Determination of an exact symplectomorphism from its primitive function.

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Abstract. To an exact symplectomorphism on an exact symplectic manifold we can associate a *primitive function*, which is a primitive of a certain closed (in fact, exact) 1-form. This function is also known by many authors as generating function, but as we remark, this function does not generate our symplectomorphism. In fact, the primitive function generates a family of symplectomorphisms and we need some additional information in order to determine one of them.

We can relate this *determination problem* with the *interpolation problem*, that is to say, to get a time-dependent Hamiltonian whose 1-time flow be our symplectomorphism. As we shall see, these problems are related with a derivation on the Lie algebra of functions (endowed with the Poisson bracket). We shall need to integrate respect to this derivative.

Our exact symplectic manifold will be the *cotangent bundle* of a manifold, endowed with the canonical symplectic form given by the differential of the *Liouville form*. We shall suppose that our exact symplectomorphism fixes the zero-section. We shall work in the analytic category.

1 Setting the problems.

First of all, we are going to introduce the elementary definitions and results about exact symplectomorphisms and Hamiltonian flows. Then, we shall state the two main problems of this paper: the *determination problem* and the *interpolation problem*.

For the sake of simplicity, all the objects (manifolds, functions, diffeomorphisms, ...) will be, at least, C^∞. See the end of this paper for a summary of definitions on Differential Geometry.

1.1 Exact symplectic manifolds.

In accordance with the standard definitions, a *symplectic structure* on a manifold $\mathcal{N}$ is given by a differential 2-form ω verifying the following two properties:

- $\forall z \in \mathcal{N}$ $\omega_z$is non-degenerate ($\forall X_z \in T_z\mathcal{N} \setminus \{0\}$ $\exists Y_z \in T_z \mid \omega_z(X_z, Y_z) \neq 0$),
- ω is closed ($\mathrm{d}\omega = 0$).

We say that $(\mathcal{N}, \omega)$ is a *symplectic manifold* and that ω is a *symplectic form*. The non-degeneracy condition implies that $\mathcal{N}$ has even dimension ($\dim \mathcal{N} = 2d$) and the map

$$\begin{array}{rccl} \omega^\flat: & T\mathcal{N} & \to & T^*\mathcal{N} \\ & X & \to & {}^\flat X = -\mathbf{i}_X\omega \end{array}$$

is an isomorphism of vector bundles (its inverse is denoted by $\omega^\sharp$: ${}^\flat X = \rho \Leftrightarrow {}^\sharp\rho = X$). If, moreover, ω is exact ($\omega = \mathrm{d}\alpha$, for some Pfaffian form α on $\mathcal{N}$), we say that $(\mathcal{N}, \omega = \mathrm{d}\alpha)$ is an *exact symplectic manifold* and α is an *action form*.

The most important examples of symplectic manifolds correspond to exact symplectic manifolds, but there are many non-exact symplectic manifolds as, for instance, any orientable and compact surface with the symplectic 2-form given by the oriented area.

Examples:

1) The standard symplectic structure on $\mathbb{R}^{2d} = \mathbb{R}^d \times \mathbb{R}^d$, endowed with the position-momentum coordinates $(x, y) = (x_1, \dots, x_d, y_1, \dots, y_d)$, is given by $\omega = dy \wedge dx = \sum_{i=1}^d dy_i \wedge dx_i$. It is exact, with $\alpha = y\ dx = \sum_{i=1}^d y_i\ dx_i$. This is a local model of all the symplectic manifolds of dimension $2d$, according to *Darboux's Theorem* (see, for instance, [4], p. 463). That is to say, if $(\mathcal{N}, \omega)$ a symplectic manifold and $z \in \mathcal{N}$, there is a local coordinate chart about z, given by (x, y) (the *canonical coordinates* or *symplectic coordinates*), in which $\omega = dy \wedge dx$.

2) In classical mechanics, the cotangent bundle (*phase space*) of a manifold $\mathcal{M}$ (*configuration space*) is the more celebrated example [1]. The dimension of $\mathcal{M}$ is known as the number of degrees of freedom, d. Recall that the differentiable structure on $T^*\mathcal{M}$ is defined by means of the *cotangent charts* $\mathcal{U} \times \mathbb{R}^d$, where each $\mathcal{U}$ is a chart of $\mathcal{M}$. We write the corresponding coordinates as (x, y).

   In order to define an exact symplectic structure on $T^*\mathcal{M}$ we firstly define a 1-form α (called the *Liouville form*) on each 'point' $\rho_x \in T^*\mathcal{M}$ such that for any $X_{\rho_x} \in T_{\rho_x}T^*\mathcal{M}$

   $$\alpha_{\rho_x}(X_{\rho_x}) = \rho_x(q_*(\rho_x)X_{\rho_x}),$$

   where $q : T^*\mathcal{M} \to \mathcal{M}$ is the standard projection. Moreover, α is the unique Pfaffian form on $T^*\mathcal{M}$ which satisfies $\rho^*\alpha = \rho$ for any 1-form ρ on $\mathcal{M}$. Then, $\omega = \mathrm{d}\alpha$ is the *canonical* symplectic structure on $T^*\mathcal{M}$.

   In cotangent coordinates $T^*\mathcal{M}$, $(x, y) \in \mathcal{U} \times \mathbb{R}^d$, these forms are:

   $$\alpha = y\ dx,\ \ \omega = dy \wedge dx.$$

   Notice that the Liouville form vanishes on the zero-section $z : \mathcal{M} \to T^*\mathcal{M}$, which is represented in cotangent coordinates by $\{y = 0\}$. $\triangleleft$

1.2 Exact symplectomorphisms.

Let $F : \mathcal{N} \to \mathcal{N}$ be a diffeomorphism, where $(\mathcal{N}, \omega = \mathrm{d}\alpha)$ is an exact symplectic manifold. We say that

- F is a *symplectomorphism* iff $F^*\omega = \omega$ (and $F^*\alpha - \alpha$ is a closed 1-form);
- F is an *exact symplectomorphism* iff $F^*\alpha - \alpha$ is exact;
- F is an *actionmorphism* iff $F^*\alpha = \alpha$.

Definition: Given a symplectomorphism F, we shall refer to the first-order partial differential equation on $\mathcal{N}$

$$F^*\alpha - \alpha \;\; = \;\; \mathrm{d}S$$

as the **exactness equation** of F. The necessary condition of existence of solutions of this equation is satisfied.

Definition: If F is exact the exactness equation is solvable, that is, $F^*\alpha - \alpha = \mathrm{d}S$ for a certain function S, and we shall say that S is a **primitive function** of F. Although S is defined up to constants, we shall write $\mathrm{pf}(F) = S$.

Remarks:

i) We can do the definitions with different symplectic manifolds, and the maps need to be immersions. The definitions are similar.

ii) (Exact) symplectomorphisms are also called *(globally) canonical transformations*, while actionmorphisms are also known by *homogeneous canonical transformations* or *Mathieu transformations* (see [3], p.181).

iii) In the literature, the primitive function is often called generating function. As we shall see, this function does not generate the symplectomorphism. This is the reason we have followed the nomenclature used in [2]. The primitive function and the different kinds of generating functions are related by certain p.d.e..

iv) Exact symplectomorphism are closed by composition and inversion. Given two exact symplectomorphisms $F, G : \mathcal{N} \to \mathcal{N}$, with $\mathrm{pf}(F) = S$ and $\mathrm{pf}(G) = T$, we have the next composition formulae:

- $\mathrm{pf}(G \circ F) = S + T \circ F$,
- $\mathrm{pf}(F^{-1}) = -S \circ F^{-1}$,
- $\mathrm{pf}(G^{-1} \circ F \circ G) = S \circ G + T - T \circ G^{-1} \circ F \circ G$.

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Examples:

1) Let $F = (f, g) : \mathbb{R}^{2d} \to \mathbb{R}^{2d}$ be a diffeomorphism given by

$$\left\{ \begin{array}{l} \bar{x} = f(x, y), \\ \bar{y} = g(x, y). \end{array} \right.$$

F is a symplectomorphisms iff $\forall z \in \mathbb{R}^{2d}$ $\mathrm{D}F(z)^\top J \mathrm{D}F(z) = J$, where J is the symplectic matrix

$$J = \begin{pmatrix} 0 & I_d \\ -I_d & 0 \end{pmatrix}$$

($^\top$ denotes the transpose of a matrix and I_d is the d-identity matrix). The condition $F^*\alpha - \alpha = \mathrm{d}S$, where $S : \mathbb{R}^{2d} \to \mathbb{R}$, is translated to the *exactness equations*:

$$\left\{ \begin{array}{l} \dfrac{\partial S}{\partial x}(x, y) = g(x, y)^\top \dfrac{\partial f}{\partial x}(x, y) - y^\top, \\ \\ \dfrac{\partial S}{\partial y}(x, y) = g(x, y)^\top \dfrac{\partial f}{\partial y}(x, y). \end{array} \right.$$

(we consider the vectors as columns). We can check that, if F is symplectic, the conditions of local resolvability of this Pfaffian system are satisfied (thanks to the topological properties of $\mathbb{R}^{2d}$ these conditions are also of global resolvability).

Now, we wonder if we can recover f and g from a function S. Obviously, this is not possible if we have not more information, because we have to solve a system of p.d.e.. For example, if $S = 0$, and $f(x, y) = \phi(x)$, then the first equation gives

$$g(x, y) = \mathrm{D}\phi(x)^{-\top} y$$

(where $^{-\top}$ means the transpose of the inverse), and we obtain many exact symplectomorphisms with null (in fact, constant) primitive function. They are known as *lifts*.

2) Previous example can be generalized to any cotangent bundle $T^*\mathcal{M}$ of a manifold $\mathcal{M}$, where we consider the canonical exact symplectic structure given by the differential of the Liouville form α. So, if $\phi : \mathcal{M} \to \mathcal{M}$ is a diffeomorphism on the base $\mathcal{M}$ then $\hat{\phi} : T^*\mathcal{M} \to T^*\mathcal{M}$ defined by

$$\hat{\phi} = (\phi^{-1})^*$$

preserves not only the symplectic form $\omega = \mathrm{d}\alpha$ but also the Liouville form. In fact, all the actionmorphisms on the whole cotangent bundle can be obtained in this way (see, for instance [11]).

