Simultaneous binary collisions in the planar four body problem

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Abstract

We consider simultaneous binary collisions in the general planar four body problem. We prove they are regularizable in the sense of continuity with respect initial conditions using a blow up of the singularity. Furthermore, numerical evidence is given that the differentiability of the regularization should be, in general, less than $C^{8/3}$. A simple example, the double isosceles four body problem, displays that kind of behaviour.

1 Introduction

Consider the planar n body problem with masses m_i, for $i = 1, 2, \ldots, n$. Let $\mathbf{q}_i \in \mathbb{R}^2$ and $\mathbf{p}_i \in \mathbb{R}^2$ be the position coordinates and momenta of the mass m_i. The equations which describe the motion of the masses according to the laws of classical mechanics are

$$\begin{aligned} \dot{\mathbf{q}}_i &= \frac{1}{m_i}\,\mathbf{p}_i, \\ \dot{\mathbf{p}}_i &= \nabla_i U(\mathbf{q}), \quad i = 1, ..., n, \quad \text{where} \quad U(\mathbf{q}) = \sum_{1 \le i < j \le n} \frac{m_i m_j}{|\mathbf{q}_i - \mathbf{q}_j|}. \end{aligned} \tag{1}$$

The system above is defined everywhere except at the collision points, that is, the points such that $|\mathbf{q}_i - \mathbf{q}_j| = 0$ for some $i \neq j$. If $|\mathbf{q}_i - \mathbf{q}_j| = 0$ we say that masses m_i and m_j collide at the same point. A binary collision occurs when only two masses collide. If all the masses coincide at the same point we get a total collision. Between these two cases one can get different types of collisions, for instance, triple collision, simultaneous binary collisions, simultaneous binary and triple collisions, etc.

If a solution of (1) is defined on some maximal interval $[0, t_c)$ with $t_c < \infty$, it is said that it has a singularity. Moroever if the masses approach a limiting position as t tends to t_c, it can be proved that the limiting position must be a collision point. It is said that the solution ends at a collision.

The extension of a collision solution for $t \geq t_c$ has been studied in several cases. If the extension is possible in some sense it is said that the collision is regularizable.

Anyhow the extension has been considered in different ways. Sundman [19] and Siegel [10, 11] look for a continuation of the collision solution as an analytic function of time. We call that an analytic regularization.

A more interesting approach is the geometric viewpoint adopted by Easton [2] who asks for an extension of the collision solutions continuous with respect to nearby initial conditions.

It is well known that a binary collision is regularizable in all senses (see [18, 8, 2, 1]).

Concerning triple collision, Siegel [10] found that most of the solutions can not be extended through triple collision as analytic functions of some transformed time variable. More concretely, the set of masses for which triple collision is regularizable in Siegel sense is a zero measure dense set [12] . McGehee [6] studied the triple collision in the collinear three body problem using a transformation which blows up the collision to an invariant manifold called the "triple collision manifold". The flow collision orbits are asymptotic (for positive or negative time) to one of these rest points. The study of the flow on the "triple collision manifold" allows to prove that triple collision cannot be regularized in the sense of Easton except, at most, for a zero measure set of masses [13]. This is a consequence of the fact that the unstable invariant manifold of one rest point on the "triple collision manifold" do not coincides with the stable one of the other rest point. This has been a useful method to prove non regularizability of collision singularities, mainly triple collision in the three body problem. Anyhow the McGehee transformation allows to study the behaviour of the solutions passing near triple collision. Many authors used this approach to study singularities due to collision [1, 14, 7]. We refer also to [4] for a general study of regularization of differential systems.

In the four body problem besides binary, triple and total collision, simultaneous binary collisions can happen. Saari [9] proved that this singularity is regularizable using analytic regularization. In [16] Simó and Lacomba considered the regularization in the sense of Easton with an additional condition concerning the preservation of time. They studied simultaneous binary collisions by looking at the solutions near the singularity as perturbations of the solutions of m-uncoupled two body problems. They proved that regularization is possible, but without giving any geometric insight and using only continuity arguments.

In this paper we study the simultaneous binary collisions in the planar four body problem and we prove that they are regularizable in the same sense introduced in [16] but using McGehee's approach. With a new set of variables we introduce a "collision manifold" for this singularity and prove that the unstable and stable invariant manifolds of the rest points on the collision manifold coincide. This allows us to prove, using some additional reasoning, that the simultaneous binary collision is regularizable. The use of the collision manifold, and the variables introduced in this paper, provides a more clear geometric insight on the problem. We note here that the coincidence of the manifolds is a necessary but not sufficient condition for regularization [15]. The new variables have been also very useful to detect numerically that the regularization can not be analytical. In order to precise the results we need some definitions.

Consider equations (1) for $n = 4$. Let us denote by $\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4) \in \mathbb{R}^8$, $\mathbf{p} = (\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4) \in \mathbb{R}^8$, $\mathbf{z} = (\mathbf{q}, \mathbf{p})$ and $\varphi(t; \mathbf{z})$ the solution of (1) such that $\varphi(0; \mathbf{z}) = \mathbf{z}$.

Definition 1 *We say that a solution $\varphi(t; \mathbf{z}_0) = (\mathbf{q}(t), \mathbf{p}(t))$ of the planar four body problem goes to a simultaneous binary collision (s.b.c.) at $t = t_c$ if there is a binary collision between m_1 and m_2 and a simultaneous binary collision between m_3 and m_4, that is, $\mathbf{q}_1(t_c) = \mathbf{q}_2(t_c)$ and $\mathbf{q}_3(t_c) = \mathbf{q}_4(t_c)$, but $\mathbf{q}_1(t_c) \neq \mathbf{q}_3(t_c)$.*

Definition 2 *We say that a s.b.c. in the planar four body problem is regularizable if for any solution $\varphi_-(t; \mathbf{z}_-)$ which goes to s.b.c. at some time $t_c > 0$ ($t \to t_c^-$) there is a unique solution $\varphi_+(t; \mathbf{z}_+)$ going to a s.b.c. at $t = -t_c$ ($t \to -t_c^+$) such that, for any $t_f \in (t_c, 2t_c]$ the following statement holds:*

For any choice of sequences of time $\{t_n\} \to t_f$ *and initial conditions* $\{\mathbf{z}_n\} \to \mathbf{z}_-$ *such that* $\varphi(t;\mathbf{z}_n)$ *does not go to s.b.c. for* $t \in [0, t_f]$, $\lim_{n\to\infty} \varphi(t_n;\mathbf{z}_n) = \varphi_+(t_f - 2t_c;\mathbf{z}_+)$.

In that case we say that $\varphi_+(t;\mathbf{z}_+)$ *is a continuous extension of the collision solution* $\varphi_-(t;\mathbf{z}_-)$.

In the new variables any solution $\varphi_-(t;\mathbf{z}_-)$ which goes to *s.b.c.* for positive time belongs to the stable invariant manifold of some rest point, P_-, on the "collision manifold". We shall prove that $\varphi_-(t;\mathbf{z}_-)$ has a unique continuous extension, $\varphi_+(t;\mathbf{z}_+)$ which belongs to the unstable invariant manifold of an equilibrium point P_+. Both, P_- and P_+ correspond to *s.b.c.* (for positive and negative time respectively) at the same point on the physical phase space. The meaning of the rest points will be clarified in section 2. We get the following result.

Theorem 1 *For any value of the masses* m_1, m_2, m_3 *and* m_4 *the s.b.c. in the planar four body problem is regularizable.*

In sections 6 and 7 we take $m_1 = m_2$ and we restrict to the invariant subproblem which is obtained when m_3 and m_4 move on a straight line and m_1 and m_2 are placed symmetrically with respect this line in such a way that $\mathbf{q}_2 - \mathbf{q}_1$ is orthogonal to $\mathbf{q}_4 - \mathbf{q}_3$ (see figure 1 *b*). We call it the double isosceles four body problem. We choose a solution $\varphi_-(t;\mathbf{z}_-)$ going to *s.b.c.* and consider two sections of the flow before and after the *s.b.c.* but close to it. Let Q_c be the intersection point of $\varphi_-(t;\mathbf{z}_-)$ with the first section. We compute numerically the map following the flow between the two sections for different initial points in a small neigbourhood of Q_c and, in particular, along an arc containing Q_c. In that way we get numerical evidence to support the following conjecture

Conjecture *The order of differentiability of the regularization of the s.b.c. in the double isosceles problem is strictly less than* 8/3.

2 Equations of motion

Consider (1) for $n = 4$. Using the integrals of the center of masses we can assume $\sum m_i \mathbf{q}_i = 0$, and $\sum \mathbf{p}_i = 0$. As we are interested in the simultaneous binary collisions between m_1, m_2 and m_3, m_4, it is more convenient to reduce the equations by introducing $\mathbf{Q}_1 = \mathbf{q}_2 - \mathbf{q}_1$, $\mathbf{Q}_2 = \mathbf{q}_4 - \mathbf{q}_3$, and $\mathbf{Q} = \mathbf{q}_{34} - \mathbf{q}_{12}$, where

$$\mathbf{q}_{12} = \frac{m_1\mathbf{q}_1 + m_2\mathbf{q}_2}{\mu_1}, \quad \mathbf{q}_{34} = \frac{m_3\mathbf{q}_3 + m_4\mathbf{q}_4}{\mu_2}$$

with $\mu_1 = m_1 + m_2$, $\mu_2 = m_3 + m_4$. Let be $M = \mu_1 + \mu_2$.

Then the equations of motion can be written as

$$\begin{aligned} \ddot{\mathbf{Q}}_i &= -\mu_i \frac{\mathbf{Q}_i}{|\mathbf{Q}_i|^3} + \mathbf{F}_i, \quad i = 1, 2, \\ \ddot{\mathbf{Q}} &= \mathbf{F}_3, \end{aligned} \tag{2}$$

where the functions $\mathbf{F}_1, \mathbf{F}_2$ and $\mathbf{F}_3$ are given in the Appendix I.

First integrals due to the energy and the angular momentum conservation are

$$h = \frac{m_1m_2}{2\mu_1}|\dot{\mathbf{Q}}_1|^2 + \frac{m_3m_4}{2\mu_2}|\dot{\mathbf{Q}}_2|^2 + \frac{\mu_1\mu_2}{2M}|\dot{\mathbf{Q}}|^2 - U(\mathbf{q}), \tag{3}$$

$$\omega = \frac{m_1m_2}{\mu_1}\dot{\mathbf{Q}}_1 \wedge \mathbf{Q}_1 + \frac{m_3m_4}{\mu_2}\dot{\mathbf{Q}}_2 \wedge \mathbf{Q}_2 + \frac{\mu_1\mu_2}{M}\dot{\mathbf{Q}} \wedge \mathbf{Q}, \tag{4}$$

where

$$U(\mathbf{q}) = \frac{m_1m_2}{|\mathbf{Q}_1|} + \frac{m_3m_4}{|\mathbf{Q}_2|} + U_1(\mathbf{q}), \quad U_1(\mathbf{q}) = \frac{m_1m_3}{r_{13}} + \frac{m_1m_4}{r_{14}} + \frac{m_2m_3}{r_{23}} + \frac{m_2m_4}{r_{24}}$$

with $r_{ij} = |\mathbf{q}_i - \mathbf{q}_j|$.

Equations (2) with the first integrals (3) and (4) give a "classical mechanical" description of the problem. Now we shall use an intermediate set of "blown up" variables (cf. (5)) which allow us to define the *s.b.c.* manifold. Later on a final set of variables which are suitable for the purpose of the paper will be introduced (cf. (8)).

Let A be the 4×4 real matrix $A = \begin{pmatrix} \alpha_1 Id_2 & 0 \\ 0 & \alpha_2 Id_2 \end{pmatrix}$, with $\alpha_1 = m_1m_2/\mu_1$, $\alpha_2 = m_3m_4/\mu_2$, and define the vectors in $\mathbb{R}^4$, $\mathbf{Q}_a = (\mathbf{Q}_1, \mathbf{Q}_2)$, $\mathbf{s} = (\mathbf{s}_1, \mathbf{s}_2)$ and $\mathbf{z} = (\mathbf{z}_1, \mathbf{z}_2)$. We introduce the McGehee coordinates to blow up the singularity due to double binary collision:

$$r^2 = \mathbf{Q}_a^T \; A \; \mathbf{Q}_a = \alpha_1|\mathbf{Q}_1|^2 + \alpha_2|\mathbf{Q}_2|^2, \qquad \mathbf{s} = r^{-1}\mathbf{Q}_a, \qquad \mathbf{z} = r^{1/2}A\dot{\mathbf{Q}}_a$$

and a rescaling of time given by $dt = r^{3/2}d\tau$. We remark that $\mathbf{s}$ belongs to the ellipsoid $\mathcal{E}$ defined by $\mathbf{s}^T A\mathbf{s} = 1$. We also define variables x, y, ζ and ρ as follows (the constants $k_1 > 0$ and $k_2 > 0$ being arbitrary)

$$\mathbf{Q} = \frac{k_1}{x^2}\vec{\zeta}, \quad \dot{\mathbf{Q}} = \Omega\;(y, k_2x^2\rho)^T, \text{ where } \vec{\zeta} = (\cos\zeta, \sin\zeta)^T, \quad \Omega = \begin{pmatrix} \cos\zeta & -\sin\zeta \\ \sin\zeta & \cos\zeta \end{pmatrix}.$$

The new equations of motion are

$$\begin{aligned}
r' &= rv, \\
\mathbf{s}' &= -v\mathbf{s} + A^{-1}\mathbf{z}, \\
\mathbf{z}' &= \frac{1}{2}v\mathbf{z} + \nabla V(\mathbf{s}) + r^2 A\mathbf{F}, \\
x' &= -\frac{1}{2k_1}r^{3/2}x^3y, \\
y' &= r^{3/2}\left(-\frac{M}{k_1^2}x^4 + \frac{k_2^2}{k_1}x^6\rho^2 + G_1\right), \\
\rho' &= \frac{1}{k_2x^2}r^{3/2}\vec{\zeta}^T J^T \mathbf{F}_3, \\
\zeta' &= \frac{k_2}{k_1}r^{3/2}x^4\rho,
\end{aligned} \tag{5}$$

where $v = \mathbf{s}^T\mathbf{z}$, $\mathbf{F} = (\mathbf{F}_1, \mathbf{F}_2)$, $G_1 = \vec{\zeta}^T\mathbf{F}_3 + Mx^4/k_1^2$, $' = \frac{d}{d\tau}$, $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and

$$V(\mathbf{s}) = \frac{m_1m_2}{s_1} + \frac{m_3m_4}{s_2}, \quad \nabla V(\mathbf{s}) = \left(-\frac{m_1m_2}{s_1^3}\mathbf{s}_1, -\frac{m_3m_4}{s_2^3}\mathbf{s}_2\right) \quad \text{with} \quad s_i = |\mathbf{s}_i|.$$

Using the expression of $\mathbf{F}_3$ given in the Appendix I the term $\vec{\zeta}^T J^T \mathbf{F}_3$ appearing in the equation of ρ' is $O(x^6)$.

We note that a *s.b.c.* occurs if and only if $r = 0$.

