2)^2)^{1/2}$.

Remarks.

1. Since $\operatorname{Tr} A = 0$, it follows that $A \in sl(2,\mathbb{R})$.
2. By [7] it follows that the function $v(t)$ and hence both the coefficient matrix $A(t)$ and the transformation $C(t)$ are almost automorphic.
3. Even in the quasi–periodic coefficients case, the spirit of Johnson's Floquet theory is such that one should generally expect almost automorphic reducibility rather than almost periodic reducibility (see also [5]). The present example, with almost automorphic coefficients, also meets with this expectation.

Proof. Differentiation of (3) gives

$$\begin{aligned} \dot M(t;t_0) &= \left[\dot C \begin{pmatrix} e^{\lambda t} & 0 \\ 0 & e^{-\lambda t} \end{pmatrix} + C \begin{pmatrix} \lambda e^{\lambda t} & 0 \\ 0 & \lambda e^{-\lambda t} \end{pmatrix}\right] C(t_0)^{-1} \\ &= \left[\dot C + C \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}\right] \begin{pmatrix} e^{\lambda t} & 0 \\ 0 & e^{-\lambda t} \end{pmatrix} C(t_0)^{-1}. \end{aligned}$$

Using the explicit form (4) of C we so obtain

$$\dot M = \begin{pmatrix} \dot v_1 + \lambda v_1 & -\dot v_2 + \lambda v_2 \\ \dot v_2 + \lambda v_2 & \dot v_1 - \lambda v_1 \end{pmatrix} \begin{pmatrix} e^{\lambda t} & 0 \\ 0 & e^{-\lambda t} \end{pmatrix} C(t_0)^{-1}.$$

Next use that, by construction,

$$\begin{pmatrix} e^{\lambda t} & 0 \\ 0 & e^{-\lambda t} \end{pmatrix} = C^{-1}(t) M(t;t_0) C(t_0),$$

so to obtain

$$\dot M = \begin{pmatrix} \dot v_1 + \lambda v_1 & -\dot v_2 + \lambda v_2 \\ \dot v_2 + \lambda v_2 & \dot v_1 - \lambda v_1 \end{pmatrix} \begin{pmatrix} v_1 & v_2 \\ -v_2 & v_1 \end{pmatrix} M.$$

It only remains to determine the coefficient matrix $A = A(t)$ as the product

$$A = \begin{pmatrix} \dot v_1 + \lambda v_1 & -\dot v_2 + \lambda v_2 \\ \dot v_2 + \lambda v_2 & \dot v_1 - \lambda v_1 \end{pmatrix} \begin{pmatrix} v_1 & v_2 \\ -v_2 & v_1 \end{pmatrix}.$$

Using that by (4) we have $v_1^2 + v_2^2 \equiv 1$ and therefore both

$$v_1 \dot v_1 + v_2 \dot v_2 = 0 \text{ and } |\dot v_2 v_1 - \dot v_1 v_2| = |\dot v|,$$

we immediately arrive at the desired form. □

Lemma 2 *1. For fixed t, the map $M(t;t_0)$, as an area preserving map of $\mathbb{R}^2$, has Lyapunov exponent λ.*

2. If $C(t)$ rotates with average angular velocity ρ, then the rotation number of $M(t;t_0)$ exists and is equal to ρ.

The proof of both claims is elementary.

2 Laplace construction of C

We now further specify the matrix $C = C(t)$, using a construction to study the well known Laplace problem on the mean motion of the perihelia of the planets, compare [1, 3]. We describe this specification in a number of steps.

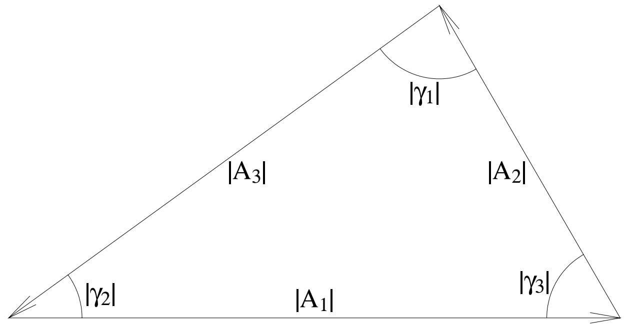

Figure 1: The triangle with sides $|A_1|$, $|A_2|$ and $|A_3|$ and corresponding opposite angles $|\gamma_1|$, $|\gamma_2|$ and $|\gamma_3|$.