In fact, we can improve this well known result by saying that any actionmorphism defined from an open tubular neighbourhood of the zero-section $\mathcal{N}_1$ onto another one $\mathcal{N}_2$ is the lift of a diffeomorphism on the base. In some sense, this is equivalent to say that the only homogeneous functions of degree 1 that are regular in the origin are the lineal ones. $\triangleleft$

The determination problem. As the primitive function does not determine our exact symplectomorphism, we wonder about

> What additional information do we need to determine an exact symplectomorphism?

1.3 Hamiltonian vector fields.

Let $(\mathcal{N}, \omega = \mathrm{d}\alpha)$ be an exact symplectic manifold.

Definition: It is well known that to a function $H : \mathcal{N} \to \mathbb{R}$ we associate a vector field

$$X_H = \ ^{\sharp}\mathrm{d}H,$$

called the **Hamiltonian vector field** of **Hamiltonian function** H. So then, X_H is uniquely determined by $\mathbf{i}_{X_H}\omega = -\mathrm{d}H$.

Definition: The **Poisson bracket** between two functions K,H is defined by

$$\{K, H\} = \omega(X_K, X_H) = -\mathrm{d}K(X_H) = \mathrm{d}H(X_K).$$

Recall that these operations are natural with respect to pull back by symplectomorphisms, that is, if $F : \mathcal{N} \to \mathcal{N}$ is a symplectomorphism then:

- $F^*X_H \stackrel{\text{def}}{=} (F^{-1})_* X_H \circ F = X_{H \circ F}$,
- $\{K, H\} \circ F = \{K \circ F, H \circ F\}$.

In symplectic coordinates (x, y), we write

$$X_H = \begin{pmatrix} \dfrac{\partial H}{\partial y} & -\dfrac{\partial H}{\partial x} \end{pmatrix}^{\top}, \ \{K, H\} = \frac{\partial K}{\partial y} \cdot \frac{\partial H}{\partial x} - \frac{\partial K}{\partial x} \cdot \frac{\partial H}{\partial y},$$

where $\cdot$ is the inner product.

Let H_t be a time-dependent Hamiltonian function (where the subscript t means the dependence on time), and X_{H_t} be the corresponding time-dependent Hamiltonian vector field.

It is well known that the time-t flow from t_0, φ_{t,t_0}, is an exact symplectomorphism (for the sake of simplicity, we shall suppose completeness,that is, the flow is defined for all the values of t_0 and t). Recall the proof.

Proof: Applying an elementary result about time-dependent vector fields and forms (see, for instance, [4] p. 307), we have

$$\frac{d}{dt}(\varphi^*_{t,t_0}\alpha) = \varphi^*_{t,t_0}\mathbf{L}_{X_{H_t}}\alpha = \varphi^*_{t,t_0}(\mathbf{i}_{X_{H_t}}\mathrm{d}\alpha + \mathrm{d}\mathbf{i}_{X_{H_t}}\alpha) = \mathrm{d}\varphi^*_{t,t_0}(\mathbf{i}_{X_{H_t}}\alpha - H_t).$$

Then

$$\phi^*_{t,t_0}\alpha - \alpha = \mathrm{d}\int_{t_0}^{t}(\mathbf{i}_{X_{H_s}}\alpha - H_s)\circ\varphi_{s,t_0}\ ds$$

and, finally, the primitive function of φ_{t,t_0} is

$$S_{t,t_0} \quad = \quad \int_{t_0}^{t}(\mathbf{i}_{X_{H_s}}\alpha - H_s)\circ\varphi_{s,t_0}\ ds.$$

□

We summarize this result in the following proposition.

Proposition 1 *Let H_t be a time-dependent Hamiltonian function and φ_{t,t_0} be the corresponding flow (that we suppose complete). Then, the time-t flow from t_0, φ_{t,t_0}, is an exact symplectomorphism whose primitive function is*

$$S_{t,t_0} = \int_{t_0}^{t}\mathbf{\Lambda}(H_s)\circ\varphi_{s,t_0}\ ds,$$

where $\mathbf{\Lambda}$ is the operator on $\mathcal{F}(\mathcal{N})$ given by $\mathbf{\Lambda}(H) = \alpha(X_H) - H$.

Remarks:

i) If we want to compute numerically the primitive function of a Hamiltonian flow, we just only need to add to our Hamiltonian equations the equation

$$\frac{\partial S_{t,t_0}}{\partial t} = \mathbf{\Lambda}(H_t),$$

and to our set of initial conditions the value $S_{t_0,t_0} = 0$. Then we can integrate the whole equations with our favourite numerical method.

ii) In particular, if $\mathbf{\Lambda}(H_t) = \lambda(t)$ does not depend on the phase variables, then $S_{t,t_0} = \int_{t_0}^{t}\lambda(s)\ ds$ does not depend on them either, and φ_{t,t_0} preserves not only ω but also the action form α (in the domain where it is defined, of course).

iii) By means of the Lie series method, we can obtain an expression for the primitive function of the time-t flow φ_t of a time-independent Hamiltonian H, if we suppose analyticity, of course. It is

$$S_t = \sum_{k\geq 1}\frac{t^k}{k!}\mathbf{\Lambda}_k(H),$$

where the $\mathbf{\Lambda}$-functions are defined by the recurrence

$$\begin{cases} \mathbf{\Lambda}_1(H) = \mathbf{\Lambda}(H), \\ \mathbf{\Lambda}_k(H) = \{H, \mathbf{\Lambda}_{k-1}(H)\} \quad (k > 1). \end{cases}$$

For instance, if H is an eigenfunction of $\mathbf{\Lambda}$, that is, if there exists a number $\lambda \in \mathbb{R}$ (the corresponding eigenvalue) such that $\mathbf{\Lambda}(H) = \lambda H$, we get that its time-$t$ flow has as primitive function

$$S_t = t\ \mathbf{\Lambda}(H) = t\lambda\ H.$$

◁

The interpolation problem. As the time-1 flow of a time-dependent Hamiltonian vector field is an exact symplectomorphism, we wonder if

> Given an exact symplectomorphism, is it the time-1 flow of a time-dependent Hamiltonian vector field?

1.4 The derivation $\mathbf{\Lambda}$.

On one hand, recall that the space of functions $\mathcal{F}(\mathcal{N})$ endowed with the Poisson bracket is a Lie algebra, and the relation between the Lie bracket and the Poisson bracket is given by

$$X_{\{K,H\}} = [X_K, X_H].$$

On the other hand, we have defined an operator in this space, given by $\mathbf{\Lambda}(H) = \alpha(X_H) - H$, which will be important in the sequel. We have the next proposition.

Proposition 2 *The operator $\mathbf{\Lambda}$ associated to an action form α of our exact symplectic form $\omega = \mathrm{d}\alpha$ in $\mathcal{N}$ is a derivation in the Lie algebra $\mathcal{F}(\mathcal{N})$:*

- *$\mathbf{\Lambda}$ is linear,*
- $\mathbf{\Lambda}\{H_1, H_2\} = \{\mathbf{\Lambda}(H_1), H_2\} + \{H_1, \mathbf{\Lambda}(H_2)\}$.

Moreover, it satisfies:

- $\mathrm{d}(\mathbf{\Lambda}(H)) = \mathbf{L}_{X_H}\alpha$.

Finally, it is natural with respect to pull back by actionmorphisms:

- *If $L^*\alpha = \alpha$ and $H \in \mathcal{F}(\mathcal{N})$ then $\mathbf{\Lambda}(H)\circ L = \mathbf{\Lambda}(H\circ L)$.*

Proof: The proof of the linearity is straightforward. Before proving the product rule we shall prove the second formula:

$$\begin{aligned} \mathbf{L}_{X_H}\alpha &= \mathrm{d}\circ\mathbf{i}_{X_H}\alpha + \mathbf{i}_{X_H}\circ\mathrm{d}\alpha = \mathrm{d}(\alpha(X_H)) + \mathbf{i}_{X_H}\omega \\ &= \mathrm{d}(\alpha(X_H)) - \mathrm{d}H = \mathrm{d}(\mathbf{\Lambda}(H)). \end{aligned}$$

Therefore:

$$\begin{aligned} \mathbf{\Lambda}(\{H_1, H_2\}) &= \alpha([X_{H_1}, X_{H_2}]) - \{H_1, H_2\} \\ &= \mathrm{d}(\alpha(X_{H_2}))X_{H_1} - \mathbf{L}_{X_{H_1}}\alpha\ X_{H_2} - \{H_1, H_2\} \\ &= \{H_1, \alpha(X_{H_2})\} - \mathrm{d}(\mathbf{\Lambda}(H_1))X_{H_2} - \{H_1, H_2\} \\ &= \{H_1, \mathbf{\Lambda}(H_2)\} + \{\mathbf{\Lambda}(H_1), H_2\}. \end{aligned}$$

Finally, if $L^*\alpha = \alpha$ and $H \in \mathcal{F}(\mathcal{N})$:

$$\alpha(X_H)\circ L = L^*(\mathbf{i}_{X_H}\alpha) = \mathbf{i}_{L^*X_H}L^*\alpha = \mathbf{i}_{X_{H\circ L}}\alpha = \alpha(X_{H\circ L}).$$

□

In symplectic coordinates (x, y), the derivation $\mathbf{\Lambda}$ is written

$$\mathbf{\Lambda}(H)(x, y) = y \cdot \nabla_y H(x, y) - H(x, y).$$

2 Determination of an exact symplectomorphism.

As we have seen, an exact symplectomorphism $F : \mathcal{N} \to \mathcal{N}$ on $(\mathcal{N}, \mathrm{d}\alpha)$ is not determined by its primitive function S. In general, we can obtain more exact symplectomorphisms with such primitive function composing F on the left with actionmorphisms L:

$$(L\circ F)^*\alpha - \alpha = F^* L^* \alpha - \alpha = F^*\alpha - \alpha = \mathrm{d}S.$$

In fact, all the exact symplectomorphisms with that primitive function can be constructed in this way. If $G : \mathcal{N} \to \mathcal{N}$ is another exact symplectomorphism with primitive function S, then $L = G\circ F^{-1}$ is an actionmorphism.

So then, in order to answer the question

> What additional information do we need in order to determine an exact symplectomorphism?

we have to look for the actionmorphisms of our exact symplectic manifold, because

> An exact symplectomorphism is determined by its primitive function save an actionmorphism.

