SINGULAR SPLITTING OF SEPARATRICES FOR THE PERTURBED MCMILLAN MAP

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The problem

In this lecture, we consider the following family of planar standard-like maps

$$F(x,y) = \left(y, -x + \frac{2\mu_0 y}{1+y^2} + \varepsilon V'(y) \right),$$

where $V(y) = \sum_{n\geq 1} V_n y^{2n}$ is an even entire function.

Provided that $\mu_0 + V_1\varepsilon > 1$, the origin $O = (0,0)$ is a hyperbolic fixed point with $\mathrm{Spec}[\,\mathrm{d}F(O)] = \left\{ \mathrm{e}^{\pm h} \right\}$, and its *characteristic exponent* $h > 0$ is given by $\cosh h = \mu_0 + V_1\varepsilon$.

For $\varepsilon = 0$, F is an integrable map (called *McMillan map*), whose stable and unstable invariant curves to the origin coincide, giving rise to a *separatrix*. Thus, the map F can be considered as a perturbation of the McMillan map, ε being the *perturbation strength*.

These two parameters, $h > 0$ and ε, will be considered the intrinsic parameters of the map F under study.

Our goal is to show that for $\varepsilon \neq 0$ and for a general perturbation, the stable and unstable invariant curves of the perturbed map intersect transversally along exactly two *primary homoclinic orbits* in the first quadrant; in particular, the unperturbed separatrix splits. The term primary means that the homoclinic orbits persist for all ε small enough.

The pieces of the perturbed invariant curves between two consecutive homoclinic points enclose a region called *lobe*. Our measure of the splitting size will be the area of this lobe. This *lobe area* is a homoclinic symplectic invariant, that is, it does not depend on the symplectic coordinates used, and all the lobes have the same area. Lobe areas also measure the flux along the homoclinic tangle, which is related to the study of transport.

Both parameters, $h > 0$ and ε, will be small "enough", but the exact interpretation of this sentence is crucial for understanding the different kinds of results to be presented. Specifically, we are going to deal with the following situations:

1. *The regular case*: fixed $h > 0$, and $\varepsilon \to 0$.
2. *The singular case*: $h \to 0^+$. In its turn this case subdivides in two sub-cases:
   (a) *The non-perturbative case:* ε fixed and $h \to 0^+$.
   (b) *The perturbative case:* $\varepsilon = \mathrm{O}(h^p)$ and $h \to 0^+$, for some $p > 0$.

The analytical results here presented are expressed in terms of the *Melnikov potential* of the problem, which gives *explicit* formulae for our map. This is the reason for our choice of the perturbed McMillan map as a model, instead of more celebrated maps like the Hénon map or the Taylor-Chirikov map. (See [6] for results concerning those maps.)

The name "singular" for the case $h \to 0^+$, is due to the fact that the lobe areas are exponentially small in h. The measure of such small quantities requires a very careful treatment, both from a numerical and an analytical point of view.

The model

The family of standard-like maps under study is given by

$$F(x,y) = (y, -x + U'(y)), \qquad U(y) = \mu_0 \log(1+y^2) + \varepsilon V(y), \tag{1}$$

where $V(y) = \sum_{n\geq 1} V_n y^{2n}$ is an even entire function. For

$$\mu := \mu_0 + \varepsilon V_1 > 1,$$

the origin $O = (0,0)$ is a hyperbolic fixed point with $\mathrm{Spec}[\,\mathrm{d}F(O)] = \left\{\mathrm{e}^{\pm h}\right\}$, where the *characteristic exponent* $h > 0$ is determined by $\cosh h = \mu$.

We will consider the characteristic exponent h and the *perturbation strength* ε as the intrinsic parameters of our model. Accordingly, for every $h > 0$ and every real ε, we rewrite the map (1) in the form

$$\begin{aligned} F(x,y) &= (y, -x + U'(y)), \quad U(y) = U_0(y) + \varepsilon U_1(y), \\ U_0(y) &= \mu \log(1+y^2), \qquad\quad U_1(y) = V(y) - V_1 \log(1+y^2). \end{aligned} \tag{2}$$