1. One ingredient consists of three complex numbers A_1, A_2 and A_3, such that for all permutation, (j, k, ℓ), of the set $\{1, 2, 3\}$ one has

$$|A_j| < |A_k| + |A_\ell|.$$

This condition simply means that $|A_1|$, $|A_2|$ and $|A_3|$ can serve as the lengths of the sides of a triangle. Let $|\gamma_1|$, $|\gamma_2|$ and $|\gamma_3|$ be the corresponding opposite angles in this triangle, see figure 1. Also let φ_j be the argument $\arg A_j$, for $j = 1, 2, 3$.

2. Next we need three real numbers (frequencies) ω_1, ω_2 and ω_3, such that for all $k \in \mathbb{Z}^3 \setminus \{0\}$ one has $k_1\omega_1 + k_2\omega_2 + k_3\omega_3 \neq 0$. Consider the sum

$$S_{\varphi;\omega}(t) = \sum_{j=1}^{3} A_j e^{i\omega_j(t-t_0)},$$

depending on the initial phases φ. Then there is a 'bad' set of initial phases

$$B_\omega := \{\varphi \in \mathbb{T}^3 \mid \exists t \neq t_0 : S_{\varphi;\omega}(t) = 0\}$$

which is dense in $\mathbb{T}^3$ with measure 0.

3. The definition of $C = C(t)$ now is completed as follows. Writing $v = v_1 + iv_2$ we identify $\mathbb{R}^2 \approx \mathbb{C}$. Then, for $\varphi \notin B_\omega$, we specify $C(t)$ by choosing

$$v_{\varphi;\omega}(t) := \frac{S_{\varphi;\omega}(t)}{|S_{\varphi;\omega}(t)|}, \tag{5}$$

By the Laplace result it follows that $v(t)$ then rotates with average speed

$$\rho = \frac{1}{\pi} \sum_{j=1}^{3} \gamma_j \omega_j.$$

3 Study of the singularities of the example

We have constructed a linear differential equation with t-dependent coefficients, as a $Sl(2,\mathbb{R})$ matrix equation (1) with an equivalent as a second order ordinary differential equation of Hill format (2). The coefficients clearly are real analytic in t, provided $\varphi \notin B_\omega$. Standard operation procedure would now be to lift $t \in \mathbb{R}$ over $\theta \in \mathbb{T}^3$, writing $A(t) = \mathcal{A}(t\omega)$, or equivalently, $q(t) = \mathcal{Q}(t\omega)$. In reversed order:

$$\begin{array}{rcl} \dot{\theta} &=& \omega, \\ \dot{x} &=& y, \\ \dot{y} &=& -\mathcal{Q}(\theta)x, \end{array} \qquad \begin{array}{rcl} \dot{\theta} &=& \omega, \\ \dot{M} &=& \mathcal{A}(\theta)M, \end{array} \tag{6}$$

where $\theta = \varphi$ at $t = t_0$.

What is the problem with this? For $\theta \in B_\omega \subset \mathbb{T}^3$, the coefficients $\mathcal{A}$ and $\mathcal{Q}$ are not defined. So at the level of Cauchy problems, equation (1) is well–defined for $\varphi \notin B_\omega$. But, in this context, usually there is a continuum of lifts like (6) projecting down to a Cauchy problem for the original system. Presently, however, this is not the case.

3.1 Regularity

The question is how regular $\mathcal{A}$ (or $\mathcal{Q}$) is as a function on $\mathbb{T}^3$. It is surely not continuous.

Lemma 3 *For any $p < 2$, $\mathcal{A}$ belongs to $L^p(\mathbb{T}^3, sl(2,\mathbb{R}))$.*

Remark. According to the Veech structure theorem, see [6], it moreover can be shown that $\mathcal{A} \in C(X, sl(2,\mathbb{R}))$, where X is a fractal 3–torus which an almost 1–cover of $\mathbb{T}^3$.

Proof. We first reduce everything to $\mathbb{T}^2$, by introducing a co–rotating frame. Indeed, replacing θ_1 by $\theta_1 - \omega_1(t-t_0) - \varphi_1$, fixes A_1 to the real axis, while for $j = 2,3$ the point A_j moves with angular velocity $\omega_j - \omega_1$. Then

$$S_{\varphi;\omega}(t) = 0 \Longleftrightarrow \begin{array}{rclcl} \psi_2 &:=& \varphi_2 - \varphi_1 + (\omega_2-\omega_1)(t-t_0) &=& \pi - |\gamma_3| \bmod (2\pi) \text{ and} \\ \psi_3 &:=& \varphi_3 - \varphi_1 + (\omega_3-\omega_1)(t-t_0) &=& \pi + |\gamma_2| \bmod (2\pi), \end{array}$$

that is, in the co–rotating frame, the situation displayed in the figure 1 has to be avoided. Also the mirror–symmetric case with respect to an horizontal line and, hence, $\psi_2 = \pi + |\gamma_3| \bmod (2\pi)$ and $\psi_3 = \pi - |\gamma_2| \bmod (2\pi)$, has to be avoided.

