Resonance tongues in Hill's equations: a geometric approach

Henk Broer * and Carles Simó †

Abstract

The geometry of resonance tongues is considered in, mainly reversible, versions of Hill's equation, close to the classical Mathieu case. Hill's map assigns to each value of the multiparameter the corresponding Poincaré matrix. By an averaging method, the geometry of Hill's map locally can be understood in terms of cuspoid Whitney singularities. This adds robustness to the result. The algorithmic nature of the averaging method enables a pull–back to the resonance tongues of the original system.

1 Introduction, main result

This paper deals with the resonances in Mathieu's equation

$$\ddot{x} + (a + b\,p(t))\,x = 0,\; p(t + 2\pi) \equiv p(t), \tag{1}$$

where a and b are real parameters. As is well–known, in the (a, b)–plane, for all $k \in \mathbb{N}$, resonance tongues emanate from the points $(a, b) = ((\frac{k}{2})^2, 0)$, as shown in figure 1. Inside these tongues, or instability domains, the trivial periodic solution $x = \dot{x} = 0$ is unstable. Compare Van der Pol and Strutt [20], Stoker [23], Hochstadt [16], Keller and Levy [17], Magnus and Winkler [21] or Arnol′d [2, 3, 4]. For related work on nonlinear parametric forcing, see Hale [15] and Broer et al. [6, 7, 8, 10, 13]. For nonlinear discrete versions see [18, 19]. Classical choices for the periodic function p are

$$p(t) = \begin{cases} \cos t & \text{(classical Mathieu case)} \\ \operatorname{sgn}\,\cos t & \text{(square Hill's case)}. \end{cases}$$

In the classical Mathieu case the tongue boundaries at the k–th resonance are tangent of order $k-1$, i.e. with a contact of order k. Also it is known [17, 3, 13] that these boundaries are transversal as soon as the k–th harmonic of the function p does not vanish. In Broer and Levi [9], for preliminary work also see [1, 13], a geometrical explanation was offered

*Dept. of Mathematics and Computing Science, University of Groningen, Blauwborgje 3, 9747 AC Groningen, The Netherlands

†Dept. de Matemàtica Aplicada i Anàlisi, Universitat de Barcelona, Gran Via 585, 08007 Barcelona, Spain

for the instability pockets as they appear in square Hill's case or in perturbations of the classical Mathieu case like with $p(t) = \cos t + c_2 \cos 2t$, where c_2 is a small parameter. See figure 1. In the latter case it turns out that, near the 2–nd resonance $(a, b) = (1, 0)$, one instability pocket occurs for $0 \neq |c_2| \ll 1$, which can be explained in terms of a Whitney fold in a 2–dimensional map, cf. [9]. One could say that the parameter c_2 versally deforms the singularity in the classical Mathieu equation for $k = 2$.

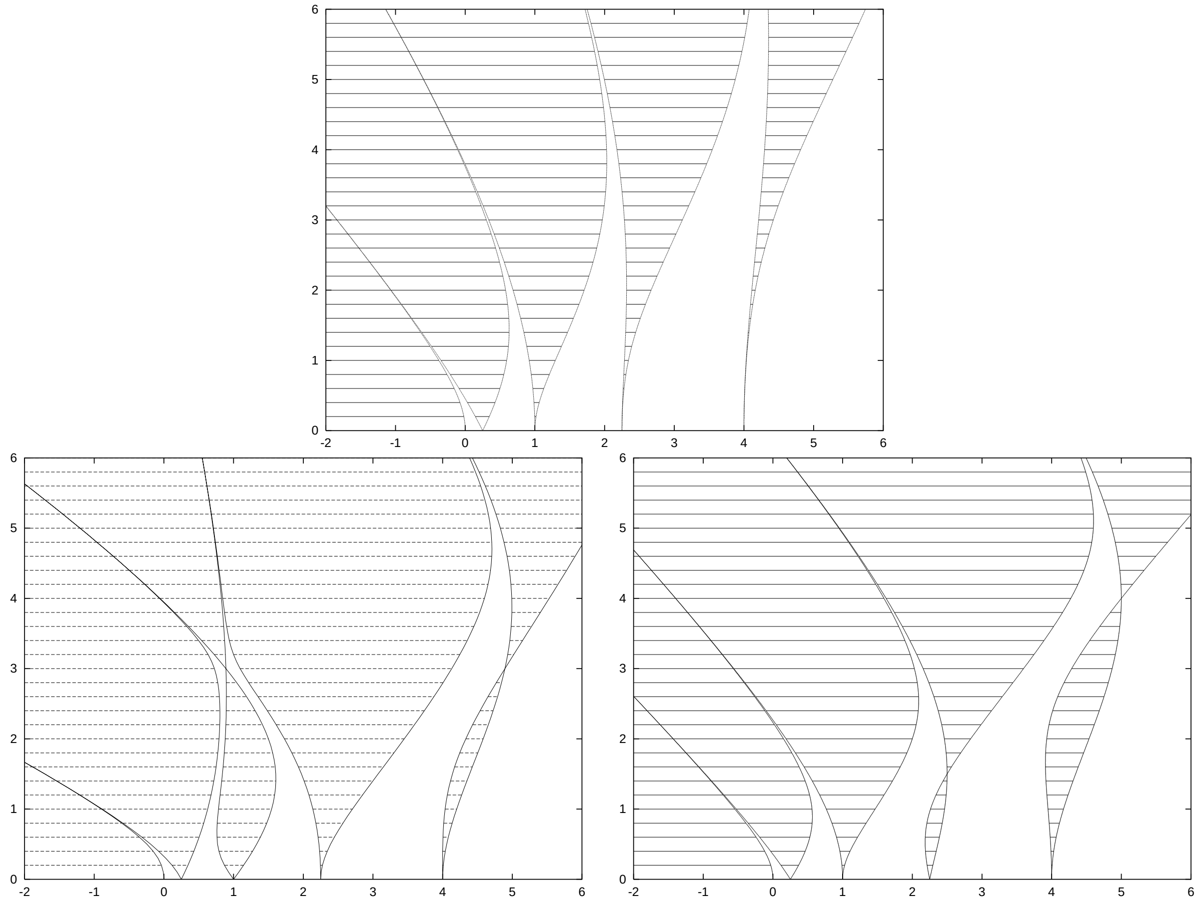

Figure 1: Resonance tongues. Top: the classical Mathieu case. Bottom left: near the Mathieu case ($c_2 = 1.5$ has been used, see later); right: the square Hill's case.

Presently the problem is to generalize this result to the k–th resonance. It turns out that more parameters are needed then. For the function p we take

$$\begin{aligned} p(t) = \cos t \quad &+ \quad c_2 \cos(2t) + \cdots + c_k \cos(kt) + \ldots \\ &+ \quad s_2 \sin(2t) + \cdots + s_k \sin(kt) + \ldots , \end{aligned} \tag{2}$$

where the c_j and s_j are considered small parameters. Our main concern, however, is with the reversible case where $p(-t) \equiv p(t)$, so where $s_j \equiv 0$.

As in [9], a main tool for these investigations is Hill's map, which assigns to each value of (a, b) the corresponding Poincaré or period matrix P :

$$\mathcal{H} : (a, b) \in \mathbb{R}^2 \mapsto P_{a,b} \in SP(1).$$

Here $SP(1)$ is the 3–dimensional Lie group of symplectic 2×2–matrices. In the reversible case range$(\mathcal{H})$ is 2–dimensional. Hill's map $\mathcal{H} = \mathcal{H}_{c,s}$ is considered in dependence of sufficiently many parameters $(c,s) = (c_1, c_2, \ldots, s_1, s_2, \ldots)$.

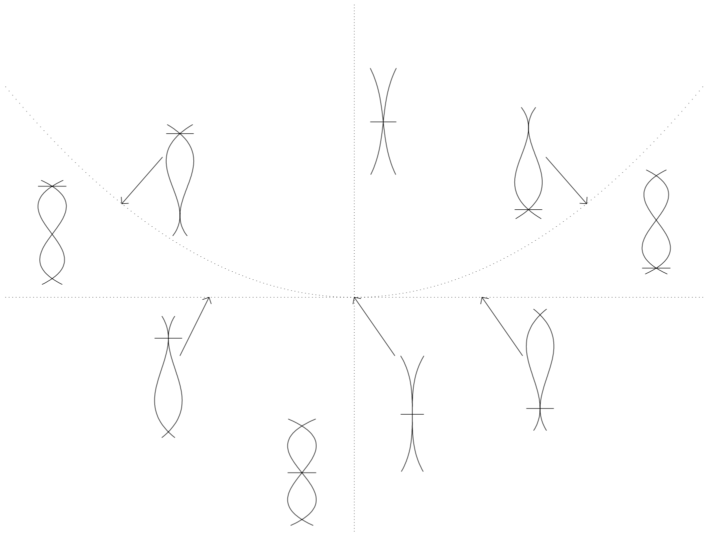

Figure 2: The geometry of tongues in the cuspoid family $\mathcal{C}_\mu$ for $k=3$.The parameter c_2 (resp. c_3) in the horizontal (resp. vertical) axis.

We now describe our approach. The tongues in the (a,b)–plane are the $\mathcal{H}$–preimages of the unstable matrices in its range, i.e., with real eigenvalues λ and $1/\lambda\,(\neq \pm 1)$. The stable matrices, i.e., with eigenvalues λ, $\overline{\lambda}\,(\neq \pm 1)$ on the unit circle, form a pair of cones with $\pm$Id as vertices. The problem then is how the family $\mathcal{H}_c$ maps the (a,b)–plane to its 2–dimensional range. In particular, at the k–th resonance we have for all values of c that $\mathcal{H}_c((\frac{k}{2})^2, 0) = (-1)^k\mathrm{Id}$, and our interest is with the corresponding singularity and its unfoldings. The main result of this paper runs as follows.

Theorem 1 (Cuspoid normal form in reversible case) *Fixing a resonance $k\in\mathbb{N}$, consider the reversible near Mathieu equation (1)*

$$\ddot{x} + (a + b\,p(t))\,x = 0,$$

with

$$p_c(t) = \cos t + c_2\cos(2t) + \cdots + c_k\cos(kt).$$

In the rescaling

$$\begin{aligned} b &= \delta B \\ c_j &= \delta^{j-1} d_j,\ j\geq 2, \end{aligned} \tag{3}$$

consider $\mathcal{H}_d = \mathcal{H}_d(a,B)$ for $(a,B)\approx((\frac{k}{2})^2, 0)$, the multiparameter $d=(d_2,\ldots,d_k)$ being $O(1)$ with respect to δ. Then, for sufficiently small $|\delta|$ the family $\mathcal{H}_d$ is left–right equivalent to the cuspoid family $\mathcal{C}_\mu$, given by

$$\mathcal{C}_\mu : (a,B)\mapsto (a_1, B_1) = (a, B^k + \sum_{j=1}^{k-1}\mu_j B^j),$$

where $\mu = (\mu_1, \ldots, \mu_{k-1})$. *Moreover the tongue boundaries of (1) correspond to the* $\mathcal{C}_\mu$*–preimage of* $B_1 = \pm a_1$.

A proof is given in §3. A left–right equivalence consists of smooth coordinate transformations both on the image and on the range, together with a diffeomorphic reparametrization. This normal form from singularity theory has a simple structure. The tongue boundaries at the k–th resonance thus become the zero–set of polynomials $B^k + \sum_{j=1}^{k-1} \mu_j B^j \pm a$. Thus its geometry in the reversible case is completely understood by theorem 1, compare figure 2 for the resonance $k = 3$. In particular this implies that one can produce any number of instability pockets not exceeding $k - 1$ and all intermediate tangencies (counting multiplicities), by an appropriate choice of the parameters $c = (c_2, \ldots, c_k)$.

By the algorithmic nature of our approach, which heavily rests on an averaging procedure [13, 14], the pull–back to the family $\mathcal{H}_c$ can be traced, compare figure 3. A major problem is to prove the main result for *any* resonance $k \in \mathbb{N}$. For this a k–fold averaging is needed. This leads to a computational normal form of Hill's map, which is another main result of the present paper. Here, next to the averaging of [14], we also use direct information on the Floquet matrix of the trivial periodic solution.

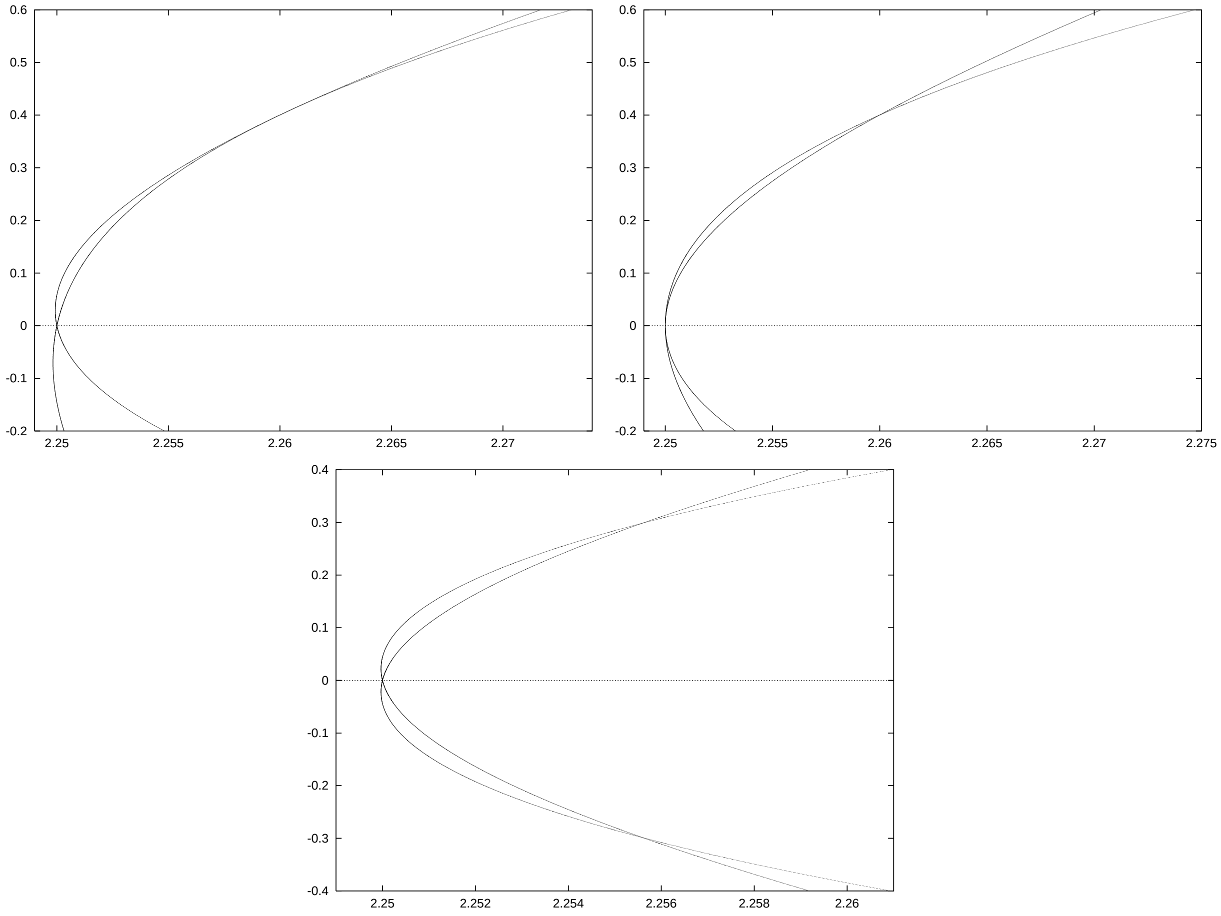

Figure 3: Some tongues in the reversible family $\mathcal{H}_c$ for $k = 3$. On the top the values $c_2 = 0.1\,, c_3 = 0.0101$ (left) and $c_2 = 0.05\,, c_3 = 0$ (right) have been used. Bottom: $c_2 = 0\,, c_3 = -0.005625\,.$

In the nonreversible case the range of $\mathcal{H}_{c,s}$ is 3–dimensional and therefore folding no longer is generic. Generally the instability pockets tend to disappear, compare figure 5. This follows from the general (nonreversible) version of the computational normal form. For an example in the case $k = 2$ see [9]. One question is whether disconnected tongues can occur. This turns out not to be not the case.