We shall study the particular case in which our exact symplectic manifold is the cotangent bundle of a manifold endowed with its canonical exact symplectic structure.

2.1 On the cotangent bundle.

Let $T^*\mathcal{M}$ be our exact symplectic manifold, if we consider on it the differential of the Liouville form α. Let F be an exact symplectomorphism defined on $T^*\mathcal{M}$ (or a tubular neighbourhood of the zero-section, $\mathcal{N}$).

Suppose that our exact symplectomorphism fixes the zero-section. This is not so restrictive, due to the Weinstein' theorems [15, 11]. They let us to send via a symplectomorphism a certain neighbourhood of any Lagrangian manifold onto a neighborhood of the zero-section of its cotangent bundle. Moreover, using a generalized Poincaré's lemma, he also proved that if our Lagrangian manifold is exact then the symplectomorphism is also exact (between two different manifolds, of course). We have some restrictions on the primitive function.

Proposition 3 *Let $F : \mathcal{N}_1 \to \mathcal{N}_2$ be an exact symplectomorphism from a tubular neighbourhood of the zero-section on $T^*\mathcal{M}$, $\mathcal{N}_1$, onto another one $\mathcal{N}_2$. Let $S : \mathcal{N}_1 \to \mathbb{R}$ be its primitive function. Then,*

$$F \textit{ fixes the zero-section } \Leftrightarrow \forall x \in \mathcal{M} \;\; \mathrm{d}S(0_x) = 0.$$

Proof: Let $x \in \mathcal{M}$ be any point on the zero-section. Since $\alpha(0_x) = 0$ and

$$(F^*\alpha)(0_x) = \alpha(F(0_x))\circ F_*(0_x) = F(0_x)\circ q_*(F(0_x))\circ F_*(0_x) = F(0_x)\circ(q\circ F)_*(0_x),$$

then

$$\mathrm{d}S(0_x) = F(0_x)\circ(q\circ F)_*(0_x).$$

Finally, as $(q\circ F)_*$ is an epimorphism in all points, we reach to the result. □

Remark: In particular, an actionmorphism defined in a neighbourhood of the zero-section fixes the zero-section. $\triangleleft$

As the zero-section is fixed, we can define a diffeomorphism $\phi = q \circ F \circ z$ on it, by restriction of our symplectomorphism to that invariant manifold. Suppose we are able to find an exact symplectomorphism $\bar{F}$ that fixes all the points of the zero-section and whose primitive function is just S. Then, $\hat{\phi} \circ \bar{F}$ is *another* symplectomorphism that fixes the zero-section, whose dynamics is ϕ and whose primitive function is also S. We shall see that this symplectomorphism is F.

Firstly, we have to see that the only actionmorphism $L : \mathcal{N}_1 \to \mathcal{N}_2$, which fixes the zero-section and whose dynamics is given by ϕ, is the corresponding lift. Composing on the left with the inverse of such a lift, is it enough to prove that the only actionmorphism which fixes all the points of the zero-section is in fact the identity map. This is a well known result in the case that $\mathcal{N}_1 = \mathcal{N}_2$ is the whole cotangent bundle (see, for instance, [11]).

Proposition 4 *Let $I : \mathcal{N}_1 \to T^*\mathcal{M}$ be an actionmorphism from a tubular neighbourhood of the zero-section $\mathcal{N}_1$ onto its image $\mathcal{N}_2 = I(\mathcal{N}_1)$. Suppose I fixes all the points of the zero-section. Then, I is the identity map.*

Proof: Take cotangent coordinates $(x, y) \in \mathcal{U} \times \mathbb{R}^d$ around each point of the zero-section, which is represented by $\{y = 0\}$. We write I by

$$\bar{x} = f(x, y), \bar{y} = g(x, y).$$

As I fixes all the points of the zero-section, we have

$$f(x, 0) = x, g(x, 0) = 0.$$

The condition $I^*\alpha = \alpha$ is written as

$$0 = g(x, y)^\top \frac{\partial f}{\partial x}(x, y) - y^\top, 0 = g(x, y)^\top \frac{\partial f}{\partial y}(x, y).$$

We shall also write

$$A(x, y) = \frac{\partial f}{\partial x}(x, y), B(x, y) = \frac{\partial f}{\partial y}(x, y),$$

and the symplectic character implies that $AB^\top = BA^\top$.

Hence, we have

$$\begin{aligned} 0 &= (g(x, y)^\top A(x, y) - y^\top) B(x, y)^\top = g(x, y) B(x, y) A(x, y)^\top - y^\top B(x, y)^\top \\ &= -y^\top B(x, y)^\top, \end{aligned}$$

and then each x-component f_i satisfies the Euler's equation (of order 0) in the y-variables

$$\sum_{j=1}^{d} y_j \frac{\partial f_i}{\partial y_j} = 0.$$

By regularity on the zero-section, these functions f_i are constant respect to the y-variables: $f(x, y) = f(x, 0) = x$. Finally, $g(x, y) = y$, because A is the identity matrix. $\square$

2.2 Effective computation.

We shall take a formal point of view, in order to understand the problem. Then, we assume that:

- our manifold $\mathcal{M}$ is $\mathbb{R}^d$;
- the primitive function S is a formal series in y:

$$S(x,y) = \sum_n s_n(x) y^n$$

  where the s_n are x-functions (we use multi-indices $n = (n_1, \ldots, n_d) \in \mathbb{N}^d$),
- and all the points of the zero-section are fixed:

$$f(x,0) = x, \;\; g(x,0) = 0.$$

Hence, we want to recover our symplectomorphism $F = (f,g)$ looking for expressions of the form:

$$f(x,y) = \sum_n f_n(x) y^n \;\; , \;\; g(x,y) = \sum_n g_n(x) y^n,$$

where $f_n = (f_n^1, \ldots, f_n^d)^\top$, $g_n = (g_n^1, \ldots, g_n^d)^\top$ are vector functions, being $f_0 = x$ and $g_0 = 0$.

2.2.1 Iterative process.

From the exactness equations we can obtain the relationship between the sequences of f,g and s. In the following, $\sum_i$ means $\sum_{i=1}^d$ and $u, v \in \mathbb{N}^d$ are multi-indices. Moreover, $e_i = (0, \ldots, 1_i, \ldots, 0)$. Firstly, as

$$\begin{aligned}
\frac{\partial S}{\partial x_j}(x,y) &= \sum_n \frac{\partial s_n}{\partial x_j}(x) y^n = \sum_i g^i(x,y) \frac{\partial f^i}{\partial x_j}(x,y) - y_j \\
&= \sum_i \left(\left(\sum_n g_n^i(x) y^n \right) \left(\sum_n \frac{\partial f_n^i}{\partial x_j}(x) y^n \right) \right) - y_j \\
&= \sum_n \sum_i \sum_{u+v=n} \left(\frac{\partial f_u^i}{\partial x_j}(x) g_v^i(x) \right) y^n - y_j,
\end{aligned}$$

then $\forall n \in \mathbb{N}^d$, $\forall j = 1 \div d$

$$\frac{\partial s_n}{\partial x_j}(x) = \sum_i \sum_{u+v=n} \frac{\partial f_u^i}{\partial x_j}(x) g_v^i(x) \; - \delta_{n e_j},$$

where δ is the Kronecker's delta. Secondly, as

$$\begin{aligned}
\frac{\partial S}{\partial y_j}(x,y) &= \sum_n (n_j + 1) s_{n+e_j}(x) y^n = \sum_i g^i(x,y) \frac{\partial f^i}{\partial y_j}(x,y) \\
&= \sum_i \left(\sum_n g_n^i(x) y^n \right) \left(\sum_n (n_j+1) f_{n+e_j}^i(x) y^n \right) \\
&= \sum_n \left(\sum_i \sum_{u+v=n} (u_j+1) f_{u+e_j}^i(x) g_v^i(x) \right) y^n,
\end{aligned}$$

then

$$(n_j+1)s_{n+e_j}(x) = \sum_i \left(\sum_{u+v=n} (u_j+1) f^i_{u+e_j}(x) g^i_v(x) \right).$$

So then:

- the function s_0 is constant, and we can suppose that this constant is zero;
- the functions s_{e_i} vanish.

Therefore, we recover that the primitive function verifies $\mathrm{D}S(x,0)=0$ and, in particular, is constant on $\{y=0\}$. We take this constant equal to zero.

In order to find the x-functions f_n and g_n, we have to solve these equations recurrently by increasing orders. The order 1 equations are, $\forall i,j = 1 \div d$

$$\begin{cases} g^j_{e_i} = \delta_{ij}, \\ \\ f^j_{e_i} = (1+\delta_{ij})\ s_{e_i+e_j}. \end{cases}$$

Now, suppose that we already know the terms of order $k-1$ and we have to obtain the terms of order k. The equations are, $\forall |n| = k,\ \forall j = 1 \div d$:

$$\begin{cases} g^j_n = G^j_n, \\ \\ n_j f^j_n + (n_j+1) \sum_{i \neq j} f^i_{n-e_i+e_j} = F^j_n, \end{cases}$$

where each G_n is computed from terms of lower order and each F_n depends on, moreover, the g_n. We have obtained a linear system with natural coefficients for the f. We are going to solve it.

2.2.2 Solving the linear systems.

Let $N = (N_1, \ldots, N_d)$ be a multi-index subscript (with $|N| = k$) and let $J = 1 \div d$ be a superscript. We want to know how many equations contain the corresponding f^J_N. Each equation is identified by a subscript n and a superscript j, and it is written as

$$\sum_i (n_j - \delta_{ij} + 1) f^i_{n-e_i+e_j} = F^j_n.$$

Since

$$f^i_{n-e_i+e_j} = f^J_N \ \Rightarrow\ i = J,\ n = N + e_J - e_j,$$

then f^J_N only appears at the d equations

$$\sum_i (N_j + \delta_{jJ} - \delta_{ij}) f^i_{N+e_J-e_i} = F^j_{N+e_J-e_j},$$

where $j = 1 \div d$.

