HOMOCLINIC ORBITS TO INVARIANT TORI IN HAMILTONIAN SYSTEMS*

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Abstract. We consider a perturbation of an integrable Hamiltonian system which possesses invariant tori with coincident whiskers (like some rotators and a pendulum). Our goal is to measure the splitting distance between the perturbed whiskers, putting emphasis on the detection of their intersections, which give rise to homoclinic orbits to the perturbed tori. A geometric method is presented which takes into account the Lagrangian properties of the whiskers. In this way, the splitting distance is the gradient of a splitting potential. In the regular case (also known as a priori-unstable: the Lyapunov exponents of the whiskered tori remain fixed), the splitting potential is well-approximated by a Melnikov potential. This method is designed as a first step in the study of the singular case (also known as a priori-stable: the Lyapunov exponents of the whiskered tori approach to zero when the perturbation tends to zero).

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1. Introduction and results.

1.1. Nearly-integrable Hamiltonians. The Hamiltonian dynamical systems that are close to integrable ones appear in a natural way as models of a wide class of real systems, and thus constitute a very active field of research in classical mechanics. The behavior of these systems is far from being completely understood, and one of the most relevant questions is the stability or instability of their trajectories. This problem remains unsolved for systems with more than 2 degrees of freedom.

The unperturbed rôle is played by a (completely) integrable Hamiltonian with n degrees of freedom. The Liouville–Arnold theorem (see for instance [3]) establishes, under certain hypotheses, the existence on some region of the phase space of canonical *action–angle variables* $(\varphi, I) = (\varphi_1, \dots, \varphi_n, I_1, \dots, I_n) \in \mathbb{T}^n \times G \subset \mathbb{T}^n \times \mathbb{R}^n$, in which the Hamiltonian only depends on the action variables: $h(I)$. The associated Hamiltonian equations for a trajectory $(\varphi(t), I(t))$ are

$$\dot{\varphi} = \omega(I), \qquad \dot{I} = 0,$$

where $\omega = \partial_I h$. Hence the dynamics is very simple: every n-dimensional torus $I = \text{const}$ is invariant, with linear flow $\varphi(t) = \varphi(0) + \omega(I)t$, and thus

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all trajectories are stable. The motion on a torus is called quasiperiodic, with associated *frequencies* given by the vector $\omega(I) = (\omega_1(I), \dots, \omega_n(I))$.

Every n-dimensional invariant torus can be nonresonant or resonant, according to whether its frequencies are rationally independent or not. A nonresonant torus is densely filled by any of its trajectories. On the other hand, a resonant torus is foliated into a family of lower dimensional tori.

A *nearly-integrable* Hamiltonian can be written in the form

$$H(\varphi, I) = h(I) + \varepsilon f(\varphi, I), \tag{1.1}$$

where ε is a small perturbation parameter. Then the Hamiltonian equations are

$$\dot{\varphi} = \omega(I) + \varepsilon \partial_I f(\varphi, I), \qquad \dot{I} = -\varepsilon \partial_\varphi f(\varphi, I).$$

For $\varepsilon \neq 0$, the dynamics can be very complicated: there can exist, in principle, chaotic trajectories or even unstable (in the sense that they can wander very far from their initial conditions). Note that the variation of the actions $|I(t) - I(0)|$ could grow slowly but unboundedly.

Although there may exist unstable trajectories, two very relevant results on stability have been established for nearly-integrable Hamiltonian systems. These are the *KAM theorem* and the *Nekhoroshev theorem*. We are going to give a brief description of both of them, putting emphasis on the fact that they lead to two different notions of stability.

1.2. KAM theorem. We can say that KAM (Kolmogorov–Arnold–Moser) theory is concerned with the preservation of quasiperiodic motions under small perturbations: the theorem states, under a suitable nondegeneracy condition, that most of the n-dimensional invariant tori $I = \text{const}$ of the integrable Hamiltonian h survive in (1.1), with some deformation, for $|\varepsilon|$ small enough. In fact, KAM theorem guarantees the preservation of the tori having "sufficiently" nonresonant frequencies, for which the influence of the *small divisors* $\langle k, \omega(I)\rangle$ can be overcome. This is expressed by means of a *Diophantine condition* on the frequency vector $\omega(I)$: for some constants τ and γ,

$$|\langle k, \omega(I)\rangle| \geq \gamma\, |k|^{-\tau} \quad \forall k \in \mathbb{Z}^n \setminus \{0\},$$

where $|k| = \sum_{j=1}^n |k_j|$. The vectors satisfying this condition for given $\tau > n-1$ and $\gamma > 0$ fill a Cantorian set of relative measure $1 - \mathrm{O}(\gamma)$ in $\mathbb{R}^n$. In this way, one obtains *perpetual stability* in the perturbed Hamiltonian, but only for initial conditions on the perturbed or *KAM tori* (the surviving ones), which form a Cantorian set not containing any open subset although its measure is large.

The KAM theorem was "discovered" in the 50's by Kolmogorov [30], and constituted one of the most relevant advances about stability in nearly-integrable Hamiltonian systems. In his first theorem, Kolmogorov established the preservation of only one fixed Diophantine torus (see also [4]).

Some years later, Arnold [1] proved, in the analytic case, the existence of a large family of invariant tori, giving an estimate for the measure of their complementary set (see also [48]). Also, Moser [42] proved an analogous theorem, in the differentiable case, for area-preserving maps satisfying a twist condition. As general references on KAM theory, see [9, 11].

The nondegeneracy condition required in KAM theorem is imposed on the frequency map $\omega = \partial_I h$ to ensure that a large set of actions I have Diophantine frequencies $\omega(I)$. There are two usual types of conditions:

$$\det\left(\partial_I\omega(I)\right) \neq 0, \qquad \det\begin{pmatrix} \partial_I\omega(I) & \omega(I) \\ \omega(I)^\top & 0 \end{pmatrix} \neq 0,$$

(for all $I \in G$), that can be called, respectively, *Kolmogorov nondegeneracy* and *isoenergetic nondegeneracy.* The second one is also called Arnold nondegeneracy. Under any of these conditions, for a given $\varepsilon > 0$ the invariant tori whose frequencies satisfy the Diophantine condition with $\tau > n-1$ and $\gamma = \mathrm{O}\left(\sqrt{\varepsilon}\right)$ are preserved, and the measure of the complement is small: $\mathrm{O}\left(\sqrt{\varepsilon}\right)$ (see [48]).

Although the two versions (Kolmogorov and isoenergetic) of KAM theorem are known since the works by Kolmogorov, Arnold and Moser, there are only a few published proofs of the isoenergetic version. Complete proofs are given in [23, 10], indirectly from the Kolmogorov version. A direct proof of the isoenergetic KAM theorem, without using the Kolmogorov version, can be found in [19].

It has to be pointed out that the isoenergetic condition is more significant from the point of view of the stability, since it ensures the existence of a large family of KAM tori on each energy level $H = \text{const}$. For $n = 2$, one deduces stability even for trajectories which do not lie on these tori, because these 2-dimensional invariant tori separate the 3-dimensional energy levels. This reasoning does not hold for $n > 2$, and one cannot guarantee stability. So in this case there could exist unstable trajectories, although it seems clear that the KAM tori should constitute stronger barriers to instability.

1.3. Nekhoroshev theorem and effective stability. The other relevant result to be commented here is Nekhoroshev theorem, which leads to the concept of *effective stability.* Nekhoroshev theorem [44], first stated in 1977, establishes for all the trajectories of the system (1.1) that the variation of the action variables remains small for a very long time, exponentially large with respect to the parameter $\varepsilon > 0$. For every initial condition $(\varphi(0), I(0))$ one has an estimate of the type

$$|I(t) - I(0)| \leq r_0\,\varepsilon^b \qquad \text{for } \ |t| \leq T_0 \,\exp\left\{(\varepsilon_0/\varepsilon)^a\right\}.$$

The constants $a, b > 0$ are called *stability exponents.*

For the validity of Nekhoroshev theorem one imposes a *steepness condition* (see [44]) on the integrable Hamiltonian h. This is a very general condition, avoiding the quick escape of trajectories along certain directions related to resonances of the frequency vector $\omega(I) = \partial_I h(I)$. The simplest case is that of *quasiconvex* functions: one says h to be quasiconvex if, for any $I \in G$ and $v \in \mathbb{R}^n$,

$$\langle v, \omega(I) \rangle = 0 \quad \Longrightarrow \quad \left\langle v, \partial_I^2 h(I)\, v \right\rangle \neq 0.$$

For a perturbation of a quasiconvex Hamiltonian, the estimate of Nekhoroshev theorem holds with the stability exponents

$$a = b = \frac{1}{2n}.$$

The proof of the theorem with these exponents has been given in [37, 49]. However, Chirikov [14] had predicted several years before that the exponent $a = 1/2n$ should be optimal, a fact that was overlooked until it was explained in Lochak's survey [34] (see also [35]), where the exponent $a = 1/(2n+1)$ was already obtained.

Comparing Nekhoroshev theorem to KAM theorem, we see that perpetual stability has been replaced by finite time stability, but on the other hand the estimates are valid for all trajectories in the phase space. Since (despite the theoretical importance of KAM theorem) it is not possible to know whether a given trajectory lies on a KAM torus, the effective stability estimates are of obvious interest from the point of view of the applications. Some recent results are intending to fill the gap between KAM and Nekhoroshev theorems. These results concern the "stickiness" of KAM tori [46], as well as the remarkable results about "superexponential stability" [40], or the existence of "quasi-invariant tori" [19].