Remarks.

1. For the 1–st resonance $\mathcal{H}$ is a local diffeomorphism, which explains why the first tongue in figure 1 has transverse boundaries. The case $k = 2$ involves a family of Whitney folds, see [9]. Also compare figure 1 top and bottom left.
2. The fact that $\mathcal{H}_c$ is considered as a (local) family means that all harmonics up to order k are present. By these a versal deformation is obtained for the singularity in the classical Mathieu case. By C^∞–stability of the cuspoid family, the result has some robustness. In particular this holds for the addition of higher harmonics, see (2), the coefficients of which, by smoothness or analyticity, decay sufficiently fast.

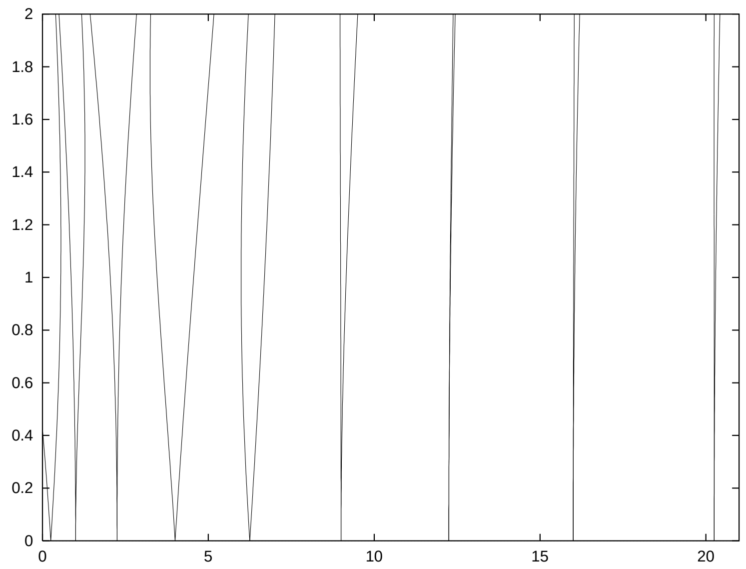

Figure 4: Asymptotics at the tongue tips for $p_0(t) = \cos t + A\cos 4t + B\cos 5t$. The simple values $A = B = 1$ have been used here.

An immediate generalization of our approach leads to the following problem. Consider the case where, instead of $p_0(t) = \cos t$, for the unperturbed case (or central singularity) p_0 an arbitrary trigonometric polynomial is taken. Simply by averaging, one easily obtains the orders of contact of all tongue tips. For an example see figure 4. In the figure the resonance tongues up to $k = 9$ are seen. The value $b = 0$ is a zero of the with of the tongue of multiplicity 1, 2, 2, 1, 1, 2, 3, 4, 2, respectively. In general, unless some particular choice of A, B is done, to obtain the multiplicity one has to look for the minimum value $m > 0$

such that $m_1 + 4m_4 + 5m_5 = k$, $|m_1| + |m_4| + |m_5| = m$. Also compare [17, 3, 13, 19]. In the light of the present paper the question is how to unfold these cases by addition of appropriate harmonics. For more related work we refer to the conclusive section.

We end the introduction by briefly outlining this paper. Section 2 first presents the local geometry of the manifold containing the Poincaré matrix, related to the k–th resonance of Mathieu's equation (1). By the exponential map our investigations are transferred to the Floquet matrix. Second, we formulate the averaging (normal form) theorem, which at the k–th resonance provides approximations of the Poincaré and the Floquet matrix in terms of the normal form coefficients.

Then, section 3 contains a proof of the main theorem 1. Indeed, we first describe how the Poincaré and Floquet matrix depend on the parameters $c = (c_2, \ldots, c_k)$, so arriving at Hill's and Floquet's map. These considerations are summarized in a computational normal form theorem. Second, we discuss how to obtain the cuspoid normal form of theorem 1.

In section 4 the nonreversible case near the classical Mathieu equation is briefly dealt with. One thing we show is that disconnected tongues cannot occur locally in the near Mathieu setting. To this and some other ends a computational normal form theorem is given of this case. Finally, section 5 contains a brief summary.

A proof of the averaging theorem is postponed to appendix A. Appendix B displays certain lower order normal form coefficients, relevant to us. In appendix C we consider the intersections of the tongue boundaries in these cases. This helps us to find the appropriate and quite natural scaling needed in the proofs of §§3 and 4. Finally in appendix D it is shown that disconnected tongues cannot occur in the present setting.

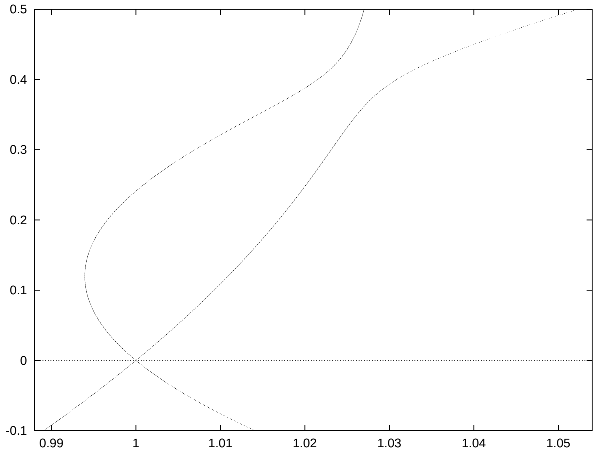

Figure 5: A tongue in the nonreversible case near resonances $k = 2$. Parameters: $c_2 = 0.2\,, s_2 = 0.025\,$.

2 Poincaré and Floquet matrix, averaging

In this section the preliminaries are set. First we give a simple local exposition of the symplectic group, at the same time passing from the Poincaré to the Floquet (monodromy) matrix. Next we obtain an expression for this Floquet matrix by a normalizing (averaging) procedure.

2.1 Geometry of the matrix spaces

For local results near the k–th resonance instead of Hill's map $\mathcal{H}$ we can resort to the Floquet's map $\mathcal{F}$ by

$$\mathcal{H} = (-\mathrm{Id})^k \exp(2\pi\mathcal{F}), \text{ or, in matrix form, } P = (-\mathrm{Id})^k \exp(2\pi F)$$

Here F is a Floquet matrix of the trivial periodic solution. The reason is that the exponential map is a local chart $(sp(1), 0) \to (SP(1), \mathrm{Id})$. Here $sp(1) = sl(2)$ is the 3–dimensional linear space of trace zero 2×2–matrices.

Reversibility is expressed as follows. Putting $\dot{x} = y$, we consider the involution $R = \mathrm{diag}\,\{1, -1\}$. The Poincaré matrix P then is reversible if $RPR = P^{-1}$. For the Floquet matrix F the corresponding infinitesimal condition reads $FR = -RF$. Let us denote the corresponding matrix spaces by $SR(1) \subset SP(1)$ and $sr(1) \subset sp(1)$.

Next let us consider the domain of stability. To this end we take the basis

$$\left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$$

of $sp(1) = sl(2)$, with corresponding coordinates (X, Y, Z).

Let us consider the spaces of 2×2–matrices $sr(1) \subset sp(1)$: the reversible symplectic matrices contained in the symplectic ones. The following result reveals the local geometry.

Proposition 2 [9] **(Stability of reversible symplectic matrices)**

The stable matrices in $sp(1)$ lie inside the cone $\mathcal{S}$ given by

$$X^2 \geq Y^2 + Z^2 \tag{4}$$

while $sr(1) \subset sp(1)$ is the symplectic reversible plane given by

$$Z = 0. \tag{5}$$

In the symplectic reversible plane $Z = 0$ we take coordinates (X, Y) in wich case stability is determined by $X^2 > Y^2$ and the stability boundaries by $Y = \pm X$.

2.2 Canonical transformations, normalization

We perform several canonical transformations to get the general form of the Floquet map. It turns out to be convenient to consider the following, slightly more general, version of Hill's equation

$$\ddot{x} + \left(\left(\frac{k}{2}\right)^2 + \alpha_0 + \sum_{j=1}^{n} \alpha_j \cos(jt) + \sum_{j=2}^{n} \beta_j \sin(jt) \right) x = 0, \tag{6}$$

where $(\alpha, \beta) = (\alpha_0, \ldots, \alpha_n, \beta_2, \ldots, \beta_n)$ are small parameters.

Remarks.

1. This is the general expression including harmonics up to order n. Indeed, the term in $\sin t$ can be cancelled by a time shift, which reduces the first harmonic to only a cos term and allows us to take $\beta_1 = 0$. The reversible case (as before) is obtained by taking $\beta = 0$.
2. The correspondence to (1) near the k–th resonance $(a, b) = ((\frac{k}{2})^2, 0)$ runs as follows:

$$\begin{aligned} \alpha_0 &= a - \left(\frac{k}{2}\right)^2, \\ \alpha_j &= bc_j, \\ \beta_j &= bs_j, \end{aligned} \tag{7}$$

where $j = 1, 2, 3, \ldots, n$. Note that we need that $n \geq k$. In the proof of the normal form theorem we shall take $n = k$. Also note that in (1) we have $c_1 = 1$, hence $\alpha_1 = b$ and also $s_1 = \beta_1 = 0$.

We proceed by writing the equation (6) in the Hamiltonian form

$$\dot{x} = y, \quad \dot{y} = -\frac{\partial H}{\partial x}, \quad \dot{t} = 1,$$

with Hamilton function

$$\begin{aligned} H(x, y, t) &= \frac{1}{2}(y^2 + \left(\frac{k}{2}\right)^2 x^2) \\ &+ \frac{1}{2}\left(\alpha_0 + \sum_{j=1}^{n} \alpha_j \cos(jt) + \sum_{j=2}^{n} \beta_j \sin(jt)\right) x^2. \end{aligned}$$

Splitting $H \equiv H_0 + H_1$, where $H_0(x, y) = \frac{1}{2}\left(y^2 + (\frac{k}{2})^2 x^2\right)$ and $H_1(x, y) = O(\alpha, \beta)$, we first introduce the traditional canonical change of variables

$$\xi = \sqrt{\frac{k}{2}}\, x \text{ and } \eta = \sqrt{\frac{2}{k}}\, y,$$

giving H_0 the rotationally symmetric form

$$H_0(\xi, \eta) = \frac{1}{2}\frac{k}{2}(\xi^2 + \eta^2).$$

A second canonical change of variables passes to complex variables

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix}, \tag{8}$$

introducing φ as a (shifted) time t and J as a variable canonically conjugate to φ. Furthermore we write $z = e^{i\varphi}$. This yields

$$\begin{aligned} H_0 &= \frac{k}{2} iqp + J, \\ H_1 &= \frac{1}{2k}\left(q^2 - p^2 + 2iqp\right)\left(\alpha_0 + \sum_{j=1}^{n} \frac{\alpha_j}{2}\left(z^j + z^{-j}\right)\sum_{j=2}^{n} \frac{\beta_j}{2i}\left(z^j - z^{-j}\right)\right). \end{aligned} \tag{9}$$

The third canonical transformation aims to cancel the dependence of H_1 on φ, except for resonant terms. This is a normalizing (averaging) transformation, based on a Lie series method. Compare, e.g., [13, 14]. In the Appendix we shall describe the method of Giorgilli–Galgani [14] in detail, here we just present the general format of the result.

Theorem 3 (Complex normal form) *For any fixed resonance $k \in \mathbb{N}$ and for any order $N \in \mathbb{N}$, by a canonical change of coordinates the Hamiltonian $H = H_0 + H_1$ can be reduced to the form*

$$\overline{H}_N = H_0 + \gamma_{2,N} q^2 z^{-k} + \gamma_{1,N} qp + \gamma_{0,N} p^2 z^k + O((\alpha, \beta)^{N+1}),$$

where the coefficients $\gamma_{2,N}, \gamma_{1,N}$ and $\gamma_{0,N}$ are functions of the parameters α, β. Moreover, $\gamma_{0,N} = -\overline{\gamma}_{2,N}$, while $\gamma_{1,N}$ is purely imaginary. Finally, $\gamma_{\ell,N+1}(\alpha,\beta) = \gamma_{\ell,N}(\alpha,\beta) + O(\alpha,\beta)^{N+1}$, for $\ell = 0, 1, 2$.

Remarks.

1. From now on we suppress the dependence on the order N as far as possible.
2. As usual, reversibility is preserved by the normalizing transformations. In the format $\overline{H}$ this is characterized by the property that γ_0, and hence also γ_2, is *real*.
3. We here note that in the Giorgilli–Galgani algorithm, the formal power series G is neither a *generating function* in the classical sense, nor the Hamiltonian of an *infinitesimal generator*. See [14] for additional properties.

Corollary 4 (Approximate Floquet matrix) *Under the conditions of theorem 3 the Floquet map $\mathcal{F}$, in dependence of the parameters α, β up to order N, expressed in the coordinates (X, Y, Z) on $sp(1)$, cf. proposition 2, gets the form*

$$X = \sigma_1,\ Y = -2\sigma_2,\ Z = 2\rho_2, \tag{10}$$

where

$$\gamma_1 = i\sigma_1 \text{ and } \gamma_2 = \sigma_2 + i\rho_2. \tag{11}$$

The reversible case is characterized by $\rho_2 = 0$, while the stability boundary is given by $\sigma_1^2 = 4(\sigma_2^2 + \rho_2^2)$.