From now on, the subscript "0" will denote an unperturbed quantity, that is, $\varepsilon = 0$, and the following notations will be used without further comment:

$$\mu = \cosh h, \qquad \gamma = \sinh h.$$

Setting $\varepsilon = 0$ in (2), we obtain the so-called *McMillan map*

$$F_0(x,y) = (y, -x + U_0'(y)) = \left(y, -x + \frac{2\mu y}{1+y^2}\right),$$

which is an *integrable* map, with a polynomial first integral given by

$$I_0(x,y) = x^2 - 2\mu xy + y^2 + x^2y^2.$$

The phase space associated to F_0 is rather simple, since it is foliated by the level curves of the first integral I_0, which are symmetric with respect to the origin. As $\mu > 1$, the zero level of I_0 is a lemniscate, whose loops are *separatrices* to the origin. From now on, we will concentrate on the separatrix Λ in the quadrant $\{x, y > 0\}$, which can be parameterized by

$$z_0(t) = (x_0(t), y_0(t)) = (\xi_0(t - h/2), \xi_0(t + h/2)), \qquad \xi_0(t) = \gamma \,\mathrm{sech}\, t. \tag{3}$$

This parameterization is called *natural* since $F_0(z_0(t)) = z_0(t+h)$, a fact that can be checked simply by noting that $\xi_0(t)$ is a homoclinic solution of the difference equation

$$\xi_0(t+h) + \xi_0(t-h) = U_0'(\xi_0(t)).$$

A natural parameterization is unique except for a translation in the independent variable. To determine it, it is worth looking at the reversors of the map.

Indeed, the involution $R_0^+(x,y) := (y,x)$ is a *reversor* of the McMillan map F_0, that is, $F_0^{-1} = R_0^+ F_0 R_0^+$. The separatrix Λ is R_0^+-symmetric, i.e., $R_0^+\Lambda = \Lambda$, and intersects transversely the fixed set $C_0^+ := \{z : R_0^+ z = z\}$ of R_0^+ in one point z_0^+. The parameterization (3) of Λ has been chosen to satisfy $z_0(0) = z_0^+$.

Moreover, the involution $R_0^- := F_0 R^+$ is another reversor of F_0. The separatrix Λ is also R_0^--symmetric and intersects transversely the fixed set C_0^- of R_0^- in one point z_0^-, and it turns out that $z_0(h/2) = z_0^-$. The associated orbits $\mathcal{O}_0^+ := \{z_0(nh) : n \in \mathbb{Z}\}$, $\mathcal{O}_0^- := \{z_0(h/2+nh) : n \in \mathbb{Z}\}$, are called *symmetric* homoclinic orbits, since $R_0^\pm \mathcal{O}_0^\pm = \mathcal{O}_0^\pm$.

For $\varepsilon \neq 0$, the phase portrait of the exact map (2) looks more intricate. The origin is a hyperbolic fixed point with the *same* characteristic exponent h, since the perturbation $\varepsilon U_1'(y) = \mathrm{O}(y^3)$ does not not contain linear terms at the origin. We denote by $\mathcal{W}^{\mathrm{u,s}}$ its unstable and stable invariant curves with respect to F. Since the map (2) is odd, the invariant curves are symmetric with respect to the origin, so that we concentrate only on the positive quadrant $\{x, y > 0\}$.

By the form of the perturbation, $R^+ := R_0^+$ is also a reversor of F, as well as the involution $R^- := FR^+$, which is given by $R^-(x,y) = (x, -y + U'(x))$. Their fixed sets $C^\pm = \{z : R^\pm z = z\}$ are important because $R^\pm \mathcal{W}^{\mathrm{u}} = \mathcal{W}^{\mathrm{s}}$. Consequently, any point in the intersection $C^\pm \cap \mathcal{W}^{\mathrm{u}}$ is a homoclinic point, and gives rise to a symmetric homoclinic orbit.

Since the separatrix Λ intersects transversely the unperturbed curve $C_0^\pm$ at the point $z_0^\pm$, there exists a point $z^\pm = z_0^\pm + \mathrm{O}(\varepsilon) \in C^\pm \cap \mathcal{W}^{\mathrm{u}}$ and, therefore, there exist at least two symmetric homoclinic orbits on the quadrant $\{x, y > 0\}$, for $|\varepsilon|$ small enough. They are called *primary* since they exist for arbitrary small $|\varepsilon|$.

The Melnikov theory

We now recall some perturbative results [1, 2] to detect the existence of transverse primary homoclinic orbits for exact maps. For simplicity, we will assume that all the objects are smooth and we shall restrict the discussion to maps on the plane with the usual symplectic structure: the area.