Function v will play an important role in the following, so we note that

$$v' = -\frac{v^2}{2} + \mathbf{z}^T A^{-1}\mathbf{z} - V(\mathbf{s}) + r^2 \mathbf{s}^T A\mathbf{F}.$$

In the new variables (3) and (4) can be written as

$$\begin{aligned} hr &= \frac{1}{2}\mathbf{z}^T\, A^{-1}\, \mathbf{z} - V(\mathbf{s}) + r\left(\frac{\mu_1\mu_2}{2M}|\dot{\mathbf{Q}}|^2 - U_1(\mathbf{q})\right), \\ \omega &= r^{1/2}(\mathbf{z}\wedge\mathbf{s}) + \frac{\mu_1\mu_2}{M}\dot{\mathbf{Q}}\wedge\mathbf{Q}. \end{aligned} \tag{6}$$

Let $\Delta_{13}, \Delta_{14}, \Delta_{23}$, and Δ_{24} be defined as

$$\Delta_{ij} = \{(r, \mathbf{s}, x, \zeta) \mid r > 0, \mathbf{s}\in\mathcal{E}, x > 0, \zeta\in[0, 2\pi],\ r_{ij} = 0\},$$

and, for $i = 1, 2$,

$$\Delta_i = \{(\mathbf{s}, x, \zeta) \mid \mathbf{s}\in\mathcal{E}, x > 0, \zeta\in[0, 2\pi],\ |\mathbf{s}_i| = 0\}.$$

Consider the collision sets $\Delta_r = \Delta_{13}\cup\Delta_{14}\cup\Delta_{23}\cup\Delta_{24}$, and $\Delta_s = \Delta_1\cup\Delta_2$.

For a fixed value of the energy h we define

$$\begin{aligned} \bar{M}(h) = \ & \{(r, \mathbf{s}, \mathbf{z}, x, y, \rho, \zeta) \in (0,\infty)\times\mathcal{E}\times(\mathbb{R}^2)^2\times(0,\infty)\times\mathbb{R}\times\mathbb{R}\times[0, 2\pi] \mid \\ & (r, \mathbf{s}, x, \zeta)\notin\Delta_r,\ (\mathbf{s}, x, \zeta)\notin\Delta_s,\ \text{and (6) holds}\}, \end{aligned}$$

which is invariant under the flow given by (5).

$\bar{M}(h)$ has two natural boundaries. The first one, $\bar{\mathcal{C}}$, is obtained when $r = 0$. We call it the *s.b.c.* manifold. We get a second boundary of $\bar{M}(h)$, $\bar{\mathcal{I}}$, when $x = 0$, called the infinity manifold. The equations (5) are defined for $r = 0$ and $x = 0$, and $\bar{\mathcal{C}}$ and $\bar{\mathcal{I}}$ are invariant. The orbits on $\bar{\mathcal{C}}$ and $\bar{\mathcal{I}}$ are not physical solutions of the four body problem, but they carry information on nearby orbits. In section 4 we study the flow restricted to $\bar{\mathcal{C}}$.

Moeckel [7] has studied the triple collision in the planar three body problem using the McGehee coordinates. The equations (5) for $r, \mathbf{s}$ and $\mathbf{z}$ have the same structure as the ones given in [7] except for the term $r^2 A\mathbf{F}$, which appears in (5) but not in the three body problem. Of course, the potentials $V(\mathbf{s})$ in [7] and (5) are different. Anyway, some results obtained in [7] can be easily translated to our problem. In particular, the same proof of proposition 3.1 in [7] holds for (5) on $\bar{\mathcal{C}}$. So we get

Proposition 1 *If $r = 0$, the function $v(\tau; \mathbf{s}_0, \mathbf{z}_0)$ on the orbits is strictly increasing unless $(\mathbf{s}_0, \mathbf{z}_0)$ is a restpoint.*

The equilibrium points of system (5) satisfy $r = 0$. Also, for any solution, $\mathbf{s}\in\mathcal{E}$, of the equation

$$\nabla V(\mathbf{s}) + V(\mathbf{s})A\mathbf{s} = 0, \tag{7}$$

there exist two equilibrium points corresponding to $v = \pm\sqrt{2V(\mathbf{s})}$ and $\mathbf{z} = vA\mathbf{s}$.

After (7), the points $s \in \mathcal{E}$ corresponding to equilibrium points can be seen as critical points of the restriction of V to the ellipsoid $\mathcal{E}$ in the same way as central configurations but for a different potential.

It is easy to see that the solutions $\mathbf{s}$ of (7) on the ellipsoid $\mathcal{E}$ are given by $s_1^2 = m_1m_2/c_1$ and $s_2^2 = k^2s_1^2$ where $k^3 = \mu_2/\mu_1$ and $c_1 = \alpha_1(\mu_1\alpha_1 + \mu_2\alpha_2/k)$. So, there is a two dimensional torus in $\mathcal{E}$ giving solutions of (7). However recovering the physical coordinates we have $s_2^2/s_1^2 = |\mathbf{Q}_2|^2/|\mathbf{Q}_1|^2$ and then, *s.b.c.* can be achieved if and only if $|\mathbf{Q}_2|^2/|\mathbf{Q}_1|^2$ tends to $(\mu_2/\mu_1)^{2/3}$.

Next we introduce a new set of variables $(r, v, \theta, u, w_1, w_2, \varphi_1, \varphi_2, x, y, \rho, \zeta)$. First, for $i = 1, 2$, we define ρ_i, ζ_i, x_i and y_i as

$$\mathbf{s}_i = \rho_i\vec{\zeta_i}, \quad \vec{\zeta_i} = (\cos\zeta_i, \sin\zeta_i)^T, \quad \mathbf{z}_i = \sqrt{\alpha_i}\,\Omega(\zeta_i)\vec{\pi}_i, \quad \vec{\pi}_i = (y_i, x_i)^T,$$

where $\Omega(\zeta_i)$ stands for a rotation of angle ζ_i, that is, $\Omega(\zeta_i) = \begin{pmatrix} \cos\zeta_i & -\sin\zeta_i \\ \sin\zeta_i & \cos\zeta_i \end{pmatrix}$. Then $\alpha_1\rho_1^2 + \alpha_2\rho_2^2 = 1$. This suggests to introduce new variables, $\theta \in [0, \pi/2]$ such that

$$\sqrt{\alpha_1}\,\rho_1 = \cos\theta, \qquad \sqrt{\alpha_2}\,\rho_2 = \sin\theta,$$

and v, u, w_1 and w_2 defined by

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \Omega(\theta)\begin{pmatrix} v \\ u \end{pmatrix}, \qquad \begin{array}{rcl} w_1 &=& -x_1\cos\theta, \\ w_2 &=& -x_2\sin\theta. \end{array}$$

It is easy to check that w_1 and w_2 are the angular momenta (in the blow up variables) of the subsystems formed by the masses m_1, m_2 and m_3, m_4 respectively, that is,

$$w_i = \mathbf{z}_i \wedge \mathbf{s}_i, \quad i = 1, 2.$$

The relations above define a diffeomorphism

$$\begin{array}{ccl} \bar{\mathcal{E}} \times \mathbb{R}^4 & \to & \mathbb{R} \times (0, \pi/2) \times \mathbb{R} \times [0, 2\pi) \times [0, 2\pi) \times \mathbb{R}^2 \\ (\mathbf{s}, \mathbf{z}) & \to & (v, \theta, u, \zeta_1, \zeta_2, w_1, w_2) \end{array}$$

where $\bar{\mathcal{E}} = \{\mathbf{s} \in \mathcal{E} \mid |\mathbf{s}_1| \neq 0,\ |\mathbf{s}_2| \neq 0\}$. We remark that $\theta = 0$ and $\theta = \pi/2$ correspond to single binary collisions between m_3, m_4 and m_1, m_2, respectively.

It is not difficult to check that the system of equations depends on the differences $\zeta_i - \zeta$, $i = 1, 2$, but does not depend on ζ. Therefore, if we define

$$\varphi_i = \zeta_i - \zeta, \qquad i = 1, 2,$$

ζ can be eliminated. In that way we can reduce the dimension of the system by one unit. That reduction corresponds to the well known elimination of the node.

In the new variables we get the following system of equations

$$\begin{array}{rcl} r' &=& rv, \\ v' &=& \dfrac{v^2}{2} + u^2 + \dfrac{w_1^2}{\cos^2\theta} + \dfrac{w_2^2}{\sin^2\theta} - V(\theta) + r^2\mathbf{s}^TA\mathbf{F}, \end{array}$$

$$
\begin{aligned}
\theta' &= u, \\
u' &= -\frac{1}{2}vu + W(\theta) - \tan\theta\frac{w_1^2}{\cos^2\theta} + \frac{w_2^2}{\tan\theta\sin^2\theta} + r^2U, \\
w_1' &= -\frac{1}{2}vw_1 + r^2\left(\sqrt{\alpha_1\alpha_2}C_0rx^6\sin\theta\cos\theta\sin(\varphi_1-\varphi_2) - k_1\sqrt{\alpha_1}C_1x^4\cos\theta\sin\varphi_1\right), \\
w_2' &= -\frac{1}{2}vw_2 + r^2\left(-\sqrt{\alpha_1\alpha_2}C_0rx^6\sin\theta\cos\theta\sin(\varphi_1-\varphi_2) - k_1\sqrt{\alpha_2}C_2x^4\sin\theta\sin\varphi_2\right), \\
\varphi_1' &= -\frac{w_1}{\cos^2\theta} - \frac{k_2}{k_1}r^{3/2}x^4\rho, \qquad (8)\\
\varphi_2' &= -\frac{w_2}{\sin^2\theta} - \frac{k_2}{k_1}r^{3/2}x^4\rho, \\
x' &= -\frac{1}{2k_1}r^{3/2}x^3y, \\
y' &= r^{3/2}\left(-\frac{M}{k_1^2}x^4 + \frac{k_2^2}{k_1}x^6\rho^2 + G_1\right), \\
\rho' &= \frac{1}{k_2x^2}r^{3/2}\vec{\zeta}^TJ^T\mathbf{F}_3, \\
\zeta' &= \frac{k_2}{k_1}r^{3/2}x^4\rho,
\end{aligned}
$$

where

$$
\begin{aligned}
V(\theta) &= \frac{\mu_1\alpha_1^{3/2}}{\cos\theta} + \frac{\mu_2\alpha_2^{3/2}}{\sin\theta}, \\
\mathbf{s}^TA\mathbf{F} &= -rx^6\left(\frac{C_4}{\mu_1}\cos^2\theta + \frac{C_5}{\mu_2}\sin^2\theta\right) + \sqrt{\alpha_1\alpha_2}C_0rx^6\sin(2\theta)\cos(\varphi_1-\varphi_2) - \\
&\quad -k_1x^4\left[\sqrt{\alpha_1}C_1\cos\theta\cos\varphi_1 + \sqrt{\alpha_2}C_2\sin\theta\cos\varphi_2\right] \qquad (9)\\
&= \frac{rx^6}{k_1^3}\left[\mu_2\cos^2\theta(-1+3\cos^2\varphi_1) + \mu_1\sin^2\theta(-1+3\cos^2\varphi_2) + O_1\right], \\
U &= rx^6\sin\theta\cos\theta\left(\frac{C_4}{\mu_1} - \frac{C_5}{\mu_2}\right) + \sqrt{\alpha_1\alpha_2}C_0rx^6\cos(\varphi_1-\varphi_2)\cos(2\theta) - \\
&\quad -k_1x^4\left(-\sqrt{\alpha_1}C_1\sin\theta\cos\varphi_1 + \sqrt{\alpha_2}C_2\cos\theta\cos\varphi_2\right) \qquad (10)\\
&= \frac{rx^6}{k_1^3}\left[\sin\theta\cos\theta\left(\mu_2(1-3\cos^2\varphi_1) - \mu_1(1-3\cos^2\varphi_2)\right) + O_1\right], \\
G_1 &= \frac{M}{k_1^2}x^4 - \frac{M}{\mu_1\mu_2}x^4\left[k_1C_3 + rx^2\left(\sqrt{\alpha_1}C_1\cos\theta\cos\varphi_1 + \sqrt{\alpha_2}C_2\sin\theta\cos\varphi_2\right)\right] \\
&= O_1, \qquad (11)\\
\vec{\zeta}^TJ^T\mathbf{F}_3 &= -\frac{M}{\mu_1\mu_2}rx^6\left[\alpha_1C_1\sin\varphi_1 + \alpha_2C_2\sin\varphi_2\right] \qquad (12)\\
&= \frac{Mrx^6}{\mu_1\mu_2}\left[\frac{3rx^2}{k_1^4}\left(\mu_2\sqrt{\alpha_1}\cos\theta\sin\varphi_1\cos\varphi_1 + \mu_1\sqrt{\alpha_2}\sin\theta\sin\varphi_2\cos\varphi_2\right) + O_2\right],
\end{aligned}
$$

$W(\theta) = V'(\theta)$ is the derivative of $V(\theta)$ with respect to θ, and the functions C_i are given in the Appendix I. In (9), (10), (11) and (12), O_p stands for terms of order p in rx^2.

The energy integral is given by

$$\bar{h}r = \frac{1}{2}\left(v^2 + u^2 + \frac{w_1^2}{\cos^2\theta} + \frac{w_2^2}{\sin^2\theta}\right) - V(\theta), \quad \text{where} \tag{13}$$
$$\bar{h} = h - \frac{\mu_1\mu_2}{2M}\left(y^2 + k_2^2x^4\rho^2\right) + x^2V_1, \quad V_1 = \frac{U_1(\mathbf{q})}{x^2}$$

and the angular momentum can be written as

$$\omega = r^{1/2}(w_1 + w_2) - \frac{\mu_1\mu_2k_1k_2}{M}\rho.$$

As we said before, the first 11 equations in (8) do not depend on ζ. So, ζ can be obtained by a quadrature from the last equation.

In fact, we have performed the usual reduction based on the invariance of the equations of the planar four body problem under rotation in $\mathbb{R}^2$ and the classical elimination of the node. For a fixed value of the energy, h, we denote by $M(h)$ the reduced invariant manifold of $\bar{M}(h)$. In the same way, $\mathcal{C}$, $\mathcal{I}$ stand for the reduced *s.b.c.* and infinity manifolds respectively.

We note that the system (8) has singularities at $\theta = 0$ and $\theta = \pi/2$ which correspond to single binary collisions between m_3, m_4 and m_1, m_2 respectively. It is well known that single binary collisions can be regularized (see also Appendix III). Other singularities of (8) are obtained when some distance r_{13}, r_{14}, r_{23} or r_{24} goes to zero. Anyhow, as it was said before, our interest is to study a neigbourhood of two *s.b.c.* which take place when m_1 and m_2 collide at the same time as m_3 and m_4 do and at a positive distance of m_1 and m_2. So, we can assume that the distances r_{13}, r_{14}, r_{23} and r_{24} are bounded from below by some constant greater than zero.

We remark that system (8) is well defined for $w_1 = 0$, $w_2 = 0$ and $(v, \theta, u) \in \mathbb{R} \times (0, \pi/2) \times \mathbb{R}$.

We recover the equilibrium points of the system in the new variables

$$r = 0, \ v = \pm v^*, \ \theta = \theta^*, \ u = w_1 = w_2 = 0, \ \text{where}$$
$$\tan\theta^* = \left(\frac{\mu_2\alpha_2^{3/2}}{\mu_1\alpha_1^{3/2}}\right)^{1/3} \quad \text{and} \quad v^* = \sqrt{2V(\theta^*)}, \tag{14}$$

and $\varphi_1, \varphi_2 \in [0, 2\pi]$. Therefore, for fixed values x^*, y^* and ρ^* of x, y and ρ, we get two 2-dimensional tori of equilibrium points depending on the sign of v. In what follows, $P_{+(-)}(\varphi_1^*, \varphi_2^*, x^*, y^*, \rho^*)$ will denote the equilibrium point with $v > 0$ ($v < 0$) for fixed values $\varphi_1 = \varphi_1^*$, $\varphi_2 = \varphi_2^*$, $x = x^*$, $y = y^*$ and $\rho = \rho^*$, and $P_{+,(-)}$ generically stands for an equilibrium point with $v > 0$ ($v < 0$).