Notice that all the f_n in these equations are of the type $f^i_{N+e_J-e_i}$, with $i = 1 \div d$. Hence, the f_n appear in $d \times d$-blocks, and we have to solve the corresponding linear sub-systems. If any subscript has a negative component, we assume that the corresponding F is equal to 0, and we also deduce that the corresponding f is equal to 0. Adding the d equations,

$$\sum_j F^j_{N+e_J-e_j} = \sum_i \sum_j (N_j + \delta_{jJ} - \delta_{ij}) f^i_{N+e_J-e_i} = |N| \sum_i f^i_{N+e_J-e_i},$$

where $|N| = \sum_i N_i$, and we obtain that

$$S^J_N \stackrel{\text{def}}{=} \sum_i f^i_{N+e_J-e_i} = \frac{1}{|N|} \sum_j F^j_{N+e_J-e_j}.$$

Finally, as

$$F^J_N = \sum_i (N_J + 1 - \delta_{iJ}) f^i_{N+e_J-e_i} = (N_J+1) S^J_N - f^J_N,$$

we get $f^J_N = (N_J+1) S^J_N - F^J_N$.

2.2.3 The result.

We have proved the next theorem.

Theorem 1 *Let F be a 'formal' symplectomorphism on $\mathbb{R}^{2d}$ given by*

$$\begin{cases} f(x,y) = x + \sum_{|n|\geq 1} f_n(x) y^n, \\ \\ g(x,y) = \sum_{|n|\geq 1} g_n(x) y^n, \end{cases}$$

being

$$S(x,y) \quad = \quad \sum_n s_n(x) y^n$$

its primitive function. We take $f_0(x) = x$ and $g_0(x) = 0$.

- *The function s_0 is constant and the functions s_{e_i} vanish: $\mathrm{D}S(x,0) = 0$.*
- *We can recover the x-functions f_n and g_n from the s_n by means of the next recurrence:*
  - *Step 1: $\forall i, j = 1 \div d$:*

$$g^j_{e_i} = \delta_{ij} \ , \ f^j_{e_i} = (1+\delta_{ij}) \ s_{e_i+e_j}.$$

  - *Step $k > 1$: $\forall |n| = k$, $\forall j = 1 \div d$:*

$$g^j_n = G^j_n \ , \ f^j_n = (n_j+1) S^j_n - F^j_n,$$

  where

$$\begin{aligned} G^j_n &= \frac{\partial s_n}{\partial x_j}(x) - \sum_i \sum_{\substack{u+v=n \\ u \neq 0, n}} \frac{\partial f^i_u}{\partial x_j}(x) g^i_v(x), \\ F^j_n &= (n_j+1) s_{n+e_j}(x) - \sum_i \sum_{\substack{u+v=n \\ |v| > 1}} (u_j+1) f^i_{u+e_j}(x) g^i_v(x), \\ S^j_n &= \frac{1}{k} \sum_i F^i_{n+e_j-e_i}. \end{aligned}$$

Examples:

1) If we work on $\mathcal{M} = \mathbb{T}^d$, then the functions s_n are 1-periodic in all its variables, and so are the functions f_n and g_n ($|n| > 0$). An example corresponds to the case in which the toral zero-section is invariant and its dynamics is given by a shift $\phi(x) = x + \omega$.

2) Another example is the case in which the dynamics on the zero-section is given by $\phi(x) = \Lambda x$, where $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_d)$ and, for instance, $\forall i = 1 \div d$ $|\lambda_i| < 1$ (that is, the origin is a hyperbolic fixed point, whose eigenvalues satisfy a certain non-resonance condition, and the zero-section is the corresponding stable manifold). $\triangleleft$

Remarks:

i) The condition $\forall x \in \mathbb{R}^d$ $\mathrm{D}S(x, 0) = 0$ is necessary and sufficient for the zero-section be fixed. A similar condition also works for an exact Lagrangian invariant graph for an exact symplectomorphism defined on a cotangent bundle [7, 8]. In particular, we associate a conserved quantity to such a graph, and we can define variational principles for these invariant graphs.

ii) If the points of the zero-section are not fixed, and we want to solve the problem directly, then the linear systems are more difficult. Let $\phi(x) = f(x, 0)$ be the dynamics on the zero-section. The order 1 equations are, $\forall i, j = 1 \div d$

$$\begin{cases} \displaystyle\sum_l \frac{\partial \phi_l}{\partial x_j} g^l_{e_i} = \delta_{ij}, \\ \\ \displaystyle\sum_l f^l_{e_i} g^l_{e_j} = (1 + \delta_{ij})\ s_{e_i + e_j}. \end{cases}$$

The order k equations are $\forall |n| = k, \forall j = 1 \div d$

$$\begin{cases} \displaystyle\sum_i \frac{\partial \phi_i}{\partial x_j} g^i_n = G^j_n, \\ \\ \displaystyle\sum_{i,l} (n_j - \delta_{lj} + 1) f^i_{n - e_i + e_j} g^i_{e_l} = F^j_n. \end{cases}$$

In order to solve these equations, we need, of course, $|\mathrm{D}\phi(x)| \neq 0$.

iii) Following the calculations we can also obtain the known normal forms around invariant tori and hyperbolic points (taking profit of that theirs stable and unstable manifolds are exact Lagrangian). We can also obtain normal forms around lower dimensional hyperbolic tori, but we need some reducibility hypothesis.

iv) We have to show the analyticity of the expansions. Instead of to do this, we shall obtain the symplectomorphism as the time-1 flow of an analytic Hamiltonian. This is the next subject. $\triangleleft$

3 Interpolation of an exact symplectomorphism.

As the time-1 flow of a Hamiltonian vector field is exact symplectic, a natural question arises:

Given an exact symplectomorphism, is it the time-1 flow of a time-dependent Hamiltonian vector field?

In such a case, we shall say that our symplectomorphism is *homologous to the identity.* Once we have interpolated our exact symplectomorphism by a time-dependent Hamiltonian flow, next question is:

Can we get our Hamiltonian be 1-periodic in time?

This subject has been studied for many authors, and it has many variants. It is a particular case of the more general problem of *inclusion of a map into a flow.* Moser [13] already dealt with this problem when he proven the analyticity of the Birkhoff normal form around a hyperbolic fixed point of an area preserving map. Douady [6] solved the problem in the smooth symplectic case provided our map is given by a generating function and Conley and Zehnder [5] solved it for smooth diffeomorphism of a torus which leave the center of mass fixed. On the other side, Douady [6], Kuksin [9] and Kuksin and Pöschel [10] solved the problem in analytic set up for maps which are close to integrable ones, but in a non-constructive way.

We shall consider that our exact symplectic manifold is $\mathcal{N} = T^*\mathcal{M}$ (or a tubular neighbourhood of the zero-section), and that our exact symplectomorphism $F : \mathcal{N} \to \mathcal{N}$, whose primitive function is S, fixes the zero-section. As we know, the function S has to satisfy $\mathrm{d}S{\circ}z = 0$ and we shall take $S{\circ}z = 0$ (without loss of generality). If we can get a Hamiltonian flow interpolating F we are solving in fact the determination problem. As a matter of fact, it is enough to solve the problem if all points of the zero-section are fixed.

So then, we shall deal with the first part of the problem. In order to get periodicity in time then we can apply, in some cases, a theorem by Pronin and Treschev [14]. We remark that they studied the analytic case and applied a constructive method to obtain the Hamiltonian, a kind of averaging method, and they began from a non-periodic Hamiltonian. That is to say, the began from a homologous to the identity symplectomorphism, which is that we shall obtain. Moreover, they worked on a compact symplectic manifold, and we can apply their theorem in a relatively compact tubular neighbourhood of the zero-section, which have to be compact (say, a torus).

We also work in analytic set up, and our proof is also constructive.

3.1 Set up.

Let $F : \mathcal{N} \to \mathcal{N}$ be an exact symplectomorphism, with $\mathrm{pf}(F) = S$. We shall try to look for the exact symplectomorphism as the time-1 flow of a time-dependent Hamiltonian vector field.

3.1.1 The homotopy method.

Let $H : \mathcal{N} \times \mathbb{R} \to \mathbb{R}$ be the Hamiltonian function, X_{H_t} be the corresponding vector field and φ_t be the corresponding flow from $t_0 = 0$ (that is $\varphi_t = \varphi_{t,0}$). We would like

$$\varphi_1^*\alpha - \alpha = \mathrm{d}S.$$

In fact, we impose 'a little more', that $\forall t$

$$\varphi_t^*\alpha - \alpha = t\ \mathrm{d}S$$

(this is the idea of a *homotopy method*). That is to say, we want that $S_{t,0} = t\ S$ (with the notation of 1.3). Hence, if we derive respect to the time the previous homotopy formula we shall see that it is enough to impose $S = \mathbf{\Lambda}(H_t){\circ}\varphi_t$ (notice that to impose $\mathbf{\Lambda}(H_t){\circ}\varphi_t = S + s(t)$, for any t-function s, also works, because primitive functions are defined up to constants). Therefore, if H_0 satisfies $S = \mathbf{\Lambda}(H_0)$ and

$$\frac{d}{dt}(\mathbf{\Lambda}(H_t)\circ\varphi_t) = 0,$$

then H_t is a time-dependent Hamiltonian whose time-1 flow is an exact symplectomorphism with primitive function equalizing S. Finally, as

$$\begin{aligned}\frac{d}{dt}(\mathbf{\Lambda}(H_t)\circ\varphi_t) &= \mathrm{d}(\mathbf{\Lambda}(H_t))(\varphi_t)\,\frac{\partial\varphi_t}{\partial t} + \frac{\partial}{\partial t}(\mathbf{\Lambda}(H_t))\circ\varphi_t \\ &= \{H_t, \mathbf{\Lambda}(H_t)\}\circ\varphi_t + \frac{\partial}{\partial t}(\mathbf{\Lambda}(H_t))\circ\varphi_t\end{aligned}$$

we obtain the next proposition.

Proposition 5 *Let $S : \mathcal{N} \to \mathbb{R}$ be a function. Then, the time-t flow $\varphi_t = \varphi_{t,t_0}$ of a time-dependent Hamiltonian $H_t : \mathcal{N} \to \mathbb{R}$ which satisfies the* **evolution problem**

$$\begin{cases}\dfrac{d}{dt}(\mathbf{\Lambda}(H_t)) = -\{H_t, \mathbf{\Lambda}(H_t)\}, \\ S = \mathbf{\Lambda}(H_0),\end{cases}$$

is exact symplectic and its primitive function is $t\ S$. That is: $\mathrm{pf}(\varphi_t) = t\ S$ *(in the domain in which this flow is defined, of course).*

3.1.2 Generating solutions from a particular one.

For the moment, if we are able of finding a solution of the previous evolution problem, we only can assure that its time-1 flow has primitive function S, and this does not determine our symplectomorphism. It is translated to our derivation $\mathbf{\Lambda}$ is not invertible (there exists integration 'constants', i.e., functions with vanishing $\mathbf{\Lambda}$-derivative).