Given the symplectic form $\omega = \mathrm{d}x \wedge \mathrm{d}y$ on the plane $\mathbb{R}^2$, a map $F : \mathbb{R}^2 \to \mathbb{R}^2$ is called *exact* if there exists some function $S : \mathbb{R}^2 \to \mathbb{R}$ such that $F^*(y\,\mathrm{d}x) - y\,\mathrm{d}x = \mathrm{d}S$. The function S is called the *generating function* of F and, except for an additive constant, it is uniquely determined.

Let $F_0 : \mathbb{R}^2 \to \mathbb{R}^2$ be an integrable exact diffeomorphism with a separatrix Λ to a hyperbolic fixed point z_∞. Next, consider a family of exact diffeomorphisms $F_\varepsilon = F_0 + \varepsilon F_1 + \mathrm{O}(\varepsilon^2)$, as a general perturbation of the situation above, and let $S_\varepsilon = S_0 + \varepsilon S_1 + \mathrm{O}(\varepsilon^2)$ be the generating function of F_ε.

We introduce the *Melnikov potential* of the problem as the smooth real-valued function

$$L : \Lambda \to \mathbb{R}, \qquad L(z) = \sum_{n \in \mathbb{Z}} \widehat{S}_1(z_n), \qquad z_n = {F_0}^n(z), \qquad z \in \Lambda, \tag{4}$$

where $\widehat{S}_1 : \mathbb{R}^2 \to \mathbb{R}$ is defined by $\widehat{S}_1 = S_1 - y\,\mathrm{d}x(F_0)[F_1]$. (In components, writing $F_0 = (X_0, Y_0)$, $F_1 = (X_1, Y_1)$, $\widehat{S}_1$ is simply given by $\widehat{S}_1 = S_1 - Y_0 X_1$.) In order to get an absolutely convergent series (4), $\widehat{S}_1$ is determined by imposing $\widehat{S}_1(z_\infty) = 0$.

The differential of L is a geometrical object which gives the $\mathrm{O}(\varepsilon)$-distance between the perturbed invariant curves $\mathcal{W}^{\mathrm{u,s}}$. More precisely, let (t, e) be some *cotangent coordinates* adapted to Λ—that is, in these coordinates the separatrix Λ is given locally by $\{e = 0\}$ and the symplectic form ω reads as $\mathrm{d}t \wedge \mathrm{d}e$—and let $\{e = E^{\mathrm{u,s}}(t)\}$ be a part of $\mathcal{W}^{\mathrm{u,s}}$. (Let us recall that cotangent coordinates can be defined in neighborhoods of Lagrangian sub-manifolds.) Then, in [2] it is shown that

$$E^{\mathrm{u}}(t) - E^{\mathrm{s}}(t) = \varepsilon L'(t) + \mathrm{O}(\varepsilon^2),$$

and that the construction above does not depend on the cotangent coordinates used.

The following theorem is a straightforward corollary of this geometric construction.

Theorem 1 *Under the above notations and hypotheses, the non-degenerate critical points of L are associated to perturbed transverse homoclinic orbits. Moreover, when all the critical points of L are non-degenerate, all the primary homoclinic orbits arising from Λ are found in this way. Finally, if z and z' are consecutive (in the internal order of the separatrix) non-degenerate critical points of L, their associated perturbed homoclinic orbits determine a lobe with area*

$$A = \varepsilon[L(z) - L(z')] + \mathrm{O}(\varepsilon^2).$$

The regular case

We are now ready to apply the theory above to our model. Along this section, the characteristic exponent $h > 0$ will be considered *fixed*, and then $\varepsilon \to 0$.

It is worth noting that the knowledge of the natural parameterization (3) of the unperturbed separatrix Λ will be the crucial point to compute explicitly the Melnikov potential (4).

The map $F = F_0 + \varepsilon F_1 + \mathrm{O}(\varepsilon^2)$ given in (2) is exact with generating function $S(x, y) = -xy + U_0(y) + \varepsilon U_1(y)$. Writing its expression in components $F_0 = (X_0, Y_0)$, $F_1 = (X_1, Y_1)$, it turns out that $X_1 = 0$, and consequently $\widehat{S}_1(x, y) = S_1(x, y) = U_1(y)$.

The parameterization (3) allows us to write the Melnikov potential (4) of our problem as

$$L(t) := L(z_0(t)) = \sum_{n\in\mathbb{Z}} U_1(y_0(t + hn)) = \sum_{n\in\mathbb{Z}} [f(t + hn) - g(t + hn)],$$

where $f(t) := V\left(\xi_0(t + h/2)\right)$ and $g(t) := V_1 \log\left(1 + \xi_0(t + h/2)^2\right)$.