If P is an equilibrium point, the linear part of the vector field $\mathbf{F}$ defined by the first 11 equations of (8) is

$$\mathrm{D}\mathbf{F}(P) = \begin{pmatrix} B_1 & 0 & 0 & 0 \\ 0 & B_2 & 0 & 0 \\ 0 & B_3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{with} \quad B_1 = \begin{pmatrix} v_P & 0 & 0 & 0 \\ 0 & v_P & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & V''(\theta^*) & -v_P/2 \end{pmatrix},$$
$$B_2 = \begin{pmatrix} -v_P/2 & 0 \\ 0 & -v_P/2 \end{pmatrix}, \qquad B_3 = \begin{pmatrix} -1/\cos^2\theta^* & 0 \\ 0 & -1/\sin^2\theta^* \end{pmatrix},$$

where $v_P = \pm v^*$ is the value of v at the equilibrium point P.

Therefore 0 is an eigenvalue of $\mathrm{D}\mathbf{F}(P)$ with multiplicity 5, $\lambda_0 = v_P$ and $\lambda_1 = -v_P/2$ have multiplicity (at least) 2, and $\lambda_\pm = \left(-v_P \pm \sqrt{v_P^2 + 16V''(\theta^*)}\right)/4$ are the additional eigenvalues. A short computation shows that

$$\begin{array}{lllllllll} \lambda_+ & = & \frac{3}{2}v^*, & \lambda_- & = & -v^*, & \text{at} & P_-^*, & \text{and} \\ \lambda_+ & = & v^*, & \lambda_- & = & -\frac{3}{2}v^*, & \text{at} & P_+^*. & \end{array} \tag{15}$$

$W^{u(s)}(P)$ stands for the unstable (stable) invariant manifold of P. According to the signs of the corresponding eigenvalues we get dim $W^{u(s)}(P_\pm) = 3$, while the center parts have dimension 5.

Any orbit which goes to a *s.b.c.* for positive time belongs to $W^s(P_-(\varphi_1^*, \varphi_2^*, x^*, y^*, \rho^*))$ for some values $\varphi_1^*, \varphi_2^*, x^*, y^*$ and ρ^*. This means that the *s.b.c.* take place at a definite point of the phase space given by x^*, y^* and ρ^*. The values φ_1^* and φ_2^* give the limit directions for the binary collisions between m_1, m_2 and m_3, m_4, respectively. A similar assertion holds for the orbits which go to *s.b.c.* for negative time with respect to some restpoint P_+.

We note that in a neighbourhood of P_-, v is negative. So, $r(\tau)$ is a decreasing function of τ. Then, an orbit contained in the physical space, that is $r > 0$, can not tend to P_- for negative time τ. This means that $W^u(P_-)$ is contained in $\mathcal{C}$. In a similar way, $W^s(P_+)$ is also in $\mathcal{C}$.

We introduce e_1, e_2 as the energies (in the blow up variables) of the binary subsystems given by m_1, m_2 and m_3, m_4 respectively, that is,

$$e_i = \frac{1}{r}\left[\frac{1}{2\mu_i\alpha_i^2}|\mathbf{z}_i|^2 - \frac{1}{|\mathbf{s}_i|}\right], \quad i = 1, 2. \tag{16}$$

The functions $h_i = re_i, i = 1, 2$ will play an important role in the following. It is easy to check that

$$e_i' = \frac{1}{\mu_i\alpha_i} r\mathbf{z}_i^T\mathbf{F}_i, \quad h_i' = vh_i + \frac{1}{\mu_i\alpha_i} r^2\mathbf{z}_i^T\mathbf{F}_i, \quad i = 1, 2.$$

Using the last set of variables we get from (16)

$$e_1 = \frac{h_1}{r} = \frac{1}{2\mu_1\alpha_1 r}\left[(v\cos\theta - u\sin\theta)^2 + \frac{w_1^2}{\cos^2\theta} - \frac{2\mu_1\alpha_1^{3/2}}{\cos\theta}\right], \tag{17}$$

$$e_2 = \frac{h_2}{r} = \frac{1}{2\mu_2\alpha_2 r}\left[(v\sin\theta + u\cos\theta)^2 + \frac{w_2^2}{\sin^2\theta} - \frac{2\mu_2\alpha_2^{3/2}}{\sin\theta}\right], \tag{18}$$

and (13) can also be written as

$$\bar{h} = \mu_1\alpha_1 e_1 + \mu_2\alpha_2 e_2. \tag{19}$$

3 Subproblems

Several subproblems are obtained by taking $w_1 = 0$, $w_2 = 0$ and $\rho = 0$, that is, when the angular momentum of the four body problem is zero.

For any value of the masses the collinear four body problem corresponds to $\sin\varphi_1 = 0$ and $\sin\varphi_2 = 0$. This means that vectors $\mathbf{s}_i$, $i = 1, 2$ and $\mathbf{Q}$ are collinear.

If $m_1 = m_2$, $m_3 = m_4$ and $\cos\varphi_1 = \cos\varphi_2 = 0$, we get the trapezoidal four body problem (see figure 1a)). In that case it is easy to check that $C_1 = C_2 = 0$ in (8).

For any masses m_3, m_4 and $m_1 = m_2$ there is a third invariant subproblem when $w_1 = 0, w_2 = 0$ and $\rho = 0$, which is obtained if $\cos\varphi_1 = 0$ and $\sin\varphi_2 = 0$. We call it the *double isosceles four body problem* (see figure 1 b)). Obviously, an analogous problem is obtained when $m_3 = m_4$ if we take $\cos\varphi_2 = 0$ and $\sin\varphi_1 = 0$.

The equations of these subproblems are given in Appendix II. In sections 6 and 7 we shall consider several examples using the double isosceles four body problem.

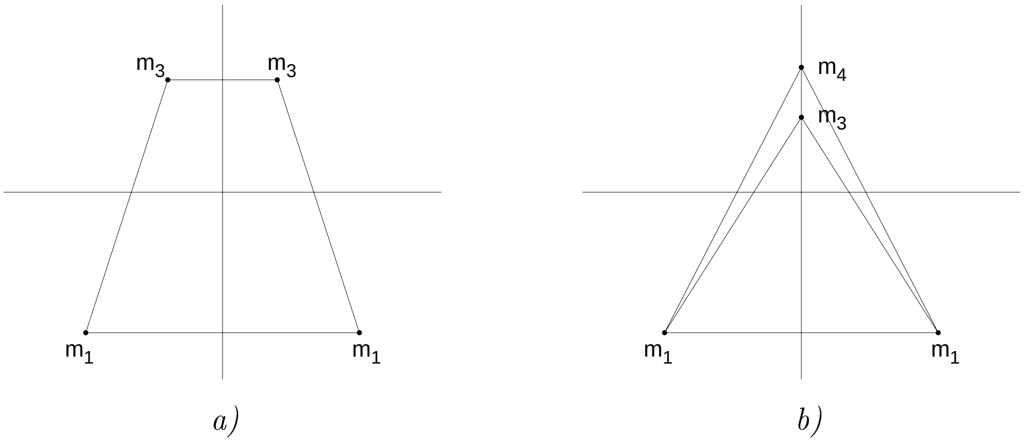

Figure 1: *a)* The trapezoidal four body problem. *b)* The double isosceles four body problem.

4 Simultaneous binary collision

The flow restricted to $\mathcal{C}$ is obtained by taking $r = 0$ in (8) . Then $r' = x' = y' = \rho' = \zeta' = 0$ and all the other equations are independent of φ_1 and φ_2. Therefore we only need to consider the following system of equations

$$\begin{aligned}
v' &= \frac{v^2}{2} + u^2 + \frac{w_1^2}{\cos^2\theta} + \frac{w_2^2}{\sin^2\theta} - V(\theta), \\
\theta' &= u, \\
u' &= -\frac{1}{2}vu + W(\theta) - \tan\theta\frac{w_1^2}{\cos^2\theta} + \frac{w_2^2}{\tan\theta\sin^2\theta}, \\
w_1' &= -\frac{1}{2}vw_1, \\
w_2' &= -\frac{1}{2}vw_2.
\end{aligned} \tag{20}$$

From (13) we also have

$$0 = \frac{1}{2}\left(v^2 + u^2 + \frac{w_1^2}{\cos^2\theta} + \frac{w_2^2}{\sin^2\theta}\right) - V(\theta). \tag{21}$$

In that way, orbits on $\mathcal{C}$ are obtained from the solutions of (20) and subsequent integration of the equations for φ_1 and φ_2 restricted to $\mathcal{C}$, that is,

$$\varphi_1' = -\frac{w_1}{\cos^2\theta}, \quad \varphi_2' = -\frac{w_2}{\sin^2\theta}. \tag{22}$$

Of course, x, y and ρ remain constant on $\mathcal{C}$. So, we can restrict ourselves to consider the projection of $\mathcal{C}$ under the projection map

$$\Pi : (r, v, \theta, u, w_1, w_2, \varphi_1, \varphi_2, x, y, \rho) \to (v, \theta, u, w_1, w_2).$$

We denote by $\mathcal{C}_0 = \Pi(\mathcal{C})$ the projection of $\mathcal{C}$. Then

$$\mathcal{C}_0 = \{\ (v, \theta, u, w_1, w_2) \in \mathbb{R} \times (0, \pi/2) \times \mathbb{R}^3 \mid (21) \ \text{ holds}\ \}.$$

The system (20) has two equilibrium points $Q_\pm = (v, \theta, u, w_1, w_2)$ with v, θ, u, w_1 and w_2 given by (14). We note that $Q_\pm$ are the projections of the equilibrium points of the global system (8), that is, for any fixed values $\varphi_1^*, \varphi_2^*, x^*, y^*$ and ρ^* we have $Q_\pm = \Pi(P_\pm(\varphi_1^*, \varphi_2^*, x^*, y^*, \rho^*))$ where $P_\pm(\varphi_1^*, \varphi_2^*, x^*, y^*, \rho^*)$ are the equilibrium points of (8). On $\mathcal{C}_0$, $Q_\pm$ are hyperbolic points. The corresponding eigenvalues of the linearization of the system are obtained from (15) as

$$\begin{array}{llllll} \lambda_1 = \frac{1}{2}v^*, & \lambda_+ = \frac{3}{2}v^*, & \lambda_- = -v^*, & \text{at} & Q_-, & \text{and} \\ \lambda_1 = -\frac{1}{2}v^*, & \lambda_+ = v^*, & \lambda_- = -\frac{3}{2}v^*, & \text{at} & Q_+, & \end{array}$$

where λ_1 has multiplicity 2 and $\lambda_\pm$ have multiplicity 1.

If we denote by $W_0^{u(s)}(Q)$ the unstable (stable) invariant manifold of Q on $\mathcal{C}_0$, we have that dim $W_0^u(Q_-) = 3 =$ dim $W_0^s(Q_+)$ and dim $W_0^u(Q_+) = 1 =$ dim $W_0^s(Q_-)$. However from $W_0^s(Q_-)$ we can recover $W^s(P_-(\varphi_1^*, \varphi_2^*, x^*, y^*, \rho^*)) \cap \mathcal{C}$ by adding to each orbit of $W_0^u(Q_-)$ the behaviour of φ_1 and φ_2 obtained by integration of (22) with the condition $\lim_{\tau\to\infty} \varphi_i(\tau) = \varphi_i^*, i = 1, 2$, and x, y and ρ equal to the constant values x^*, y^* and ρ^* respectively. In a similar way from $W_0^s(Q_+)$ and $W_0^u(Q_\pm)$ we can recover the intersection of $\mathcal{C}$ with the corresponding unstable and stable invariant manifolds for the global system. In that way we can get exactly $W^u(P_-)$ and $W^s(P_+)$ for any equilibrium point $P_\pm$ because, as it was shown in the section 2, these two invariant manifolds are contained in $\mathcal{C}$.

Proposition 2 *For any value of the masses, m_1, m_2, m_3 and m_4, $W_0^u(Q_-) \equiv W_0^s(Q_+)$.*

Proof Using (17), (18) and (21) we get for the scaled energies h_1, h_2 of the two binary subsystems

$$\mu_1\alpha_1 h_1 + \mu_2\alpha_2 h_2 = 0. \tag{23}$$

On $\mathcal{C}_0$, h_1 and h_2 satisfy $h_1' = vh_1$ and $h_2' = vh_2$ and, hence, $h_1 = 0$ and $h_2 = 0$ are invariant sets on $\mathcal{C}_0$. Also at the equilibrium points $Q_\pm$, $h_1 = h_2 = 0$ holds.

Let us consider an orbit $\Gamma(\tau) \in W_0^u(Q_-)$ and define $h_1(\tau) := h_1(\Gamma(\tau))$. Then $h_1(\tau) \to 0$ when $\tau \to -\infty$. Let us assume that for $\tau < 0$ and $|\tau|$ big enough one has $h_1(\tau) > 0$.

In a neighbourhood of Q_- v is negative. Hence, $h_1' = vh_1 < 0$ and so, for these values of τ, $h_1(\tau)$ is a decreasing function of τ. This gives a contradiction with the fact that $h_1(\tau) > 0$ tends to 0 when τ tends to $-\infty$. A similar argument shows that $h_1(\tau)$ is non negative for $\tau < 0$ with $|\tau|$ big enough. Therefore, on $\Gamma(\tau)$ one has $h_1 = 0$ and using (23) we conclude that $h_2(\Gamma(\tau)) = 0$. The same argument shows that $W_0^s(Q_+)$ is contained in the invariant set defined by $h_1 = 0$ and $h_2 = 0$. From (17) and (18), if $h_1 = h_2 = 0$ we get

$$2\mu_1\alpha_1^{3/2}\cos\theta - w_1^2 \geq 0, \qquad 2\mu_2\alpha_2^{3/2}\sin\theta - w_2^2 \geq 0, \tag{24}$$

and

$$v = \pm\left[2\mu_1\alpha_1^{3/2}\cos\theta - w_1^2\right]^{1/2} \pm \left[2\mu_2\alpha_2^{3/2}\sin\theta - w_2^2\right]^{1/2}, \tag{25}$$

$$u = \mp\left[2\mu_1\alpha_1^{3/2}\cos\theta - w_1^2\right]^{1/2}\frac{\sin\theta}{\cos\theta} \pm \left[2\mu_2\alpha_2^{3/2}\sin\theta - w_2^2\right]^{1/2}\frac{\cos\theta}{\sin\theta}. \tag{26}$$

The expressions above show that if $h_1 = 0$ and $h_2 = 0$, v is bounded. Then, if $\Gamma(\tau)$ is an orbit in $W_0^u(Q_-)$, we have that $v(\tau) := v(\Gamma(\tau))$ is bounded for all real τ. Using that the flow on $\mathcal{C}_0$ is gradient-like with respect to v we conclude that the $\omega-$limit set of $\Gamma(\tau)$ is Q_+. A similar argument shows that Q_- is the $\alpha-$limit set of the orbits $\Gamma(\tau) \in W_0^s(Q_+)$.♣

Remark 1 The proof of the proposition shows that the invariant manifolds $W_0^u(Q_-)$ and $W_0^s(Q_+)$ are obtained as the intersection of the invariant sets $h_1 = 0$ and $h_2 = 0$.