1.4. Arnold diffusion: whiskered tori, splitting, and transition chains. Nothing is said in KAM theorem about the stability of the trajectories close to unperturbed resonant invariant tori. Nekhoroshev theorem does not exclude the existence of unstable trajectories, but predicts for them an exponentially long stability time. This extremely slow phenomenon of instability is called *Arnold diffusion,* and a first description of it was given in 1964 by Arnold [2] by means of his famous example. As noticed in section 1.2, diffusion can only take place for more than 2 degrees of freedom.

The *Arnold's example* proposed in [2] is a nonautonomous Hamiltonian, periodic in the time variable t: in canonical variables $(x, y, \varphi_1, I_1) \in \mathbb{T} \times \mathbb{R} \times \mathbb{T} \times \mathbb{R}$,

$$H(x, y, \varphi_1, I_1, t) = \frac{1}{2}\left(y^2 + I_1^2\right) + \varepsilon(\cos x - 1)\left(1 + \mu(\sin\varphi_1 + \cos t)\right),$$

with ε and μ as two independent parameters. Taking t as a new angular variable, the Hamiltonian becomes the following 3-degrees-of-freedom

Hamiltonian: in canonical variables $(x, y, \varphi, I) \in \mathbb{T} \times \mathbb{R} \times \mathbb{T}^2 \times \mathbb{R}^2$,

$$\begin{aligned} H(x,y,\varphi,I) = {} & \frac{1}{2}\left(y^2 + I_1^2\right) + I_2 \\ & + \varepsilon(\cos x - 1)\left(1 + \mu(\sin\varphi_1 + \cos\varphi_2)\right). \end{aligned} \tag{1.2}$$

We shall ignore the variable I_2 which does not take part in the Hamiltonian equations, considering a 5-dimensional phase space.

The frequency vector $\omega(y, I_1) = (y, I_1, 1)$ has, on $y = 0$, a *single resonance* (double for rational I_1), and for $\varepsilon = 0$ the 3-dimensional tori $y, I_1 = \text{const}$ are foliated into 2-dimensional invariant tori. On the resonance, KAM theorem cannot be applied.

A first perturbative step is to consider $\varepsilon > 0$ but $\mu = 0$, still having an integrable Hamiltonian (1.2) which is an *uncoupled* system formed by one pendulum (variables (x, y)) and 2 rotors (variables $(\varphi_1, \varphi_2, I_1)$). The 3-dimensional tori on the resonance $y = 0$ have been destroyed, but a 1-parameter family of 2-dimensional tori $\mathcal{T}_{I_1}$ still remains along the resonance: $x = y = 0$, $I_1 = \text{const}$, $(\varphi_1, \varphi_2) \in \mathbb{T}^2$, with associated frequencies $(I_1, 1)$. These tori come from the hyperbolic equilibrium point $x = y = 0$ of the pendulum, and hence they are *hyperbolic invariant tori,* also called *whiskered tori,* with associated 3-dimensional *stable* and *unstable whiskers* (invariant manifolds) $\mathcal{W}^{\pm}_{I_1}(\varepsilon, 0)$, inherited from the *separatrices* of the pendulum. Since these separatrices constitute a homoclinic connection given by the equation $y^2/2 + \varepsilon(\cos x - 1) = 0$, it turns out that for $\varepsilon > 0$, $\mu = 0$, the whiskers coincide: $\mathcal{W}^{+}_{I_1}(\varepsilon, 0) = \mathcal{W}^{-}_{I_1}(\varepsilon, 0)$. Notice that the characteristic exponents of the origin of the pendulum (and therefore the Lyapunov exponents of the whiskered tori $\mathcal{T}_{I_1}$) are $\pm\sqrt{\varepsilon}$, and therefore the whiskered tori are *weakly hyperbolic* for $|\varepsilon|$ small. Notice also that there are elliptic invariant tori on $x = \pi$, $y = 0$, which come from the elliptic point of the pendulum.

For $\varepsilon > 0$ and $\mu \neq 0$, the Hamiltonian (1.2) is no longer integrable. The *same* tori $\mathcal{T}_{I_1}$ are still whiskered tori, independently of μ, but their stable and unstable whiskers $\mathcal{W}^{\pm}_{I_1}(\varepsilon, \mu)$ do depend on μ and are not expected to coincide for $\mu \neq 0$. Indeed, Arnold [2] applied the Poincaré's perturbative method [47, §19] for detecting *splitting* of separatrices and showed that, for μ small enough: $0 < |\mu| < \mu_0(\varepsilon)$, the whiskers do not coincide and intersect transversely. The intersections are obtained as zeros of a distance function between the whiskers of the Hamiltonian (1.2). The maximum of this distance can be estimated as

$$\Delta \sim \mu \exp\left(-1/\sqrt{\varepsilon}\right) + \mathrm{O}\left(\mu^2\right), \tag{1.3}$$

where $\mu \exp\left(-1/\sqrt{\varepsilon}\right)$ is a bound of some integrals provided by Poincaré's method, which are usually known as *Poincaré–Melnikov integrals* (or, more briefly, *Melnikov integrals*). An important feature is that the Melnikov integral is exponentially small in ε, as a result of the weak hyperbolicity of the unperturbed whiskers (the power $1/2$ of ε in the exponential is due to the fact that the Hamiltonian (1.2) has only a finite number of harmonics in φ), and is the dominant term in (1.3) only if μ is chosen exponentially small with respect to ε:

$$\mu = \mathrm{o}\left(\exp\left(-1/\sqrt{\varepsilon}\right)\right).$$

The intersections above between whiskers are called *homoclinic* because they involve only one torus. These homoclinic intersections can be extended to *heteroclinic* intersections between the unstable whisker of a given torus $\mathcal{T}_{I_1}$ and stable whiskers of other tori $\mathcal{T}_{I_1'}$ in a neighborhood, as long as they are chosen exponentially close with respect to ε (see (1.3)):

$$|I_1' - I_1| \preceq \mu \exp\left(-1/\sqrt{\varepsilon}\right).$$

It is then possible to construct a *transition chain* along the resonance $y = 0$. This notion was introduced in [2], and means a finite sequence of tori $\mathcal{T}_{I_1^{(0)}}, \ldots, \mathcal{T}_{I_1^{(N)}}$ having *nonresonant* frequencies, with $\left|I_1^{(N)} - I_1^{(0)}\right| = \mathrm{O}(1)$, and such that the unstable whisker of each torus intersects transversely the stable whisker of the next one. Besides, it was asserted in [2] that the tori in the chain are *transition tori*: for every torus, arbitrarily small neighborhoods of points in its stable and unstable whiskers are connected by trajectories of the Hamiltonian. From these facts, Arnold concluded that diffusion can take place along neighborhoods of all the successive tori of a transition chain.

A feature that makes Arnold's example very particular is the fact that the perturbation and its derivatives vanish on the whiskered tori $\mathcal{T}_{I_1}$. This implies that all these tori survive in the perturbation for $\mu \neq 0$, remaining even unchanged. This is very convenient for the purpose of finding transition chains of whiskered tori, since it makes possible to choose two of them as close as necessary.

For a general perturbation, the survival of all the tori associated to a single resonance will not be true, and an adapted version of KAM theorem (see section 1.7 for references) has to be applied. This *hyperbolic KAM theorem* only ensures the preservation of those whiskered tori with Diophantine frequencies, constituting a Cantorian set near the resonance. Thus, the set of surviving whiskered tori has a large relative measure but also "many gaps", which are due to the presence of double resonances. These gaps and the fact that the splitting is expected to be exponentially small can make the unstable whisker of a torus with frequency on this given set not to intersect the stable whisker of the "next" surviving torus, and prevents the construction of a transition chain in this way.

A generalization of Arnold's example to the case of an arbitrary number of degrees of freedom, was introduced by Lochak [33, 34] in order to point out the main difficulties concerning the detection of the diffusion. In canonical variables $(x, y, \varphi, I) \in \mathbb{T} \times \mathbb{R} \times \mathbb{T}^n \times \mathbb{R}^n$, *Lochak's example* is:

$$H(x, y, \varphi, I) = \frac{1}{2}\left(y^2 + \langle I, I\rangle\right) + \varepsilon(\cos x - 1) + \varepsilon\mu F(x, \varphi), \tag{1.4}$$

where F is an analytic function on a complex neighborhood of $\mathbb{T}^{n+1}$. As a new feature (see [34, V§2]), the fact that the function F is allowed to have arbitrarily high harmonics in the angles φ brings out the problem of the small divisors inside the Melnikov integrals, as an essential ingredient in order to decide which the relevant harmonics are in the study of these integrals. It turns out that the Melnikov integrals can be estimated only for Diophantine whiskered tori, and they are again exponentially small in ε. Their asymptotic behavior is no longer $\exp(-1/\sqrt{\varepsilon})$, but depends on the arithmetic properties of the frequencies of the invariant tori, as was noticed in [34, page 119], where the connection between the size of the splitting and the speed of diffusion was first discussed. (See [53, 17, 18, 52] for some explicit examples.)

Recently, it has become a point of great interest the question of the validity of the predictions of the splitting given by the Melnikov integrals for a power-like relation $\mu = \varepsilon^p$, with $p > 0$. Unfortunately, this is not an easy problem because of the exponentially small character of these integrals with respect to ε.

We remark that Arnold's and Lochak's examples are important not only as specific examples designed to detect Arnold diffusion but, mainly, because they are special cases of the general model near a single resonance of a nearly integrable Hamiltonian system. We suspend here this report of results related to Arnold diffusion to deduce such model. In passing, the actual kind of relation $\mu = \varepsilon^p$ needed for the real applications will become transparent.