This gives us two points p^*, q^* on the 2–torus $\{\psi_2, \psi_3\}$, which should be missed by the linear flow with velocity

$$\begin{pmatrix} \omega_2 - \omega_1 \\ \omega_3 - \omega_1 \end{pmatrix},$$

which also describes the bad set B_ω in these terms.

We analyze the situation close to the point p^*, that is, in the situation depicted in the figure 1. The other case is similar. Shifting the origin of time, we can suppose we are in this situation for $t - t_0 = 0$, and denote the new time again as t. As in the co–rotating frame we have

$$S_{\varphi;\omega}(t) = |A_1| + |A_2| e^{i((\omega_2-\omega_1)t+\pi-|\gamma_3|)} + |A_3| e^{i((\omega_3-\omega_1)t+\pi+|\gamma_2|)}, \tag{7}$$

the differentiation of (7) at $t=0$ gives a vector,

$$i(\omega_2-\omega_1)|A_2|e^{i(\pi-|\gamma_3|)}+i(\omega_3-\omega_1)|A_3|e^{i(\pi+|\gamma_2|)},$$

which is different from zero. This gives an open transversality condition.

A suitable deformation and scaling of time gives that the vector $S_{\varphi;\omega}(t)$ behaves, locally at p^*, as

$$S_{\varphi;\omega}(t)=\left(\begin{array}{c}\psi_2=t\\ \psi_3=c\end{array}\right),$$

where c is constant with respect to t and we have used again $\{\psi_2,\psi_3\}$ as local coordinates in the 2–torus around p^*. Therefore, we get for the vector v the expression

$$v(t)=\left(\begin{array}{c}t(t^2+c^2)^{-1/2}\\ c(t^2+c^2)^{-1/2}\end{array}\right). \qquad (8)$$

This gives immediately the value $w=-c(t^2+c^2)^{-1}$. Going back to the $\{\psi_2,\psi_3\}$ variables, we see that the 'worst' term in $\mathcal{A}(\theta)$ is of the form

$$\frac{\psi_3}{\psi_2^2+\psi_3^2}, \qquad (9)$$

the remaining contributions being bounded (and even analytic everywhere except at p^*). The function in (9) is clearly in L^p for any $p<2$, as it is immediate to check by looking for integrability in polar coordinates around $(\psi_2,\psi_3)=(0,0)$.

To go back to $\mathbb{T}^3$, we just note that in any leave, corresponding to any argument of $A_1e^{i(\omega_1(t-t_0))}$, we have exactly the same situation, and then we apply Fubini's theorem. $\Box$

3.2 Summary

We have a linear system with quasi–periodic coefficients of 3 frequencies, that is reducible, has positive Lyapunov exponent and varying rotation number. However, its lift to $\mathbb{T}^3\times Sl(2,\mathbb{R})$, or $\mathbb{T}^3\times\mathbb{R}^2$, only is of class L^p for any $p<2$.

One often considers the corresponding Schrödinger operator

$$\frac{d^2}{dt^2}+Q(\varphi+\omega t),$$

studying its spectral properties. Compare [4] for a discrete analogue with 2 frequencies. For the present example these spectral properties amount to hyperbolicity and exponential dichotomy.

Acknowledgements

The authors thank Russell Johnson and Yingfei Yi for their valuable comments on this work. The research of the second author has been supported by DGICYT grant PB 94–0215 (Spain). Partial support of the EC grant ERBCHRXCT940460, and the catalan grant CIRIT 1998S0GR-00042 is also acknowledged.

References

[1] Arnol′d, V.I.: *Mathematical Methods of Classical Mechanics*. Springer–Verlag 1980.

[2] Arnol′d, V.I.: *Geometrical Methods in the Theory of Ordinary Differential Equations*. Springer–Verlag 1983.

[3] Arnol′d, V.I. and Avez, A.: *Ergodic Problems of Classical Mechanics*. Benjamin 1968.

[4] Chulaevsky, V. and Sinai, Ya.: Anderson localization for the 1-D discrete Schrödinger operators with two–frequency potential. *Comm. Math, Phys.*, **125** (1989), 91–112.

[5] Sell, G.R., Shen, W. and Yi, Y.: Toplogical dynamics and differential equations. In: Topological Dynamics and Applications, *Contempory Math.* **215**, (1997), 279–297

[6] Shen, W. and Yi, Y.: Almost automorphic and almospt periodic dynamics in skew–product semiflows. *Mem. Amer. Math. Soc.* **136** (647), (1998), 93 pp.

[7] Veech, W.A.: On a theorem of Bochner, *Ann. Math.* (2), **86**, (1967), 117–137.