Remark 2 From (26) we see that u is not bounded when $\theta = 0$ or $\theta = \pi/2$ which correspond to binary collisions. We can regularize these singularities by introducing a new variable $w = u\sin\theta\cos\theta/\sqrt{F(\theta)}$, where $F(\theta) = \sin\theta\cos\theta V(\theta)$ (which is bounded, from above and from below by positive quantities, $\forall\,\theta \in [0, \pi/2]$), and a new parametrization of time given by $d\tau = \left(\sin\theta\cos\theta/\sqrt{F(\theta)}\right)d\bar{\tau}$ which allows to extend analytically the flow to $\theta = 0$ and $\theta = \pi/2$ (see Appendix III). Using w (21) can be written as

$$\frac{w^2F}{\sin\theta\cos\theta} + w_1^2\tan\theta + \frac{w_2^2}{\tan\theta} = 2F(\theta) - v^2\sin\theta\cos\theta$$

and then w tends to zero when θ tends to 0 or $\pi/2$, because the left hand side is a sum of non negative terms and the right hand side tends to zero.

Furthermore for the invariant manifold $W_0^u(P_-)$ we get from (26)

$$w = \mp\left[2\mu_1\alpha_1^{3/2}\cos\theta - w_1^2\right]^{1/2}\frac{\sin^2\theta}{\sqrt{F}} \pm \left[2\mu_2\alpha_2^{3/2}\sin\theta - w_2^2\right]^{1/2}\frac{\cos^2\theta}{\sqrt{F}},$$

which is well defined for $\theta = 0$ and $\theta = \pi/2$.

Remark 3 In fact, (25) and (26) give the invariant manifolds $W_0^u(Q_-)$ and $W_0^s(Q_+)$ decomposed in four branches $\Gamma_1, \Gamma_2, \Gamma_3$ and Γ_4 depending on the signs of v, that is,

$$\begin{aligned}
\Gamma_1 &: \ v = +\left[2\mu_1\alpha_1^{3/2}\cos\theta - w_1^2\right]^{1/2} + \left[2\mu_2\alpha_2^{3/2}\sin\theta - w_2^2\right]^{1/2}, \\
\Gamma_2 &: \ v = +\left[2\mu_1\alpha_1^{3/2}\cos\theta - w_1^2\right]^{1/2} - \left[2\mu_2\alpha_2^{3/2}\sin\theta - w_2^2\right]^{1/2}, \\
\Gamma_3 &: \ v = -\left[2\mu_1\alpha_1^{3/2}\cos\theta - w_1^2\right]^{1/2} + \left[2\mu_2\alpha_2^{3/2}\sin\theta - w_2^2\right]^{1/2}, \\
\Gamma_4 &: \ v = -\left[2\mu_1\alpha_1^{3/2}\cos\theta - w_1^2\right]^{1/2} - \left[2\mu_2\alpha_2^{3/2}\sin\theta - w_2^2\right]^{1/2}.
\end{aligned}$$

We note that only the branches Γ_2 and Γ_3 cut the plane $v = 0$. Γ_2 is contained in the region $u < 0$ and Γ_3 in $u > 0$. On the other hand $v < 0$ on Γ_4 and $v > 0$ on Γ_1.

The restrictions (24) imply that, if we consider an orbit $\Gamma(\tau) \in W_0^u(Q_-)$ such that $\theta(\hat{\tau}) = \pi/2$ for some $\hat{\tau}$, then $w_1(\hat{\tau}) = 0$. Therefore $w_1(\tau) = 0\ \forall\, \tau \in \mathbb{R}$ because $w_1 = 0$ is invariant on $\mathcal{C}_0$. In a similar way, if $\theta(\hat{\tau}) = 0$ for some $\hat{\tau}$ then $w_2(\tau) = 0\ \forall\, \tau \in \mathbb{R}$. Therefore, for any orbit $\Gamma(\tau) \in W_0^u(Q_-)$, there exist $0 \le \theta_1 < \theta_2 \le \pi/2$ such that $\theta_1 \le \theta(\tau) \le \theta_2$ and $\theta_1 = 0$ ($\theta_2 = \pi/2$) if and only if $w_2 = 0$ ($w_1 = 0$). Figure 2 shows the projection on the plane (θ, v) of some orbits of $W_0^u(P_-)$ obtained by numerical integration.

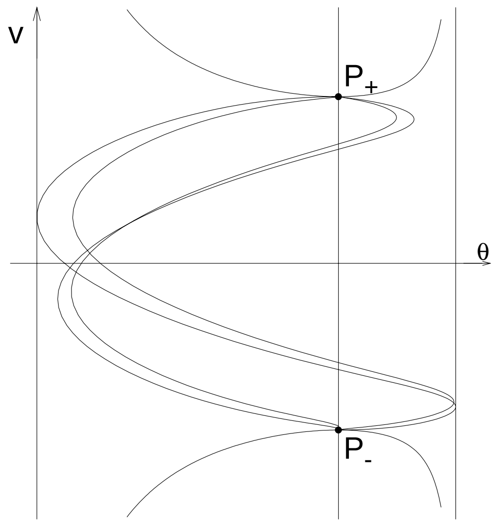

Figure 2: Orbits of $W_0^u(P_-)$ for masses $m_1 = 1, m_2 = 2, m_3 = 3, m_4 = 4$.

Theorem 2 *For any value of the masses m_1, m_2, m_3 and m_4, and $\varphi_i^* \in [0, 2\pi]$, $i = 1, 2$, $x^* \ge 0, y^* \in \mathbb{R}$, $\rho^* \in \mathbb{R}$, $W^u(P_-(\varphi_1^*, \varphi_2^*, x^*, y^*, \rho^*)) \equiv W^s(P_+(\varphi_1^*, \varphi_2^*, x^*, y^*, \rho^*))$.*

Proof We recall that for any equilibrium point P_- of (8), the unstable invariant manifold is contained in the *s.b.c.* manifold $\mathcal{C}$, that is, $r = 0$.

Let be γ an orbit of $W^u(P_-(\varphi_1^*, \varphi_2^*, x^*, y^*, \rho^*))$. Then we can write $\gamma(\tau) = (v(\tau), \theta(\tau),$ $u(\tau), w_1(\tau), w_2(\tau), \varphi_1(\tau), \varphi_2(\tau), x^*, y^*, \rho^*))$ where $\gamma_0(\tau) = (v(\tau), \theta(\tau), u(\tau), w_1(\tau), w_2(\tau))$

is a solution of (20) such that $\gamma_0 \in W_0^u(Q_-)$, and $\varphi_i(\tau)$, $i = 1, 2$ are obtained by integration of (22). Using proposition 2 we only need to prove that $\varphi_i(\tau) \to \varphi_i^*$ (mod 2π), $i = 1, 2$, when $\tau \to \infty$.

If $w_1(0) = 0$ ($w_2(0) = 0$) then $w_1(\tau) = 0$ ($w_2(\tau) = 0$) for all τ and $\varphi_1(\tau)$ ($\varphi_2(\tau)$) is constant. Therefore the statement of the theorem holds.

Let us assume that $w_1(0) \neq 0$. After remark 3, for γ_0 one has $\theta(\tau) < \pi/2\ \forall\, \tau \in \mathbb{R}$.

We introduce

$$v_1 = \alpha_1^{-3/4}\sqrt{\cos\theta}\,(v\cos\theta - u\sin\theta), \qquad u_1 = -\alpha_1^{-3/4}\frac{w_1}{\sqrt{\cos\theta}}.$$

It is not difficult to check that if $h_1 = 0$ then

$$\frac{dv_1}{d\varphi_1} = \frac{u_1}{2}, \qquad \frac{du_1}{d\varphi_1} = -\frac{v_1}{2}. \tag{27}$$

The solutions $(v_1(\varphi_1), u_1(\varphi_1))$ of the system above are $4\pi-$periodic in φ_1. We can write $u_1 = d_1 \sin(\varphi_1/2 + d_2)$ for some constants d_1, d_2, and then on γ_0

$$-\frac{\alpha_1^{-3/4} w_1(\tau)}{\sqrt{\cos\theta(\tau)}} = u_1(\tau) = d_1 \sin\left(\frac{\varphi_1(\tau)}{2} + d_2\right), \qquad \tau \in \mathbb{R}. \tag{28}$$

We know that on $\mathcal{C}$, $w_1 = 0$ is invariant. Therefore, $w_1(\tau)$ does not change the sign and then, from (22), $\varphi_1(\tau)$ is an increasing or a decreasing function of τ depending on the sign of w_1. However, $w_1(\tau) \to 0$ and $\theta(\tau) \to \theta^*$ when $\tau \to \pm\infty$. Taking the limits of (28) when $\tau \to \infty$ and $-\infty$, we get that $\lim_{\tau\to\infty}\varphi_1(\tau) = \lim_{\tau\to-\infty}\varphi_1(\tau) \pm 2\pi = \varphi_1^* \pm 2\pi$.

If $w_2(0)$ is different from zero we can introduce

$$v_2 = \alpha_2^{-3/4}\sqrt{\sin\theta}\,(v\sin\theta + u\cos\theta), \qquad u_2 = -\alpha_2^{-3/4}\frac{w_2}{\sqrt{\sin\theta}}.$$

Now, on $h_2 = 0$, we get

$$\frac{dv_2}{d\varphi_2} = \frac{u_2}{2}, \qquad \frac{du_2}{d\varphi_2} = -\frac{v_2}{2}$$

and, in the same way as before, we can prove that $\lim_{\tau\to\infty}\varphi_2(\tau) = \varphi_2^* \pm 2\pi$. ♣

The variables v_1, u_1, v_2 and u_2 introduced in the proof of the theorem 2 allow us to give an interpretation of the orbits on the *s.b.c.* manifold, $\mathcal{C}$, in terms of two uncoupled two body problems. To see that we recall (see [1]) that the binary collision in the planar two body problem described by the differential equation $\ddot{\mathbf{q}}_1 = -k_1\mathbf{q}_1/|\mathbf{q}_1|^3$, $k_1 > 0$, can be studied using polar coordinates (r_1, φ_1) and the radial and angular velocities, v_1 and u_1, respectively. After a rescaling of the physical time t, given by $dt = r_1^{3/2}d\tau_1$, the equations can be written as

$$\begin{array}{rcl} r_1' &=& r_1 v_1, \\ v_1' &=& \frac{1}{2}\, v_1^2 + u_1^2 - k_1, \\ \theta_1' &=& u_1, \\ u_1' &=& -\frac{1}{2} v_1 u_1, \end{array} \tag{29}$$

and the energy integral

$$\frac{1}{2}(v_1^2 + u_1^2) - k_1 = r_1 h_1. \tag{30}$$

We remark that the solutions of (29) can be obtained by integrating the equations for v_1, θ_1 and u_1 and then r_1 is obtained by integrating the first equation. We get binary collision by taking $r_1 = 0$. Now the collision manifold defined by (30) when $r_1 = 0$ is a two dimensional torus. On that torus the solutions of (29) satisfy the following equations

$$\frac{dv_1}{d\theta_1} = \frac{u_1}{2}, \qquad \frac{du_1}{d\theta_1} = -\frac{v_1}{2}.$$

For a fixed value of the energy the physical orbits which go to collision are exactly the ones contained on the invariant plane $u_1 = 0$. In fact, if $h_1 < 0$ we get ejection-collision orbits. For positive values of h_1, the orbits become unbounded (see figure 3). In order to simplify the exposition all the orbits on $u_1 = 0$, except the rest points, will be called homothetic orbits.

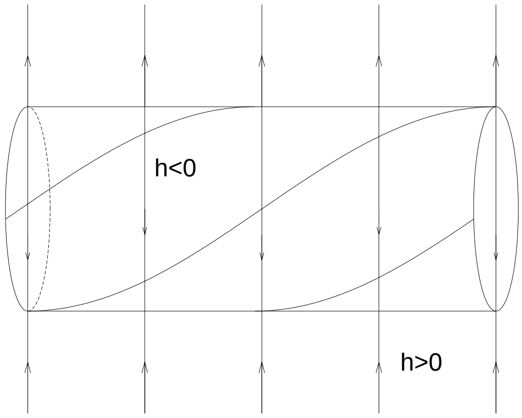

Figure 3: The collision manifold and some homothetic orbits for the two body problem.

Now we return to the planar four body problem. The variables v_1 and u_1, introduced in the proof of theorem 2, are the radial and angular velocities of the two body problem defined by m_1 and m_2, and φ_1 is the angular coordinate when that problem is written in polar coordinates. The variables v_2, u_2 and φ_2 are the corresponding ones for the two body problem defined by m_3 and m_4. In fact, from (20) and (22) we get the following equations for $v_1, u_1, v_2, u_2, \varphi_1$ and φ_2

$$\begin{aligned}
v_1' &= \frac{\alpha_1^{3/4}}{\cos^{3/2}\theta}\left(\tfrac{1}{2}v_1^2 + u_1^2 - \mu_1\right), & v_2' &= \frac{\alpha_2^{3/4}}{\sin^{3/2}\theta}\left(\tfrac{1}{2}v_2^2 + u_2^2 - \mu_2\right),\\
u_1' &= -\frac{\alpha_1^{3/4}}{2\cos^{3/2}\theta}\, v_1 u_1, & u_2' &= -\frac{\alpha_2^{3/4}}{2\sin^{3/2}\theta}\, v_2 u_2,\\
\varphi_1' &= \frac{\alpha_1^{3/4}}{\cos^{3/2}\theta}\, u_1, & \varphi_2' &= \frac{\alpha_2^{3/4}}{\sin^{3/2}\theta}\, u_2.
\end{aligned}$$

Then, using two different scales of time given by $\alpha_1^{-3/4}\cos^{3/2}\theta$ and $\alpha_2^{-3/4}\sin^{3/2}\theta$ respectively, the equations above can be seen as a product of two uncoupled two body problems. Moroever, for the scaled energies h_1 and h_2 we have

$$h_1\cos\theta = \frac{\alpha_1^{1/2}}{\mu_1}\left[\frac{1}{2}(v_1^2+u_1^2)-\mu_1\right], \quad h_2\sin\theta = \frac{\alpha_2^{1/2}}{\mu_2}\left[\frac{1}{2}(v_2^2+u_2^2)-\mu_2\right],$$

and if we restrict to $h_1 = 0$ and $h_2 = 0$, the equations reduce to

$$\begin{aligned} \frac{dv_1}{d\varphi_1} &= \frac{1}{2}\,u_1, & \frac{dv_2}{d\varphi_2} &= \frac{1}{2}\,u_2, \\ \frac{du_1}{d\varphi_1} &= -\frac{1}{2}\,v_1, & \frac{du_2}{d\varphi_2} &= -\frac{1}{2}\,v_2. \end{aligned}$$

Therefore after proposition 2 the orbits which belong to $W^u(Q_-)$ or, equivalently, to $W^s(Q_+)$, can be seen as the product of the orbits on the binary collision tori of two uncoupled two body problems.