1.5. One step of normal form near a single resonance. Near a single resonance, under some generic hypotheses, we are going to carry out *one* step of (resonant) normal form procedure and try to obtain a new expression of the Hamiltonian that generalizes the examples considered in section 1.4. To reach a *hyperbolic normal form*, we will have to impose some nondegeneracy conditions on the perturbation f, since no hyperbolicity can be obtained from the unperturbed Hamiltonian h. Essentially, we follow Eliasson [25] and Niederman [45] (see also [51]).

To simplify the notation, the number of degrees of freedom in (1.1) will be, from now on, $n+1$ instead of n. Thus, the angle–action variables are $(\varphi, I) \in \mathbb{T}^{n+1} \times \mathbb{R}^{n+1}$.

Having selected an action $I^* = 0$, the unperturbed Hamiltonian h in (1.1) can be written (disregarding the additive constant) as:

$$h(I) = \langle \lambda^*, I\rangle + \frac{1}{2}\langle QI, I\rangle + \mathrm{O}_3(I).$$

Assume, for the selected action, that the associated frequency vector $\lambda^* = \partial_I h(0) \in \mathbb{R}^{n+1}$ has a *single resonance* ($\langle k^*, \lambda^* \rangle = 0$ for a certain $k^* \in \mathbb{Z}^{n+1} \setminus \{0\}$ and $\langle k, \lambda^* \rangle \neq 0$ for any $k \in \mathbb{Z}^{n+1}$ not co-linear to k^*). By a classical algebraic result, we can assume λ^* of the form

$$\lambda^* = (0, \omega^*),$$

where $\omega^* \in \mathbb{R}^n$ is nonresonant. In fact, we shall assume a Diophantine condition on ω^*.

Rewrite $(\varphi, I) \in \mathbb{T}^{n+1} \times \mathbb{R}^{n+1}$ as $(x, y, \varphi, I) \in \mathbb{T} \times \mathbb{R} \times \mathbb{T}^n \times \mathbb{R}^n$, and the matrix $Q = \partial_I^2 h(0)$ as

$$\begin{pmatrix} \beta^2 & q^\top \\ q & Q \end{pmatrix},$$

where we have put $\beta^2 > 0$ in order to fix ideas, $q \in \mathbb{R}^n$, and the new matrix Q is $n \times n$. We will assume $\beta = 1$; this can be achieved replacing y, I by y/β, I/β (changing in this way the time scale by a factor β), and rewriting ω^*/β, q/β^2, Q/β^2 as ω^*, q, Q respectively, and redefining also the function f. Then, we can write our Hamiltonian in the form

$$\begin{aligned} H(x, y, \varphi, I) &= h(y, I) + \varepsilon f(x, y, \varphi, I), \\ h(y, I) &= \langle \omega^*, I \rangle + \frac{y^2}{2} + \langle q, I \rangle\, y + \frac{1}{2} \langle QI, I \rangle + \mathrm{O}_3(y, I). \end{aligned}$$

We now perform *one* step of resonant normal form procedure: following the Lie method, we seek for functions $S(x, \varphi)$ and $R(x, y, \varphi, I) = \mathrm{O}(y, I)$ such that

$$\{S, h\} + V + R = f, \tag{1.5}$$

where $V(x)$ is the periodic function obtained by averaging with respect to the angles φ:

$$V(x) = \overline{f(x, 0, \cdot, 0)} = \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} f(x, 0, \varphi, 0)\,\mathrm{d}\varphi, \qquad x \in \mathbb{T}.$$

The construction of S and R is easily carried out: one first solves the equation

$$\langle \omega^*, \partial_\varphi S \rangle + V = f(\cdot, 0, \cdot, 0)$$

with the help of standard small divisors estimates, and then one takes R simply by fitting equation (1.5). The time-1 symplectic flow Φ of the generating Hamiltonian εS leads to

$$H \circ \Phi = H + \{H, \varepsilon S\} + \mathrm{O}\left(\varepsilon^2\right) = h + \varepsilon(V + R) + \mathrm{O}\left(\varepsilon^2\right) = H_0 + H_1,$$

with

$$H_0(x,y,I;\varepsilon) = \langle \omega^*, I\rangle + \frac{y^2}{2} + \varepsilon V(x) + \langle q, I\rangle\, y + \frac{1}{2}\langle QI, I\rangle\,,$$
$$H_1(x,y,\varphi,I;\varepsilon) = \varepsilon R(x,y,\varphi,I) + \mathrm{O}_3(y,I) + \mathrm{O}\left(\varepsilon^2\right).$$

This expression generalizes that of Lochak's example (1.4), with H_0 (the truncated normal form) playing the rôle of the integrable Hamiltonian, and H_1 being the perturbation of some size $\varepsilon\mu$ where μ is going to be determined in terms of ε.

It can be assumed, except for degenerate cases, that the function $V(x)$ has a unique and nondegenerate maximum x_0; we denote $\alpha^2 = -V''(x_0) > 0$. Then, for $\varepsilon > 0$, the 1-degree-of-freedom Hamiltonian

$$P(x,y;\varepsilon) = \frac{y^2}{2} + \varepsilon V(x),$$

has a hyperbolic point in $(x_0, 0)$, with (homoclinic) separatrices. The case $\varepsilon < 0$ is analogous, provided one considers a minimum instead of a maximum. Then, the Hamiltonian H_0 has whiskered tori with coincident whiskers associated to this hyperbolic point. Note that H_0 constitutes a Hamiltonian situated between the unperturbed Hamiltonian h and the perturbed one H, which possesses hyperbolic invariant tori but their whiskers still coincide. Note also that, in general, H_0 is not an uncoupled Hamiltonian because of the *coupling term* $\langle q, I\rangle\, y$. This seems an additional complication, but it cannot be avoided if one starts with an arbitrary unperturbed Hamiltonian h.

The Lyapunov exponents of the hyperbolic point of P are $\pm\sqrt{\varepsilon}\alpha$, which tend to zero for $\varepsilon \to 0^+$. To have fixed Lyapunov exponents, we can perform the following (non-canonical) linear change: we replace y, I by $\sqrt{\varepsilon}y$, $\sqrt{\varepsilon}I$. The new system is still Hamiltonian if we divide the Hamiltonian by ε (making in this way a change of time scale by a factor $\sqrt{\varepsilon}$). We obtain for $H = H_0 + H_1$ the expressions:

$$H_0 = \langle \omega, I\rangle + \frac{y^2}{2} + V(x) + \langle q, I\rangle\, y + \frac{1}{2}\langle QI, I\rangle\,, \tag{1.6}$$

$$H_1 = R\left(x, \sqrt{\varepsilon}y, \varphi, \sqrt{\varepsilon}I\right) + \frac{1}{\varepsilon}\mathrm{O}_3\left(\sqrt{\varepsilon}y, \sqrt{\varepsilon}I\right) + \mathrm{O}\left(\varepsilon\right) = \mathrm{O}(\mu), \tag{1.7}$$

where

$$\omega = \frac{\omega^*}{\sqrt{\varepsilon}}, \qquad \mu = \sqrt{\varepsilon}.$$

We close this section by remarking that an essential point in the approach described in this work is to have an integrable H_0. A similar procedure could have been carried out for double resonances (or higher multiplicity) instead of single ones, but then H_0 would not be, in general, integrable.

1.6. Regular and singular hyperbolic Hamiltonians. For $\varepsilon \to 0^+$, the study of the Hamiltonian (1.6–1.7) is a singular perturbation problem, due to the *fast frequencies* $\omega = \omega^*/\sqrt{\varepsilon}$ in the unperturbed Hamiltonian H_0. We are thus confronted with a *singular* system, often referred to as *weakly hyperbolic*, and also called *a-priori stable* according to the terminology introduced in [13]. In fact, this case can be referred to as *totally singular*, because *all* the frequencies are fast.

The singular problem can be avoided if one considers independent parameters, namely a *fixed* $\varepsilon > 0$ (that is, a *fixed* ω in (1.6)) and $\mu \to 0$. In such a case, the system (1.6–1.7) has the property that the hyperbolicity and the homoclinic orbits are present in the unperturbed Hamiltonian ($\mu = 0$), and are simply perturbed for $|\mu|$ small. In this case, we are confronted with a *regular* or *strongly hyperbolic* system, or also *a-priori unstable.*

This strategy of keeping $\varepsilon > 0$ fixed and letting $\mu \to 0$ (having in this way a regular system) was introduced by Arnold in order to avoid dealing with a singular perturbation problem. As already remarked in section 1.4, the exponentially small splitting of separatrices predicted by the Melnikov integrals could then be justified only for μ exponentially small in ε.

This regular situation ($\varepsilon > 0$ fixed and μ exponentially small in ε) has been considered in some papers through the construction of appropriate Melnikov integrals [56, 39], but their applicability to the existence of Arnold diffusion [29, 58, 59] has overlooked the existence of the gaps between the whiskered tori, as well as the transition properties needed for the invariant tori. (This transition property has been recently studied in [24, 38, 16, 45].) Also inside this regular situation there is a recent approach [6, 5, 7], reminiscent of the parallel shooting method in numerical analysis, which is based on looking directly for the connecting paths of diffusion, using variational methods.

Considering it only from the point of view of bifurcation theory, the relation μ exponentially small in ε is anything but a generic relation between these two parameters. Moreover, from the analysis of the previous section, a power-like relation $\mu = \varepsilon^p$, with $p > 0$, is the natural one from the point of view of perturbation theory (the smaller p, the better). In this singular case, the problem of detecting the splitting of separatrices from the Melnikov integrals is much more intricate because of its exponentially small character with respect to ε.

Concerning the singular problem for 2 degrees of freedom ($n = 1$ in section 1.5), or equivalently area preserving maps, the first results [43, 31] appeared in 1984, and now it seems fairly well understood (see, for instance, [22] for a survey of results).