The corollary reduces our job to studying the map

$$\mathcal{N} : (\alpha, \beta) \in \mathbb{R}^{2n} \mapsto (\sigma_1, \sigma_2, \rho_2) \in \mathbb{R}^3,$$

which provides the normal form coefficients. The components σ_1, σ_2 and ρ_2 of $\mathcal{N}$ can be obtained to any suitable order by an ad hoc formula manipulator. The results for $n = 3$, and $k = 1, 2, 3$ up to order $N = 4$ are displayed in Appendix B.

Remarks.

1. In complex parameters the stability boundary is given by $\gamma_1^2 = 4\gamma_0\gamma_2$.
2. Let us compare with the results from [9, 13], where the second resonance $k = 2$ was considered in the perturbed Mathieu equation (1). So consider the reversible example

   $$\ddot{x} + (a + b(\cos t + c_2 \cos 2t))\, x = 0,$$

   where the parameters a, b and c_2 are expressed in α_0, α_1 and α_2 by the scaling (7), i.e., where $a = 1 + \alpha_0$, $b = \alpha_1$ and $bc_2 = \alpha_2$.
   In the case $k = 2$ we obtain (up to order 3 in α)

   $$\begin{aligned} \gamma_1 &= \frac{1}{2}\alpha_0 - \frac{1}{8}\alpha_0^2 - \frac{1}{12}\alpha_1^2 - \frac{1}{64}\alpha_2^2, \\ 2\gamma_0 &= -\frac{1}{4}\alpha_2 + \frac{1}{8}\alpha_0\alpha_2 + \frac{1}{8}\alpha_1^2, \\ 2\gamma_2 &= \frac{1}{4}\alpha_2 - \frac{1}{8}\alpha_0\alpha_2 - \frac{1}{8}\alpha_1^2, \end{aligned}$$

   compare Appendix B. A change of variables leads to the 'diagonal' form obtained in [9, 13].

Proof. (of corollary 4). We approximate by truncating the O–terms. The corresponding Hamiltonian system is given by

$$\dot{q} = \left(\frac{ki}{2} + \gamma_1\right) q + 2\gamma_0 e^{ikt} p, \qquad \dot{p} = -\left(\frac{ki}{2} + \gamma_1\right) p - 2\gamma_2 e^{-ikt} q,$$

which is linear with periodic coefficients of period $2\pi/k$. This system easily provides us with the Floquet matrix of the trivial solution. Indeed, passing to co–rotating coordinates (u, v) by

$$q = e^{\frac{ki}{2}t} u \quad \text{and} \quad p = e^{-\frac{ki}{2}t} v,$$

we obtain

$$\dot{u} = \gamma_1 u + 2\gamma_0 v\,, \qquad \dot{v} = -2\gamma_2 u - \gamma_1 v.$$

So, introducing the complex version of the Floquet matrix

$$\tilde{F} = \begin{pmatrix} \gamma_1 & 2\gamma_0 \\ -2\gamma_2 & -\gamma_1 \end{pmatrix}, \tag{12}$$

the evolution of the pair (u, v) over time t reads

$$\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = \exp(t\tilde{F}) \begin{pmatrix} u(0) \\ v(0) \end{pmatrix}.$$

From this we directly obtain the corresponding evolution of the complex coordinates (q, p). Indeed, taking $t = 2\pi$ we find for Hill's map of the equation (6) in terms of the parameters γ and in the complex (q, p)–coordinates

$$\tilde{\mathcal{H}} = (-\mathrm{Id})^k \exp(2\pi\tilde{\mathcal{F}}).$$

A direct translation of the equation (12) in terms of σ and ρ yields

$$\tilde{F} = \begin{pmatrix} i\sigma_1 & -2\sigma_2 + 2i\rho_2 \\ -2\sigma_2 - 2i\rho_2 & -i\sigma_1 \end{pmatrix},$$

where we are still in the (q, p)–coordinates. Returning to (ξ, η) by the coordinate change (8), we denote

$$T = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}, \text{ hence } T^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}$$

and simply have to compute $F = T\tilde{F}T^{-1}$, which gives the desired result. □

2.3 At the tongue boundaries

In this subsection we present some results regarding the Floquet and Poincaré matrix as well as their behavior on the tongue boundaries.

2.3.1 Floquet matrix at the tongue boundary

Again fixing the resonance k, we consider the Floquet matrix at the tongue boundary. The reversible case is easiest. One branch of the boundary is given by $\sigma_1 = 2\sigma_2$. By corollary 4 it follows that here F is a lower diagonal matrix with only one non–zero element $-2\sigma_1$, which is off diagonal. (Similarly for the other branch $\sigma_1 = -2\sigma_2$, now F is upper diagonal.)

Next consider the general case. Putting $\gamma_2 = \frac{1}{2}re^{i\varphi}$, implying that $r = 2(\sigma_2^2 + \rho_2^2)^{1/2}$, the equation of the tongue boundaries becomes $\sigma_1 = \pm r$. A brief computation now reveals that

$$F = r \begin{pmatrix} \sin\varphi & \pm 1 - \cos\varphi \\ \mp 1 - \cos\varphi & -\sin\varphi \end{pmatrix} = 2r \begin{pmatrix} \sin\psi\cos\psi & \sin^2\psi \\ -\cos^2\psi & -\sin\psi\cos\psi \end{pmatrix},$$

where $\varphi = 2\psi$. In the final expression we restricted to the upper sign. For the lower sign one has to interchange the off diagonal terms. Note that in the reversible case $\varphi = 0$ and at the tongue boundary $\psi = 0$.

In figure 6 we compare normal form approximations of the tongue boundaries for the resonance $k = 3$, in the reversible near Mathieu case of equation (1), with numerical computations, compare Appendix B. Figure 6 shows the stability diagram in the (a, b)–plane, in a case with 2 instability pockets. Compare figure 3. We also show the size of the off–diagonal terms of the Floquet matrix F_c at the tongue–boundaries is depicted as a function of b. Finally, the formal computations up to order $N = 8$, are compared to the numerical ones.

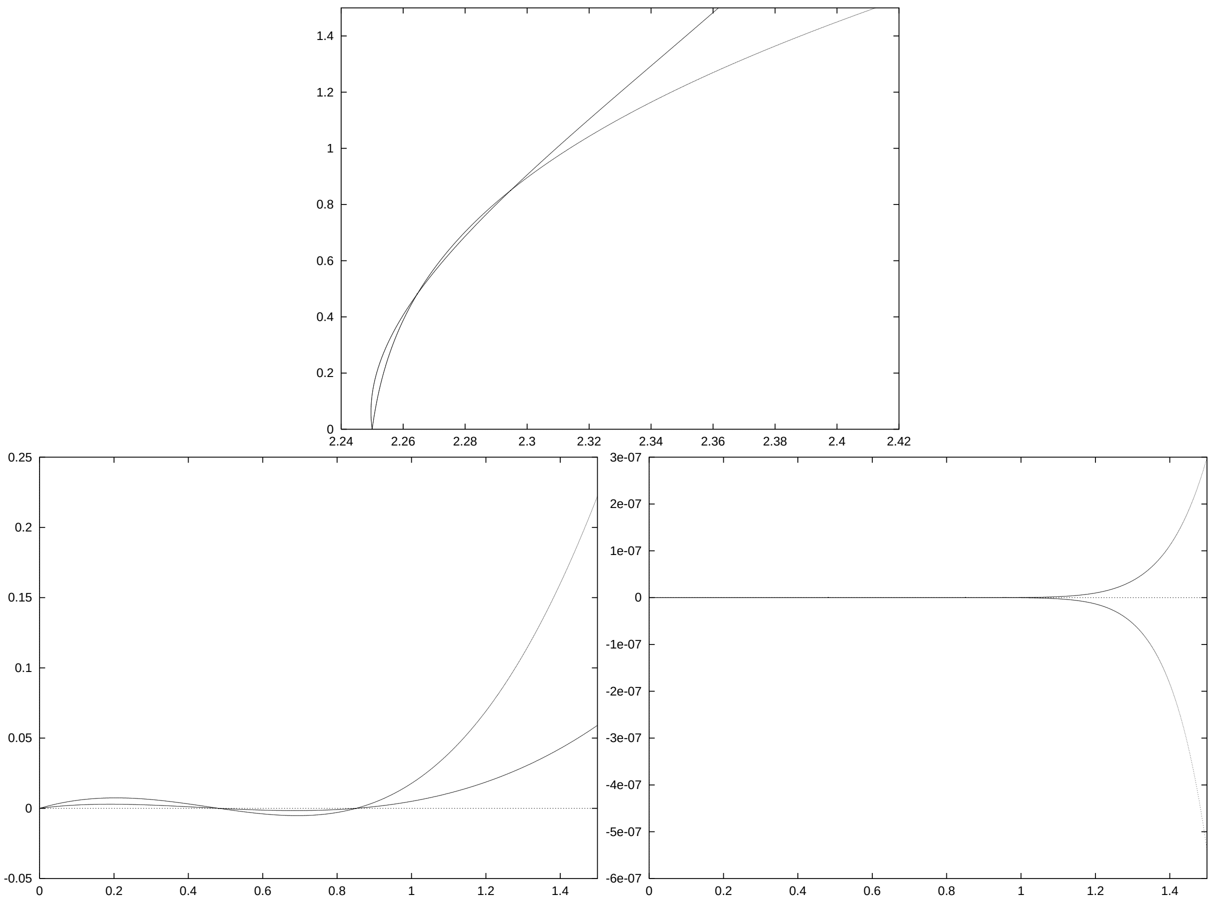

Figure 6: Top: Reversible stability diagram in the near Mathieu case near the resonance $k = 3$, showing two instability pockets. Specification: $c_2 = 0.1625,\ c_3 = 0.025,$. Bottom left: The off–diagonal elements of the Floquet matrix as a function of b for both tongue–boundaries. Bottom right: Accuracy of the above results: The errors when using the normal form formula for the tongue boundary up to order 8, as a function of b. Up to $b = 1$ the errors are less than 3.2×10^{-10}.

2.3.2 Poincaré matrix at the tongue boundary

Let us now compute the Poincaré matrix P explicitly. From F as given by (10) one has $F^2 = (r^2 - \sigma_1^2)\,\mathrm{Id}$. Let $\Delta^2 = (2\pi)^2(r^2 - \sigma_1^2)$. Then it follows

$$P = (-\mathrm{Id})^k \exp(2\pi F) = (-1)^k \left[\cosh\Delta\,\mathrm{Id} + 2\pi\frac{\sinh\Delta}{\Delta}\,F\right].$$

In the reversible case this simplifies to

$$P = (-1)^k \begin{pmatrix} 1 & 4\pi(-\sigma_2 \pm |\sigma_2|) \\ 4\pi(-\sigma_2 \pm |\sigma_2|) & 1 \end{pmatrix}.$$

At the tongue boundaries one has $\Delta = 0$ and hence

$$P = (-1)^k \left[\mathrm{Id} + 2\pi F\right],$$

the matrix F now being nilpotent. The latter also follows from §2.3.1, since at the tongue boundary

$$F = 2r \begin{pmatrix} \sin\psi \\ -\cos\psi \end{pmatrix} \begin{pmatrix} \cos\psi & \sin\psi \end{pmatrix},$$

similar for the other case.

3 Proof of theorem 1

We now proceed by giving a proof of the cuspoidal normal form theorem 1. Since in [9] the case $k = 2$ was treated we here restrict to $k > 2$. As said before, we restrict to the case where $n = k$. So, fixing the resonance $k > 2$, we consider $(a, b) \approx ((\frac{k}{2})^2, 0)$, thereby restricting to the reversible case of (1)

$$\ddot{x} + (a + b\, p_c(t))\, x = 0,\ p_c(t) = \cos t + c_2 \cos(2t) + \cdots + c_k \cos(kt),$$

so with all harmonics of order $2, \ldots, k$ included, with small parameters as coefficients. Our aim is to approximate the local Floquet map $\mathcal{F}_c = \mathcal{F}_c(a, b)$ in dependence of these parameters c.

There are two scalings to deal with. Firstly, scaling (7) in this reversible case reads

$$\begin{aligned} \alpha_0 &= a - \left(\frac{k}{2}\right)^2, \\ \alpha_1 &= b \text{ or } c_1 = 1, \\ \alpha_j &= bc_j \text{ for } j \geq 2. \end{aligned}$$

Secondly, we employ scaling (3):

$$\begin{aligned} b &= \delta B, \\ c_1 &= d_1 = 1, \\ c_j &= \delta^{j-1} d_j, \text{ for } j \geq 2, \end{aligned}$$

where δ is a small parameter.

Remark. Many different notations are of course confusing. In the following we shall mostly express things either in α_0, b and the c_j, $j \geq 2$, or in α_0, B and the d_j, $j \geq 2$, where δ is small.

3.1 Preliminaries and structure of the map $\mathcal{N}$

In the present reversible case, the map $\mathcal{N}$, which expresses the normal form coefficients in terms of the parameters, has the form $\mathcal{N} = (\sigma_1, \sigma_2, 0)$, and we have to determine σ_1 and σ_2 to sufficient order in the parameters. To this purpose we start with the averaging process, see theorem 3 and §A.1. The initial Hamiltonian has the form (9)

$$H = \frac{k}{2} iqp + J + \frac{1}{2k}\left(q^2 - p^2 + 2iqp\right)\left(\alpha_0 + \sum_{j=1}^{k} \frac{\alpha_j}{2}\left(z^j + z^{-j}\right)\right),$$

recalling that $\alpha_1 = b$. Averaging once, see theorem 3 and §A.1, we find

$$\overline{H}_1 = \frac{1}{2k}\left(q^2 \frac{\alpha_k}{2} z^{-k} - p^2 \frac{\alpha_k}{2} z^k + 2iqp\alpha_0\right).$$

Therefore, the lowest order contributions to σ_1 and σ_2 are

$$\begin{aligned} \sigma_1 &= \frac{\alpha_0}{k} \\ \sigma_2 &= \frac{bc_k}{4k} = \frac{1}{4k} B d_k \delta^k. \end{aligned}$$

It will be necessary to know the general format of the monomials in the map $\mathcal{N}$. We shall consider the general case $\mathcal{N} : (\alpha, \beta) \mapsto (\sigma_1, \sigma_2, \rho_2)$. The monomials in the components of $\mathcal{N}$ indeed satisfy simple rules, that can be checked by induction from the Lie series procedure. Compare §A.1.