Now we consider $W_0^u(Q_+)$ and an orbit $\Gamma(\tau) \in W_0^u(Q_+)$. Then $\Gamma(\tau) \to Q_+$ when $\tau \to -\infty$. In particular, $w_i(\tau) \to 0$, $i = 1, 2$, when $\tau \to -\infty$. From (20), and using the fact that the flow on $\mathcal{C}$ is gradient like with respect to v, on $\Gamma(\tau)$ we have $v(\tau) > 0\;\forall\,\tau \in \mathbb{R}$ and then, for $i = 1, 2$ one has $w_i' < 0$ if $w_i > 0$ and $w_i' > 0$ if $w_i < 0$. This gives a contradiction with the fact that, for $i = 1, 2$, one has $w_i(\tau) \to 0$ when $\tau \to -\infty$. We conclude that the orbits of $W_0^u(Q_+)$ are contained in the invariant set defined by $w_1 = 0$ and $w_2 = 0$. The same is true for the orbits on $W_0^s(Q_-)$. On this invariant set (20) and (21) reduce to

$$\begin{aligned} v' &= \frac{v^2}{2} + u^2 - V(\theta), \\ \theta' &= u, \\ u' &= -\frac{1}{2}vu + W(\theta) \end{aligned}$$

and $\frac{1}{2}(v^2+u^2) - V(\theta) = 0$ holds.

$W_0^u(Q_+)$ ($W_0^s(Q_-)$) has two branches. One branch, Γ_1, leaves a neigbourhood of Q_+ with $\theta > \theta^*$ and then v becomes unbounded at the same time as θ approaches $\pi/2$. On the second branch, Γ_0, v increases at the same time as θ approaches to zero. As $W_0^s(Q_+)$ is given by $h_1 = 0$ and $h_2 = 0$, we get that on the branches Γ_0 and Γ_1 the signs of the scaled energies, h_1 and h_2, are opposite and it is not difficult to check that $h_1 < 0$ and $h_2 > 0$ on Γ_1, while $h_1 > 0$ and $h_2 < 0$ on Γ_0.

Moroever on $\mathcal{C}_0$, $h_i' = vh_i$, $i = 1, 2$. Therefore, for the orbits of $W_0^u(Q_+)$, $v(\tau) \geq v^* > 0$ with v^* defined in (14), and we get

$$\begin{aligned} h_1(\tau) &\leq h_1(0)e^{v^*\tau}, \quad \text{if} \quad h_1(0) < 0, \quad \text{and} \\ h_1(\tau) &\geq h_1(0)e^{v^*\tau}, \quad \text{if} \quad h_1(0) > 0. \end{aligned}$$

Therefore on Γ_1 (Γ_0) we get $h_1(\tau) \to -\infty(\infty)$ when $\tau \to \infty$ and using (23), $h_2(\tau) \to \infty(-\infty)$.

For fixed values $\varphi_1^*, \varphi_2^* \in [0, 2\pi]$, $x^* \geq 0, y^* \in \mathbb{R}$ and $\rho^* \in \mathbb{R}$, we have $w_1 \equiv 0$ and $w_2 \equiv 0$ on each orbit of $W_0^u(P_+(\varphi_1^*, \varphi_2^*, x^*, y^*, \rho^*))$. Using the variables v_1, u_1, v_2 and u_2

this means that these orbits are contained in the invariant planes defined by $u_1 = 0$ and $u_2 = 0$. So, they can be seen as the product of two homothetic orbits of two uncoupled two body problems (see figure 3), one of them, with negative energy, corresponding to the subsystem m_1 and m_2, and the other to the subsystem m_3, m_4, with positive energy. The radial velocity of the last one will be unbounded.

5 Regularization

The purpose of this section is to prove the theorem 1. For fixed values of $\varphi_1^*, \varphi_2^*, x^*, y^*$ and ρ^* we will denote as $P_\pm^* = P_\pm(\varphi_1^*, \varphi_2^*, x^*, y^*, \rho^*)$. In order to give an idea of the key points of the proof, we note first that the orbits which pass near *s.b.c.* will enter in a neigbourhood of some equilibrium point, P_-^*, and go out by following one of the two branches (i.e., the branch which leaves the rest point with $u > 0$ or the one with $u < 0$) of the unstable invariant manifold of the equilibrium point. After a finite time they should enter a neighbourhood of P_+^* and leave it near $W^u(P_+^*)$. Furthermore we can take a small arc $\gamma(s)$, with $s \in [-1, 1]$, such that $\gamma(0) \in W^s(P_-^*)$, the orbits with initial conditions equal to $\gamma(s)$ going out from a neighbourhood of P_-^* by following different branches of $W^u(P_-^*)$, depending on the sign of s. This means that these orbits will enter a neighbourhood of P_+^* in two different ways. So, we need to ensure that the orbits with initial conditions equal to $\gamma(s)$ will leave the neighbourhood of P_+^* near one orbit of $W^u(P_+^*)$ which is independent of the sign of s, that is, independent of which branch of $W^u(P_-^*)$ they followed. To do that we shall prove that, for any orbit of $W^s(P_-^*)$ ($W^u(P_+^*)$), the energies e_1 and e_2 of the subsystems have definite limits when τ tends to ∞ ($-\infty$) and these limits are different for different orbits. Therefore the continuous extension of the collision orbit corresponding to $\gamma(0)$ will be the unique orbit of $W^u(P_+^*)$ which has as energy limits for τ tending to $-\infty$ the same ones that the collision orbit has for τ going to ∞.

Let us consider an equilibrium point P_-^*. We denote by $\mathbf{x}$ the set of variables $(r, v, \theta, u, w_1, w_2, \varphi_1, \varphi_2, x, y, \rho)$. For a small positive value δ such that $\delta < v^*$ we define $N_\delta = \{\mathbf{x} \in M(h) \mid |\mathbf{x} - P_-^*| < \delta\}$ and $W^s_{\rm loc}(P_-^*) = \{\mathbf{x} \in N_\delta \cap W^s(P_-^*) \mid \varphi(\tau; \mathbf{x}) \in N_\delta \;\; \forall \tau \geq 0\}$. That is, $W^s_{\rm loc}(P_-^*)$ contains the orbits of $W^s(P_-^*)$ which remain in N_δ for all positive τ. In a similar way we can define $W^u_{\rm loc}(P_-^*)$.

Let be $P \in W^s_{\rm loc}(P_-^*)$. The orbit $\varphi(\tau; P)$ tends to P_-^* when $\tau \to \infty$. So, in the physical space it corresponds to a solution going to a double binary collision at some time $t = t_c$. As it is usual we can bound the collision time t_c in the following way.

Using that $\varphi(\tau; P) \in N_\delta$ for all $\tau \geq 0$ we get $r(\tau) < \delta$ and $v(\tau) < \delta_0$, where $\delta_0 = -v^* + \delta < 0$. Then the first equation of (8) gives $r(\tau) < \delta \exp(\delta_0 \tau)$ $\forall \tau \geq 0$. We recover the physical time t as

$$t = \int_0^\tau r^{3/2}(\tau) d\tau < \frac{2\delta^{3/2}}{3\delta_0} \left[\exp\left(\frac{3}{2}\delta_0 \tau\right) - 1\right],$$

where the last expression tends to $\frac{2\delta^{3/2}}{3|\delta_0|}$ when τ tends to ∞. This gives an upper bound for the collision time t_c.

In a similar way, for $\delta > 0$ small enough, we define $M_\delta = \{\mathbf{x} \in M(h) \mid |\mathbf{x} - P_+^*| < \delta\}$, $W^u_{\rm loc}(P_+^*)$ and $W^s_{\rm loc}(P_+^*)$.

The following lemmas concern to the limiting behaviour of e_1 and e_2 for the orbits which tend to double binary collision for positive or negative time.

Lemma 1 *Let us consider equilibrium points* $P_-^* = P_-(\varphi_1^*, \varphi_2^*, x^*, y^*, \rho^*)$ *and* $P_+^* = P_+(\varphi_1^*, \varphi_2^*, x^*, y^*, \rho^*)$ *for some values* $x^* \geq 0$, $y^* \in \mathbb{R}$, $\rho^* \in \mathbb{R}$, *and* $\varphi_i^* \in [0, 2\pi]$, $i = 1, 2$.

(a) Let be $P \in W^s_{\text{loc}}(P_-^*)$ *and* $e_i(\tau) = e_i(\varphi(\tau; P))$, *for* $i = 1, 2$, *the energies of the two binary subsystems on the orbit* $\varphi(\tau; P)$. *Then* $\lim_{\tau\to\infty} e_i(\tau)$ *for* $i = 1, 2$ *exist.*

(b) In a similar way, if $\hat{P} \in W^u_{\text{loc}}(P_+^*)$ *and* $\hat{e}_i(\tau) = e_i(\varphi(\tau; \hat{P}))$ *then* $\lim_{\tau\to-\infty} \hat{e}_i(\tau)$ *for* $i = 1, 2$ *exist.*

Proof For the derivatives of e_1 and e_2 with respect to τ we can write

$$\begin{aligned} e_1' &= \frac{r^2 x^6}{\mu_1 \alpha_1}\left(\frac{\mu_2}{k_1^3} v \cos^2\theta(-1 + 3\cos^2\varphi_1) + F_1\right), \\ e_2' &= \frac{r^2 x^6}{\mu_2 \alpha_2}\left(\frac{\mu_1}{k_1^3} v \sin^2\theta(-1 + 3\cos^2\varphi_2) + F_2\right), \end{aligned}$$

where F_1 and F_2 are functions of order $O(rx^2)$. Therefore, in a neighbourhood N_δ of P_-^* we have that $|e_i'| < C_i r^2$, $i = 1, 2$, for some constants C_1 and C_2. However, if $P \in W^s_{\text{loc}}(P_-^*)$, then $r(\tau)$ tends to 0 exponentially when τ tends to ∞. Hence, $\lim_{\tau\to\infty} e_i(\tau)$ exists for $i = 1, 2$.

The part (b) of the lemma is proved using a similar argument for negative time.

Lemma 2 *Let us consider equilibrium points* P_-^* *and* P_+^* *as in lemma 1.*

(a) Let P *and* Q *be two points in* $W^s_{\text{loc}}(P_-^*)$ *corresponding to different orbits, and* $e_{i,P} = \lim_{\tau\to\infty} e_i(\varphi(\tau; P))$, $e_{i,Q} = \lim_{\tau\to\infty} e_i(\varphi(\tau; Q))$ *for* $i = 1, 2$. *Then* $e_{i,P} \neq e_{i,Q}$, *for* $i = 1, 2$.

(b) In a similar way, if P *and* Q *are two points in* $W^u_{\text{loc}}(P_+^*)$ *corresponding to different orbits, and* $e_{i,P} = \lim_{\tau\to-\infty} e_i(\varphi(\tau; P))$, $e_{i,Q} = \lim_{\tau\to-\infty} e_i(\varphi(\tau; Q))$ *for* $i = 1, 2$, *then* $e_{i,P} \neq e_{i,Q}$, *for* $i = 1, 2$.

Proof Let us denote by $\bar{\theta} := \theta - \theta^*$, $\bar{w} := (w_1, w_2)$, $\bar{a} := (\varphi_1, \varphi_2, x, y, \rho)$ and $\lambda = -v^*$ the negative eigenvalue of the linear part of the vector field (8) at P_-^*. We recall that it has multiplicity 3. The local stable invariant manifold $W^s_{\text{loc}}(P_-^*)$ is representable as a graph in the following way

$$\begin{aligned} W^s_{\text{loc}}(P_-^*) = &\{(r, v, \theta, u, \bar{w}, \bar{a}) |\, u = \lambda\bar{\theta} + g_u(r, v, \theta),\ \bar{w} = g_{\bar{w}}(r, v, \theta),\ \bar{a} = \bar{a}^* + g_{\bar{a}}(r, v, \theta), \\ &\text{with } Dg_u(0, -v^*, \theta^*) = 0,\ Dg_{\bar{w}}(0, -v^*, \theta^*) = 0,\ Dg_{\bar{a}}(0, -v^*, \theta^*) = 0\}. \end{aligned} \quad (31)$$

For a fixed value of the energy h we can write (13) as

$$E(r, v, \theta, u, \bar{w}, \bar{a}) := \bar{h}r - \frac{1}{2}\left(v^2 + u^2 + \frac{w_1^2}{\cos^2\theta} + \frac{w_2^2}{\sin^2\theta}\right) + V(\theta) = 0,$$

where $\bar{h}$ is given in (13). Let $W^s_{\text{loc},h}(P_-^*)$ be the intersection of $W^s_{\text{loc}}(P_-^*)$ with the energy level h. Then, on $W^s_{\text{loc},h}(P_-^*)$ we get

$$E(r, v, \theta, \lambda\bar{\theta} + g_u, g_{\bar{w}}, \bar{a}^* + g_{\bar{a}}) = 0, \quad (32)$$

and $\frac{\partial E}{\partial v}(P_-^*) = v^* \neq 0$.

Therefore, using the implicit function theorem, in a neigbourhood of P_-^*, there is a function $v = v(r,\theta)$ satisfying (32) such that

$$v(0,\theta^*) = -v^*, \quad \frac{\partial v}{\partial r}(0,\theta^*) = -\frac{\bar{h}(P_-^*)}{v^*} \quad \text{and} \quad \frac{\partial v}{\partial \theta}(0,\theta^*) = 0, \tag{33}$$

where

$$\bar{h}(P_-^*) = h - \frac{\mu_1\mu_2}{2M}\left[(y^*)^2 + k_2^2(x^*)^4(\rho^*)^2\right] + (x^*)^2 V_1(P_-^*).$$

Hence, $W^s_{\mathrm{loc},h}(P_-^*)$ can be parametrized by r and θ.

For a fixed real value, c, the intersection of the manifold $W^s_{\mathrm{loc},h}(P_-^*)$ with the level set $e_1(r,v,\theta,u,w_1) = c$ is given by the zeroes of the following function

$$\begin{aligned} F(r,\theta) &= 2\mu_1\alpha_1 r\left[e_1(r,v,\theta,\lambda\bar{\theta}+g_u,g_{w_1}) - c\right] = \\ &= \left(v\cos\theta - (\lambda\bar{\theta}+g_u)\sin\theta\right)^2 + \frac{g_{w_1}^2}{\cos^2\theta} - \frac{2\mu_1\alpha_1^{3/2}}{\cos\theta} - 2\mu_1\alpha_1 c r = 0, \end{aligned}$$

where $v = v(r,\theta)$ is given by (33). We compute

$$\frac{\partial F}{\partial \theta}(0,\theta^*) \quad = \quad -3\mu_1\alpha_1^{3/2}\left(1 + \frac{\mu_2^{2/3}\alpha_2}{\mu_1^{2/3}\alpha_1}\right) \neq 0.$$

Using the implicit function theorem we have that, in a small neigbourhood of $(r,\theta) = (0,\theta^*)$, there is a function $\theta = \theta(r;c)$ such that for any real c

$$\theta(0;c) = \theta^* \qquad \text{and} \qquad F(r,\theta(r;c)) = 0.$$

We also can compute

$$\frac{d\theta}{dr}(0;c) = \frac{-\mu_1\alpha_1 c + \bar{h}(P_-^*)\cos^2\theta^*}{\sin\theta^*\left[2(v^*)^2 + \mu_1\alpha_1^{3/2}/\cos^2\theta^*\right]}.$$

As a function of c, $\frac{d\theta}{dr}(0;c)$ is a monotonically decreasing function. This means that in a small neigbourhood of P_-^* two different level sets of e_1, for values c_1, c_2, $c_1 \neq c_2$, have a unique point of intersection, which corresponds to the equilibrium point P_-^*. Also we conclude that for any real m, there is a unique level set of e_1 such that the intersection of this level set with $W^s_{\mathrm{loc},h}(P_-^*)$ has slope m at the equilibrium point P_-^*.