For more than 2 degrees of freedom, the existence of intersection between the whiskers can be detected by geometrical and topological methods [25, 8], but the effective measure of such intersections involve, in general, small divisors into the expressions for the Melnikov integrals, and arithmetic problems take place in giving asymptotics or lower bounds for such integrals.

As a way to avoiding such small divisors, one can consider different time scales in the frequencies, like for instance $\omega = (\omega_1^*, \omega_2^*/\sqrt{\varepsilon})$ in (1.6). These different time scales take place in some problems of celestial mechanics, and were first considered in [13, 12], where the splitting was estimated by direct Lindstedt expansions, and was wrongly stated as some power of ε, due to some mistaken computations in the third order of such expansions. This error was recently corrected in [27] for for the case of *one* fast frequency $\omega_2^*/\sqrt{\varepsilon}$ (one 'singular' frequency), and an *even* perturbation $F(x,\varphi)$ with only a finite number of harmonics in the angles x, φ. In this case, the splitting of the separatrices turns out to be $\mathrm{O}(\exp(-1/\sqrt{\varepsilon}))$, as in the Arnold's example, and no problems related with small divisors appear in the exponent of this expression.

The difficult "totally singular" case has been lately receiving special attention. The first formal discussion about the relation between the size of the splitting and the speed of diffusion can be found in [34]. Upper bounds appear in [26], and the first explicit computations showing the arithmetic problems for a concrete frequency were carried out in [53] for fast quasiperiodically forced Hamiltonians ("isochronous systems").

The paradigmatic example of a pendulum under fast quasiperiodic forcing (the Hamiltonian (1.6–1.7) for $q = 0$, $Q = 0$, $V(x) = \cos x - 1$, and $H_1 = \varepsilon^p m(\varphi)\cos x$, $p > 3$) was studied in [17] for the case of a nonresonant two-dimensional external frequency $\omega^* = \big(1, (\sqrt{5}+1)/2\big)$, and an asymptotic behavior for the splitting of the type $\Delta \sim \varepsilon^{p/2-1}\exp\big(-c(\log\varepsilon)/\varepsilon^{1/4}\big)$ was proved, where $c(u)$ is a positive periodic function of u that depends on the arithmetic properties of ω^* and the analyticity properties of $m(\varphi)$.

Lower bounds for the anisochronous Lochak's example (1.4) with the *same* exponents as in the Nekhoroshev theorem have been recently stated in [52] for $\mu = \varepsilon^p$ and $p > 0$ big enough, assuming an *even* perturbation $F(x,\varphi)$ (ensuring in this way the existence of a homoclinic orbit), but with a proof containing essential errors.

For the obtainment of asymptotic or lower bounds for the splitting of separatrices, the perturbations H_1 and F are assumed in [17, 52] to possess arbitrarily high harmonics in the angles φ, to ensure that the estimated dominant Fourier mode of the Melnikov integral is not zero. When the perturbation has only a finite number of harmonics (for instance, in the Arnold's example [2]), the Poincaré–Melnikov theory applied to the original system cannot give the correct asymptotics, since the phenomenum of small divisors does not show up in the Melnikov function. In this case, before applying the results of [17, 52], at least one step of averaging is required to produce a new unperturbed Hamiltonian and a new perturbation with arbitrarily high harmonics (see [53] for an explicit example).

Finally, let us point out that an *essential* assumption in [26, 52, 27] is the reversibility of the perturbation. This hypothesis is required to ensure the existence of a perturbed homoclinic orbit, to begin the iterative approximation of the whiskers through the study of variational-like equations. This assumption is not required in [17], where instead an invariant splitting function is defined in some flow-box coordinates provided by a normal form, convergent in a full neighborhood of the whiskered torus, which is constructed with a precise statement of the loss of complex domain in the variable φ.

We stop here this account of results, and refer to Lochak [36] for a better organized compendium of results concerning Arnold diffusion.

1.7. Description of the results. According to section 1.5, we concentrate our efforts on the Hamiltonian H given in (1.6–1.7), including the coupling term $\langle q, I\rangle\, y$.

The main goal of this note is to introduce a *geometrical* method to measure the splitting of the whiskers of a hyperbolic invariant torus of the anisochronous Hamiltonian (1.6–1.7), relating it with the Melnikov function. Our main interest is the singular case $\mu = \varepsilon^p$ for some $p > 0$, but for the presentation of the method in this note, we will be mostly concerned with the regular case $\varepsilon > 0$ fixed, $\mu \to 0$. The proofs of the announced results are not included, and will appear elsewhere [20].

In our first results, in section 3, we establish the preservation of the whiskered torus having frequency vector ω. This is ensured by means of a version of the KAM theorem, adapted to the preservation of whiskered tori under perturbations. We call it *hyperbolic KAM theorem.*

We point out that most papers about preservation of whiskered tori, like [28, 60, 32, 55], deal mainly with a local point of view. A significantly new approach was introduced by Eliasson [25], by expressing the hyperbolic KAM theorem in the original variables, global in $x \in \mathbb{T}$, instead of some local variables around the whiskered torus. Our version of this result (theorem 3.2) can be considered a refinement of the one given in [25]. Thus, we provide a symplectic transformation Φ such that the new Hamiltonian $\widetilde{H} = H \circ \Phi$ (the *normal form*) has a simpler expression and has clearly an invariant torus with frequencies ω.

Moreover, to deal in the next future with the singular case $\mu = \varepsilon^p$, we obtain a quantitative normal form in Theorem 3.1, with a precise statement of the loss of complex domain in the variable φ, which can be applied later on an extension theorem like in [22, 17, 52].

This global point of view allowed Eliasson to introduce in a very natural way a (vector) *splitting function* measuring the distance between the perturbed whiskers. In fact, the paper [25] is concerned with the existence of (at least $n+1$) homoclinic intersections between the whiskers, in both regular and singular systems, although the splitting is not computed. This is deduced from the use of *exact symplectic* transformations to normal form in the hyperbolic KAM theorem.

Proceeding as in [25], but going farther with the use of exact symplectic transformations we see that the splitting function $\mathcal{M}(\varphi)$ is the gradient of a (scalar) function $\mathcal{L}(\varphi)$, which is called *splitting potential* (theorem 4.1). It is an important remark that, to get this result, the splitting function is expressed in the normal form variables provided by the hyperbolic KAM theorem, where its properties appear transparently, instead of using the original variables. Besides, we point out that the existence of the splitting potential is also a reflection of the Lagrangian properties of the whiskers.

In section 5, we will follow Treschev [56] for the Poincaré–Melnikov method, to obtain first order approximations for both the functions $\mathcal{L}(\varphi)$ and $\mathcal{M}(\varphi)$ in Theorem 5.1. The first order approximations $L(\varphi)$ and its gradient $M(\varphi)$ are called, respectively, the *Melnikov potential* (like in [21], although it deserves the name of Poincaré function [47, §19]), and the *Melnikov function.* In particular, when the Melnikov potential is a Morse function (a generic situation), there exist 2^n transverse homoclinic orbits for $|\mu|$ small enough (in the *regular* case).

The Melnikov potential $L(\varphi)$ and its gradient $M(\varphi)$ are given by absolutely convergent integrals, and are designed in order to be useful also for a coupled Hamiltonian H_0 (with a nonzero coupling term). To illustrate this, a computable coupled example is shown in section 6.

2. Setup and the unperturbed Hamiltonian. In view of the analysis made in section 1.5, we will consider a (real analytic) perturbation of a hyperbolic integrable Hamiltonian, with $n+1 \geq 3$ degrees of freedom, that in canonical variables $z=(x,y,\varphi,I) \in D \subset \mathbb{T}\times\mathbb{R}\times\mathbb{T}^n\times\mathbb{R}^n$, is of the form

$$H(x,y,\varphi,I;\mu) = H_0(x,y,I) + \mu H_1(x,y,\varphi,I), \tag{2.1}$$

$$H_0(x,y,I) = \langle \omega, I\rangle + \frac{y^2}{2} + V(x) + \langle q, I\rangle\, y + \frac{1}{2}\langle QI, I\rangle, \tag{2.2}$$

where μ is a perturbation parameter. The given ingredients of H_0 are the vectors $\omega, q \in \mathbb{R}^n$, a symmetric $(n\times n)$-matrix Q, and a function $V(x)$ of $x\in\mathbb{T}$ that is required to have a unique and nondegenerate global maximum. To fix ideas, we require

$$\begin{aligned} &V(0)=0, \qquad V'(0)=0, \qquad V''(0)<0,\\ &V(x)<0 \quad \forall x\neq 0 \pmod{2\pi}, \end{aligned} \tag{2.3}$$

and we denote

$$\alpha = \sqrt{-V''(0)}. \tag{2.4}$$

We also assume the following nondegeneracy condition:

$$\det\begin{pmatrix} 1 & q^\top \\ q & Q \end{pmatrix} \neq 0. \tag{2.5}$$

Let us write

$$P(x,y) = \frac{y^2}{2} + V(x).$$

On every plane $I = \text{const}$, the Hamiltonian H_0 reduces to the 1-degree-of-freedom Hamiltonian

$$h(x, y; I) = P(x, y + \langle q, I\rangle) = \frac{(y + \langle q, I\rangle)^2}{2} + V(x).$$

This is a *generalized pendulum* (the standard pendulum being given by $V(x) = \cos x - 1$). It has $(x, y) = (0, -\langle q, I\rangle)$ as a hyperbolic equilibrium point, with (homoclinic) separatrices given by $y + \langle q, I\rangle = \pm\sqrt{-2V(x)}$.