As an example consider the term containing z^k. The coefficient of z^k is obtained by adding the multiplication of several terms containing powers z^ℓ. The powers z^ℓ in the initial Hamiltonian have coefficients α or β with index equal to the absolute value of ℓ. This simple fact, together with the parity of the occurrence of β–factors in σ_1 and σ_2 (even) and in ρ_2 (odd) can be summarized as

Proposition 5 (Format of the map $\mathcal{N}$) *Let ν denote any of the components σ_1, σ_2, ρ_2 of $\mathcal{N}$. Then $\nu = \sum_{m\geq 1} \nu_m$, where ν_m is a homogeneous polynomial of degree m in (α, β), which can be expressed as*

$$\nu_m = \sum \mathrm{coef} \prod_{j=0}^{k} \alpha_j^{e_j + \bar{e}_j} \prod_{j=2}^{k} \beta_j^{f_j + \bar{f}_j}.$$

Here coef *denotes the coefficients, which are rational numbers and the summation extends over the set of nonnegative exponents $e_j, \bar{e}_j, f_j, \bar{f}_j$ with constraints*

$$\sum_j e_j + \bar{e}_j + f_j + \bar{f}_j = m, \quad \sum_j j(e_j - \bar{e}_j) + j(f_j - \bar{f}_j) = \begin{cases} k & \text{for } \sigma_2, \rho_2 \\ 0 & \text{for } \sigma_1, \end{cases}$$

$$\sum_{j=2}^{k} f_j + \bar{f}_j = \begin{cases} \text{even} & \text{for } \sigma_1, \sigma_2 \\ \text{odd} & \text{for } \rho_2. \end{cases}$$

Note that eventually some of the rational 'coef' may be 0, due to cancellations. Also observe that in the present, reversible case $f_j \equiv 0 \equiv \bar{f}_j$.

3.2 Computation of σ_1

We now focus on σ_1, proceeding with a second averaging yielding the Hamiltonian $\overline{H}_2$. An elementary computation provides the 'generator' G_1. In $\overline{H}_2$ we have to inspect the terms q^2z^{-k}, p^2z^k, qp. Since

$$\overline{H}_2 = [G_1, H_{1,0}] + \frac{1}{2}[G_1, H_{0,1}] + [G_2, H_{0,0}]$$

and $H_{0,1} = \overline{H}_1 - H_{1,0}$, one has to compute

$$\frac{1}{2}\left([G_1, \overline{H}_1] + [G_1, H_{1,0}]\right).$$

The first term gives no contribution to $\overline{H}_2$, but the second contributes to σ_1 giving the following result, recalling that we assumed $k > 2$.

Lemma 6 **(Form of σ_1)**

$$\begin{aligned}\sigma_1 &= \frac{\alpha_0}{k} - \frac{1}{2k(k^2-1)}b^2 + O(b^3) \\ &= \frac{\alpha_0}{k} - \frac{\delta^2}{2k(k^2-1)}B^2 + O(\delta^3).\end{aligned}$$

Remark. Hence for the equation $\sigma_1 = 0$, see Appendix C, the dominant terms give

$$\alpha_0 = \frac{b^2}{2(k^2-1)}.$$

3.3 Computation of σ_2

We proceed with σ_2, again referring to §A.1 From G_1 and $H_{1,0}$, using that $k > 2$, one obtains the contribution

$$-\frac{1}{4k(k-1)}\alpha_1\alpha_{k-1} = -\frac{1}{4k(k-1)}b^2c_{k-1} = -\frac{1}{4k(k-1)}B^2d_{k-1}\delta^k$$

as well as terms of the form $\alpha_2\alpha_{k-2},\ \alpha_3\alpha_{k-3},\ldots$, that is, $b^2O(c_2c_{k-2}, c_3c_{k-3},\ldots)$.

Now we use proposition 5 and the scaling (3). From theorem 3 and the Appendices A and C we recall that σ_2 occurs as the coefficient of q^2z^{-k} in the final normal form, or (up to a change of sign) as the coefficient of p^2z^k. Note that no term containing α_0 as a factor can appear among the dominant terms of σ_2, because skipping this factor would give another term giving a larger contribution to the coefficient of z^k. The terms which remain in σ_2 are of the form

$$\sum_{m\geq 1}\sum \mathrm{coef}\prod_{j=1}^{k}\left(Bd_j\delta^j\right)^{e_j+\bar{e}_j},$$

where $d_1 = 1$, with $\sum_{j=1}^{k} j(e_j - \bar{e}_j) = k$. The lowest power δ^k appears only if $\bar{e}_j \equiv 0$. Hence we conclude

Lemma 7 (Form of σ_2) *There exist rational numbers* coef_j, $j = 3, \ldots, k$ *and polynomials* $P_j(d)$, $j = 2, \ldots, k-2$ *in* $d = (d_2, \ldots, d_k)$ *with rational coefficients, such that*

$$\begin{aligned}\frac{\sigma_2}{\delta^k} = \frac{1}{4k} d_k B - \left(\frac{1}{4k(k-1)} d_{k-1} + P_2(d_2, \ldots, d_{k-2})\right) B^2 \\ + \left(\mathrm{coef}_3\, d_{k-2} + P_3(d_2, \ldots, d_{k-3})\right) B^3 + \left(\mathrm{coef}_4\, d_{k-3} + P_4(d_2, \ldots, d_{k-4})\right) B^4 \\ + \cdots + \left(\mathrm{coef}_{k-2}\, d_3 + P_{k-2}(d_2)\right) B^{k-2} + \mathrm{coef}_{k-1}\, d_2 B^{k-1} + \mathrm{coef}_k\, B^k + O(\delta).\end{aligned}$$

Remarks.

1. In view of the above one might say that

$$\mathrm{coef}_1 = \frac{1}{4k} \text{ and } \mathrm{coef}_2 = -\frac{1}{4k(k-1)}.$$

2. The form of the polynomials $P_j(d)$ is readily seen. Indeed, the terms in B^j containing $d_{q_1}, \ldots, d_{q_r}$, $2 \leq q_1 \leq q_2 \leq \ldots \leq q_r$ satisfy $\sum_{s=1}^{r} q_s = k - (j - r)$, since the factor $d_1 = 1$ should appear $j - r$ times. The factor of B^j containing d_{k-j+1} has been explicitely given in front of P_j (it is the only term with $r = 1$). If $r \geq 2$ we have

$$q_r \leq k - j + r - \sum_{s=1}^{r-1} q_s \leq k - j + r - 2(r-1) = k - j - r + 2 \leq k - j.$$

The rational coefficients coef_j are obtained during the averaging process. They will play an essential role, as we shall see now.

Let us rewrite the expression in lemma 7 as

$$\sigma_2 = Q_k(B)\delta^k + O(\delta^{k+1}),$$

then $Q_k(B)$ is a k–th degree polynomial in B, satisfying $Q_k(0) = 0$ and with coefficients depending on $d = (d_2, \ldots, d_k)$. Let $B_1 = 0, B_2, \ldots, B_k$ be the zeros of $Q_k(B)$, then

$$Q_k(B) = \mathrm{coef}_k \prod_{s=1}^{k} (B - B_j) = \mathrm{coef}_k\, B^k + \sum_{j=1}^{k-1} w_j B^j,$$

where the w_j are the symmetric polynomials in the B_j. By comparison of coefficients it follows

$$w_j = \mathrm{coef}_j\, d_{k+1-j} + P_j(d_2, \ldots, d_{k-j}). \tag{13}$$

Lemma 8 (Deformation) *If* $\mathrm{coef}_j \neq 0$ *for* $j = 3, \ldots, k$*, the mapping*

$$(d_2, \ldots, d_k) \in (\mathbb{R}^{k-1}, 0) \mapsto (w_1, \ldots, w_{k-1}) \in (\mathbb{R}^{k-1}, 0)$$

is a local diffeomorphism.

Indeed, from the 'triangular' form of (13) we conclude that the smooth map $d \mapsto w$ locally has a smooth inverse. The conclusion of the lemma implies that the 'singularity' $\mathrm{coef}_k\, B^k$ is 'deformed versally' by the parameters d.

Lemma 9 (Coefficients coef_j, for $j = 3, \dots, k$)

$$\begin{aligned}
\mathrm{coef}_k &= \frac{(-1)^{k-1}}{2^{k+1}k!(k-1)!}, \\
\mathrm{coef}_j &= \frac{(-1)^{j-1}}{2^{j+1}k!(k-1)!}\sum_{m=0}^{j-1}\frac{(k-j+m)!\,(k-1-m)!}{m!\,(j-1-m)!}, \ \textit{for } 3 \le j \le k-1.
\end{aligned}$$

Proof. First, to show that $\mathrm{coef}_k \neq 0$ we only neeed to consider the Mathieu case at exact resonance

$$H = \frac{k}{2}iqp + J + \frac{b}{2k}\left(q^2 - p^2 + 2iqp\right)\frac{1}{2}(z + z^{-1}).$$

Similarly, to show that $\mathrm{coef}_j \neq 0,\ j \in [2, k-1]$ we only need to consider

$$H = \frac{k}{2}iqp + J + \frac{b}{2k}\left(q^2 - p^2 + 2iqp\right)\left(\frac{1}{2}(z + z^{-1}) + \frac{c_{\bar{j}}}{2}(z^{\bar{j}} + z^{-\bar{j}})\right),$$

where $\bar{j} = k + 1 - j$, since only powers of α_1 times, at most, one of the $\alpha_{\bar{j}},\ \bar{j} \in [2, k-1]$ are relevant.

Instead of using the Hamiltonian formulation it is more convenient to turn directly to the differential equations. Furthermore, as we are interested in the highest power in z, instead of $\cos t = \frac{1}{2}(z + z^{-1})$ we may use simply z. A further simplification consists in replacing t by $2t$. Therefore the differential equation in the case of coef_k reads

$$\ddot{x} + (k^2 + \varepsilon z^2)x = 0,$$

where $z = e^{it}$ (with the new time) and where ε is used instead of $b/2$. It is moreover convenient to use z as independent variable and so we arrive at the linear differential equation

$$-z^2x'' - zx' + k^2x + \varepsilon z^2 x = 0,$$

where $'$ denotes the derivation with respect to z. The solution is expanded in powers of $\varepsilon\ :\ x = x_0 + \varepsilon x_1 + \cdots + \varepsilon^k x_k + \cdots$. For x_0 we can take one of the two fundamental solutions of the case $\varepsilon = 0$. Taking $x_0 = z^{-k}$, it follows by comparison of powers of ε, that

$$-z^2{x_m}'' - z{x_m}' + k^2x_m + \varepsilon z^2 x_{m-1} = 0,\ m = 1, \dots, k.$$

Particular solutions of these equations, if $m < k$, are of the form $x_m = \nu_m z^{-k+2m}$, where the values of ν_m are obtained recurrently as

$$\nu_m = -\frac{1}{k^2 - (k-2m)^2}\nu_{m-1},$$

starting with $\nu_0 = 1$. The case $m = k$ is different, because logarithmic terms appear. Looking for x_k as $\nu_k z^k \log z$ we obtain $\nu_k = \frac{1}{2k}\nu_{k-1}$. Hence, a solution, say $x^{(1)}$, is given by

$$\begin{aligned} x = &\quad z^{-k} + \nu_1 \varepsilon z^{-k+2} + \cdots + \nu_{k-1}\varepsilon^{k-1} z^{-k+2(k-1)} + \nu_k \varepsilon^k z^k \log z + O(\varepsilon^{k+1}), \\ x' = &\quad -kz^{-k-1} + (-k+2)\nu_1 \varepsilon z^{-k+1} + \cdots + (k-2)\nu_{k-2}\varepsilon^{k-1} z^{k-3} \\ &\quad + \nu_k \varepsilon^k (kz^{k-1}\log z + z^{k-1}) + O(\varepsilon^{k+1}). \end{aligned}$$

Considering the initial and final values of time (now $t = 0$ and $t = 4\pi$) and keeping in mind that $\log z = it$, one sees that the initial and final values of x, x', after one period, are of the form

$$\begin{pmatrix} x(0) \\ x'(0) \end{pmatrix} = \begin{pmatrix} 1 + \nu_1 \varepsilon + \cdots + 0 \cdot \varepsilon^k + O(\varepsilon^{k+1}) \\ -k + (-k+2)\nu_1 \varepsilon + \cdots + \nu_k \varepsilon^k O(\varepsilon^{k+1}) \end{pmatrix}$$

and

$$\begin{pmatrix} x(4\pi) \\ x'(4\pi) \end{pmatrix} = \begin{pmatrix} 1 + \nu_1 \varepsilon + \cdots + \nu_k 4\pi i \varepsilon^k + O(\varepsilon^{k+1}) \\ -k + (-k+2)\nu_1 \varepsilon + \cdots + \nu_k \varepsilon^k + \nu_k k 4\pi i \varepsilon^k O(\varepsilon^{k+1}) \end{pmatrix}.$$

Note that these expressions differ only in the $O(\varepsilon^k)$ contributions arising from the logarithmic terms in the solution.

We can now proceed with the other fundamental solution, say $x^{(2)}$, starting with the zero order term z^k. However, it is better to look at that solution for the conjugate equation $\ddot{x} + (k^2 + \varepsilon z^{-2})x = 0$. The same recurrence as before is found for the computation of the coefficients $\nu_m,\ m < j$, now with $\nu_k = -\frac{1}{2k}\nu_{k-1}$. By taking $\frac{1}{2}(x^{(1)} \pm x^{(2)})$ as fundamental solutions we obtain the Poincaré matrix as

$$\begin{pmatrix} 1 & 4\pi i \nu_k \varepsilon^k \\ 4\pi i k \nu_k \varepsilon^k & 1 \end{pmatrix} + O(\varepsilon^{k+1}).$$

A comparison with the expression of the Poincaré matrix in the reversible case, if we do not take into account σ_1, shows that ν_k for the present equation and σ_2 for the initial formulation, are proportional. From the recurrence we have

$$\nu_k = (-1)^{k-1} \frac{1}{2k} \prod_{m=1}^{k-1} \frac{1}{k^2 - (k-2m)^2}.$$

From this and the change of parameter (ε instead of $b/2$) one obtains the first result of the lemma.

Now we proceed to the computation of $\text{coef}_j,\ j = 2, \ldots, k-1$. Here, the relevant linear equation is

$$\ddot{x} + (k^2 + \varepsilon z^2 + \rho z^{2\bar{j}})x = 0,$$

where ρ abreviates $\frac{1}{2}bc_{\bar{j}}$. We are after the coefficient of $\varepsilon^{k-\bar{j}}\rho$ in the z^k coefficient of the solution $x^{(1)} = z^{-k} + \ldots$. It is possible to proceed recurrently as before, but now in the iterative process we can take $k - \bar{j}$ times the term in εz^2 and then the term $\rho z^{2\bar{j}}$, or one can use this term at some intermediate step or at the first one. Recalling that $\bar{j} = k+1-j$ the second result of the lemma is immediate. □

Remark. The same approach can be applied to obtain explicit expressions for the coefficients of terms involving several d_j (using multiple summations), but they are not needed for our purposes. From the computation it is immediate that all of these are different from zero. In particular $\text{coef}_j \neq 0$, as desired. See lemma 9, where all terms of the sum have equal sign.

3.4 Conclusion

We partially summarize the results of this section as follows.