We can obtain many families of exact symplectomorphisms satisfying similar conditions from a particular one. Let $\bar{H}_t$ be a solution of the evolution problem, and $\varphi_t = \varphi_{t,0}$ its corresponding flow, that is

$$\begin{cases}\dfrac{d\varphi_t}{dt} = X_{\bar{H}_t}(\varphi_t), \\ \varphi_0 = \mathrm{id}.\end{cases}$$

If we compose φ_t on the left with a family of actionmorphisms connected to the identity, L_t, generated by a time-dependent Hamiltonian h_t satisfying $\mathbf{\Lambda}(h_t) = 0$, then we obtain a family of exact symplectomorphism ψ_t connected to the identity such that $\mathrm{pf}(\psi_t) = t\ S$. That is, if $\forall t\ L_t^*\alpha = \alpha$ and $L_0 = \mathrm{id}$ then the new family $\psi_t = L_t\circ\varphi_t$ satisfies $\mathrm{pf}(\psi_t) = t\ S$ and $\psi_0 = \mathrm{id}$. We are going to see that this new family is also generated by a new Hamiltonian: $H_t = h_t + \bar{H}_t\circ L_t^{-1}$. In fact, H_t is also a solution of the evolution problem.

Proposition 6 *Given $S : \mathcal{N} \to \mathbb{R}$ a function, let $\bar{H}_t$ be a solution of the corresponding evolution problem, and $\varphi_t = \varphi_{t,0}$ be its time-t flow. Let L_t be a family of actionmorphisms, which is given by the time-t flow of a time-dependent Hamiltonian h_t satisfying $\mathbf{\Lambda}(h_t) = 0$. Consider the new family of exact symplectomorphisms $\psi_t = L_t\circ\varphi_t$, which is connected to the identity. Then:*

- $\mathrm{pf}(\psi_t) = \mathrm{pf}(\varphi_t) = t\ S$;
- *ψ_t is the time-t flow of the Hamiltonian $H_t = h_t + \bar{H}_t \circ L_t^{-1}$;*
- *H_t is also a solution of the evolution problem.*

Proof: First point is straightforward, because each L_t preserves the action form (and has primitive function zero). For the second one:

$$\begin{aligned}
\frac{d\psi_t}{dt} &= \frac{d}{dt}(L_t \circ \varphi_t) = \frac{\partial L_t}{\partial t} \circ \varphi_t + (L_t)_*(\varphi_t)\ \frac{d\varphi_t}{dt} \\
&= X_{h_t} \circ L_t \circ \varphi_t + ((L_t)_*\ X_{\bar{H}_t}) \circ \varphi_t = X_{h_t} \circ \psi_t + (L_t)_*\ X_{\bar{H}_t} \circ L_t^{-1} \circ \psi_t \\
&= X_{h_t} \circ \psi_t + (L_t^{-1})^* X_{\bar{H}_t} \circ \psi_t = X_{h_t} \circ \psi_t + X_{\bar{H}_t \circ L_t^{-1}} \circ \psi_t \\
&= X_{h_t + H_t \circ L_t^{-1}} \circ \psi_t .
\end{aligned}$$

And for the third one:

$$\mathbf{\Lambda}(H_0) = \mathbf{\Lambda}(h_0 + \bar{H}_0 \circ L_0^{-1}) = \mathbf{\Lambda}(\bar{H}_0) = S$$

because $L_0 = \mathrm{id}$, and

$$\begin{aligned}
\frac{d}{dt}(\mathbf{\Lambda}(H_t)) &= \frac{d}{dt}(\mathbf{\Lambda}(\bar{H}_t \circ L_t^{-1})) = \frac{d}{dt}(\mathbf{\Lambda}(\bar{H}_t) \circ L_t^{-1}) \\
&= \frac{\partial}{\partial t}(\mathbf{\Lambda}(H_t)) \circ L_t^{-1} + \mathrm{d}(\mathbf{\Lambda}(\bar{H}_t))(L_t^{-1})\ \frac{\partial L_t^{-1}}{\partial t} \\
&\overset{*}{=} -\{\bar{H}_t, \mathbf{\Lambda}(\bar{H}_t)\} \circ L_t^{-1} - \mathrm{d}(\mathbf{\Lambda}(\bar{H}_t))(L_t^{-1})\ (L_t^{-1})_*\ X_{h_t} \\
&= -\{\bar{H}_t \circ L_t^{-1}, \mathbf{\Lambda}(\bar{H}_t) \circ L_t^{-1}\} - \mathrm{d}(\mathbf{\Lambda}(\bar{H}_t) \circ L_t^{-1}) X_{h_t} \\
&= -\{\bar{H}_t \circ L_t^{-1}, \mathbf{\Lambda}(\bar{H}_t \circ L_t^{-1})\} - \{h_t, \mathbf{\Lambda}(\bar{H}_t \circ L_t^{-1})\} \\
&= -\{h_t + \bar{H}_t \circ L_t^{-1}, \mathbf{\Lambda}(\bar{H}_t \circ L_t^{-1})\} \\
&= -\{H_t, \mathbf{\Lambda}(H_t)\}.
\end{aligned}$$

In the equality $*$ we have applied that, since $L_t \circ L_t^{-1} = \mathrm{id}$, then

$$0 = (L_t)_* \circ L_t^{-1}\ \frac{\partial L_t^{-1}}{\partial t} + \frac{\partial L_t}{\partial t}(L_t^{-1}) = (L_t^{-1})_*^{-1}\ \frac{\partial L_t^{-1}}{\partial t} + X_{h_t}$$

and, finally

$$\frac{\partial L_t^{-1}}{\partial t} = -(L_t^{-1})_*\ X_{h_t}.$$

□

Remark: Second point of the proposition could be proved using some identities about globally Hamiltonian isotopies [12]. In fact, it would be enough for our purposes, but we have proved something more. ◁

3.1.3 Fixing an evolution problem.

We are going to fix a particular solution of the evolution problem. Suppose that $\mathcal{F} = \mathcal{F}(\mathcal{N})$ splits as

$$\mathcal{F} = \ker \mathbf{\Lambda} \oplus \mathbf{\Lambda}(\mathcal{F}),$$

that is, $\mathbf{\Lambda}_{|} \stackrel{\text{def}}{=} \mathbf{\Lambda}_{|\mathbf{\Lambda}(\mathcal{F})} : \mathbf{\Lambda}(\mathcal{F}) \to \mathbf{\Lambda}(\mathcal{F})$ is an isomorphism. If we work inside $\mathbf{\Lambda}(\mathcal{F})$, that is, if we look for a solution $\bar{H}_t \in \mathbf{\Lambda}(\mathcal{F})$, then we can translate our evolution problem in another one. If we define $S_t = \mathbf{\Lambda}(H_t) \in \mathbf{\Lambda}(\mathcal{F})$, then S_t have to be a solution of the 'new' evolution problem

$$\begin{cases} \dfrac{dS_t}{dt} = -\{\mathbf{\Lambda}_{|}^{-1}(S_t), S_t\}, \\ \\ S_0 = S. \end{cases}$$

Of course, we need that $\mathbf{\Lambda}(\mathcal{F})$ be invariant under these operations. If we are able of solving this new problem, then $\bar{H}_t = \mathbf{\Lambda}_{|}^{-1}(S_t)$ is a solution of the original one.

An iterative method. Suppose we know S_0 and we want to search for S_t as a development in powers of the time t: $S_t = \sum_{k\geq 0} S_k t^k$. Then, $\forall k \geq 0$

$$S_{k+1} = \frac{-1}{k+1} \sum_{u+v=k} \{\mathbf{\Lambda}_{|}^{-1}(S_u), S_v\}.$$

3.2 The Liouville derivative.

We are going to work in $\mathcal{N} = T^*\mathcal{M}$, endowed with the canonical symplectic structure given by the differential of the Liouville form α. In this context, the derivation $\mathbf{\Lambda}$ associated to the action form can be written by means of the *Liouville vector field*, which is the only vector field Z^* on $T^*\mathcal{M}$ that satisfies the relation

$$\mathbf{i}_{Z^*}\mathrm{d}\alpha = \alpha.$$

Proposition 7 *The derivation $\mathbf{\Lambda}$ associated to the Liouville form on $T^*\mathcal{M}$ satisfies the next relations:*

- $\mathbf{\Lambda}(H) = \mathrm{d}H(Z^*) - H,$
- $X_{\mathbf{\Lambda}(H)} = [Z^*, X_H].$

Proof: Firstly,

$$\alpha(X_H) = \mathbf{i}_{Z^*}\mathrm{d}\alpha\ X_H = -\mathbf{i}_{X_H}\mathrm{d}\alpha\ Z^* = \mathrm{d}H(Z^*),$$

and secondly,

$$\begin{aligned} \mathbf{i}_{[Z^*,X_H]}\omega &= \mathbf{L}_{Z^*}\mathbf{i}_{X_H}\omega - \mathbf{i}_{X_H}\mathbf{L}_{Z^*}\omega = -\mathbf{L}_{Z^*}\mathrm{d}H - \mathbf{i}_{X_H}\omega \\ &= -\mathbf{L}_{Z^*}\mathrm{d}H + \mathrm{d}H = \mathrm{d}(H - \mathbf{L}_{Z^*}H) = -\mathrm{d}(\mathbf{\Lambda}(H)). \end{aligned}$$

□

So then, in this context we shall refer to the $\mathbf{\Lambda}$-derivative as *Liouville derivative.* $\mathbf{\Lambda}(H)$ is also known as the *elementary action* associated with the Hamiltonian H, because it is used in order to define a variational principle for its orbits (see, for instance, [1, 11]). Moreover, it is very close to the *Legendre transformation*.