We are now confronted to the computation of $L(t)$. Let $\sum_{n\in\mathbb{Z}} v_n(h)\tau^{2n}$ be the Laurent expansion around $\tau = 0$ of the function $\tau \mapsto f(-h/2 + \pi\mathrm{i}/2 - \mathrm{i}h\tau)$, and

$$\Theta^0(h) := 8\pi \sum_{n\geq 1} \frac{(2\pi)^{2n-1}}{(2n-1)!} v_{-n}(h) = 8\pi\widehat{V}(2\pi) + \mathrm{O}(h^2), \tag{5}$$

where $\widehat{V}(\xi) := \sum_{n\geq 1} V_n \xi^{2n-1}/(2n-1)!$ is the so-called Borel transform of $V(y)$. If $V(y)$ is a polynomial, $\Theta^0(h)$ can be explicitly computed in a finite number of steps. For instance,

$$\Theta^0(h) = \begin{cases} 8\pi^2\gamma^2 h^{-2} & \text{for } V'(y) = y \\ \frac{8}{3}\pi^2\gamma^4 h^{-2}[1 + \pi^2 h^{-2}] & \text{for } V'(y) = y^3 \end{cases}.$$

It turns out that $\Theta^0(h)$ is an even entire function such that

$$L(t) = \text{constant} + \mathrm{e}^{-\pi^2/h} \cos(2\pi t/h) \left[-\Theta^0(h)/2 + \mathrm{O}(\mathrm{e}^{-2\pi^2/h})\right]. \tag{6}$$

We refer to [3] for the details.

From the formula (5), it is clear that if $\widehat{V}(2\pi) \neq 0$ and h is small enough, the set of critical points of the Melnikov potential (6) is $h\mathbb{Z}/2$. All of them are non-degenerate, and parameterize the two unperturbed, symmetric, primary homoclinic orbits $\mathcal{O}_0^{\pm}$. Now, the following result is a corollary of theorem 1.

Theorem 2 *Assume that $\widehat{V}(2\pi) \neq 0$. Then, for any small enough (but fixed) characteristic exponent $h > 0$, there exists a positive constant $\varepsilon^* = \varepsilon^*(h)$ such that the map (2) has exactly two transverse, symmetric, primary homoclinic orbits $\mathcal{O}^\pm$ in the quadrant $\{x, y > 0\}$, for $0 < |\varepsilon| < \varepsilon^*$. These orbits determine a lobe with area $A = \varepsilon A_{\mathrm{Mel}} + \mathrm{O}(\varepsilon^2)$, where the first order in ε approximation A_{Mel} is given by*

$$A_{\mathrm{Mel}} = L(h/2) - L(0) = \mathrm{e}^{-\pi^2/h}\left[\Theta^0(h) + \mathrm{O}(\mathrm{e}^{-2\pi^2/h})\right].$$

We note that $\varepsilon A_{\mathrm{Mel}}$ is the dominant term for the Melnikov formula of the lobe area A only if $|\varepsilon| < \varepsilon^*(h) = \mathrm{o}(\exp(-\pi^2/h))$. Otherwise, in the case $\varepsilon = \mathrm{O}(h^p)$, the Melnikov theory as described is not useful, since it only gives the very coarse estimate $A = \mathrm{O}(h^{2p})$, and not the desired exponentially small asymptotic behavior.

The singular non-perturbative case

The limit $h \to 0^+$ in (2) is highly singular, since all the interesting dynamics is contained in a $\mathrm{O}(h)$ neighborhood of the origin, which becomes a parabolic point of the map for $h = 0$. To see clearly this behavior, we perform the following linear change of variables:

$$z = Cw, \qquad C = h\begin{pmatrix} \lambda^{-1/2} & \lambda^{1/2} \\ \lambda^{1/2} & \lambda^{-1/2} \end{pmatrix}, \qquad z = (x, y), \qquad w = (u, v),$$

that is, we diagonalize the linear part of (2) at the origin and we scale by a factor h. Then, $C^{-1}(F(Cw)) = w + hX^0(w) + \mathrm{O}(h^2)$, where

$$X^0(u, v) = \left(u - \eta(u+v)^3, -v + \eta(u+v)^3\right), \qquad \eta = 1 - (V_1 + 2V_2)\varepsilon, \tag{7}$$

is a Hamiltonian vector field, with associated Hamiltonian

$$H^0(u, v) = uv - \eta(u+v)^4/4. \tag{8}$$

This shows clearly that $C^{-1}FC$ is $\mathrm{O}(h)$-close to the identity, and that, after the change of variables $z = Cw$, the map (2) asymptotes to the Hamiltonian flow associated to the vector field (7) when $h \to 0^+$. When such situation takes place, it is known that the map has homoclinic points to its (weakly) hyperbolic fixed point for $h \to 0^+$, if and only if the limit Hamiltonian flow has a homoclinic orbit to its hyperbolic equilibrium point.