Using (31), we get the restriction of (8) to $W^s_{\mathrm{loc},h}(P_-^*)$

$$\begin{aligned} r' &= rv(r,\theta), \\ \theta' &= \lambda(\theta-\theta^*) + g_u(r,v(r,\theta),\theta), \end{aligned} \tag{34}$$

where $v(r,\theta)$ is given in (33). The system above has an equilibrium point at $(r,\theta) = (0,\theta^*)$. The linearization at that point has eigenvalues $-v^*$ and $\lambda = -v^*$. So, the equilibrium point is a linear degenerate stable node. However the linearization diagonalizes and, furthermore, the system (34) is analytic in a neigbourhood of $(0,\theta^*)$. This means that for any possible direction, there is a unique orbit which tends to the equilibrium point with this direction. In that way we have proved statement (a) for e_1. Using (19) the same result follows for e_2.

The same argument for negative time proves part (b). ♣

Proof of Theorem 1 Let be $P \in \partial N_{\delta/2} \cap W^s(P_-^*)$ and t_c the collision time for the orbit of P. We take an arc $\gamma(s) \in \partial N_{\delta/2}$ parametrized by $s \in [0,1]$ with $\gamma(0) = P$, such that $\gamma(s)$ does not belong to $W^s_{\text{loc}}(P_-^*)$ nor $W^u_{\text{loc}}(P_-^*)$ for any $s \in (0,1]$. If $\gamma(s)$ is sufficiently close to P, the orbit $\varphi(\tau;\gamma(s))$ goes out of $N_{\delta/2}$ following closely the unstable invariant manifold of P_-^*, $W^u(P_-^*)$. Therefore, for any $s \in (0,1]$ there exists

$$\tau_-(s) = \min\{\tau_1 > 0|\ \varphi(\tau_1,\gamma(r)) \in \partial N_\delta \ \text{ and } \ \varphi(\tau,\gamma(s)) \in N_\delta \ \text{ for all } \ 0 < \tau < \tau_1\},$$

which satisfies that $\tau_-(s) \to \infty$ when $s \to 0$.

We consider the arc of points $\Gamma(s) \in \partial N_\delta$ defined by $\Gamma(s) = \{\varphi(\tau_-(s);\gamma(s)) \mid 0 < s \leq 1\}$. It is clear that if $s \to 0$, then $\Gamma(s)$ tends to a well defined point on $\partial N_\delta \cap W^u(P_-^*)$. We denote that point by $\Gamma(0)$. Moroever, using lemma 1, if $e_{1,P}, e_{2,P}$ stand for the limit values of the energies e_1 and e_2, respectively, of the collision orbit, $\varphi(\tau;P)$, then one has $\lim_{s\to 0} e_i(\Gamma(s)) = e_{i,P}$ for $i = 1,2$.

Let $T : \partial N_\delta \to \partial M_{\delta/2}$ be the Poincaré map obtained following the orbits for positive τ. Theorem 2 and the continuity of the solutions with respect to initial conditions, show that the map T is well defined in a neighbourhood of $\partial N_\delta \cap W^u_{\text{loc}}(P_-^*)$. Then there exists $s_0 > 0$ such that $\gamma^+(s) := T(\Gamma(s))$ for $s \in [0,s_0]$ is an arc on $\partial M_{\delta/2}$ with an endpoint $\gamma^+(0) = T(\Gamma(0)) \in W^s(P_+^*)$. Let $\tau_0(s)$ be the time such that $\gamma^+(s) = \varphi(\tau_0(s);\Gamma(s))$. Moroever, $e_1(\tau)$ and $e_2(\tau)$ remain constant on $\mathcal{C}$ and so $\lim_{s\to 0} e_i(\gamma^+(s)) = e_{i,P}$ for $i = 1,2$. The orbits passing through $\gamma^+(s)$, $s \in (0,s_0]$ leave the neighbourhood M_δ of P_+ following $W^u(P_+^*)$. Then, if

$$\tau_+(s) = \min\{\tau_1 > 0 \mid \varphi(\tau_1,\gamma^+(s)) \in \partial M_\delta \text{ and } \varphi(\tau,\gamma^+(s)) \in M_\delta \text{ for all } 0 < \tau < \tau_1\},$$

$\Gamma^+(s) = \{\varphi(\tau_+(s);\gamma^+(s)) \mid 0 < s < s_0\} \in \partial M_\delta$ is an arc such that $\Gamma^+(s)$ tends to a point Q, which is the unique point on $\partial M_\delta \cap W^u_{\text{loc}}(P_+^*)$ such that $\lim_{\tau\to-\infty} e_i(\varphi(\tau;Q)) = e_{i,P}$ for $i = 1,2$. We remark that the point Q is independent of the initial arc γ. Now we can take the orbit of Q as the natural continuation of the collision orbit $\varphi(\tau;P)$.

Furthermore it is clear that for any $s \in (0,s_0]$ one has $\Gamma^+(s) = \varphi(\tau_p(s);\gamma(s))$, where $\tau_p(s) = \tau_-(s) + \tau_0(s) + \tau_+(s)$, both $\tau_-(s)$ and $\tau_+(s)$ tend to ∞ and $\tau_0(s)$ tends to a finite value when $s \to 0$. The physical time required for a point $\gamma(s)$, $s \in (0,s_0]$ to go out of a neighbourhood of $s.b.c.$ is $t(s) = t_-(s) + t_0(s) + t_+(s)$ with $t_i(s) = \int_0^{\tau_i(s)} r^{3/2}(\tau)d\tau$, where the subindex i stands for $-, 0$ or $+$. It is clear now that if $s \to 0$, then $t_-(s)$ and $t_+(s)$ tend to t_c and $t_0(s) \to 0$. ♣

6 Numerical computations

In order to compute some orbits passing near $s.b.c.$ we consider the double isosceles four body problem introduced in section 3. We recall that in this case $w_1 = w_2 = \rho = 0$ and so we can reduce (8) to a system of 6 equations which are given in Appendix II. We regularize the single double collisions by introducing a new variable $w = u\sin\theta\cos\theta/\sqrt{F(\theta)}$ where $F(\theta) = \sin\theta\cos\theta V(\theta)$, and a new parametrization of time given by $d\tau = \left(\sin\theta\cos\theta/\sqrt{F(\theta)}\,\right) d\bar{\tau}$. Then the equations (taking the arbitrary constant k_1 equal to 1) are written as

$$\begin{aligned}
\dot{r} &= \frac{\sin\theta\cos\theta}{\sqrt{F(\theta)}} rv, \\
\dot{v} &= \sqrt{F(\theta)} + \frac{\sin\theta\cos\theta}{\sqrt{F(\theta)}}\left(-\frac{1}{2}v^2 + 2\bar{h}r + r^2\mathbf{s}^T A\mathbf{F}\right), \\
\dot{\theta} &= w, \\
\dot{w} &= -w^2\frac{F'(\theta)}{2F(\theta)} - \frac{1}{2}vw\frac{\sin\theta\cos\theta}{\sqrt{F(\theta)}} + \frac{\sin^2\theta\cos^2\theta}{F(\theta)}\left(W + r^2U\right) \\
&\quad + \cos(2\theta)\left[2 + \frac{\sin\theta\cos\theta}{F(\theta)}\left(2\bar{h}r - v^2\right)\right], \\
\dot{x} &= -\frac{\sin\theta\cos\theta}{2\sqrt{F(\theta)}}\, r^{3/2}x^3y, \\
\dot{y} &= \frac{\sin\theta\cos\theta}{\sqrt{F(\theta)}}\, r^{3/2}\left(-Mx^4 + G_1\right).
\end{aligned} \tag{35}$$

We remark that given some values $x^* \in \mathbb{R}$, $x^* > 0$ and $y^* \in \mathbb{R}$, the system (35) has two equilibrium points $P_{\pm}(x^*, y^*)$. Of course, they are on $r = 0$. Note that ρ and the angles φ_1 and φ_2 defined for the general problem remain constant for the double isosceles problem. It is easy to check that the unstable invariant manifold of an equilibrium point $P_-(x^*, y^*)$ is one dimensional.

Let us consider an orbit passing near a *s.b.c.* The corresponding solution of (35), $\varphi(\bar{\tau}; Q)$, will pass near some equilibrium point, $P_-\ (x^*, y^*)$. Then it follows closely the unstable manifold $W^u(P_-(x^*, y^*))$ until it reaches a neigbourhood of $P_+(x^*, y^*)$. Moroever, let us assume that at the initial point Q, $r = r_1$ for some value $r_1 > 0$ sufficiently small. First, $r(\bar{\tau})$ decreases until $\varphi(\bar{\tau}; Q)$ reaches the plane $v = 0$. When v is positive $r(\bar{\tau})$ increases and at some time $\bar{\tau}_f$, $r(\bar{\tau}_f) = r_1$. Let be $Q_f = \varphi(\bar{\tau}_f; Q)$. For a small fixed value $r_1 > 0$, we can consider a map T such that for any point Q in a small neigbourhood of $P_-(x^*, y^*)$ with $r = r_1$, $T(Q) = Q_f$ if Q_f exists.

We are interested in computing $T(Q)$ for several Q points in order to get some information about the differentiability of the map T.

A first test has been made as follows. Using $m_1 = m_2 = 2, m_3 = 1, m_4 = 3$, we have taken as initial conditions $r_0 = 10^{-10}, v_0 = v^*, \theta_0 = \theta^*, w_0 = 0, x_0 = 0.3$ and $y_0 = 0.3$. Then a computation backwards in time is started until r reaches the value 0.3. In this way we obtain a point $r_1 = 0.3, v_1, \ldots, y_1$, of the stable invariant manifold entering the fixed point in the r direction. Then, the following set of initial conditions is chosen:

$$r = r_1, \quad v = v_1, \quad \theta = \theta_1 + \rho\cos(\gamma), \quad w = w_1 + \rho\sin(\gamma), \quad x = x_1, \quad y = y_1. \tag{36}$$

The values of ρ have been chosen equal to $j \times 10^{-6}$ for $j = 1, \ldots, 20$ and the angle γ takes 400 equiespaced values in $[0, 2\pi]$. Then we take each one of these initial points and we integrate for positive time. First it approaches the collision manifold and then it escapes. The integration is stopped when the current value of r is again 0.3. The orbit which escapes of the upper critical point in the r direction has been computed. It intersects

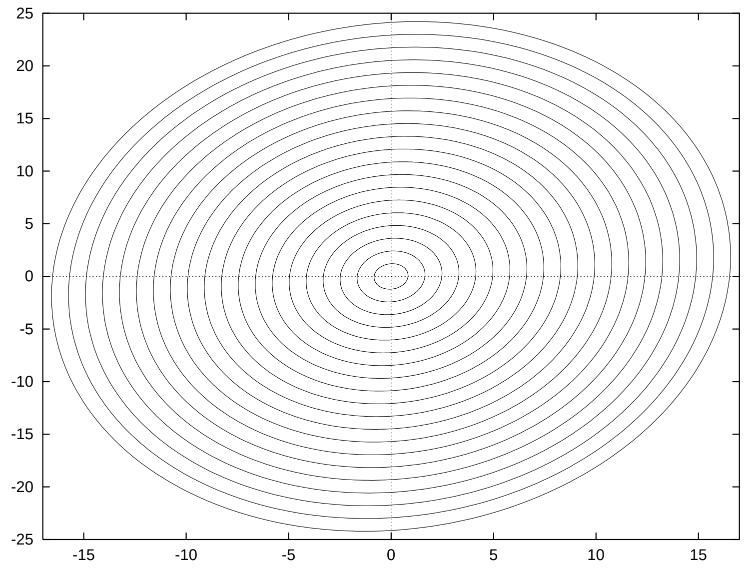

Figure 4: An illustration of the image of a set of circles after the passage close to the manifold of simultaneous binary collision. See the text for details.

$r = 0.3$ in some point. This point has been used as the origin to which the images of the 20 circles (of the form given by (36)) are refered. In figure 4 we display the (θ, w) projections of the images of the 20 circles. The coordinates have been multiplied by 10^6. This gives an evidence of the smoothness of the map T. How smooth is that map is the question we study in section 7. On this section it will become apparent that a good accuracy in the numerical integration of system (35) is essential. The computational details are given in Appendix IV.

7 On the numerical evidence of a finite order of differentiability in the regularization of s.b.c.

The next simulations in the case of the double isosceles problem have been carried out with $m_1 = m_2 = 2$, $m_3 = 1$ and $m_4 = 5$. The initial values have been taken as $r_0 = 10^{-20}$, $v_0 = v^*$, $\theta_0 = \theta^*$, $w_0 = 0$, $x_0 = 1$, $y_0 = 1$ and the selected value of r_1 is 1. All the computations have been done in quadruple precision. The initial points in $r = r_1$ have been taken as given by (36) with $\gamma = 0$ (fixed) and putting $\rho = l^3$, where the values $l_j = j \times 2.5 \times 10^{-4}, j \in J = \{j \in \mathbb{Z}, |j| \leq 800, j \neq 0\}$, have been used. Furthermore, in contrast with (36), the value of v_1 has been modified, depending on j, so that all these initial points are on the level of energy of the point where the backwards integration is stopped. Here we shall present the numerical results for this case, but many other choices of masses and initial conditions have been tested, leading to the same conclusions.

The image, under the map T, of this set of points, which discretizes an arc parametrized by l, "looks" quite smooth. However, to have a quantitative measure of the smoothness, the following test has been made. Let z denote any of the variables $\{v, \theta, w, x, y\}$. Assume we have computed a table of values $\{(l_j, z_j)\}_{j\in J}$. Let Q_N the polynomial approximation of degree N minimizing $\sum_{j\in J}(z_j - Q_N(l_j))^2$. These polynomials are easily obtained using orthogonal polynomials over the discrete set $\{l_j\}_{j\in J}$.

The reason why we have used parametrizations in terms of l instead of ρ is guided by a local analysis near the points $P_{\pm}$, which shows the role of the cubic root of some components of the (local) coordinates.

The observed results are as follows: Let $Q_N = \sum_{k=0}^{N} q_{N,k} l^k$. When N increases, the values of $q_{N,k}$ stabilize quickly for $k \leq 8$. Furthermore, for k different from a multiple of 3, the values obtained can be considered as zero. This is not true for $k = 9$. The figure 5 shows this behaviour for the variables v, θ, w and y for $N = 24, 36, 48, \ldots, 96$.

An amazing fact is that $q_{N,9}$ has a behaviour of the form $\alpha + \beta N$, α and β constants, almost exactly. The same behaviour has been observed also for the variable x and for the "physical" time t, but the values of the slope, β, are smaller. For the variables shown in figure 5 the values of $q_{N,9}$ go away from zero when increasing N.