In cotangent coordinates $(x, y) \in \mathcal{U} \times \mathbb{R}^d$ the Liouville vector field is

$$Z^*(x, y) = y \, \frac{\partial}{\partial y}$$

and the Liouville derivative of a function $H = H(x, y)$ is

$$\mathbf{\Lambda}(H)(x, y) = \mathrm{d}H(y\frac{\partial}{\partial y}) - H(x, y) = y \cdot \nabla_y H(x, y) - H(x, y).$$

As Z^* is a vertical vector field, then $\mathbf{\Lambda}$ is a vertical operator, because the value of $\mathbf{\Lambda}(H)$ on a fiber only depends on the value of H on such fiber. Notice that if the $\mathbf{\Lambda}$-derivative of a Hamiltonian function is constant, then the flows of the corresponding Hamiltonian vector field and Z^* commute. Of course, the flow of Z^* is fiberwised (in fact, is exponential in each fiber) and the flow of X_H is given by a parametric family of lifts.

To integrate with respect to the Liouville derivative is to solve the problem

> Given a function $S \in \mathcal{F}(T^*\mathcal{M})$, which are the functions $H \in \mathcal{F}(T^*\mathcal{M})$ such that $\mathbf{\Lambda}(H) = S$?

As $\mathbf{\Lambda}$ is a vertical operator, we can restrict our attention to each fiber, where is easy to work. On each fiber (fixing x) we have a linear operator which transforms y-valued functions. Its properties will be inherited by our Liouville derivative.

3.2.1 Solving the p.d.e. $y \cdot \nabla_y H - H = S$.

Let $\mathcal{U} \subset \mathbb{R}^d$ be an open star-shaped neighbourhood of the origin in $\mathbb{R}^d$, with coordinates $y = (y_1, \ldots, y_d)$, and let $S : \mathcal{U} \to \mathbb{R}$ be a function. We want to solve the p.d.e.

$$y \cdot \nabla_y H(y) - H(y) = S(y).$$

If we derive the previous equation we get a necessary condition to solve it: $\nabla_y S(0) = 0$. We shall consider the case $S(0) = 0$. If not, we define $\bar{S}(y) = S(y) - S(0)$, $\bar{H}$ a solution of the p.d.e with $\bar{S}$ and, finally, $H(y) = \bar{H}(y) - S(0)$ will be a solution of the original p.d.e.. Notice that the case $S = 0$ is the well known *Euler's equation* for the homogeneous functions of degree 1.

Since $S(0) = 0$ and $\nabla_y S(0) = 0$ we can write

$$S(y) = \sum_{i,j=1 \div d} y_i y_j s_{ij}(y)$$

where

$$s_{ij}(y) = \int_{[0,1]^2} t \cdot \frac{\partial^2 S}{\partial y_i \partial y_j}(sty) \; d(s, t).$$

By the equation, we have $H(0) = 0$, and we shall look for an expression

$$H(y) = \sum_{i=1}^{d} y_i h_i(y).$$

Since

$$y \cdot \nabla_y H(y) - H(y) = \sum_{i,j=1\div d} y_i y_j \frac{\partial h_i}{\partial y_j}(y)$$

we impose $\forall i, j = 1 \div d$

$$\frac{\partial h_i}{\partial y_j}(y) = s_{ij}(y)$$

and we obtain $\forall i = 1 \div d$

$$h_i(y) = \frac{\partial u}{\partial y_i} = a_i + \int_{[0,1]^2} \frac{1}{s} \frac{\partial S}{\partial y_i}(sty)\ d(s,t),$$

where $a = (a_1, \ldots, a_d) \in \mathbb{R}^d$. Finally

$$\begin{aligned} H(y) &= \sum_{i=1}^{d} y_i h_i(y) = \sum_{i=1}^{d} y_i \left(a_i + \int_{[0,1]^2} \frac{1}{s} \frac{\partial S}{\partial y_i}(sty)\ d(s,t) \right) \\ &= a \cdot y + \int_{[0,1]^2} \frac{1}{s}\ y \cdot \nabla_y S(sty)\ d(s,t) = a \cdot y + \int_0^1 \frac{1}{s^2} S(sy)\ ds. \end{aligned}$$

In summary, we have the next proposition.

Lemma 1 *Let $\mathcal{U} \subset \mathbb{R}^d$ be an open star-shaped set centered in the origin and $S : \mathcal{U} \to \mathbb{R}$ be a function satisfying $S(0) = 0, \nabla_y S(0) = 0$. Then, the solutions of the p.d.e.*

$$y \cdot \nabla_y H(y) - H(y) = S(y)$$

are

$$H(y) = a \cdot y + \int_0^1 \frac{1}{t^2} S(ty)\ dt$$

where $a \in \mathbb{R}^d$.

Remarks:

i) The function H is defined up a constant vector a, and $\nabla_y H(0) = a$. Moreover, the functions $H(y) = a \cdot y$ satisfy the equation for $S = 0$, that is, they are the homogeneous functions of degree 1.

ii) If S has the form $S(y) = y^k s_k(y)$, with $|k| \geq 2$, then H has the same form $H(y) = y^k h_k(y)$ (we choose $a = 0$), with

$$h_k(y) = \int_0^1 t^{|k|-2} s_k(ty)\ dt.$$

iii) Finally, formally speaking, if

$$S(y) = \sum_n s_n y^n$$

then we can recover H by

$$H(y) = a \cdot y + \sum_n \frac{s_n}{|n| - 1} y^n.$$

and we need $s_{e_i} = 0$ for all the $e_i = (0, \ldots, 1_i, \ldots 0)$. We see that if S is an analytic function in a certain polydisk, so H is. $\triangleleft$

3.2.2 A splitting lemma.

As a corollary of the previous results we obtain the next lemma.

Lemma 2 *The space of functions defined on $T^*\mathcal{M}$ (or in a tubular neighborhood of its zero-section), $\mathcal{F}$, splits as*

$$\mathcal{F} \quad = \quad \ker \mathbf{\Lambda} \oplus \mathbf{\Lambda}(\mathcal{F}).$$

Moreover, the functions of $\mathbf{\Lambda}(\mathcal{F})$ are those that have vertical derivatives vanishing on the zero-section, and the functions of $\ker \mathbf{\Lambda}$ *are the fiberwise homogeneous functions of degree* 1*.*

We shall write $\mathbf{\Lambda}_| = \mathbf{\Lambda}_{|\mathbf{\Lambda}(\mathcal{F})}$. Hence $\mathbf{\Lambda}_|$ is an isomorphism in $\mathbf{\Lambda}(\mathcal{F})$.

Remark: In fact, an eigenfunction of $\mathbf{\Lambda}$, with eigenvalue $\lambda \in \mathbb{R}$, is a fiberwise homogeneous function of degree $\lambda + 1$. ◁

3.3 Convergence of the expansions.

Since we have the previous splitting, we are going to solve the evolution problem

$$\begin{cases} \dfrac{dS_t}{dt} = -\{\mathbf{\Lambda}_|^{-1}(S_t), S_t\}, \\ \\ S_0 = S, \end{cases}$$

working inside $\mathbf{\Lambda}(\mathcal{F})$. Recall that $\mathrm{d}S{\circ}z = 0$, and in particular $S \in \mathbf{\Lambda}(\mathcal{F})$. We shall use expansions in powers of t. If $S_t = \sum_{k\geq 0} S_k t^k$ is the expansion of S_t (where $S_0 = S$), then we can compute all the terms by the recurrence

$$S_{k+1} = \frac{-1}{k+1} \sum_{u+v=k} \{\mathbf{\Lambda}_|^{-1}(S_u), S_v\},$$

thanks to the next lemma.

Lemma 3 *Let S, T be two functions defined in $T^*\mathcal{M}$ (or in a neighborhood of its zero-section), such that $\mathrm{d}S{\circ}z = 0$ and $\mathrm{d}T{\circ}z = 0$. Then,*

$$\mathrm{d}(\mathbf{\Lambda}_|^{-1}(S)){\circ}z = 0 \ , \ \mathrm{d}\{S, T\}{\circ}z = 0.$$

This lemma can be easily proved using cotangent coordinates. Hence, all the terms of the expansion verify $\mathrm{d}S_k{\circ}z = 0$ (and, in particular, belong to $\mathbf{\Lambda}(T^*\mathcal{M})$). In fact, as the function $S_0 = S$ has y-order 2, the y-orders of the S_k increase: the y-order of S_k is $k+2$. This is the key point in order to prove the convergence of the expansions.

We see that if our manifold is analytic (differentiable) and the initial term S is analytic (differentiable), so are all the terms in the expansion. The problem is to get that the complete series be analytic (differentiable) in the 'spatial' variables, at least until a time $t > 1$ in a neighborhood of the zero-section. Now, the analysis is local, and we shall prove the analytic case using majorant estimates.

3.3.1 Majorant estimates.

Recall that for any two functions $f(z)$, $g(z)$ $(z=(z_1,\ldots,z_m))$ analytic at $z=0$:

$$f(z)=\sum_n f_n z^n,\ g(z)=\sum_n g_n z^n$$

(using multi-index notation), we say that g is a majorant for f ($f \ll g$) iff $\forall n\ |f_n| \leq g_n$.

Next lemma in very standard.

Lemma 4 *The relation $\ll$ satisfies the following properties:*

1. $f_1 \ll g_1,\ f_2 \ll g_2\ \Rightarrow\ f_1+f_2 \ll g_1+g_2,\ f_1 g_1 \ll g_1 g_2;$
2. $f \ll g\ \Rightarrow\ \dfrac{\partial f}{\partial z_i} \ll \dfrac{\partial g}{\partial z_i}\ (i=1\div m);$
3. $\forall t \in [a,b]\ f_t \ll g_t\ \Rightarrow\ \displaystyle\int_a^b f_t(z)dt \ll \int_a^b g_t(z)dt.$

Let w_b be the product $w_b(z)=\prod_{i=1}^m (b-z_i)$, where $b>0$. Hence:

4. $\forall i,k=1\div m$

$$1 \ll \frac{b}{b-z_i}\ ,\ \frac{z_i}{w_b} \ll \frac{b}{w_b}\ ,\ \frac{1}{(b-z_1)\ldots(b-z_k)} \ll \frac{b^{m-k}}{w_b};$$

5. $\forall z \mid \|z\|_\infty < b,\ |f(z)|<c,\ \Rightarrow\ f \ll \dfrac{cb^m}{w_b}.$

We complete the previous lemma with a more specific one.