From the expression (8), we see that the zero level $\{H^0(u, v) = 0\}$ contains homoclinic connections to the origin if and only if $\eta > 0$, i.e., if and only if

$$(V_1 + 2V_2)\varepsilon < 1. \tag{9}$$

Assuming $\eta > 0$, the homoclinic orbit of the Hamiltonian (8) is given by

$$w^0(t) = \eta^{-1/2}\left(\frac{\cosh t - \sinh t}{2\cosh^2 t}, \frac{\cosh t + \sinh t}{2\cosh^2 t}\right),$$

which is analytic on the strip $\{t \in \mathbb{C} : |\Im t| < d := \pi/2\}$. In this situation, it is also well-known [5] that the splitting size is $\mathrm{O}(\exp(-\beta/h))$, for all $\beta < 2\pi d = \pi^2$. We summarize this result in the following theorem.

Theorem 3 *For any real ε verifying (9), and any $\beta \in (0, \pi^2)$, there exists a constant $N = N(\varepsilon, \beta) \geq 0$ such that the area of the lobe between the invariant curves of the map (2) satisfies:*

$$|A| \leq N\,\mathrm{e}^{-\beta/h} \qquad (\varepsilon \text{ fixed}, \ h \to 0^+).$$

The singular perturbative case

The previous theorem gives only an upper bound for the lobe area and not an asymptotic one (the constant $N(\varepsilon, \beta)$ can blow up when $\beta \to \pi^2$). On the other hand, it does not exclude the case $A = 0$, that is, it cannot detect effective splitting of separatrices. In the perturbative case $\varepsilon = \mathrm{O}(h^p)$, for $p > 6$, the following result gives an asymptotic expression for the lobe area in terms of the Melnikov potential, and establishes transversal splitting of separatrices.

Theorem 4 *Assume that* $\varepsilon = \mathrm{O}(h^p)$, $p > 6$. *Then, if* $\widehat{V}(2\pi) \neq 0$, *there exists* $h^* > 0$ *such that the map (2) has exactly two transverse, symmetric, primary homoclinic orbits in the first quadrant, for all* $0 < h < h^*$. *Moreover, they enclose a lobe with area*

$$A = \varepsilon\, \mathrm{e}^{-\pi^2/h} \left[8\pi \widehat{V}(2\pi) + \mathrm{O}(h^2) \right] \qquad (h \to 0^+).$$

If $\widehat{V}(2\pi) = 0$, *there may exist more primary homoclinic orbits, but the area of any lobe is* $\mathrm{O}(\varepsilon h^2\, \mathrm{e}^{-\pi^2/h})$.

The proof of this theorem is contained in [3, 4]. It is based on the study of the perturbed invariant curves of the maps (1) for complex values of the discret time t, as close as possible to the singularities of the unperturbed natural parameterization $z^0(t)$ given in (3). This approach was suggested by V.I. Lazutkin several years ago, for the case of the Taylor-Chirikov map.

To the best of our knowledge, this theorem is the first analytical result about asymptotics for singular separatrix splitting for a map with a complete and rigorous proof.

A numerical study for singular cases can be found in [4]. The numerical results suggest that the lobe area A is given by

$$A = \varepsilon\, \mathrm{e}^{-\pi^2/h} \left[\Theta^\varepsilon(h) + \mathrm{O}(\mathrm{e}^{-2\pi^2/h}) \right] \qquad (\varepsilon \text{ fixed}, \; h \to 0^+),$$

where $\Theta^\varepsilon(h)$ is an even Gevrey-1 function such that the radius of convergence of its Borel transform is $2\pi^2$, and $\Theta^\varepsilon(h) = \Theta^0(h) + \mathrm{O}(\varepsilon)$, uniformly in $h \in [0, 1]$.

We finish this lecture by remarking that the numerical computation of the lobe areas for singular cases requires the use of an expensive multiple-precision arithmetic and to expand the invariant curves $\mathcal{W}^{\mathrm{u,s}}$ up to an optimal order, which is very large, see [4].

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