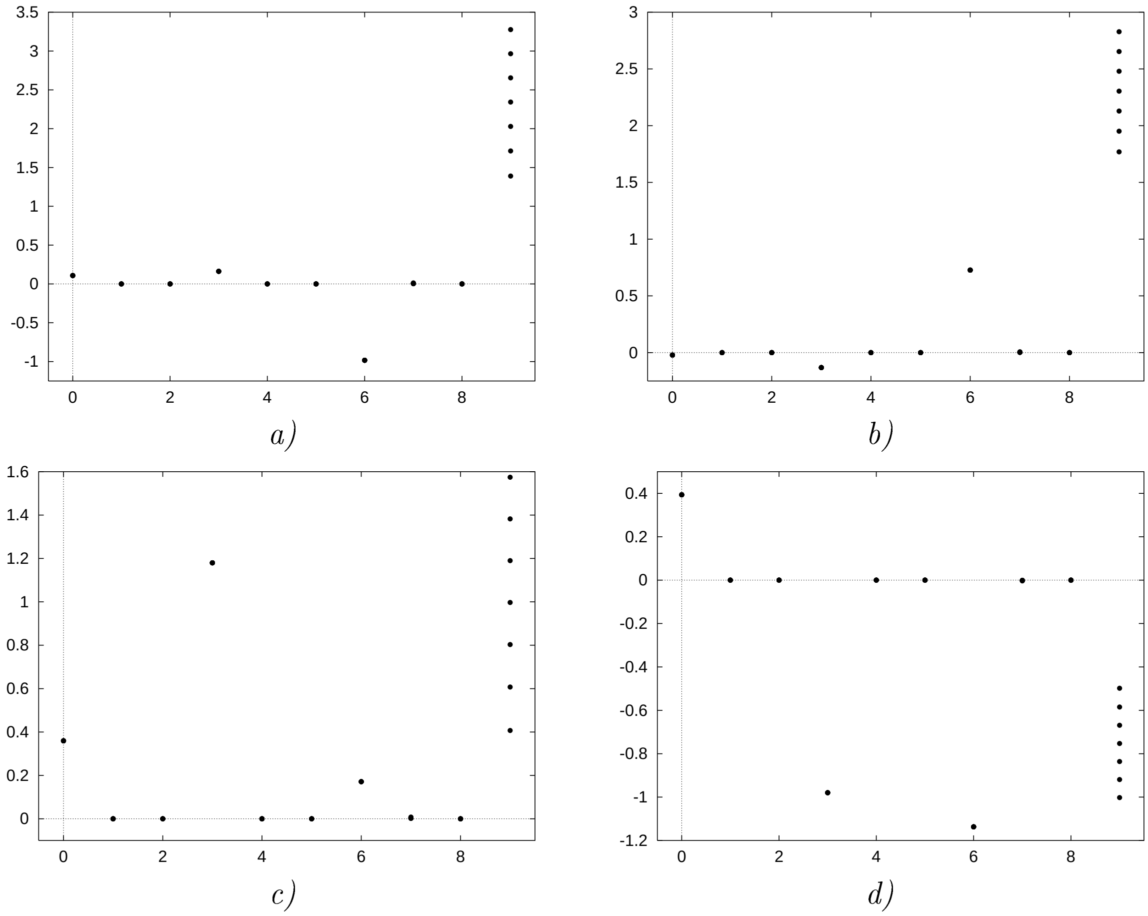

Figure 5: The results of minimum squares fittings to the image, under the map T, of a segment through a point going to $P_-(x^*, y^*)$, using polynomials of different degrees. See the text for details. The plots a), b), c) and d) display, respectively, the coefficients of the variables v, θ, w and y as a function of the degree in l.

A tentative explanation of this behaviour is that the image of the initial arc, which is analytic with respect to the parameter ρ, is of the form $Z_0 + Z_1\rho + Z_2\rho^2 + Z_3\rho^3(1 + o(1)) + Z_{8/3}\text{sign}(\rho)\rho^{8/3}$, around $\rho = 0$, the Z_r being numerical coefficients. To make this plausible, let us consider what would be the result of computing polynomials Q_N, as before, and then to look for $\{q_{N,0}\}$ as a function of N if we take, as $z(l)$, the delta distribution concentrated at 0, δ_0. For simplicity we consider the continuous case instead of the discrete one. For instance, we can take the Legendre polynomials, $\{P_n\}$, orthogonal in $[-1,1]$. Then $Q_N=\sum_{j=0}^{N}\frac{(P_j,\delta_0)}{(P_j,P_j)}P_j$ and, hence, $q_{N,0}=Q_N(0)=\sum_{j=0}^{N}\frac{P_j(0)^2}{(P_j,P_j)}$, where $(\,,\,)$ denotes here the inner product in $[-1,1]$. As $P_j(0)=0$ for j odd, it is enough to consider j even. For $j=2m$ we get $\frac{P_j(0)^2}{(P_j,P_j)}=\frac{4m+1}{2\pi}\left[\frac{\Gamma(m+1/2)}{\Gamma(m+1)}\right]^2$ which tends to $2/\pi$ as $m\to\infty$. This gives the desired behaviour of $q_{N,0}$ as a function of N.

Going back to our problem, the fact that the derivative of order 9 with respect to l has a behaviour like δ_0 shows that the derivative of order 8 behaves like the function sign, except by multiplication by constants and the eventual addition of a constant, which in the present case seems to be zero. All this evidence suggests the conjecture given in section 1.

Geometrically this behaviour can be interpreted as a change of sign in the higher order terms when points which come close to P_- go to a vicinity of P_+ following the right or left branches of $W^u_{P_-}$.

We remark that the computations above concern the differentiability of the map T, that is, with respect to initial conditions. So, the lack of differentiability observed numerically has nothing to do with the parametrization used for the orbits, in particular with the factor $r^{3/2}$ introduced in the equations in section 2.

A research on the proof of the observed behaviour is in progress [5].

8 Additional remarks

For fixed values $\varphi_i^*\in[0,2\pi]$, $i=1,2$, and real y^* and ρ^*, consider the equilibrium points $P_-^*=P_-(\varphi_1^*,\varphi_2^*,0,y^*,\rho^*)$ and $P_+^*=P_+(\varphi_1^*,\varphi_2^*,0,y^*,\rho^*)$. We denote by $W^s_{\mathcal{I}}(P_-^*)$ ($W^u_{\mathcal{I}}(P_+^*)$) the stable (unstable) invariant manifold of P_-^* (P_+^*) restricted to $\mathcal{I}$.

The flow on the infinity manifold $\mathcal{I}$ is obtained by taking $x=0$ in (8). In particular the equations for w_i, $i=1,2$, become $w_i'=-vw_i/2$, for $i=1,2$. Therefore $w_i=0$, $i=1,2$, are invariant sets on $\mathcal{I}$. The same argument used in the proof of proposition 2 for w_i instead of h_i shows that for any orbit on $\mathcal{I}$ with $r>0$ which tends to P_-^* (P_+^*) for positive (negative) time, $w_i\equiv 0$, for $i=1$ and 2. Then we have the following result.

Lemma 3 *For any value of the masses, on the orbits of $W^s_{\mathcal{I}}(P_-^*)$ and $W^u_{\mathcal{I}}(P_+^*)$, one has $w_i\equiv 0$ for $i=1,2$.*

We note that the problem restricted to $\mathcal{I}$ corresponds to a pair of uncoupled two body problems. In order to have a solution which goes to a *s.b.c.*, a necessary condition is that the angular momenta of both two body problems be zero. This is exactly what lemma 3 states.

The system of equations on $\mathcal{I}$ restricted to $w_1=w_2=0$ is given by

$$\begin{aligned} r' &= rv\,, \\ v' &= \frac{v^2}{2}+u^2-V(\theta), \\ \theta' &= u, \\ u' &= -\frac{1}{2}vu+V'(\theta). \end{aligned} \tag{37}$$

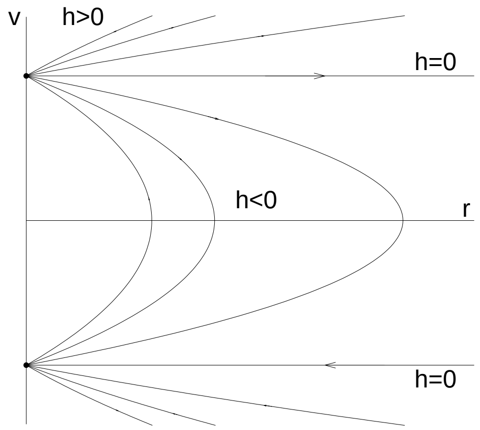

Figure 6: Homothetic orbits for $x = 0$.

The energy integral reduces to

$$\bar{h}r = \frac{1}{2}(v^2 + u^2) - V(\theta). \tag{38}$$

Using that $V'(\theta^*) = 0$, if $\theta = \theta^*$ and $u = 0$, (37) and (38) reduce to

$$\begin{aligned} r' &= rv, \\ v' &= \frac{v^2}{2} - V(\theta^*) \end{aligned} \tag{39}$$

and $\bar{h}r = v^2/2 - V(\theta^*)$. The orbits of (39) are represented in figure 6 for different values of $\bar{h}$. They are orbits which end in a *s.b.c.* for positive time, negative time or both, but θ remains constant and equal to θ^*. We call these orbits homothetic with respect to *s.b.c.* in the sense that if we recover the physical coordinates we have that $|\mathbf{Q}_2|/|\mathbf{Q}_1| \equiv \mu_2^{1/3}/\mu_1^{1/3}$ for all time.

For a fixed value of $\bar{h}$, the orbits of (39) belong to $W^u_{\mathcal{I}}(P^*_+) \cup W^s_{\mathcal{I}}(P^*_-)$. Using theorem 5 of [17] we get that $W^u_{\mathcal{I}}(P^*_+)$ meets transversally $W^s_{\mathcal{I}}(P^*_-)$ along the homothetic orbit with respect to *s.b.c.* Consider a fixed value of the energy h. For any x^* sufficiently small, $y^* \in \mathbb{R}$, $\rho^* \in \mathbb{R}$ and $\varphi_i^* \in [0, 2\pi]$, for $i = 1, 2$, $W^u(P^*_+)$ and $W^s(P^*_-)$ meet transversally along an orbit which goes to *s.b.c.* for positive and negative time.

Acknowledgements

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Appendix I

The functions $\mathbf{F}_1, \mathbf{F}_2$ and $\mathbf{F}_3$ which appear in the equations (2) are given by

$$\mathbf{F}_1 = \frac{1}{m_2}\mathbf{A}_2 - \frac{1}{m_1}\mathbf{A}_1, \qquad \mathbf{F}_2 = \frac{1}{m_4}\mathbf{A}_4 - \frac{1}{m_3}\mathbf{A}_3,$$

$$
\begin{aligned}
\mathbf{F}_3 &= \frac{1}{\mu_2}(\mathbf{A}_3+\mathbf{A}_4)-\frac{1}{\mu_1}(\mathbf{A}_1+\mathbf{A}_2), \quad \text{where}\\
\mathbf{A}_1 &= -m_1m_3\frac{\mathbf{r}_{13}}{r_{13}^3}-m_1m_4\frac{\mathbf{r}_{14}}{r_{14}^3}, \quad \mathbf{A}_2=-m_2m_3\frac{\mathbf{r}_{23}}{r_{23}^3}-m_2m_4\frac{\mathbf{r}_{24}}{r_{24}^3},\\
\mathbf{A}_3 &= m_1m_3\frac{\mathbf{r}_{13}}{r_{13}^3}+m_2m_3\frac{\mathbf{r}_{23}}{r_{23}^3}, \quad \mathbf{A}_4=m_1m_4\frac{\mathbf{r}_{14}}{r_{14}^3}+m_2m_4\frac{\mathbf{r}_{24}}{r_{24}^3},\\
\mathbf{r}_{13} &= \mathbf{q}_1-\mathbf{q}_3=-\frac{m_2}{\mu_1}\mathbf{Q}_1+\frac{m_4}{\mu_2}\mathbf{Q}_2-\mathbf{Q}, \quad \mathbf{r}_{14}=\mathbf{q}_1-\mathbf{q}_4=-\frac{m_2}{\mu_1}\mathbf{Q}_1-\frac{m_3}{\mu_2}\mathbf{Q}_2-\mathbf{Q},\\
\mathbf{r}_{23} &= \mathbf{q}_2-\mathbf{q}_3=\frac{m_1}{\mu_1}\mathbf{Q}_1+\frac{m_4}{\mu_2}\mathbf{Q}_2-\mathbf{Q}, \quad \mathbf{r}_{24}=\mathbf{q}_2-\mathbf{q}_4=\frac{m_1}{\mu_1}\mathbf{Q}_1-\frac{m_3}{\mu_2}\mathbf{Q}_2-\mathbf{Q},
\end{aligned}
$$

and $r_{ij}=|\mathbf{r}_{ij}|$.

The functions $C_i, i=0,\dots 5$ which appear in the equations (8) have the following expressions

$$
\begin{aligned}
C_0 &= \frac{1}{x^6}\left[\left(\frac{1}{r_{13}^3}-\frac{1}{r_{23}^3}\right)+\left(\frac{1}{r_{24}^3}-\frac{1}{r_{14}^3}\right)\right]=\frac{1}{k_1^3}O_2,\\
C_1 &= \frac{1}{x^6}\left[m_3\left(\frac{1}{r_{13}^3}-\frac{1}{r_{23}^3}\right)+m_4\left(\frac{1}{r_{14}^3}-\frac{1}{r_{24}^3}\right)\right]=-\frac{1}{k_1^4}\left[\frac{3\mu_2}{\sqrt{\alpha_1}}\cos\theta\cos\varphi_1 rx^2+O_2\right],\\
C_2 &= \frac{1}{x^6}\left[m_1\left(\frac{1}{r_{14}^3}-\frac{1}{r_{13}^3}\right)+m_2\left(\frac{1}{r_{24}^3}-\frac{1}{r_{23}^3}\right)\right]=-\frac{1}{k_1^4}\left[\frac{3\mu_1}{\sqrt{\alpha_2}}\sin\theta\cos\varphi_2 rx^2+O_2\right],\\
C_3 &= \frac{1}{x^6}\left[\frac{m_1m_3}{r_{13}^3}+\frac{m_1m_4}{r_{14}^3}+\frac{m_2m_3}{r_{23}^3}+\frac{m_2m_4}{r_{24}^3}\right]=\frac{\mu_1\mu_2}{k_1^3}(1+O_1),\\
C_4 &= \frac{1}{x^6}\left[\frac{m_1m_3}{r_{23}^3}+\frac{m_1m_4}{r_{24}^3}+\frac{m_2m_3}{r_{13}^3}+\frac{m_2m_4}{r_{14}^3}\right]=\frac{\mu_1\mu_2}{k_1^3}(1+O_1),\\
C_5 &= \frac{1}{x^6}\left[\frac{m_1m_3}{r_{14}^3}+\frac{m_2m_3}{r_{24}^3}+\frac{m_1m_4}{r_{13}^3}+\frac{m_2m_4}{r_{23}^3}\right]=\frac{\mu_1\mu_2}{k_1^3}(1+O_1),
\end{aligned}
$$

where O_p stands for terms of order p in rx^2.