Theorem 10 (Computational normal form, reversible case) *Fixing the resonance $k > 2$ in the reversible near Mathieu equation (1), with $p_c(t) = \cos t + \sum_{j=2}^{k} c_k \cos(jt)$, the map $\mathcal{N} = (\sigma_1, \sigma_2, 0)$ in the scaling (3) has the form*

$$\begin{aligned}\sigma_1(\alpha_0, B; d; \delta) &= \frac{\alpha_0}{k} - \frac{\delta^2}{2k(k^2-1)} B^2 + O(\delta^3), \\ \sigma_2(\alpha_0, B; d; \delta) &= Q_k(B; d)\delta^k + O(\delta^{k+1}).\end{aligned}$$

Here $Q_k(B;d)$ is a k–th degree polynomial in B. The parameters $d = (d_2, \ldots, d_k)$ versally deform the classical Mathieu case where

$$Q_k(B;0) = \frac{(-1)^{k-1}}{2^{k+1} k!(k-1)!} B^k.$$

As indicated in the above theorem, we consider (α_0, B) as the variables and $d = (d_2, \ldots, d_k)$ as parameters. Furthermore δ just serves to control the size of the perturbation. From here a proof of the main theorem 1 is quite standard in singularity theory, compare, e.g., Bröcker and Lander [5]. Indeed, one first applies the submersion theorem thereby turning the variable α_0 into a kind of parameter and reducing to the B–direction. Second, in the B–direction singularity theory for functions of one variable applies, which gives the cuspoid normal form of theorem 1. By versality this local normal form holds for sufficiently small δ.

Remarks.

1. Finally we go back to the scaling (3), where $c_j = \delta^{j-1} d_j$. We now show that there is no other choice. Indeed, assume we do not scale the parameter c_j. Then σ_2 can be written as

$$\begin{aligned}\sigma_2 = {} & \text{coef}_k \prod_{s=1}^{k} (\delta B - \delta B_s)(1 + O(\delta)) \\ = {} & \text{coef}_k\, \delta^k B^k + \text{coef}_{k-1}\, c_2 \delta^{k-1} B^{k-1} \\ + {} & (\text{coef}_{k-2}\, c_3 + P_{k-2}(c_2))\delta^{k-2} B^{k-2} + \cdots \\ + {} & (\text{coef}_3\, c_{k-2} + P_3(c_2, \ldots, c_{k-3}))\, \delta^3 B^3 \\ + {} & (\text{coef}_2\, c_{k-1} + P_2(c_2, \ldots, c_{k-2}))\, \delta^2 B^2 + \text{coef}_1\, c_k \delta B + O(\delta^{k+1})\end{aligned}$$

and the choice $c_j = d_j\delta^{j-1}$, for some finite d_j, appears as inevitable.

2. It may be clear that the situation is completely different if some harmonics, c_j, $1 < j \leq k$ are missing. For instance, if $c_k = 0$ then $B = 0$ is always a double zero, but if $c_k = c_{k-1} = 0$ there are terms in B^2 containing the products $c_2c_{k-2}, c_3c_{k-3}, \ldots$ and, in general, we shall not have a triple zero at $B = 0$. If $c_k = c_{k-1} = 0$ but there is an harmonic $c_{k+1} \neq 0$, then terms in $c_{k+1}B^2$ shall appear and, if we scale by $c_{k+1} = \delta^{k-2}d_{k+1}$, d_{k+1} finite, the behavior will be similar to the case $c_k = 0, c_{k-1} \neq 0$.

4 The general near Mathieu case for any $k > 2$.

In this section we give some brief remarks about the general case, which is not necessarily reversible. We recall the general fact that the k–th tongue boundaries meet transversal at the resonance precisely if the k–th harmonic does not vanish, cf. [17, 3, 13]. Also we bring to mind that, since the range $Sp(1)$ of $\mathcal{H}$ is 3–dimensional, generically no nontrivial foldings (crossings) appear. One of the things we shall show is that disconnected tongues locally cannot exist in the near Mathieu case. For a more general discussion see appendix D.

To be more precise we now take

$$\begin{aligned} p_{c,s}(t) = \cos t \quad &+ \quad c_2\cos(2t) + \cdots + c_k\cos(kt) \\ &+ \quad s_2\sin(2t) + \cdots + s_k\sin(kt), \end{aligned}$$

with small parameters $(c, s) = (c_2, \ldots, c_k, s_2, \ldots, s_k) \in \mathbb{R}^{2(k-1)}$. To the earlier scalings here add

$$\begin{aligned} \beta_1 &= s_1 = 0, \\ \beta_j &= bs_j \text{ for } j \geq 2 \end{aligned}$$

and

$$\begin{aligned} s_1 &= t_1 = 0, \\ s_j &= \delta^{j-1}t_j \text{ for } j \geq 2, \end{aligned}$$

where δ is a small parameter as before and things are expressed either in α_0, b and the c_j, s_j, $j \geq 2$, or in α_0, B, δ and the d_j, t_j, $j \geq 2$. In these circumstances we have, following the same line of argument as in §3:

Theorem 11 (Computational normal form, general case) *Fixing the resonance $k > 2$ in the near Mathieu equation (1), with $p_{c,s}(t)$ as above, the map $\mathcal{N} = (\sigma_1, \sigma_2, \rho_2)$ in the above scalings has the form*

$$\begin{aligned} \sigma_1(\alpha_0, B; d, t; \delta) &= \frac{\alpha_0}{k} - \frac{\delta^2}{2k(k^2-1)}B^2 + O(\delta^3), \\ \sigma_2(\alpha_0, B; d, t; \delta) &= Q_k(B; d, t)\delta^k + O(\delta^{k+1}), \\ \rho_2(\alpha_0, B; d, t; \delta) &= R_k(B; d, t)\delta^k + O(\delta^{k+1}). \end{aligned}$$

Here $Q_k(B;d,t)$ and $R_k(B;d,t)$ are k–th degree polynomials in B, the parameters now being $(d,t) = (d_2,\dots,d_k,t_2,\dots,t_k)$. The number of t–factors in each monomial of Q_k is even, while for R_k this number is odd. Moreover,

$$Q_k(B;0,0) = \frac{(-1)^{k-1}}{2^{k+1}k!(k-1)!}B^k, \text{ while } R_k(B;0,0) = 0.$$

The fact that the leading term of R_k is missing implies that the reparametrization $(c,s) \mapsto (d,t)$, unlike in the case of lemma 8, is not a local diffeomophism. In fact, the reparametrization is neither injective nor surjective. This can be seen by following the line of the proof in §3. Indeed, it turns out not to be possible to prescribe zeroes $B_2,\dots,B_k$ of the Floquet map, or equivalently, of $\mathcal{N}$. Here we recall that always $B_1 = 0$, while $d_1 = 1$ and $t_1 = 0$. Such zeroes correspond to intersection points of the tongue boundaries of the k–th resonance. Note the general property that the equation $\sigma_1 = 0$, by the special form of $\sigma_1(\alpha_0,B;d,t;\delta)$, then determines $\alpha_0 = \alpha_0(d,t;\delta)$. Now let us suppress one of the prescribed values, say B_k. Then both $Q_k(B)$ and $R_k(B)$ are proportional to $\prod_{m=1}^{k-1}(B-B_m)$. From this it follows that two of the parameters $(d,t) = (d_2,\dots,d_k,t_2,\dots,t_k)$ are free to choose.

Another option is the following, again fixing the resonance $k > 2$. Assume that, apart from (α_0,b) and $(\alpha,\beta) = (\alpha_2,\dots,\alpha_k,\beta_2,\dots,\beta_k)$, we also introduce the coefficient β_{2k-1} as a parameter. Moreover, we change the above scaling by putting

$$\begin{aligned} s_j &= \delta^j t_j \text{ for } j = 2,\dots,k \\ s_{2k-1} &= \delta\, t_{2k-1} \end{aligned}$$

instead. It turns out that in this case the form of theorem 11 does not change, but that now in the polynomial $R_k(B)$ the term B^k has a nonzero coefficient. This yields that it is possible to have any prescribed set of k values of B (including $B = 0$), for which the Floquet map covers zero. For the corresponding value of α_0 we again use the special form of σ_1. We then have a set of possible choices of the parameters of dimension 1. So now the reparametrization is surjective, but not injective. For the cases $k = 2,3$ this can be checked from appendix B.

Finally we consider the problem of the disconnected tongues. Concerning the points (α_0,b) at the boundaries of the tongue for small b, the following holds. Firstly recall the equation $\sigma_1^2 = 4(\sigma_2^2 + \rho_2^2)$ for the tongue boundary and, secondly, again the dominant terms in σ_1. Then, apart from the values of b where the Floquet map covers zero, for any small b always two different values of α_0 occur. Therefore, *disconnected* tongues cannot occur in the near Mathieu case.

Remarks.

1. In view of the general expression for σ_1, which always contains $\frac{1}{k}\alpha_0$ as the dominant term containing α_0, it follows that under the sole assumption that the parameters α and β are small, disconnected tongues cannot occur.
2. From the geometry of the stability domains inside $Sp(1)$, compare [9], by a connectedness argument, it even seems plausible that disconnected tongues cannot occur for any values of a and b. This is confirmed by our considerations in appendix D.

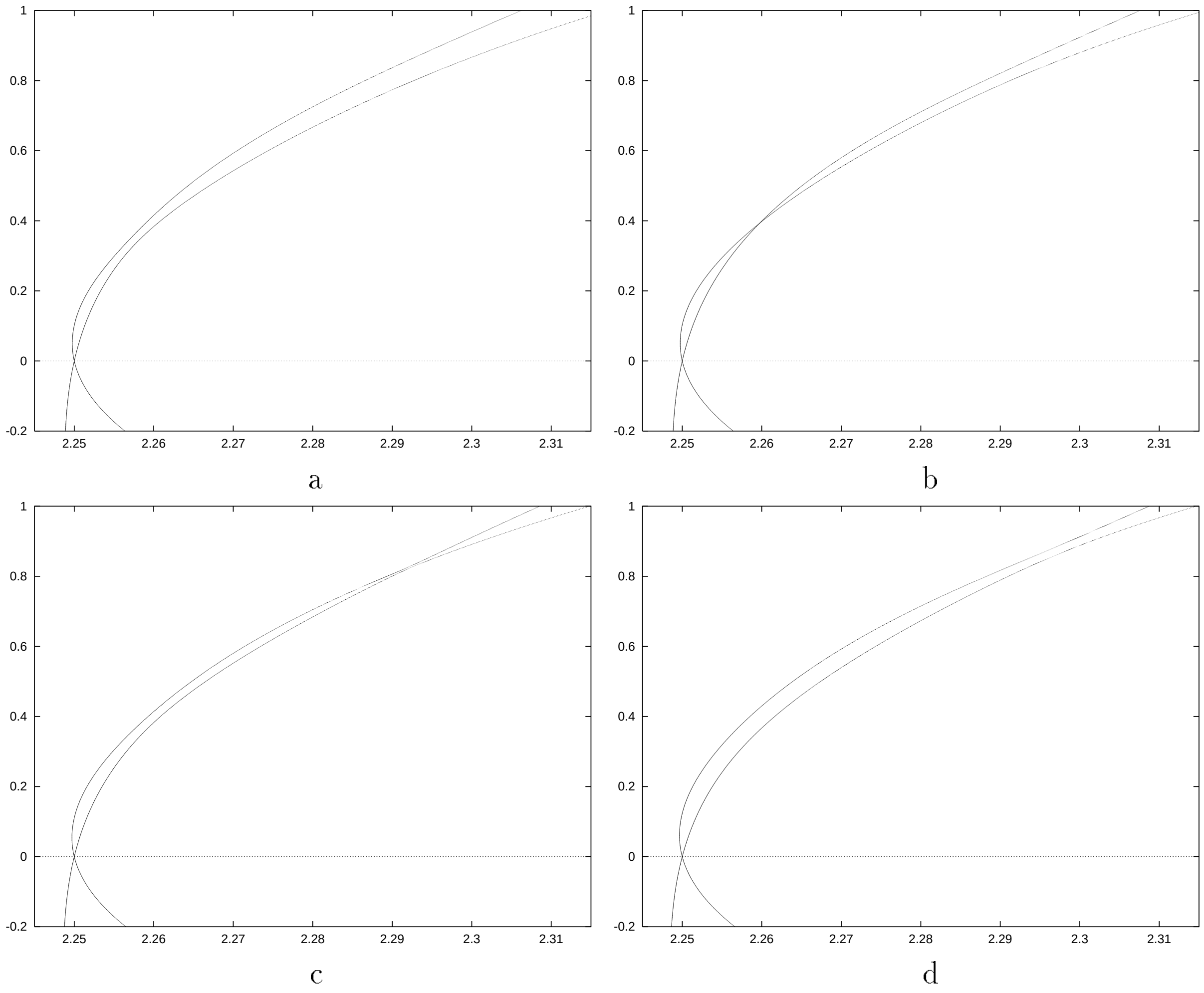

Figure 7: Near reversible resonance tongues for $k = 3$, following from §B.3. For the four examples displayed here the values $c_2 = 0.15$, $c_3 = 0.02$, $s_2 = 0.02$ have been used. The values of s_3 from left to right and top to bottom are 0, 0.004, 0.008 and 0.012. The reversible case is very similar to the one displayed in figure 6, with two contacts of the boundaries for positive values of b. In case b above the situation is quite close to keep the first contact, while in case c it is the second one which is kept.

5 Conclusions

We studied the geometry of the resonance tongues of Hill's equation near the classical Mathieu case,

$$\ddot{x} + (a + bp(t))x = 0, \;\; p(t + 2\pi) \equiv p(t),$$

mainly restricting to the reversible case, where $p(-t) \equiv p(t)$. It turns out that the classical Mathieu equation with $p(t) = \cos t$ is quite degenerate, since in the (a, b)–plane the tongue boundaries at the k–th resonance $(a, b) = ((\frac{k}{2})^2, 0)$ are tangent of order $k-1$. From [17, 3, 13] it was already known that these boundaries at the k–th resonance meet transversally if and only if the k–th harmonic of $p(t)$ does not vanish. It turned out that this singularity can be versally deformed by adding all lower harmonics as small parameters. This result in theorem 1 was phrased in terms of a (Whitney) cuspoid normal form of Hill's map $\mathcal{H}$, which assigns to each parameter point (a, b) the corresponding Poincaré matrix $P_{a,b}$ in a 2–dimensional range. This theorem implies that for the deforming family $p_c(t) = \cos t + \sum_{j=2}^{k} c_j \cos(jt)$, for appropriate small values of $c = (c_2, \ldots, c_k)$, any number up to $k - 1$ of instability pockets can occur, as well as any tangency of the tongue boundaries up to order $k - 1$. As an example, in the figures 2, 3 the organisation is shown of all this both in the cuspoid normal form and in the (c_2, c_3)–plane for the case $k = 3$. This result has some robustness in the class of all C^∞–functions p, in any case for the addition of small higher harmonics. The basis of these considerations is a computational normal form, obtained by averaging both for the reversible and the nonreversible case. The latter result enables a perturbation analysis of the tongue–geometry away from reversibility, see figures 5 and 7.