Lemma 5 *Take $m=2d, z=(x_1,\ldots,x_d,y_1,\ldots,y_d), w(z)=\prod_{i=1}^d (b-x_i)(b-y_i)$. Then:*

6. $f \ll w^{-k}, g \ll w^{-l} \Rightarrow \{f,g\} \ll 2db^{2d-2}klw^{-(1+k+l)};$
7. $f \ll \displaystyle\prod_{t=1}^u y_{i_t} w^{-k}, g \ll \prod_{t=1}^v y_{j_t} w^{-l}\ (u>0, v>0) \Rightarrow$

$$\{f,g\} \ll b^{2d-1}\left(\begin{array}{c} c_{uvkl}\displaystyle\prod_{t=1}^u y_{i_t}\cdot\sum_{s=1}^v \prod_{t=1,t\neq s}^v y_{j_t} \\ + \\ c_{uvlk}\displaystyle\prod_{t=1}^v y_{j_t}\cdot\sum_{s=1}^u \prod_{t=1,t\neq s}^u y_{i_t} \end{array}\right) w^{-(1+k+l)},$$

where $c_{uvkl}=k+\frac{2dkl}{u+v}$;

8. $f \ll \displaystyle\prod_{t=1}^u y_{i_t} w^{-k}\ (u>1) \Rightarrow \mathbf{\Lambda}_{|}^{-1}(f) \ll \frac{1}{u-1}\prod_{t=1}^u y_{i_t} w^{-k}.$

Proof:

6. As $f \ll w^{-k}, g \ll w^{-l}$ and $\{f,g\} = \sum_{i=1}^{d} \left(\frac{\partial f}{\partial y_i}\frac{\partial g}{\partial x_i} - \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial y_i} \right)$ then

$$\{f,g\} \ll 2kl \sum_{i=1}^{d} \frac{1}{(b-x_i)(b-y_i)} \; w^{-(k+l)} \ll 2db^{2(d-1)} kl w^{-(1+k+l)}.$$

7. As

$$\sum_{j=1}^{d} \frac{\partial f}{\partial x_j}\frac{\partial g}{\partial y_j} \quad \ll \quad \sum_{j=1}^{d} \left(\prod_{t=1}^{u} y_{i_t} \frac{kw^{-k}}{b-x_j} \left(\frac{\partial}{\partial y_j} \left(\prod_{t=1}^{v} y_{j_t} \right) w^{-l} + \prod_{t=1}^{v} y_{j_t} \frac{lw^{-l}}{b-y_j} \right) \right)$$

and

$$\sum_{j=1}^{d} \frac{\partial}{\partial y_j} \left(\prod_{t=1}^{v} y_{j_t} \right) = \sum_{s=1}^{v} \prod_{t=1, t \neq s}^{v} y_{j_t},$$

then

$$\{f,g\} \ll \left(\left(\begin{array}{c} l \prod_{t=1}^{v} y_{j_t} \cdot \sum_{s=1}^{u} \prod_{t=1, t\neq s}^{u} y_{i_t} \\ + \\ k \prod_{t=1}^{u} y_{i_t} \cdot \sum_{s=1}^{v} \prod_{t=1, t\neq s}^{v} y_{j_t} \end{array} \right) b^{2d-1} + 2db^{2d-2} kl \prod_{t=1}^{u} y_{i_t} \prod_{t=1}^{v} y_{j_t} \right) w^{-(1+k+l)}.$$

Using that

$$\prod_{t=1}^{u} y_{i_t} \prod_{t=1}^{v} y_{j_t} \quad = \quad \frac{1}{u+v} \left(\begin{array}{c} \prod_{t=1}^{u} y_{i_t} \cdot \sum_{s=1}^{v} \left(\prod_{t=1, t\neq s}^{v} y_{j_t} \cdot y_{j_s} \right) \\ + \\ \prod_{t=1}^{v} y_{j_t} \cdot \sum_{s=1}^{u} \left(\prod_{t=1, t\neq s}^{u} y_{i_t} \cdot y_{i_s} \right) \end{array} \right),$$

we obtain, finally:

$$\{f,g\} \quad \ll \quad b^{2d-1} \left(\begin{array}{c} c_{uvkl} \prod_{t=1}^{u} y_{i_t} \cdot \sum_{s=1}^{v} \prod_{t=1, t\neq s}^{v} y_{j_t} \\ + \\ c_{uvlk} \prod_{t=1}^{v} y_{j_t} \cdot \sum_{s=1}^{u} \prod_{t=1, t\neq s}^{u} y_{i_t} \end{array} \right) w^{-(1+k+l)}.$$

8. As

$$\mathbf{\Lambda}_{|}^{-1}(f)(x,y) \quad = \quad \int_0^1 t^{-2} f(x,ty)dt,$$

and, $\forall t \in [0,1]$

$$f(x,ty) \ll \prod_{s=1}^{u} y_{i_s} t^u w^{-k}(x,ty) \; , \; w^{-k}(x,ty) \ll w^{-k}(x,y),$$

then

$$\mathbf{\Lambda}_{|}^{-1}(f) \quad \ll \quad \frac{1}{u-1} \prod_{s=1}^{u} y_{i_s} w^{-k}.$$

□

3.3.2 Proof of the convergence of the expansions.

Proposition 8 *Let $\mathcal{M}$ be an analytic d-manifold, and $\mathcal{N} = T^*\mathcal{M}$ its cotangent bundle (or a tubular neighbourhood of its zero-section). Let $S : \mathcal{N} \to \mathbb{R}$ be an analytic function, with* $\mathrm{d}S{\circ}z = 0$.

Then, there exists a tubular neighbourhood of the zero-section where the solution of the evolution problem S_t is defined until a time $t > 1$.

Proof: We can use cotangent coordinates (x, y) in a neighbourhood of each point of the zero-section. It is sufficient to prove that we can get a small neighbourhood of zero where the series $\sum_{k\geq 0} S_k(x,y)t^k$ is defined for $t < T$ and $T \geq 1$.

$S(x, 0)$ is constant, and we suppose that this constant is 0. Moreover, since $\forall x\ \frac{\partial S}{\partial y}(x, 0) = 0$ we can write

$$S(x, y) = \sum_{i,j} y_i y_j s_{ij}(x, y),$$

where de functions s_{ij} are analytic (and $s_{ij} = s_{ji}$). Fixing a radius $b > 0$, let c be the maximum of the sup-norms of the functions s_{ij} on $\{\|(x,y)\|_\infty < b\}$. So then, $\forall i, j = 1 \div d\ s_{ij} \ll c b^{2d} w^{-1}$, where $w(x, y) = \prod_{i=1}^{d}(b - x_i)(b - y_i)$, and

$$S_0 \ll c b^{2d} \sum_{i_1,i_2} y_{i_1} y_{i_2}\ w^{-1},$$

Suppose that $\forall u \leq n$

$$S_u \ll \gamma_u \sum_{i_1,\ldots,i_{u+2}} \prod_{t=1}^{u+2} y_{i_t}\ w^{-(2u+1)},$$

where $\gamma_0 = cb^{2d}, \gamma_1, \ldots, \gamma_n$ are constants. We want to estimate S_{n+1}. So then, applying the majorant estimates of the previous lemma, we obtain

$$\begin{aligned} S_{n+1} &= \frac{-1}{n+1} \sum_{u+v=n} \{\mathbf{\Lambda}_{|}^{-1}(S_u), S_v\} \\ &\ll \frac{b^{2d-1}}{n+1} \sum_{\substack{u+v=n \\ i_1,\ldots,i_{u+2} \\ j_1,\ldots,j_{v+2}}} \frac{\gamma_u \gamma_v}{u+1} \left(\begin{array}{c} \hat{c}_{uv} \prod_{t=1}^{u+2} y_{i_t} \cdot \sum_{s=1}^{v+2} \prod_{t=1, t\neq s}^{v+2} y_{j_t} \\ + \\ \hat{c}_{vu} \prod_{t=1}^{v+2} y_{j_t} \cdot \sum_{s=1}^{u+2} \prod_{t=1, t\neq s}^{u+2} y_{i_t} \end{array} \right) w^{-(2n+3)}, \end{aligned}$$

where $\hat{c}_{uv} = c_{(u+2),(v+2),(2u+1),(2v+1)} = (2u+1) + \frac{2d(2u+1)(2v+1)}{u+v+4}$. Applying that

$$\begin{aligned}\sum_{\substack{i_1,\dots,i_{u+2}\\ j_1,\dots,j_{v+2}}} \left(\prod_{t=1}^{u+2} y_{i_t} \cdot \sum_{s=1}^{v+2} \prod_{t=1,t\neq s}^{v+2} y_{j_t}\right) &= \sum_{s=1}^{v+2} \sum_{\substack{i_1,\dots,i_{u+2}\\ j_1,\dots,j_{v+2}}} \left(\prod_{t=1}^{u+2} y_{i_t} \prod_{t=1,t\neq s}^{v+2} y_{j_t}\right)\\ &= d(v+2) \sum_{k_1,\dots,k_{n+3}} \prod_{t=1}^{n+3} y_{k_t},\end{aligned}$$

we reach to

$$S_{n+1} \ll \gamma_{n+1} \sum_{k_1,\dots,k_{n+3}} \prod_{t=1}^{n+3} y_{k_t}\ w^{-(2n+3)},$$

where

$$\gamma_{n+1} = \frac{db^{2d-1}}{n+1} \sum_{u+v=n} \frac{\gamma_u \gamma_v}{u+1} C_{uv}$$

and

$$\begin{aligned}C_{uv} &= \hat{c}_{uv}(v+2) + \hat{c}_{vu}(u+2) = (2u+1)(v+2) + (2v+1)(u+2) + 2d(2u+1)(2v+1)\\ &= (4+8d)uv + (5+4d)(u+v) + (4+2d) \le 4(1+2d)(u+1)(v+1).\end{aligned}$$

Thus

$$\gamma_{n+1} \le \frac{4(1+2d)db^{2d-1}}{n+1} \sum_{u+v=n} (v+1)\gamma_u\gamma_v \le 4d(1+2d)b^{2d-1} \sum_{u+v=n} \gamma_u\gamma_v.$$

Hence, we have majored the sequence of S_n by

$$S_n \ll \gamma_n \sum_{i_1,\dots,i_{n+2}} \prod_{t=1}^{n+2} y_{i_t}\ w^{-(2n+1)},$$

where the sequence of 'new' γ verifies the recurrence

$$\begin{cases} \gamma_0 = cb^{2d},\\ \gamma_{n+1} = K \sum_{u+v=n} \gamma_u\gamma_v, \end{cases}$$

where $K = 4d(1+2d)b^{2d-1}$.