If we take $q = 0$ in (2.2), we have an *uncoupled* unperturbed Hamiltonian H_0, which is somewhat simpler since it is formed by one pendulum and n rotors. However, our approach is designed for the more general *coupled* case, for whose study we have given the motivation in section 1.5.

We see that the Hamiltonian H_0 has an n-parameter family of n-dimensional hyperbolic invariant tori given by the equations $I = \text{const}$, $y = -\langle q, I\rangle$, $x = 0$. The associated stable and unstable whiskers of each torus coincide, and hence this (unique) whisker is made of homoclinic orbits, biasymptotic to the torus. Our aim is to give conditions for the splitting of the whiskers for $\mu \neq 0$.

We will focus our attention on a concrete hyperbolic torus, that we assume located at the origin: $I = 0$, $x = y = 0$, with flow $\dot\varphi = \omega$. The frequency vector ω will be assumed to satisfy a Diophantine condition: for some $\tau \geq n - 1$ and $\gamma > 0$,

$$|\langle k, \omega\rangle| \geq \gamma\, |k|^{-\tau} \quad \forall k \in \mathbb{Z}^n \setminus \{0\}\,. \tag{2.6}$$

When the frequency vector ω is fixed, we say that the system (2.1–2.2) is *regular*. Eventually, we can allow ω to depend on an additional parameter: $\omega = \omega(\varepsilon)$ (for instance, $\omega = \omega^*/\sqrt{\varepsilon}$ according to (1.6), and then $\gamma = \gamma^*/\sqrt{\varepsilon}$). Then, if we let $\varepsilon \to 0^+$ and consider $\mu = \varepsilon^p$, it becomes *singular*, but if we keep ε fixed and let $\mu \to 0$, the system is still regular.

We denote $\mathcal{T}_0$ the n-dimensional hyperbolic invariant torus, of frequency vector ω, of our unperturbed Hamiltonian (2.2). Since this torus is given by the equations $I = 0$, $x = y = 0$, we can obviously parameterize it as

$$\mathcal{T}_0 : \qquad z_0^*(\varphi) = (0, 0, \varphi, 0), \qquad \varphi \in \mathbb{T}^n.$$

Note that the trajectories on $\mathcal{T}_0$ are $t \mapsto z_0^*(\varphi + \omega t)$ for any given φ.

As mentioned in section 2, the stable and unstable whiskers of the torus $\mathcal{T}_0$ coincide; this homoclinic whisker is given by the equations $I = 0$, $P(x, y) = 0$. We denote $\mathcal{W}_0$ the positive part ($y > 0$) of the whisker. To give a suitable parameterization for $\mathcal{W}_0$, we consider the 1-degree-of-freedom Hamiltonian $P(x, y)$, and denote $(x_0(t), y_0(t))$ the associated homoclinic trajectory, biasymptotic (for $t \to \pm\infty$) to the hyperbolic point $(0, 0)$, with $x_0(0) = \pi$, $y_0(0) > 0$. We can then give the whisker $\mathcal{W}_0$ the parameterization

$$\mathcal{W}_0 : \qquad z_0(t, \varphi) = (x_0(t), y_0(t), \varphi + \delta_0(t) + \omega t, 0), \qquad \varphi \in \mathbb{T}^n, \quad t \in \mathbb{R},$$

where

$$\delta_0(t) = \left(\int_0^t y_0(s) \mathrm{d}s \right) q = (x_0(t) - \pi) q.$$

Note that $z_0(\cdot, \varphi)$ is a trajectory on $\mathcal{W}_0$ for any given φ. Besides, this trajectory is biasymptotic to the invariant torus $\mathcal{T}_0$, and $\delta_0(t)$ is just the *phase shift* undergone by any trajectory when traveling along $\mathcal{W}_0$. Since $\lim_{t \to \pm\infty} \delta_0(t) = \pm \pi q$, we have

$$\lim_{t \to \pm\infty} \left[z_0(t, \varphi) - z_0^*(\varphi \pm \pi q + \omega t) \right] = 0, \tag{2.7}$$

with exponentially decreasing bounds.

3. On the hyperbolic KAM theory: the perturbed hyperbolic torus and its local whiskers. We recall in this section the KAM theorem for hyperbolic invariant tori, which states that the torus $\mathcal{T}_0$ of H_0 survives, with its local stable and unstable whiskers, in the perturbation H for μ small enough. This result, which we call the hyperbolic KAM theorem, follows from a KAM-like iterative scheme providing, after a symplectic transformation, a *convergent local normal form* in a neighborhood of the hyperbolic torus. A first version was proved in [28] for a regular hyperbolic Hamiltonian. Subsequent papers, like [55, 13, 51], provide proofs that hold also for a singular hyperbolic Hamiltonian.

We consider here the approach of [25, 45], which are the closest to our purpose of carrying out a global control of the whiskers in order to study their splitting. Moreover, the use of exact symplectic transformations in the construction of the normal form will be very suitable in order to put the splitting function as the gradient of a splitting potential.

To formulate the hyperbolic KAM theorem, it is convenient to consider local canonical variables in a neighborhood of the torus, in such a way that the unperturbed local whiskers become coordinate planes. The new variables $w = (u, v, \varphi, I)$, referred from now on as *hyperbolic variables,* are defined by an exact symplectic transformation $z = \Psi(w)$, $\Psi = \Psi_1 \circ \Psi_2$, with

$$\Psi_1(x, y, \varphi, I) = (x, y - \langle q, I \rangle, \varphi + qx, I), \tag{3.1}$$

$$\Psi_2(u, v, \varphi, I) = (\psi(u, v), \varphi, I), \tag{3.2}$$

where ψ is a local (exact) symplectic transformation taking the pendulum P to a function of uv:

$$P(\psi(u, v)) = g(uv) = -\alpha uv + g_{\geq 2}(uv),$$

with $g_{\geq 2} = \mathrm{O}_2(uv)$ (recall that α is the Lyapunov exponent, defined in (2.4)). The existence of this transformation and its analyticity in a neighborhood of the origin is a well-known Moser's result [41] on the convergence of the Birkhoff normal form in a neighborhood of a hyperbolic equilibrium point (see also [13, §A3]).

We denote G_0, G, the Hamiltonians H_0, H, expressed in the hyperbolic variables:

$$G_0(u,v,I) = H_0 \circ \Psi = \langle \omega, I\rangle - \alpha uv + g_{\geq 2}(uv) + \frac{1}{2}\left\langle \widehat{Q} I, I\right\rangle, \tag{3.3}$$

$$G(u,v,\varphi,I;\mu) = H \circ \Psi = H_0 \circ \Psi + \mu\, H_1 \circ \Psi = G_0 + \mu G_1, \tag{3.4}$$

where

$$\widehat{Q} = Q - qq^\top.$$

Working in these hyperbolic variables, the hyperbolic KAM theorem will provide us a transformation $\widehat{\Phi}$ to normal form, and we will denote $\widetilde{G} = G \circ \widehat{\Phi}$ the new Hamiltonian.

In order to formulate quantitative statements, we introduce some notations. As a neighborhood of the whiskered torus, we define for $s, r, \rho > 0$ the complex domain

$$B_{s,r,\rho} = \{(u,v,\varphi,I) : |u|, |v| \leq s, |I| \leq r, |\operatorname{Im}\varphi| \leq \rho\}.$$

For a function f analytic on some bounded domain D and continuous on its boundary, we consider the norm

$$|f|_D = \sup_D |f(u,v,\varphi,I)|.$$

The notation $f_1 \equiv f_2$ means that $f_1 - f_2 = \text{const}$ (so the associated Hamiltonian equations are the same). In general, our functions depend also on μ as an additional parameter; the word "constant" will not exclude dependence on μ.

A statement of the hyperbolic KAM theorem to be applied in this work is given below. In fact this is a standard hyperbolic KAM theorem, with some minor changes due to that one is seeking for an exact symplectic transformation to normal form. The exactness is important in order to detect intersections between the whiskers, with the help of a generating function, as we show in section 4. The proof is a refinement of Eliasson's proof (see [20]).

THEOREM 3.1 (hyperbolic KAM theorem). *Let $G = G_0 + \mu G_1$ as in (3.3–3.4), analytic on $B_{s,r,\rho}$. Assume the frequency vector ω satisfies the Diophantine condition (2.6) for some $\tau > n-1$ and $\gamma > 0$. Assume also the nondegeneracy conditions*

$$\alpha \neq 0, \qquad \det \widehat{Q} \neq 0.$$

Let $0 < \eta < s$, $0 < \zeta < r$ and $0 < \delta < \rho$ given, and define

$$\mu_0 = \frac{\delta^{4\tau+4}\gamma^4}{C_1}. \tag{3.5}$$

Then, for $|\mu| \leq \mu_0$ *there exist an exact symplectic transformation* $\widehat{\Phi}$: $B_{s-\eta,r-\zeta,\rho-\delta} \longrightarrow B_{s,r,\rho}$ *(depending on* μ*), and constants* $a = a(\mu)$, $b = b(\mu)$ *such that* $\widetilde{G} = G \circ \widehat{\Phi}$ *takes the form*

$$\begin{aligned}\widetilde{G} &\equiv \langle \omega, I - a(\mu) \rangle - b(\mu)\alpha uv + \widehat{R}(u,v,\varphi,I;\mu),\\ \widehat{R} &= \mathrm{O}_2(uv, I - a(\mu)).\end{aligned}$$

The transformation $\widehat{\Phi}$ *and the constants* a, b *are analytic on their domain with respect to* (u,v,φ,I) *and* μ. *For any* $|\mu| \leq \mu_0$ *the following estimates hold:*

$$\begin{aligned}\left|\widehat{\Phi} - \mathrm{id}\right|_{B_{s-\eta,r-\zeta,\rho-\delta}} &\leq \frac{C_2}{\gamma^2 \delta^{2\tau+2}} |\mu|,\\ |a|, |b-1| &\leq \frac{C_3}{\gamma \delta^{\tau+1}} |\mu|.\end{aligned}$$

The constants C_j *do not depend on* μ, δ, ω.