The question of whether it is possible to have *simultaneously* $k - 1$ pockets, emerging from the k–th order resonance, for all $k \in \mathbb{N}$ is addressed in [12], where some striking relation with a class of Ince equations and interesting global phenomena are displayed.

A related problem concerns the setting where the forcing term, instead of periodic, is quasi–periodic. An example of this with two frequencies is the equation $\ddot{x} + (a + b\, p_c(t))x = 0$, with forcing term $p_c(t) = \cos t + \cos \gamma t + c \cos(1 + \gamma)t)$, where $\gamma = \frac{1}{2}(1 + \sqrt{5})$, the golden number. (Another terminology here speaks of the Schrödinger equation with potential p_c.) In [11] it was discovered that the resonance phenomena for small b are largely the same as here: also in the quasi–periodic case resonance tongues and instability pockets show up with the same complexity as in the present periodic case. More globally in the (a, b)–plane, however, there turns out to be a huge difference with the periodic case, also compare [11, 22].

Acknowledgements

The authors thank Jack Hale, Igor Hoveijn, Mark Levi, Floris Takens and Gert Vegter for their valuable comments. The research of the second author has been supported by DGICYT grant PB 94–0215 (Spain). Partial support of the EC grant ERBCHRXCT-940460, and the catalan grant CIRIT 1996S0GR-00105 also is acknowledged. Each of the authors is indebted to the host institution of the other for hospitality.

References

[1] Afsharnejad, Z.: Bifurcation geometry of Mathieu's equation. *Indian J. Pure Appl. Math.* **17** (1986), 1284–1308.

[2] Arnol′d, V.I.: *Mathematical Methods of Classical Mechanics*. Springer–Verlag 1980.

[3] Arnol′d, V.I.: Remarks on the perturbation theory for problems of Mathieu type. *Russ. Math. Surv.* **38** (1983), 215–233.

[4] Arnol′d, V.I.: *Geometrical Methods in the Theory of Ordnary Differential Equations*. Springer–Verlag 1983.

[5] Bröcker, Th. and Lander, L.: *Differentiable Germs and Catastrophes*. Cambridge Unversity Press, 1976.

[6] Broer, H.W., Hoveijn, I., Lunter, G.A. and Vegter, G.: Resonances in a Spring–Pendulum: algorithms for equivariant singularity theory, *Nonlinearity*, **11**, (1998), 1–37.

[7] Broer, H.W., Hoveijn, I. and Noort, M. van: A reversible bifurcation analysis of the inverted pendulum. *Physica D*, **112**, (1997), 50–63.

[8] Broer, H.W., Hoveijn, I., Noort, M. van Vegter, G.: The inverted pendulum: a singularity theory approach, submitted for publication.

[9] Broer, H.W. and Levi, M.: Geometrical aspects of stability theory for Hill's equations. *Arch. Rat. Mech. An.* **131** (1995), 225-240.

[10] Broer, H.W., Lunter, G.A. and Vegter, G.: Equivariant singularity theory with distinguished parameters: two case–studies of resonant Hamiltonian systems. *Physica D*, **112**, (1997), 64–80.

[11] Broer, H.W. and Simó, C.: Hill's equation with quasi–periodic forcing: resonance tongues, instability pockets and global phenomena. To appear *Bol. Soc. Bras. Mat.*

[12] Broer, H.W. and Simó, C.: The Ince equation, infinitely many instability pockets. In preparation.

[13] Broer, H.W. and Vegter, G.: Bifurcational aspects of parametric resonance. *Dynamics Reported, New Series* **1** (1992), 1–51.

[14] Giorgilli, A. and Galgani, L.: Formal integrals for an autonomous Hamiltonian system near an equilibrium point. *Celest. Mech.* **17** (1978), 267–280.

[15] Hale, J.K.: *Oscillations in Nonlinear Systems*. McGraw–Hill 1963; Dover 1992.

[16] Hochstadt, H.: *Differential equations: a modern approach*. Holt, Rinehart and Winston 1964.

[17] Levy, D.M. and Keller, J.B.: Instability intervals of Hill's equation. *Comm. Pure Appl. Math.* **16** (1963), 469–479.

[18] Olvera, A. and Simó, C.: Normal forms close to invariant circles of twist maps. In C. Alsina, J. Llibre, Ch. Mira, C. Simó, G. Targonski, and R. Thibault, editors, *European Conference on Iteration Theory (ECIT 87)*, 438–443, World Scientific, Singapore, 1989.

[19] Olvera, A. and Simó, C.: Normal forms of twist maps in a resonant annulus. Preprint, 1998.

[20] Van der Pol, B. and Strutt, M.J.O.: On the stability of the solutions of Mathieu's equation. *The London, Edinburgh and Dublin Phil. Mag. 7th series* **5** (1928), 18–38.

[21] Magnus W. and Winkler S.: *Hill's Equation.* Wiley 1966; Dover 1979.

[22] Rychlik, M: Renormalization of cocycles and linear ODE with almost periodic coefficients. *Invent. Math.* **110** (1992), 173–206.

[23] Stoker, J.J., *Nonlinear Vibrations in Mechanical and Electrical Systems.* Interscience 1950; Wiley 1992.

A The Giorgilli–Galgani averaging method

We here describe the normalizing (averaging) transformation leading to the normal form theorem 3 in detail [14]. The search is for the Taylor–Fourier series of a generating function $G = G_1 + G_2 + \dots$, where the subscript refers to the order, in this case in (α, β). This series will be found inductively, starting with (9)

$$H_{0,0} \equiv H_0 \text{ and } H_{1,0} \equiv H_1.$$

Let us now recurrently compute the functions $H_{j,m}$, $j = 0, 1$ and $m \geq 1$ by

$$H_{j,m} = \sum_{\ell=1}^{m} \frac{\ell}{m} [G_\ell, H_{j,m-\ell}],$$

where $[\ ,\ ]$ denotes the Poisson bracket $[f, g] = \frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial q} + \frac{\partial f}{\partial \varphi}\frac{\partial g}{\partial J} - \frac{\partial f}{\partial J}\frac{\partial g}{\partial \varphi}$. The Hamiltonian then is transformed to $\overline{H} = \sum_m \overline{H}_m$, where $\overline{H}_m = H_{0,m} + H_{1,m-1}$ gathers all the terms of degree m.

At this stage we do not yet know the terms G_ℓ. Suppose then, that $G_1, G_2, \dots, G_{m-1}$ are already known. Then the expression $H_{1,m-1}$ can be obtained completely and also $H_{0,m}$, except for the term $[G_m, H_{0,0}]$. We can write

$$M_m + [G_m, H_{0,0}] = \overline{H}_m,$$

where the M_m are known from the previous normalisations. More concretely

$$M_m = H_{1,m-1} + \sum_{\ell=1}^{m-1} \frac{\ell}{m} [G_\ell, H_{0,m-\ell}].$$

To solve this *homological* equation for G_m we just note that, if G_m contains terms of the form $q^r p^s z^n$, then

$$[q^r p^s z^n, H_{0,0}] = i\left(\frac{m}{2}(r-s)+n\right) q^r p^s z^n.$$

Therefore, all terms in M_m can be cancelled by a suitable choice of the corresponding terms in G_m, and hence do not show up in $\overline{H}_m$, except if $\frac{m}{2}(r-s)+n=0$. To determine G_1 we take $M_1 = H_{1,0}$, which is quadratic in (q,p). Since none of the $G_\ell, H_{j,m}$, except $H_{0,0}$, depend on J, all brackets in effect are brackets in (q,p) only. Therefore all terms are quadratic in (q,p). Returning to the $q^r p^s z^n$–term, the possibilities for (r,s) are $(2,0)$, $(1,1)$ and $(0,2)$. Hence only resonances appear when (r,s,n) takes one of the values $(2,0,-m)$, $(1,1,0)$ or $(0,2,m)$. This directly gives the format of theorem 3.

B The map $\mathcal{N}$ up to order four in (α,β)

We here display the result of the formula manipulator mentioned in §2.2.

B.1 The case $k=1$

$$\sigma_1 = \alpha_0 - \alpha_0^2 - \frac{1}{8}\alpha_1^2 + \frac{1}{6}\alpha_2^2 + \frac{1}{16}\alpha_3^2 + \frac{1}{6}\beta_2^2 + \frac{1}{16}\beta_3^2$$
$$+2\alpha_0^3 + \frac{1}{8}\alpha_0\alpha_1^2 - \frac{1}{9}\alpha_0\alpha_2^2 - \frac{3}{32}\alpha_0\alpha_3^2 - \frac{1}{9}\alpha_0\beta_2^2 - \frac{3}{32}\alpha_0\beta_3^2 - \frac{11}{48}\alpha_1^2\alpha_2 + \frac{1}{72}\alpha_1\alpha_2\alpha_3 + \frac{1}{72}\alpha_1\beta_2\beta_3$$
$$-5\alpha_0^4 + \frac{1}{4}\alpha_0^2\alpha_1^2 + \frac{23}{27}\alpha_0^2\alpha_2^2 + \frac{21}{64}\alpha_0^2\alpha_3^2 + \frac{23}{27}\alpha_0^2\beta_2^2 + \frac{21}{64}\alpha_0^2\beta_3^2 + \frac{17}{72}\alpha_0\alpha_1^2\alpha_2 + \frac{199}{2592}\alpha_0\alpha_1\alpha_2\alpha_3$$
$$+\frac{199}{2592}\alpha_0\alpha_1\beta_2\beta_3 + \frac{49}{384}\alpha_1^4 - \frac{239}{3456}\alpha_1^3\alpha_3 - \frac{815}{5184}\alpha_1^2\alpha_2^2 - \frac{61}{17280}\alpha_1^2\alpha_3^2 - \frac{815}{5184}\alpha_1^2\beta_2^2$$
$$-\frac{61}{17280}\alpha_1^2\beta_3^2 - \frac{253}{4320}\alpha_1\alpha_2^2\alpha_3 - \frac{253}{2160}\alpha_1\alpha_2\beta_2\beta_3 + \frac{253}{4320}\alpha_1\alpha_3\beta_2^2 - \frac{1}{1080}\alpha_2^4 - \frac{1}{432}\alpha_2^2\alpha_3^2 - \frac{1}{540}\alpha_2^2\beta_2^2$$
$$-\frac{1}{432}\alpha_2^2\beta_3^2 - \frac{87}{35840}\alpha_3^4 - \frac{1}{432}\alpha_3^2\beta_2^2 - \frac{87}{17920}\alpha_3^2\beta_3^2 - \frac{1}{1080}\beta_2^4 - \frac{1}{432}\beta_2^2\beta_3^2 - \frac{87}{35840}\beta_3^4,$$

$$\sigma_2 = \frac{1}{4}\alpha_1 - \frac{1}{2}\alpha_0\alpha_1 + \frac{1}{8}\alpha_1\alpha_2 + \frac{1}{24}\alpha_2\alpha_3 + \frac{1}{24}\beta_2\beta_3$$
$$+\alpha_0^2\alpha_1 - \frac{3}{16}\alpha_0\alpha_1\alpha_2 - \frac{11}{144}\alpha_0\alpha_2\alpha_3 - \frac{11}{144}\alpha_0\beta_2\beta_3 - \frac{7}{64}\alpha_1^3 + \frac{5}{192}\alpha_1^2\alpha_3 - \frac{5}{144}\alpha_1\alpha_2^2$$
$$-\frac{55}{2304}\alpha_1\alpha_3^2 - \frac{5}{144}\alpha_1\beta_2^2 - \frac{55}{2304}\alpha_1\beta_3^2 + \frac{1}{48}\alpha_2^2\alpha_3 + \frac{1}{24}\alpha_2\beta_2\beta_3 - \frac{1}{48}\alpha_3\beta_2^2$$
$$-\frac{5}{2}\alpha_0^3\alpha_1 + \frac{9}{16}\alpha_0^2\alpha_1\alpha_2 + \frac{245}{1296}\alpha_0^2\alpha_2\alpha_3 + \frac{245}{1296}\alpha_0^2\beta_2\beta_3 + \frac{17}{32}\alpha_0\alpha_1^3 - \frac{41}{864}\alpha_0\alpha_1^2\alpha_3$$
$$+\frac{107}{648}\alpha_0\alpha_1\alpha_2^2 + \frac{19}{216}\alpha_0\alpha_1\alpha_3^2 + \frac{107}{648}\alpha_0\alpha_1\beta_2^2 + \frac{19}{216}\alpha_0\alpha_1\beta_3^2 - \frac{5}{864}\alpha_0\alpha_2^2\alpha_3 - \frac{5}{432}\alpha_0\alpha_2\beta_2\beta_3$$
$$+\frac{5}{864}\alpha_0\alpha_3\beta_2^2 - \frac{149}{1152}\alpha_1^3\alpha_2 - \frac{431}{6912}\alpha_1^2\alpha_2\alpha_3 - \frac{431}{6912}\alpha_1^2\beta_2\beta_3 - \frac{1}{48}\alpha_1\alpha_2^3 + \frac{13}{23040}\alpha_1\alpha_2\alpha_3^2$$

$$-\frac{1}{48}\alpha_1\alpha_2\beta_2^2+\frac{13}{23040}\alpha_1\alpha_2\beta_3^2-\frac{31}{4320}\alpha_2^3\alpha_3-\frac{31}{4320}\alpha_2^2\beta_2\beta_3-\frac{289}{69120}\alpha_2\alpha_3^3-\frac{31}{4320}\alpha_2\alpha_3\beta_2^2$$
$$-\frac{289}{69120}\alpha_2\alpha_3\beta_3^2-\frac{289}{69120}\alpha_3^2\beta_2\beta_3-\frac{31}{4320}\beta_2^3\beta_3-\frac{289}{69120}\beta_2\beta_3^3,$$

$$\rho_2=\frac{1}{8}\alpha_1\beta_2+\frac{1}{24}\alpha_2\beta_3-\frac{1}{24}\alpha_3\beta_2$$
$$-\frac{3}{16}\alpha_0\alpha_1\beta_2-\frac{11}{144}\alpha_0\alpha_2\beta_3+\frac{11}{144}\alpha_0\alpha_3\beta_2+\frac{5}{192}\alpha_1^2\beta_3-\frac{1}{48}\alpha_2^2\beta_3+\frac{1}{24}\alpha_2\alpha_3\beta_2+\frac{1}{48}\beta_2^2\beta_3$$
$$+\frac{9}{16}\alpha_0^2\alpha_1\beta_2+\frac{245}{1296}\alpha_0^2\alpha_2\beta_3-\frac{245}{1296}\alpha_0^2\alpha_3\beta_2-\frac{41}{864}\alpha_0\alpha_1^2\beta_3+\frac{5}{864}\alpha_0\alpha_2^2\beta_3-\frac{5}{432}\alpha_0\alpha_2\alpha_3\beta_2$$
$$-\frac{5}{864}\alpha_0\beta_2^2\beta_3-\frac{311}{3456}\alpha_1^3\beta_2-\frac{119}{20736}\alpha_1^2\alpha_2\beta_3+\frac{119}{20736}\alpha_1^2\alpha_3\beta_2-\frac{1}{48}\alpha_1\alpha_2^2\beta_2+\frac{13}{23040}\alpha_1\alpha_3^2\beta_2$$
$$-\frac{1}{48}\alpha_1\beta_2^3+\frac{13}{23040}\alpha_1\beta_2\beta_3^2-\frac{31}{4320}\alpha_2^3\beta_3+\frac{31}{4320}\alpha_2^2\alpha_3\beta_2-\frac{289}{69120}\alpha_2\alpha_3^2\beta_3-\frac{31}{4320}\alpha_2\beta_2^2\beta_3$$
$$-\frac{289}{69120}\alpha_2\beta_3^3+\frac{289}{69120}\alpha_3^3\beta_2+\frac{31}{4320}\alpha_3\beta_2^3+\frac{289}{69120}\alpha_3\beta_2\beta_3^2.$$