Let $\rho \in [0,1[$ be a ratio we shall choose later. If $\|x\|_\infty \le \rho b$ and $\|y\|_\infty \le \rho b$, then

$$|S_n(x,y)| \le \gamma_n (d\rho b)^{n+2} (b(1-\rho))^{-2d(2n+1)}.$$

We call the right term in this formula β_n. Therefore, we have bounded all the terms of the expansion in a certain domain of x, y:

$$\sum_{n\ge 0} |S_n(x,y)| t^n \le \sum_{n\ge 0} \beta_n t^n.$$

We want that the convergence radius of this series be greater than 1. As

$$\lim_n \frac{\beta_{n+1}}{\beta_n} = \frac{d\rho}{b^{4d-1}(1-\rho)^{4d}} \lim_n \frac{\gamma_{n+1}}{\gamma_n}.$$

we have to compute the convergence radius of $\sum_{n\geq 0} \gamma_n t^n$. We can write $\gamma_n = K^n \gamma_0^{n+1} a_n$ where a_n is the sequence of natural numbers given by

$$\begin{cases} a_0 = 1, \\ a_{n+1} = \sum_{u+v=n} a_u a_v. \end{cases}$$

The elements of this sequence are the coefficients of the Taylor series of the function

$$f(t) = \frac{1-\sqrt{1-4t}}{2t}$$

and, hence $\lim_n \frac{a_{n+1}}{a_n} = 4$. Finally:

$$\lim_n \frac{\beta_{n+1}}{\beta_n} = \frac{d\rho}{b^{4d-1}(1-\rho)^{4d}} \lim_n \frac{\gamma_{n+1}}{\gamma_n} = \frac{4K\gamma_0 d\rho}{b^{4d-1}(1-\rho)^{4d}} < 1$$

if ρ is small enough. □

Remark: Of course, if we use a better sequence of γ, we can improve the factor 4 in the last formula. For example, we can change it by e if we consider the sequence of γ satisfying the recurrence

$$\gamma_{n+1} = \frac{K}{n+1} \sum_{u+v=n} (v+1)\gamma_u \gamma_v.$$

◁

3.4 Solving the interpolation problem.

Once we have solve the 'new' evolution problem, we can solve the interpolation problem. Let S_t be the solution, that belongs to $\mathbf{\Lambda}(\mathcal{F})$. Then $\bar{H}_t = \mathbf{\Lambda}_{|}^{-1}(S_t)$ is a Hamiltonian whose flow $\varphi_t = \varphi_{t,0}$ has primitive function $t\ S$. Since $\mathrm{d}S_t \circ z = 0$ then $\mathrm{d}\bar{H}_t \circ z = 0$, and all the points of the zero-section are fixed. Hence, φ_1 is 'the' exact symplectomorphism whose primitive function is S and fixes all the points of the zero-section. We have proved the analyticity of the expansions in 2.2.3.

We are going to prove something more. If the dynamics *on* the zero-section can be interpolated by a flow, then the dynamics *around* the zero-section can be interpolated by a Hamiltonian flow.

Theorem 2 *Let $\mathcal{M}$ be an analytic manifold, and $\mathcal{N} = T^*\mathcal{M}$ be its cotangent bundle (or a tubular neighbourhood of its zero-section). Let $F : \mathcal{N} \to \mathcal{N}$ be an analytic exact symplectomorphism, such that the zero-section is invariant, being $f : \mathcal{M} \to \mathcal{M}$ its dynamics. Suppose that f is interpolated by the flow $f_t = f_{t,0}$ of an analytic time-dependent vector field $X_{H_t} \in \mathcal{X}(\mathcal{M})$, that is, $f = f_1$.*

Then, F is (analytically) homologous to the identity (at least in a tubular neighbourhood of the zero-section).

Proof: Let S_t be the solution of the 'new' evolution problem associated to S. It is well known (see, for instance, [11]) that the family of lifts $L_t = \hat{f}_t$ is generated by the Hamiltonian lift of X_{H_t}: $h_t(\rho_x) = \rho_x(X_{H_t}(x))$. Notice that $h_t \in \ker(\mathbf{\Lambda})$. Then, applying 3.1.2, 3.1.3 and the previous proposition, it is sufficient to choose $H_t = h_t + \mathbf{\Lambda}_{|}^{-1}(S_t) \circ L_t^{-1}$. □

Conclusion

As we have seen, the primitive function of an exact symplectomorphism gives us some information about it, but not all the information. In order to obtain all the information we need to know, from a geometrical point of view, where and how our symplectomorphism sends an exact Lagrangian manifold. Recall that although our symplectic objects do not be exact, sometimes it is possible that they turn into exact by lifting to suitable covering spaces.

Thanks to the Weinstein' theorems, we only worry about the case our exact symplectic manifold is a cotangent bundle and our exact symplectomorphism fixes the zero-section. We have work in analytic set up, and the differentiable case rest open.

On the other side, we have seen that the primitive function can be useful from a methodological point of view, because it let us obtain *all* the symplectic dynamics around exact Lagrangian manifolds. Dynamics which are not necessarily generated by generating functions.

As a corollary of our results, the dynamics of a symplectomorphism around an invariant torus whose dynamics is conjugated to an ergodic translation is homologous to the identity, and the Hamiltonian can be chosen periodic in time [14].

Appendix: Notes on Differential Geometry

We shall use the next standard notations and results of Differential Geometry (see, for instance, [4]). For the sake of simplicity, all the objects are C^∞.
Let $\mathcal{M}$,$\mathcal{N}$ be manifolds of dimensions m and n, respectively.

- $C^\infty(\mathcal{M},\mathcal{N})$ is the set of (smooth) maps from $\mathcal{M}$ to $\mathcal{N}$, and $\mathcal{F}(\mathcal{M}) = C^\infty(\mathcal{M},\mathbb{R})$ is the set of (smooth) *functions* on $\mathcal{M}$.

- $\mathrm{Diff}(\mathcal{M},\mathcal{N})$ is the set of *diffeomorphisms* from $\mathcal{M}$ onto $\mathcal{N}$, and $\mathrm{Diff}(\mathcal{M}) = \mathrm{Diff}(\mathcal{M},\mathcal{M})$.

- $\mathcal{X}(\mathcal{M})$ is the set of *vector fields* on $\mathcal{M}$. A vector field $X \in \mathcal{X}(\mathcal{M})$ is a section of the *tangent bundle* $T\mathcal{M}$.

- $\mathcal{X}^*(\mathcal{M}) = \Omega^1(\mathcal{M})$ is the set of 1-*forms* or *Pfaffian forms* on $\mathcal{M}$. A 1-form is a section of the *cotangent bundle* $T^*\mathcal{M}$.

- $\Omega(\mathcal{M}) = \bigoplus_{k=0}^m \Omega^k(\mathcal{M})$ is the set of all *exterior differential forms* on $\mathcal{M}$. A *k-form* is an element of $\Omega^k(\mathcal{M})$, and it is a section of the vector bundle of *exterior k-forms on the tangent space* of $\mathcal{M}$, $\Lambda^k(\mathcal{M}) \stackrel{\text{def}}{=} \Lambda^k(T\mathcal{M})$.

- A differentiable map $F : \mathcal{M} \to \mathcal{N}$ induces a tangent map $F_* = TF : T\mathcal{M} \to T\mathcal{N}$ and k cotangent maps $F^* = T^*F \stackrel{\text{def}}{=} (TF)^* : \Lambda^k(\mathcal{N}) \to \Lambda^k(\mathcal{M})$.

In the sequel, $k \in \mathbb{N}$, $f \in \mathcal{F}(\mathcal{M})$, $X, Y, X_1, \dots, X_k \in \mathcal{X}(\mathcal{M})$, $\alpha \in \Omega^k(\mathcal{M})$ and $F \in \mathrm{Diff}(\mathcal{M})$ (for the sake of symplicity).

- **Action of a vector field on a function**: $X(f) = \mathbf{L}_X f = Tf(X)$.

- **Action of a diffeomorphism:**
  - **Push forward:** $F_* f = f \circ F^{-1}$, $F_* X = TF \circ X \circ F^{-1}$, $F_* \alpha = T^*F^{-1} \circ \alpha \circ F^{-1}$.
  - **Pull back:** $F^* f = f \circ F$, $F^* X = TF^{-1} \circ X \circ F$, $F^* \alpha = T^*F \circ \alpha \circ F$.

- **Lie bracket:** $\mathbf{L}_X Y = [X,Y] \in \mathcal{X}(\mathcal{M})$ is given by
$$[X,Y](f) = X(Y(f)) - Y(X(f)).$$
- **Interior product:** $\mathbf{i}_X\alpha \in \Omega^{k-1}(\mathcal{M})$ is given by
$$\mathbf{i}_X\alpha(X_2,\ldots,X_k) = \alpha(X,X_2,\ldots,X_k).$$
- **Lie derivative:** $\mathbf{L}_X\alpha \in \Omega^k(\mathcal{M})$ is given by
$$\mathbf{L}_X\alpha(X_1,\ldots,X_k) = X(\alpha(X_1,\ldots,X_k)) - \sum_{i=1}^{k}\alpha(X_1,\ldots,[X,X_i],\ldots,X_k).$$
- **Exterior derivative:** $\mathrm{d}\alpha \in \Omega^{k+1}(\mathcal{M})$ is given by
$$\begin{aligned}\mathrm{d}\alpha(X_0,\ldots,X_k) &= \sum_{i=0}^{k}(-1)^i X_i(\alpha(X_0,\ldots,\hat{X}_i,\ldots,X_k)) + \\ &\quad \sum_{i<j}(-1)^{i+j}\alpha([X_i,X_j],X_0,\ldots,\hat{X}_i,\ldots,\hat{X}_j,\ldots,X_k),\end{aligned}$$
where $\hat{X}_l$ denotes that X_l is deleted. We have $\mathrm{dd}\alpha = 0$ and $F^*\mathrm{d} = \mathrm{d}F^*$.
- **Cartan's formula:** $\mathbf{L}_X\alpha = \mathrm{d}\mathbf{i}_X\alpha + \mathbf{i}_X\mathrm{d}\alpha$.
- **Poincaré's lemma:** If $\mathrm{d}\alpha = 0$, then α is locally exact. That is, there exist a neighbourhood about each point on which $\alpha = \mathrm{d}\beta$.

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