Notice that the threshold given in (3.5) is proportional to γ^4, as in [45], differing from the one given in [51], proportional to γ^2. This seems due to the different methods of proof that follow the Kolmogorov's and Arnold's approaches to KAM theorem.

An additional remark is that the validity of this version of KAM theorem in the singular case can be established by regarding how the involved parameters (mainly the threshold μ_0) depend on ω. In this way one can rewrite ω as $\omega^*/\sqrt{\varepsilon}$ without affecting, for $\varepsilon \to 0^+$, the smallness condition and the estimates (or rather making them better).

The next statement is a more refined version of Eliasson's result [25, p. 65]. From the hyperbolic variables w, this result comes back to the original variables z through the transformation Ψ introduced in (3.1–3.2). The transformation to normal form is then $\Phi = \Psi \circ \widehat{\Phi} \circ \Psi^{-1}$ and the new Hamiltonian $\widetilde{H} = \widetilde{G} \circ \Psi^{-1} = H \circ \Phi$. This strategy of expressing the KAM theorem in the original variables will be very useful to our purposes. Concerning the domains, notice that, for any $s, r, \rho > 0$,

$$\begin{aligned}\Psi_1(B_{s,r,\rho}) = \{(x,y,\varphi,I) : |x| \leq s, |y + \langle q, I \rangle| \leq s,&\\ |I| \leq r, |\mathrm{Im}(\varphi - qx)| \leq \rho\}&.\end{aligned}$$

THEOREM 3.2. *Let* $H = H_0 + \mu H_1$ *as described in (2.1–2.3), analytic on* $\Psi_1(B_{s,r,\rho})$. *Assume the frequency vector* ω *satisfies the Diophantine condition (2.6) for some* $\tau > n-1$ *and* $\gamma > 0$. *Assume also the nondegeneracy condition (2.5). Let* $0 < \delta < \rho$ *given, and define*

$$\mu_0 = \frac{\delta^{4\tau+4}\gamma^4}{C_1}.$$

Then, for $|\mu| \leq \mu_0$ *and some* $0 < \kappa < 1/2$ *there exist an exact symplectic transformation* $\Phi : \Psi_1\left(B_{\kappa s, r/2, \rho-\delta}\right) \longrightarrow \Psi_1(B_{s,r,\rho})$ *(depending on* μ*), and* *constants* $a = a(\mu)$, $b = b(\mu)$ *such that* $\widetilde{H} = H \circ \Phi$ *takes the form*

$$\begin{aligned}&\widetilde{H} \equiv \langle \omega, I - a(\mu) \rangle + b(\mu) P(x, y + \langle q, I \rangle) + R(x, y, \varphi, I; \mu),\\&R = \mathrm{O}_2(P(x, y + \langle q, I \rangle), I - a(\mu)).\end{aligned}$$

The transformation Φ *and the constants* a, b *are analytic on their domain with respect to* (x, y, φ, I) *and* μ. *For any* $|\mu| \leq \mu_0$ *the following estimates hold:*

$$\begin{aligned}&|\Phi - \mathrm{id}|_{\Psi_1(B_{\kappa s, r/2, \rho - \delta})} \leq \frac{C_2}{\gamma^2 \delta^{2\tau+2}} |\mu|,\\&|a|, |b - 1| \leq \frac{C_3}{\gamma \delta^{\tau+1}} |\mu|.\end{aligned}$$

The constants C_j *do not depend on* μ, δ, ω.

The most important point about theorem 3.2 is, in our opinion, the fact that the local normal form $\widetilde{H}$ is put in terms of $P(x, y + \langle q, I \rangle)$. Using this expression, it is constructed in [25] a "global" Hamiltonian (in x) having exactly the same hyperbolic torus and its (local) whiskers as $\widetilde{H}$, as well as the dynamics on them. These whiskers are global (and coincident) in this new Hamiltonian. We are going to use intensively this feature in section 4.

From now on, in order to keep a more readable notation, we shall usually omit the μ-dependence of our functions.

It is clear that the Hamiltonian $\widetilde{H}$ has a hyperbolic invariant torus of frequencies ω. This torus and its associated local whiskers can be parameterized as follows:

$$\begin{aligned}\widetilde{\mathcal{T}} = \widetilde{\mathcal{T}}(\mu):&\quad \tilde{z}^*(\varphi) = (0, -\langle q, a \rangle, \varphi, a), \quad \varphi \in \mathbb{T}^n,\\\widetilde{\mathcal{W}}^{\pm}_{\mathrm{loc}} = \widetilde{\mathcal{W}}^{\pm}_{\mathrm{loc}}(\mu):&\quad \tilde{z}(t, \varphi) = (x_0(bt), y_0(bt) - \langle q, a \rangle, \varphi + \delta_0(bt) + \omega t, a),\\&\qquad \varphi \in \mathbb{T}^n, \quad \pm t \geq t_0,\end{aligned}$$

with a suitable $t_0 = t_0(s)$. We then have, for the original Hamiltonian H, a hyperbolic torus and its associated local stable and unstable whiskers (for $|\mu| \leq \mu_0$):

$$\begin{aligned}\mathcal{T} = \mathcal{T}(\mu):&\quad z^*(\varphi) = \Phi(\tilde{z}^*(\varphi)), \qquad \varphi \in \mathbb{T}^n,\\\mathcal{W}^{\pm}_{\mathrm{loc}} = \mathcal{W}^{\pm}_{\mathrm{loc}}(\mu):&\quad z^{\pm}(t, \varphi) = \Phi(\tilde{z}(t, \varphi)), \quad \varphi \in \mathbb{T}^n, \quad \pm t \geq t_0.\end{aligned}$$

These parameterizations of the whiskers $\mathcal{W}^{\pm}_{\mathrm{loc}}$ can be extended to further values of t in a natural way, as trajectories associated to our Hamiltonian H. We denote $\mathcal{W}^{\pm} = \mathcal{W}^{\pm}(\mu)$ the extended or *global whiskers,* and our aim is to measure the distance between them.

4. The splitting potential and the splitting function. As mentioned in the previous section, the special formulation of theorem 3.2 allows us to carry out a more global control on the perturbed whiskers. We introduce as in [25] the following integrable Hamiltonian:

$$N(x,y,I;\mu) = \langle \omega, I-a\rangle + bP(x, y+\langle q,a\rangle) + b\,\langle q, I-a\rangle\,(y+\langle q,a\rangle) \\ +\frac{b}{2}\,\langle Q(I-a), I-a\rangle$$

(recall that a and b depend on μ, although this is not made explicit). This Hamiltonian is defined globally in the variable $x\in\mathbb{T}$, and has the same hyperbolic torus $\widetilde{\mathcal{T}}$ and its whiskers $\widetilde{\mathcal{W}}^{\pm}_{\mathrm{loc}}$, and the dynamics on them, as the local normal form $\widetilde{H}$. The only difference is that the parameterization $\tilde z(t,\varphi)$ (of the whiskers in $\widetilde{H}$) can now be defined for any $t\in\mathbb{R}$; in this way the local whiskers $\widetilde{\mathcal{W}}^{\pm}_{\mathrm{loc}}$ can be extended to a (unique) global homoclinic whisker of N. Following [25], we are going to take advantage of the fact that the "artificial" Hamiltonian N is integrable.

We assume our starting Hamiltonian H in (2.1) analytic on a complex domain

$$D_{\sigma,s,r,\rho} = \left\{(x,y,\varphi,I) : |\mathrm{Im}\,x|\le\sigma, |y|\le s, |I|\le r, |\mathrm{Im}\,\varphi|\le\rho\right\},$$

with s big enough in order to contain a neighborhood of the (global) unperturbed whiskers. We denote Υ_0^t, Υ^t, Υ_N^t the time-t flows of the Hamiltonians H_0, H, N respectively.

We choose two points $\left(x^0,y^0\right)$, $\left(x^1,y^1\right)$ on the positive $(y^0,y^1>0)$ homoclinic orbit of the pendulum P, with $x^0>\pi$ and $x^1<\pi$, and such that the associated φ-sections $\left(x^j, y^j-\langle q,a\rangle, \mathbb{T}^n, a\right)$, $j=0,1$, are contained in the domain of the normal form $\widetilde{H}$. Consider $T>0$ such that

$$\Upsilon_N^T\left(x^1, y^1-\langle q,a\rangle, \mathbb{T}^n, a\right) = \left(x^0, y^0-\langle q,a\rangle, \mathbb{T}^n, a\right).$$

Comparing the flows of N and H we can define, in a neighborhood Ω of $\Phi\left(x^0, y^0-\langle q,a\rangle, \mathbb{T}^n, a\right)$, the map

$$\Theta = \Upsilon^T\circ\Phi\circ\Upsilon_N^{-T}\circ\Phi^{-1}.$$

This map is exact symplectic, and takes $\mathcal{W}^+_{\mathrm{loc}}\cap\Omega$ (a subset of the local stable whisker) into $\mathcal{W}^-$ (the global unstable whisker). Besides, it is easy to check that the map Θ gives a correspondence between our parameterizations of the whiskers:

$$\Theta\left(z^+(t,\varphi)\right) = z^-(t,\varphi), \tag{4.1}$$

for any t, φ such that $z^+(t,\varphi)\in\Omega$. Therefore, the difference $\Theta-\mathrm{id}$ constitutes a measure for the splitting on $\mathcal{W}^+_{\mathrm{loc}}\cap\Omega$. We stress that a formula like (4.1) cannot be expected outside $\mathcal{W}^+_{\mathrm{loc}}$, since we need the fact that the Hamiltonians $\widetilde{H}$ and N have the same dynamics on the whiskers.