Appendix II

In this Appendix we give the equations corresponding to the subproblems introduced in section 3. For each of them $w_1=w_2=\rho=0$ and so w_1, w_2, φ_1 and φ_2 remain constant. For the remaining variables, equations (8) are written as

$$
\begin{aligned}
r' &= rv,\\
v' &= \frac{v^2}{2}+u^2-V(\theta)+r^2\mathbf{s}^T A\mathbf{F},\\
\theta' &= u,\\
u' &= -\frac{vu}{2}+W(\theta)+r^2U,\\
x' &= -\frac{1}{2k_1}r^{3/2}x^3y,\\
y' &= r^{3/2}\left(-\frac{M}{k_1^2}x^4+G_1\right),
\end{aligned}
$$

where $\mathbf{s}^T A\mathbf{F}$, U and G_1 are given by (9), (10) and (11), respectively. In the collinear case we have $\sin\varphi_1 = \sin\varphi_2 = 0$ and the signs of $\cos\varphi_1$ and $\cos\varphi_2$ depend on the order of the masses. For the trapezoidal and the double isosceles problems the expressions (9), (10) and (11) can be simplified as follows.

• Trapezoidal four body problem

Using that $m_1 = m_2$, $m_3 = m_4$, $\cos\varphi_i = 0$, $i = 1, 2$, we get $r_{13} = r_{24}$, $r_{14} = r_{23}$ and then $C_1 = C_2 = 0$, $C_4 = C_5$ and

$$\begin{aligned}
\mathbf{s}^T A\mathbf{F} &= -rx^6 C_4\left(\frac{\cos^2\theta}{\mu_1} + \frac{\sin^2\theta}{\mu_2}\right) + \sqrt{\alpha_1\alpha_2}C_0 rx^6\cos(\varphi_1-\varphi_2)\sin(2\theta),\\
U &= rx^6\sin\theta\cos\theta C_4\left(\frac{1}{\mu_1} - \frac{1}{\mu_2}\right) + \sqrt{\alpha_1\alpha_2}C_0 rx^6\cos(\varphi_1-\varphi_2)\cos(2\theta),\\
G_1 &= \frac{M}{k_1^2}x^4 - \frac{Mk_1}{\mu_1\mu_2}x^4 C_3.
\end{aligned}$$

• Double isosceles four body problem

In that case, $m_1 = m_2$, $\cos\varphi_1 = \sin\varphi_2 = 0$. Therefore $r_{13} = r_{23}$, $r_{14} = r_{24}$ and then $C_0 = C_1 = 0$ and

$$\begin{aligned}
\mathbf{s}^T A\mathbf{F} &= -rx^6\left(\frac{C_4\cos^2\theta}{\mu_1} + \frac{C_5\sin^2\theta}{\mu_2}\right) - k_1x^4\sqrt{\alpha_2}C_2\cos\varphi_2\sin\theta,\\
U &= rx^6\sin\theta\cos\theta\left(\frac{C_4}{\mu_1} - \frac{C_5}{\mu_2}\right) - k_1x^4\sqrt{\alpha_2}C_2\cos\varphi_2\cos\theta,\\
G_1 &= \frac{M}{k_1^2}x^4 - \frac{M}{\mu_1\mu_2}x^4\left[k_1C_3 + rx^2\sqrt{\alpha_2}C_2\sin\theta\cos\varphi_2\right].
\end{aligned}$$

In the three cases the integrals of the energy and angular momentum reduce to

$$hr = \frac{1}{2}(v^2+u^2) - V(\theta) + r\left[\frac{\mu_1\mu_2}{2M}(y^2 + k_2^2x^4\rho^2) - x^2V_1\right], \quad \omega = -\frac{\mu_1\mu_2k_1k_2}{M}\rho.$$

Appendix III

This appendix concerns the problem of the regularization of a single binary collision in the blow up variables.

In order to regularize the singularities of system (8) for $\theta = 0$ and $\theta = \pi/2$, we can introduce a new variable $w = u\sin\theta\cos\theta/\sqrt{F(\theta)}$ where $F(\theta) = \sin\theta\cos\theta V(\theta)$ and a new parametrization of time given by $d\tau = (\sin\theta\cos\theta/\sqrt{F(\theta)})d\bar{\tau}$. The new equations are given explicitely in section 6 for the double isosceles problem. Similar equations can be obtained for the other subproblems. It is not difficult to get the new system for the general case. We only write the equations corresponding to the angles φ_1 and φ_2, which seem to have a singularity at the single binary collisions corresponding to $\theta = 0$ and $\theta = \pi/2$

$$\begin{aligned}
\frac{d\varphi_1}{d\bar{\tau}} &= -\frac{\sin\theta}{\sqrt{F}}\frac{w_1}{\cos\theta} - \frac{k_2\sin\theta\cos\theta}{k_1\sqrt{F}}r^{3/2}x^4\rho,\\
\frac{d\varphi_2}{d\bar{\tau}} &= -\frac{\cos\theta}{\sqrt{F}}\frac{w_2}{\sin\theta} - \frac{k_2\sin\theta\cos\theta}{k_1\sqrt{F}}r^{3/2}x^4\rho.
\end{aligned}$$

Anyhow, it is well known that in a neighbourhood of a single binary collision the distance between the two particles which collide is of the order of $t^{2/3}$ assuming that collision takes place at $t=0$. Now, t stands for the physical time. We recall that w_1 and w_2 are the scaled angular momenta of the subsystems that is, $r^{1/2} w_i = \dot{\mathbf{Q}}_i \wedge \mathbf{Q}_i$ for $i=1,2$. Therefore $w_i, i=1,2$ are of the order of $t^{5/3}$. Furthermore, the relation between $\bar{\tau}$ and t is given by $\bar{\tau} = \int_0^t \frac{\sqrt{F}}{r^{3/2}\sin\theta\cos\theta}\,dt$ and so the physical time t is of the order of $\bar{\tau}^3$ near $\theta=\pi/2$. Therefore, $w_1/\cos\theta$ will be of the order of $\bar{\tau}^3$ for θ near $\pi/2$ and then it does not causes any trouble in the equation of $d\varphi_1/d\bar{\tau}$. A similar reasoning holds for $w_2/\sin\theta$ near a single binary collision corresponding to $\theta=0$.

The regularization of the single binary collisions is a problem which also appears in the blow up of the triple and higher multiplicity collision. In the case of triple collision, for subproblems like the collinear or the isosceles planar three body problems, the regularization of single binary collisions can be easely done explicitely. This is also true in the present four body problem when we restrict to the collinear, the double isosceles or the trapezoidal problem. The source of the difficulties in planar general problems comes from the fact that close to a single binary collision the angle between the position and velocity vectors changes too quickly: in the limit it jumps by π. It is well known that the regularization of the single binary collisions can be done in the general n-body problem (see, e.g. [20, 21, 7], and also [3] where an analysis based on orders of magnitude is carried out). We consider worth to give here a discussion for the present problem.

Next we give an alternative change of variables to blow up the *s.b.c.* Let us consider the physical position vectors $\mathbf{Q}_1, \mathbf{Q}_2$ and $\mathbf{Q}$ defined in section 2 and the corresponding momenta $\mathbf{P}_i = \mu_i \dot{\mathbf{Q}}_i, i=1,2$ and $\mathbf{P} = M\dot{\mathbf{Q}}$. First, we introduce a Levi–Civita transformation to regularize the single binary collisions. It is well known that this type of transformation eliminates the problem of the "jump of angles" mentioned before. To do that we consider the two-dimensional vectors $\mathbf{Q}_i, \mathbf{P}_i, i=1,2, \mathbf{Q}$ and $\mathbf{P}$ in the complex plane (identifying $\mathbb{R}^2$ and $\mathbb{C}$) and introduce complex variables B_1, B_2, c_1 and c_2 given by $\mathbf{Q}_i = B_i^2, i=1,2$, with the product in $\mathbb{C}$, and $c_i = \dot{B}_i/U_c, i=1,2$, where $U_c = m_1m_2/|\mathbf{Q}_1| + m_3m_4/|\mathbf{Q}_2|$. In the definition of each one of the B_i two determinations are possible. Both of them are admissible and give the same result. Now we perform a blow up of the *s.b.c.* using the variables $r\in\mathbb{R}, s_1, s_2 \in \mathbb{C}$ defined by

$$r^2 = m_1m_2|B_2|^2 + m_3m_4|B_1|^2, \qquad s_i = \frac{B_i}{r}, \quad i=1,2.$$

The energies e_1 and e_2 of the subsystems will be also considered as variables. We introduce a new time given by $d\tau = (U_c/r)dt$ where in the new variables $U_c = 1/(r^2|s_1|^2|s_2|^2)$. Then the equations can be written as

$$\begin{aligned}
r' &= r[m_1m_2(s_2,c_2) + m_3m_4(s_1,c_1)],\\
c_1' &= \frac{1}{2}\left[r^4|s_1|^2|s_2|^4\frac{\mathbf{F}_1}{\mu_1}\times\bar{s}_1 + \mu_1e_1r^2|s_2|^4s_1 + 4m_3m_4\left((s_2,c_2)\frac{|s_1|^2}{|s_2|^2}c_1 - (s_1,c_1)c_1\right)\right],\\
c_2' &= \frac{1}{2}\left[r^4|s_1|^4|s_2|^2\frac{\mathbf{F}_2}{\mu_2}\times\bar{s}_2 + \mu_2e_2r^2|s_1|^4s_2 + 4m_1m_2\left((s_1,c_1)\frac{|s_2|^2}{|s_1|^2}c_2 - (s_2,c_2)c_2\right)\right],\\
s_i' &= c_i - [m_1m_2(s_2,c_2) + m_3m_4(s_1,c_1)]s_i, \quad i=1,2, \qquad\qquad (40)\\
e_i' &= \frac{2r^2}{\mu_i^2}(s_i\times c_i, \mathbf{F}_i), \quad i=1,2,
\end{aligned}$$

$$
\begin{aligned}
\mathbf{Q}' &= \frac{r}{MU_c}\mathbf{P}, \\
\mathbf{P}' &= \frac{r}{U_c}\mathbf{F}_3,
\end{aligned}
$$

where $'$ stands for the derivative with respect to τ, $\bar{s}_i$ denotes the complex conjugate of s_i, $\times$ means the product in $\mathbb{C}$, $(\ ,\ )$ denotes the scalar product in $\mathbb{R}^2$, that is, $(z, w) = \frac{1}{2}(z\bar{w}+\bar{z}w)$, for $z, w \in \mathbb{C}$, and $\mathbf{F}_1, \mathbf{F}_2$ and $\mathbf{F}_3$ are given in the Appendix I. The introduction of the new time slows down the unbounded velocity (either $\dot{\mathbf{Q}}_i$ or $\dot{B}_i$) to a finite limit velocity, c_i, at the binary collision of the i-couple. Of course, this slow down has the side effect that the other "velocity", c_j if $j = 3 - i$, goes to zero at the moment of the i-collision. But this is a kind of "artificial" effect.

From the energy integral we get

$$
h = m_1 m_2 e_1 + m_3 m_4 e_2 + \frac{\mu_1\mu_2}{2M^3}|\mathbf{P}|^2 - U_1(\mathbf{Q}),
$$

where $U_1(\mathbf{Q})$ is regular at single and simultaneous binary collisions, and the relations defining e_1 and e_2 can be written in the new variables as

$$
\frac{|c_2|}{|s_1|^2} = \left(\frac{\mu_1}{2}(1 + e_1 r^2 |s_1|^2)\right)^{1/2}, \qquad \frac{|c_1|}{|s_2|^2} = \left(\frac{\mu_2}{2}(1 + e_2 r^2 |s_2|^2)\right)^{1/2}. \tag{41}
$$

We note that the only non obvious regular behaviour of the flow associated to the equations above can come from the term $(s_2, c_2)c_1|s_1|^2/|s_2|^2$ in the equation of c_1' (and the corresponding term in c_2'). It is clear that this "difficulty" appears in c_1' not at the collision of the subsystem 1 but at the collision of the other binary. Using (40) the term above becomes

$$
(s_2, c_2)\frac{c_1}{|c_1|}|s_1|^2 \left(\frac{\mu_2}{2}(1 + e_2 r^2 |s_2|^2)\right)^{1/2}.
$$

Let us define $z_1 = c_1/|c_1|$. We consider z_1 as an additional variable in our system. Using (40) and (41) we get for z_1

$$
\begin{aligned}
z_1' = \frac{r^2}{\mu_2(1 + e_2 r^2 |s_2|^2)} \Bigg(& \left[\frac{1}{\mu_1} r^2 |s_1|^2 \mathbf{F}_1 \times \bar{s}_1 + \mu_1 e_1 s_1\right] |c_1| - \\
& \left[\frac{1}{\mu_1} r^2 |s_1|^2 (c_1, \mathbf{F}_1 \times \bar{s}_1) + \mu_1 e_1 (c_1, s_1)\right] z_1 \Bigg).
\end{aligned}
$$

The equation above is well defined at single binary collisions if r is sufficiently small. A similar argument shows that the term $c_2/|c_2|$ which appears in the equation of c_2' is regular at single binary collisions.

Appendix IV

To be able to integrate numerically the system (35) with high accuracy, the only possible way is to use methods of very high order. In the simulations done in quadruple precision the order used is around 36. Methods with variable step and variable order also take advantage of the behaviour of the vector field in different domains on a vicinity of the collision manifold.

The simplest method is a Taylor series method. The integration of $\dot{x} = f(x)$, assuming f analytic, makes use of

$$x(t+h) = x(t) + \sum_{j\geq 1} x^{(j)}(t)h^j, \qquad \text{where} \quad x^{(j)}(t) = \frac{1}{j!}\frac{d^j x}{dt^j}. \tag{42}$$

The components of $x^{(j)}(t)$ are evaluated *numerically* in a recurrent way. An inspection of (35) shows that, beyond sums and multiplications by constants, all the operations appearing in the equations are: products, powers (including the reciprocal, square root, etc) and trigonometric functions. A repeated use of Leibnitz rule allows to compute $x^{(j)}$, $j = 1, \ldots, N$ with an effort $O(N^2)$. Indeed:

1. To compute $c^{(j)}$, where $c = ab$, if $a^{(i)}$, $b^{(i)}$, $i \leq j$ are available, just use $c^{(j)} = \sum_{i=0}^{j} a^{(i)} b^{(j-i)}$.

2. In the case $c = a^\alpha$, by logarithmic derivation the relation $c^{(1)}a^{(0)} - \alpha\, c^{(0)}a^{(1)} = 0$ holds. Then $\sum_{i=0}^{j} [i - \alpha(j-i)]\; c^{(i)}a^{(j-i)} = 0$ follows. From this the recurrence $c^{(j)} = -\frac{1}{ja^{(0)}} \sum_{i=0}^{j-1} [i - \alpha(j-i)]\; c^{(i)}a^{(j-i)}$ is obtained.

3. For the trigonometric operations, let $c = \cos(a)$, $s = \sin(a)$. Then $c^{(1)} = -s^{(0)}a^{(1)}$, $s^{(1)} = c^{(0)}a^{(1)}$, and the recurrences $c^{(j)} = -\sum_{i=0}^{j-1} s^{(i)}a^{(j-i+1)}(j-i+1)$ and $s^{(j)} = \sum_{i=0}^{j-1} c^{(i)}a^{(j-i+1)}(j-i+1)$ follow.

4. For other elementary functions, as exp and log, similar rules apply.

Assume (42) is an analytic function of h with radius of convergence r and that the cost up to order N is $O(N^2)$. Then, elementary considerations of computational efficiency, show that the optimal stepsize is $\frac{r}{\exp(2)}$. Of course, this is not true if some of the terms $x^{(j)}(t)h^j$ become too large compared to $x(t)$, because of the loss of significant digits. This is the case if (42) is entire with respect to h. The value of r can be estimated from the experimental behaviour of $||x^{(j)}||^{1/j}$.

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