B.2 The case $k=2$

$$\sigma_1=\frac{1}{2}\alpha_0-\frac{1}{8}\alpha_0^2-\frac{1}{12}\alpha_1^2-\frac{1}{64}\alpha_2^2+\frac{1}{20}\alpha_3^2-\frac{1}{64}\beta_2^2+\frac{1}{20}\beta_3^2$$
$$+\frac{1}{16}\alpha_0^3+\frac{11}{72}\alpha_0\alpha_1^2+\frac{1}{256}\alpha_0\alpha_2^2+\frac{3}{200}\alpha_0\alpha_3^2+\frac{1}{256}\alpha_0\beta_2^2+\frac{3}{200}\alpha_0\beta_3^2-\frac{1}{192}\alpha_1^2\alpha_2$$
$$-\frac{107}{1440}\alpha_1\alpha_2\alpha_3-\frac{107}{1440}\alpha_1\beta_2\beta_3$$
$$-\frac{5}{128}\alpha_0^4-\frac{203}{864}\alpha_0^2\alpha_1^2+\frac{1}{512}\alpha_0^2\alpha_2^2+\frac{123}{4000}\alpha_0^2\alpha_3^2+\frac{1}{512}\alpha_0^2\beta_2^2+\frac{123}{4000}\alpha_0^2\beta_3^2+\frac{73}{2304}\alpha_0\alpha_1^2\alpha_2$$
$$+\frac{10237}{259200}\alpha_0\alpha_1\alpha_2\alpha_3+\frac{10237}{259200}\alpha_0\alpha_1\beta_2\beta_3-\frac{43}{6912}\alpha_1^4+\frac{23}{2880}\alpha_1^3\alpha_3+\frac{7117}{207360}\alpha_1^2\alpha_2^2-\frac{43}{43200}\alpha_1^2\alpha_3^2$$
$$+\frac{7117}{207360}\alpha_1^2\beta_2^2-\frac{43}{43200}\alpha_1^2\beta_3^2-\frac{7}{6912}\alpha_1\alpha_2^2\alpha_3-\frac{7}{3456}\alpha_1\alpha_2\beta_2\beta_3+\frac{7}{6912}\alpha_1\alpha_3\beta_2^2+\frac{49}{49152}\alpha_2^4$$
$$-\frac{23911}{12096000}\alpha_2^2\alpha_3^2+\frac{49}{24576}\alpha_2^2\beta_2^2-\frac{23911}{12096000}\alpha_2^2\beta_3^2+\frac{93}{64000}\alpha_3^4-\frac{23911}{12096000}\alpha_3^2\beta_2^2+\frac{93}{32000}\alpha_3^2\beta_3^2$$
$$+\frac{49}{49152}\beta_2^4-\frac{23911}{12096000}\beta_2^2\beta_3^2+\frac{93}{64000}\beta_3^4,$$

$$\sigma_2=\frac{1}{8}\alpha_2-\frac{1}{16}\alpha_0\alpha_2-\frac{1}{16}\alpha_1^2+\frac{1}{24}\alpha_1\alpha_3$$
$$+\frac{1}{32}\alpha_0^2\alpha_2+\frac{3}{32}\alpha_0\alpha_1^2-\frac{1}{144}\alpha_0\alpha_1\alpha_3-\frac{5}{288}\alpha_1^2\alpha_2-\frac{5}{192}\alpha_1\alpha_2\alpha_3-\frac{5}{192}\alpha_1\beta_2\beta_3-\frac{7}{2048}\alpha_2^3$$
$$+\frac{11}{7200}\alpha_2\alpha_3^2-\frac{7}{2048}\alpha_2\beta_2^2+\frac{11}{7200}\alpha_2\beta_3^2$$

$$-\frac{5}{256}\alpha_0^3\alpha_2 - \frac{15}{128}\alpha_0^2\alpha_1^2 + \frac{77}{5184}\alpha_0^2\alpha_1\alpha_3 + \frac{29}{10368}\alpha_0\alpha_1^2\alpha_2 + \frac{109}{3456}\alpha_0\alpha_1\alpha_2\alpha_3 + \frac{109}{3456}\alpha_0\alpha_1\beta_2\beta_3$$
$$+\frac{17}{4096}\alpha_0\alpha_2^3 + \frac{1579}{432000}\alpha_0\alpha_2\alpha_3^2 + \frac{17}{4096}\alpha_0\alpha_2\beta_2^2 + \frac{1579}{432000}\alpha_0\alpha_2\beta_3^2 - \frac{1}{576}\alpha_1^4 + \frac{1}{3456}\alpha_1^3\alpha_3$$
$$+\frac{853}{36864}\alpha_1^2\alpha_2^2 + \frac{17}{7200}\alpha_1^2\alpha_3^2 + \frac{1847}{110592}\alpha_1^2\beta_2^2 + \frac{17}{7200}\alpha_1^2\beta_3^2 - \frac{13357}{1382400}\alpha_1\alpha_2^2\alpha_3 - \frac{2593}{518400}\alpha_1\alpha_2\beta_2\beta_3$$
$$+\frac{19}{86400}\alpha_1\alpha_3^3 - \frac{19327}{4147200}\alpha_1\alpha_3\beta_2^2 + \frac{19}{86400}\alpha_1\alpha_3\beta_3^2 - \frac{119}{36864}\alpha_2^2\alpha_3^2 + \frac{119}{36864}\alpha_2^2\beta_3^2 - \frac{119}{9216}\alpha_2\alpha_3\beta_2\beta_3$$
$$+\frac{119}{36864}\alpha_3^2\beta_2^2 - \frac{119}{36864}\beta_2^2\beta_3^2,$$

$$\rho_2 = \frac{1}{8}\beta_2 - \frac{1}{16}\alpha_0\beta_2 + \frac{1}{24}\alpha_1\beta_3$$
$$+\frac{1}{32}\alpha_0^2\beta_2 - \frac{1}{144}\alpha_0\alpha_1\beta_3 - \frac{5}{288}\alpha_1^2\beta_2 - \frac{5}{192}\alpha_1\alpha_2\beta_3 + \frac{5}{192}\alpha_1\alpha_3\beta_2 - \frac{7}{2048}\alpha_2^2\beta_2 + \frac{11}{7200}\alpha_3^2\beta_2$$
$$-\frac{7}{2048}\beta_2^3 + \frac{11}{7200}\beta_2\beta_3^2$$
$$-\frac{5}{256}\alpha_0^3\beta_2 + \frac{77}{5184}\alpha_0^2\alpha_1\beta_3 + \frac{29}{10368}\alpha_0\alpha_1^2\beta_2 + \frac{109}{3456}\alpha_0\alpha_1\alpha_2\beta_3 - \frac{109}{3456}\alpha_0\alpha_1\alpha_3\beta_2 + \frac{17}{4096}\alpha_0\alpha_2^2\beta_2$$
$$+\frac{1579}{432000}\alpha_0\alpha_3^2\beta_2 + \frac{17}{4096}\alpha_0\beta_2^3 + \frac{1579}{432000}\alpha_0\beta_2\beta_3^2 + \frac{1}{3456}\alpha_1^3\beta_3 + \frac{89}{13824}\alpha_1^2\alpha_2\beta_2 - \frac{19327}{4147200}\alpha_1\alpha_2^2\beta_3$$
$$-\frac{2593}{518400}\alpha_1\alpha_2\alpha_3\beta_2 + \frac{19}{86400}\alpha_1\alpha_3^2\beta_3 - \frac{13357}{1382400}\alpha_1\beta_2^2\beta_3 + \frac{19}{86400}\alpha_1\beta_3^3 - \frac{119}{18432}\alpha_2^2\alpha_3\beta_3$$
$$+\frac{119}{18432}\alpha_2\alpha_3^2\beta_2 - \frac{119}{18432}\alpha_2\beta_2\beta_3^2 + \frac{119}{18432}\alpha_3\beta_2^2\beta_3.$$

B.3 The case $k = 3$

$$\sigma_1 = \frac{1}{3}\alpha_0 - \frac{1}{27}\alpha_0^2 - \frac{1}{48}\alpha_1^2 - \frac{1}{30}\alpha_2^2 - \frac{1}{216}\alpha_3^2 - \frac{1}{30}\beta_2^2 - \frac{1}{216}\beta_3^2$$
$$+\frac{2}{243}\alpha_0^3 + \frac{13}{864}\alpha_0\alpha_1^2 + \frac{23}{675}\alpha_0\alpha_2^2 + \frac{1}{1944}\alpha_0\alpha_3^2 + \frac{23}{675}\alpha_0\beta_2^2 + \frac{1}{1944}\alpha_0\beta_3^2 + \frac{1}{80}\alpha_1^2\alpha_2$$
$$-\frac{1}{270}\alpha_1\alpha_2\alpha_3 - \frac{1}{270}\alpha_1\beta_2\beta_3$$
$$-\frac{5}{2187}\alpha_0^4 - \frac{47}{5184}\alpha_0^2\alpha_1^2 - \frac{301}{10125}\alpha_0^2\alpha_2^2 + \frac{1}{8748}\alpha_0^2\alpha_3^2 - \frac{301}{10125}\alpha_0^2\beta_2^2 + \frac{1}{8748}\alpha_0^2\beta_3^2 - \frac{137}{7200}\alpha_0\alpha_1^2\alpha_2$$
$$+\frac{4873}{583200}\alpha_0\alpha_1\alpha_2\alpha_3 + \frac{4873}{583200}\alpha_0\alpha_1\beta_2\beta_3 - \frac{227}{138240}\alpha_1^4 + \frac{1}{640}\alpha_1^3\alpha_3 - \frac{13}{10800}\alpha_1^2\alpha_2^2 + \frac{5377}{1632960}\alpha_1^2\alpha_3^2$$
$$-\frac{13}{10800}\alpha_1^2\beta_2^2 + \frac{5377}{1632960}\alpha_1^2\beta_3^2 + \frac{11}{2016}\alpha_1\alpha_2^2\alpha_3 + \frac{11}{1008}\alpha_1\alpha_2\beta_2\beta_3 - \frac{11}{2016}\alpha_1\alpha_3\beta_2^2 + \frac{83}{189000}\alpha_2^4$$
$$+\frac{27281}{11664000}\alpha_2^2\alpha_3^2 + \frac{83}{94500}\alpha_2^2\beta_2^2 + \frac{27281}{11664000}\alpha_2^2\beta_3^2 + \frac{49}{839808}\alpha_3^4 + \frac{27281}{11664000}\alpha_3^2\beta_2^2 + \frac{49}{419904}\alpha_3^2\beta_3^2$$
$$+\frac{83}{189000}\beta_2^4 + \frac{27281}{11664000}\beta_2^2\beta_3^2 + \frac{49}{839808}\beta_3^4,$$

$$\sigma_2 = \frac{1}{12}\alpha_3 - \frac{1}{54}\alpha_0\alpha_3 - \frac{1}{24}\alpha_1\alpha_2$$
$$+\frac{1}{243}\alpha_0^2\alpha_3 + \frac{13}{432}\alpha_0\alpha_1\alpha_2 + \frac{1}{192}\alpha_1^3 - \frac{55}{6912}\alpha_1^2\alpha_3 + \frac{11}{10800}\alpha_2^2\alpha_3 - \frac{7}{15552}\alpha_3^3 + \frac{11}{10800}\alpha_3\beta_2^2$$
$$-\frac{7}{15552}\alpha_3\beta_3^2$$
$$-\frac{5}{4374}\alpha_0^3\alpha_3 - \frac{217}{11664}\alpha_0^2\alpha_1\alpha_2 - \frac{5}{864}\alpha_0\alpha_1^3 + \frac{167}{46656}\alpha_0\alpha_1^2\alpha_3 - \frac{1}{288}\alpha_0\alpha_1\alpha_2^2 + \frac{1}{288}\alpha_0\alpha_1\beta_2^2$$
$$-\frac{2467}{729000}\alpha_0\alpha_2^2\alpha_3 + \frac{17}{69984}\alpha_0\alpha_3^3 - \frac{2467}{729000}\alpha_0\alpha_3\beta_2^2 + \frac{17}{69984}\alpha_0\alpha_3\beta_3^2 + \frac{1}{13824}\alpha_1^3\alpha_2$$
$$+\frac{181}{115200}\alpha_1^2\alpha_2\alpha_3 + \frac{41}{38400}\alpha_1^2\beta_2\beta_3 - \frac{11}{21600}\alpha_1\alpha_2^3 + \frac{2357}{518400}\alpha_1\alpha_2\alpha_3^2 - \frac{11}{21600}\alpha_1\alpha_2\beta_2^2$$
$$+\frac{13111}{4665600}\alpha_1\alpha_2\beta_3^2 + \frac{4051}{2332800}\alpha_1\alpha_3\beta_2\beta_3 + \frac{1}{864}\alpha_2^3\alpha_3 + \frac{1}{288}\alpha_2^2\beta_2\beta_3 - \frac{1}{288}\alpha_2\alpha_3\beta_2^2 - \frac{1}{864}\beta_2^3\beta_3,$$

$$\rho_2 = \frac{1}{12}\beta_3 - \frac{1}{54}\alpha_0\beta_3 - \frac{1}{24}\alpha_1\beta_2$$
$$+\frac{1}{243}\alpha_0^2\beta_3 + \frac{13}{432}\alpha_0\alpha_1\beta_2 - \frac{55}{6912}\alpha_1^2\beta_3 + \frac{11}{10800}\alpha_2^2\beta_3 - \frac{7}{15552}\alpha_3^2\beta_3 + \frac{11}{10800}\beta_2^2\beta_3 - \frac{7}{15552}\beta_3^3$$
$$-\frac{5}{4374}\alpha_0^3\beta_3 - \frac{217}{11664}\alpha_0^2\alpha_1\beta_2 + \frac{167}{46656}\alpha_0\alpha_1^2\beta_3 - \frac{1}{144}\alpha_0\alpha_1\alpha_2\beta_2 - \frac{2467}{729000}\alpha_0\alpha_2^2\beta_3 + \frac{17}{69984}\alpha_0\alpha_3^2\beta_3$$
$$-\frac{2467}{729000}\alpha_0\beta_2^2\beta_3 + \frac{17}{69984}\alpha_0\beta_3^3 + \frac{1}{13824}\alpha_1^3\beta_2 + \frac{181}{115200}\alpha_1^2\alpha_2\beta_3 - \frac{41}{38400}\alpha_1^2\alpha_3\beta_2 - \frac{11}{21600}\alpha_1\alpha_2^2\beta_2$$
$$+\frac{4051}{2332800}\alpha_1\alpha_2\alpha_3\beta_3 + \frac{13111}{4665600}\alpha_1\alpha_3^2\beta_2 - \frac{11}{21600}\alpha_1\beta_2^3 + \frac{2357}{518400}\alpha_1\beta_2\beta_3^2 - \frac{1}{864}\alpha_2^3\beta_3$$
$$+\frac{1}{288}\alpha_2^2\alpha_3\beta_2 + \frac{1}{288}\alpha_2\beta_2^2\beta_3 - \frac{1}{864}\alpha_3\beta_2^3.$$