Note also that (4.1) tells us that the map Θ does not depend on the points (x^0, y^0), (x^1, y^1) chosen. What does depend strongly on these points is the neighborhood Ω where Θ is defined.

To continue, it is convenient to express the map Θ in the normal form hyperbolic variables $w = (u, v, \varphi, I)$ introduced in section 3, in which the local whiskers become coordinate planes, and the global unstable whisker can be seen as a graphic over the local stable one. Recall that $\widetilde{G} = \widetilde{H} \circ \Psi$ is our local normal form expressed in the hyperbolic variables. The local whiskers for $\widetilde{G}$ are given by

$$\widehat{\mathcal{W}}^+_{\mathrm{loc}} = \Psi^{-1}\left(\widetilde{\mathcal{W}}^+_{\mathrm{loc}}\right) = \{v = 0, I = a\},$$
$$\widehat{\mathcal{W}}^-_{\mathrm{loc}} = \Psi^{-1}\left(\widetilde{\mathcal{W}}^-_{\mathrm{loc}}\right) = \{u = 0, I = a\}.$$

The exact symplectic map

$$\widehat{\Theta} = (\Phi \circ \Psi)^{-1} \circ \Theta \circ (\Phi \circ \Psi)$$

is defined in the neighborhood $\widehat{\Omega} = (\Phi \circ \Psi)^{-1}(\Omega)$ of $(u^0, 0, \mathbb{T}^n, a)$, where $u^0 > 0$ is defined by $(x^0, y^0) = \psi(u^0, 0)$. It is adequate to define

$$\widehat{\mathcal{W}}^- = \widehat{\Theta}\left(\widehat{\mathcal{W}}^+_{\mathrm{loc}} \cap \widehat{\Omega}\right)$$

as an invariant manifold of $\widetilde{G}$, which is the equivalent in the hyperbolic variables for (a piece of) the global unstable whisker $\mathcal{W}^-$.

In the hyperbolic variables, the problem of measuring the splitting has a simpler formulation, since $\widehat{\mathcal{W}}^-$ can be seen as a graphic over the local whisker $\widehat{\mathcal{W}}^+_{\mathrm{loc}}$. Let us parameterize:

$$\begin{aligned}
\widehat{\mathcal{W}}^+_{\mathrm{loc}}: \qquad & \hat{w}^+(t, \varphi) = \Psi^{-1}(\tilde{z}(t, \varphi)) = (u_0(bt), 0, \varphi - \pi q + \omega t, a), \\
\widehat{\mathcal{W}}^-: \qquad & \hat{w}^-(t, \varphi) = (\Phi \circ \Psi)^{-1}\left(z^-(t, \varphi)\right),
\end{aligned}$$

where $u_0(t)$ is defined by $\psi(u_0(t), 0) = (x_0(t), y_0(t))$. In components, we will write $\hat{w}^{\pm} = \left(\hat{u}^{\pm}, \hat{v}^{\pm}, \hat{\varphi}^{\pm}, \hat{I}^{\pm}\right)$. Then the splitting is given by the difference $\hat{I}^- - \hat{I}^+ = \hat{I}^- - a$. We remark that we do not need to consider the difference $\hat{v}^- - \hat{v}^+ = \hat{v}^-$ because $\hat{I}^-(t, \varphi) = a$ implies that $\hat{v}^-(t, \varphi) = 0$; this is related to the fact that the whiskers are contained in energy levels of $\widetilde{G}$.

The exactness of the symplectic map $\widehat{\Theta}$ and the fact that $\widehat{\Theta} - \mathrm{id} = \mathrm{O}(\mu)$, imply that for μ small enough there exists a *generating function* $\theta\left(u, \check{v}, \varphi, \check{I}\right)$ such that $\widehat{\Theta}: \left(\check{u}, \check{v}, \check{\varphi}, \check{I}\right) \mapsto (u, v, \varphi, I)$ is given by

$$u = \check{u} - \partial_{\check{v}}\theta, \qquad \check{v} = v - \partial_u \theta, \qquad \varphi = \check{\varphi} - \partial_{\check{I}}\theta, \qquad \check{I} = I - \partial_{\varphi}\theta.$$

We define the *splitting potential* as the following periodic function:

$$\mathcal{L}(\varphi)=\mathcal{L}(t_0,\varphi)=\theta(u_0(bt_0),0,\varphi-\pi q+\omega t_0,a), \qquad \varphi\in\mathbb{T}^n,$$

with t_0 such that $\hat{w}^+(t_0,\mathbb{T}^n)\subset\widehat{\Omega}$. The generating function θ is determined up to an additive constant, that can be chosen to ensure that $\mathcal{L}$ has zero average. Note that $\mathcal{L}$ depends on μ and on the choice of t_0. The gradient $\mathcal{M}(\varphi)=\partial_\varphi\mathcal{L}(\varphi)$ will be called the (vector) *splitting function,* and the following theorem states that, on a suitable φ-section, the function $\mathcal{M}$ gives the splitting distance.

THEOREM 4.1. *There exist functions $\tau(\varphi)=\mathrm{O}(\mu)$ and $\Xi(\varphi)=\varphi+\mathrm{O}(\mu)$, periodic in φ, such that, for all $\varphi\in\mathbb{T}^n$,*

$$\mathcal{M}(\varphi)=\hat{I}^-\left(t_0+\tau(\varphi),\Xi(\varphi)\right)-a.$$

For our fixed t_0, note that

$$\hat{I}^-(t_0,\varphi)-a=\partial_\varphi\theta\left(\hat{u}^-(t_0,\varphi),0,\hat{\varphi}^-(t_0,\varphi),a\right). \tag{4.2}$$

One can think that this should be the most natural election for what we call "splitting function". Nevertheless, it does not come from formula (4.2) that this is the gradient of a scalar function, although this obstruction has been easily overcome with the change of variables given by the periodic functions τ and Ξ.

An important consequence of theorem 4.1 is the existence of at least $n+1$ effective intersections between the whiskers $\mathcal{W}^\pm$, giving rise to at least $n+1$ trajectories biasymptotic to the invariant torus $\mathcal{T}$. These intersections are obtained as critical points of a function on $\mathbb{T}^n$, and the minimum number of them comes from Lyusternik–Schnirelman theory (see [15, sect. 2.12]). This constitutes the main result contained in [25]. Note also that, in nondegenerate cases, the number of intersections becomes at least 2^n, as one deduces from Morse theory.

5. First order approximations for the splitting: Melnikov integrals. We now use Poincaré–Melnikov theory to give a first order approximation (in μ) for the splitting potential introduced in the previous section. We define the (scalar) *Melnikov potential* and the (vector) *Melnikov function* as, respectively, the following periodic functions:

$$L(\varphi)=-\int_{-\infty}^{\infty}\left(H_1-\overline{H_1}-\{\chi,H_0\}\right)(z_0(t,\varphi))\,\mathrm{d}t, \tag{5.1}$$

$$M(\varphi)=\partial_\varphi L(\varphi)=-\int_{-\infty}^{\infty}\left[\partial_\varphi\left(H_1-\{\chi,H_0\}\right)\right](z_0(t,\varphi))\,\mathrm{d}t. \tag{5.2}$$

In these formulas, $\chi(x,y,\varphi,I)=\chi(\varphi)$ is the (zero average) function solving the small divisors equation

$$\langle\omega,\partial_\varphi\chi\rangle+\overline{H_1(0,0,\cdot,0)}=H_1(0,0,\cdot,0). \tag{5.3}$$

It is not difficult to check that the integrals in (5.1–5.2) are absolutely convergent (in contrast with [29, 50, 57], where conditionally convergent Melnikov integrals are introduced for $n \geq 2$). This absolute convergence is due to the incorporation of the function $\chi(\varphi)$ (introduced by Treschev in [56]), which is closely related to the shift suffered by the perturbed whiskered torus $\mathcal{T}$ with respect to the unperturbed torus $\mathcal{T}_0$ in applying KAM theorem. Indeed, writing the parameterization of the torus $\mathcal{T}$ in components: $z^* = (x^*, y^*, \varphi^*, I^*)$, the following lemma gives a first order approximation for $I^*(\varphi)$.

LEMMA 5.1. *The I-component of $z^*(\varphi)$ satisfies*

$$I^*(\varphi) = \mu\left(\mathrm{const} - \partial_\varphi \chi(\varphi)\right) + \mathrm{O}\left(\mu^2\right).$$

We note that $L(\varphi)$, as defined in (5.1), has zero average ($\overline{L} = 0$), and that $L(\varphi)$ and $M(\varphi)$ do not depend on μ.

We stress that our formula for $L(\varphi)$ is quite compact, and useful in the coupled case ($q \neq 0$), as we show in section 6. It is also worth remarking that (5.1) it is equivalent to the formula appearing in [13, §4], but much simpler.

We now show some alternative expressions for $L(\varphi)$ and $M(\varphi)$. Using that

$$\int_{t_1}^{t_2} \left\{\chi, H_0\right\}\left(z_0(t,\varphi)\right) \mathrm{d}t = \chi\left(z_0(t_2,\varphi)\right) - \chi\left(z_0(t_1,\varphi)\right) \tag{5.4}$$

for any t_1, t_2, and applying (2.7), we get

$$L(\varphi) = \lim_{T\to\infty} \left[-\int_{-T}^{T} \left(H_1 - \overline{H_1}\right)\left(z_0(t,\varphi)\right)\mathrm{d}t \right.$$
$$\left. + \chi(\varphi + \pi q + \omega T) - \chi(\varphi - \pi q - \omega T) \right].$$

Taking a φ-derivative, we obtain

$$M(\varphi) = \lim_{T\to\infty} \left[-\int_{-T}^{T} \partial_\varphi H_1\left(z_0(t,\varphi)\right)\mathrm{d}t \right.$$
$$\left. + \partial_\varphi\chi(\varphi + \pi q + \omega T) - \partial_\varphi\chi(\varphi - \pi q - \omega T) \right].$$

In the uncoupled case $q = 0$, we can provide simpler (and perhaps more classical) expressions. Proceeding like in (5.4) and using that $\langle \omega, \partial_\varphi \chi \rangle = \left\{\chi, H_0\right\}(0,0,\cdot,0)$, we have

$$\chi(\varphi + \omega t_2) - \chi(\varphi + \omega t_1) = \int_{t_1}^{t_2} \left\{\chi, H_0\right\}(0,0,\varphi + \omega t, 0)\mathrm{d}t$$

$$= \int_{t_1}^{t_2} \left(H_1 - \overline{H_1}\right)(0,0,\varphi+\omega t,0)\mathrm{d}t.$$

Thus, for the special case $q = 0$ we obtain the following expressions (compare with formula (2.15) of [21]):

$$L(\varphi) = -\int_{-\infty}^{\infty} \left[\left(H_1 - \overline{H_1}\right)(z_0(t,\varphi)) - \left(H_1 - \overline{H_1}\right)(0,0,\varphi+\omega t,0)\right] \mathrm{d}t,$$

$$M(\varphi) = -\int_{-\infty}^{\infty} \left[\partial_\varphi H_1\left(z_0(t,\varphi)\right) - \partial_\varphi H_1\left(0,0,\varphi+\omega t,0\right)\right] \mathrm{d}t.$$

Coming again to the general case, the following standard lemma shows that a first order approximation for the difference $I^-(t_0,\cdot) - I^+(t_0,\cdot)$ is given by the Melnikov function $M(\varphi)$ (we denote $z^\pm = (x^\pm, y^\pm, \varphi^\pm, I^\pm)$ in the parameterizations of the whiskers $\mathcal{W}^\pm$). In fact, an analogous expression is given in [56] for $F_0\left(z^-(t_0,\cdot)\right) - F_0\left(z^+(t_0,\cdot)\right)$, where F_0 is any given first integral of the unperturbed Hamiltonian H_0. Our statement concerns the case $F_0 = I_j$, $j = 1, \ldots, n$.

LEMMA 5.2. *For any fixed* $t_0 \in \mathbb{R}$,

$$I^-(t_0,\varphi) - I^+(t_0,\varphi) = \mu M(\varphi) + \mathrm{O}(\mu^2). \tag{5.5}$$

Finally, we see that the splitting potential $\mathcal{L}(\varphi)$ introduced in theorem 4.1 can be approximated by the Melnikov potential $L(\varphi)$. This requires to show that the approximation (5.5), expressed in the original variables, remains true after changing to the normal form variables in which the splitting potential $\mathcal{L}(\varphi)$ had to be defined.

THEOREM 5.1. *For the splitting function and the splitting potential introduced in theorem 4.1, one has:*

$$\mathcal{L}(\varphi) = \mu L(\varphi) + \mathrm{O}\left(\mu^2\right), \qquad \mathcal{M}(\varphi) = \mu M(\varphi) + \mathrm{O}\left(\mu^2\right).$$

The proof shows that the Melnikov function $M(\varphi)$ remains valid as a first order approximation for the splitting function $\mathcal{M}(\varphi)$, in spite of the fact that $\mathcal{M}(\varphi)$ was defined in theorem 4.1 using the normal form variables.

We remark that the first order approximations for $\mathcal{M}(\varphi)$ and $\mathcal{L}(\varphi)$ do not depend on t_0 at first order in μ but they do at higher orders.

Since both the splitting potential and the Melnikov potential are defined on $\mathbb{T}^n$, a direct application of Morse theory implies the existence, for $|\mu|$ small enough, of at least 2^n transverse homoclinic orbits to the whiskered torus $\mathcal{T}$, as long as the Melnikov potential is a Morse function, that is, all its critical points are nondegenerate (a generic property).

6. A computable example in the coupled case. As an example, we consider a perturbation of a coupled integrable Hamiltonian ($q \neq 0$ in (2.2)), with $n+1$ degrees of freedom. In the integrable part, we choose the classical pendulum: $V(x) = \cos x - 1$. In the perturbation, we consider a function only depending on φ:

$$H_1(\varphi) = \sum_{k \in \mathbb{Z}^n} h_k \, e^{i\langle k, \varphi \rangle}.$$

We use the following well-known formulas for the homoclinic trajectory of the standard pendulum:

$$\begin{aligned} &x_0(t) = 4 \arctan e^t, \quad y_0(t) = \frac{2}{\cosh t}, \\ &\delta_0(t) = q(x_0(t) - \pi), \\ &z_0(t, \varphi) = (x_0(t), y_0(t), \varphi + \delta_0(t) + \omega t, 0) \,. \end{aligned}$$

The solution $\chi(\varphi)$ of equation (5.3) is simply

$$\chi(\varphi) = -\sum_{k \neq 0} \frac{i \, h_k}{\langle k, \omega \rangle} \, e^{i\langle k, \varphi \rangle},$$

and the Melnikov potential is given by its Fourier series

$$L(\varphi) = \sum_{k \neq 0} L_k \, e^{i\langle k, \varphi \rangle},$$

where each coefficient is given by the following integral:

$$L_k = \frac{h_k \; \langle k, q \rangle \, e^{-i\pi\langle k, q \rangle}}{\langle k, \omega \rangle} \int_{-\infty}^{\infty} e^{i\langle k, \omega \rangle t} \, e^{i\langle k, q \rangle x_0(t)} \, y_0(t) \, \mathrm{d}t.$$

Then, for the Melnikov function $M(\varphi) = \sum_{k \neq 0} M_k \, e^{i\langle k, \varphi \rangle}$, it is clear that $M_k = ikL_k$.

For the sake of simplicity, we assume that q is "half-integer" (i.e. $2q \in \mathbb{Z}^n$). In this way, we can easily compute the Melnikov integrals using residue theory since all the singularities of the functions involved are poles. Roughly, we get:

$$|L_k| \sim |h_k| \; \frac{e^{\pm \frac{\pi}{2} \langle k, \omega \rangle}}{|\sinh(\pi \langle k, \omega \rangle)|} \; |\langle k, \omega \rangle|^{|\langle k, 2q \rangle| - 1} \,.$$

It is very interesting to sketch an analysis of this expression in the case of *fast* frequencies. Thus, we consider $\omega = \omega^*/\sqrt{\varepsilon}$ with ω^* Diophantine, and introduce $\gamma = \gamma^*/\sqrt{\varepsilon}$ in (2.6), for some $\tau \geq n-1$. We also assume exponentially decreasing coefficients for the perturbation, like $|h_k| \sim e^{-|k|\rho}$ for every k (the parameter ρ is the width of analyticity of $H_1(\varphi)$). Then we can make the following estimate: the dominant harmonic $L_k e^{i\langle k,\varphi\rangle}$ in the Fourier series of $L(\varphi)$ is given for the indexes k such that

$$|\langle k, \omega^*\rangle| \sim \gamma^* |k|^{-\tau}, \qquad |k| \sim \left(\frac{\pi}{2}\frac{\gamma^*}{\rho\sqrt{\varepsilon}}\right)^{1/(\tau+1)},$$

(assuming that the corresponding coefficients h_k are nonvanishing), and we deduce that the Melnikov potential $L(\varphi)$ is exponentially small in ε: $-\log L = \mathrm{O}\left(\varepsilon^{-1/2(\tau+1)}\right)$, as well as its gradient $M(\varphi)$.

A more precise asymptotic behavior can be obtained for the case of *two* rotators ($n = 2$ in (2.2)), since then one can apply the theory of continued fractions to the ratio of frequencies ω_2^*/ω_1^*. For instance, in the case of a *quadratic* number ω_2^*/ω_1^* like the 'golden mean' $(1+\sqrt{5})/2$, one can find, as in [17], that $\log L \sim -c(\log\varepsilon)\varepsilon^{-1/4}$, where $c(u)$ is a positive periodic function with period $2\log(\omega_2^*/\omega_1^*)$. A direct application of Theorem 5.1 ensures that the splitting distance is as predicted by the Melnikov function, but requires $\mu = \mathrm{o}\left(\exp\left(-c(\log\varepsilon)\varepsilon^{-1/4}\right)\right)$.

A justification of the case $\mu = \varepsilon^p$ for some $p > 0$ is immediate as long as one has a significant refinement of Theorem 5.1:

$$\mathcal{L}(\varphi - t_0\omega^*/\sqrt{\varepsilon}) = \mu L(\varphi - t_0\omega^*/\sqrt{\varepsilon}) + \mathrm{O}\left(\mu^2\varepsilon^{-p}\right),$$

for φ, t_0 on the *complex* strip $|\mathrm{Im}\,\varphi| \le \rho - \varepsilon^{1/4}$, $|\mathrm{Im}\,t_0| \le \pi/2 - \varepsilon^{1/4}$. Such kind of result is a byproduct of an extension theorem (see, for instance, [17, 52]), which is currently being researched by the authors.

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