C The averaging approach for low resonances

In this section we consider the near Mathieu case of equation (1). For the lower resonances $k \leq 6$ we directly inspect the tongue boundaries, particularly their intersection points, using the outcome of the averaging as presented in Appendix B. Apart from the checks this gives, it also provides important ideas for the proofs in §§3 and 4. We recall from (12) that the Floquet map $\mathcal{F}$ is given by

$$X = \sigma_1,\ Y = -2\sigma_2,\ Z = 2\rho_2,$$

and from (7) the scaling

$$\begin{aligned} \alpha_0 &= a - \left(\frac{k}{2}\right)^2, \\ \alpha_j &= bc_j, \\ \beta_j &= bs_j, \end{aligned}$$

for $j \geq 1$, recalling that $c_1 = 1$. We consider the stability diagram in the (α_0, b)–plane. Tongue boundaries meet whenever the Floquet map $\mathcal{F}$ covers the zero–matrix, which is expressed by the equations

$$\sigma_1 = \sigma_2 = \rho_2 = 0. \tag{14}$$

Note that by construction (parametric forcing) always $(\alpha_0, b) = (0, 0)$ is a solution of this. In the forthcoming subsections expressions like

$$\sum \text{monomials} = 0$$

mean that

$$\sum \text{monomials} \times (1 + O(\alpha_0, b, c_2, c_3, s_2, s_3)) = 0.$$

Recall that in the reversible case always $s = 0$ and $\rho_2 = 0$.

C.1 The case $k = 1$

Note that now $\alpha_0 = a - \frac{1}{4}$. Inspection of Appendix B, §B.1, reveals that the equations (14) here amount to

$$\alpha_0 = \frac{1}{8}b^2 \text{ and } b = 0.$$

This leads to the unique, isolated solution $(\alpha_0, b) = (0, 0)$, compatible with the transversality of the present tongue boundaries, cf. [9].

C.2 The case $k = 2$

Now $\alpha_0 = a - 1$. The equations (14) by §B.2 yield

$$\begin{aligned}
\sigma_1 = 0 \quad &\Leftrightarrow \quad \alpha_0 = \frac{1}{6}b^2, \\
\sigma_2 = 0 \quad &\Leftrightarrow \quad \frac{1}{8}c_2 - \frac{1}{16}b = 0 \text{ or } b = 0, \\
\rho_2 = 0 \quad &\Leftrightarrow \quad \frac{1}{8}s_2 + \frac{1}{24}bs_3 = 0 \text{ or } b = 0,
\end{aligned}$$

leading to the cases given by $b = 0$ and $b = 2c_2$, $3s_2 = -bs_3$. This again fits with [9].

Remark. Perturbing away from the reversible case $s = 0$, we find the following. The familiar quadratic tangency of the tongue boundaries occurs for $c_2 = 0$. Then, for $c_2 \neq 0$ one instability pocket arises. This can be kept for a suitable choice of s, but generically will open, compare figure 5. For a singularity theory treatment, see [9].

C.3 The case $k = 3$

Here $\alpha_0 = a - \frac{9}{4}$, while (14) by §B.3 gives

$$\begin{aligned}
\sigma_1 = 0 \quad &\Leftrightarrow \quad 16\alpha_0 = b^2, \\
\sigma_2 = 0 \quad &\Leftrightarrow \quad 16c_3 - 8bc_2 + b^2 = 0 \text{ or } b = 0, \\
\rho_2 = 0 \quad &\Leftrightarrow \quad 2s_3 - bs_2 = 0 \text{ or } b = 0,
\end{aligned}$$

leading to the 3 cases $b = 0$ or $b = 4(c_2 \pm \sqrt{c_2^2 - c_3})$, $s_3 = bs_2/2$. Of course, some solutions may coincide or dissapear into the complex plane. This (partly) explains the complexity of figures 2, 3 and 6. Indeed, in the classical Mathieu case, corresponding to $c_3 = 0$, the tongue boundaries have third order contact. Cf. figure 1. For $c_3 < c_2^2$ this leads to 2 instability pockets. The transversal intermediate crossing becomes tangent for $c_3 = c_2^2$. check

C.4 The reversible cases $k = 4, k = 5$ and $k = 6$

Presently we restrict to the reversible case, where $\rho_2 = 0$. In the equations (14) the dominant terms in σ_1 for $k = 4, 5$ and 6 give, respectively,

$$30\alpha_0 = b^2, \ 48\alpha_0 = b^2 \text{ and } 70\alpha_0 = b^2.$$

For σ_2 things are more interesting: we obtain for $k = 4, 5$ and 6, respectively,

$$\frac{1}{16}c_4 b - \left(\frac{1}{48}c_3 + \frac{1}{128}c_2^2\right) b^2 + \frac{5}{1152}c_2 b^3 - \frac{1}{4608}b^4 = 0,$$

$$\frac{1}{20}c_5 b - \left(\frac{1}{80}c_4 + \frac{1}{120}c_2 c_3\right) b^2 + \left(\frac{7}{3840}c_3 + \frac{1}{720}c_2^2\right) b^3 - \frac{1}{4608}c_2 b^4 + \frac{1}{184320}b^5 = 0$$

and

$$\frac{1}{24}c_6 b - \left(\frac{1}{120}c_5 + \frac{1}{192}c_2 c_4 + \frac{1}{432}c_3^2\right) b^2 + \left(\frac{3}{3200}c_4 + \frac{11}{8640}c_2 c_3 + \frac{1}{6144}c_2^3\right) b^3$$
$$- \left(\frac{7}{86400}c_3 + \frac{259}{2764800}c_2^2\right) b^4 + \frac{7}{1105920}c_2 b^5 - \frac{1}{11059200}b^6 = 0.$$

As should be, one of the solutions is given by $b = 0$. Other solutions near $b = 0$ are found by the scaling (3)

$$\begin{aligned} b &= \delta B \\ c_j &= \delta^{j-1} d_j, \end{aligned}$$

for all relevant values of j, where δ is taken small. Indeed, this scaling was suggested to us by these considerations. The new equations now read

$$\frac{1}{16}d_4 - \left(\frac{1}{48}d_3 + \frac{1}{128}d_2^2\right) B + \frac{5}{1152}d_2 B^2 - \frac{1}{4608}B^3 = 0,$$

$$\frac{1}{20}d_5 - \left(\frac{1}{80}d_4 + \frac{1}{120}d_2 d_3\right) B + \left(\frac{7}{3840}d_3 + \frac{1}{720}d_2^2\right) B^2 - \frac{1}{4608}d_2 B^3 + \frac{1}{184320}B^4 = 0,$$

and

$$\frac{1}{24}d_6 - \left(\frac{1}{120}d_5 + \frac{1}{192}d_2 d_4 + \frac{1}{432}d_3^2\right) B + \left(\frac{3}{3200}d_4 + \frac{11}{8640}d_2 d_3 + \frac{1}{6144}d_2^3\right) B^2$$
$$- \left(\frac{7}{86400}d_3 + \frac{259}{2764800}d_2^2\right) B^3 + \frac{7}{1105920}d_2 B^4 - \frac{1}{11059200}B^5 = 0.$$

The form of these equations is such that we can prescribe any polynomial in B of degree $k-1$ and, then recurrently determine the d_j from d_2 to d_k to produce this polynomial. In particular we can obtain $k-1$ instability pockets or multiple fixed points. Compare the complexity predicted by the cuspoid normal form theorem 1 and the remarks following this. Also see the proof of §3.

D Disconnected tongues do not exist

This appendix deals with the near–Mathieu equation

$$\ddot{x} + (a + b\,p(t) + \varepsilon q(t))\,x = 0. \tag{15}$$

The functions p and q are 2π–periodic in t, while p is even. The 'perturbation' q, however, is not necessarily even. Consider a parameter point (a, b), with $b \neq 0$ where the boundaries of a given resonance tongue meet for $\varepsilon = 0$. Note that by changing $\bar{\varepsilon} = \varepsilon/b$, we may rewrite $b\,p(t) + \varepsilon q(t) = b(p(t) + \bar{\varepsilon} q(t))$. The univalence of Hill's map implies that $\mathcal{H}_{\varepsilon=0}(a, b) = \pm\mathrm{Id}$. For simplicity we here restrict to the plus–sign, now studying the local behaviour of the tongue boundaries for $|\varepsilon| \ll 1$. In particular we address the general question whether the perturbation can change the crossing into a disconnected tongue. The local situation for small $|b|$ was treated in the main text with a negative answer. In fact for our proof of this we do not need p to be even, only that the crossing is transverse, for a sketch of the situation see figure 8.

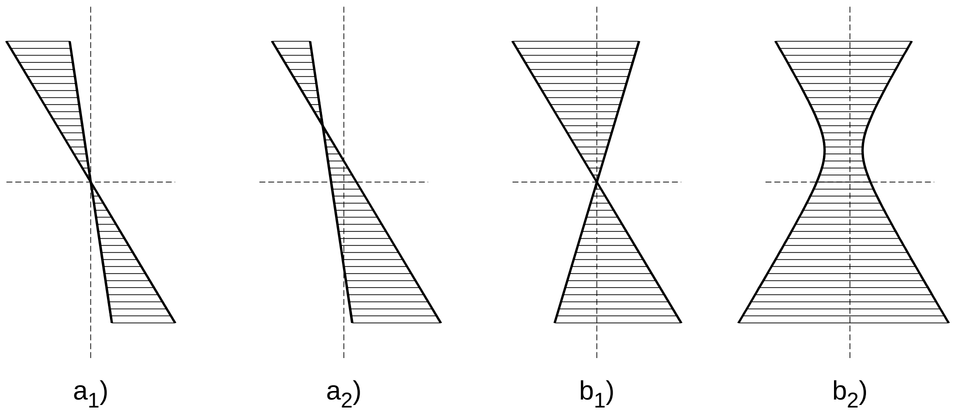

Figure 8: A resonance tongue near a transversal crossing: Several perturbation scenarios, where shading means unstability. In the cases a) q is even; in the cases b) q is general. The cases a1) and b1) are unperturbed ($\varepsilon = 0$). For case a) we fixed c_1, c_2, c_4, c_5, c_6 positive, c_3 negative and the remaining c's zero. For case b) we chose c_1, c_2, c_4, c_6, c_9 positive, c_3, c_5 negative and the remaining c's zero. See the text for details.

Proposition 12 (No disconnectedness of tongues) *Assume that the equation (15) for $\varepsilon = 0$ has a resonance tongue with a* transversal *crossing at (a, b), $b \neq 0$. Then for any perturbation function q and $|\varepsilon|$ sufficiently small, the resonance tongue of (15) near (a, b) remains connected.*

Proof. Consider the matrix equivalent

$$M' = A(t)M \tag{16}$$

of (15), where

$$M = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \text{ and } A(t) = \begin{pmatrix} 0 & 1 \\ a_{2,1}(t) & 0 \end{pmatrix},$$

with $a_{2,1}(t) = -(a + bp(t) + \varepsilon q(t))$. Taking all parameters into account we consider a solution $M_{a,b;\varepsilon}(t)$ such that $M_{a,b;\varepsilon}(0) = \mathrm{Id}$. Then $M_{a,b;\varepsilon}(2\pi) = \mathcal{H}_\varepsilon(a,b)$. The assumption that the tongue boundaries for $\varepsilon = 0$ cross at (a,b) therefore means that $M_{a,b;0}(2\pi) = \mathrm{Id}$. We now turn to the trace $\mathrm{Tr}\,(M_{a+\Delta a,b+\Delta b;\varepsilon}(2\pi))$, for small values of $\Delta a, \Delta b$ and ε. For simplicity δ denotes any of these small quantities. Let M_δ satisfy the variational equation

$$M_\delta' = A_\delta M + AM_\delta$$

of (16). After some computations, where it is convenient to use a splitting $M_\delta = MN_\delta$, it follows that

$$\begin{aligned}\mathrm{Tr}\,(M_{a+\Delta a,b+\Delta b;\varepsilon}(2\pi)) &= 2 - (c_1\Delta a + c_2\Delta b + c_3\varepsilon)(c_4\Delta a + c_5\Delta b + c_6\varepsilon) \\ &\quad + (c_7\Delta a + c_8\Delta b + c_9\varepsilon)^2 + O_3(\Delta a, \Delta b, \varepsilon), \qquad (17)\end{aligned}$$

where

$$\begin{array}{lll} c_1 = \int_0^{2\pi} x_1^2\,dt\,, & c_2 = \int_0^{2\pi} p\,x_1^2\,dt\,, & c_3 = \int_0^{2\pi} q\,x_1^2\,dt\,, \\ c_4 = \int_0^{2\pi} x_3^2\,dt\,, & c_5 = \int_0^{2\pi} p\,x_3^2\,dt\,, & c_6 = \int_0^{2\pi} q\,x_3^2\,dt\,, \\ c_7 = \int_0^{2\pi} x_1x_3\,dt\,, & c_8 = \int_0^{2\pi} p\,x_1x_3\,dt\,, & c_9 = \int_0^{2\pi} q\,x_1x_3\,dt. \end{array}$$

Note that always $c_1 > 0$ and $c_4 > 0$, while e.g. for p and q both even one has $c_7 = c_8 = c_9 = 0$. To see this it is enough to recall that x_1, x_4 are even and x_2, x_3 are odd, in case p is even. The assertion of the proposition now follows by inspection of the formula (17). Main observation is that by positivity of c_1 and c_3 the directions of the tongue boundaries at (a,b) have a nonzero vertical component. This is persistent for small ε. □

Remarks.

1. For generic q the transverse crossing disappears and the tongue locally changes into an open domain of unstability with smooth (analytic) boundaries. See figure 8, case d).
2. What can one say when the crossing for $\varepsilon = 0$ is nontransverse? The case where both p and q are even is completely similar to the general theory as presented in the main text, which implies that the tongue remains connected.

   For more general p and q, the crossing usually will disappear as in figure 8, case d).